현대대수학

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현대대수학

todo : abelian group ? → 곱셈에대해같쳐?
associative ..? ?

Section 18. Ring and Fields (환과체)

Q 에서 +, · 각각 보여야!

Def) A ring $(R, +, \cdot)$ is a nonempty set R with two binary operations $+$ and $\cdot$ such that

1. $(R, +)$ is an abelian group. 교환법칙
2. $\cdot$ is associative on R : $(a\cdot b)\cdot c = a\cdot(b\cdot c)$
3. $a\cdot(b+c) = \underline{a\cdot b} + a\cdot c$ [left distributive]. → $a\cdot(b+c) = a\cdot c + b\cdot c$

   $a\cdot(b+c)$ b+c 연산먼저하나 분배법칙을 하나 같다

   $(a+b)\cdot c = a\cdot c + b\cdot c$ [right distributive].

   for all $a, b, c \in R$

Recall) $(G, +)$ is an abelian gp.

① $+$ is an binary operation i.e. $a+b \in G$ $\forall a, b \in G$

② associate i.e. $(a+b)+c = a+(b+c)$ $\forall a, b, c \in G$

③ identity elt i.e $\exists e \in G$ s.t. $a+e = e+a = a$ $\forall a \in G$

④ inverse elt i.e, $\exists -a \in G$ s.t. $a+(-a) = (-a)+a = e$ $\forall a \in G$

⑤ $a+b = b+a$ $\forall a, b \in G$

Ex) $(\mathbb{Z}, +)$ is an abelian group.

① $2+3 = 5 \in \mathbb{Z}$

② $(2+3)+4 = 2+(3+4)$

③ $2+0 = 0+2 = 2$

④ $2+(-2) = (-2)+2 = 0$

Ex) $(\mathbb{Z}, +, \cdot)$ is a ring, $(\mathbb{Z} \neq \emptyset)$

$+, \cdot$ are binary operations. $2+3 \in \mathbb{Z}$, $2\cdot 3 \in \mathbb{Z}$

① $(\mathbb{Z}, +)$: abelian group.

② $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\forall a, b, c \in \mathbb{Z}$ $(2\cdot 3)\cdot 4 = 2\cdot(3\cdot 4)$

③ $(a+b)\cdot c = a\cdot c + b\cdot c$ $\forall a, b, c \in \mathbb{Z}$ $(2+3)\cdot 4 = 2\cdot 4 + 3\cdot 4$

가환환.

Def) A ring $(R, +, \cdot)$ is a commutative ring if $a\cdot b = b\cdot a$ = $\forall a, b \in R$

Ex) $(\mathbb{Z}, +, \cdot)$ is a commutative ring.

Ex) $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ → comm ring

pf) $\frac{b}{a} \in \mathbb{Q}$, $a \neq 0$ → 증명해보기.

example 18.3) Let $\mathbb{R}$: ring

define $M_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \middle| \, a_{ij} \in R \right\}$

Ex) $\mathbb{Z}$: ring

$M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \right\}$

$(M_n(R), +, \cdot)$ is ring → matrix multiplication.

① $(M_2(\mathbb{Z}), +)$ is an abelian gp.

id elt : $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

② $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$

$(A\cdot B)\cdot C = A\cdot(B\cdot C)$

위의 행렬에 대한 ring 을 matrix ring 이라한다.

↳ 작성해보기.

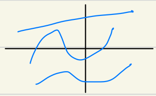
identity element → id elt

Todo : cancellation property.

additive identity : 덧셈에 대한 항등원.

Ex 18.4) $f : \mathbb{R} \to \mathbb{R}$ function.

Define $F = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$



→ 옆과같은 함수의 집합.

$(F, +, \cdot)$

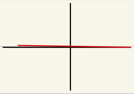
define by $(f+g)(x) = f(x) + g(x)$

$f \cdot g = f(x) \cdot g(x)$

then $(F, +, \cdot)$ is ring.

$(F \cdot +)$: abelian g.p.

id elt : zero map. $(f(x) = 0)$



inverse of $f(x)$ is $-f(x)$

Ex 18.6) $(\mathbb{Z}_n, +, \cdot)$ is a ring (comm ring)

$\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$.

$(\mathbb{Z}, +, \cdot)$, $(M_n(\mathbb{R}), +, \cdot)$, $(\mathbb{Z}_n, +, \cdot)$ → 이 3개는 현대대수학에서 중요하게 사용된다.

