

# Construction of a quasi-geodesic on a linear sphere by piece

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In 1949, Alexandre Pogorelov proves the existence of a simple and closed quasi-geodesic on any convex polyhedron<sup>1</sup>. We propose to describe an algorithm allowing to build such a quasi-geodesic on any linear sphere by pieces, not necessarily convex. We first prove the existence of a quasi-geodesic of length less than a bound depending on a triangulation of the faces. This gives us an upper bound on the combinatorics of the quasi-geodesic (order of meeting of the vertices and edges) and allows us to draw up a list of the paths to be tested.

**Definition 0.0.1** (Quasi-geodesic)

A quasi-geodesic on a piecewise linear surface  $S$  is a continuous path  $\gamma : I \rightarrow \partial S$ , where  $I$  is an interval of  $\mathbb{R}$ , satisfying four conditions :

- It is uniform rectilinear on the faces of  $S$  that it crosses.
- It forms on the passage of an edge two angles equal to  $\pi$ .
- It forms on the passage of a vertex of positive curvature two angles less than or equal to  $\pi$ .
- It forms on the passage of a vertex of negative curvature two angles greater than or equal to  $\pi$ .

**Notation** : Given a linear sphere by piece  $S$  of vertices  $p_1, \dots, p_n$ , we consider a triangulation  $\{T_1, \dots, T_l\}$  which is based on the vertices and which be shellable<sup>2</sup>, that is to say :

$$\forall i = 1 \dots l, \bigcup_{k=1}^{k=i} T_k \simeq D^2 \text{ et } \bigcup_{k=1}^{k=l} T_k \simeq S^2$$

From now on, we can work from the metric of  $T_k$  and their combinatorics within  $S$ . We denote by  $a_1, \dots, a_m$  the edges of this triangulation (old and new edges combined), or else if necessary,  $a_{ij}$  the edge which connects the vertices  $p_i$  and  $p_j$ . We call *hat* of vertex  $p_i$ , denoted by  $\mathcal{C}_i$ , the union of the triangles  $T_k$  having  $p_i$  for common vertex. We specify that  $T_i$  and  $T_{i+1}$  are not necessarily adjacent. We optionally rename the vertices of  $P$  to have  $p_0 \in T_1$  et  $p_n \in T_l$ .

**Theorem 0.0.1** (Existence)

Let  $S$  be a linear sphere by piece as above. There is a simple and closed quasi-geodesic  $\gamma : S^1 \rightarrow \partial P$  whose length does not exceed  $M = \sum_i L(a_i)$ . We can also ask, even if it means dragging  $\gamma$  in parallel along the edges it crosses, that a point of  $S^1$  be sent to a vertex  $p_\star$ .

*Démonstration.* We start by describing a sweep of  $S$  by simple loops, bounded in length. We sweep  $T_1$  (resp.  $T_l$ ) by segments  $\sigma_s^1$  (resp.  $\sigma_s^l$ ), parallel to the side opposite to  $p_0$  (resp.  $p_n$ ). Then, for  $i$  from 2 to  $l-1$  :

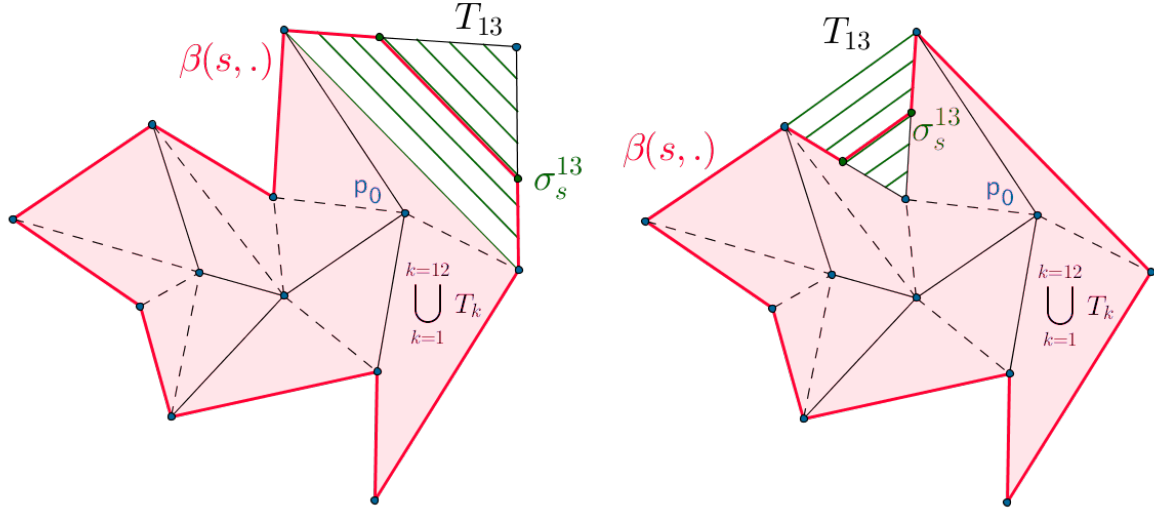
- If  $T_i$  shares a single side with  $\bigcup_{k=1}^{k=i-1} T_k$ , we sweep  $T_i$  by segments  $\sigma_s^i$  parallel to this side.
- If  $T_i$  shares two sides with  $\bigcup_{k=1}^{k=i-1} T_k$ , we sweep  $T_i$  by segments  $\sigma_s^i$  parallel to the third side.

