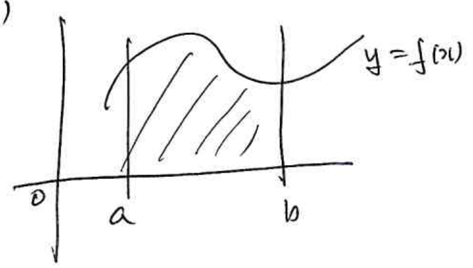
— 정적분의 응용 —

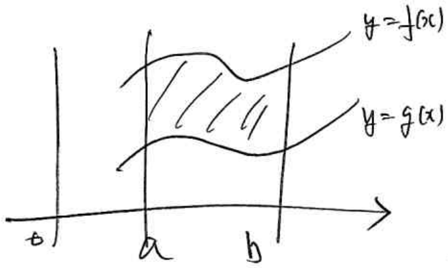
1. 평면 도형의 면적.

(1)



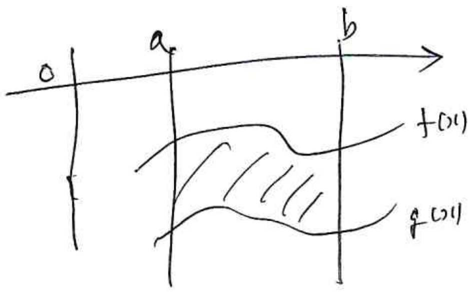
$$S = \int_a^b f(x)\,dx = \int_a^b y\,dx.$$

(2)



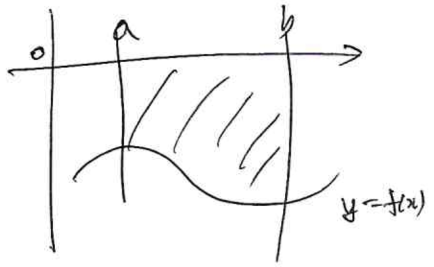
$$S = \int_a^b (f(x) - g(x))\,dx.$$

(3)



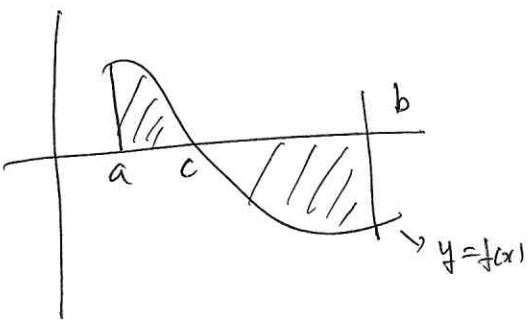
$$S = \int_a^b (f(x) - g(x))\,dx.$$

(4)



$$S = -\int_a^b f(x)\,dx.$$

(5)



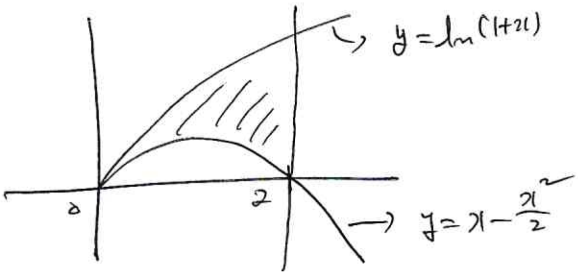
$$S = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx.$$

exam $y=\sin x$ $(0\leq x\leq \pi)$ 와 x축으로 둘러싸인 면적.

<sol> $0\leq x\leq\pi$ 에서 $\sin x\geq 0$.

$$\therefore\ S=\int_0^{\pi}\sin x\,dx=\Big[-\cos x\Big]_0^{\pi}=-\cos\pi+\cos 0=2.$$

exam $y=x-\dfrac{x^2}{2}$, $y=\ln(1+x)$, $x=2$ 로 둘러싸인 면적.



