Canadian travellers minimize the traversed distance: definitions, bounds and heuristics

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1 Introduction

The *Canadian Traveller Problem* (CTP) was introduced by Papadimitriou and Yannakakis [**?**]. This is a generalization of the *Shortest Path Problem*. Given an undirected weighted graph $G = (V, E, \omega)$ and two nodes $s, t \in V$, the objective is to design a strategy in order to make a traveller walk from s to t through the graph G, knowing that some edges can be blocked. The traveller ignores which edges are blocked when he begins and discover them when he visits an adjacent node. The k-*Canadian Traveller Problem* (k-CTP) is the parameterized variant of CTP where we specify an upper bound for the total number of blocked edges. Both CTP and k-CTP are PSPACE-complete [**?**, **?**].

State-of-the-art: Several strategies have been designed and studied through the competitive analysis, which is a way to assess the quality of an online algorithm. A first class of strategies to be considered are deterministic. Westphal proved that there is no deterministic algorithm that can achieve a better competitive ratio than $2k + 1$ where k is an upper bound of the number of blockages and that this ratio is achieved by REPOSITION algorithm [**?**]. However, in practice (for example in the case of an urban network), returning to node s everytime the traveller is blocked does not seem to be an efficient strategy. This is why Xu et al. introduced the GREEDY algorithm for the CTP which achieves a $2^{k+1} - 1$ ratio [**?**]. For grids, they showed that the GREEDY strategy achieves a $\mathcal{O}(1)$ ratio, independent of k, under realistic hypotheses. Both GREEDY and REPOSITION strategies are executed in polynomial time.

A second class of strategies are randomized. We evaluate these strategies by supposing that an oblivious adversary is setting the blocked edges. Westphal proved that there is no randomized algorithm that can achieve a ratio lower than $k+1$ [**?**]. Recently, two randomized algorithms have been proposed. Demaine et al. designed a strategy for arbitrary graphs with a ratio $\left(1 + \frac{\sqrt{2}}{2}\right) k + 1$. This is executed in time $\mathcal{O}\left(k\mu^2 |E|^2\right)$ where μ is an parameter that can potentially be exponential [**?**]. However, for a rather large class of graph (graphs with a reasonable number of groups of path, where a group of path is a set of paths which have the same length), this strategy offers better results than deterministic methods and is executed in polynomial time. Furthermore, Bender et al. studied the specific case of node-disjoint-paths graphs and proposed a strategy of ratio $(k + 1)$ and with a polynomial running time for this kind of graphs [**?**]. Their method consists in a randomized REPOSITION: we assign a probability p_i to each path P_i and execute a draw: the traveller crosses the path and, if he is blocked, returns to s and restarts the process.

2 Preliminaries

2.1 Notations and definitions with a single traveller

The traveller traverses an undirected weighted graph $G = (V, E, \omega)$, $n = |V|$ and $m = |E|$. He starts his walk at source $s \in V$. His objective is to reach target $t \in V$ with a minimum cost (also called distance), which is the sum of the weights of edges traversed. Set E_* contains blocked edges, which means that when the traveller reaches an endpoint of one of these edges, he discovers that he cannot pass through it. A pair (G, E_*) is called a *road map*. From now on, we suppose that any road map (G, E_*) is feasible, *i.e.* s and t are always connected in graph $G \backslash E_*$.

We remind the definition of the competitive ratio introduced in [?]. Let $\omega_A(G, E_*)$ be the distance traversed by the traveller guided by a given strategy A on graph G from source s to target t with blocked edges E_*. The shortest (s,t)-path in $G \backslash E_*$ is called the *optimal offline path* of map (G, E_*) and its cost, noted $\omega_{\min}(G, E_*)$, is the optimal offline cost of map (G, E_*). Strategy A is c_A-competitive if, for any road map (G, E_*):

$$\omega_A(G, E_*) \leq c_A \omega_{\min}(G, E_*) + \eta,$$

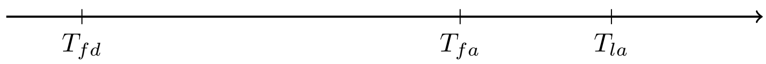
where η is constant. For randomized strategies, it becomes:

