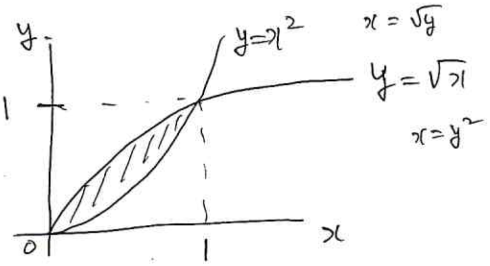
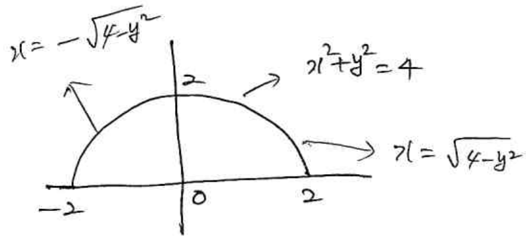
Exam $y=x^2$, $y=\sqrt{x}$ $(x\ge 0, y\ge 0)$ 으로 둘러싸인 D 위의 곡면 $f(x,y)=x^2+y^2$ 아래에 놓인 입체의 부피를 구하라.



$$\iint_D (x^2+y^2)\,dy\,dx$$
$$=\int_0^1\int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy\,dx.$$
$$=\int_0^1 \left[x^2y+\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx.$$
$$=\int_0^1 \left(x^2\sqrt{x}+\frac{x^{\frac{3}{2}}}{3}-x^4-\frac{x^6}{3}\right)dx.$$
$$=\left[\frac{2}{7}x^{\frac{7}{2}}+\frac{2}{15}x^{\frac{5}{2}}-\frac{1}{5}x^5-\frac{1}{21}x^7\right]_0^1$$
$$=\frac{2}{7}+\frac{2}{15}-\frac{1}{5}-\frac{1}{21}=\frac{1}{105}(30+14-21-5)$$
$$=\frac{18}{105}$$

$$\iint_D (x^2+y^2)\,dx\,dy=\int_0^1\int_{y^2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy$$ 로도 가능함

Exam x축과 반원 $x^2+y^2=4$, $y\ge 0$ 으로 둘러싸인 영역 위에서 함수 $f(x,y)=x^2y$ 의 이중적분값을 계산하자

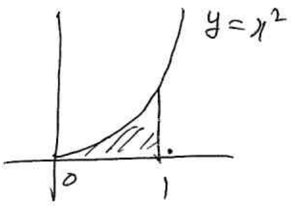


$$\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} x^2y\,dy\,dx.=\frac{64}{15}$$
$$=\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y\,dx\,dy.=\frac{64}{15}$$

Rmk 적분순서 교환은 교환했을때 계산을 좀더 간단히 할 수 있을 경우만 사용해야한다.

예제 $I=\int_0^1\int_{\sqrt{y}}^1 e^{\frac{y}{x}}\,dx\,dy$ 값을 계산하라.

$D=\{\sqrt{y}\le x\le 1\ ,\ 0\le x\le 1\}$



$\int e^{\frac{y}{x}}\,dx$ 가 적분불능 이므로 적분순서 교환.

$$I=\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dy\,dx = \int_0^1 \left[x e^{\frac{y}{x}}\right]_0^{x^2} dx$$

$$=\int_0^1 (x\cdot e^x - x)\,dx = \left[xe^x - e^x - \frac{x^2}{2}\right]_0^1$$

$$=\left(e-e-\frac{1}{2}+1\right)=\frac{1}{2}$$

— 중적분에서의 이상적분 —

D 에서 $f(x,y)$ 가 불연속이거나 유계가 아닌 부분이 있을 경우. 아닌부분을 D 제외한 부분영역 D' 를 잡아 (단 $D'\to D$)

$\iint_D f(x,y)\,dy\,dx = \lim_{D'\to D}\iint_{D'} f(x,y)\,dy\,dx$. 로 정의한다.

예제 ~~예제~~ $D=\{(x,y)\mid 0\le y<x\le 1\}$

$\iint_D (x-y)^{-\alpha}\,dx\,dy$. , $0<\alpha<1$

$y=x$ 이면 불연속 ⇒ $D'=\{(x,y)\mid \varepsilon\le x\le 1\ ,\ 0\le y\le x-\varepsilon\}$