쉬운 예이므로 기억해 둘 것.

Ex) $(M_2(\mathbb{Z}_2), +, \cdot)$ $\mathbb{Z}_2$

$M_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$.

$(M_2(\mathbb{Z}_2), +, \cdot)$ is a non commutative ring.

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

$A \cdot B \neq B \cdot A$ → non commutative ring.

Ex 18.7)

Thm 18.8) Let $(R, +, \cdot)$ be a ring with additive identity 0, that for any $a, b \in R$.

(1) $0 \cdot a = a \cdot 0 = 0$

(2) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

(3) $(-a) \cdot (-b) = ab$.

pf) (1) $a \cdot 0 + a \cdot 0 = a(0+0) = a \cdot 0$ and

by the cancellation property of $(R, +)$, $a \cdot 0 = 0$

(2) $a \cdot (-b) + ab = a \cdot (-b+b) = a \cdot 0 = 0$ by (1)

$\therefore a \cdot (-b) = -ab$. similarly $(-a) \cdot b = -ab$.

(3) $(-a) \cdot (-b) = ab$

$(-a) \cdot (-b) = -(a \cdot (-b))$ by (2) and

$-(a \cdot (-b)) = -(-(a \cdot b))$ by (2)

Then $(-a) \cdot (-b) = -(-(a \cdot b)) = a \cdot b$.

→ add inverse of (add inverse ab)

additive identity (identity) : 0 → 항등원.

multiplitive identity (unity) : 1.

Remark) Ring with unity.

모든 Ring은 identity는 항상 가지지만 identity는 가지지 않을수도 있다.

$|M_2(\mathbb{Z}_2)| = 16$.

identity elt $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

unity elt $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Def) For ring R and R'

a map $\phi : R \to R'$ is a ring homomorphism

① $\phi(a+b) = \phi(a) + \phi(b)$ | group homo

② $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ | ring homo

Ex) $\phi : \mathbb{Z} \to \mathbb{Z}'$

Ex) $\phi : \mathbb{Z} \to \mathbb{Z}_5$ by $\phi(n) = r$ when $n = 5q + r$, $0 \le r < 5$

$\{0, 1, 2, 3, 4\}$

Then ϕ is a ring homomorphism. → 증명해보기.

Q : ϕ ?

Ex) $\phi : \mathbb{Z} \to \mathbb{Z}$ (ring ring)

$\phi(a) = r$ where $a = nq + r$ $0 \le r < n$

is a ring homomorphism

$\phi : \mathbb{Z} \to \mathbb{Z}$

$\phi(a) = 5q + r$ $0 \le r < 5$ $\left(\begin{array}{l} 5 \mapsto 0 \\ 6 \mapsto 1 \\ 8 \to 3 \ \cdots \end{array} \right)$.

sol) $a, b \in \mathbb{Z}$ and let

$a = bq_1 + r_1$, $b = nq_2 + r_2$

when $0 \le r_1 < n$, $0 \le r_2 < n$

① $\phi(a+b) = \phi(nq_1 + r_1 + aq_2 + r_2)$

$= \phi(n(q_1 + q_2) + r_1 + r_2) = r_1 + r_2$

$\phi(a) = \phi(nq_1 + r_1) = r_1$

$\phi(b) = \phi(nq_2 + r_2) = r_2$

$\therefore \phi(a+b) = \phi(a) + \phi(b)$

② $a \cdot b = (nq_1 + r_1)(nq_2 + r_2)$

$= n^2 q_1 q_2 + n(q_1 r_2 + q_2 r_1) + r_1^2$

$\phi(a \cdot b) = \phi(n(q_1 q_2 + q_1 r_2 + q_2 r_1) + r_1 r_2)$

$= r_1 \cdot r_2$

$\phi(a) \cdot \phi(b) = \phi(nq_1 + r_1)\, \phi(nq_2 + r_2)$

$= r_1 \cdot r_2$

$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$

초기조건이 중요!

Let $a, b \in \mathbb{Z}$ and $a = nq_1 + r_1$, $b = nq_2 + r_2$

when $0 \le r_1 < n$, $0 \le r_2 < n$

〈시험예제〉 try!