$$\mathbb{E}[\omega_A(G, E_*)] \leq c_A \omega_{\min}(G, E_*) + \eta.$$

2.2 Notations and definitions with L travellers

The L travelers traverses the same undirected weighted graph $G = (V, E, \omega)$, $n = |V|$ and $m = |E|$. They all start their walk at source $s \in V$. Their objective is to reach target $t \in V$ with a minimum cost (also called distance), which is the sum of the weights of edges traversed. We call d_i the distance travelled by the i^{th} traveler. We consider that the L travelers can communicate with each other. Set E_* contains blocked edges, which means that when a traveller reaches an endpoint of one of these edges, he discovers that he cannot pass through it and communicate this information to the other travelers. A pair (G, E_*) is called a *road map*. From now on, we suppose that any road map (G, E_*) is feasible, *i.e.* s and t are always connected in graph $G \backslash E_*$.

We consider the following timeline which brings forth three times we will need to study this problem.



$$T_{fd} : \text{time of the first departure}$$
$$T_{fa} : \text{time of the first arrived}$$
$$T_{la} : \text{time of the last arrived}$$

We want to study the minimization of the distances travelled by all the travelers from the departure point s and the arrival point t. We have two main problems to consider, we will try to minimize the distances travelled by all travelers:

- Before the first arrived :

$$\sum_{i=1}^{L} d_i \qquad \text{between } T_{fd} \text{ and } T_{fa}$$

- After the last arrived :

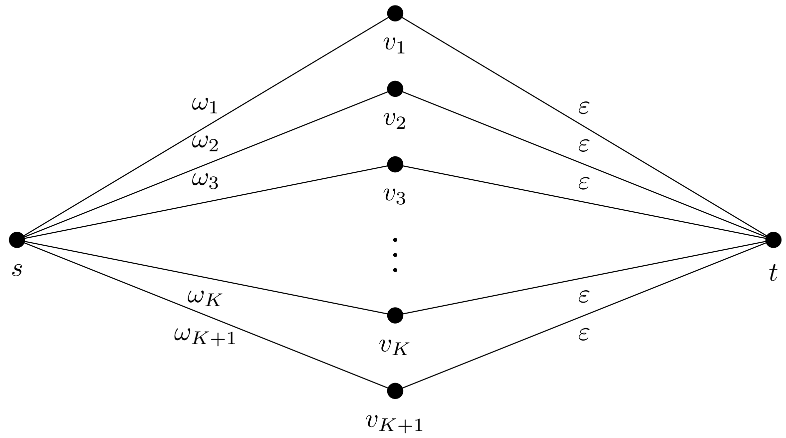
$$\sum_{i=1}^{L} d_i \qquad \text{between } T_{fd} \text{ and } T_{la}$$

3 Bounds of competitiveness with multiple travellers with complete communication: deterministic algorithm

We consider the following deterministic and online algorithm for K blocked edges and L travellers with complete communication: we launch the travellers one at a time. If the traveller meets a blocked edge, it has to stay there and we launch another traveller who will have to take the second shortest path possible. Once again when this one meets a blocked edge, it stays there and another is launched. Let's call this strategy the abandonment strategy.

Lemma 3.1: The cost of the abandonment strategy on the Westphal graph is $2(K+1) - min(K+1, L)$ between T_{fd} and T_{fa} and $2K + L$ between T_{fd} and T_{la}.

Proof: We consider the following Westphal graph, where all the ways from the departure point to the arrival point have the same cost $1 + \epsilon$. We are searching the cost for the worst case scenario: the only unblocked route is the one we choose last (s, v_{K+1}, t).



We consider that $w_1 = w_2 = ... = w_{K+1} = 1$, that ε is very small, that for $1 \leq i \leq K$, (v_i, t) is blocked and that t_{i+1} is a time after the i^{th} movement.

