Learning to time (LeT): A tutorial

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1 States dynamics

I based this manuscript in paper by Machado, Malheiro and Erlhagen (2009).

LeT have three components:

- A series of (behavioral) states.
- Associative links connecting states to a operant response.
- The operant response

The states advance at a rate λ, which is a normal random variable.

$$\lambda \sim \mathcal{N}(\mu, \sigma),\ \lambda > 0$$

The symbol $\sim$ means "distributed as", and $\mathcal{N}$ is for the normal distribution with parameters μ and σ, not to be confused with the *cumulative distribution function* [1].

Every i-th trial, a value λ_i is sampled from $\mathcal{N}$ and the current state at t is $N(t) = \lambda t$, that is, the state at time t is the product of the time passed and rate λ (which is states/second). Because N must be an integer, the authors use a ceiling function to round λt. They define a ceiling function, denoted by $\lceil x \rceil$, like the *smallest integer greater than* x. That is,

$$\lceil \lambda T \rceil = \lambda T + \varepsilon$$

So, for example, $\lceil 1.2 \rceil = 2$. The trial advance from $t = 1$ until T, the time of reinforcement, so that the reinforced state is $N(t = T)$. The states transitions form the random process $N(t) = \lceil \lambda t \rceil$. The N states starts at 1 and are activated serially.

There are no upper bound of the states that can be activated, so the limits of $N(t)$ are determined by μ and σ. For example, if $\mu = 1$ [2] and $\sigma = 0.1$, the states will be very close to the true T, but if $\sigma = 0.6$, there will be more reinforced states that are far from T, and $N(t)$ spans over a broader set of values. Figure 1 shows a simulation with $\mu = 1, \sigma = 0.2$. Note that if $\mu > 1$ the mean of the reinforced states (dotted line in the histogram of Figure 1) will be shifted above T, or below if $\mu < 1$.

Thus, the reinforced states $N(t = T)_i$ (or $N(T)_i$ for short) of the i-th trials, is the vector $\mathbf{N}^* = \boldsymbol{\lambda} T$.

[1] The authors use a different notation. For example, $N(x, \mu, \sigma)$ for the normal density function evaluated at x. This is a bit confusing since every density evaluated at x is 0

[2] Which means the rate λ most probable value is 1 state per second.

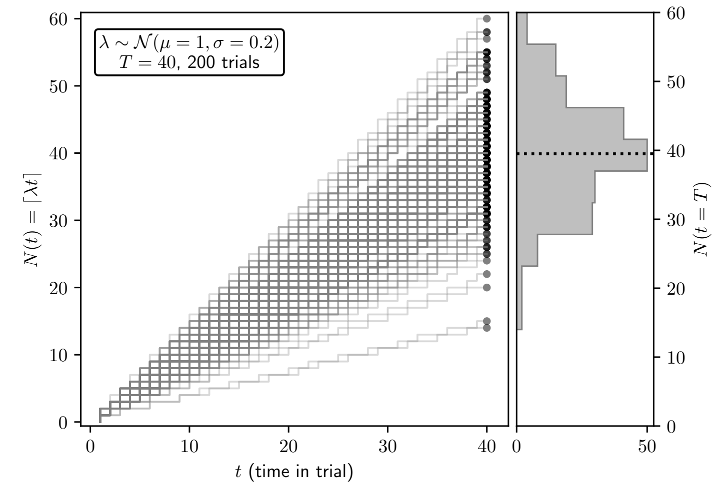

Figure 1: Example of a process of 400 trials. The left panel shows the state transitions for every λ sampled from the normal distribution (see the textbox inset). At $t = 40$, the reinforced states spread around the true value of T (40 s), so they approximate a normal distribution with mean $\mu \times t$ and standard deviation $\sigma \times t$ (right panel).

The states below $N(t)$ are extinguished, and the states above are inactive. There is a $q(n, T)$ probability that the state n is extinguished at T, and a probability $p(n, T)$ that it is reinforced. Also, there is a $r(n, T)$ probability that the n state will be inactive (i.e., the trial will end before T, and state n will not be reinforced nor extinguished). All need sum to 1, so that $p(n, T) = 1 - q(n, T) - r(n, T)$.

Because n is extinguished if and only if (*iif*) $n < N(T)$, so $q(n, T) = p(n < N(T))$, and $N(T) = \lceil \lambda T \rceil$, we have

$$q(n, T) = P\Big(\lambda > \frac{n}{T}\Big) = 1 - P(\lambda < \frac{n}{T}) = 1 - \Phi\Big(\frac{n}{T}, \mu, \sigma\Big)$$

On which $\Phi(\cdot)$ is the *cumulative distribution function (CDF)*, or just *distribution function*:

$$\Phi\Big(\frac{n}{T}, \mu, \sigma\Big) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{n}{T}} \exp^{-(x-\mu)/2\sigma^2} dx$$

