

Canadian travellers minimize the traversed distance: definitions, bounds and heuristics

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1 Introduction

The *Canadian Traveller Problem* (CTP) was introduced by Papadimitriou and Yannakakis [5]. This is a generalization of the *Shortest Path Problem*. Given an undirected weighted graph $G = (V, E, \omega)$ and two nodes $s, t \in V$, the objective is to design a strategy in order to make a traveller walk from s to t through the graph G , knowing that some edges can be blocked. The traveller ignores which edges are blocked when he begins and discover them when he visits an adjacent node. The *k-Canadian Traveller Problem* (*k*-CTP) is the parameterized variant of CTP where we specify an upper bound for the total number of blocked edges. Both CTP and *k*-CTP are PSPACE-complete [1, 5].

State-of-the-art: Several strategies have been designed and studied through the competitive analysis, which is a way to assess the quality of an online algorithm. A first class of strategies to be considered are deterministic. Westphal proved that there is no deterministic algorithm that can achieve a better competitive ratio than $2k + 1$ where k is an upper bound of the number of blockages and that this ratio is achieved by REPOSITION algorithm [6]. However, in practice (for example in the case of an urban network), returning to node s everytime the traveller is blocked does not seem to be an efficient strategy. This is why Xu et al. introduced the GREEDY algorithm for the CTP which achieves a $2^{k+1} - 1$ ratio [7]. For grids, they showed that the GREEDY strategy achieves a $\mathcal{O}(1)$ ratio, independent of k , under realistic hypotheses. Both GREEDY and REPOSITION strategies are executed in polynomial time.

A second class of strategies are randomized. We evaluate these strategies by supposing that an oblivious adversary is setting the blocked edges. Westphal proved that there is no randomized algorithm that can achieve a ratio lower than $k + 1$ [6]. Recently, two randomized algorithms have been proposed. Demaine et al. designed a strategy for arbitrary graphs with a ratio $\left(1 + \frac{\sqrt{2}}{2}\right)k + 1$. This is executed in time $\mathcal{O}\left(k\mu^2 |E|^2\right)$ where μ is an parameter that can potentially be exponential [4]. However, for a rather large class of graph (graphs with a reasonable number of groups of path, where a group of path is a set of paths which have the same length), this strategy offers better results than deterministic methods and is executed in polynomial time. Furthermore, Bender et al. studied the specific case of node-disjoint-paths graphs and proposed a strategy of ratio $(k + 1)$ and with a polynomial running time for this kind of graphs [2]. Their method consists in a randomized REPOSITION: we assign a probability p_i to each path P_i and execute a draw: the traveller crosses the path and, if he is blocked, returns to s and restarts the process.

2 Preliminaries

2.1 Notations and definitions with a single traveller

The traveller traverses an undirected weighted graph $G = (V, E, \omega)$, $n = |V|$ and $m = |E|$. He starts his walk at source $s \in V$. His objective is to reach target $t \in V$ with a minimum cost (also called distance), which is the sum of the weights of edges traversed. Set E_* contains blocked edges, which means that when the traveller reaches an endpoint of one of these edges, he discovers that he cannot pass through it. A pair (G, E_*) is called a *road map*. From now on, we suppose that any road map (G, E_*) is feasible, *i.e.* s and t are always connected in graph $G \setminus E_*$.

We remind the definition of the competitive ratio introduced in [3]. Let $\omega_A(G, E_*)$ be the distance traversed by the traveller guided by a given strategy A on graph G from source s to target t with blocked edges E_* . The shortest (s, t) -path in $G \setminus E_*$ is called the *optimal offline path* of map (G, E_*) and its cost, noted $\omega_{\min}(G, E_*)$, is the optimal offline cost of map (G, E_*) . Strategy A is c_A -competitive if, for any road map (G, E_*) :

$$\omega_A(G, E_*) \leq c_A \omega_{\min}(G, E_*) + \eta,$$

where η is constant. For randomized strategies, it becomes:

$$\mathbb{E}[\omega_A(G, E_*)] \leq c_A \omega_{\min}(G, E_*) + \eta.$$

2.2 Notations and definitions with L travellers

To be completed...

3 Bounds of competitiveness with multiple travellers

We consider the following determinist and online algorithm for K blocked edges and L travellers with complete communication: we launch the travellers one at a time. If the traveller meets a blocked edge, it has to stay there and we launch another traveller who will have to take the second shortest path possible. Once again when this one meets a blocked edge, it stays there and another is launched. Let's call this strategy the abandonment strategy.

Lemma 3.2: Lemme Aurelie

Proof: Preuve Aurelie.

Lemma 3.2: The abandonment strategy has the optimal cost on the Westphal graph.

Proof: This Lemma can be proved using recurrence.

Recurrence hypothesis: For K (blocked edges), for all L the optimal cost is : $K + 1$ if $L \nless K$ and $2K - L + 1$ if $L \less K$.

Initialization: Let's take the example of 1 traveller. As shown in the existing literature we already have the cost equal to $K + 1$.

