$\mathbb{R}^2$ $+, \cdot$ scalar multiplications.

└→ 두 수의 복소수 合 · 정의

$(\mathbb{R}^2, +, \cdot)$ complex number

Notation. In $\mathbb{R}^2$ $(x, y) \in \mathbb{R}$

① $(1,0) = i$ $(1,0) = 1$

$z = x + iy$, $w = a + ib$

$zw = ax - yb + i(xb + ay)$

$\Rightarrow i^2 = i \cdot i = (-1, 0) = -1$

$\mathbb{C}$ is a field

복소수 $x+iy$ 의 곱셈에대한 역원은 항상 존재한다.

For $z = x + iy \neq 0$, $\exists w \in \mathbb{C}$ s.t $z \cdot w = 1$

$zw = ax - yb + i(xb + ay) = 1$

$ax - yb = 1$ $xb + ay = 0$

$$\begin{array}{l} ax^2 - xyb = x \\ + \ \ bxy + ay^2 = 0 \\ \hline a(x^2+y^2) = x \end{array}$$

$\therefore a = \dfrac{x}{x^2+y^2}$ $(x + iy \neq 0)$

$ax - yb = 1$ $bx + ay = 0$

$$\begin{array}{l} axy - by^2 = y \\ - \ \ bx^2 + axy = 0 \\ \hline -b(x^2+y^2) = y \end{array}$$

$\therefore b = -\dfrac{y}{x^2+y^2}$

$w = \dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2} i$

w is the inverse of z for ·

THM $\forall z_1, z_2 \in \mathbb{C}$

① $z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$

② $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

③ $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

pf $z_1 z_2 = z_2 z_1$

let $z_1 = a + bi$ $z_2 = c + di$

$z_1 \cdot z_2 = (a+bi)(c+di) = (ac - bd) + (ad + bc)i$

$z_2 \cdot z_1 = (c+di)(a+bi) = (ac - bd) + (ad + bc)i$

NOTATION

$z = x + iy$ $\operatorname{Re} z = x$ $\operatorname{Im} z = y$

복소수 z의 실수부

복소수 z의 허수부.

$\bar{z} = x - iy$

THM. $z = x + iy$, $z_1 = x_1 + iy_1$ $z_2 = x_2 + iy_2$

$\Rightarrow$ ① $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$

② $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$

③ $|z| \leq |x| + |y| \leq \sqrt{2}|z|$

④ $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ $\overline{\left(\dfrac{z_2}{z_1}\right)} = \dfrac{\bar{z}_2}{\bar{z}_1}$

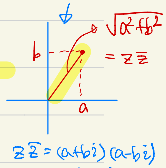
⑤ $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$

$|z|$: z의 크기 → 좌표평면에서 원점과의 거리

pf) ① let $z = a + bi$

$\operatorname{Re} z = a \leq |a| \leq |z| = \sqrt{a^2+b^2}$.

