Construction of a quasi-geodesic on a linear sphere by piece

Jean Chartier

9 septembre 2021

In 1949, Alexandre Pogorelov proves the existence of a simple and closed quasi-geodesic on any convex polyhedron [1]. We propose to describe an algorithm allowing to build such a quasi-geodesic on any linear sphere by pieces, not necessarily convex. We first prove the existence of a quasi-geodesic of length less than a bound depending on a triangulation of the faces. This gives us an upper bound on the combinatorics of the quasi-geodesic (order of meeting of the vertices and edges) and allows us to draw up a list of the paths to be tested.

Definition 0.0.1 (Quasi-geodesic)
A quasi-geodesic on a piecewise linear surface S is a continuous path $\gamma : I \longrightarrow \partial S$, where I is an interval of $\mathbb{R}$, satisfying four conditions :

- *It is uniform rectilinear on the faces of S that it crosses.*
- *It forms on the passage of an edge two angles equal to π.*
- *It forms on the passage of a vertex of positive curvature two angles less than or equal to π.*
- *IIt forms on the passage of a vertex of negative curvature two angles greater than or equal to π.*

Notation : Given a linear sphere by piece S of vertices $p_1, \dots, p_n$, we consider a triangulation $\{T_1, \dots, T_l\}$ which is based on the vertices and which be shellable [2], that is to say :

$$\forall i = 1 \dots l, \bigcup_{k=1}^{k=i} T_k \simeq D^2 \text{ et } \bigcup_{k=1}^{k=l} T_k \simeq S^2$$

From now on, we can work from the metric of T_k and their combinatorics within S. We denote by $a_1, \dots, a_m$ the edges of this triangulation (old and new edges combined), or else if necessary, a_{ij} the edge which connects the vertices p_i and p_j. We call *hat* of vertex p_i, denoted by $\mathcal{C}_i$, the union of the triangles T_k having p_i for common vertex. We specify that T_i and T_{i+1} are not necessarily adjacent. We optionally rename the vertices of P to have $p_0 \in T_1$ et $p_n \in T_l$.

Theorem 0.0.1 (Existence)
Let S be a linear sphere by piece as above. There is a simple and closed quasi-geodesic $\gamma : S^1 \longrightarrow \partial P$ whose length does not exceed $M = \sum_i L(a_i)$. We can also ask, even if it means dragging γ in parallel along the edges it crosses, that a point of S^1 be sent to a vertex $p_\star$.

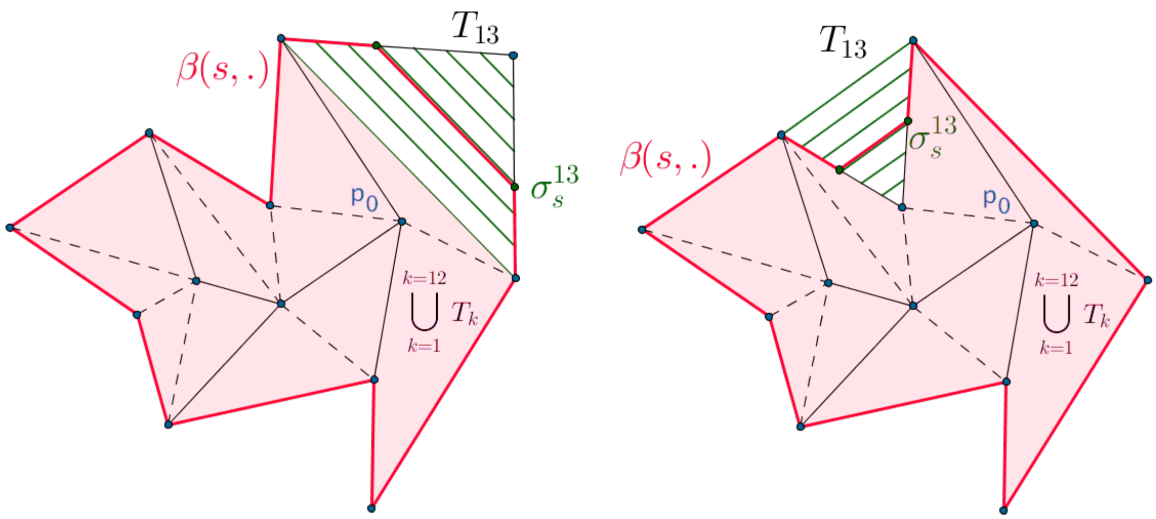
Démonstration. We start by describing a sweep of S by simple loops, bounded in length. We sweep T_1 (resp. T_l) by segments σ_s^1 (resp. σ_s^l), parallel to the side opposite to p_0 (resp. p_n). Then, for i from 2 to $l-1$:

