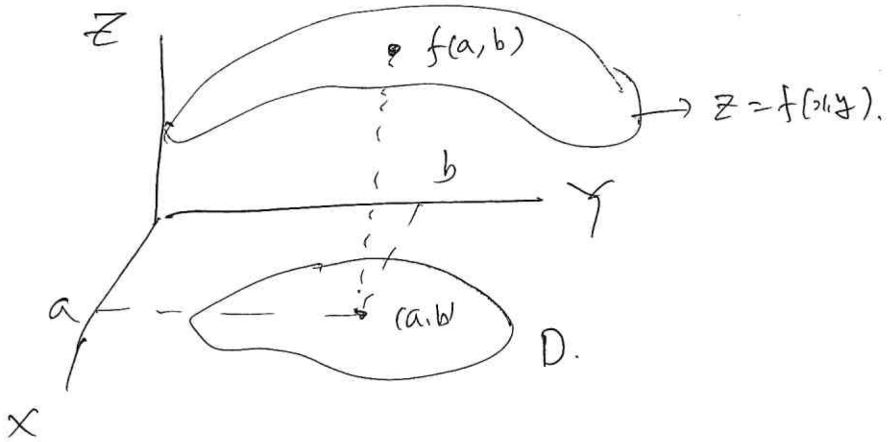
— 편미분법 — (교과서 12장)

1. 이변수 함수의 극한과 연속.

$f: R \times R \longrightarrow R$.

$D(f) \subset R \times R$, $R(f) \subset R$

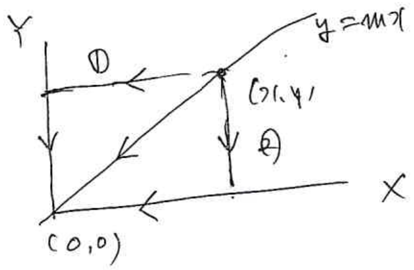


<Def> $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = C$ or $\lim\limits_{(x,y) \to (a,b)} f(x,y) = C$.

$\Leftrightarrow \forall_{\varepsilon > 0}, \exists \delta > 0$ s.t. $|x-a| < \delta$, $|y-b| < \delta$.

$|f(x,y) - C| < \varepsilon$.

exm $(x,y) \longrightarrow (0,0)$



① $x \to 0$, $y \to 0$

② $y \to 0$, $x \to 0$

③ $y = mx$ 에 따라 $(x,y) \to (0,0)$

위의 세가지 경우 극한값이 같을 때.

exm $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ $(x,y) \longrightarrow (0,0)$ i.e. $\lim\limits_{(x,y) \to (0,0)} f(x) = ?$

① $\lim\limits_{y \to 0} \left(\lim\limits_{x \to 0} \dfrac{x^2 - y^2}{x^2 + y^2} \right) = \lim\limits_{y \to 0} \dfrac{-y^2}{y^2} = -1$.

② $\lim\limits_{x \to 0} \left(\lim\limits_{y \to 0} \dfrac{x^2 - y^2}{x^2 + y^2} \right) = \lim\limits_{x \to 0} \dfrac{x^2}{x^2} = 1$

③ $y = mx$ 를 대입. $\dfrac{x^2 - m^2 x^2}{x^2 + m^2 x^2} = \dfrac{1 - m^2}{1 + m^2}$ m 값에 따라 극한값이 달라지므로 수렴하지 않는다.

<Def> $z=f(x,y)$: conti. at $(a,b)\in D$.

$\Leftrightarrow$ ① $\exists\, f(a,b)$: 정의 되어 있다.

② $\exists \lim_{(x,y)\to(a,b)} f(x,y)$

③ $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$\Leftrightarrow$ $\forall \varepsilon>0$, $\exists\, \delta>0$ s.t. $|x-a|<\delta$, $|y-b|<\delta$,

$|f(x,y)-f(a,b)|<\varepsilon$.

exm (1) $f(x,y)=x^2+y^2$ 은 연속인가.