그 잡으면

$$\iint_D (x-y)^{-\alpha}\,dx\,dy = \lim_{\varepsilon\to 0}\iint_{D'}(x-y)^{-\alpha}\,dy\,dx$$

$$=\lim_{\varepsilon\to 0}\int_\varepsilon^1\int_0^{x-\varepsilon}(x-y)^{-\alpha}\,dy\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1\left[\frac{-1}{1-\alpha}(x-y)^{1-\alpha}\right]_0^{x-\varepsilon}dx.$$

$$=\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{-1}{1-\alpha}\left(\varepsilon^{1-\alpha}-x^{1-\alpha}\right)dx. = \lim_{\varepsilon\to 0}\frac{-1}{1-\alpha}\left[\varepsilon^{1-\alpha}x-\frac{1}{2-\alpha}x^{2-\alpha}\right]_0^1$$

$$=\lim_{\varepsilon\to 0}\frac{-1}{1-\alpha}\left(\varepsilon^{1-\alpha}-\frac{1}{2-\alpha}\right) = \frac{-1}{1-\alpha}\cdot\frac{-1}{2-\alpha} = \frac{1}{(1-\alpha)(2-\alpha)}$$

— 직교좌표에서 극좌표 로의 변환 —

$x=x(u,v)$, $y=y(u,v)$ 에 의해 xy-평면상의 영역 D.
에서 uv-평면상의 집합 D' 으로의 1대1 대응이 주어지고
$x(u,v)$, $y(u,v)$ 가 u,v 에 관하여 미분가능이고 그 도함수가 연속일때.
Jacobian J 가

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\neq 0 .$$

이면

$$\iint_D f(x,y)\,dy\,dx=\iint_{D'} f(x(u,v),y(u,v))\,|J|\,du\,dv .$$

이다.

<Rmk> $x=r\cos\theta$ $y=r\sin\theta$

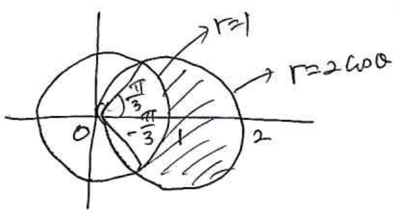
$$\Rightarrow |J|=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta$$
$$=r.$$

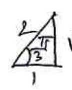
$$\therefore \iint_D f(x,y)\,dy\,dx=\iint_{D'} f(r\cos\theta,r\sin\theta)\cdot r\,dr\,d\theta.$$

<u>Exam</u> $D=\{(x,y)\mid x^2+y^2-2x\leq 0,\ x^2+y^2\geq 1\}$ 일 때

$$\iint_D\sqrt{x^2+y^2}\,dy\,dx.$$

<Sol>. $x^2+y^2-2x=(x-1)^2+y^2-1=0$ i.e. $(x-1)^2+y^2=1$





$x=r\cos\theta$, $y=r\sin\theta$ ⇒ 대입.

$r=1$, $r=2\cos\theta$. ⇒ 연립 $2\cos\theta=1$

$\therefore \cos\theta=\frac{1}{2}$ $\theta=\pm\frac{\pi}{3}$

$\sqrt{x^2+y^2}=\sqrt{r^2}=r.$ $|J|=r.$

$$\therefore \iint_D\sqrt{x^2+y^2}\,dy\,dx=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\cdot r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left[\frac{r^3}{3}\right]_1^{2\cos\theta}d\theta=\frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(8\cos^3\theta-1)\,d\theta.$$

$$=\frac{1}{3}\left\{8\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos\theta(1-\sin^2\theta)\,d\theta-[\theta]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\right\}=\frac{1}{3}\left\{8\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}-\frac{2}{3}\pi\right\}$$

$$=\frac{1}{3}\left(8\cdot\left(\frac{\sqrt{3}}{2}-\frac{1}{3}\cdot\frac{3\sqrt{3}}{8}+\frac{\sqrt{3}}{2}-\frac{1}{3}\cdot\frac{3\sqrt{3}}{8}\right)-\frac{2}{3}\pi\right)$$

$$= \frac{1}{3}\left\{8\left(\sqrt{3}-\frac{\sqrt{3}}{4}\right)-\frac{2}{3}\pi\right\}$$

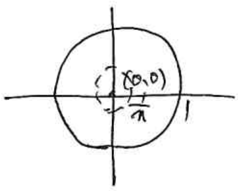
$$= \frac{8}{3}\left(\sqrt{3}-\frac{\sqrt{3}}{4}\right)-\frac{2}{9}\pi .$$

exam $\quad D=\{(x,y)\mid x^2+y^2\le 1\}$.

$$I=\iint_D \frac{1}{(x^2+y^2)^{\frac{\alpha}{2}}}\,dy\,dx \qquad (0<\alpha<2)$$

<sol>

(0,0) 에서 불연속 이므로



$D'=\{(x,y)\mid \frac{1}{n^2}\le x^2+y^2\le 1\}$ 로 두면 $\lim_{n\to\infty} D'=D$.

