— 정적분 —

강의 순서

1. 정적분의 정의
2. 정적분에 관한 기본정리
3. 이상적분
4. Gamma ft.
5. Beta ft.

— 정적분의 응용

1. 평면도형의 면적.
2. 곡선의 길이
3. 입체의 체적.
4. 회전체의 표면적

1. 정적분의 정의.

$f(x)$: conti. on $[a, b]$

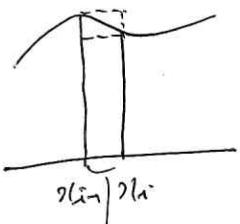
$a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

put $\Delta x_i = x_i - x_{i-1}$

Take $c_i \in [x_{i-1}, x_i]$, $i = 1, 2, \cdots, n$.

$f(x)$: conti. on $[x_{i-1}, x_i]$

$\exists \; M_i = \max_{x \in [x_{i-1}, x_i]} f(x)$, $m_i = \min_{x \in [x_{i-1}, x_i]} f(x)$ $\quad i = 1, 2, \cdots, n$.



$|x_i - x_{i-1}| = \Delta x_i$

$$m_i \le f(c_i) \le M_i$$

$$m_i \Delta x_i \le f(c_i)\Delta x_i \le M_i \Delta x_i$$

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(c_i)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i$$

put $\Delta x = \max\{\Delta x_i \mid i = 1, 2, \cdots, n\}$

$$\underbrace{\lim_{\Delta x \to 0} \sum_{i=1}^{n} m_i \Delta x_i}_{= s} \le \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \le \underbrace{\lim_{\Delta x \to 0} \sum_{i=1}^{n} M_i \Delta x_i}_{= S}$$

If $s = S$

$$\Rightarrow \exists \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$$

; $f(x)$ 의 $[a,b]$ 에서의 정적분.

(*) (1) $f(x)$ 가 연속이면 $s = S$ 이다.

(2) 연속이 아닌 경우에도 정적분을 정의 할수 있다.

이 경우 최대치, 최소값 대신 상한과 하한을 이용한다.

2. 정적분에 관한 기본정리.

Thm (1) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a<c<b).$

(2) $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx. \quad$ (k: 상수)

(3) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Thm $f(x) \le g(x)$ on $[a,b]$

$$\Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Coro $g(x) \ge 0$ on $[a,b]$

$$\Rightarrow \int_a^b g(x)\,dx \ge 0.$$

Thm $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx. \quad (a<b)$

Thm (적분의 평균값 정리)

$f(x)$: conti. on $[a,b]$

$$\Rightarrow \exists\, c \in [a,b] \text{ s.t. } \int_a^b f(x)\,dx = f(c)(b-a).$$

<pf> since $f(x)$: conti. on $[a,b]$,

$\exists\ M = \max_{x\in[a,b]} f(x) \qquad m = \min_{x\in[a,b]} f(x)$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

$$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

중간값 정리에 의해.

$\exists\ c \in [a,b]$ s.t. $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$

$$\therefore \quad \int_a^b f(x)\,dx = f(c)(b-a)$$

Thm $f(x)$: cont. on $[a,b]$

$F(x) = \int_a^x f(t)\,dt. \qquad (a \le x \le b)$

$$\Rightarrow \quad \frac{d}{dx}F(x) = f(x).$$

Thm. $G(x)$: $f(x)$ 의 원시함수 $\left(\int f(x)\,dx = G(x) \right.$

$$\Rightarrow \quad \int_a^b f(x)\,dx = G(b) - G(a) = [G(x)]_a^b$$

exm $\int_0^{\frac{\pi}{2}} \sin x\,dx. = [-\cos x]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 1$

exm. $S_n = \frac{1}{n} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+(n-1)^2}$

$\Rightarrow \lim_{n\to\infty} S_n.$

<sol> $S_n = \frac{1}{n}\left\{ \frac{1}{1+0^2} + \frac{1}{1+(\frac{1}{n})^2} + \frac{1}{1+(\frac{2}{n})^2} + \cdots + \frac{1}{1+(\frac{n-1}{n})^2} \right\}.$

$\Rightarrow f(x) = \frac{1}{1+x^2}$ 구간: $[0,1]$ $\qquad \Delta x_i = \frac{1}{n}$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$$

$\tan^{-1} 1 = x \qquad 1 = \tan x \qquad \therefore x = \frac{\pi}{4} \qquad \tan^{-1} 0 = x \Rightarrow 0 = \tan x.\ x=0.$

3. 이상적분.