($\because$) $\forall a,b\in R$, $f(a,b)=a^2+b^2$

$\lim_{\substack{x\to a\\ y\to b}} f(x,y)=a^2+b^2=f(a,b)$

$\therefore$ $f(x,y)$는 R^2에서 연속이다.

(2) $f(x,y)=\begin{cases} \dfrac{x+y-1}{x-y+1}, & (x-y+1\neq 0) \\ 0 & (x=0,\ y=1) \end{cases}$

($\because$)

$f(x,y)$ 는 $x-y+1\neq 0$ i.e. $x\neq y-1$ 인 모든점에서 연속.

$f(0,1)=0$ $\quad \lim_{\substack{x\to 0\\ y\to 1}} f(x,y)$. ¤

$\lim_{x\to 0}\left(\lim_{y\to 1} \dfrac{x+y-1}{x-y+1}\right)=\lim_{x\to 0} \dfrac{x}{x}=1$

$\lim_{y\to 1}\left(\lim_{x\to 0} \dfrac{x+y-1}{x-y+1}\right)=\lim_{y\to 1} \dfrac{y-1}{-y+1}=-1$

$\therefore$ 극한값이 존재하지 않는다.

$\therefore$ $f(x,y)$ 는 $(0,1)$ 에서 불연속이다.

<u>Thm</u> $f(x,y)$, $g(x,y)$: conti. on D.

$\Rightarrow$ $f+g$, $f\cdot g$, f/g $(g\neq 0)$, $\max\{f,g\}$, $\min\{f,g\}$

: conti. on D.

<u>Thm</u> $f(x,y)$: conti. on D.

D : bounded closed set

$\Rightarrow$ $\exists \max\limits_{(x,y)\in D} f(x,y)$

$\exists \min\limits_{(x,y)\in D} f(x,y)$

— 편도함수 —

$f(x,y)$; D 상의 함수.

$(a,b)\in D$.

① $\exists \lim\limits_{\Delta x\to 0} \dfrac{f(a+\Delta x,b)-f(a,b)}{\Delta x}$ y 값은 고정하고 x 값만 움직일 때.

$= \dfrac{\partial f(a,b)}{\partial x} = f_x(a,b)$: f의 (a,b) 에서의 x에 관한 <u>편미분계수</u>.

② $\exists \lim\limits_{\Delta y\to 0} \dfrac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$ x 값은 고정하고 y 값만 움직일 때.

$= \dfrac{\partial f(a,b)}{\partial y} = f_y(a,b)$: f의 (a,b) 에서의 y에 관한 편미분계수.

<u>Rmk</u> $\exists f_x(a,b)$, $f_y(a,b)$ $\Rightarrow$ (a,b) 에서 편미분가능하다

$\forall (x,y)\in D$,

$\exists \lim\limits_{\Delta x\to 0} \dfrac{f(x+\Delta x,y)-f(x,y)}{\Delta x} = f_x(x,y) = \dfrac{\partial f(x,y)}{\partial x}$: x에 관한 편도함수.

$\exists \lim\limits_{\Delta y\to 0} \dfrac{f(x,y+\Delta y)-f(x,y)}{\Delta y} = f_y(x,y) = \dfrac{\partial f(x,y)}{\partial y}$: y에 관한 편도함수.

; $f(x,y)$ 에 관한 1차 편도함수.

exam (1) $f(x,y) = \tan^{-1}\dfrac{y}{x} \Rightarrow f_x,\ f_y,\ f_x(4,-3),\ f_y(4,-3)$

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{-y}{x^2}$$

$$= \frac{-y}{x^2+y^2} \qquad (x,y)\neq(0,0)$$

$$f_x(4,-3) = \frac{3}{16+9} = \frac{3}{25}$$

$$f_y(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f_y(4,-3) = \frac{4}{25}$$