$= z\bar{z}$



② $\operatorname{Im} z = y \leq |y| = |\operatorname{Im} z|$

$\leq \sqrt{x^2+y^2} = |z|$

$z\bar{z} = (a+bi)(a-bi)$
$= a^2 + b^2$
$= |z|^2$

③ claim : $(|x|+|y|)^2 - |z|^2$

$= x^2+y^2+2|x||y| - (x^2+y^2)$

$= 2|x||y| \geq 0$

$\therefore |z| \leq |x|+|y|$

claim; $(\sqrt{2}\,|z|)^2 - (|x|+|y|)^2$

$= 2(x^2+y^2) - (x^2+y^2+2|x||y|)$

$= x^2+y^2-2|x||y| = (|x|-|y|)^2 \geq 0$

$\therefore |x|+|y| \leq \sqrt{2}\,|z|$

④ $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

$\overline{x_1+y_1 i + x_2+y_2 i} = \overline{x_1+y_1 i} + \overline{x_2+y_2 i}$

$\overline{(x_1+x_2)+i(y_1+y_2)} = x_1-y_1 i + x_2 - y_2 i$

$x_1+x_2-(y_1+y_2)i = x_1+x_2-(y_1+y_2)i$

$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

$\overline{z_1 z_2} = \overline{(x_1+y_1 i)(x_2+y_2 i)}$

$= \overline{x_1x_2-y_1y_2+(x_1y_2+x_2y_1)i}$

$= x_1x_2-y_1y_2-(x_1y_2+x_2y_1)i$

$\bar{z}_1\cdot\bar{z}_2 = \overline{x_1+y_1 i}\cdot\overline{x_2+y_2 i}$

$= (x_1-y_1 i)(x_2-y_2 i)$

$= x_1x_2-y_1y_2-(x_1y_2+x_2y_1)i$

⑤ let $z = a+bi$, then $\bar{z} = a-bi$

$\mathrm{Re}\, z = a$

$\frac{z+\bar{z}}{2} = \frac{a+\cancel{bi}+a-\cancel{bi}}{2} = \frac{2a}{2} = a$

$\therefore \mathrm{Re}\, z = \frac{z+\bar{z}}{2}$

$z = x+iy = \mathrm{Re}\, z + i\,\mathrm{Im}\, z \cdots ①$

$\bar{z} = x-iy = \mathrm{Re}\, z - i\,\mathrm{Im}\, z \cdots ②$

①+② $\Rightarrow z+\bar{z} = 2\mathrm{Re}\, z \quad \therefore \mathrm{Re}\, z = \frac{z+\bar{z}}{2}$

①−② $\Rightarrow z-\bar{z} = 2i\,\mathrm{Im}\, z \quad \therefore \mathrm{Im}\, z = \frac{z-\bar{z}}{2i}$

THM ① $|z_1 z_2| = |z_1||z_2|$

② $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

③ $|z_1+z_2| \leq |z_1|+|z_2|$

④ $||z_1|-|z_2|| \leq |z_1-z_2|$

pf) ① $|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2})$

$= (z_1z_2)(\bar{z}_1\bar{z}_2)$

$= z_1\bar{z}_1 \cdot z_2\bar{z}_2 = |z_1|^2|z_2|^2$

$\therefore |z_1z_2| = |z_1||z_2|$

② $\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|}$

$\left|\frac{z_2}{z_1}\right|^2 = \left(\frac{z_2}{z_1}\right)\left(\overline{\frac{z_2}{z_1}}\right) = \left(\frac{z_2}{z_1}\right)\left(\frac{\bar{z}_2}{\bar{z}_1}\right)$

$= \frac{z_2\bar{z}_2}{z_1\bar{z}_1} = \frac{|z_2|^2}{|z_1|^2}$

$\therefore \left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|}$

③ $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$

$= |z_1|^2+|z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$

$= |z_1|^2 + 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2$

$\because \mathrm{Re}\, z \leq |\mathrm{Re}\, z| \leq |z|$

$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$

$= (|z_1|+|z_2|)^2$

$\therefore |z_1+z_2| \leq |z_1|+|z_2|$

④ $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$

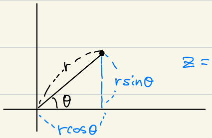
$$|z_1| - |z_2| \le |z_1 - z_2|$$

$$|z_2| = |z_2 - z_1 + z_1| \le |z_1 - z_2| + |z_1|$$

$$-|z_1 - z_2| \le |z_1| - |z_2|$$

$$\big||z_1| - |z_2|\big| \le |z_1 - z_2|$$

Polar Coordinate



$z = r\cos\theta + i\, r\sin\theta$.

$\arg z = \underbrace{2n\pi}_{\text{크는 바뀌수}} + \operatorname{Arg} z \qquad -\pi \le \operatorname{Arg} z \le \pi$

$A = \cos\theta_1 + i\sin\theta_1 \qquad B = \cos\theta_2 + i\sin\theta_1$

$$AB = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

THM $\quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow$ ① $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

② $z_1^{-1} = \frac{1}{r_1}(\cos(-\theta_1) + i\sin(-\theta_1))$

③ $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$