— 상한과 하한 —

$\mathbb{R}$: 실수 집합.

<Def> (1) $M(\subset \mathbb{R})$: lower bounded (밑으로 유계)

$\iff \forall x \in M, \exists a \in \mathbb{R}$ s.t. $a \le x$.

여기서 a : M의 하계 (lower bound. of M).

(2) $M(\subset \mathbb{R})$: upper bounded (위로 유계)

$\iff \forall x \in M, \exists a \in \mathbb{R}$ s.t. $a \ge x$.

여기서 a : M의 상계 (upper bound of M)

(3) $M(\subset \mathbb{R})$: bounded. (유계)

$\iff \forall x \in M, \exists a \in \mathbb{R}$ s.t $|x| \le a$.

$\iff$ M : 위로 유계 이고 밑으로유계 인 경우.

(4) $\inf M$: a greatest lower bound of M $\sup M$: a smallest upper bound of M.

Thm Let a : lower bound of M. (M의 하계)

$a = \inf M$

$\iff \forall \varepsilon > 0, \ [a, a+\varepsilon) \cap M \neq \phi$.

<proof> ($\Longrightarrow$) Since a : lower bound of M, $(-\infty, a) \cap M = \phi$.

$a = \inf M$.

Assume $\exists \varepsilon > 0$ s.t. $[a, a+\varepsilon) \cap M = \phi$.

$\Rightarrow (-\infty, a+\varepsilon) \cap M = \phi$.

$\therefore$ $a + \frac{\varepsilon}{2}$: lower bound of M.

$\varepsilon > 0$ 이므로, $a < a + \frac{\varepsilon}{2}$.

$\therefore$ 모순이다.

$\therefore$ $a = \inf M \Rightarrow [a, a+\varepsilon) \cap M \neq \phi$ 이다. ($\forall \varepsilon > 0$).

$(\Leftarrow)$ $[a, a+\varepsilon) \cap M \neq \phi$, a : lower bound of M.

Assume $a \neq \inf M$.

$\Rightarrow \exists\ a'$; lower bound of M s.t. $a < a'$

$\Rightarrow (-\infty, a') \cap M = \phi$. $a' - a > 0$.

$\therefore$ Take $\varepsilon > 0$ s.t. $0 < \varepsilon < a' - a$.

$\Rightarrow a' > a + \varepsilon$.

$\therefore \{[a, a+\varepsilon) \cap M\} \subset (-\infty, a') \cap M$.

$\therefore$ 모순

$\therefore a = \inf M$

Thm Let a : upperbound of M.
$a = \sup M$
$\Leftrightarrow \forall \varepsilon > 0, \ (a-\varepsilon, a] \cap M \neq \phi$.

<pf> ($\Rightarrow$)
Since a : upper bound of M, $(a, \infty) \cap M = \phi$.
$a = \sup M$.
Assume $\exists \ \varepsilon > 0$ s.t. $(a-\varepsilon, a] \cap M = \phi$.

$\Rightarrow (a-\varepsilon, \infty) \cap M = \phi$.

$\therefore a - \frac{\varepsilon}{2}$: upper bound of M.

$\varepsilon > 0, \quad a - \frac{\varepsilon}{2} < a$.

∴ 모순

$\therefore a = \sup M \Rightarrow (a-\varepsilon, a] \cap M \neq \phi$. $(\forall \varepsilon > 0)$

($\Leftarrow$) $(a-\varepsilon, a] \cap M \neq \phi$, $(a, \infty) \cap M = \phi$. (i.e. a : upper bound of M)
Assume $a \neq \sup M$.

$\Rightarrow \exists \ a'$: upper bound of M. s.t. $a' < a$.

$\Rightarrow (a', \infty) \cap M = \phi$. $a - a' > \varepsilon > 0$.

$\Rightarrow a - \varepsilon > a'$

$\therefore \{(a-\varepsilon, a] \cap M.\} \subset \{(a', \infty) \cap M.\}$.

∴ 모순 $\therefore a = \sup M$.

exm. $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n} \cdots\} \Rightarrow \inf A = 0$.

$= \{\frac{1}{n} \mid n$: 자연수 $\}$.

<sol>

$\forall x \in A, \quad 0 < x. \quad \therefore$ 0 : lower bound of A.

우리가 보여야 할것. $[0, \varepsilon) \cap A \neq \phi \Rightarrow$ 정리에 의해 $0 = \inf M$.