(2) $f(x,y) = \ln(xy) \Rightarrow f_x,\ f_y$

$$f_x(x,y) = \frac{1}{xy}\cdot y = \frac{1}{x}$$

$$f_y(x,y) = \frac{1}{xy}\cdot x = \frac{1}{y}$$

(3) $f(x,y) = e^{x^2y}$

$$f_x = e^{x^2y}\cdot 2xy$$

$$f_y = e^{x^2y}\cdot x^2$$

(4) $f(x,y) = \sin^{-1}(x^2y)$

$$f_x = \frac{2xy}{\sqrt{1-(x^2y)^2}} \qquad f_y = \frac{x^2}{\sqrt{1-(x^2y)^2}}$$

(5) $f(x,y) = \dfrac{1}{xy}$

$$f_x = \frac{-y}{(xy)^2} \qquad f_y = \frac{-x}{(xy)^2}$$

<Rmk> (1) $f(x,y)$ 가 편미분가능이면 ∃ f_x, f_y (1차편도함수)

(2) $f_x(x,y)$, $f_y(x,y)$: 편미분가능이면 각각에 대하여 x와 y에 관한 편도함수가 존재한다. 이를 2차편도함수라 한다.

$$\left.\begin{matrix}(f_x)_x, & (f_x)_y \\ (f_y)_x, & (f_y)_y\end{matrix}\right\} \Rightarrow f_{xx}, f_{xy}, f_{yx}, f_{yy}$$

: 이차편도함수.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

이와 같은 방법으로 하여 3차, 4차 ... 편도함수를 정의 할 수 있고 2차편도함수 이상을 고차 편도함수라 한다.

Exam (1) $f(x,y) = x^3y^2 - 2x^2y + 3x$ 의 2차편도함수를 구하라.

$f_x = 3x^2y^2 - 4xy + 3$ $\qquad f_y = 2x^3y - 2x^2$.

$f_{xx} = 6xy^2 - 4y$. $\qquad f_{yy} = 2x^3$

$f_{xy} = 6x^2y - 4x$. $\qquad f_{yx} = 6x^2y - 4x$.

(2) $f(x,y) = y^2e^x + y \Rightarrow f_{xyy}$.

$f_x = y^2e^x \qquad f_{xy} = 2ye^x \qquad f_{xyy} = 2e^x$.

Thm. $z = f(x,y)$, ∃ $f, f_x, f_y, f_{xy}, f_{yx}$: conti.

$\Rightarrow f_{xy} = f_{yx}$.

<Rmk> 고차편도함수가 연속이면 편미분 순서를 변경할 수 있다.

i.e. $f_{xxy} = f_{xyx} = f_{yxx}$.

i.e. x에 관한 미분갯수와 y에 관한 미분갯수만 같으면 되고 순서는 관계없다.

— 합성함수의 편미분법 —

<u>Thm</u> — chain rule —

$z = f(u, v)$, $u = \varphi(x, y)$, $v = \phi(x, y)$.

f_u, f_v : conti. , u, v : 편미분 가능 이면.

$$\Rightarrow (1)\ \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$(2)\ \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

<pf> (1) put

$$\Delta u = \varphi(x+\Delta x, y) - \varphi(x, y)$$
$$\Delta v = \phi(x+\Delta x, y) - \phi(x, y)$$
$$\Delta z = f(u+\Delta u, v+\Delta v) - f(u, v).$$

$$\Rightarrow \Delta z = \underline{f(u+\Delta u, v+\Delta v) - f(u, v+\Delta v)} + \underline{f(u, v+\Delta v) - f(u, v)}.$$

평균값 정리.

$$= \Delta u \cdot f_u(u+\theta_1 \Delta u, v+\Delta v) + \Delta v f_v(u, v+\theta_2 \Delta v)$$

$$0 < \theta_1, \theta_2 < 1$$

$$\therefore \frac{\Delta z}{\Delta x} = f_u(u+\theta_1\Delta u, v+\Delta v) \cdot \frac{\Delta u}{\Delta x} + f_v(u, v+\theta_2 \Delta v) \frac{\Delta v}{\Delta x}$$

$\Delta x \to 0 \Rightarrow \Delta u \to 0, \Delta v \to 0$.

f_u, f_v : conti

$$\therefore \Delta x \to 0 \Rightarrow f_u(u+\theta_1\Delta u, v+\Delta v) \longrightarrow f_u(u, v)$$
$$f_v(u, v+\theta_2\Delta v) \longrightarrow f_v(u, v)$$

u, v가 편미분가능이므로.