At each segment  $\sigma_s^i$  so pulled through a certain  $T_i$ , we continuously associate (in  $s$  then in  $i$ ) the parameterization of a loop formed by the edge of the disc  $\bigcup_{k=1}^{k=i-1} T_k$ , deprived of its intersection with  $T_i$  and linked to  $\sigma_s^i$  by two portions of the edge of  $T_i$  (see figure below). For  $i = 1$  (resp.  $i = l$ ), rather we connect  $\sigma_s^i$  to  $p_0$  (resp.  $p_n$ ) by two portions of the  $T_i$ 's boundary. We have thus constructed a sweep of  $S$  by circles  $\beta : S^2 \rightarrow P$  – where  $S^2$  is seen as the quotient of the cylinder  $[0, 1] \times S^1$  by the relation which identifies the circles  $(0, S^1)$  and  $(1, S^1)$  to two points – which is of degree 1 and which satisfies :

- Each fiber  $\beta(s, \cdot) : S^1 \rightarrow P$  is a single broken line on the surface of  $S$ .
- There is a sequence of values :  $0 < s_1 < \dots < s_{l-1} < 1$  such that  $\beta([0, s_i] \times S^1) = \bigcup_{k=1}^{k=i} T_k$ .
- $\beta(0, \cdot)$  is the constant loop in  $p_0$  and  $\beta(1, \cdot)$  is the constant loop in  $p_n$ .
- $\forall s \in ]0, 1[, \beta(s, \cdot)$  is a polygon of length  $l_s$  drawn on  $S$ , traveled at constant speed  $c_s = l_t/2\pi$ .

1. We give in the appendix the proof that there does not always exist a straight quasi-geodesic on a convex polyhedron, that is to say if necessary, which crosses the vertices forming two equal angles.

2. ref



Thus constructed,  $\beta$  has no fiber  $\beta(s, \cdot)$  Whose length does not exceed  $M$ .

From now on, we can distinguish between the hats *convex* - for which the sum of the angles adjacent to  $p_i$  is less than  $2\pi$  - and *concave* - for which the sum of the adjacent angles to  $p_i$  is greater than  $2\pi$ .

### Shortening by discs.

We define here a disk flow protocol, that is to say an iterative shortening of the  $\beta$ 's fibers located in  $\mathcal{C}_i$ . In this case, we can speak of *straightening* of the fibers, brought locally into portions of quasi-geodesics. We are inspired here by the work of Hass and Scott [HS]. Gaps opened by the flow, especially around vertices and *beaks* (see below), will need to be stitched up with new fibers, smaller than the longest fiber of the initial sweep.

Let  $0 \leq i \leq n$  and  $\mathcal{C}_i$  be a hat. We call *arc* of  $\mathcal{C}_i$  a connected component of :

$$\mathcal{C}_{ij} \cap \text{Im}(\beta(s, \cdot))$$

The different cases to be treated depend on the following three criteria :