Recurrence: We know that there is a natural integer n number of travellers for which the cost is: if $n \nless K$, the cost is $K + 1$, and if $n \less K$, the cost is $2K - L + 1$. Let's prove it for $n + 1$. If $n + 1 \less K$, since we are in the worst case scenario, the n travellers have to discover the K blocked edges before reaching the end. To do so, the cost is at least of K . Then 1 is added to reach the end. However, our strategy has a cost of $K + 1$ also, therefore once again, for $n + 1 \less K$, the optimal cost is $K + 1$.

Now if $n + 1 \nless K$, we launch the first traveller. He discovers one blocked edge and stays there. We are now facing a problem of n travellers facing $K-1$ blockage. By recurrence hypothesis we know that for that case the cost is $2(K-1) - n + 1$ because $n \nless K - 1$. To that cost we add the cost of discovering the first blocked edge. We have $2(K + 1) - n + 2$ which is equal to $2K - (n + 1) - 1$. This shows the second part of the recurrence.

We can therefore conclude that the optimal cost for the Westphal graph is : $K + 1$ if $L \nless K$ and $2K - L + 1$ if $L \less K$. This is the cost of the abandonment strategy. Then this proves Lemma 3.2.

4 Bounds of competitiveness with multiple travellers: randomized algorithms

Now we consider the randomized online algorithms for K blocs and L travellers with complete communication: we assign a probability p_i to each path P_i and execute a draw: let the travellers cross the path one by one, if the first traveller is blocked, leave him at the bloc point and let the second traveller to start the process; if all the travellers are blocked, then let the first traveller return to s and restarted the process. We can get:

Lemma 4.1: If we have more travellers than blocs, there is no randomized online algorithm with competitive ratio less than:

$$\frac{k+2}{2}$$

Proof: With K smaller than L , it is certain that we do not have to let a traveller return to s before we find all the blocs. If the algorithm is successful on its $(l+1)$ th try, only the $(l+1)$ th try cost $(1+\eta)$ and so the total cost will be:

$$(l-1) \cdot 1 + (1+\eta) = l + \eta$$

and the probability will be:

$$\frac{1}{l+1}$$

Hence, its expected cost is at least:

$$\sum_{l=1}^{k+1} (l+\eta) \cdot \frac{1}{l+1} = \frac{K+2}{2} + \eta$$

As we know, the expected optimal offline cost is $1 + \hat{\mu}$, so the competitive ratio is $\frac{K+2}{2}$.

Lemma 4.2: If we have less travellers than blocs ($K < L$), there is no randomized online algorithm with competitive ratio less than:

$$K+2-L + \frac{L(L-1)}{2(K+1)}$$

Proof: Similar with the proof of Lemma 4.1, if the algorithm is successful on its $(l+1)$ th try, the probability will be:

$$\frac{1}{l+1}$$

But to calculate the cost, we have to consider two possibilities: if $l < L$, which means we have found all the blocs without letting any traveller to return to s , the cost will be

$$(l-1) \cdot 1 + 1 + \eta = l + \eta;$$

if $l > L$, it means that we have not found all the blocs after letting all the travellers depart s , so we have to let $(l-L)$ travellers return to s and restart the process. In this case, there will be L travellers with cost 1, $(l-L-1)$ travellers with cost 2 and the last traveller with cost $(2+\eta)$. So the total cost will be:

$$L \cdot 1 + (l-L-1) \cdot 2 + 2 + \eta = 2l - L + \eta.$$

So, its expected cost is at least:

$$\sum_{l=1}^L (l+\eta) \cdot \frac{1}{l+1} + \sum_{l=L+1}^{K+1} (2l-L+\eta) \cdot \frac{1}{l+1} = K+2-L + \frac{L(L-1)}{2(K+1)} + \eta$$

As we know, the expected optimal offline cost is $1 + \hat{\mu}$, so the competitive ratio is $K+2-L + \frac{L(L-1)}{2(K+1)}$.

If we consider the special case $L=1$ which means only one traveller, the competitive ratio becomes:

$$K+2-L + \frac{L(L-1)}{2(K+1)} = K+2-1 = K+1$$

This result corresponds to the randomized case of Bender et al. [2].

References

- [1] Amotz Bar-Noy and Baruch Schieber. The Canadian Traveller Problem. In *Proc. of ACM/SIAM SODA*, pages 261–270, 1991.
- [2] Marco Bender and Stephan Westphal. An optimal randomized online algorithm for the k-Canadian Traveller Problem on node-disjoint paths. *J. Comb. Optim.*, 30(1):87–96, 2015.
- [3] Allan Borodin and Ran El-Yaniv. *Online computation and competitive analysis*. Cambridge University Press, 1998.
- [4] Erik D. Demaine, Yamming Huang, Chung-Shou Liao, and Kunihiro Sadakane. Canadians Should Travel Randomly. *Proc. of ICALP*, pages 380–391, 2014.
- [5] Christos Papadimitriou and Mihalis Yannakakis. Shortest paths without a map. *Theor. Comput. Sci.*, 84(1):127–150, 1991.
- [6] Stephan Westphal. A note on the k-Canadian Traveller Problem. *Inform. Proces. Lett.*, 106(3):87–89, 2008.
- [7] Yinfeng Xu, Maolin Hu, Bing Su, Binhai Zhu, and Zhijun Zhu. The Canadian traveller problem and its competitive analysis. *J. Comb. Optim.*, 18(2):195–205, 2009.