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \qquad \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

(2) put
$$\Delta u = \varphi(x, y+\Delta y) - \varphi(x,y)$$
$$\Delta v = \phi(x, y+\Delta y) - \phi(x,y)$$
$$\Delta z = f(u+\Delta u, v+\Delta v) - f(u,v)$$

$\Rightarrow$ $$\Delta z = f(u+\Delta u, v+\Delta v) - f(u, v+\Delta v) + f(u, v+\Delta v) - f(u,v)$$
$$= \Delta u\, f_u(u+\theta_1 \Delta u, v+\Delta v) + \Delta v\, f_v(u, v+\theta_2 \Delta v)$$

$$\frac{\Delta z}{\Delta y} = f_u(u+\theta_1\Delta u, v+\Delta v)\cdot\frac{\Delta u}{\Delta y} + f_v(u, v+\theta_2\Delta v)\frac{\Delta v}{\Delta y}$$

$\Delta y \to 0$ $\Rightarrow$ $\Delta u \to 0$, $\Delta v \to 0$. u, v 가 편미분 가능

$$\therefore \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

<coro> ~~exam~~

$z=f(u,v)$, f_u, f_v : conti. $u=u(x)$, $v=v(x)$: differentiable

$$\Rightarrow \frac{dz}{dx} = \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx}$$

exam $z=f(x,y)$, f_x, f_y : conti., $x=r\cos\theta$, $y=r\sin\theta$.

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

(증)
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta.$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$
$$= \frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\cos\theta\frac{\partial z}{\partial y}\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$
$$+\frac{1}{r^2}\left(\left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2r^2\frac{\partial z}{\partial x}\cos\theta\frac{\partial z}{\partial y}\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta\right)$$
$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

<u>Exam</u> $z=\cos\frac{x}{y}$, $x=e^t$, $y=t^2$ $\Rightarrow$ $\frac{dz}{dt}$.

(풀이) $\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$.

$$=\left(-\frac{1}{y}\sin\frac{x}{y}\right)e^t+\left(-\sin\frac{x}{y}\cdot\frac{-x}{y^2}\right)\cdot 2t.$$

$$=-\frac{e^t}{y}\sin\frac{x}{y}+\frac{2tx}{y^2}\sin\frac{x}{y}$$

$$=\frac{2tx-e^ty}{y^2}\sin\frac{x}{y}=\frac{2te^t-e^tt^2}{t^4}\sin\frac{e^t}{t^2}$$

$$=\frac{(2-t)e^t}{t^3}\sin\frac{e^t}{t^2}.$$

<Rmk>

(1) $z=f(x,y)$, $x=a+ht$ $y=b+kt$.

$$\Rightarrow\quad \frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$=h\frac{\partial z}{\partial x}+k\frac{\partial z}{\partial y}\ :=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)=\frac{\partial}{\partial x}\left(\frac{dz}{dt}\right)\cdot\frac{dx}{dt}+\frac{\partial}{\partial y}\left(\frac{dz}{dt}\right)\frac{dy}{dt}.$$

$$=\left(h\frac{\partial^2}{\partial x^2}+k\frac{\partial^2}{\partial x\partial y}\right)z\cdot h+\left(h\frac{\partial^2}{\partial y\partial x}+k\frac{\partial^2}{\partial y^2}\right)z\cdot k.$$

$$=\left(h^2\frac{\partial^2z}{\partial x^2}+2hk\frac{\partial^2z}{\partial x\partial y}+k^2\frac{\partial^2z}{\partial y^2}\right)$$

$$:=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2z.$$

$$\vdots$$

$$\frac{d^nz}{dt^n}:=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^nz$$

(2) 삼변수 이상인 함수 에서도 Chain Rule 은 그대로 확장된다.