We have in fact 4 problems to consider: for each of the 2 main problems (between T_{fd} and T_{fa} and between T_{fd} and T_{la}) we have two secondary problems, when $K < L$ and $K \geq L$.

- $1^{st}case$: between T_{fd} and T_{fa} and $K < L$

  At the time t_1, we send a first traveler through the edge (s, v_1), who discovers that the edge (v_1, t) is blocked and communicates it to the other travelers.

  Then we send at the times t_i with $1 < i \leq K$ a traveler through the edge (s, v_i) and discovers that the edge (v_i, t) is blocked and communicates it to the other travelers.

  At the time t_{K+1}, we send a traveler through the edges (s, v_{K+1}) and then reaches the destination by the edge (v_{K+1}, t).

  Therefore we have:
$$\sum_{i=1}^{K} \omega_i + \omega_{K+1} + \varepsilon = \sum_{i=1}^{K} 1 + 1 + \varepsilon = K + 1 + \varepsilon$$

- $2^{nd}case$: between T_{fd} and T_{la} and $K < L$

  We redo the three first steps of the 1st case and at T_{pa} the first traveler communicate the unblocked route to the target and therefore that the edge (v_{K+1}, t) is not blocked.

  At the time t_{K+2} the K travelers blocked on the nodes $v_1 ... v_K$ come back to the point s and then take the unblocked route (s, v_{K+1}, t). The $L - K - 1$ travelers who have not left, take the unblocked route (s, v_{K+1}, t).

  Each traveler at the point v_i will then arrive at the destination with a cost of $w_i + w_{K+1} + \varepsilon$. And each traveler still at the departure point will arrive at destination with a cost of $w_{K+1} + \varepsilon$.

  Therefore, we have:
$$\begin{aligned} &\sum_{i=1}^{K} \omega_i + \omega_{K+1} + \varepsilon + \sum_{i=1}^{K} (\omega_i + \omega_{K+1} + \varepsilon) + (L - K)(\omega_{K+1} + \varepsilon) \\ &= 2\sum_{i=1}^{K} \omega_i + L(\omega_{K+1} + \varepsilon) = 2\sum_{i=1}^{K} 1 + L(1 + \varepsilon) \\ &= 2K + L(1 + \varepsilon) \end{aligned}$$

- $3^{rd}case$: between T_{fd} and T_{fa} and $K \geq L$

  At the time t_1, we send a first traveler through the edge (s, v_1), who discovers that the edge (v_1, t) is blocked and communicates it to the other travelers.

Then we send at the times t_i with $1 < i \leq L$ a traveler through the edge (s, v_i) and discovers that the edge (v_i, t) is blocked and communicates it to the other travelers.

At the time t_{L+1}, we send the traveler which is at the point v_l through the edges (v_L, s) and then (s, v_{L+1}). He discovers that the edge (v_{L+1}, t) is blocked and communicates it to the other travelers.

Then we send at the times t_i with $L+1 < i \leq K$ a traveler through the route (v_{i-1}, s, v_i) and discovers that the edge (v_i, t) is blocked and communicates it to the other travelers.

At the time t_{K+1}, we send a traveler through the route (v_K, s, v_{K+1}) and then reaches the destination by the edge (v_{K+1}, t).

Therefore we have:

$$\begin{aligned}
&\sum_{i=1}^{L} \omega_i + \sum_{i=L+1}^{K} (\omega_{i-1} + \omega_i) + \omega_K + \omega_{K+1} + \varepsilon \\
&= \sum_{i=1}^{L} 1 + \sum_{i=1}^{K-L} 2 + 1 + 1 + \varepsilon \\
&= L + 2(K-L) + 2 + \varepsilon \\
&= 2(K+1) - L + \varepsilon
\end{aligned}$$

- 4^{th}*case* : between T_{fd} and T_{la} and $K \geq L$

  We redo the five first steps of the third case and at T_{pa} the first traveler will communicate the unblocked route to the target and therefore that the edge (v_{K+1}, t) is not blocked.

