

# Learning to time (LeT): A tutorial

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## 1 States dynamics

I based this manuscript in the paper by Machado, Malheiro and Erlhagen (2009).

LeT have three components:

- A series of (behavioral) states.
- Associative links connecting states to a operant response.
- The operant response.

The states advance at a rate  $\lambda$ , which is a normal random variable.

$$\lambda \sim \mathcal{N}(\mu, \sigma), \lambda > 0$$

The symbol  $\sim$  means "distributed as", and  $\mathcal{N}$  is for the normal distribution with parameters  $\mu$  and  $\sigma$ , not to be confused with the *cumulative distribution function*<sup>1</sup>.

Every  $i$ -th trial, a value  $\lambda_i$  is sampled from  $\mathcal{N}$  and the current state at  $t$  is  $N(t) = \lambda t$ , that is, the state at time  $t$  is the product of the time elapsed and rate  $\lambda$  (which is states/second). Because  $N$  must be an integer, the authors use a ceiling function to round  $\lambda t$ . They define a ceiling function, denoted by  $\lceil x \rceil$ , like the *smallest integer greater than  $x$* . That is,

$$\lceil \lambda T \rceil = \lambda T + \varepsilon$$

So, for example,  $\lceil 1.2 \rceil = 2$ . The trial advance from  $t = 1$  until  $T$ , the time of reinforcement, so that the reinforced state is  $N(t = T)$ . The states transitions form the random process  $N(t) = \lceil \lambda t \rceil$ . The  $N$  states starts at 1 and are activated serially.

There are no upper bound of the states that can be activated, so the limits of  $N(t)$  are determined by  $\mu$  and  $\sigma$ . For example, if  $\mu = 1$ <sup>2</sup> and  $\sigma = 0.1$ , the states will be very close to the true  $T$ , but if  $\sigma = 0.6$ , there will be more reinforced states that are far from  $T$ , and  $N(t)$  spans over a broader set of values. Figure 1 shows a simulation with  $\mu = 1, \sigma = 0.2$ . Note that if  $\mu > 1$  the mean of the reinforced states (dotted line in the histogram of Figure 1) will be shifted above  $T$ , or below if  $\mu < 1$ .

Thus, the reinforced states  $N(t = T)_i$  (or  $N(T)_i$  for short) of the  $i$ -th trials, is the vector  $\mathbf{N}^* = \lambda T$ .

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<sup>1</sup>The authors use a different notation. For example,  $N(x, \mu, \sigma)$  for the normal density function evaluated at  $x$ . This is a bit confusing since every density evaluated at  $x$  is 0

<sup>2</sup>Which means the rate  $\lambda$  most probable value is 1 state per second.

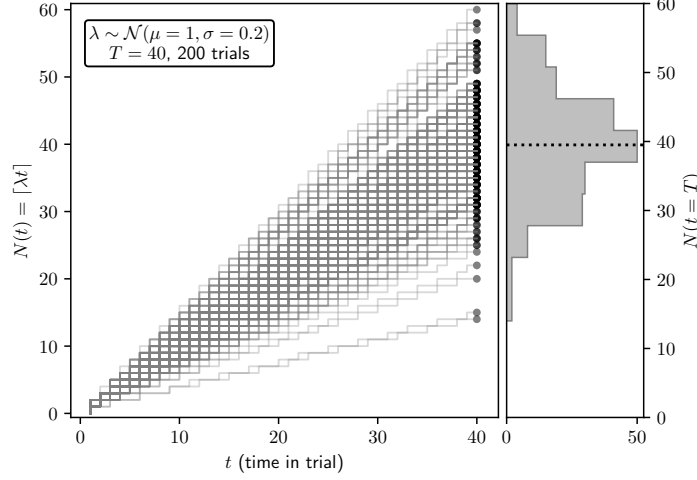


Figure 1: Example of a process of 400 trials. The left panel shows the state transitions for every  $\lambda$  sampled from the normal distribution (see the textbox inset). At  $t = 40$ , the reinforced states spread around the true value of  $T$  (40 s), so they approximate a normal distribution with mean  $\mu \times t$  and standard deviation  $\sigma \times t$  (right panel).

The states below  $N(t)$  are extinguished, and the states above are inactive. There is a  $q(n, T)$  probability that the state  $n$  is extinguished at  $T$ , and a probability  $p(n, T)$  that it is reinforced. Also, there is a  $r(n, T)$  probability that the  $n$  state will be inactive (i.e., the trial will end before  $T$ , and state  $n$  will not be reinforced nor extinguished). All need sum to 1, so that  $p(n, T) = 1 - q(n, T) - r(n, T)$ .