① $z=f(u,v,w,\cdots)$, $u=u(x)$, $v=v(x)$, $w=w(x)$, $\cdots$

$$\Rightarrow\quad \frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}+\frac{\partial z}{\partial w}\cdot\frac{dw}{dx}+\cdots$$

② $X = f(u, v, w, \cdots)$, $u = u(x, y, z, \cdots)$, $v = v(x, y, z, \cdots)$, $w = w(x, y, z, \cdots)$

$$\Rightarrow \frac{\partial X}{\partial x} = \frac{\partial X}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial X}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial X}{\partial w}\cdot\frac{\partial w}{\partial x} + \cdots$$

$$\frac{\partial X}{\partial y} = \frac{\partial X}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial X}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial X}{\partial w}\cdot\frac{\partial w}{\partial y} + \cdots$$

$$\frac{\partial X}{\partial z} = \frac{\partial X}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial X}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial X}{\partial w}\cdot\frac{\partial w}{\partial z} + \cdots$$

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Exm (1) $u = e^{x}(y - z)$, $x = t$, $y = \sin t$, $z = \cos t \Rightarrow \frac{du}{dt}$

(풀이) $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$

$$= e^{x}(y-z)\cdot 1 + e^{x}\cdot\cos t + (-e^{x})\cdot(-\sin t)$$
$$= e^{t}(\sin t - \cos t) + e^{t}\cos t + e^{t}\sin t$$
$$= 2e^{t}\sin t$$

(2) $z = u^2 - v^2$, $u = 2x + y + 1$, $v = x - 3y + 2$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$= 2u\cdot 2 + (-2v)\cdot 1 = 4u - 2v$$
$$= 4(2x + y + 1) - 2(x - 3y + 2)$$
$$= 8x + 4y + 4 - 2x + 6y - 4$$
$$= 6x + 10y$$

— Taylor 정리의 확장 —

Thm (이변수 함수의 Taylor 전개)

$f(x)$; (a, b) 근방에서 연속인 n차 편도함수를 가지면

$$f(a+h, b+k) = f(a, b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a, b)$$
$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) + \cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n$$

여기서 $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(a + \theta h, b + \theta k)$, $0 < \theta < 1$

위의 정리에서

put $h = x-a$, $k = y-b$ 라 두면

$$f(x,y) = f(a,b) + \left\{ (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right\} f(a,b)$$

$$+ \frac{1}{2!}\left\{ (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right\}^2 f(a,b)$$

$$+ \cdots$$

$$+ \frac{1}{(n-1)!}\left\{ (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right\}^{n-1} f(a,b) + R_n$$

$$R_n = \frac{1}{n!}\left\{ (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right\}^{n} f(a+\theta(x-a),\, b+\theta(y-b))$$

$$0 < \theta < 1.$$

exm (1) $f(x,y) = x^2 + 4xy + 3y^2 + 2x + 4y + 1$ 을 $(x-1)$, $(y-2)$ 의 멱급수로 표현하라.

최고차수가 2 이므로 3차 편도함수는 모두 0

$$f(x,y) = f(1,2) + \left\{ (x-1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y} \right\} f(1,2)$$

$$+ \frac{1}{2!}\left\{ (x-1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y} \right\}^2 f(1,2)$$

$f_x = 2x + 4y + 2$ $\quad f_y = 4x + 6y + 4$

$f_{xx} = 2$ $\quad f_{xy} = 4$ $\quad f_{yy} = 6$

$f(1,2) = 1 + 8 + 12 + 2 + 8 + 1 = 32$

$f_x(1,2) = 2 + 8 + 2 = 12$ $\quad f_y(1,2) = 4 + 12 + 4 = 20$

$$f(x,y) = 32 + \{12(x-1) + 20(y-2)\} + \frac{1}{2!}\left(2\cdot(x-1)^2 + 2\cdot 4(x-1)(y-2) + 6\cdot(y-2)^2\right)$$

$$= 32 + 12(x-1) + 20(y-2) + (x-1)^2 + 4(x-1)(y-2) + 3(y-2)^2.$$