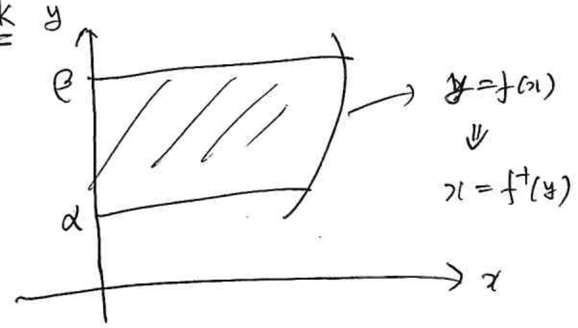
$$S=\int_0^2\left(\ln(1+x)-\left(x-\frac{x^2}{2}\right)\right)dx.$$

$$\int\ln(1+x)\,dx=(1+x)\ln(1+x)-\int(1+x)\cdot\frac{1}{1+x}\,dx=(1+x)\ln(1+x)-x.$$

$$=\left[(1+x)\ln(1+x)-x-\frac{x^2}{2}+\frac{x^3}{6}\right]_0^2$$

$$=3\ln 3-2-\frac{4}{2}+\frac{8}{6}=3\ln 3-\frac{8}{3}$$

Rmk

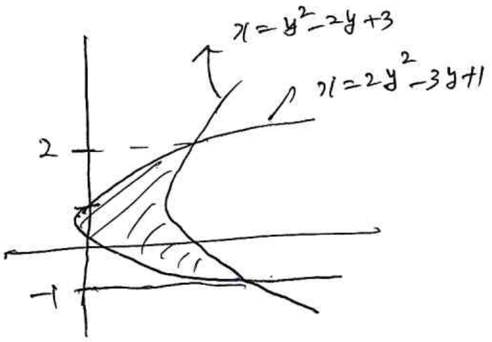


$$S=\int_\alpha^\beta x\,dy=\int_\alpha^\beta f^{-1}(y)\,dy.$$

exam. $2y^2-3y=x-1$, $y^2-2y=x-3$ 으로 둘러싸인 부분의 면적.

<sol> $x=2y^2-3y+1$, $x=y^2-2y+3$ 에서 교점을 구한다.

$2y^2-3y+1=y^2-2y+3$ $\quad y^2-y-2=(y-2)(y+1)=0$ $\quad\therefore y=2,\ y=-1$



$$S=\int_{-1}^{2}\left\{(y^2-2y+3)-(2y^2-3y+1)\right\}dy.$$

$$=\int_{-1}^{2}(-y^2+y+2)\,dy$$

$$=\left[-\frac{y^3}{3}+\frac{y^2}{2}+2y\right]_{-1}^{2}=\frac{9}{2}.$$

— 매개변수 함수의 적분법 —

(1) $x = g(t)$, $y = f(t)$. $x = a$, $x = b$.

$$S = \int_a^b y\, dx. \qquad dx = g'(t)\, dt. \qquad t = g^{-1}(x).$$

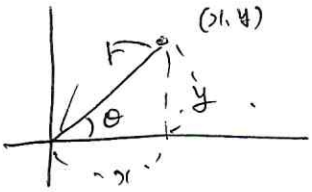
$$= \int_{g^{-1}(a)}^{g^{-1}(b)} f(t)\, g'(t)\, dt.$$

(2) $x = g(t)$, $y = f(t)$, $y = a$, $y = b$.

$$S = \int_a^b x\, dy. \qquad dy = f'(t)\, dt \qquad t = f^{-1}(y).$$

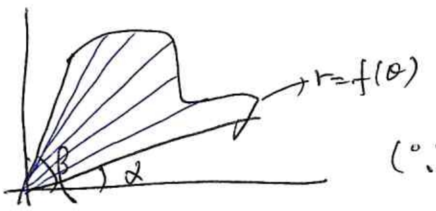
$$= \int_{f^{-1}(a)}^{f^{-1}(b)} g(t)\, f'(t)\, dt.$$

— 극좌표에 의한 면적 구하는 법 —



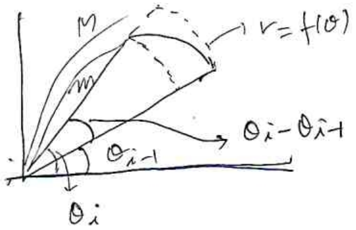
$x = r\cos\theta$, $y = r\sin\theta$.

