Limites de funções compostas - Mudança de variável

Nosso objetivo nesta parte da aula é trabalhar com o limite de expressões do tipo: $\lim\limits_{t \to p} f(g(t))$.

$f(t) = t^2$, $g(t) = \sin t$

$h(t) = t^2 + \sin t$ $\quad$ $h(t) = t^2 \cdot \sin(t)$

$$\lim_{t \to 0} \frac{\sin(t)}{t^2} = \lim_{t \to 0} \underbrace{\frac{1}{t}}_{\nexists} \cdot \underbrace{\frac{\sin t}{t}}_{\substack{\text{L.T.F.} \\ \to 1}} = ?$$

$$\lim_{t \to 0^+} \underbrace{\frac{1}{t}}_{\to +\infty} \cdot \underbrace{\frac{\sin t}{t}}_{\substack{\text{L.T.F.} \\ \to 1}} = +\infty \quad \text{"} 1 \cdot \infty \text{"}$$

$$\lim_{t \to 0^-} \underbrace{\frac{1}{t}}_{\to -\infty} \cdot \underbrace{\frac{\sin t}{t}}_{\substack{\text{L.T.F.} \\ \to 1}} = -\infty. \quad \text{"} 1 \cdot \infty \text{"}$$

$$\Rightarrow \nexists \lim_{t \to 0} \frac{\sin t}{t^2}$$

$2 \cdot \frac{1}{2} = 1$

$f(t) = t$

$g(t)$

$$\lim_{t \to 0} \sin(t^2).$$

$g \circ f(t) = g(t)$

$\begin{cases} f(t) = e^t \\ g(t) = \ln t \end{cases}$

$g \circ f(t) = g(f(t))$

$$g \circ f(t) = g(f(t))$$

$$= \ln(e^t) = t \cdot \ln e$$
$$= t \cdot 1 = t.$$

Teorema: Se $f: I \subset \mathbb{R} \to J \subset \mathbb{R}$ é contínua em $p \in I$ e $g: J \subset \mathbb{R} \to \mathbb{R}$ é contínua em $q \in J$, com $q = f(p)$. Então:

$$\lim_{t \to p} g(f(t)) = g\left(\lim_{t \to p} f(t)\right)$$

$$= \lim_{u \to q} g(u)$$

Mudança de variáveis $u = f(t)$

Ex.: $$\lim_{t \to 0} \sin(t^2) = \sin\left(\lim_{t \to 0} t^2\right)$$

Solução 1

$$= \sin(0) = 0.$$

Solução 2 Mudança de variáveis:

$$u = f(t) = t^2$$

$t \to 0$

$u \to 0$

$$\lim_{u \to 0} \sin(u) = 0$$

Ex:

$$\lim_{t \to +\infty} \frac{2}{\sqrt{t^2+2}+t} = \lim_{t \to +\infty} g(f(t))$$

$$g(t) = \frac{2}{t} \quad , \quad f(t) = \sqrt{t^2+2}+t$$

$$g \circ f(t) = g(f(t)) = \frac{2}{\sqrt{t^2+2}+t}$$

$$u = f(t) = \sqrt{t^2+2}+t$$

$$t \to +\infty \implies u = f(t) \to +\infty$$

$$\lim_{t \to +\infty} \sqrt{t^2+2}+t = \lim_{t \to +\infty} t \cdot \overbrace{\Big(\underbrace{\sqrt{1+\underbrace{2/t^2}_{0}}}_{1}+1\Big)}^{2} = +\infty$$

$$\hookrightarrow \text{"}2 \cdot \infty\text{"}$$

$$\lim_{t \to +\infty} \frac{2}{\sqrt{t^2+2}+t} = \lim_{t \to +\infty} g(f(t)) = \lim_{u \to +\infty} g(u)$$

$$= \lim_{u \to +\infty} \frac{2}{u} = 0$$

Ex:

$\to$ $$\lim_{\ldots \to 0} \frac{1}{\ldots} = \ldots = 0$$

$\rightarrow \lim_{t \to p} \frac{1}{f(t)} = \ldots\ldots = 0$

$\lim_{t \to p} f(t) = +\infty$

Exercício... Justifiquem via M.D.V.

Ex.:

$\rightarrow \lim_{t \to 1} \frac{\sqrt[3]{t} - 1}{t - 1}$

$u = \sqrt[3]{t} \quad \therefore \quad t = u^3$

$t \to 1 \Rightarrow u \to 1$

$\rightarrow \lim_{u \to 1} \frac{u - 1}{u^3 - 1} = \lim_{u \to 1} \frac{\cancel{u - 1}}{\cancel{(u-1)} \cdot (u^2 + u + 1)}$

$= \lim_{u \to 1} \frac{1}{u^2 + u + 1} = \frac{1}{3}.$

$u^3 - 1 = (u - 1) \cdot (u^2 + u + 1)$

Briot-Ruffini

1	1	0	0	-1
	1	1	1	0

Ex.: $\lim_{t \to 0} \frac{e^t - 1}{t} \rightarrow$ "$\frac{0}{0}$"

$\log_a b^c = c \cdot \log_a b$

$= \log b$

$u = e^t - 1 \qquad t \to 0 \Rightarrow u \to 0$

$u = e^t - 1$ $\quad t \to 0 \Rightarrow u \to 0$ $\quad = \log_{a^{1/c}} b$

$\therefore\ u+1 = e^t \quad t = \ln(u+1)$

$$\to \lim_{u \to 0} \frac{u}{\ln(u+1)} = \lim_{u \to 0} \frac{1}{\frac{1}{u} \cdot \ln(1+u)} = \lim_{u \to 0} \frac{1}{\ln \underbrace{(1+u)^{1/u}}_{e}} = \frac{1}{\ln e} = 1$$

Obs: Limite exponencial fundamental (L.E.F.).

$$\lim_{t \to +\infty} \underbrace{\left(1 + \frac{1}{t}\right)}_{1}^{t \to +\infty} = e \qquad \text{“} 1^{\infty} \text{”}$$

t	$\left(1 + \frac{1}{t}\right)^t$
1	2
2	2,25
3	2,37037...
$\vdots$	
10	2,5937...
$\vdots$	
1000	2,71814...

Exercício: Sabendo que $\lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^t = e$,

verifique que:

1) $\lim_{t \to -\infty} \left(1 + \frac{1}{t}\right)^t = e$

$$\lim_{t \to -\infty} \left(1 + \frac{1}{t}\right)^{t} = e$$

2) $$\lim_{h \to 0} (1 + h)^{1/h} = e$$

3) $$\lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^{2t} = e^2$$

4) $$\lim_{t \to +\infty} \left(1 + \frac{1}{2t}\right)^{t} = \sqrt{e}$$

5) $$\lim_{t \to 0} \frac{e^t - 1}{t} = 1$$