Learning to time (LeT): A tutorial

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1 States dynamics

LeT have three components:

- A series of (behavioral) states.
- Associative links connecting states to a operant response.
- The operant response

The states advance at a rate λ, which is a normal random variable.

$$\lambda \sim \mathcal{N}(\mu, \sigma),\ \lambda > 0$$

The symbol $\sim$ means "distributed as", and $\mathcal{N}$ is for the Normal Density Function, but often referred as *normal distribution*, not to be confused with the *cumulative distribution function*. Every i-th trial, a value λ_i is sampled and the trial advance from $t = 1$ until T, the time of reinforcement, so that the reinforced state is $N(t = T)$. The states transitions form the random process $N(t) = \lceil \lambda t \rceil$. The N states starts at 1 and are activated serially.

There are no upper bound of the states that can be activated, so the limits of $N(t)$ are determined by μ and σ. For example, if $\mu = 1$ [1] and $\sigma = 0.1$, the states will be very close to the true T, but if $\sigma = 0.6$, there will be more reinforced states that are far from T, and $N(t)$ spans over a broader set of values. Figure 1 shows a simulation with $\mu = 1, \sigma = 0.2$. Note that if $\mu > 1$ the mean of the reinforced states (dotted line in the histogram of Figure 1) will be shifted above T, or below if $\mu < 1$.

Thus, the reinforced states $N(t = T)_i$ (or $N(T)_i$ for short) of the i-th trials, is the vector $\mathbf{N}^* = \boldsymbol{\lambda} T$.

[1] Which means the rate λ most probable value is 1 state per second.

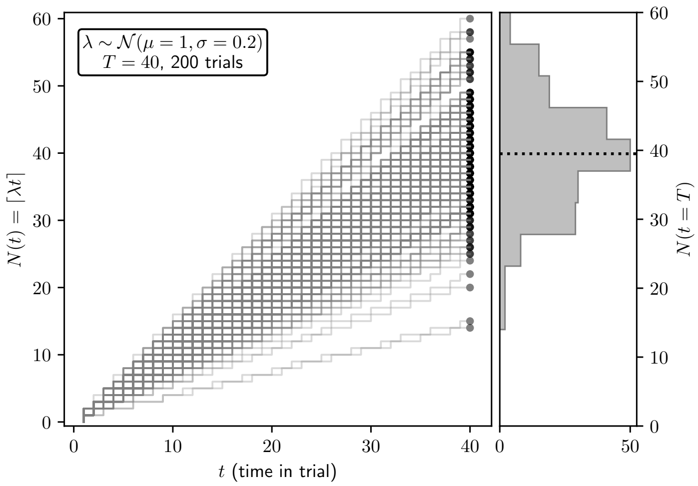

Figure 1: Example of a process of 400 trials. The left panel shows the state transitions for every λ sampled from the normal distribution (see the textbox inset). At $t = 40$, the reinforced states spread around the true value of T (40 s), so they approximate a normal distribution with mean $\mu \times t$ and standard deviation $\sigma \times t$ (right panel).

The states below $N(t)$ are extinguished, and the states above are inactive. There is a $q(n, T)$ probability that the state n is extinguished at T, and a probability $p(n, T)$ that it is reinforced. Also, there is a $r(n, T)$ probability that the n state will be inactive (i.e., the trial will end before T, and state n will not be reinforced nor extinguished). All need sum to 1, so that $p(n, T) = 1 - q(n, T) - r(n, T)$.

Because n is extinguished if and only if (*iif*) $n < N(T)$, so $q(n, T) = p(n < N(T))$, and $N(T) = \lceil \lambda T \rceil$, we have

$$q(n, T) = P\Big(\lambda > \frac{n}{T}\Big) = 1 - P(\lambda < \frac{n}{T}) = 1 - \Phi\Big(\frac{n}{T}, \mu, \sigma\Big)$$

On which $\Phi(\cdot)$ is the *cumulative distribution function (CDF)*, or just *distribution function*:

$$\Phi\Big(\frac{n}{T}, \mu, \sigma\Big) = \frac{1}{\sigma\sqrt{2\pi}} \int\limits_{-\infty}^{\frac{n}{T}} \exp^{-(x-\mu)/2\sigma^2} dx$$

But since $t > 0$, we are just interested in evaluate the integral in the domain $(0, \frac{n}{T}]$ [2]. Φ doesn't have a closed form solution, and it's often approximated by Taylor expansion with

$$\exp^u = \sum_{\infty}^{k=0} \frac{u^k}{k!}, \text{ for } u = \frac{-(x-\mu)}{2\sigma^2}$$

But we are modern people and will take advantage of the software. Most statistical packages can compute CDFs. Let's see an example. If $\mu = 1, \sigma = 0.2, n = 15$ and $T = 40$, $q(15, 40) = P\Big(\lambda > \frac{15}{40}\Big)$ or $1 - \Phi(\frac{15}{40}, 1, 0.2)$. Using Python:

[2] Of course, we should subtract first the integral of the domain $[-\infty, 0]$, but is a very small quantity, so it's negligible.

```
from scipy.stats import norm
1 - norm.cdf(n/T,loc=1,scale=0.2)
```

Returns 0.99911, that is, the probability (actually, the density, or the area under the curve, etc) of state 15 being a extinguished state is very high. We can also interpret that quantity in terms of expected proportion of trials with state 15 not reinforced. See Figure 2 for a graphical representation of this probability just calculated.

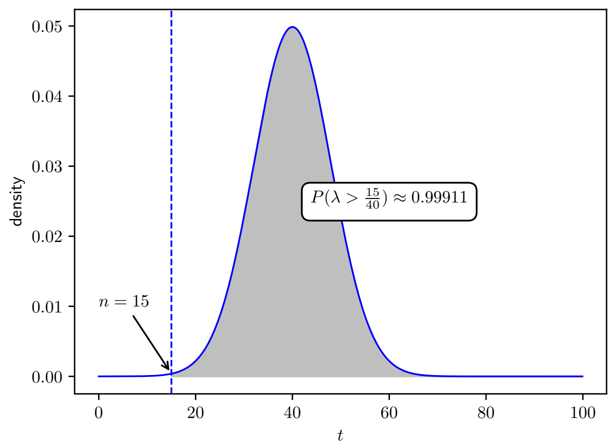

Figure 2: Probability density function of a normal distribution with $\mu = 1, \sigma = 0.2$ for $t = 1, 2, 3.., 40$. Shaded area shows the density $P(\lambda > 15/40)$, which means that is *highly* probable that state 15 is a extinguished state.

The state n is inactive *iif* $n > N(T)$, and, again, $N(T) = \lceil \lambda T \rceil$. Because $\lceil \lambda T \rceil$ is a quantity greater than λT, we can express $n > \lceil \lambda T \rceil$ as $n - 1 > \lambda T$ to avoid that rounding $\lceil \lambda T \rceil$ give us n. Then, the probability that n is inactive, or $r(n, T)$, is

$$r(n, T) = P(n > \lceil \lambda T \rceil) = P(n - 1 > \lambda T) = \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big)$$

Now, the expresion for the probability that n will be reinforced can be expressed as

$$p(n, T) = \Phi\Big(\frac{n}{T}, \mu, \sigma\Big) - \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big)$$

In Machado, Malheiro and Erlhagen (2009) the authors provide an approximation of $p(n, T)$ without saying how they achieved that solution. The approximation is

$$p(n, T) \approx \frac{1}{T} \mathcal{N}\Big(\frac{n}{T}, \mu, \sigma\Big)$$