# Autoregressive Methods

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# Goals for this lecture

- Stochastic processes with memory
- Understanding Autoregressive processes
  - $AR(1)$
- Understanding continuous autoregressive processes
  - $CAR(1)$
- Unmodeled fits
- Forecasting

# How do we analyze correlated time-series of unknown stochasticity?

# Markov process

- Markov chain : A stochastic process whose dynamics depends only on the current position.

$$P(X_n = x_n | X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, .. , X_1 = x_1) = P(X_n = x_n | X_{n-1} = x_{n-1})$$

- "Memory" only of the current state
- The transition probabilities define the process entirely.
    - Eg: Random walk or drunk walk
        - Uniform transition probabilities.
- Usage
    - Markov chain Monte Carlo samplers
    - Hidden Markov Model

# Autoregressive process

- A random process that retains memory.

$$\psi_i = \sum_{j=1}^{n} c_j \psi_{i-j} + \omega_i$$

- $\{c_j\}$ are deterministic coefficients.
- $\{\omega_i\}$ are independent random variables
- An autoregressive process of order p — $AR(p)$ — will have memory of the current state and $p - 1$past states.
- $AR(p)$ processes are continuous and differentiable unto order $p - 1$

# $AR(1)$ Process

$$\psi_i = c_i \psi_{i-1} + \omega_i$$

Let $\omega_i \sim \mathcal{N}(0, \sigma)$

Means

$$\langle \psi_1 \rangle = \langle \omega_1 \rangle = 0$$

$$\langle \psi_2 \rangle = c_1 \langle \psi_1 \rangle + \langle \omega_1 \rangle = 0$$

In fact by induction $\langle \psi_n \rangle = 0$

Covariance matrix

$$\langle \psi_{i+n} \psi_n \rangle = \left( \prod_{k=n+1}^{i+n} c_k \right) \langle \psi_n^2 \rangle$$

$$\langle \psi_i \psi_i \rangle = c_i^2 \langle \psi_{i-1}^2 \rangle + \sigma^2$$

# $CAR(1)$ Process

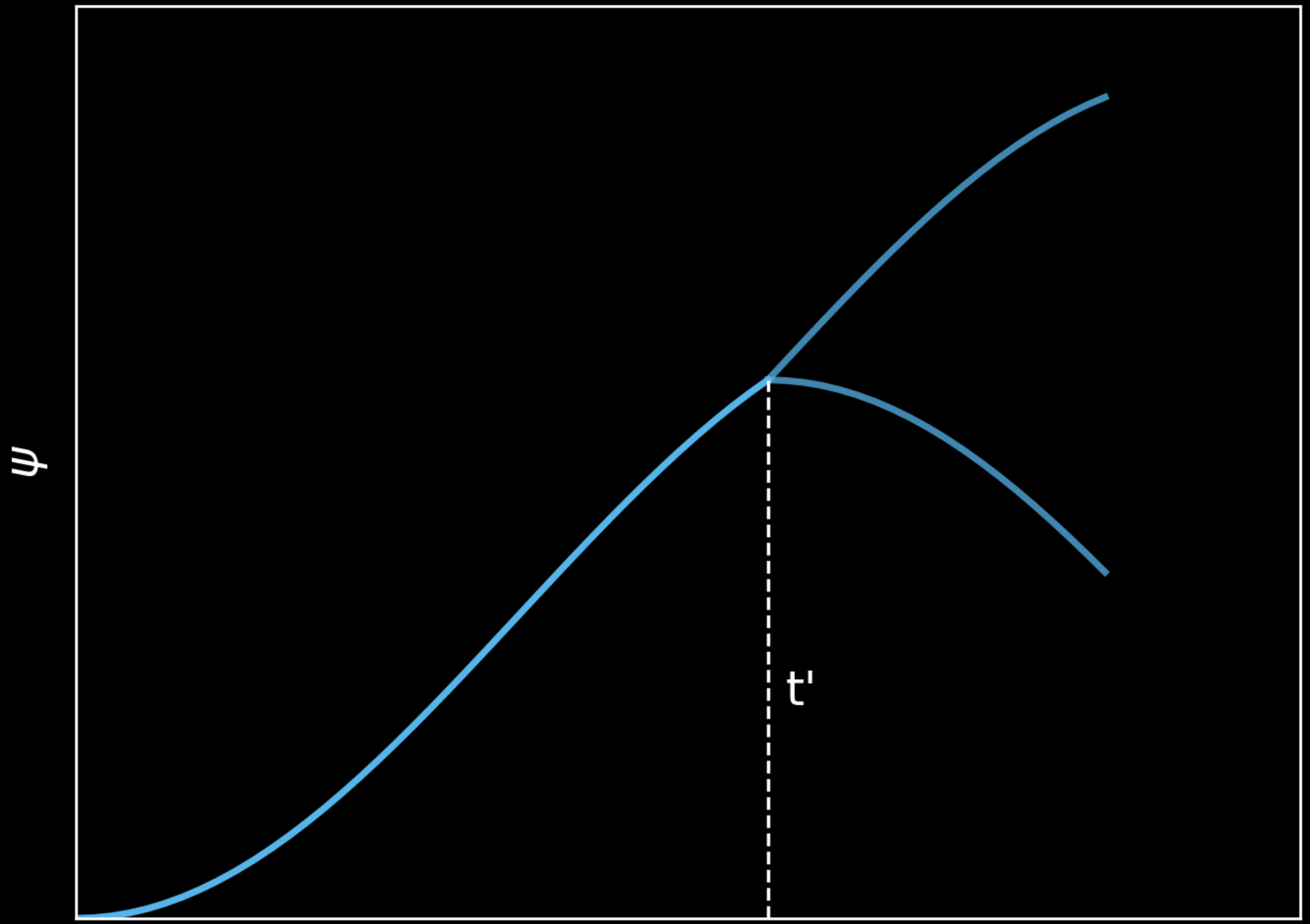
The continuous autoregressive process is a solution of the stochastic differential equation