- If T_i shares a single side with $\bigcup_{k=1}^{k=i-1} T_k$, we sweep T_i by segments σ_s^i parallel to this side.
- If T_i shares two sides with $\bigcup_{k=1}^{k=i-1} T_k$, we sweep T_i by segments σ_s^i parallel to the third side.

At each segment σ_s^i so pulled through a certain T_i, we continuously associate (in s then in i) the parameterization of a loop formed by the edge of the disc $\bigcup_{k=1}^{k=i-1} T_k$, deprived of its intersection with T_i and linked to σ_s^i by two portions of the edge of T_i (see figure below). For $i = 1$ (resp. $i = l$), rather we connect σ_s^i to p_0 (resp. p_n) by two portions of the T_i's boundary. We have thus constructed a sweep of S by circles $\beta : S^2 \longrightarrow P$ – where S^2 is seen as the quotient of the cylinder $[0,1] \times S^1$ by the relation which identifies the circles $(0, S^1)$ and $(1, S^1)$ to two points – which is of degree 1 and which satisfies :

- Each fiber $\beta(s, .) : S^1 \longrightarrow P$ is a single broken line on the surface of S.
- There is a sequence of values : $0 < s_1 < \dots < s_{l-1} < 1$ such that $\beta([0, s_i] \times S^1) = \bigcup_{k=1}^{k=i} T_k$.
- $\beta(0, .)$ is the constant loop in p_0 and $\beta(1, .)$ is the constant loop in p_n.
- $\forall s \in]0,1[, \beta(s, .)$ is a polygon of length l_s drawed on S, traveled at constant speed $c_s = l_t/2\pi$.

1. We give in the appendix the proof that there does not always exist a straight quasi-geodesic on a convex polyhedron, that is to say if necessary, which crosses the vertices forming two equal angles.
2. ref



Thus constructed, β has no fiber $\beta(s,.)$ Whose length does not exceed M.

From now on, we can distinguish between the hats *convex* - for which the sum of the angles adjacent to p_i is less than 2π - and *concave* - for which the sum of the adjacent angles to p_i is greater than 2π.

Shortening by discs.

We define here a disk flow protocol, that is to say an iterative shortening of the β's fibers located in $\mathcal{C}_i$. In this case, we can speak of *straightening* of the fibers, brought locally into portions of quasi-geodesics. We are inspired here by the work of Hass and Scott [HS]. Gaps opened by the flow, especially around vertices and *beaks* (see below), will need to be stitched up with new fibers, smaller than the longest fiber of the initial sweep.

Let $0 \leq i \leq n$ and $\mathcal{C}_i$ be a hat. We call *arc* of $\mathcal{C}_i$ a connected component of :

$$\mathcal{C}_{ij} \cap Im(\beta(s,.))$$

The different cases to be treated depend on the following three criteria :

— Presence or not of a fiber entirely included in $\mathcal{C}_i$.