  At the time t_{K+2} the L-1 travelers blocked on the nodes $v_1 ... v_{L-1}$ come back to the point s and then take the unblocked route (s, v_{K+1}, t).

  Each traveler at the point v_i will then arrive at the destination with a cost of $w_i + w_{K+1} + \varepsilon$.

  Therefore we have:

$$\begin{aligned}
&2(K+1) - L + \varepsilon + \sum_{i=1}^{L-1} (\omega_i + \omega_{K+1} + \varepsilon) \\
&= 2K + L(1 + \varepsilon)
\end{aligned}$$

Therefore we have the following costs for the Westphal graph:

	between T_{fd} and T_{fa}	between T_{fd} and T_{la}
$K < L$	$K + 1$	$2K + L$
$K \geq L$	$2(K+1) - L$	$2K + L$

Lemma 3.2: The abandonment strategy has the optimal cost on the Westphal graph.

Proof: This Lemma can be proved using recurrence.

Recurrence hypothesis: For K (blocked edges), for all L the optimal cost is : K + 1 if L > K and 2K - L + 1 if L < K.

Initialization: Let's take the example of 1 traveller. As shown in the existing literature we already have the cost equal to K + 1.

Recurrence: We know that there is a natural integer n number of travellers for which the cost is: if n > K, the cost is K + 1, and if n < K, the cost is 2K - L + 1. Let's prove it for n + 1. If n + 1 < K, since we are in the worst case scenario, the n travellers have to discover the K blocked edges before reaching the end. To do so, the cost is at least of K. Then 1 is added to reach the end. However, our strategy has a cost of K + 1 also, therefore once again, for n + 1 < K, the optimal cost is K + 1.

Now if n + 1 > K, we launch the first traveller. He discovers one blocked edge and stays there. We are now facing a problem of n travellers facing K-1 blockage. By recurrence hypothesis we know that for that case the cost is 2(K-1) - n + 1 because n > K - 1. To that cost we add the cost of discovering the first blocked edge. We have 2(K + 1) - n + 2 which is equal to 2K - (n + 1) - 1. This shows the second part of the recurrence.

We can therefore conclude that the optimal cost for the Westphal graph is : K + 1 if L > K and 2K - L + 1 if L < K. This is the cost of the abandonment strategy. Then this proves Lemma 3.2.

4 Bounds of competitiveness with multiple travellers with complete communication: randomized algorithms

Now we consider the randomized online algorithms for K blocked edges and L travellers with complete communication: we assign a probability pi to each path Pi and execute a draw: let the travellers crosses the path one by one, if the first traveller is blocked, leave him at the blocked point and let the second traveller to start the process; if all the travellers are blocked, then let the first traveller return to s and restarted the process. We can get:

Lemma 1 *If we have more travellers than blocked edges, there is no randomized online algorithm with competitive ratio less than:*

$$\frac{k+2}{2}$$

Proof. With K smaller than L, it is certain that we do not have to let a traveller return to s before we find all the blocked edges. If the algorithm is successful on its $(l+1)^{\text{th}}$ try, only the $(l+1)^{\text{th}}$ try cost $(1+\eta)$ and so the total cost will be:

$$(l-1)\cdot 1 + (1+\eta) = l+\eta$$

and the probability will be:

$$\frac{1}{l+1}$$

Hence, its expected cost is at least:

$$\sum_{l=1}^{k+1}(l+\eta)\cdot\frac{1}{l+1} = \frac{K+2}{2}+\eta$$

As we know, the expected optimal offline cost is $1+\eta$, so the competitive ratio is

$$\frac{K+2}{2}.$$

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Lemma 2 *If we have less travellers than blocked edges ($K > L$), there is no randomized online algorithm with competitive ratio less than:*

$$K+2-L+\frac{L(L-1)}{2(K+1)}$$

Proof. Similar with the proof of 1, if the algorithm is successful on its $(l+1)^{\text{th}}$ try, the probability will be:

$$\frac{1}{l+1}$$

But to calculus the cost, we have to consider two possibilities: if $l < L$, which means we have find all the blocked edges without letting any traveller to return to s, the cost will be