But since $t > 0$, we are just interested in evaluate the integral in the domain $(0, \frac{n}{T}]$ [3]. Φ doesn't have a closed form solution, and it's often approximated by Taylor expansion with

$$\exp^u = \sum_{\infty}^{k=0} \frac{u^k}{k!}, \text{ for } u = \frac{-(x-\mu)}{2\sigma^2}$$

But we are modern people and will take advantage of the software. Most statistical packages can compute CDFs. Let's see an example. If $\mu = 1, \sigma = 0.2, n = 15$ and $T = 40$, $q(15, 40) = P\Big(\lambda > \frac{15}{40}\Big)$ or $1 - \Phi(\frac{15}{40}, 1, 0.2)$. Using Python:

[3] Of course, we should subtract first the integral of the domain $[-\infty, 0]$, but is a very small quantity, so it's negligible.

```
from scipy.stats import norm
1 - norm.cdf(n/T,loc=1,scale=0.2)
```

Returns 0.734014, that is, the probability (actually, the density, or the area under the curve, etc) of state 35 being an extinguished state is about 74 %. We can also interpret that quantity in terms of expected proportion of trials in which state 35 is *not reinforced*. See Figure 2 for a graphical representation of this probability just calculated.

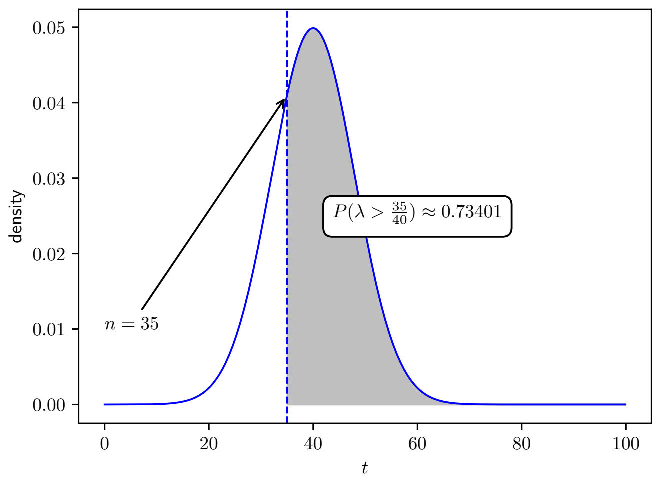

Figure 2: Probability density function of a normal distribution with $\mu = 1, \sigma = 0.2$ for $t = 1, 2, 3.., 40$. Shaded area shows the density $P(\lambda > 35/40)$, which means that is moderately probable that state 35 is a extinguished state.

The state n is inactive *iif* $n > N(T)$, and, again, $N(T) = \lceil \lambda T \rceil$. Because $\lceil \lambda T \rceil$ is a quantity greater than λT, we can express $n > \lceil \lambda T \rceil$ as $n - 1 > \lambda T$ [4]. Then, the probability that n is inactive, or $r(n, T)$, is

$$r(n, T) = P(n > \lceil \lambda T \rceil) = P(n - 1 > \lambda T) = \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big)$$

Now, the expresion for the probability that n will be reinforced can be expressed as

$$p(n, T) = \Phi\Big(\frac{n}{T}, \mu, \sigma\Big) - \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big)$$

In Machado, Malheiro and Erlhagen (2009) the authors provide an approximation of $p(n, T)$ without saying how they achieved that solution. The approximation is

$$p(n, T) \approx \frac{1}{T}\phi_{\mu,\sigma}\Big(\frac{n}{T}\Big)$$

On which $\phi_{\mu,\sigma}(\frac{n}{T})$ is the normal probability density function (pdf) evaluated at $\frac{n}{T}$. One way to get into this is by the following reasoning. The CDFs above can be rewritten as a definite integral

[4] This follows from the definition of the ceiling function.

$$
\begin{aligned}
p(n,T) &= \Phi\Big(\frac{n}{T},\mu,\sigma\Big) - \Phi\Big(\frac{n-1}{T},\mu,\sigma\Big) \\
&= \int_{\frac{n-1}{T}}^{\frac{n}{T}} \phi_{\mu,\sigma}(x)dx
\end{aligned}
$$

If we approximate $\phi_{\mu,\sigma}$ by taking the upper value of the definite integral, $x = \frac{n}{T}$, and since this is a constant (the pdf evaluated ad the upper limit): we have

$$
\begin{aligned}
p(n,T) &= \int_{\frac{n-1}{T}}^{\frac{n}{T}} \phi_{\mu,\sigma}\Big(\frac{n}{T}\Big)dx \\
&= \phi_{\mu,\sigma}\Big(\frac{n}{T}\Big)\int_{\frac{n-1}{T}}^{\frac{n}{T}} dx \\
&= \phi_{\mu,\sigma}\Big(\frac{n}{T}\Big)\Big(\frac{n}{T} - \frac{n-1}{T}\Big) \\
\text{Rearenging:} \\
p(n,T) &= \Big(\frac{1}{T}\Big)\phi_{\mu,\sigma}\Big(\frac{n}{T}\Big) \\
\text{Q.E.D}
\end{aligned}
$$

From the above equation, the authors derive the scalar property.

$$cv$$