

Limites de funções compostas - Mudança de variável

Nosso objetivo nesta parte da aula é

trabalhar com o limite de expressões

do tipo: $\lim_{t \rightarrow p} f(g(t))$

$$f(t) = t^2, \quad g(t) = \sin t$$

$$h(t) = t^2 + \sin t \quad \left\{ \quad h(t) = t^2 \cdot \sin(t) \right.$$

$$\lim_{t \rightarrow 0} \frac{\sin(t)}{t^2} = \lim_{t \rightarrow 0} \cancel{\frac{1}{t}} \cdot \frac{\sin t}{t} = ?$$

L.T.F. $\rightarrow 1$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \cdot \frac{\sin t}{t} = +\infty$$

"1.∞" $\rightarrow +\infty$ L.T.F. $\rightarrow 1$

$$\lim_{t \rightarrow 0^-} \frac{1}{t} \cdot \frac{\sin t}{t} = -\infty$$

"1.∞" $\rightarrow -\infty$ L.T.F. $\rightarrow 1$

$$\Rightarrow \cancel{\lim_{t \rightarrow 0} \frac{\sin t}{t^2}}$$

$$2 \cdot \frac{1}{2} = 1$$

$$f(t) = t$$
$$g(t)$$

$$\lim_{t \rightarrow 0} \sin(t^2)$$

$$g \circ f(t) = g(f(t))$$

$$\begin{cases} f(t) = e^t \\ g(t) = \ln t \end{cases}$$

$$g \circ f(t) = g(f(t))$$

$$(g \circ f)(t) = g(f(t))$$

$$= \ln(e^t) = t \cdot \ln e \\ = t \cdot 1 = t.$$

Teorema: Se $f: I \subset \mathbb{R} \rightarrow J \subset \mathbb{R}$ é contínua em $p \in I$ e $g: J \subset \mathbb{R} \rightarrow \mathbb{R}$ é contínua em $q \in J$, com $q = f(p)$. Então:

$$\lim_{t \rightarrow p} g(f(t)) = g\left(\lim_{t \rightarrow p} f(t)\right)$$

$$= \lim_{u \rightarrow q} g(u) \quad \begin{array}{l} \text{Mudança de} \\ \text{variáveis} \\ u = f(t) \end{array}$$

$$\text{Ex: } \lim_{t \rightarrow 0} \sin(t^2) = \sin\left(\lim_{t \rightarrow 0} t^2\right)$$

$$= \sin(0) = 0.$$

Solução 1 ↗

Solução 2 ↘ Mudança de variáveis:

$$u = f(t) = t^2$$

$$t \rightarrow 0$$

$$u \rightarrow 0$$

$$\lim_{u \rightarrow 0} \sin(u) = 0$$

Exi:

$$\lim_{t \rightarrow +\infty} \frac{2}{\sqrt{t^2+2} + t} = \lim_{t \rightarrow +\infty} g(f(t))$$

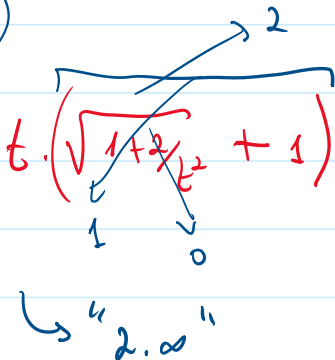
$$g(t) = \frac{2}{t} \quad , \quad f(t) = \sqrt{t^2+2} + t$$

$$g \circ f(t) = g(f(t)) = \frac{2}{\sqrt{t^2+2} + t}$$

$$u = f(t) = \sqrt{t^2+2} + t$$

$$t \rightarrow +\infty \Rightarrow u = f(t) \rightarrow +\infty$$

$$\lim_{t \rightarrow +\infty} \sqrt{t^2+2} + t = \lim_{t \rightarrow +\infty} t \cdot \left(\sqrt{1 + \frac{2}{t^2}} + 1 \right) = +\infty$$