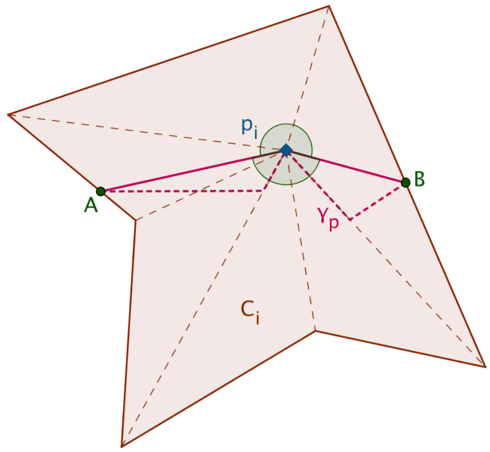
— Presence or not of a *beac* in $\mathcal{C}_i$, namely an arc δ in $\mathcal{C}_i$ such that $\partial\mathcal{C}_i \cap \delta$ admits more than two connected components. The number of connected components beyond two defines the degree of the beak.

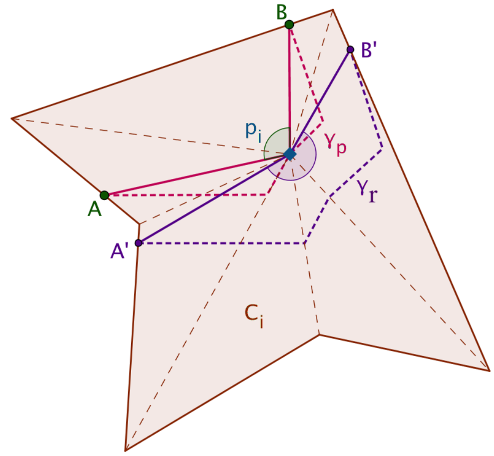
— Convexity or concavity of $\mathcal{C}_i$.

Case 1 - We treat the case where no fiber $\beta(t,.)$ is included in $\mathcal{C}_i$, no beak crosses $\mathcal{C}_i$, with $\mathcal{C}_i$ convex. We test one of the fibers passing through p_i (there may be a bunch). We denote by γ_p this fiber and A, B its entry and exit points of $\mathcal{C}_i$. The broken lineAp_iB split $\mathcal{C}_i$ in two quarters $\mathcal{C}_i^l$ and $\mathcal{C}_i^r$.

— If the two angles [3] formed by Ap_iB are $\leq \pi$, then γ_p is straightened out Ap_iB, via a decreasing homotopy for the length. Therefore and in the same homotopy ballet, all the other fibers of $\mathcal{C}_i$ can be straightened on a shorter path in $\mathcal{C}_i$, on the side of Ap_iB where they take roots, namely $\mathcal{C}_i^l$ or $\mathcal{C}_i^r$. Remember that no fiber up to now has been cut transversely. Note : a hat is not necessarily convex – it is at least starred –, then the shortest path between two point of its boundary can come to rest against it.

3. **All zenith projections are color coded for the angles :**
$\leq \pi \longrightarrow$ **green** | $> \pi \longrightarrow$ **blue** | $= \pi \longrightarrow$ **purple**





— If the angle formed by Ap_iB on the side of $\mathcal{C}_i^l$ (resp. $\mathcal{C}_i^r$) is $> \pi$ (necessarily, the other will be $< \pi$), we are looking for a fiber γ_l (resp. γ_r) which can be straightened in $\mathcal{C}_i^l$ (resp. $\mathcal{C}_i^r$) passing through p_i, to form an angle equal to π : it realizes on this side of Ap_iB a shorter path. Therefore, we straighten all the fibers of $\mathcal{C}_i$ on either side of γ_l (resp. γ_r), as before.

Case 2 - We treat the case where no fiber $\beta(t,.)$ is included in $\mathcal{C}_i$, no beak crosses $\mathcal{C}_i$, with $\mathcal{C}_i$ concave. We test one of the fibers passing through p_i, by taking the previous notations.