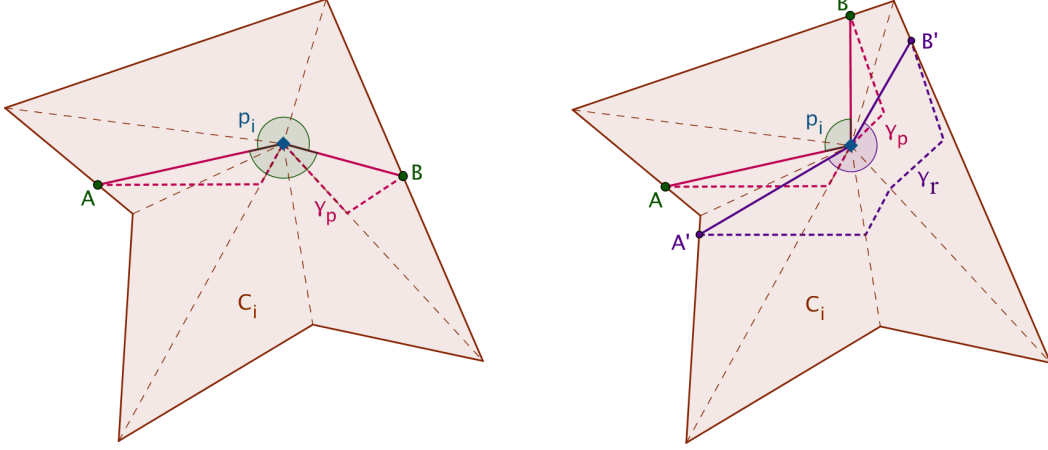
- Presence or not of a fiber entirely included in  $\mathcal{C}_i$ .
- Presence or not of a *beac* in  $\mathcal{C}_i$ , namely an arc  $\delta$  in  $\mathcal{C}_i$  such that  $\partial\mathcal{C}_i \cap \delta$  admits more than two connected components. The number of connected components beyond two defines the degree of the beak.
- Convexity or concavity of  $\mathcal{C}_i$ .

**Case 1** - We treat the case where no fiber  $\beta(t, \cdot)$  is included in  $\mathcal{C}_i$ , no beak crosses  $\mathcal{C}_i$ , with  $\mathcal{C}_i$  convex. We test one of the fibers passing through  $p_i$  (there may be a bunch). We denote by  $\gamma_p$  this fiber and  $A, B$  its entry and exit points of  $\mathcal{C}_i$ . The broken line  $Ap_iB$  split  $\mathcal{C}_i$  in two quarters  $\mathcal{C}_i^l$  and  $\mathcal{C}_i^r$ .

- If the two angles<sup>3</sup> formed by  $Ap_iB$  are  $\leq \pi$ , then  $\gamma_p$  is straightened out  $Ap_iB$ , via a decreasing homotopy for the length. Therefore and in the same homotopy ballet, all the other fibers of  $\mathcal{C}_i$  can be straightened on a shorter path in  $\mathcal{C}_i$ , on the side of  $Ap_iB$  where they take roots, namely  $\mathcal{C}_i^l$  or  $\mathcal{C}_i^r$ . Remember that no fiber up to now has been cut transversely. Note : a hat is not necessarily convex - it is at least starred -, then the shortest path between two point of its boundary can come to rest against it.

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3. All zenith projections are color coded for the angles :  
 $\leq \pi \rightarrow \text{green} \mid > \pi \rightarrow \text{blue} \mid = \pi \rightarrow \text{purple}$



- If the angle formed by  $Ap_iB$  on the side of  $\mathcal{C}_i^l$  (resp.  $\mathcal{C}_i^r$ ) is  $> \pi$  (necessarily, the other will be  $< \pi$ ), we are looking for a fiber  $\gamma_l$  (resp.  $\gamma_r$ ) which can be straightened in  $\mathcal{C}_i^l$  (resp.  $\mathcal{C}_i^r$ ) passing through  $p_i$ , to form an angle equal to  $\pi$  : it realizes on this side of  $Ap_iB$  a shorter path. Therefore, we straighten all the fibers of  $\mathcal{C}_i$  on either side of  $\gamma_l$  (resp.  $\gamma_r$ ), as before.

**Case 2** - We treat the case where no fiber  $\beta(t, \cdot)$  is included in  $\mathcal{C}_i$ , no beak crosses  $\mathcal{C}_i$ , with  $\mathcal{C}_i$  concave. We test one of the fibers passing through  $p_i$ , by taking the previous notations.

Attention, the presence of a beak, in this district of  $\mathcal{C}_i$  can compromise the existence of such a fiber.  $\square$

**Theorem 0.0.2** (Theorem 2)

*There is an algorithm making it possible to construct such a quasi-geodesic.*

*Proof of Theorem 2.* Let  $\mathcal{E} = \{p_1, \dots, p_n, a_1, \dots, a_m\}$  be the set of vertices and open edges of  $S$ . To a closed curve  $c : I \rightarrow P$ , we associate the word  $\mathcal{E}(c)$  whose successive letters (grouped under a bar) are those which designate the elements of  $\mathcal{E}$  met by  $c(t)$  when  $t$  travels  $S^1$ . We want to give a bound  $\eta$  on the combinatorial of  $\gamma$ , that is to say a bound on the length of  $\mathcal{E}(\gamma)$ . To each edge  $a_{ij}$ , we associate a meeting coefficient  $k(a_{ij})$  measured as follows :

$$k(a_{ij}) = \max_{\substack{A \in a_{ij} \\ B \in \partial(\mathcal{C}_i \cup \mathcal{C}_j)}} \frac{\#([AB], \mathcal{E}) - 1}{L(AB)}$$

Where  $[AB]$  is the shortest way on  $S$  between  $A$  and  $B$  and  $\#([AB], \mathcal{E})$  denotes the number of elements of  $\mathcal{E}$  met by  $[AB]$ , transversely or longitudinally if it is an edge.

