3. $I = \int f(x, \sqrt{ax^2+bx+c}\,)\,dx \quad (a \neq 0)$

(1) $a > 0$ 인 경우

put $\sqrt{ax^2+bx+c} = t - \sqrt{a}\,x$

$$\Rightarrow \cancel{ax^2} + bx + c = t^2 - 2\sqrt{a}\,tx + \cancel{ax^2}$$

$$(b + 2\sqrt{a}\,t)x = t^2 - c$$

$$\therefore \quad x = \frac{t^2 - c}{b + 2\sqrt{a}\,t}$$

$$dx = \frac{2t(b+2\sqrt{a}\,t) - (t^2-c)2\sqrt{a}}{(b+2\sqrt{a}\,t)^2}\,dt$$

$$= \frac{2\sqrt{a}\,t^2 + 2bt + c2\sqrt{a}}{(b+2\sqrt{a}\,t)^2}\,dt = \frac{2(\sqrt{a}\,t^2+bt+c\sqrt{a})}{(b+2\sqrt{a}\,t)^2}\,dt.$$

$$\sqrt{ax^2+bx+c} = t - \sqrt{a}\,x = t - \sqrt{a}\,\frac{t^2-c}{b+2\sqrt{a}\,t}$$

$$= \frac{2\sqrt{a}\,t^2 + bt - \sqrt{a}\,t^2 + \sqrt{a}\,c}{b+2\sqrt{a}\,t} = \frac{\sqrt{a}\,t^2 + bt + \sqrt{a}\,c}{b+2\sqrt{a}\,t}$$

$$\therefore \quad I = \int f\left(\frac{t^2-c}{b+2\sqrt{a}\,t}, \frac{\sqrt{a}\,t^2+bt+\sqrt{a}\,c}{b+2\sqrt{a}\,t} \right) \cdot \frac{2(\sqrt{a}\,t^2+bt+c\sqrt{a})}{(b+2\sqrt{a}\,t)^2}\,dt.$$

: 유리함수.

<u>exam</u> $I = \int \frac{1}{\sqrt{x^2+a}}\,dx.$

put $\sqrt{x^2+a} = t - x \qquad x^2 + a = t^2 - 2tx + x^2$

$$x = \frac{t^2-a}{2t} \qquad dx = \frac{2t^2 - (t^2-a)}{2t^2}\,dt = \frac{t^2+a}{2t^2}\,dt.$$

$$\sqrt{x^2+a} = t - x = t - \frac{t^2-a}{2t} = \frac{t^2+a}{2t}$$

$$\therefore \quad I = \int \frac{2t}{t^2+a} \cdot \frac{t^2+a}{2t^2}\,dt = \int \frac{1}{t}\,dt = \ln|t|$$

$$= \ln|x + \sqrt{x^2+a}|$$

(2) $a<0$ 인 경우

$ax^2+bx+c=0$ 의 서로 다른 두실근 . α, β $(\alpha<\beta)$

put $\sqrt{\dfrac{x-\alpha}{\beta-x}}=t \Rightarrow \dfrac{x-\alpha}{\beta-x}=t^2$

$\Rightarrow$ $(1+t^2)x=\beta t^2+\alpha$ $\quad\therefore\ x=\dfrac{\beta t^2+\alpha}{1+t^2}$

$$dx=\frac{2\beta t(\cancel{1+t^2})-(\cancel{\beta t^2+\alpha})2t}{(1+t^2)^2}dt=\frac{2(\beta-\alpha)t}{(1+t^2)^2}dt.$$

$$\sqrt{ax^2+bx+c}=\sqrt{a(x-\alpha)(x-\beta)}=\sqrt{-a(x-\alpha)(\beta-x)}$$

$$=\sqrt{-a}\cdot(\beta-x)\sqrt{\frac{x-\alpha}{\beta-x}}=\sqrt{-a}\left(\beta-\frac{\beta t^2+\alpha}{1+t^2}\right)\cdot t.$$

$$=\sqrt{-a}\cdot\frac{(\beta-\alpha)t}{1+t^2}$$

$$\therefore\ I=\int f\left(\frac{\beta t^2+\alpha}{1+t^2},\ \sqrt{-a}\cdot\frac{(\beta-\alpha)t}{1+t^2}\right)\frac{2(\beta-\alpha)t}{(1+t^2)^2}dt.$$

: 유리함수 적분 .