Thm $r = f(\theta)$ 와 $\theta = \alpha$, $\theta = \beta$ 에 의해 둘러싸인 면적.
(θ에 관해 연속) ($\alpha < \beta$)



$$S = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta.$$

($\because$) $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$.



$r = f(\theta)$: conti. on $[\theta_{i-1}, \theta_i]$

$\forall c_i \in [\theta_{i-1}, \theta_i]$ $i = 1, 2, \cdots, n$.

$$M = \max_{c_i \in [\theta_{i-1}, \theta_i]} f(c_i)$$

$$m = \min_{c_i \in [\theta_{i-1}, \theta_i]} f(c_i)$$

부채꼴의 면적.

$\pi r^2 \times \frac{\Delta\theta_i}{2\pi} = \frac{1}{2} r^2 \Delta\theta_i$

$\Delta\theta_i = \theta_i - \theta_{i-1}$ $i = 1, 2, \cdots, n$

$$\frac{1}{2} m_i^2 \Delta\theta_i \le \frac{1}{2} f(c_i)^2 \Delta\theta_i \le \frac{1}{2} M_i^2 \Delta\theta_i$$

$$\sum_{i=1}^{n}\frac{1}{2}m_i^2\Delta\theta_i \le \sum_{i=1}^{n}\frac{1}{2}f(c_i)^2\Delta\theta_i \le \sum_{i=1}^{n}\frac{1}{2}M_i^2\Delta\theta_i$$

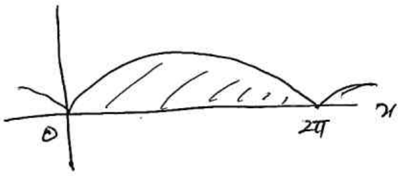
put $\Delta\theta = \max\{\Delta\theta_i \mid i=1,2,\cdots,n\}$

$$\underbrace{\lim_{\Delta\theta\to0}\sum_{i=1}^{n}\frac{1}{2}m_i^2\Delta\theta_i}_{s} \le \lim_{\Delta\theta\to0}\sum_{i=1}^{n}\frac{1}{2}f(c_i)^2\Delta\theta_i \le \underbrace{\lim_{\Delta\theta\to0}\sum_{i=1}^{n}\frac{1}{2}M_i^2\Delta\theta_i}_{S}$$

$s=S \Rightarrow S = \exists\lim_{\Delta\theta\to0}\sum_{i=1}^{n}\frac{1}{2}f(c_i)^2\Delta\theta_i$

$$= \int_\alpha^\beta \frac{1}{2}f(\theta)^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta.$$; 면적공식.

<u>exm</u> cycloid $\begin{cases} x = a(t-\sin t) & (a>0) \\ y = a(1-\cos t), & 0\le t\le 2\pi. \end{cases}$



$S = \int_0^{2\pi} y\,dx.$ $\quad dx = a(1-\cos t)dt$

$x=0 \Rightarrow t=0$

$x=2\pi \Rightarrow t=2\pi.$

$= \int_0^{2\pi} a(1-\cos t)\,a(1-\cos t)\,dt.$

$= a^2\int_0^{2\pi}(1-\cos t)^2 dt = a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt.$

$= a^2\int_0^{2\pi}\left(1-2\cos t+\frac{1+\cos 2t}{2}\right)dt.$

$= a^2\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 6\pi a^2$

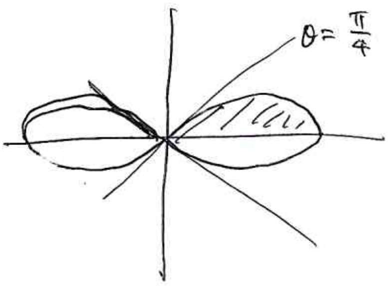
exm. 연주형 $(x^2+y^2)^2 = x^2-y^2$ 인 곡선이 만드는 부분의 면적.

