Learning to time (LeT): A tutorial

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1 States dynamics

I based this manuscript in paper by Machado, Malheiro and Erlhagen (2009).

LeT have three components:

- A series of (behavioral) states.
- Associative links connecting states to a operant response.
- The operant response

The states advance at a rate λ, which is a normal random variable.

$$\lambda \sim \mathcal{N}(\mu, \sigma),\ \lambda > 0$$

The symbol $\sim$ means "distributed as", and $\mathcal{N}$ is for the normal distribution with parameters μ and σ, not to be confused with the *cumulative distribution function* [1].

Every i-th trial, a value λ_i is sampled from $\mathcal{N}$ and the current state at t is $N(t) = \lambda t$, that is, the state at time t is the product of the time passed and rate λ (which is states/second). Because N must be an integer, the authors use a ceiling function to round λt. They define a ceiling function, denoted by $\lceil x \rceil$, like the *smallest integer greater than* x. That is,

$$\lceil \lambda T \rceil = \lambda T + \varepsilon$$

So, for example, $\lceil 1.2 \rceil = 2$. The trial advance from $t = 1$ until T, the time of reinforcement, so that the reinforced state is $N(t = T)$. The states transitions form the random process $N(t) = \lceil \lambda t \rceil$. The N states starts at 1 and are activated serially.

There are no upper bound of the states that can be activated, so the limits of $N(t)$ are determined by μ and σ. For example, if $\mu = 1$ [2] and $\sigma = 0.1$, the states will be very close to the true T, but if $\sigma = 0.6$, there will be more reinforced states that are far from T, and $N(t)$ spans over a broader set of values. Figure 1 shows a simulation with $\mu = 1, \sigma = 0.2$. Note that if $\mu > 1$ the mean of the reinforced states (dotted line in the histogram of Figure 1) will be shifted above T, or below if $\mu < 1$.

Thus, the reinforced states $N(t = T)_i$ (or $N(T)_i$ for short) of the i-th trials, is the vector $\mathbf{N}^* = \boldsymbol{\lambda} T$.

[1] The authors use a different notation. For example, $N(x, \mu, \sigma)$ for the normal density function evaluated at x. This is a bit confusing since every density evaluated at x is 0

[2] Which means the rate λ most probable value is 1 state per second.

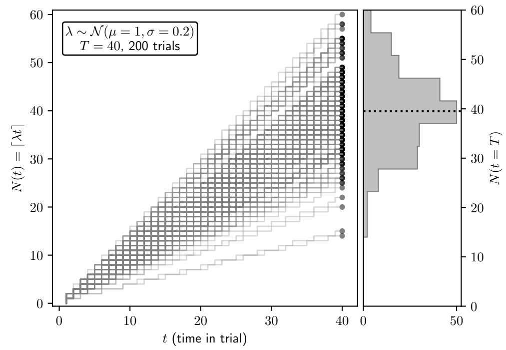

Figure 1: Example of a process of 400 trials. The left panel shows the state transitions for every λ sampled from the normal distribution (see the textbox inset). At $t = 40$, the reinforced states spread around the true value of T (40 s), so they approximate a normal distribution with mean $\mu \times t$ and standard deviation $\sigma \times t$ (right panel).

The states below $N(t)$ are extinguished, and the states above are inactive. There is a $q(n, T)$ probability that the state n is extinguished at T, and a probability $p(n, T)$ that it is reinforced. Also, there is a $r(n, T)$ probability that the n state will be inactive (i.e., the trial will end before T, and state n will not be reinforced nor extinguished). All need sum to 1, so that $p(n, T) = 1 - q(n, T) - r(n, T)$.

Because n is extinguished if and only if (*iif*) $n < N(T)$, so $q(n, T) = p(n < N(T))$, and $N(T) = \lceil \lambda T \rceil$, we have

$$q(n, T) = P\Big(\lambda > \frac{n}{T}\Big) = 1 - P(\lambda < \frac{n}{T}) = 1 - \Phi\Big(\frac{n}{T}, \mu, \sigma\Big)$$

On which $\Phi(\cdot)$ is the *cumulative distribution function (CDF)*, or just *distribution function*:

$$\Phi\Big(\frac{n}{T}, \mu, \sigma\Big) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{n}{T}} \exp^{-(x-\mu)/2\sigma^2} dx$$