$\phi : \mathbb{Z} \to \mathbb{Z}_4$

by $\phi(n) = r$, when $a = 4q + r$ by .

with $0 \le r < 4$

Ex) let $F = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$

Then for each a the evaluation map . $\phi_a : F \to \mathbb{R}$

$\phi_a(f) = f(a)$ for $f \in F$ is a ring homo

called an evaluation homomorphism.

For example

$\phi_{\sqrt{2}} : F \to \mathbb{R}$ by $\phi_{\sqrt{2}}(f) = f(\sqrt{2})$

$f(x) = x + 1$ then

$\phi_{\sqrt{2}}(f) = \underbrace{\sqrt{2} + 1}_{\in \mathbb{R}}$

$\phi_2(f) = 2 + 1 = 3$

① $\phi_a(f+g) = \phi_a(f) + \phi_a(g)$

② $\phi_a(f \cdot g) = \phi_a(f) \cdot \phi_a(g)$.

①, ② 증명해보기.

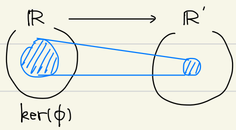
pf) ① $\phi_a(f+g) = (f+g)(a)$

$\phi_a(f)$ $\because$ (a) $\phi_a(g) = g(a)$

$\mathbb{Z}_n$ n으로나누었을 때 나머지의 집합.

Note : kernel of ϕ, dnoted. $\phi(a)=0$ 인 a의 집합.

$\ker(\phi) = \{a \in \mathbb{R} \mid \phi(a) = 0\}$ additive identity.



Ex) $\phi : \mathbb{Z} \to \mathbb{Z}_5$ $\phi : \mathbb{Z} \to \mathbb{Z}_n$

$\ker(\phi) = 5\mathbb{Z}$ $\ker(\phi) = 5n$ -

Ex) $\phi_a : F \to \mathbb{R}$

$\ker(\phi_a) = \{f \mid f(a) = 0\}$.

Def 18.2) A ring homomorphism $\phi : R \to R'$ is called an isormorphism dinoted by $R \cong R'$ if ϕ is 1-1 and onto.

곱셈에대한 역원.

Def) A multiplicative Inverse of an element a in a ring R with unity 1 is an elt a^{-1} such that

$aa^{-1} = a^{-1}a = 1$

Def) Let R be a ring with unity 1

An elt $u \in R$ is a unit if it has a multi Inverse in R

Ex) $(\mathbb{Q}, +, \cdot)$

unit ; $2^{-1} = \frac{1}{2} \to 2 \cdot \frac{1}{2} = \frac{1}{2} \cdot 2 = 1$

Ex) Im $(\mathbb{Z}, +, \cdot)$

units of $\mathbb{Z} = \{1, -1\}$ || dinote

$\mathbb{Z}^{\times} = \{1, -1\}$.

Q 어떠한 집합이 field 임을보이는 과정?

모든원소에대해 0은 역원을가지지 않는다. ($\frac{0}{0}$ ×)

Def 18.6) A ring $(R, +, \cdot)$ is a division ring if $(R - \{0\}, \cdot)$ is a group

실수집합 중 0을제외한 집합.

If $(R - \{0\}, \cdot)$ is an abelian group.

☆ R is called a field (체).

Ex) $(\mathbb{Q}, +, \cdot)$ is a infinite field.

$(\mathbb{Q} - \{0\}, \cdot)$ is an abelian group.

1 is unity.

$a \neq 0$ inverse of a is $\frac{1}{a}$

$(\frac{b}{a} \neq 0$ inverse of a is $\frac{a}{b}$ $(b \neq 0))$

Ex) $(\mathbb{Z}_p, +, \cdot)$ is a field (finite)
(p : prime)

$\mathbb{Z}_p : \{0, 1, 2, \cdots p-1\}$.

Note : $(\mathbb{Z}_n, +, \cdot)$ is not a field in general

for example (counting ex) $(\mathbb{Z}_4, +, \cdot)$ is not a field.

sol) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. (1: unity.)

$2 \cdot \boxed{} = 1$?
$1 = 2$
$2 = 0$
$3 = 2$
$\to$ $\mathbb{Z}_4$ is not field.

※ A ring $(F, +, \cdot)$ field $(F - \{0\}, \cdot)$ abelian group.

