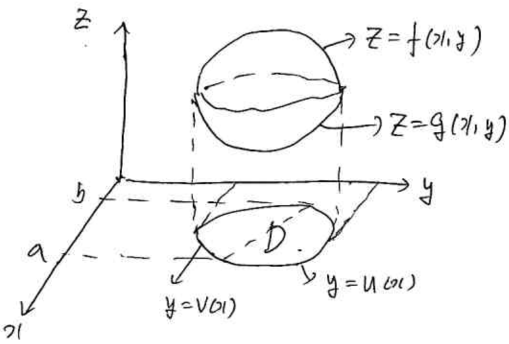
— 중적분의 응용 —

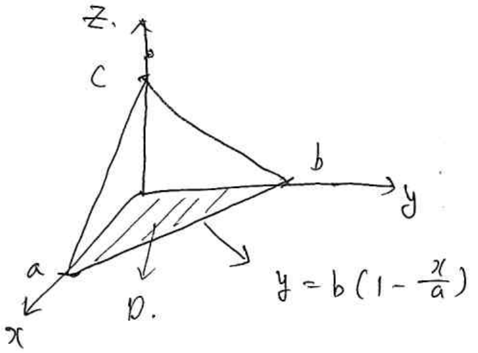
1. 부피 (체적)



$$V=\iint_D \left(f(x,y)-g(x,y)\right)dy\,dx$$

$$=\int_a^b\int_{v(x)}^{u(x)}\left(f(x,y)-g(x,y)\right)dy\,dx.$$

Ex〉 평면 $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ 과 좌표면으로 둘러싸인 입체의 부피?



$$z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$D=\left\{(x,y)\;\middle|\;0\le x\le a,\;0\le y\le b\left(1-\frac{x}{a}\right)\right\}.$$

$$V=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx.$$

$$=c\int_0^a\left[y-\frac{x}{a}y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)}dx.$$

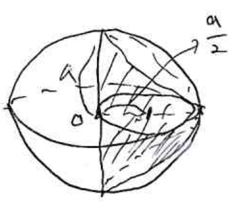
$$=c\int_0^a\left(b\left(1-\frac{x}{a}\right)-\frac{bx}{a}\left(1-\frac{x}{a}\right)-\frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right)dx.$$

$$=c\int_0^a\left(b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right)dx.$$

$$=\frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2dx.=\frac{bc}{2}\int_0^a\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)dx$$

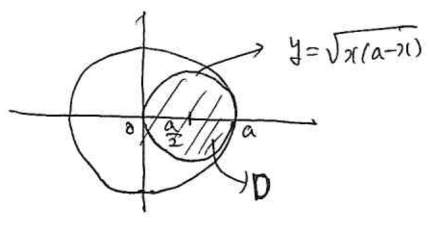
$$=\frac{bc}{2}\left[x-\frac{x^2}{a}+\frac{x^3}{3a^2}\right]_0^a=\frac{bc}{2}\left(a-a+\frac{a}{3}\right)=\frac{abc}{6}$$

Ex〉 반지름이 a 인 구와 반지름이 $\frac{a}{2}$ 인 원주가 다음 그림과같이 위치하고 있을때 이들 두입체의 공통부분의 부피?



구면 방정식 $x^2+y^2+z^2=a^2$.

원주 $y^2=x(a-x)$



$$V=4\int_0^a\int_0^{\sqrt{x(a-x)}}\sqrt{a^2-x^2-y^2}\,dy\,dx.$$

$$V = 4\int_0^a \int_0^{\sqrt{x(a-x)}} \sqrt{a^2-x^2-y^2}\, dy\, dx.$$

이 경우 직교좌표로 계산하면 상당히 복잡하므로 극좌표로 바꾸어 계산

$x = r\cos\theta \quad y = r\sin\theta \quad \Rightarrow \quad z^2 = a^2-x^2-y^2 = a^2-r^2$

