Limit Order Routing in Fragmented Markets

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Problem description

Exchanges (limit order books [2]) are the constituents of a fragmented market. They are owned by private entities with their own interests. Exchange owners aim to maximize the share of total trading happening on their exchange, because they make a profit every time a buyer and a seller agree to transact on it. Owners usually pay traders who us limit orders a rebate to attract more of these orders to their exchange.

Limit orders compete on more liquid exchanges where rebates are high. This translates in longer delays before a seller counter-party is found. On the contrary, exchanges where rebates are lower have less competing limit orders, thus execution is faster in general. [3] and [6] find empirical evidence of this phenomenon. If patient investors are in fact not so patient and a bit concerned about lengthy execution, they might seek a compromise between a faster execution and a higher rebate.

Modelling

The problem is modelled as an infinite horizon stochastic shortest path problem. A trader decides every 1 second interval how many shares to buy on each exchange.

States

The trader initially wants to buy x_0 shares. At step k, the trader has a remaining number of shares to buy (inventory) $x_k > 0$. y_k is the vector of shares remaining on each exchange at the end of the interval.

Controls

1. Shares routed to exchange 1 at step k is $u_k^{(1)}$. Rebate of this exchange is $r_1 = 0.1\%$. Order matching follows a Poisson($\lambda_1 = 2$).
2. Shares routed to exchange 2 at step k is $u_k^{(2)}$. Rebate of this exchange is $r_2 = 0.2\%$. Order matching follows a Poisson($\lambda_2 = 1$).

Costs

$$g_k(x_k, u_k, w_k) = g_k(x_k, y_k) = (x_k - y_k) \cdot (1-r)^T + \underbrace{\beta \times (y_k^{(1)} + y_k^{(2)})}_{\text{Risk aversion}}$$

Transition function

$$P[Y^{(i)} = y^{(i)} | u^{(i)}] = \begin{cases} 0 & y^{(i)} > u^{(i)} \\ \frac{\lambda_i^{u^{(i)} - y^{(i)}} e^{-\lambda_i}}{(u^{(i)} - y^{(i)})!} & u^{(i)} \geq y^{(i)} > 0 \\ 1 - \sum_{z=0}^{u^{(i)}-1} \frac{\lambda_i^z e^{-\lambda_i}}{z!} & y^{(i)} = 0 \end{cases}$$