exm. $I=\displaystyle\int\frac{1}{(x-1)\sqrt{2+x-x^2}}dx.$

<sol>

$-x^2+x+2=-(x^2-x-2)=-(x-2)(x+1)=0$

$=(2-x)(x+1)$

$x=2,\ x=-1$

$\therefore\ \sqrt{\dfrac{x+1}{2-x}}=t.\quad \dfrac{x+1}{2-x}=t^2\quad (1+t^2)x=2t^2-1$

$\therefore\ x=\dfrac{2t^2-1}{1+t^2}\qquad dx=\dfrac{4t(1+t^2)-(2t^2-1)2t}{(1+t^2)^2}dt$

$$=\frac{6t}{(1+t^2)^2}dt.$$

$$x-1=\frac{2t^2-1}{1+t^2}-1=\frac{t^2-2}{1+t^2}$$

$$\sqrt{2+x-x^2}=\sqrt{(2-x)(x+1)}=(2-x)\sqrt{\frac{x+1}{2-x}}=\left(2-\frac{2t^2-1}{1+t^2}\right)\cdot t.$$

$$=\frac{3t}{1+t^2}$$

$$\therefore\ I=\int\frac{\cancel{1+t^2}}{t^2-2}\cdot\frac{\cancel{1+t^2}}{\cancel{3t}}\cdot\frac{\overset{2}{\cancel{6t}}}{\cancel{(1+t^2)^2}}dt=\int\frac{2}{t^2-2}dt.$$

$t=\sqrt{\frac{x+1}{2-x}}$ 대입.

$$=2\cdot\frac{1}{2\sqrt2}\left(\int\left(\frac{1}{t-\sqrt2}-\frac{1}{t+\sqrt2}\right)dt\right)=\frac{1}{\sqrt2}\ln\left|\frac{t-\sqrt2}{t+\sqrt2}\right|$$

4. $I = \int x^m (ax^n+b)^{\frac{p}{q}} dx$ (m, n, p, q : 정수, $q>0$)

(1) $\frac{m+1}{n}$: 정수 인 경우.

put $ax^n+b = t^q \Rightarrow x = \left(\frac{t^q-b}{a}\right)^{\frac{1}{n}}$

$$dx = \frac{1}{n} \cdot \left(\frac{t^q-b}{a}\right)^{\frac{1}{n}-1} \cdot q t^{q-1} dt.$$

$$x^m = \left(\frac{t^q-b}{a}\right)^{\frac{m}{n}}, \quad (ax^n+b)^{\frac{p}{q}} = t^p.$$

$$\therefore I = \int \left(\frac{t^q-b}{a}\right)^{\frac{m}{n}} \cdot t^p \cdot \frac{1}{n}\left(\frac{t^q-b}{a}\right)^{\frac{1}{n}-1} \cdot q t^{q-1} dt.$$

$$= \frac{q}{n} \int t^{p+q-1} \cdot \left(\frac{t^q-b}{a}\right)^{\frac{m}{n}+\frac{1}{n}-1} dt.$$

$\frac{m}{n}+\frac{1}{n}-1 = \frac{m+1}{n}-1$: 정수. $p+q-1$: 정수

$\therefore$ I : 유리함수 적분.

(2) $\frac{m+1}{n} + \frac{p}{q}$: 정수 인 경우.

put $bx^{-n}+a = t^q \Rightarrow x = \left(\frac{t^q-a}{b}\right)^{-\frac{1}{n}}$

$$dx = -\frac{1}{n}\left(\frac{t^q-a}{b}\right)^{-\frac{1}{n}-1} \cdot \frac{q}{b} t^{q-1} dt.$$

$$x^m = \left(\frac{t^q-a}{b}\right)^{-\frac{m}{n}} \qquad ax^n+b = t^q \cdot x^n$$

$$(ax^n+b)^{\frac{p}{q}} = t^p \cdot x^{\frac{np}{q}} = t^p \cdot \left(\frac{t^q-a}{b}\right)^{-\frac{p}{q}}$$

$$\therefore I = \int \left(\frac{t^q-a}{b}\right)^{-\frac{m}{n}} \cdot t^p \left(\frac{t^q-a}{b}\right)^{-\frac{p}{q}} \cdot \left(-\frac{1}{n}\right) \cdot \left(\frac{t^q-a}{b}\right)^{-\frac{1}{n}-1} \cdot \frac{q}{b} t^{q-1} dt.$$

$$= \left(-\frac{q}{nb}\right) \int t^{p+q-1} \cdot \left(\frac{t^q-a}{b}\right)^{-\frac{m}{n}-\frac{1}{n}-\frac{p}{q}-1} dt$$