$y^2 = x(a-x) \Rightarrow r^2 = ar\cos\theta, \quad r>0$

$\therefore\ r = a\cos\theta.$

$$\therefore\ V = 4\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \sqrt{a^2-r^2}\cdot r\, dr\, d\theta.$$

$$= 4\int_0^{\frac{\pi}{2}} \left[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^{a\cos\theta} d\theta.$$

$$= 4\int_0^{\frac{\pi}{2}} \left(-\frac{1}{3}(a^2-a^2\cos^2\theta)^{\frac{3}{2}} + \frac{1}{3}a^3\right) d\theta.$$

$$= \frac{4}{3}a^3 \int_0^{\frac{\pi}{2}} (1-\sin^3\theta)\, d\theta$$

$-\sin^3\theta = -\sin\theta(1-\cos^2\theta)$

$= -\sin\theta + \cos^2\theta\sin\theta.$

$$= \frac{4}{3}a^3 \left[\theta + \cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{4}{3}a^3\left(\frac{\pi}{2} - 1 + \frac{1}{3}\right) = \frac{4}{3}a^3\left(\frac{\pi}{2} - \frac{2}{3}\right)$$

2. 회전체의 부피.

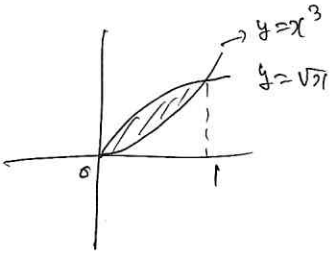
(1) $D = \{(x,y) \mid a \le x \le b,\ 0 \le v(x) \le y \le u(x)\}$ 를 x축 둘레로 회전하여 생기는 입체의 부피.

$$V = \pi\int_a^b (u(x)^2 - v(x)^2)\, dx = 2\pi\int_a^b \int_{v(x)}^{u(x)} y\, dy\, dx.$$

(2) $D = \{(x,y) \mid 0 \le v(y) \le x \le u(y),\ a \le y \le b\}$ 를 y축 둘레로 회전하여 생기는 입체의 부피

$$V = \pi\int_a^b (u(y)^2 - v(y)^2)\, dy = 2\pi\int_a^b \int_{v(y)}^{u(y)} x\, dx\, dy.$$

ex) $y=x^3$, $y=\sqrt{x}$ 로 둘러싸인 부분 D 를 x 축 둘레로 회전하여 회전체의 부피.



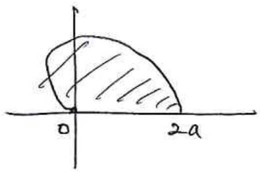
$$V = 2\pi\int_0^1\int_{x^3}^{\sqrt{x}} y\, dy\, dx.$$

$$= 2\pi\int_0^1 \left[\frac{y^2}{2}\right]_{x^3}^{\sqrt{x}} dx$$

$$= \pi\int_0^1 (x-x^6)\,dx = \pi\left[\frac{x^2}{2}-\frac{x^7}{7}\right]_0^1$$

$$= \pi\left(\frac{1}{2}-\frac{1}{7}\right) = \frac{5}{14}\pi.$$

ex) 심장형 $r=a(1+\cos\theta)$ $(a>0)$ 을 x축으로 회전하여 생기는 회전체의 부피.



$$V = \iint_D 2\pi y\, dy\, dx$$

$$= 2\pi\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \cdot r\, dr\, d\theta.$$

$$= 2\pi\int_0^{\pi}\left[\frac{1}{3}r^3\sin\theta\right]_0^{a(1+\cos\theta)} d\theta.$$

$$= \frac{2}{3}\pi a^3\int_0^{\pi}(1+\cos\theta)^3\sin\theta\, d\theta.$$

$$= \frac{2}{3}\pi a^3\left[-\frac{1}{4}(1+\cos\theta)^4\right]_0^{\pi}$$

$$= \frac{2}{3}\pi a^3\left(0-\frac{1}{4}\cdot 16\right) = \frac{8}{3}\pi a^3$$

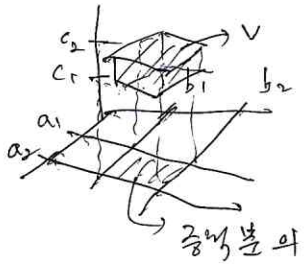
— 3중적분 —

이중적분의 개념을 확장하여 3중적분을 정의할수 있다.