But since $t > 0$, we are just interested in evaluate the integral in the domain $(0, \frac{n}{T}]$ [3]. Φ doesn't have a closed form solution, and it's often approximated by Taylor expansion with

$$\exp^u = \sum_{\infty}^{k=0} \frac{u^k}{k!}, \text{ for } u = \frac{-(x-\mu)}{2\sigma^2}$$

But we are modern people and will take advantage of the software. Most statistical packages can compute CDFs. Let's see an example. If $\mu = 1, \sigma = 0.2, n = 15$ and $T = 40$, $q(15, 40) = P\Big(\lambda > \frac{15}{40}\Big)$ or $1 - \Phi(\frac{15}{40}, 1, 0.2)$. Using Python:

```
from scipy.stats import norm
1 - norm.cdf(n/T,loc=1,scale=0.2)
```

[3] Of course, we should subtract first the integral of the domain $[-\infty, 0]$, but is a very small quantity, so it's negligible.

Returns 0.734014, that is, the probability (actually, the density, or the area under the curve, etc) of state 35 being an extinguished state is about 74 %. We can also interpret that quantity in terms of expected proportion of trials in which state 35 not reinforced. See Figure 2 for a graphical representation of this probability just calculated.

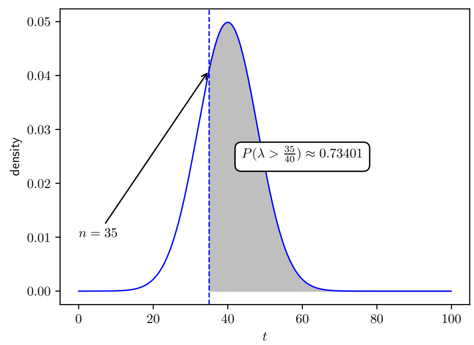

Figure 2: Probability density function of a normal distribution with $\mu = 1, \sigma = 0.2$ for $t = 1, 2, 3.., 40$. Shaded area shows the density $P(\lambda > 35/40)$, which means that is moderately probable that state 35 is a extinguished state.

The state n is inactive *iif* $n > N(T)$, and, again, $N(T) = \lceil \lambda T \rceil$. Because $\lceil \lambda T \rceil$ is a quantity greater than λT, we can express $n > \lceil \lambda T \rceil$ as $n - 1 > \lambda T$ to avoid that rounding $\lceil \lambda T \rceil$ give us n. Then, the probability that n is inactive, or $r(n, T)$, is

$$r(n,T) = P(n > \lceil \lambda T \rceil) = P(n - 1 > \lambda T) = \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big)$$

Now, the expresion for the probability that n will be reinforced can be expressed as

$$p(n,T) = \Phi\Big(\frac{n}{T}, \mu, \sigma\Big) - \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big)$$

In Machado, Malheiro and Erlhagen (2009) the authors provide an approximation of $p(n, T)$ without saying how they achieved that solution. The approximation is

$$p(n,T) \approx \frac{1}{T}\phi_{\mu,\sigma}\Big(\frac{n}{T}\Big)$$

One way to get into this is by the following reasoning. Let $f(x)$ be $\phi_{\mu,\sigma}$, the normal pdf:

$$\begin{aligned} p(n,T) &= \Phi\Big(\frac{n}{T}, \mu, \sigma\Big) - \Phi\Big(\frac{n-1}{T}, \mu, \sigma\Big) \\ &= \int\limits_{\frac{n-1}{T}}^{\frac{n}{T}} f(x)dx \end{aligned}$$

If we approximate $\phi_{\mu,\sigma}$ by taking the upper value of the definite integral, and since this is a constant (the pdf evaluated ad the upper limit): we have

$$
\begin{aligned}
p(n,T) &= \int_{\frac{n-1}{T}}^{\frac{n}{T}} f\Big(\frac{n}{T}\Big)dx \\
&= f\Big(\frac{n}{T}\Big)\int_{\frac{n-1}{T}}^{\frac{n}{T}} dx \\
&= f\Big(\frac{n}{T}\Big)\Big(\frac{n}{T}-\frac{n-1}{T}\Big) \\
&= f\Big(\frac{n}{T}\Big)\Big(\frac{1}{T}\Big)
\end{aligned}
$$

Q.E.D