$$d\psi(t) = \frac{-1}{\tau}\psi(t)dt + \sigma\sqrt{dt}\epsilon(t) + bdt \qquad \tau, \sigma, t > 0$$

$\tau$ : timescale of correlation

$\sigma$ : variance

$b$ : mean parameter




$$E\left(\psi(t)\,|\,\psi(t')\right) = e^{-\Delta t/\tau}\psi(t') + b\tau\left(1 - e^{-\Delta t/\tau}\right)$$

$$\text{Var}\left(\psi(t)\,|\,\psi(t')\right)^{\dagger} = \frac{\tau\sigma^2}{2}\left(1 - e^{-2\Delta t/\tau}\right)$$

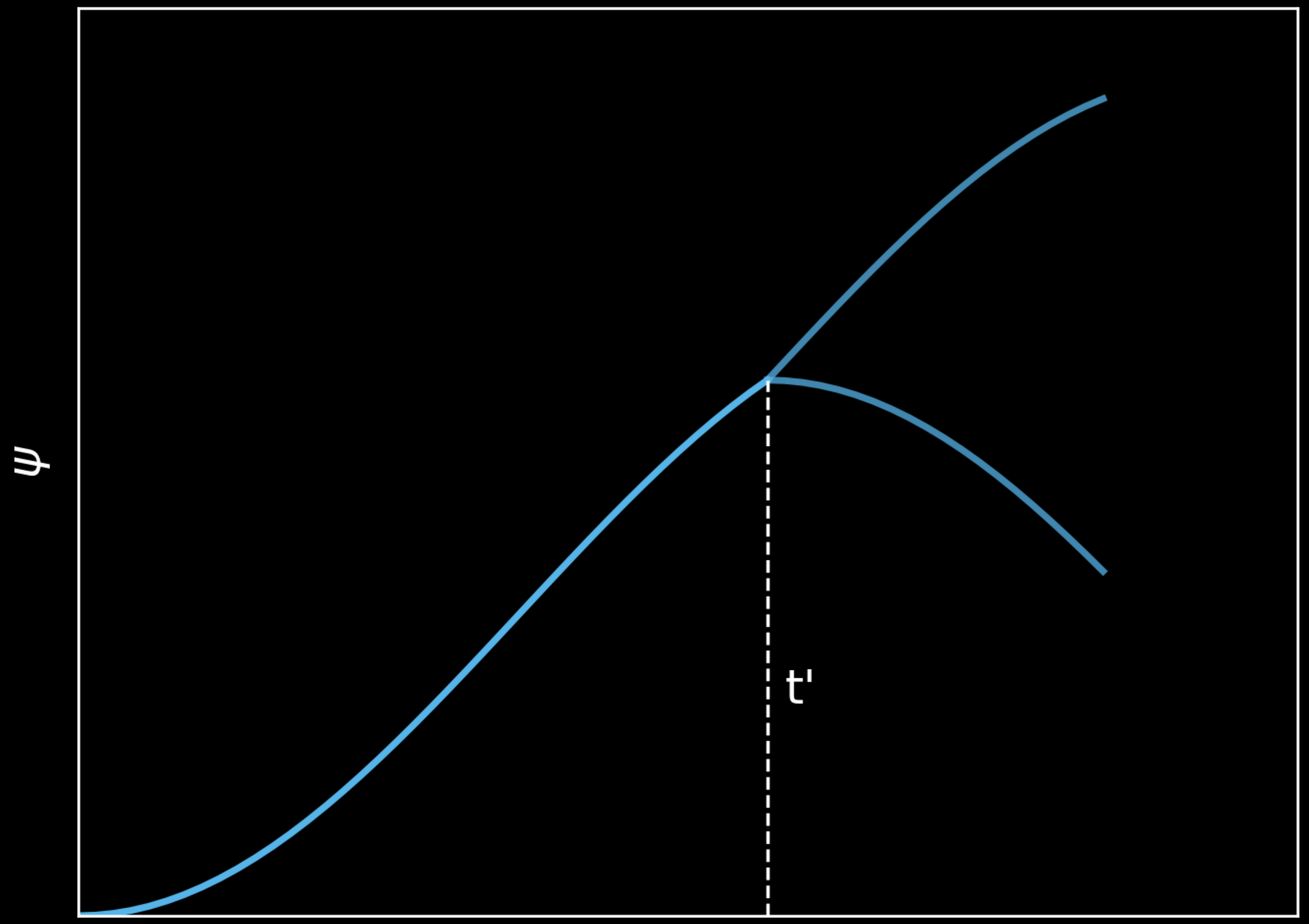


$\dagger$ not to be confused by $\langle \psi(t)\, \psi(t') \rangle$

# $CAR(1)$ Process

$$\hat{\psi}(t) \equiv E\left(\psi(t)\,|\,\psi(t')\right) - b\tau = e^{-\Delta t/\tau}\left(\psi(t') - b\tau\right)$$

$$\Omega(t) \equiv \mathrm{Var}\left(\psi(t)\,|\,\psi(t')\right)^{\dagger} = \frac{\tau\sigma^2}{2}\left(1 - e^{-2\Delta t/\tau}\right)$$




Conditional likelihood for $\psi(t)$

$$p\left(\psi(t)\,|\,\psi(t')\right) = \frac{1}{\sqrt{2\pi\Omega(t)}}\exp\left[-\frac{\left(\psi^*(t) - \hat{\psi}(t)\right)^2}{2\Omega^2(t)}\right]$$

$$\psi^*(t) = \psi(t) - b\tau$$

# Applications of $CAR(1)/AR(1)$

- Unmodeled Bayesian fitting.
- Bayesian forecasting.

- Can also do maximum likelihood versions of both

# A brief intro to Bayesian inference in practice

# Bayes Theorem

Model parameters
eg masses of binaries, spins, distance

Likelihood
Data distribution

Prior distribution

$$p(\theta | d, M) = \frac{p(d | \theta)\, \pi(\theta | M)}{Z(M)}$$

Model or hypothesis
eg binary black hole inspired

Posterior probability
density function

Bayesian Evidence
a measure of model goodness

# Bayesian inference in practice

- How do we estimate the posterior in practice?
- Make a grid in model space and estimate likelihood
  - Curse of dimensionality
  - A binary neutron star merger inference can have upto $16$ $m_1, m_2, \vec{\chi}_1, \vec{\chi}_2, \iota, t_c, d, \phi, \lambda_1, \lambda_2,$ ra, dec