$V(\subset R^3)$ 가 유계, $f: V\to R$: 유계 인경우.

적분영역을 삼차원으로 (공간도형) 으로 생각하여 정의한다.

$a_1\le x\le a_2,\quad b_1\le y\le b_2,\quad c_1\le z\le c_2$



$$\iiint_V f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dy\,dx.$$

exm

(1) $\int_2^3\int_1^2\int_2^5 xy^2\, dz\, dy\, dx$

$= \int_2^3\int_1^2 \left[xy^2z\right]_2^5 dy\, dx = \int_2^3\int_1^2 3xy^2\, dy\, dx$

$= \int_2^3 \left[xy^3\right]_1^2 dx = \int_2^3 7x\, dx = \left[\frac{7x^2}{2}\right]_2^3 = \frac{35}{2}.$

exm $\int_0^1\int_0^1\int_0^{\frac{\pi}{2}} xy\cos yz\, dz\, dy\, dx.$

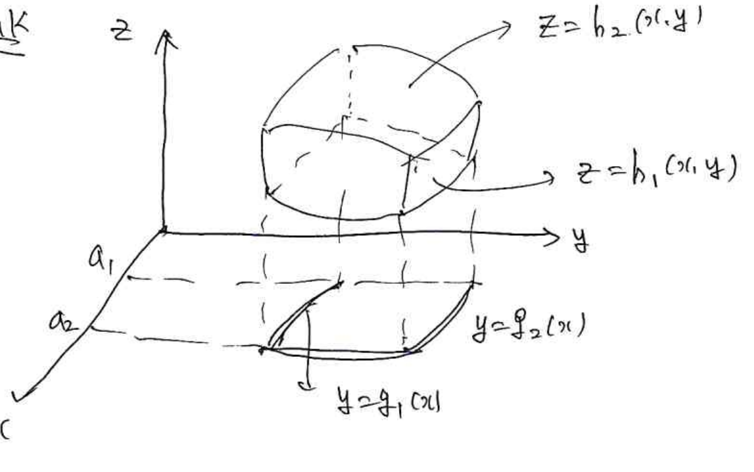
$= \int_0^1\int_0^1 \left[xy\cdot\frac{1}{y}\sin yz\right]_0^{\frac{\pi}{2}} dy\, dx.$

$= \int_0^1\int_0^1 \left(x\sin\frac{\pi}{2}y\right) dy\, dx.$

$= \int_0^1 \left[-\frac{2}{\pi}x\cos\frac{\pi}{2}y\right]_0^1$

$= \int_0^1 \frac{2}{\pi}x\, dx = \left[\frac{x^2}{\pi}\right]_0^1 = \frac{1}{\pi}$

Rmk



$V=\{(x,y,z) \mid a_1 \le x \le a_2,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\}.$

상의 $f(x,y,z)$ 의 삼중적분

$$\iiint_V f(x,y,z)\, dz\, dy\, dx = \int_{a_1}^{a_2}\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

exm $V=\{(x,y,z)\mid 0\le x\le 1,\ x^2\le y\le x,\ x-y\le z\le x+y\}$. 3-3-5.