<sol> $x = r\cos\theta$ $y = r\sin\theta$ 를 대입.

$$(r^2\cos^2\theta + r^2\sin^2\theta)^2 = r^2\cos^2\theta - r^2\sin^2\theta.$$

$$r^4 = r^2(\cos^2\theta - \sin^2\theta)$$

$r>0$ $\quad\therefore r^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta.$

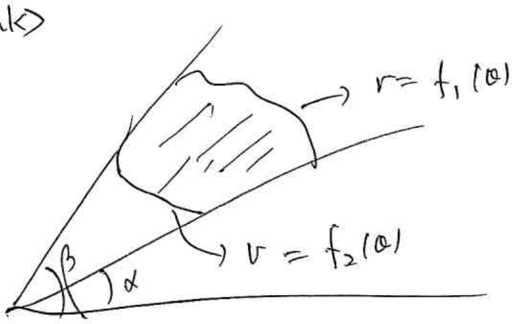


$$S = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2 d\theta.$$

$$= 2\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta = 2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

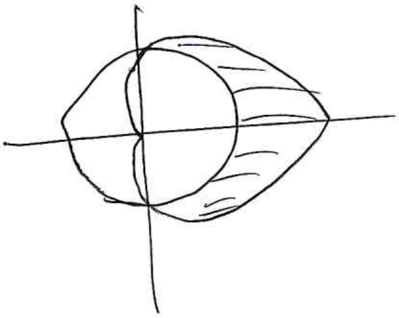
$= 1$

<Remark>



$$S = \frac{1}{2}\int_{\alpha}^{\beta} \left(f_1(\theta)^2 - f_2(\theta)^2 \right) d\theta.$$

exm 심장형 $r = a(1+\cos\theta)$ 의 내부와 원 $r=a$ 의 외부와의 공통부분 면적 ?



$$S = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(a^2(1+\cos\theta)^2 - a^2 \right) d\theta.$$

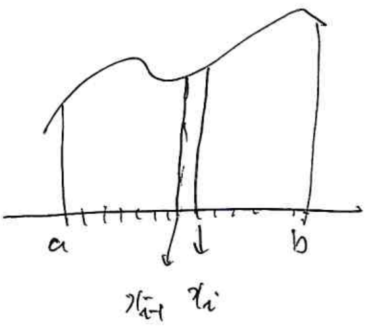
$$= a^2 \int_0^{\frac{\pi}{2}} \left(\cancel{1} + 2\cos\theta + \cos^2\theta - \cancel{1} \right) d\theta$$

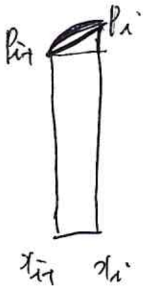
$$= a^2 \int_0^{\frac{\pi}{2}} \left(2\cos\theta + \frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= a^2 \left[+2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{2}}$$

$$= a^2 \left(2 + \frac{\pi}{4} \right)$$

2. 곡선의 길이.





$$\overline{P_{i-1}P_i} = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)^2}\ (x_i - x_{i-1})$$

평균값 정리에 의해.

