Assignment 6

Due at class time on Thursday Nov 19, 2020

1. It is stated in the notes that the Crank-Nicolson method for the one-dimensional diffusion equation:
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{D}{2}\left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2}\right)$$
is unconditionally stable. Prove this fact.

2. Consider the following possible discretization for the one-dimensional diffusion equation:
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left[(1-\alpha)\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + \alpha\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2}\right],$$
with $0 \le \alpha \le 1$. Use the von Neumann method to determine the stability features of this discretization. Check your results with the special cases $\alpha = 0$, $\frac{1}{2}$ and 1 for which you know the answer.

3. **Computational Problem**

   Write a computer code to solve by the Crank-Nicolson method over the time interval $0 \le t \le T$ the one-dimensional diffusion equation
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + F(x,t)\,,$$
   with $0 \le x \le L$ and the constant D and the function $F(x,t)$ prescribed. The initial condition is
$$u(x, t=0) = f(x)\,, \tag{1}$$
   and the boundary conditions
$$u(x=0,t) = g_0(t)\,, \qquad u(x=L,t) = g_L(t)\,, \tag{2}$$
   with $f(x)$, $g_0(t)$ and $g_L(t)$ also prescribed.

   To test your program you can consider the following situations:

   (a) $L = \pi$, $D = 0.1$, $T = 10$, $F = 0$, $g_0 = g_L = 0$, $f(x) = \sin kx$ with k an integer. The exact solution in this case is $u_{exact}(x,t) = \exp(-Dk^2 t)\,\sin kx$.

   (b) $L = \pi$, $D = 0.1$, $g_0 = \sin\omega t$, $g_L(t) = \sin\omega t\,\cos kL$, $f(x) = 0$, $F(x,t) = (\omega\cos\omega t + Dk^2\sin\omega t)\cos kx$, with k and integer. The exact solution in this case is $u_{exact}(x,t) = \sin\omega t\,\cos kx$. For a fixed Δt try different increasing values of ω, from $\omega\Delta t = 0.1$ on up. What happens to the error?

   In each case calculate the average error
$$\epsilon = \frac{1}{N}\sum_{\ell=1}^{N}\left|\frac{u(x_\ell,T) - u_{exact}(x_\ell,T)}{u_{exact}(x_\ell,T)}\right| \tag{3}$$
   where N is the number of interior nodes. Examine grid convergence and make sure that the results that you show are grid independent.

   Submit a report with some graphs comparing the exact and numerical solution, $u(x,t)$ as a function of x at a few values of t, for example $t = T/5$, $T/2$, T, and also some tables including some (only some!)

numerical values to permit a better comparison between the two solutions. Remember that you are supposed to write a report explaining what you have done and what conclusions you have drawn. The report should include only enough material to support these conclusions. Don't just "throw" graphs and tables to us – this is not what you are asked to do and this will not improve your grade!

All the code writing history should be logged using `git` version control and the *.git* folder should be submitted along with the code.