$\iiint_V 2x^3y^2z\,dz\,dy\,dx$.

$$=\int_0^1\int_{x^2}^{x}\int_{x-y}^{x+y} 2x^3y^2z\,dz\,dy\,dx.$$

$$=\int_0^1\int_{x^2}^{x}\left[x^3y^2z^2\right]_{x-y}^{x+y}dy\,dx.$$

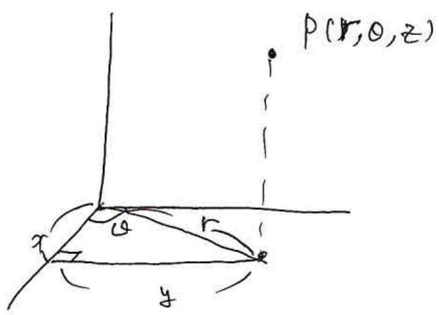
$(x+y)^2-(x-y)^2$
$=x^2+2xy+y^2-(x^2-2xy+y^2)$
$=4xy.$

$$=\int_0^1\int_{x^2}^{x}4x^4y^3\,dy\,dx$$

$$=\int_0^1\left[x^4y^4\right]_{x^2}^{x}dx=\int_0^1 x^4(x^4-x^8)\,dx=\int_0^1(x^8-x^{12})\,dx$$

$$=\left[\frac{1}{9}x^9-\frac{1}{13}x^{13}\right]_0^1=\frac{1}{9}-\frac{1}{13}$$

— 원주좌표(극좌표)와 구면좌표에 있어서의 삼중적분 —



$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z.$

$(x,y,z)\Rightarrow(r,\theta,z)$

$$|J|=\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}x_r & x_\theta & x_z\\ y_r & y_\theta & y_z\\ z_r & z_\theta & z_z\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r.$$

$V=\{(r,\theta,z)\mid \alpha\le\theta\le\beta,\ g_1(\theta)\le r\le g_2(\theta),\ h_1(r,\theta)\le z\le h_2(r,\theta)\}$

$$\iiint_V f(x,y,z)\,dz\,dy\,dx=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r,\theta)}^{h_2(r,\theta)}f(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta.$$

exm $V=\{(r,\theta,z)\mid 0\le r\le\sin\theta,\ 0\le\theta\le\frac{\pi}{3},\ 0\le z\le r\}$ 상의 함수 $f(r,\theta,z)=4r$ 의 삼중적분을 구하라.

$$\iiint_V f(r,\theta,z)\,dz\,dr\,d\theta=\int_0^{\frac{\pi}{3}}\int_0^{\sin\theta}\int_0^{r}4r\cdot r\,dz\,dr\,d\theta.$$

$$=\int_0^{\frac{\pi}{3}}\int_0^{\sin\theta}\left[4r^2\cdot z\right]_0^{r}dr\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \int_0^{\sin\theta} 4r^3 \, dr \, d\theta . = \int_0^{\frac{\pi}{3}} [r^4]_0^{\sin\theta} d\theta .$$

$$= \int_0^{\frac{\pi}{3}} \sin^4\theta \, d\theta .$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$\therefore \sin^2\theta = \frac{1-\cos 2\theta}{2} .$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta$$

$$\frac{1}{4}(1-\cos 2\theta)^2 = \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{3}} \left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right) d\theta .$$

$$\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{3}} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta .$$

$$= \frac{1}{4} \left[\frac{3}{2}\theta - \sin 2\theta + \frac{1}{2} \cdot \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{4} \left(\frac{3}{2} \cdot \frac{\pi}{3} - \sin\frac{2\pi}{3} + \frac{1}{8} \sin\frac{4}{3}\pi \right)$$

$$= \frac{1}{4} \left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right) \right) = \frac{1}{4} \left(\frac{\pi}{2} - \frac{9\sqrt{3}}{16} \right)$$

<u>exm</u> $V = \{ (x, y, z) \mid -2 \le x \le 2, \ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

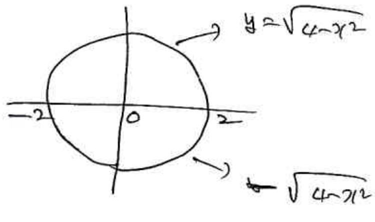
$x^2 + y^2 \le z \le 4 \ \}.$

$$\iiint_V \sqrt{x^2+y^2} \, dz \, dy \, dx .$$

원주좌표로 변환하여 계산

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \cdot r \, dz \, dr \, d\theta$$