$x=r\cos\theta,\ y=r\sin\theta$ 라 두면

$$I=\lim_{n\to\infty}\int_{\frac{1}{n}}^{1}\int_0^{2\pi}\frac{1}{r^{\alpha}}\cdot r\,d\theta\,dr.$$

$$=\lim_{n\to\infty}\int_{\frac{1}{n}}^{1} 2\pi\cdot r^{1-\alpha}\,dr.$$

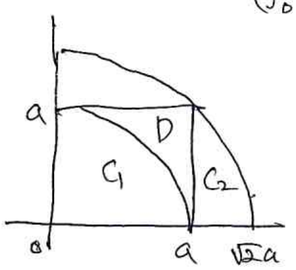
$$=\lim_{n\to\infty}2\pi\left[\frac{1}{2-\alpha}r^{2-\alpha}\right]_{\frac{1}{n}}^{1}$$

$$=\frac{2\pi}{2-\alpha}\lim_{n\to\infty}\left(1-\left(\frac{1}{n}\right)^{2-\alpha}\right)=\frac{2\pi}{2-\alpha}$$

exam $\quad I=\int_0^\infty e^{-x^2}\,dx$ 를 구하라.

$$I^2=\left(\int_0^\infty e^{-x^2}dx\right)\cdot\left(\int_0^\infty e^{-y^2}dy\right)=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx.$$

$$=\lim_{a\to\infty}\int_0^a\int_0^a e^{-(x^2+y^2)}\,dy\,dx.$$



$$e^{-(x^2+y^2)}>0$$

$$\therefore\ \iint_{C_1}e^{-(x^2+y^2)}\,dy\,dx<\iint_D e^{-(x^2+y^2)}\,dy\,dx<\iint_{C_2}e^{-(x^2+y^2)}\,dy\,dx.$$

$$\iint_{C_1} e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^a e^{-r^2}\cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[-\frac{1}{2}e^{-r^2}\right]_0^a d\theta.$$

$$= \int_0^{\frac{\pi}{2}}\left(-\frac{1}{2}e^{-a^2}+\frac{1}{2}\right)d\theta = \frac{\pi}{4}\left(1-e^{-a^2}\right)$$

$$\iint_{C_2} e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}a} e^{-r^2}\cdot r\,dr\,d\theta = \frac{\pi}{4}\left(1-e^{-2a^2}\right)$$

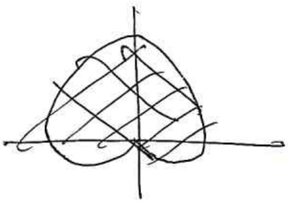
$$\therefore \quad \frac{\pi}{4}\left(1-e^{-a^2}\right) < \left\{\int_0^a e^{-x^2}dx\right\}^2 < \frac{\pi}{4}\left(1-e^{-2a^2}\right)$$

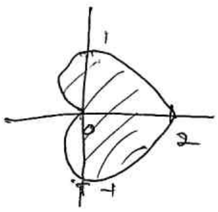
$$a\to\infty \quad \Rightarrow \quad \frac{\pi}{4} \le I^2 \le \frac{\pi}{4}$$

$$\therefore \quad I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

<u>exm</u>

$r=1+\cos\theta$ (심장형) 위의 곡면 $z=3+r$ 아래에 놓여있는 입체의 부피?





$$D=\{(r,\theta)\mid 0\le r\le 1+\cos\theta,\ 0\le\theta\le 2\pi\}.$$

$$V=\int_0^{2\pi}\int_0^{1+\cos\theta}(3+r)\cdot r\,dr\,d\theta.$$

$$=\int_0^{2\pi}\left[\frac{3r^2}{2}+\frac{r^3}{3}\right]_0^{1+\cos\theta}d\theta.$$

$$=\int_0^{2\pi}\left(\frac{3}{2}(1+\cos\theta)^2+\frac{1}{3}(1+\cos\theta)^3\right)d\theta.$$

$$=\int_0^{2\pi}\left(\frac{3}{2}\left(1+2\cos\theta+\cos^2\theta\right)+\frac{1}{3}\left(1+3\cos\theta+3\cos^2\theta+\cos^3\theta\right)\right)d\theta.$$

$$=\int_0^{2\pi}\left(\frac{11}{6}+4\cos\theta+\frac{5}{2}\cos^2\theta+\frac{1}{3}\cos^3\theta\right)d\theta.$$

$$=\int_0^{2\pi}\left(\frac{11}{6}+4\cos\theta+\frac{5}{2}\cdot\frac{1-\cos2\theta}{2}+\frac{1}{3}\cos\theta\left(1-\sin^2\theta\right)\right)d\theta.$$

$$=\left[\frac{11}{6}\theta+4\sin\theta+\frac{5}{4}\left(\theta-\frac{\sin2\theta}{2}\right)+\frac{1}{3}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)\right]_0^{2\pi}$$

$$=\frac{22\pi}{6}+\frac{10\pi}{4}=\frac{37}{6}\pi$$