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i) \qquad x_{i-1} < c_i < x_i$$

$$\therefore \sum_{i=1}^{n} \overline{P_{i-1}P_i} = \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\ \Delta x_i$$

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\ \Delta x_i = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

$$\underset{\text{곡선 길이}}{L} = \int_a^b \sqrt{1 + y'^2}\ dx$$

<Rmk>

(1) $x=f(t)$, $y=g(t)$ $\quad (t_1 \le t \le t_2)$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{g'(t)}{f'(t)} \qquad dx=f'(t)\,dt.$$

$$L=\int_a^b \sqrt{1+(y')^2}\,dx = \int_{t_1}^{t_2}\sqrt{1+\left(\frac{g'(t)}{f'(t)}\right)^2}\cdot f'(t)\,dt$$

$$=\int_{t_1}^{t_2}\sqrt{f'(t)^2+g'(t)^2}\,dt$$

$$=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

Exam $x=a\cos t$ $\quad y=a\sin t$ 의 원주를 구하라.

$$dx=-a\sin t\,dt \qquad \frac{dy}{dt}=a\cos t.$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=a.$$

$$\therefore\quad L=4\int_0^{\frac{\pi}{2}} a\,dt=4\left[at\right]_0^{\frac{\pi}{2}}=2\pi a.$$

(2) $r=f(\theta)$, $\quad (\alpha\le\theta\le\beta)$ 인 경우.

$x=r\cos\theta$, $\quad y=r\sin\theta$ $\qquad$ (여기서 주의 할 점 r도 θ 함수이다.)

$$\frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta$$

$$\frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta.$$

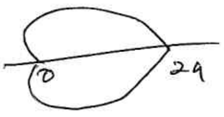
$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left(\frac{dr}{d\theta}\right)^2\cos^2\theta-2r\frac{dr}{d\theta}\sin\theta\cos\theta+r^2\sin^2\theta$$

$$+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta+2r\frac{dr}{d\theta}\sin\theta\cos\theta+r^2\cos^2\theta$$

$$=\left(\frac{dr}{d\theta}\right)^2+r^2$$

$$\therefore\quad L=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta. \quad \text{(정의)}$$

exm. $r = a(1+\cos\theta)$ $(a>0)$ 의 전체 길이

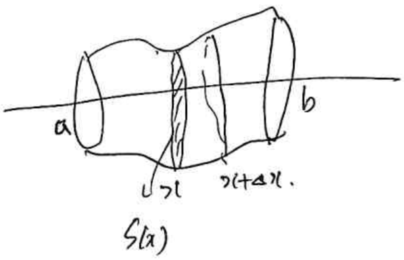


$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}$$
$$= a\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$$
$$= a\sqrt{2(1+\cos\theta)}$$
$$= a\sqrt{2\cdot 2\cos^2\frac{\theta}{2}} = 2a\cos\frac{\theta}{2}$$

$$\therefore\ L = 2\int_0^{\pi} 2a\cos\frac{\theta}{2}\,d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

3. 입체의 체적.

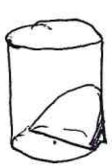


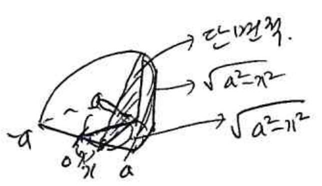
$S(x)$: 단면적.

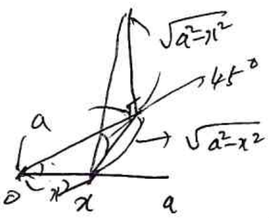
$[x, x+\Delta x]$ 에서 최대값 M
최소값 m.

$$m\Delta x \le \Delta V \le M\Delta x.$$
$$\Rightarrow\ m \le \frac{\Delta V}{\Delta x} \le M.$$
$$\Delta x \to 0 \ \Rightarrow\ m, M \longrightarrow S(x).$$
$$\therefore\ \lim_{\Delta x\to 0}\frac{\Delta V}{\Delta x} = S(x)$$
$$\therefore\ V = \int_a^b \underbrace{S(x)}_{\text{단면적}}\,dx.$$

exm 밑면의 반지름이 a인 직원기둥을 밑면의 지름을 지나고 밑면과 45° 이루는 평면으로 잘라서 되는 입체의 체적 ?