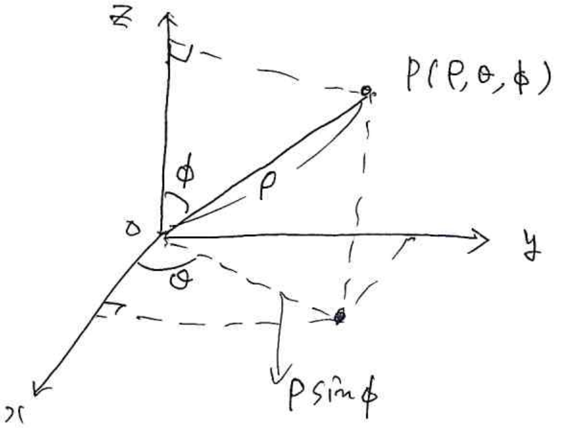
$$= \int_0^{2\pi} \int_0^2 [r^2 z]_{r^2}^4 \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 [4r^2 - r^4] \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4}{3} r^3 - \frac{r^5}{5} \right]_0^2 d\theta .$$

$$= \int_0^{2\pi} \left(\frac{32}{3} - \frac{32}{5} \right) d\theta = 2\pi \cdot \frac{64}{15} = \frac{128}{15}\pi$$

— 구면좌표 —



$$z = \rho\cos\phi.$$

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta.$$

$$\begin{cases} \rho \ge 0 \\ 0 \le \phi \le \pi \\ -\pi \le \theta \le \pi \\ (\text{or } 0 \le \theta \le 2\pi) \end{cases}$$

$$(x,y,z) \Longrightarrow (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

$$|J| = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\sin\phi$$

exm 반지름이 a 인 공의 부피.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

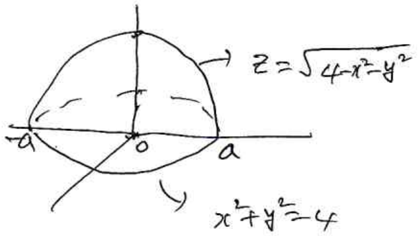
$$= \int_0^{2\pi}\int_0^{\pi} \left[\tfrac{1}{3}\rho^3\sin\phi\right]_0^a d\phi\, d\theta = \tfrac{1}{3}a^3\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta$$

$$= \tfrac{1}{3}a^3\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi} d\theta = \tfrac{1}{3}a^3\int_0^{2\pi} 2\, d\theta = \tfrac{1}{3}a^3\left[2\theta\right]_0^{2\pi}$$

$$= \tfrac{4}{3}\pi a^3$$

exm

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$



$$V = \{(x,y,z) \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \; 0 \le z \le \sqrt{4-x^2-y^2}\}.$$

$$V = \{(\rho,\phi,\theta) \mid 0 \le \rho \le 2, \; 0 \le \phi \le \tfrac{\pi}{2}, \; 0 \le \theta \le 2\pi\}$$

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

$$x^2+y^2+z^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi$$

$$= \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) + \rho^2\cos^2\phi$$

$$= \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2.$$

$$\iiint_V z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} \rho^2\cos^2\phi \cdot \rho \cdot \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} \rho^5 \cos^2\phi \sin\phi \; d\rho\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left[\frac{1}{6}\rho^6 \cos^2\phi \sin\phi\right]_0^2 d\phi\, d\theta$$

$$= \frac{32}{3}\int_0^{2\pi} \left[-\frac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{2}} d\theta.$$

$$= \frac{32}{3}\int_0^{2\pi} \frac{1}{3}\, d\theta = \frac{32}{3}\left[\frac{\theta}{3}\right]_0^{2\pi} = \frac{32}{3}\cdot\frac{2\pi}{3} = \frac{64}{9}\pi$$