We then have an expression of  $\eta$  :

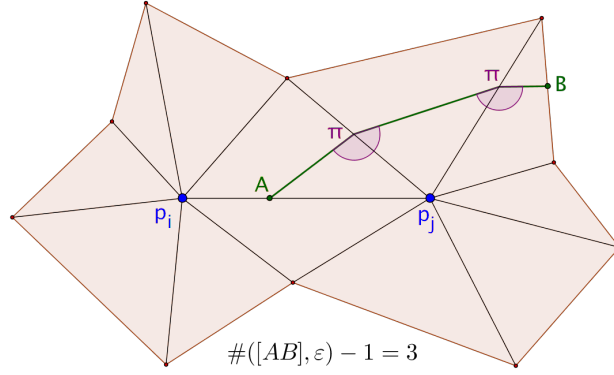
$$\eta = \left\lceil M \cdot \max_{a_{ij}} k(a_{ij}) \right\rceil$$

We consider the set<sup>4</sup> of  $(\eta + 1)$ -letter words written in the alphabet  $\mathcal{E}$ , beginning and ending with the same vertex name, noted  $p_*$ . A *valid* word must contain only three types of sequences :

- An edge crimped by its two ends.
- An edge crimped by the two opposite vertices of the triangles joined at this edge.
- A series of adjacent two by two edges (except triplets  $a_{ij}a_{jk}a_{ij}$  and  $a_{ij}a_{jk}a_{ki}$ ), preceded by the vertex opposite the first edge (but not attached to the second) and followed by the vertex opposite last edge (but not attached to the penultimate).

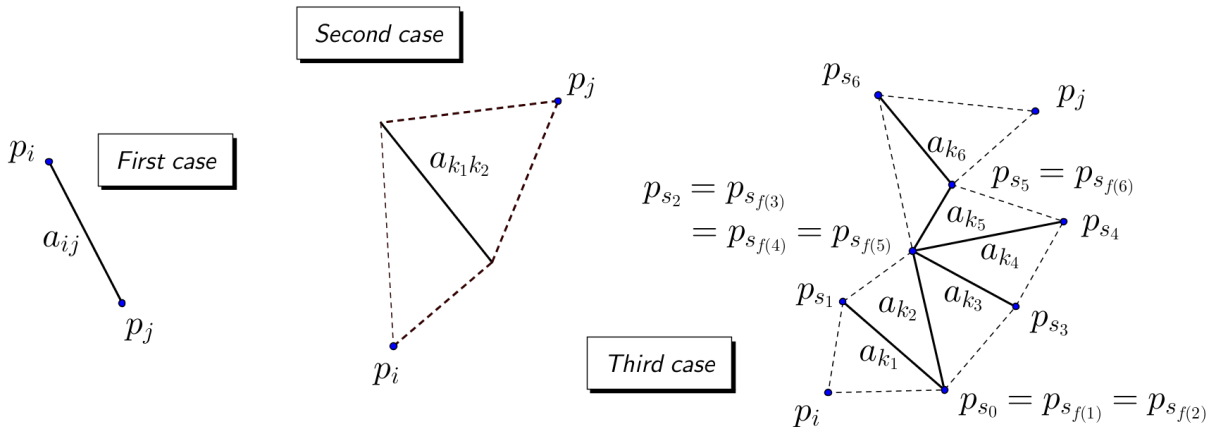
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4. Of cardinal  $n(n + m)^{\eta-1}$



To  $\gamma$  necessarily corresponds such a word, noted  $\mathcal{E}(\gamma)$ . Conversely, to a word  $x = \overline{p_\star l_1 \dots l_{\eta-1} p_\star}$  corresponds a simple and closed quasi-geodesic if it satisfies, from left to right, the following test :

