Physics 344 Assignment 3

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August, 2022

1 Introduction

Building on the work done in the previous assignment, we would like to broaden our scope to investigate not only square lattices, but also cubic and hexagonal lattices, treated identically to what was described in the previous report. We would also like to investigate scaling exponents near the bond-percolation transition line in the phase diagram previously obtained.

2 Square lattice revisited

With newly optimised code (most notably, possible edges and underlying lattices are precomputed and now only loaded once) and understanding of the system at hand, we would like to try a new, more hands-off, approach that can more easily be generalized to systems less amenable to analytic approaches. Since we have both the analytic and numerical results in the previous report, we can make a good comparison to judge the effectiveness of this new approach.

2.1 Methods

We have found previously that there are 4 relevant order parameters related to different percolation phase transitions, which yields a phase diagram with 5 different phases. We generate 10^4 points with $-2 < \mu < 3$, $-15 < \beta < 15$, which is the part of the phase diagram we previously saw captures all the interristing behaviour. We can the calculate the order parameters for each of these 10^4 points (by averaging over 128 graphs that are typical at the given point), which we can then choose a reasonable cutoff (we chose 0.45, which gave stable results), and all percolation order parameters (that are of course bound between 0 and 1) above this threshold we will take as having percolated. We can then group all of our random points acording to which types of percolation has occured there. This can then be compared to our analytic results to see how well it agrees. For an even nicer picture we can feed this classified data into a neural network to build a very basic classifier which would allow us to draw an estimated phase diagram without analytic knowledge, while also not needing to do a full computation for each pixel. The hope is that this will give a good qualitative picture of the phase diagram structure for the different underlying lattices. As a reminder, our four order parameters track

- I Size of largest component (bond percolation)
- II Size of largest connected subgraph with all vertices degree 8 (8-site percolation)
- III Size of largest connected subgraph with all vertices degree 4 (4-site percolation)
- IV Size of largest connected subgraph of the underlying lattice with all vertices degree 0 in generated lattice (0-site percolation).

We could trach n-site percolation for all $0 \leq n \leq 8$, but we have analytically shown only $n = 0, 4, 8$ are relevant for square lattices. We will now introduce the following labeling convention to differentiate different phases, based on which percolations have occured. We will use a length 4 binary string (due to our 4 order parameters), where the i'th character is 1 if the corresponding percolation has happened in that phase and 0 otherwise. This gives us the following labeling for the names given to phases in the previous assignment

- 0001 completely disconnected phase
- 0000 minimally disconnected phase
- 1000 amorphous graph phase

1010 lattice graph phase

1100 degenerate lattice graph phase.

2.2 Results

Using the abovementioned methods on 100×100 periodic square lattices, we find, as shown on the left of Figure 1 that the different phases transition in agreement with previously found analytics (shown in light-blue lines on the plot). We also get a result from the classifier discussed above that gives a good qualitative result, although the quantitative result is still lacking.