Ex) $(\mathbb{Q}, +, \cdot)$ field (infinite)

Ex) $(\mathbb{Z}_p, +, \cdot)$ (p : prime) field (finite)

Ex) $(\mathbb{Z}_p, +, \cdot)$ is field.
ring

show $(\mathbb{Z}_p - \{0\}, \cdot)$ is an abelian g.p.

① $\cdot$ is a binary operation $a \cdot b \in \mathbb{Z}_p$ for $a, b \in \mathbb{Z}_p$.

② $\cdot$ is an associative

③ identity elt.

④ inverse elt

⑤ $a \cdot b = b \cdot a$.

※ ①,②,③,⑤는 자명하므로 ④만 증명하면 된다.

Todo : GCD 세로식 구하는 거 다시 공부.

Recall from the number theory.

Thm) $(a,b)=1$ (GCD (a,b)), then $\exists\ s,t \in \mathbb{Z}$ s.t.

$as+bt=1$

Ex) $(4,7)=1$ $\exists\ s,t$ s.t. $4s+7t=1$

Let $m \leftarrow \mathbb{Z}_p - \{0\}$ then (p: prime.)

$(m,p)=1$ $\exists$ (by the thm)

$\exists\ s,t$ s.t. $ms + pt \equiv 1 \pmod p$. ($pt \bmod p = 0$.)

Then $ms \equiv 1$ then $m^{-1} \equiv s$

Ex) $(\mathbb{Z}_p, +, \cdot)$ is a field.

$\mathbb{Z}_{13} = \{0,1,2,3,\dots,12\}$.

Find 5^{-1}? (multi inverse of 5 in $\mathbb{Z}_{13}$).

sol) $(13,5)=1$ want to find s,t s.t.

$13s+5t \equiv 1 \pmod{13}$ $(\equiv 0)$

to find $(13,5)_{GCD}$.

$(13,5) \to (3,5) \to (3,2) \to (1,2) \to (1,0) = 1$

2	13	5	1
	10	3	
1	3	2	2
	2	2	
	1	0	

to find s.t.

	2	1	1	2	
s	1	0	1	−1	2
t	0	1	−2	3	−5

$13 \cdot 2 + 5(-5) = 1$

$\therefore\ 5^{-1} = -5 \equiv 8 \pmod{13}$.

Ex2) find 8^{-1}

sol) $(13,8)=1$

$13s+8t=1$

1	13	8	1
	8	5	
1	5	3	1
	3	2	
2	2	1	
	2		
	0		

$\Rightarrow$

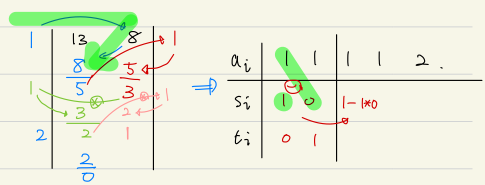
a_i	1	1	1	1	2	
s_i	1	0	1	−1	2	−3 : s
t_i	0	1	−1	2	−3	5 : t

여기는 항상 고정.

$13(-3) + 8(5) = 1$

$\therefore\ 8^{-1} = 5 \pmod{13}$.

유클리드 호제법 세로표 작성법.



연습: $\mathbb{Z}_{17}$ 에서 5^{-1}는?

3	17	5	2
	15	4	
2	2	1	
	2		
	0		

a_i	3	2	2	
s_i	1	0	1	−2 : s
t_i	0	1	−3	7 : t

$(17,5)=1$

$\underbrace{17 \cdot (-2)}_{\bmod 17 = 0} + 5 \cdot (7) = 1$

$\therefore\ 5^{-1} = 7$.

Todo: 19.3 pf 다시해보기.

p ⇒ q 일때　　Q $m\frac{n}{d} = \frac{m}{d}n = 0$ 왜?.

※ 간접증명 ~q ⇒ ~p 를 증명.

직접증명 p ⇒ q 를 바로 증명.

Ex) $\mathbb{Z}_{19}$, $5^{-1} = ?$

$(19, 5) = ?$

3	19	5	1
	15	4	
4	4	1	
	4		
	0		

a	3	1	4	
s	1	0	1	-1
t	0	1	-3	4

$\underbrace{19\cdot(-1)}_{\text{mod }19 = 0} + 5\cdot 4 = 1$.

$5^{-1} = 4$.