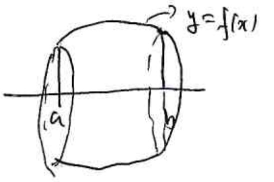


$$S(x) = \frac{1}{2}\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}$$
$$= \frac{1}{2}(a^2-x^2)$$

$$\therefore\ V = 2\int_0^a \frac{1}{2}(a^2-x^2)\,dx$$
$$= \left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{2}{3}a^3$$

< Rmk > (1) $y=f(x)$; conti. on $[a,b]$

$y=f(x)$ 와 $x=a$, $x=b$ 및 x축으로 둘러싸인 부분을
x축 둘레로 1회전시켜 생기는 회전체의 체적.

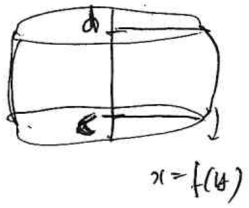


$S(x)$는 원 면적.

$\therefore \quad S(x) = \pi f(x)^2.$

$$\therefore \quad V = \int_a^b \pi f(x)^2 dx = \int_a^b \pi y^2 dx.$$

(2) $x=f(y)$: conti. $y=c$, $y=d$ 및 y축으로 회전하여
생기는 회전체의 체적.



$S(y) = \pi f(y)^2$

$$\therefore \quad V = \int_c^d \pi f(y)^2 dy = \int_c^d \pi x^2 dy.$$

(3) ① $x=f(t)$ $y=g(t)$ 일 때 x축 둘레로 회전하여 생기는 체적.

$V = \int_a^b \pi y^2 dx.$ 에서 $t=f^{-1}(a)$, $t=f^{-1}(b)$
$dx = f'(t)dt.$

$$= \int_{f^{-1}(a)}^{f^{-1}(b)} \pi g(t)^2 \cdot f'(t) dt.$$

② $x=f(t)$, $y=g(t)$ 인 때 y축 둘레로 회전하여 생기는 체적.

$V = \int_a^b \pi x^2 dy$ $t=g^{-1}(a)$ $t=g^{-1}(b)$
$dy = g'(t) dt$

$$= \int_{g^{-1}(a)}^{g^{-1}(b)} \pi f(t)^2 \cdot g'(t) dt.$$

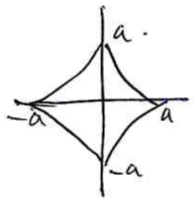
(4) $r=f(\theta)$ 를 x축 방향.

$x = r\cos\theta$ $y = r\sin\theta$
$dx = (-r\sin\theta)d\theta.$

$V = \int_a^b \pi y^2 dx$

$$= \int_\alpha^\beta \pi \, r\sin\theta \cdot (-r\sin\theta) d\theta.$$

Exam. 성망형 $x = a\cos^3 t$, $y = a\sin^3 t$ $(a>0)$ y축 둘레로 회전한 체적.



$$V = \pi \int_{-a}^{a} x^2\, dy. \qquad dy = 3a\sin^2 t \cos t\, dt$$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos^6 t \cdot 3a \sin^2 t \cos t\, dt.$$

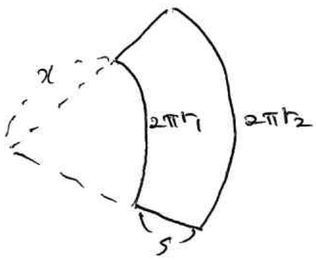
$$= 2\pi \cdot 3a^3 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^7 t\, dt. \qquad 2m-1=7 \quad m=4 \qquad 2n-1=2 \quad n=\frac{3}{2}$$

$$= 3a^3\pi\, B\left(4, \frac{3}{2}\right)$$

$$= 3a^3\pi \frac{\Gamma(4)\Gamma(\frac{3}{2})}{\Gamma(4+\frac{3}{2})} = 3a^3\pi \frac{3! \cdot \frac{1}{2} \cancel{\Gamma(\frac{1}{2})}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cancel{\Gamma(\frac{1}{2})}}$$

$$= \frac{32}{105}\pi a^3$$

4. 회전면의 면적 (회전체의 표면적).