$\forall \varepsilon > 0$, Take $n_0 \in N$ s.t. $n_0 > \frac{1}{\varepsilon}$

$\Rightarrow 0 < \frac{1}{n_0} < \varepsilon, \quad \frac{1}{n_0} \in A$

$\therefore \quad [0, \varepsilon) \cap A \neq \phi$.

$\therefore \quad \inf A = 0$.

<문제> (1) $\inf \{x \mid x^2 \geq 2, x > 0\}$

(2) $\sup \{\frac{n}{n+1} \mid n \in N\}$.

(3) $\sup \{[0, 3) \sqcup (5, 8)\}$.

(4) $\inf \{\frac{n+1}{n} \mid n \in N\}$

(2) 과 (4) 문제는 앞의 정리 두개를 이용하고.
위의 예제를 참고 하여 구하시오.

Thm (Weierstrass 의 정리)

$M (\subset R)$ 이 위로 유계 (아래로 유계) 이면
상한 (하한) 이 존재한다.

- 수열의 극한 -

Define $f : N \longrightarrow R$ by $f(i) = a_i$ $i = 1, 2, \cdots, n, \cdots$

$$f(1) = a_1$$
$$f(2) = a_2$$
$$\vdots$$
$$f(n) = a_n.$$
$$\vdots$$

$\{a_n\}$: sequence. (수열)

$\lim_{n\to\infty} a_n = \underset{\text{극한}}{\underset{\Vert}{\alpha}}$ 는 어떤 의미인가?

극한이 존재하면 수렴한다.

〃 존재하지 않으면 즉, 수렴하지 않으면 발산한다라고 한다.

<Def> $\lim_{n\to\infty} a_n = \alpha$ (or $a_n \longrightarrow \alpha$ as $n \to \infty$).

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists\, n_0 \in N$ s.t. $|a_n - \alpha| < \varepsilon$, $n \geq n_0$.

Thm. 수열이 수렴하면 그 극한은 일의적이다. (유일하다.)

<pf> Assume $\alpha \neq \beta$, $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} a_n = \beta$.

$\Rightarrow$ $\forall \varepsilon_1 > 0$, $\exists\, n_1 \in N$ s.t. $|a_n - \alpha| < \varepsilon_1$, $n \geq n_1$

$\forall \varepsilon_2 > 0$, $\exists\, n_2 \in N$ s.t $|a_n - \beta| < \varepsilon_2$, $n \geq n_2$

$$|\alpha - \beta| = |\alpha - a_n + a_n - \beta| \leq |\alpha - a_n| + |a_n - \beta| < \varepsilon_1 + \varepsilon_2$$

$$= \frac{1}{2}|\alpha - \beta| + \frac{1}{2}|\alpha - \beta| = |\alpha - \beta|$$

$\therefore$ 모순이다. $\therefore$ $\alpha = \beta$.

Thm $a_n \to \alpha$, $b_n \to \beta$ as $n \to \infty$

(1) $k a_n \to k\alpha$ as $n \to \infty$ (k: 상수)

(2) $a_n \pm b_n \to \alpha \pm \beta$ as $n \to \infty$.

(3) $a_n \cdot b_n \to \alpha \cdot \beta$ as $n \to \infty$.

(4) $\dfrac{a_n}{b_n} \to \dfrac{\alpha}{\beta}$ as $n \to \infty$ ($\beta \neq 0$)

<pf> (1) $a_n \to \alpha$ as $n \to \infty$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists n_0 \in N$ s.t. $|a_n - \alpha| < \varepsilon$, $n \geq n_0$.

$|k a_n \to k\alpha| \leq |k|\,|a_n - \alpha| < |k| \dfrac{\varepsilon}{|k|} = \varepsilon.$

(2) $a_n \to \alpha$ as $n \to \infty$

$\Rightarrow$ $\forall \varepsilon_1 > 0$, $\exists n_1 \in N$ s.t. $|a_n - \alpha| < \varepsilon_1$, $n \geq n_1$

$b_n \to \beta$ as $n \to \infty$

$\Rightarrow$ $\forall \varepsilon_2 > 0$, $\exists n_2 \in N$ s.t. $|b_n \to \beta| < \varepsilon_2$, $n \geq n_2$.

Take $n_0 = \max\{n_1, n_2\}$

$n \geq n_0$,

$$|(a_n \pm b_n) - (\alpha \pm \beta)| \leq |a_n - \alpha| + |b_n - \beta|$$
$$< \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(3) Take $n_0 = \max\{n_1, n_2\}$

$n \geq n_0$,

$$|a_n \cdot b_n - \alpha \cdot \beta| = |(a_n - \alpha) b_n + \alpha (b_n - \beta)|$$

$$\leq |a_n - \alpha|\,|b_n| + |\alpha|\,|b_n - \beta| < \varepsilon_1 \cdot |b_n| + |\alpha| \varepsilon_2$$

$$= \frac{\varepsilon}{2|b_n|} \cdot |b_n| + |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} = \varepsilon.$$

(4). $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n}$

$\therefore$ $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\beta}$ 임을 보이면 (3)에 의해 성립.