$$x_{k+1} = f_k(x_k, u_k, w_k) = f_k(x_k, y_k) = y_k^{(1)} + y_k^{(2)}$$

Recurrence

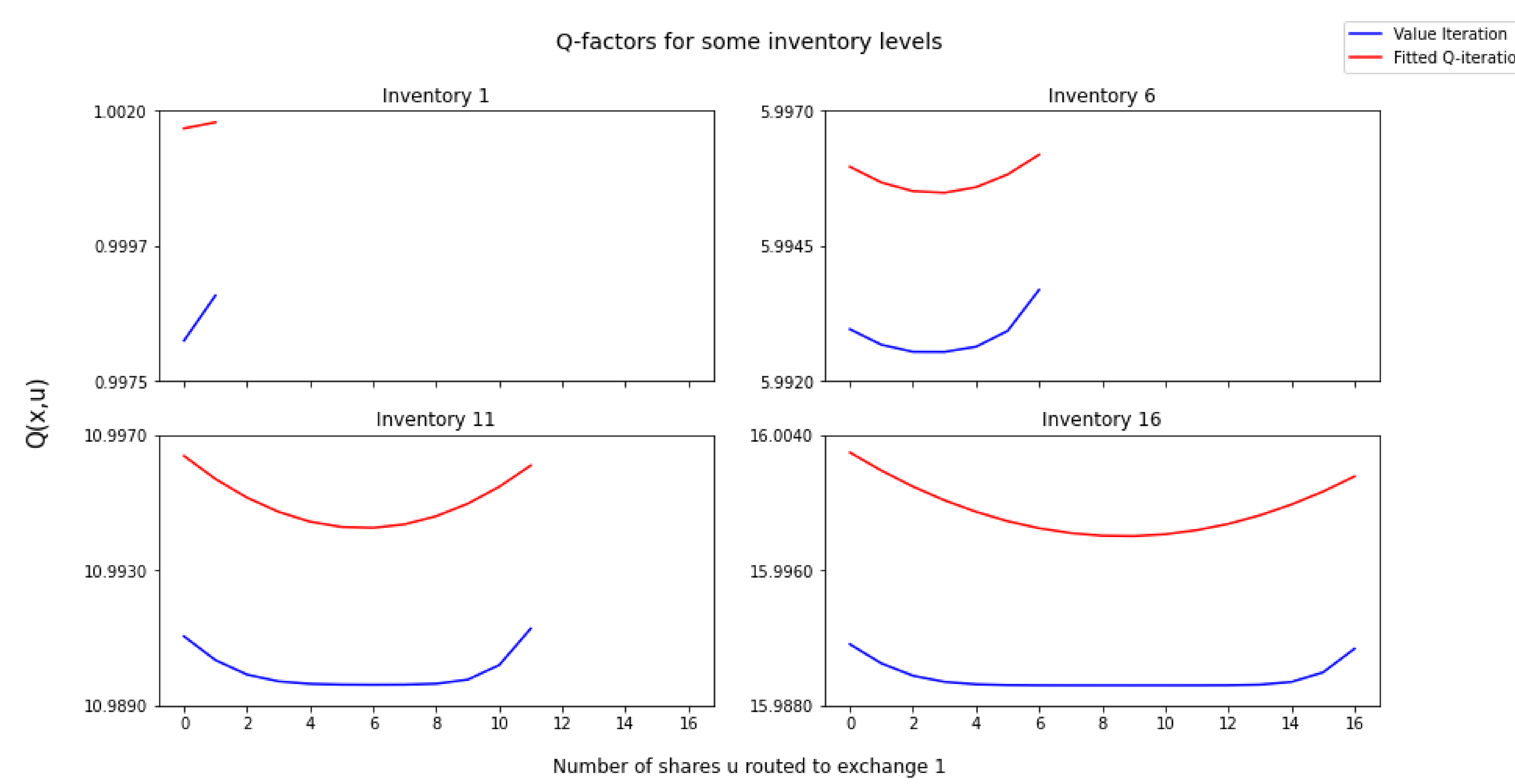
$$J_{k+1}(x) = min_{u \in U(x)} \Big[\sum_{y^{(1)}} \sum_{y^{(2)}} P[Y^{(1)} = y^{(1)} | u^{(1)}] \times P[Y^{(2)} = y^{(2)} | u^{(2)}] \times [g(x, [y^{(1)}, y^{(2)}]) + J_k(f(x, y^{(1)} + y^{(2)}))] \Big]$$