Attention, the presence of a beak, in this district of $\mathcal{C}_i$ can compromise the existence of such a fiber. □

Theorem 0.0.2 (Theorem 2)
There is an algorithm making it possible to construct such a quasi-geodesic.

Proof of Theorem 2. Let $\mathcal{E} = \{p_1, \ldots, p_n, a_1, \ldots, a_m\}$ be the set of vertices and open edges of S. To a closed curve $c : I \longrightarrow P$, we associate the word $\mathcal{E}(c)$ whose successive letters (grouped under a bar) are those which designate the elements of $\mathcal{E}$ met by $c(t)$ when t travels S^1. We want to give a bound η on the combinatorial of γ, that is to say a bound on the length of $\mathcal{E}(\gamma)$. To each edge a_{ij}, we associate a meeting coefficient $k(a_{ij})$ measured as follows :

$$k(a_{ij}) = \max_{\substack{A \in a_{ij} \\ B \in \partial(\mathcal{C}_i \cup \mathcal{C}_j)}} \frac{\#([AB], \mathcal{E}) - 1}{L(AB)}$$

Where $[AB]$ is the shortest way on S beetwen A and B and $\#([AB], \mathcal{E})$ denotes the number of elements of $\mathcal{E}$ met by $[AB]$, transversely or longitudinally if it is an edge.
We then have an expression of η :

$$\eta = \left\lceil M \, . \, \max_{a_{ij}} k(a_{ij}) \right\rceil$$

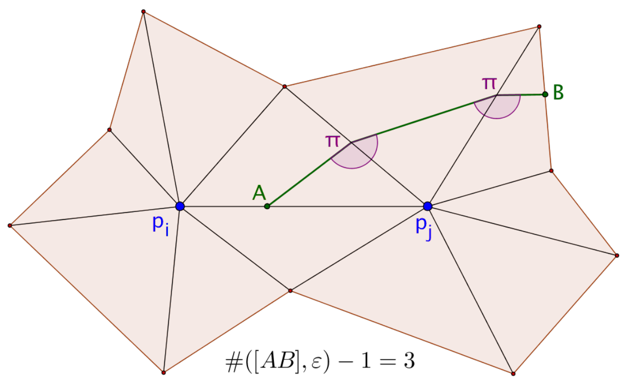
We consider the set [4] of $(\eta + 1)$-letter words written in the alphabet $\mathcal{E}$, beginning and ending with the same vertex name, noted $p_\star$. A *valid* word must contain only three types of sequences :

— An edge crimped by its two ends.

— An edge crimped by the two opposite vertices of the triangles joined at this edge.

— A series of adjacent two by two edges (except triplets $a_{ij}a_{jk}a_{ij}$ and $a_{ij}a_{jk}a_{ki}$), preceded by the vertex opposite the first edge (but not attached to the second) and followed by the vertex opposite last edge (but not attached to the penultimate).

4. Of cardinal $n(n+m)^{\eta-1}$



$\#([AB], \varepsilon) - 1 = 3$

To γ necessarily corresponds such a word, noted $\mathcal{E}(\gamma)$. Conversely, to a word $x = \overline{p_\star l_1 \dots l_{\eta-1} p_\star}$ corresponds a simple and closed quasi-geodesic if it satisfies, from left to right, the following test :

— First case : $\overline{p_i a_{ij} p_j}$ takes place in x. We save the path $[p_i, p_j]$ along a_{ij}.