$$(l-1) \cdot 1 + 1 + \eta = l + \eta;$$

if $l > L$, it means that we have not find all the blocked edges after letting all the travellers depart s, so we have to let $(l-L)$ travellers return to s and restart the process. In this case, there will be L travellers with cost 1, $(l-L-1)$ travellers with cost 2 and the last traveller with cost $(2+\eta)$. So the total cost will be:

$$L \cdot 1 + (l-L-1) \cdot 2 + 2 + \eta = 2l - L + \eta.$$

So, its expected cost is at least:

$$\sum_{l=1}^{L}(l+\eta) \cdot \frac{1}{l+1} + \sum_{L+1}^{K+1}(2l-L+\eta) \cdot \frac{1}{l+1} = K + 2 - L + \frac{L(L-1)}{2(K+1)} + \eta$$

As we know, the expected optimal off-line cost is $1+\eta$, so the competitive ratio is

$$K + 2 - L + \frac{L(L-1)}{2(K+1)}.$$

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If we consider the special case $L = 1$ which means only one traveller, the competitive ratio becomes:

$$K + 2 - L + \frac{L(L-1)}{2(K+1)} = K + 2 - 1 = K + 1$$

This result corresponds to the randomized case of Bender et al. [**?**].

5 Bounds of competitiveness with multiple travellers without communication

In the last parts, we have discussed deterministic and randomized online algorithms with total communication, where all the travellers can both send and receive messages. Now we will begin to study deterministic and randomised cases without communication.

5.1 Deterministic algorithms

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5.2 Randomized algorithms

Lemma 3 *Considering the distance taken by all travellers after the last arrived (distance between T_{fd} and T_{la}), there is no randomized online algorithm with competitive ratio less than:*

$$L(K+1)$$

Proof. According to the result of Bender et al. [**?**], in randomized case, the the competitive ratio for only one traveller is $K+1$. When there is no communication between the travellers, all the travellers make choices independently. In this case, all the L travellers have to reach the target, so the obviously the result becomes L times $K+1$. ■

Lemma 4 *Considering the distance taken by all travellers before the first arrived (distance between T_{fd} and T_{fa}), there is no randomized online algorithm with competitive ratio less than:*

$$\sum_{l=1}^{K+1} \frac{(K-t+2)^L - (K-t+1)^L}{(K+1)^L} \cdot \left(\frac{2l-2}{L} + 1\right)$$

In this case, we have made an assumption which is different from the cases with complete communication: in the beginning, instead of setting off one by one, all the travellers set off together, which is more meaningful in the real world. So when the first travellers reached the target, the travellers, who had set off together with these lucky travellers but were blocked, are on the way back to the source s.

Proof.

The probability that the algorithm is successful on its $(1)^{\text{th}}$ try is: $1-(\frac{K}{K+1})^L$;
The probability that the algorithm is successful on its $(2)^{\text{th}}$ try is: $(\frac{K}{K+1})^L \cdot [1-(\frac{K-1}{K})^L]$;
The probability that the algorithm is successful on its $(l)^{\text{th}}$ try is:

$$(\frac{K}{K+1})^L \cdot (\frac{K-1}{K})^L \cdots (\frac{K-l+2}{K-l+3})^L \cdot [1-(\frac{K-l+1}{K-l+2})^L] = \frac{(K-t+2)^L-(K-t+1)^L}{(K+1)^L}$$

And according to the assumption, the cost will be:$(1+\eta)\cdot L+2(l-1)$, so the total expected cost is:

$$\sum_{l=1}^{K+1} \frac{(K-t+2)^L-(K-t+1)^L}{(K+1)^L} \cdot ((1+\eta)\cdot L+2(l-1))$$

As the expected optimal off-line cost is $L(1+\eta)$, so the competitive ratio is

$$\sum_{l=1}^{K+1} \frac{(K-t+2)^L-(K-t+1)^L}{(K+1)^L} \cdot (\frac{2l-2}{L}+1)$$

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5.3 Conclusion

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6 Conclusion

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