$-\left(\frac{m+1}{n}+\frac{p}{q}\right)-1$: 정수 $\quad p+q-1$: 정수

$\therefore$ I : 유리함수 적분

exm $I=\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$

<sol> $I=\int (x^2+1)^{-\frac{3}{2}}\,dx.$

$m=0,\quad n=2,\quad \frac{p}{s}=-\frac{3}{2}$

$\frac{m+1}{n}+\frac{p}{s}=-1$: 정수.

put $x^{-2}+1=t^2 \qquad x=(t^2-1)^{-\frac{1}{2}} \qquad dx=-\frac{1}{2}(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt$

$$=-(t^2-1)^{-\frac{3}{2}}\cdot t\,dt.$$

$$(x^2+1)^{-\frac{3}{2}}=(t^2x^2)^{-\frac{3}{2}}=t^{-3}\cdot(t^2-1)^{\frac{3}{2}}$$

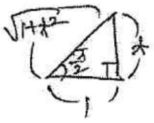
$$\therefore\quad I=\int t^{-3}\cdot(t^2-1)^{\frac{3}{2}}\left(-(t^2-1)^{-\frac{3}{2}}\cdot t\right)dt.$$

$$=-\int t^{-2}\,dt=\frac{1}{t}=\frac{1}{\sqrt{\frac{1+x^2}{x^2}}}=\frac{x}{\sqrt{1+x^2}}$$

— 삼각함수 의 적분 —

1. $I=\int f(\sin x,\cos x)\,dx$ 인 경우.

put $\tan\frac{x}{2}=t \qquad x=2\tan^{-1}t \qquad dx=\frac{2}{1+t^2}\,dt.$



$$\sin x=\sin\left(\frac{x}{2}+\frac{x}{2}\right)=2\sin\frac{x}{2}\cdot\cos\frac{x}{2}=2\cdot\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}$$

$$=\frac{2t}{1+t^2}$$

$$\cos x=\cos\left(\frac{x}{2}+\frac{x}{2}\right)=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1}{1+t^2}-\frac{t^2}{1+t^2}$$

$$=\frac{1-t^2}{1+t^2}$$

$\therefore\quad I=\int f\left(\frac{2t}{1+t^2},\ \frac{1-t^2}{1+t^2}\right)\cdot\frac{2}{1+t^2}\,dt$: 유리함수 적분.

exm $I=\int\frac{1}{\sin x+\cos x}\,dx=\int\frac{1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$$=\int\frac{2}{1+2t-t^2}\,dt=-2\int\frac{1}{t^2-2t-1}\,dt=-2\int\frac{1}{(t-1)^2-2}\,dt.$$

$$=-\frac{2}{2\sqrt{2}}\int\left(\frac{1}{(t-1)-\sqrt{2}}-\frac{1}{(t-1)+\sqrt{2}}\right)dt.$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{(t-1)+\sqrt{2}} - \frac{1}{(t-1)-\sqrt{2}} \right) dt$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{(t-1)+\sqrt{2}}{(t-1)-\sqrt{2}} \right| \qquad t = \tan\frac{x}{2}$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{\tan\frac{x}{2} - 1 + \sqrt{2}}{\tan\frac{x}{2} - 1 - \sqrt{2}} \right|$$

2. $I = \int f(e^{ax})\, dx$

put $e^{ax} = t \qquad a e^{ax} dx = dt \qquad \therefore dx \ \frac{1}{at} dt.$

$\therefore \ I = \int f(t) \cdot \frac{1}{at} dt$: 유리 함수.

exam $\quad I = \int \frac{1}{ae^x + b} dx$

put $e^x = t \qquad dx = \frac{1}{t} dt$

$$\therefore \ I = \int \frac{1}{at+b} \cdot \frac{1}{t} dt.$$

$$= \frac{1}{b} \int \left(\frac{1}{t} - \frac{a}{at+b} \right) dt.$$

$$= \frac{1}{b} \left(\ln t - \ln(at+b) \right)$$

$$= \frac{1}{b} \left(x - \ln(ae^x + b) \right)$$

3. $I = \int f^{-1}(x)\, dx$ 인 경우.