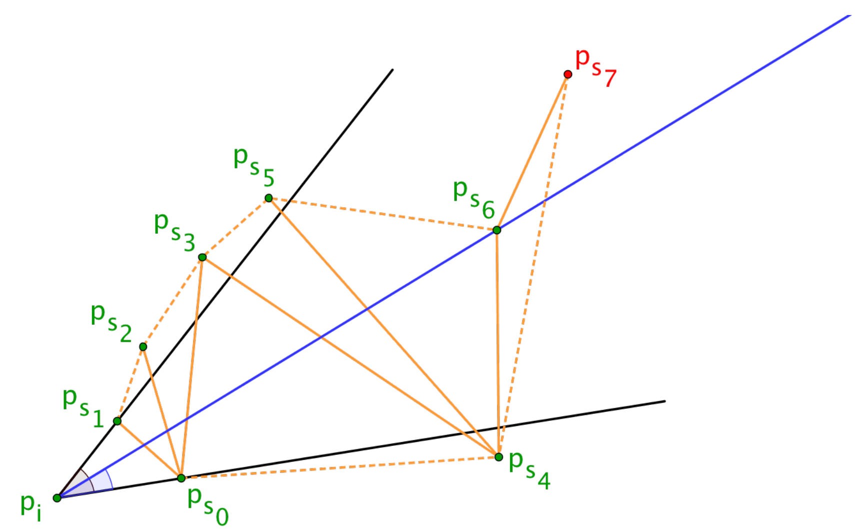
— Second case : $\overline{p_i a_{k_1 k_2} p_j}$ takes place in x, where (p_i, p_{k_1}, p_{k_2}) and (p_j, p_{k_1}, p_{k_2}) designate two adjacent triangles. We save the path $[p_i, p_j]$ through the straightening of the two triangles in question.

— Third case : two vertices p_i and p_j surround a sequence $\overline{a_{k_1} \dots a_{k_r}}$ in x. We associate with a_{k_1} its two ends noted p_{s_0}, p_{s_1} and for $l \geq 2$, we associate with each a_{k_l} the vertex p_{s_l} it does not share with $a_{k_{l-1}}$. Also, we have a decreasing function $f : l \mapsto f(l)$ such that p_{s_l} and $p_{s_{f(l)}}$ are the ends of a_{k_l}. We send the vertices $(p_i, p_{s_1}, \dots, p_{k_r} p_j)$ in an oriented plane coordinate system, so that all the triangles keep their metric and their relative orientations (we assume that $0 < (\overrightarrow{p_i p_{s_1}}, \overrightarrow{p_i p_{s_2}}) < \pi$). We endow ourselves with Boolean functions determining whether a point is located to the left (resp. the right) of an oriented line $(p_i, \overrightarrow{u})$, strictly or not : $p \mapsto (p < u)$, $p \mapsto (p \leq u)$ (resp. $p \mapsto (p > u)$, $p \mapsto (p \geq u)$). The following algorithm evaluates this syllable of x :

$u = \overrightarrow{p_i p_{k_1}}$
$v = \overrightarrow{p_i p_{k_2}}$
$\tau = 1$
For l from 3 to r and as long as $\tau == 1$:
  If $(p_{s_l} \leq v)$ with $(p_{s_{f(l)}} \leq v)$: then $\tau = 0$.
  If $(p_{s_l} \geq u)$ with $(p_{s_{f(l)}} \geq u)$: then $\tau = 0$.
  If $(p_{s_l} < u)$ and $(p_{s_l} > v)$:
    If $(p_{s_{f(l)}} \geq u)$: then $v = \overrightarrow{p_i p_{s_l}}$
    If $(p_{s_{f(l)}} \leq v)$: then $u = \overrightarrow{p_i p_{s_l}}$
If $(p_j > u)$ or $(p_j < v)$: then $\tau = 0$.
Only if $\tau == 1$, we save the path $[p_i, p_j]$ through this local pattern.

Finally, we check the angles induced by the paths saved upstream and downstream of each vertex appearing in x. If they satisfy the conditions of a quasi-geodesic, we can say that x is the word of a quasi-geodesic.

At least one of the valid words must pass the test, starting with $\overline{\gamma}$.
Thus, **the test provides the construction of a quasi-geodesic on** P. □


p_{s_7}
p_{s_5}
p_{s_6}
p_{s_3}
p_{s_2}
p_{s_1}
p_{s_4}
p_i
p_{s_0}