- First case :  $\overline{p_i a_{ij} p_j}$  takes place in  $x$ . We save the path  $[p_i, p_j]$  along  $a_{ij}$ .
- Second case :  $\overline{p_i a_{k_1 k_2} p_j}$  takes place in  $x$ , where  $(p_i, p_{k_1}, p_{k_2})$  and  $(p_j, p_{k_1}, p_{k_2})$  designate two adjacent triangles. We save the path  $[p_i, p_j]$  through the flattening of the two triangles in question, unless it is not convex, in which case the test fails.
- Third case : two vertices  $p_i$  and  $p_j$  surround a sequence  $\overline{a_{k_1} \dots a_{k_r}}$  in  $x$ . We associate with  $a_{k_1}$  its two ends noted  $p_{s_0}, p_{s_1}$  and for  $l \geq 2$ , we associate with each  $a_{k_l}$  the vertex  $p_{s_l}$  it does not share with  $a_{k_{l-1}}$ . Also, we have a decreasing function  $f : l \mapsto f(l)$  such that  $p_{s_l}$  and  $p_{s_{f(l)}}$  are the ends of  $a_{k_l}$ . We send the vertices  $(p_i, p_{s_1}, \dots, p_{k_r}, p_j)$  in an oriented plane coordinate system, so that all the triangles keep their metric and their relative orientations (we assume that  $0 < (\overrightarrow{p_i p_{s_0}}, \overrightarrow{p_i p_{s_1}}) < \pi$ ).



We endow ourselves with Boolean functions determining whether a point is located to the left (resp. the right) of an oriented line  $(p_i, \overrightarrow{u})$ , strictly or not :  $p \mapsto (p < u)$ ,  $p \mapsto (p \leq u)$  (resp.  $p \mapsto (p > u)$ ,  $p \mapsto (p \geq u)$ ). The following algorithm evaluates this syllable of  $x$  :

$$\begin{aligned} u &= \overrightarrow{p_i p_{k_1}} \\ v &= \overrightarrow{p_i p_{k_2}} \\ \tau &= 1 \end{aligned}$$

For  $l$  from 3 to  $r$  and as long as  $\tau == 1$  :

If  $(p_{s_l} \leq v)$  with  $(p_{s_{f(l)}} \leq v)$  : then  $\tau = 0$ .

If  $(p_{s_l} \geq u)$  with  $(p_{s_{f(l)}} \geq u)$  : then  $\tau = 0$ .

If  $(p_{s_l} < u)$  and  $(p_{s_l} > v)$  :

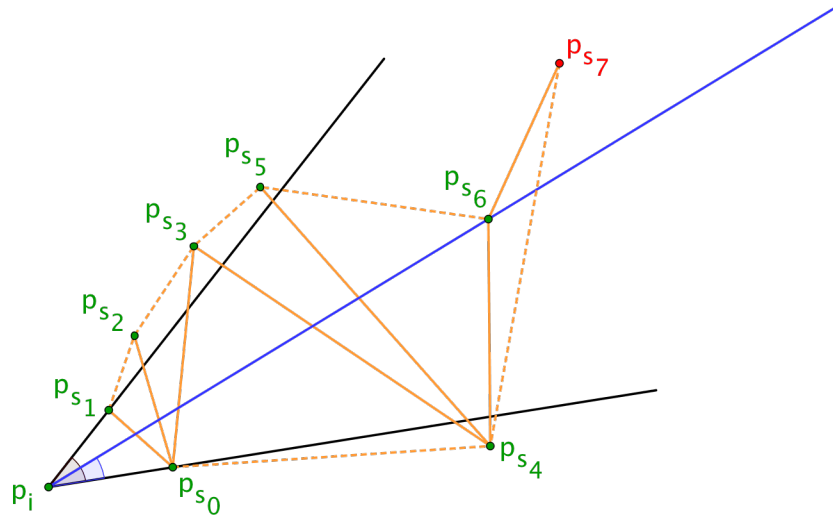
If  $(p_{s_{f(l)}} \geq u) :$  then  $v = \overrightarrow{p_i p_{s_l}}$

If  $(p_{s_{f(l)}} \leq v) : \text{ then } u = \overrightarrow{p_i p_{s_l}}$

If  $(p_j > u)$  or  $(p_j < v)$  : then  $\tau = 0$ .

Only if  $\tau == 1$ , we save the path  $[p_i, p_j]$  through this local pattern.

Finally, we check the angles induced by the paths saved upstream and downstream of each vertex appearing in  $x$ . If they satisfy the conditions of a quasi-geodesic, we can say that  $x$  is the word of a quasi-geodesic.



At least one of the valid words must pass the test, starting with  $\bar{\gamma}$ .

Thus, the test provides the construction of a quasi-geodesic on  $P$ .