put $f^{-1}(x) = t \ \Rightarrow \ x = f(t) \qquad dx = f'(t)\, dt$

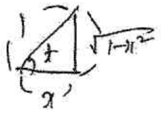
$$\therefore \ I = \int t \cdot f'(t)\, dt = t \cdot f(t) - \int f(t)\, dt.$$

$$= f^{-1}(x) \cdot x - \int f(t)\, dt.$$

exam $I=\int \cos^{-1}x\,dx$

<sol> put $\cos^{-1}x = t$ $\quad x=\cos t \quad dx=(\cos t)'dt$.

$$\therefore\ I = \int t\cdot(\cos t)'dt = t\cdot\cos t - \int \cos t\,dt$$



$$= t\cdot\cos t - \sin t = x\cdot\cos^{-1}x - \sqrt{1-x^2}$$

<Rmk> — 부정적분을 구할수 없는 예 —

$$\int\frac{1}{\ln x}dx,\ \int\frac{e^x}{x}dx,\ \int\frac{e^{-x}}{x}dx,\ \int\frac{\sin x}{x}dx,\ \int\frac{\cos x}{x}dx$$

$$\int\sin(x^2)dx,\ \int\sqrt{\sin x}\,dx,\ \int\ln(\sin x)dx,\ \int\frac{1}{\sqrt{1-x^4}}dx.$$

4. $I(m,n)=\int\sin^m x\cos^n x\,dx$. $\quad$ (m, n : 정수)

① m, n 이 짝수 ($m>n$) 또는 m은 짝수 n은 홀수인 경우.

$$I(m,n)=\int \underbrace{\sin^m x\cdot\cos x}_{g'}\cdot\underbrace{\cos^{n-1}x}_{f}\,dx$$

$$= \frac{\sin^{m+1}x\cdot\cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int\sin^{m+2}x\cos^{n-2}x\,dx.$$

$$\sin^m x\sin^2 x = \sin^m x(1-\cos^2 x) = \sin^m x - \sin^m x\cos^2 x.$$

$$= \frac{\sin^{m+1}x\cdot\cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\Big[\underbrace{\int\sin^m x\cos^{n-2}x\,dx}_{I(m,n-2)} - \underbrace{\int\sin^m x\cos^n x\,dx}_{I(m,n)}\Big]$$

$$\therefore\ \Big(1+\frac{n-1}{m+1}\Big)I(m,n) = \Big(\frac{m+n}{m+1}\Big)I(m,n) = \frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}I(m,n-2)$$

$$\therefore\ I(m,n)=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}I(m,n-2)$$ (점화식으로 표현)

n 이 홀수이면 $I(m,1)=\int\sin^m x\cos x\,dx = \frac{1}{m+1}\sin^{m+1}x$.

n 이 짝수이면

$$I(m,0) = \int \sin^m x \, dx = I_m .$$

$$I_m = \int \sin^{m-2} x \sin^2 x \, dx = \int \sin^{m-2} x (1-\cos^2 x) \, dx$$

$$= \int \sin^{m-2} x \, dx - \int \underbrace{\sin^{m-2} x \cdot \cos x}_{g'} \cdot \underbrace{\cos x}_{f} \, dx . \qquad g = \frac{\sin^{m-1} x}{m-1}$$

$$= I_{m-2} - \left[\frac{\sin^{m-1} x \cos x}{m-1} + \frac{1}{m-1} \int \sin^{m-1} x \sin x \, dx \right]$$

$$= I_{m-2} - \frac{\sin^{m-1} x \cos x}{m-1} - \frac{1}{m-1} I_m$$

$$\left(1 + \frac{1}{m-1}\right) I_m = \frac{m}{m-1} I_m = I_{m-2} - \frac{\sin^{m-1} x \cos x}{m-1}$$

$$\therefore \; I_m = \frac{m-1}{m} I_{m-2} - \frac{\sin^{m-1} x \cos x}{m}$$

② m, n 이 짝수 또는 m이 홀수 n이 짝수인 경우 $(m<n)$

위와 같은 방법으로 하면

$$I(m,n) = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I(m-2, n)$$

m 이 홀수이면

$$I(1,n) = \int \sin x \cos^n x \, dx = -\frac{1}{n+1} \cos^{n+1} x .$$

m 이 짝수이면

$$I(0,n) = \int \cos^n x \, dx = I_n$$

$$= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$