Because  $n$  is extinguished if and only if (*iff*)  $n < N(T)$ , so  $q(n, T) = p(n < N(T))$ , and  $N(T) = \lceil \lambda T \rceil$ , we have

$$q(n, T) = P\left(\lambda > \frac{n}{T}\right) = 1 - P\left(\lambda < \frac{n}{T}\right) = 1 - \Phi\left(\frac{n}{T}, \mu, \sigma\right)$$

On which  $\Phi(\cdot)$  is the *cumulative distribution function (CDF)*, or just *distribution function*:

$$\Phi\left(\frac{n}{T}, \mu, \sigma\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{n}{T}} \exp^{-(x-\mu)/2\sigma^2} dx$$

But since  $t > 0$ , we are just interested in evaluate the integral in the domain  $(0, \frac{n}{T}]$ <sup>3</sup>.  $\Phi$  doesn't have a closed form solution, and it's often approximated by Taylor expansion with

$$\exp^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}, \text{ for } u = \frac{-(x-\mu)}{2\sigma^2}$$

But we are modern people and will take advantage of the software. Most statistical packages can compute CDFs. Let's see an example. If  $\mu = 1, \sigma = 0.2, n = 15$  and  $T = 40$ ,  $q(15, 40) = P\left(\lambda > \frac{15}{40}\right)$  or  $1 - \Phi\left(\frac{15}{40}, 1, 0.2\right)$ . Using Python:

<sup>3</sup>Of course, we should subtract first the integral of the domain  $[-\infty, 0]$ , but is a very small quantity, so it's negligible.

```
from scipy.stats import norm
1 - norm.cdf(n/T, loc=1, scale=0.2)
```

Returns 0.734014, that is, the probability (actually, the density, or the area under the curve, etc) of state 35 being an extinguished state is about 74 %. We can also interpret that quantity in terms of expected proportion of trials in which state 35 is *not reinforced*. See Figure 2 for a graphical representation of this probability just calculated.

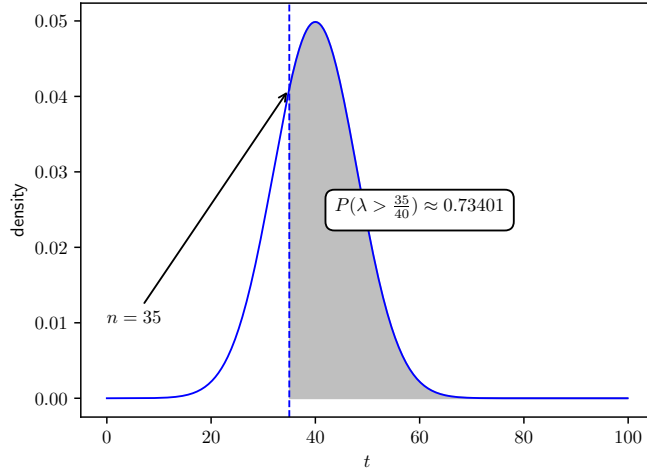


Figure 2: Probability density function of a normal distribution with  $\mu = 1, \sigma = 0.2$  for  $t = 1, 2, 3, \dots, 40$ . Shaded area shows the density  $P(\lambda > 35/40)$ , which means that is moderately probable that state 35 is a extinguished state.

The state  $n$  is inactive *iff*  $n > N(T)$ , and, again,  $N(T) = \lceil \lambda T \rceil$ . Because  $\lceil \lambda T \rceil$  is a quantity greater than  $\lambda T$ , we can express  $n > \lceil \lambda T \rceil$  as  $n - 1 > \lambda T$ <sup>4</sup>. Then, the probability that  $n$  is inactive, or  $r(n, T)$ , is

$$r(n, T) = P(n > \lceil \lambda T \rceil) = P(n - 1 > \lambda T) = \Phi\left(\frac{n - 1}{T}, \mu, \sigma\right)$$

Now, the expresion for the probability that  $n$  will be reinforced can be expressed as

$$p(n, T) = \Phi\left(\frac{n}{T}, \mu, \sigma\right) - \Phi\left(\frac{n - 1}{T}, \mu, \sigma\right)$$

In Machado, Malheiro and Erlhagen (2009) the authors provide an approximation of  $p(n, T)$  without saying how they achieved that solution. The approximation is

$$p(n, T) \approx \frac{1}{T} \phi_{\mu, \sigma}\left(\frac{n}{T}\right)$$

On which  $\phi_{\mu, \sigma}\left(\frac{n}{T}\right)$  is the normal probability density function (pdf<sup>5</sup>) evaluated at  $\frac{n}{T}$ . One way to get into this is by the following reasoning. The CDFs above can be rewritten as a definite integral

<sup>4</sup>This follows from the definition of the ceiling function.

<sup>5</sup>Note that the notation for the CDF and the pdf is different: we use bold capital letters for the CDF, and lowercase letters for the pdf

$$\begin{aligned}
p(n, T) &= \Phi\left(\frac{n}{T}, \mu, \sigma\right) - \Phi\left(\frac{n-1}{T}, \mu, \sigma\right) \\
&= \int_{\frac{n-1}{T}}^{\frac{n}{T}} \phi_{\mu, \sigma}(x) dx
\end{aligned}$$

If we approximate  $\phi_{\mu, \sigma}$  by taking the upper value of the definite integral,  $x = \frac{n}{T}$ , and since this is a constant (the pdf evaluated at the upper limit): we have

$$\begin{aligned}
p(n, T) &= \int_{\frac{n-1}{T}}^{\frac{n}{T}} \phi_{\mu, \sigma}\left(\frac{n}{T}\right) dx \\
&= \phi_{\mu, \sigma}\left(\frac{n}{T}\right) \int_{\frac{n-1}{T}}^{\frac{n}{T}} dx \\
&= \phi_{\mu, \sigma}\left(\frac{n}{T}\right) \left(\frac{n}{T} - \frac{n-1}{T}\right)
\end{aligned}$$

Rearranging:

$$p(n, T) = \left(\frac{1}{T}\right) \phi_{\mu, \sigma}\left(\frac{n}{T}\right)$$

Q.E.D