

Assignment 4 – Computing Problem

Due at class time on Tuesday Oct. 13, 2020

Discretize the interval $0 \leq x \leq L$ using N equally spaced internal points $x_1 = h, x_2 = 2h, \dots, x_N = Nh$, with $h = L/(N+1)$; the end points are $x_0 = 0$ and $x_{N+1} = L$. Write a code using the tri-diagonal algorithm to solve the Helmholtz equation

$$\frac{d^2u}{dx^2} - k^2u(x) = f(x),$$

in the range $0 \leq x \leq L$. Consider two cases in which the boundary conditions are:

1. Dirichlet type: $u(x=0) = U_0, u(x=L) = 0$; the exact solution in this case is, for $f(x) = A$, a constant,

$$u(x) = \left(\frac{\sinh[k(L-x)] + \sinh(kx)}{\sinh(kL)} - 1 \right) \frac{A}{k^2} + U_0 \frac{\sinh[k(L-x)]}{\sinh(kL)}$$

2. Neumann type: $du/dx|_{x=0} = v, u(x=L) = 0$; the exact solution is, for $f(x) = A$, a constant,

$$u(x) = \left(\frac{\cosh(kx)}{\cosh(kL)} - 1 \right) \frac{A}{k^2} - \frac{v}{k} \frac{\sinh[k(L-x)]}{\cosh(kL)}$$

Do not start with a very large number of nodes N . Start with a reasonable number (e.g. $N = 10$) and then increase it if necessary. A good way to see if more nodes are needed is to use a certain value of N , and then repeat the calculation with $2N$ nodes. If the results of the two calculations are different, it means that N nodes are too few. In this case you should compare the results for $2N$ and $4N$ and so forth until the results of two successive discretizations are about the same; this is called a *grid convergence study* and is an essential procedure for reliable computations. We have given you the exact solutions so that, *after* the grid convergence study, you can check whether your results are indeed good.

Additionally to have some idea whether the finite difference scheme has the desired order of accuracy, you can find the formal order of accuracy. To do that you have to calculate the maximum absolute errors in N and $2N$ nodes using the analytical solution. Then use following formula to find the formal order of accuracy.

$$\frac{\log\left(\frac{Error_N}{Error_{2N}}\right)}{\log(2)}$$

You should expect that, once the grid convergence criterion has been satisfied, the numerical solution and the exact solutions should more or less superpose within the thickness of the line. A significantly larger difference is a symptom of coding errors.

For numerical purposes use two values of k , namely $k = 1$ and $k = 10$ and, for each one of them, solve both problems 1 and 2. For all cases you can take $L = 1, U_0 = 1, v = 1, A = 1$.

Submit graphs comparing the exact and numerical solutions and also tables including some (only some!) numerical values to permit a better comparison between the two solutions.

All the code writing history should be logged using git version control and the *.git* folder should be submitted along with the code.