 $\hookrightarrow "2 \cdot \infty"$

$$\lim_{t \rightarrow +\infty} \frac{2}{\sqrt{t^2+2} + t} = \lim_{t \rightarrow +\infty} g(f(t)) = \lim_{u \rightarrow +\infty} g(u)$$

$$= \lim_{u \rightarrow +\infty} \frac{2}{u} = 0$$

Exi:
→

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\rightarrow \lim_{t \rightarrow p} \frac{1}{f(t)} = \dots = 0$$

$$\lim_{t \rightarrow p} f(t) = +\infty$$

↓
Exercício... Justifique
via M.D.V.

Ex.:

$$\rightarrow \lim_{t \rightarrow 1} \frac{\sqrt[3]{t} - 1}{t - 1}$$

$$u = \sqrt[3]{t} \quad \therefore t = u^3$$

$$t \rightarrow 1 \Rightarrow u \rightarrow 1$$

$$\lim_{u \rightarrow 1} \frac{u - 1}{u^3 - 1} = \lim_{u \rightarrow 1} \frac{\cancel{u - 1}}{(\cancel{u - 1}) \cdot (u^2 + u + 1)}$$

$$= \lim_{u \rightarrow 1} \frac{1}{u^2 + u + 1} = \frac{1}{3}$$

$$u^3 - 1 = (u - 1) \cdot (u^2 + u + 1)$$

Briot-Ruffini

1	1	0	0	-1
	1	1	1	0

$$\text{Ex.: } \lim_{t \rightarrow 0} \frac{e^t - 1}{t}$$

→ "0"
0

$$u = e^t - 1$$

$$t \rightarrow 0 \Rightarrow u \rightarrow 0$$

$$\log_a b^c = c \cdot \log_a b$$

$$= \log_a b$$

$$u = e^t - 1 \quad t \rightarrow 0 \Rightarrow u \rightarrow 0$$

$$= \log_b a^{1/c}$$

$$\therefore u+1 = e^t \quad t = \ln(u+1)$$

$$\rightarrow \lim_{u \rightarrow 0} \frac{u}{\ln(u+1)} = \lim_{u \rightarrow 0} \frac{1}{\frac{1}{u} \cdot \ln(1+u)} = \lim_{u \rightarrow 0} \frac{1}{\ln(1+u)^{1/u}} = \frac{1}{\ln e} = 1$$

Obs.: Limite exponencial fundamental (L.E.F.)

$$\lim_{t \rightarrow +\infty} \left(1 + \frac{1}{t}\right)^t = e$$

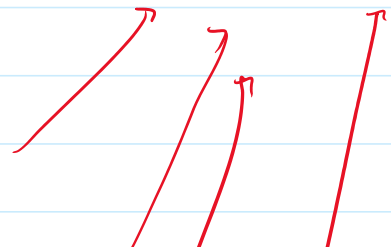
"1" "∞" "1"

t	$\left(1 + \frac{1}{t}\right)^t$
1	2
2	2,25
3	2,37037...
...	
10	2,5937...
...	
1000	2,71814...

Exercício: Sabendo que $\lim_{t \rightarrow +\infty} \left(1 + \frac{1}{t}\right)^t = e$,

verifique que:

$$1) \lim_{t \rightarrow -\infty} \left(1 + \frac{1}{t}\right)^t = e$$



$$1) \lim_{t \rightarrow -\infty} \left(1 + \frac{1}{t}\right)^t = e$$

$$2) \lim_{h \rightarrow 0} (1+h)^{1/h} = e$$

$$3) \lim_{t \rightarrow +\infty} \left(1 + \frac{1}{t}\right)^{2t} = e^2$$

$$4) \lim_{t \rightarrow +\infty} \left(1 + \frac{1}{2t}\right)^t = \sqrt{e}$$

$$5) \lim_{t \rightarrow 0} \frac{e^t - 1}{t} = 1$$