$n \geq n_0$,

$$\left| \frac{1}{b_n} - \frac{1}{\beta} \right| = \left| \frac{\beta - b_n}{b_n \cdot \beta} \right| = \frac{1}{|b_n \cdot \beta|} |b_n - \beta|$$

$$< \frac{1}{|b_n||\beta|} \varepsilon_2 = \frac{1}{|b_n||\beta|} \cdot |b_n||\beta| \cdot \varepsilon = \varepsilon.$$

$\therefore$ $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\beta}$

By (3), $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}$

<Def> (1) $\{a_n \mid a_n \le a_{n+1}\}$: monotone increasing sequence
(단조증가수열)

(2) $\{a_n \mid a_n \ge a_{n+1}\}$: monotone decreasing sequence
(단조감소수열)

<exm> ① $\{\frac{1}{n} \mid n \in N\}$: monotone decreasing sequence

② $\{\frac{n-1}{n} \mid n \in N\}$: 〃 increasing 〃

Thm. (1) $\{a_n \mid a_n \le a_{n+1}\}$: upper bounded.

$\Rightarrow a_n \to \alpha$ as $n \to \infty$

(2) $\{a_n \mid a_n \ge a_{n+1}\}$: lower bounded.

$\Rightarrow a_n \to \alpha$ as $n \to \infty$

<pf> (1) $M = \{a_n \mid a_n \le a_{n+1}\}$: upper bounded.

$\Rightarrow$ By weierstrass Thm, $\exists \sup M = \alpha$.

$\Rightarrow (\alpha - \varepsilon, \alpha] \cap M \ne \phi$

$\Rightarrow \exists n_0 \in N$ s.t. $a_{n_0} \in (\alpha-\varepsilon, \alpha]$, $a_{n_0} \in M$.

$\Rightarrow \alpha - \varepsilon < a_{n_0} \le a_n \le \alpha \quad n \ge n_0$ (단조증가)

$\therefore n \ge n_0, \quad \alpha - \varepsilon < a_n < \alpha + \varepsilon$ i.e. $|a_n - \alpha| < \varepsilon$.

$\therefore a_n \to \alpha$ as $n \to \infty$

(2) $M = \{a_n \mid a_n \ge a_{n+1}\}$: lower bounded.

$\Rightarrow$ By weierstrass Thm, $\exists \inf M = \alpha$.

$\Rightarrow [\alpha, \alpha+\varepsilon) \cap M \ne \phi$.

$\Rightarrow \exists n_0 \in N$ s.t. $a_{n_0} \in [\alpha, \alpha+\varepsilon)$, $a_{n_0} \in M$.

$\Rightarrow \alpha \le a_n \le a_{n_0} < \alpha + \varepsilon$, $n \ge n_0$. (단조감소)

$\Rightarrow n \ge n_0, \quad \alpha - \varepsilon < a_n < \alpha + \varepsilon$ (i.e. $|a_n - \alpha| < \varepsilon$, $n \ge n_0$)

$\therefore a_n \to \alpha$ as $n \to \infty$

<u>exam</u> $a_n = \left(1+\frac{1}{n}\right)^n \Rightarrow a_n \to ?$ as $n \to \infty$.

<sol> 2항정리에 의해

$$a_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\left(\frac{1}{n}\right)^3 +$$

$$\cdots + \frac{n(n-1)\cdots 2}{n!}\cdot\left(\frac{1}{n}\right)^n.$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+$$

$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+$$

$$\cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

$$1-\frac{i}{n} < 1-\frac{i}{n+1} \qquad i=1,2,\cdots$$

$\therefore\ a_n < a_{n+1},\ \forall n\in N$ $\quad\therefore \{a_n\}$: 단조증가수열.

Since $1-\frac{i}{n} < 1$, $i=1,2,\cdots,n-1$,

$$a_n \leq 1+1+\frac{1}{2!}+\cdots\frac{1}{n!}$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$$

$$< 3$$

$\therefore$ 3 : upper bound (상계) $\quad\therefore \{a_n\}$: 위로 유계.