$$2\pi r_2 : 2\pi r_1 = x+s : x.$$

$$2\pi r_2 \cdot x = 2\pi r_1 (x+s)$$

$$2\pi (r_2 - r_1) x = 2\pi r_1 s.$$

$$\therefore\ x = \frac{r_1 s}{r_2 - r_1}$$

$$x + s = \frac{r_2 s}{r_2 - r_1}$$

부채꼴 면적의 차이

$$\frac{\pi r_2^2 s}{r_2 - r_1} - \frac{\pi r_1^2 s}{r_2 - r_1} = \pi(r_1 + r_2)s = 2\pi\left(\frac{r_1 + r_2}{2}\right)s$$

$$a = x_0 < x_1 < \cdots < x_n = b.$$

$[x_{i-1}, x_i]$ 에서. $r_1 = f(x_{i-1})$, $r_2 = f(x_i)$ s는 $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ 를 연결한 직선.이라 하면

$$\frac{r_1 + r_2}{2} = \frac{f(x_i) + f(x_{i-1})}{2} = f(c_i), \quad x_{i-1} < c_i < x_i \quad (\text{중간값정리})$$

s는 곡선의 길이에서 만든 $\sqrt{1 + f'(d_i)^2}\,\Delta x_i$

겉면적 $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi f(c_i) \sqrt{1+f'(d_i)^2}\, \Delta x_i$

$= \int_a^b 2\pi f(x) \sqrt{1+f'(x)^2}\, dx.$

<u>Rmk</u> (1) $x = f(t)$, $y = g(t)$ 인 매개변수 방정식인 경우. $(t_1 \le t \le t_2)$

$$S = 2\pi \int_{t_1}^{t_2} g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(2) $r = f(\theta)$, $(\alpha \le \theta \le \beta)$ 인 경우.

$x = r\cos\theta$
$y = r\sin\theta$.

$$S = 2\pi \int_\alpha^\beta \underbrace{r\sin\theta}_{y} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

<u>exm</u> (1) 반지름이 a 인 공의 표면적을 구하라.

$x^2+y^2=a^2$ (원의 방정식) $\qquad y = \sqrt{a^2-x^2}$ (상반원)

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}} \qquad 1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$

$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{a}{\sqrt{a^2-x^2}}$

$S = 2\pi \int_{-a}^{a} \cancel{\sqrt{a^2-x^2}} \cdot \frac{a}{\cancel{\sqrt{a^2-x^2}}}\, dx.$

$= 2\pi \left[ax\right]_{-a}^{a} = 2\pi(a^2+a^2) = 4\pi a^2.$

exm. 심장형 $r = a(1+\cos\theta)$ 를 회전하여 생긴 회전체의 표면적을 구하라.

$\frac{dr}{d\theta} = -a\sin\theta. \qquad \left(\frac{dr}{d\theta}\right)^2 + r^2 = a^2\sin^2\theta + a^2(1+\cos\theta)^2$

$= a^2(\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta)$

$= a^2(2(1+\cos\theta)) = a^2(2 \cdot 2 \cdot \cos^2\frac{\theta}{2})$

$\therefore \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} = 2a\cos\frac{\theta}{2}.$

$$S = 2\pi \int_0^{\pi} r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$$= 2\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta \; 2a\cos\frac{\theta}{2}\, d\theta.$$

$$= 4\pi a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2} \; 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta.$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta.$$

$$= 16\pi a^2 \left[-\frac{2}{5}\cos^5\frac{\theta}{2}\right]_0^{\pi}$$

$$= \frac{32}{5}\pi a^2.$$