Ex) $\mathbb{Z}_{23}$　$9^{-1} = ?$

2	23	9	1
	18	5	
1	5	4	4
	4	4	
	1	0	

a	2	1	1	4	
s	1	0	1	-1	2
t	0	1	-2	3	-5

$\underbrace{23\cdot 2}_{\text{mod }23=0} + 9\cdot(-5) = 1$

$\therefore 9^{-1} = -5 = 17$.

Section 19. Integral Domains.

Def 19.2) If a and b are two non zero elts of a ring R such that $a\cdot b = 0$, then

영인자에 대한 정의.

a and b are zero divisors (or divisions of zero).

($a.b \neq 0$　$a\cdot b = 0$ 일때 $a.b$를 zero divisions 라한다.)

Ex) In $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$　$(\mathbb{Z}, +, \cdot)$ ring.

위의 정의에따라 $\mathbb{Z}_6 - \{0\}$ 에서 생각한다.

$\{1, 2, 3, 4, 5\}$.

zero divicions.　$2\cdot 3 = 0$, $3\cdot 4 = 0$

5는　$5\cdot 1 \neq 0$, $5\cdot 2 \neq 0$　$5\cdot 3 \neq 0$　$5\cdot 4 \neq 0$

1은　$1\cdot 1 \neq 0$　$1\cdot 2 \neq 0$ ...

$\therefore$ 1과 5는　non zero divisor

Thm 19.3) In the ring $\mathbb{Z}_n$,

m is a zero divisor iff $(m, n) \neq 1$

(⟸⟹)　if and only if : 필요충분조건.

ex) $(1,6) = 1$, $(5,6) = 1$, $(2,6) \neq 1$, $(3,6) \neq 1$.

pf) (⟸) Let $m \in \mathbb{Z}_n$, $m \neq 0$ and $(m,n) \neq 1$, and

$(m,n) = d > 1$ then $m\left(\frac{n}{d}\right) = \left(\frac{m}{d}\right)n = 0$

이 증명의 핵심.

(ex: $(4,6) = 2$). so $m\left(\frac{n}{d}\right) = 0$ and $\frac{n}{d} \neq 0$

Thus m is a zero divisor.

「 Not a zero divisor ; $a\cdot b = 0 \Rightarrow a = 0$ or $b = 0$ 」

(⟹) Suppose $(m,n) = 1$ If $s \in \mathbb{Z}_n$

간접증명※ ~q ⇒ ~p

$m\cdot s = 0$, then　$n \mid ms$ (n devides ms).

ms가 n의배수이다.

Since $(m,n) = 1$, $n \mid s$

so that $s = 0$ in $\mathbb{Z}_n$

참고!　Recall) From the number theory,

Thm) $n \mid ms$ and $(m,s) = 1 \Rightarrow n \mid s$

ex) $4 \mid 3\cdot 8$　$(4,3) = 1 \Rightarrow 4 \mid 8$.

In $\mathbb{Z}_n$　$(m,n) \neq 1 \iff m$ is a zero divisor.

$(m,n) = 1$　　m is a unit.

Ex) $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

unit : $\{1, 5, 7, 11\}$

곱셈에 대한역원.

zero-divisors : $\{2, 3, 4, 6, 8, 9, 10\}$

Ex) $\mathbb{Z}_{24} = \{0, 1, 2, 3, 4, \cdots, 22, 23\}$.

unit : $\{1, 5, 7, 11, 13, 17, 19, 23\}$.

zero-divisor : $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22\}$

Cancellation laws 란?

Corollary) If p is a prime, then

$\mathbb{Z}_p$ has no zero divisors.

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

Note) The cancellation laws hold in $\mathbb{R}$
(property).

if $a \cdot b = a \cdot c$ with $a \neq 0$ → $\cancel{a}b = \cancel{a}c$
$b = c$.

implies $b = c$ and $b \cdot a = c \cdot a$

with $a \neq 0$ implies $b = c$

In $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$.

$2 \cdot 3 = 4 \cdot 3 \Rightarrow 2 = 4$.

In general. Ring 에서는 cancellation laws 가 성립X.
(곱셈에대한).

Recall from the group theory,

In a group$(G, *)$ cancellations laws always hold.
Group 에서는 C·l이 항상성립.
→ group 의 elt 는 모두 역원이 존재하므로 성립.

ring 에서는 역원이 존재할수도 안할수도 있으므로 일반적으로 성립 X

$(\mathbb{Z}_n, +, \cdot)$ $m \neq 0$ $n \in \mathbb{Z}$

$(m, n) \neq 1$ → m is a zero divisor.