(1) $f(x)$ 가 c 를 제외한 $[a,b]$ 에서 연속인 함수.

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \left\{ \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right\}$$

의 극한이 존재할 때.

exm. $I = \int_0^1 \frac{1}{\sqrt{x}}\,dx$. $f(x)$ 가 0 에서 불연속.

$$\lim_{c \to 0} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c \to 0} \left[2\sqrt{x}\right]_c^1$$

$$= \lim_{c \to 0} (2 - 2\sqrt{c}) = 2.$$

$$\therefore \quad I = \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

(2) 적분 영역이 ∞ 일 때.

① $\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$

② $\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$

③ $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(x)\,dx$.

exm $\int_0^{\infty} \sin x\,dx = \lim_{a \to \infty} \int_0^a \sin x\,dx$

$$= \lim_{a \to \infty} \left[-\cos x\right]_0^a = \lim_{a \to \infty} (-\cos a + \cos 0)$$

$\nexists \lim_{a \to \infty} \cos a$ $\therefore$ 적분값이 존재하지않는다.

4. Gamma ft.

Define $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n>0)$

Rmk (1) $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$

$$= \underbrace{\left[-e^{-x}x^n\right]_0^\infty}_{=0} + n\int_0^\infty e^{-x} x^{n-1}\, dx = n\Gamma(n)$$

(2) $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1$

(*) n : 양의 정수

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2\cdot 1\cdot \Gamma(1) = n!$$

exm $n-1=7 \Rightarrow n=8$

$$I = \int_0^\infty e^{-x} x^7\, dx = \Gamma(8) = 7!$$

(*) n : 임의 양의 정수.

$$\Gamma(n) = (n-1)\Gamma(n-1) = \cdots = (n-1)\cdots(n-N)\Gamma(n-N).$$

$1 \le n-N \le 2$ Γ함수의 수표범위.

exm $n-1 = \frac{7}{2} \Rightarrow n = \frac{9}{2}$

$$I = \int_0^\infty e^{-x} x^{\frac{7}{2}}\, dx = \Gamma\left(\tfrac{9}{2}\right) = \tfrac{7}{2}\Gamma\left(\tfrac{7}{2}\right) = \tfrac{7}{2}\cdot\tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right)$$

$$= \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma\left(\tfrac{1}{2}\right)$$

(*) n : 음의 수 (정수가 아닌 경우)

$\Rightarrow \exists\, N$: 양의 정수 s.t. $1 \le n+N \le 2$.

$$\Gamma(n+N) = (n+N-1)\Gamma(n+N-1) = (n+N-1)(n+N-2)\cdots n\,\Gamma(n)$$

$$\therefore \Gamma(n) = \frac{\Gamma(n+N)}{(n+N-1)(n+N-2)\cdots n}$$

ex

$$I=\int_0^\infty e^{-x}x^{-\frac{3}{2}}\,dx. \qquad n-1=-\frac{3}{2} \Rightarrow n=-\frac{1}{2}$$

$$=\Gamma\left(-\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

Rmk (1) $n=0 \Rightarrow$ 정의 되지 않는다.