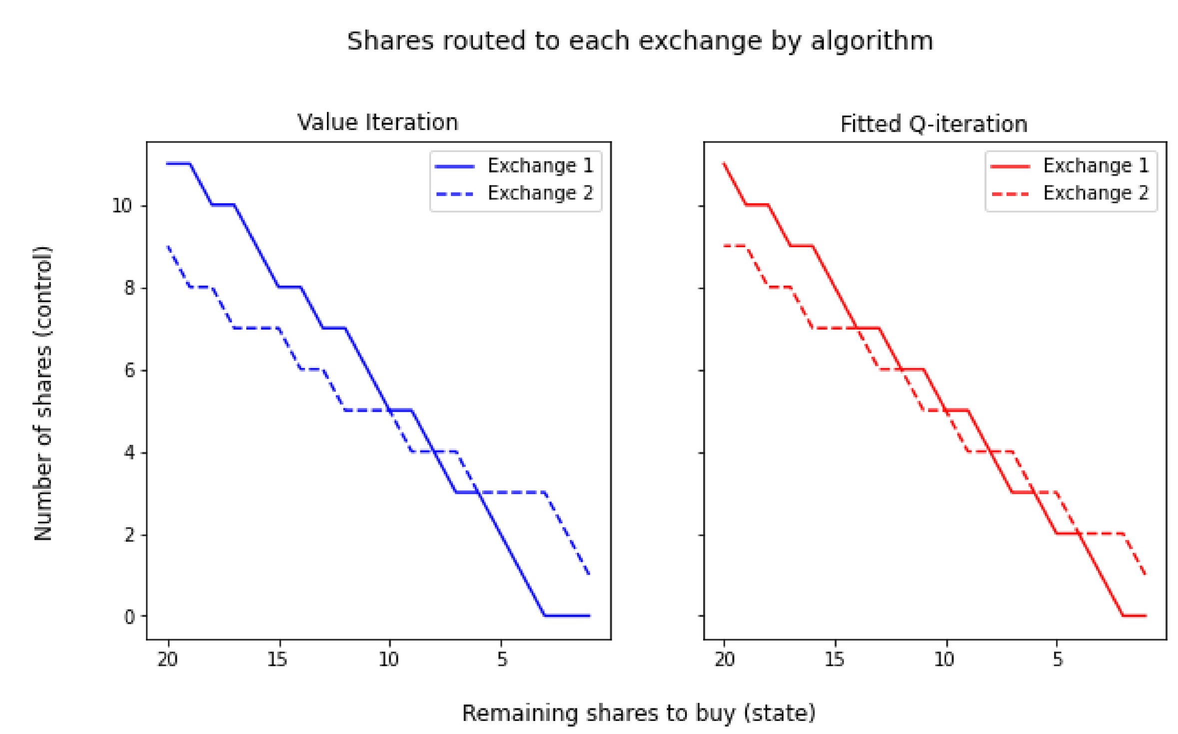
Algorithms

First, the Value Iteration [1] is used to find an exact solution. However, as the number of shares to buy grows, the number of computations necessary for the exact solution explodes and the algorithm is too slow. Hence, the Fitted Q-Iteration [4, 5] algorithm with a second order polynomial regression is used to find a good approximate solution.

Results

Both algorithms are used to find Q-factors and policies for small, medium and large instances of the problem. An initial number of shares to buy x_0 is set to 5, 10 and 20 respectively for these instances and it is the only parameter that changes. The trader has a fixed risk aversion (impatience) coefficient $\beta = .0005$. The same convergence tolerance threshold $\epsilon = 10^{-10}$ is used for the Value Iteration and the Fitted Q-iteration algorithms. Moreover, the number of simulated transitions used for Fitted Q-iteration is $n = 5000$ for all instances. Results for the large instance are presented below.





Conclusion

A model for solving the limit order routing problem in fragmented markets has been proposed. An important assumption made is that the number of shares filled when routing a limit order to an exchange depends on the arrivals of counter-party orders, which are in turn Poisson distributed with mean λ during a one second interval. With this assumption, a distribution for the number of shares filled for a limit order routed to any specific exchange has been derived. The problem has then been formulated as a stochastic shortest path problem with infinite horizon. The setting for which a trader, subject to risk aversion (impatience) initially wants to buy x_0 shares and has access to two different exchanges with different rebates and order filling processes has been considered. An exact solution has been found with the Value Iteration algorithm and an approximate method using Fitted Q-iteration has also been proposed. The approximate method finds solutions close to the optimal ones on all instances considered while being computationally faster for larger state spaces. It is faster because it relies on simulating transitions in the state-control space and estimating the Q-factors using a polynomial regression. The polynomial regression has shown to approximate well the shape of the exact Q-factors. The main characteristic of the solutions found is that the trader should choose exchanges where the rebate is lower but the execution is more likely when he/she has more shares to buy. As the trader's limit orders get filled, he/she can send a bigger proportion of shares to higher rebates and slower execution exchanges.

References

References

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