$(m, n) = 1$ → m is a unit

덧셈에대한 C·l은 성립. 우리는 곱셈에 대한 것을 본다.

Thm) cancellation laws hold in a ring R

iff R has no zero divisor,

pf) ($\Rightarrow$) suppose $a \cdot b = 0$ ⇝ $a = 0$ or $b = 0$.

If $a \neq 0$, then $a \cdot b = a \cdot 0$.

implies $b = 0$ by the cancellation laws.
$(\cancel{a} \cdot b = \cancel{a} \cdot 0)$ → punchline of this pf

Similary, $b \neq 0$ implies $a = 0$

thus, R has no zero divisor.

($\Leftarrow$) Suppose R has no zero divisor,

and suppose $ab = ac$. with $a \neq 0$

Then $ab - ac = a(b - c) = 0$ → punchline of this pf.

since $a \neq 0$ and since R has no zero divisor,

we must have $b - c = 0$ so that $b = c$

Similary $ba = ca$ with $a \neq 0$ then $b = c$.

Def 19.6) An Integral domain is a commutative ring

with unity 1 and containing no zero divisors.

Ex) $(\mathbb{Z}, +, \cdot)$ is an ID.

Ex) $(\mathbb{Z}_6, +, \cdot)$ not an ID.

Ex) $(\mathbb{Z}_p, +, \cdot)$ is an ID.

Note) $\mathbb{Z}_n$ is not an ID, in general (n이 prime 일때만 ID).

Ex 19.7) R, S be two ID.

but $R \times S$ is not an ID

$R \times S = \{(r, s) \mid r \in R, s \in S\}$

$(r, 0)$ $(0, s) \leftarrow R \times S$

but $(r, 0) \cdot (0, s) = (0, 0)$.

$M_2(\mathbb{Z}_2) = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdots, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \}$.

EX19.8) $(M_2(\mathbb{Z}_2), +, \cdot)$

$|M_2(\mathbb{Z}_2)| = 16$.

$(M_2(\mathbb{Z}_2), +, \cdot)$ non comm ring.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (non zero · nonzero = zero. ∴ not an ID.)

* < Field, ID 모두 Ring 이다. >
체

Thm 19.9) Every field is an ID.
필드이면 ID이다. → 역은성립X.

pf) Let $a, b \in F$ and $ab = 0$ $\underline{w/ \ a \neq 0 \rightsquigarrow b = 0}$.
이것만 보이면 된다.

Then $a^{-1}(ab) = a^{-1} \cdot 0 = 0$. → punchline of this pf.
field 이므로 역원을가짐.

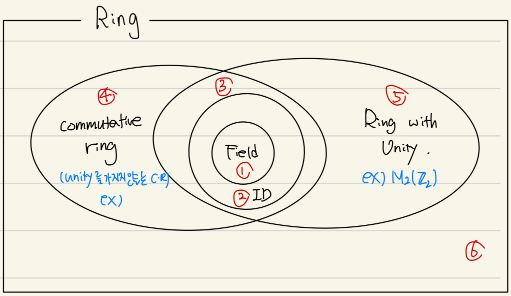
But $0 = a^{-1}(ab) = (a^{-1}a)b = 1 \cdot b = b$ thus $b = 0$.

Similary, $ab = 0$ w/ $b \neq 0$, then $a \neq 0$.

∴ Every field is an ID.

Note) Field $\underset{\times}{\rightleftarrows}$ ID

Counter ex) $(\mathbb{Z}, +, \cdot)$ is an ID but not a field.



① ex ; $\mathbb{Z}_p$
③ ex : $\mathbb{Z}_6$

시험에서 Ring 임은 생략가능

시험예제 : ③ 번의 예제를 제시하고 보여라.
$\mathbb{Z}_6$ 는 commutative ring 이며 ring with unity 이고
ID는 아니다.

⑥ 의 ex 는?

Todo : Diagram ①~⑥ 예제 구해보기.

unity : 곱셈에대한 id

unit ; elt has an inverse.

a : unit and $ab = 0$.

$a^{-1}(ab) = a^{-1}a \ b = 1 \cdot b = 0$.

∴ $b = 0$.

Thm) Every finite ID is a field.

pf) 생략

Ex) $(\mathbb{Z}_p, +, \cdot)$ is a finite ID so that a field.