$$p(\theta \,|\, d, M) = \frac{p(d \,|\, \theta)\; \pi(\theta \,|\, M)}{Z(M)}$$

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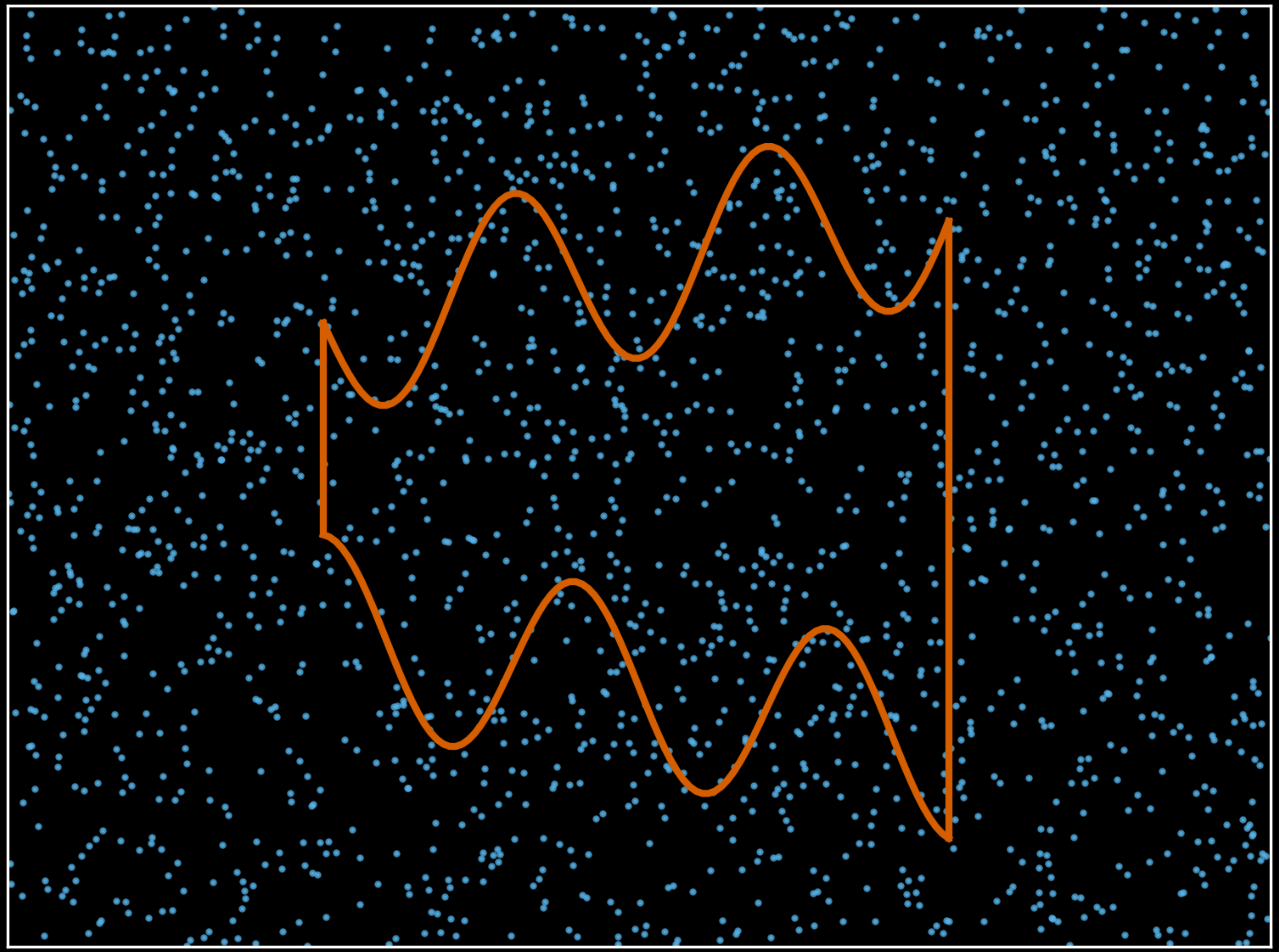
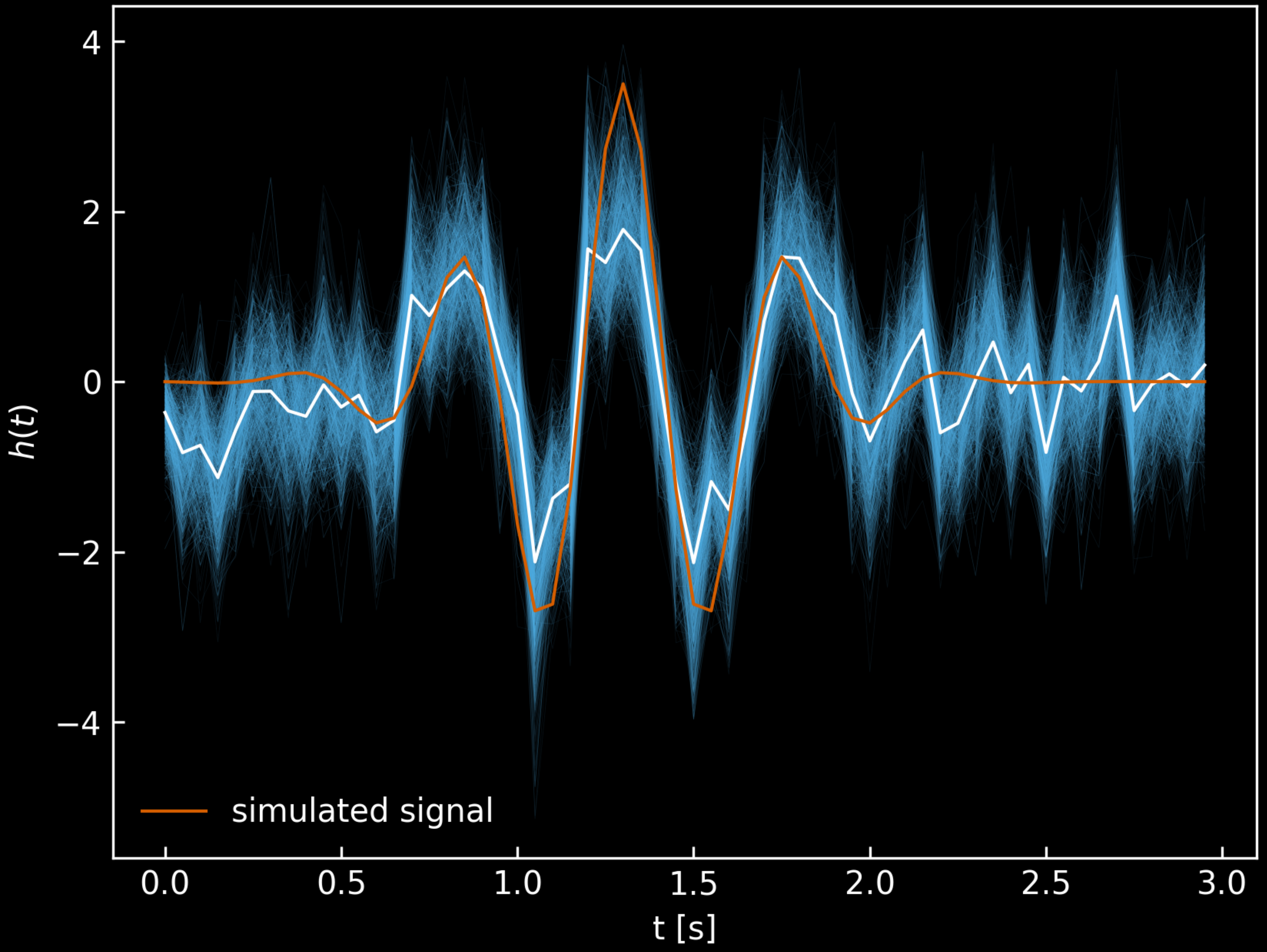
# Bayesian inference in practice