(2) n ; 음의 정수

$$\Gamma(-n)=\frac{\Gamma(0)}{(-n)(-n+1)\cdots(-2)(-1)}=(-1)^n\frac{\Gamma(0)}{n!}=\pm\infty$$

— $\Gamma\left(\frac{1}{2}\right)$ 값 —

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-x}x^{-\frac{1}{2}}\,dx=\sqrt{\pi}$$ (계산과정이 중적분을 사용해야 하므로 결과만 사용)

ex

(1) $$I=\int_0^\infty e^{-x}x^{\frac{9}{2}}\,dx \qquad n-1=\frac{9}{2}=\frac{11}{2}$$

$$=\Gamma\left(\frac{11}{2}\right)=\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

(2) $$I=\int_0^\infty e^{-x^4}\,dx.$$

put $x=y^{\frac{1}{4}}$ $\quad dx=\frac{1}{4}y^{-\frac{3}{4}}\,dy.$

$$\therefore I=\int_0^\infty e^{-y}\cdot\frac{1}{4}y^{-\frac{3}{4}}\,dy.=\frac{1}{4}\int_0^\infty e^{-y}y^{-\frac{3}{4}}\,dy \qquad n-1=-\frac{3}{4},\ n=\frac{1}{4}$$

$$=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

(3) $$I=\int_0^\infty 4x^4e^{-x^4}\,dx.$$

put $x=y^{\frac{1}{4}}$ $\quad dx=\frac{1}{4}y^{-\frac{3}{4}}\,dy.$

$$I=\int_0^\infty 4\cdot y\cdot e^{-y}\cdot\frac{1}{4}y^{-\frac{3}{4}}\,dy$$

$$=\int_0^\infty e^{-y}y^{\frac{1}{4}}\,dy \qquad n-1=\frac{1}{4} \quad n=\frac{5}{4}$$

$$=\Gamma\left(\frac{5}{4}\right)=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

5. Beta ft.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx. \qquad , m, n > 0$$

put $x = 1-y$ $\quad dx = -dy$

$$B(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1} (-1)\, dy.$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy = B(n,m).$$

put $x = \sin^2\theta$ $\quad dx = 2\sin\theta\cos\theta\, d\theta.$

$$\therefore\ B(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta.$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

— Beta ft 와 Gamma ft 의 관계 —

계산 과정에 중적분을 사용해야하므로 결과만 소개.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Rmk $\quad B(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta.$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \frac{1}{2}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

put $2m-1 = r$ $\quad 2n-1 = 0$ $\quad$ i.e. $m = \frac{r+1}{2}$ $\quad n = \frac{1}{2}$

$$\int_0^{\frac{\pi}{2}} \cos^r\theta\, d\theta = \frac{\Gamma(\frac{r+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{r}{2}+1)} = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2}+1)} \cdot \frac{\sqrt{\pi}}{2} \qquad (r > -1)$$

put $2m-1 = 0$, $\quad 2n-1 = r$

$$\int_0^{\frac{\pi}{2}} \sin^r\theta\, d\theta = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2}+1)} \cdot \frac{\sqrt{\pi}}{2} \qquad (r > -1)$$

예제

$$I = \int_0^1 \frac{1}{\sqrt{1-x^{\frac{1}{4}}}}\,dx.$$

put $\quad y = x^{\frac{1}{4}} \qquad x = y^4 \qquad dx = 4y^3\,dy.$

$$I = \int_0^1 (1-y)^{-\frac{1}{2}} \cdot 4y^3\,dy$$

$$= 4\int_0^1 y^3 (1-y)^{-\frac{1}{2}}\,dy \qquad m-1=3, \quad n-1=-\frac{1}{2}$$

$$m=4 \qquad n=\frac{1}{2}$$

$$= 4\,B\left(4, \frac{1}{2}\right) = 4 \cdot \frac{\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})}$$

$$= 4 \cdot \frac{3!\,\Gamma(\frac{1}{2})}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$= \frac{128}{35}$$

<문제> 1. $I = \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx$ 답. $\frac{1}{4} \cdot \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$

2. $I = \int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx.$ 답. $\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)}$

3. $I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos\theta}}\,d\theta.$ 답. $\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$