$\therefore\ a_n \to e$ as $n\to\infty$

e : 자연대수의 밑.

exam

(1) $1-\frac{3}{n}$

(∵) $a_n=\frac{1}{n} \Rightarrow a_n \to 0$ as $n\to\infty$

$$\therefore \lim_{n\to\infty}\left(1-\frac{3}{n}\right)=1$$

(해석적으로 보면 $a_n=1$, $b_n=\frac{1}{n}$

$\Rightarrow a_n-3\cdot b_n = 1-3\cdot 0 = 1$.

(2) $\frac{n+1}{3n-1}$

(∵) n으로 나누면

$$\frac{1+\frac{1}{n}}{3-\frac{1}{n}} \longrightarrow \frac{1}{3} \quad \text{as } n\to\infty$$

(3) $\frac{n}{n^2+1}$

(∵) n^2으로 나누면

$$\frac{\frac{1}{n}}{1+\frac{1}{n^2}} \longrightarrow \frac{0}{1}=0 \quad \text{as } n\to\infty$$

(4) $\underset{\downarrow \atop 0}{\frac{3}{n}} - \underset{\downarrow \atop 0}{\frac{8}{5^n}} \longrightarrow 0$ as $n\to\infty$

(5) $\sqrt{n^2-n}-n$

(∵) 유리화하면

n으로 나누면

$$\frac{n^2-n-n^2}{\sqrt{n^2-n}+n} = \frac{-n}{\sqrt{n^2-n}+n} = \frac{-1}{\sqrt{1-\frac{1}{n}}+1}$$

$$\longrightarrow -\frac{1}{2} \quad \text{as } n\to\infty$$

<문제> (1) $\frac{n}{1+\sqrt{n}}$

(2) $\frac{\sqrt{n+3}-\sqrt{n+2}}{\sqrt{n+2}-\sqrt{n}}$

Thm <조임의 정리>

$\{a_n\}, \{b_n\} \longrightarrow \alpha$ as $n \to \infty$

$a_n < c_n < b_n$

$\Rightarrow c_n \longrightarrow \alpha$ as $n \to \infty$

Thm $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$.

($\because$) $-|a_n| \le a_n \le |a_n|$

$\therefore \lim_{n\to\infty} a_n = 0$

exam. $a_n = \frac{1}{n}\cos n$.

($\because$) $0 \le |a_n| = \left|\frac{1}{n}\cos n\right| \le \frac{1}{n} \longrightarrow 0$.

$\therefore \lim_{n\to\infty} \frac{1}{n}\cos n = 0$.

— 함수의 극한 —

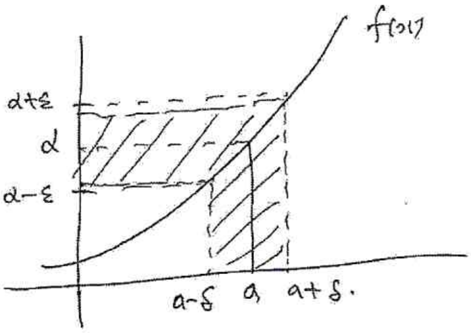
Define $f : R \longrightarrow R$

$$\underbrace{D(f)}_{\hookrightarrow \text{정의역}} \subset R, \quad \underbrace{R(f)}_{\hookrightarrow \text{치역}} \subset R.$$

$$\lim_{x \to a} f(x) = \alpha \quad \Rightarrow \quad f(x) \longrightarrow \alpha \quad \text{as} \quad x \longrightarrow a.$$

<Def> $\lim_{x \to a} f(x) = \alpha$

$\Longleftrightarrow \quad \forall \varepsilon > 0, \ \exists \delta > 0 \ \text{ s.t. } \ |x-a| < \delta, \ |f(x) - \alpha| < \varepsilon.$



$$f((a-\delta, a) \cup (a, a+\delta)) \subset (b-\varepsilon, b+\varepsilon)$$

exm. $\lim_{x \to 1} 2x = 2.$

$(\because)$ $\forall \varepsilon > 0, \ |x-1| < \delta,$

$$|2x - 2| = 2|x-1| < 2 \cdot \delta = \varepsilon \qquad \therefore \delta = \frac{\varepsilon}{2}.$$

Rmk (1) $x \to a$ 는 x 가 a 에 한없이 가까이 가는 것을 의미하며 $x \neq a$ 는 아니다.

(2)

$$\xrightarrow[x \to a-]{} \quad a \quad \xleftarrow[x \to a+]{}$$

(3) $f(x)$ 가 $x = a$ 에서 정의 되지 않아도 된다. (i.e. $\nexists f(a)$)