- How do we estimate the posterior PDF in practice?
- Use some type of Monte Carlo Sampling
  - Nested sampling
  - Markov chain Monte Carlo
- Flows (e.g. normalizing flows)
  - Learn Jacobins to transform the posterior PDF into a simple distribution

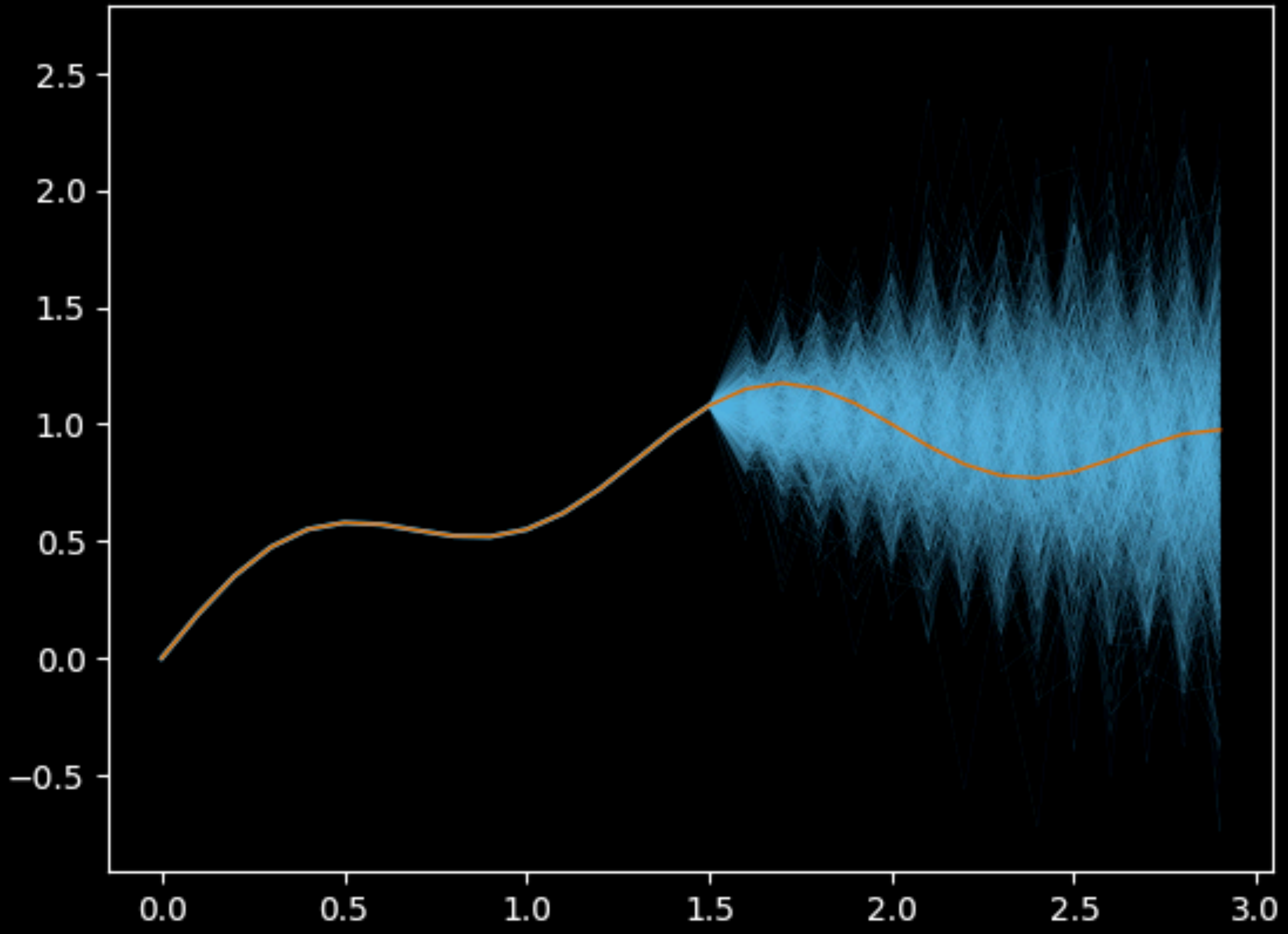


# Unmodeled Fitting

Unknown signal in known noise



# Forecasting



# References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- Introduction to Time Series and Forecasting, Peter J. Brockwell & Richard A. Davis
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