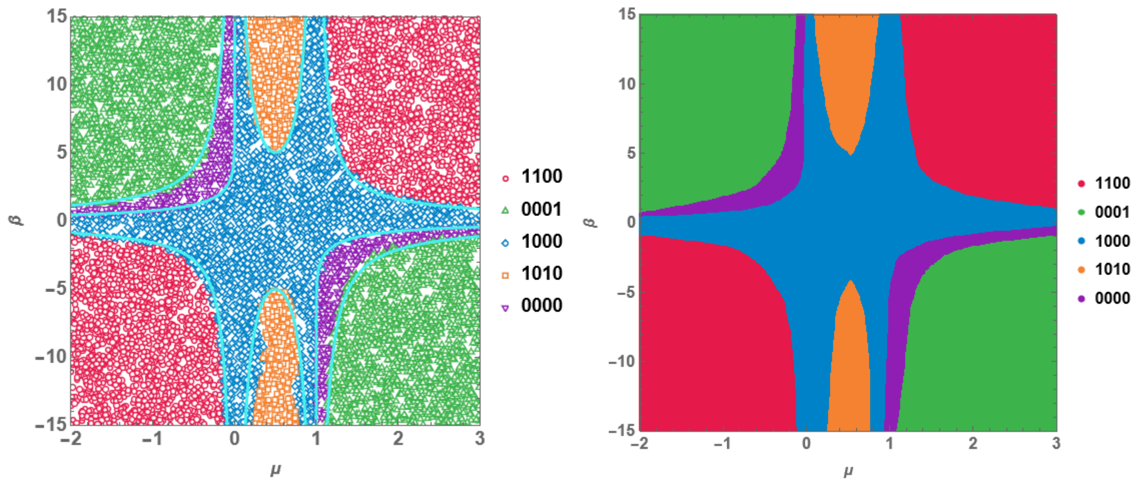

Figure 1: (Left)Random points in the $\beta - \mu$ plane colored according to which order parameters at them were found to be above the threshold. (Right)The result of applying a classifier to our data, giving a qualitative impression of the phases corresponding with previous results.

We also find that overall the transitions of the 4 relevant order parameters are already quite sharp with lattices of size 100×100, as can be seen in Figure 2.

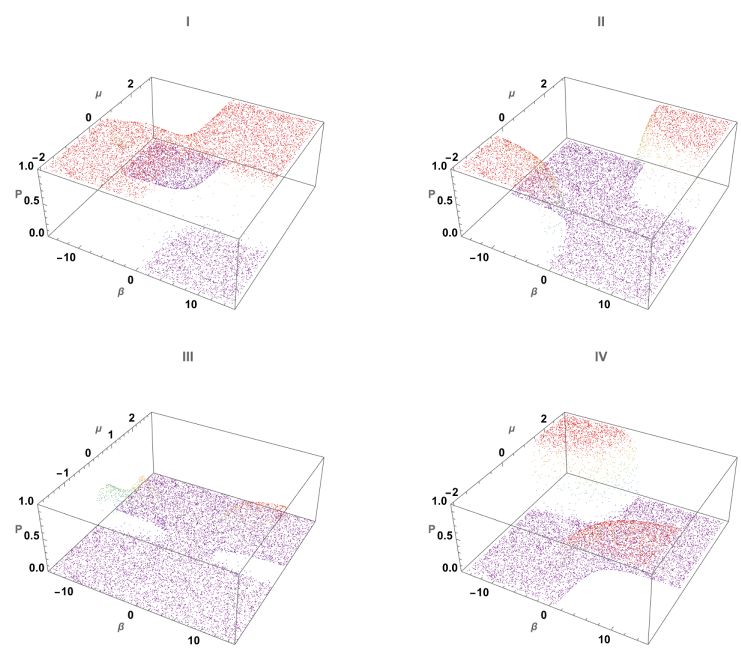

Figure 2: The 4 order parameters at random points in the $\beta - \mu$ plane, colored based on value, with purple being 0 and red 1.

3 Cubic lattice

3.1 Methods

Using the exact same points that was generated for the square lattice we can calculate (using essentially identical methods as above) all 28 possible order parameters, most of which will in all likelihood not go above the threshold, can be estimated. The 28 is made up of bond-percolation together with n-site percolation for $0 \leq n \leq 26$, where 26 is the maximum degree a vertex can have. After a threshold is chosen, the relevant order parameters can then be used to name and identify phases, for which we can estimate a phase diagram.

3.2 Results

Using threshold of 0.45 gives stable results when considering $20 \times 20 \times 20$ periodic cubic lattices. We then find the relevant order parameters are those of bond percolation, 6,8,18,20,26,0-site percolation. We can then again name the phases by using a binary string formed in that order. The resultant classified data can be seen on the left of Figure 3, and the qualitative result of the neural net classifier can be seen on the right. There are many resemblences with the square lattice result, but instead

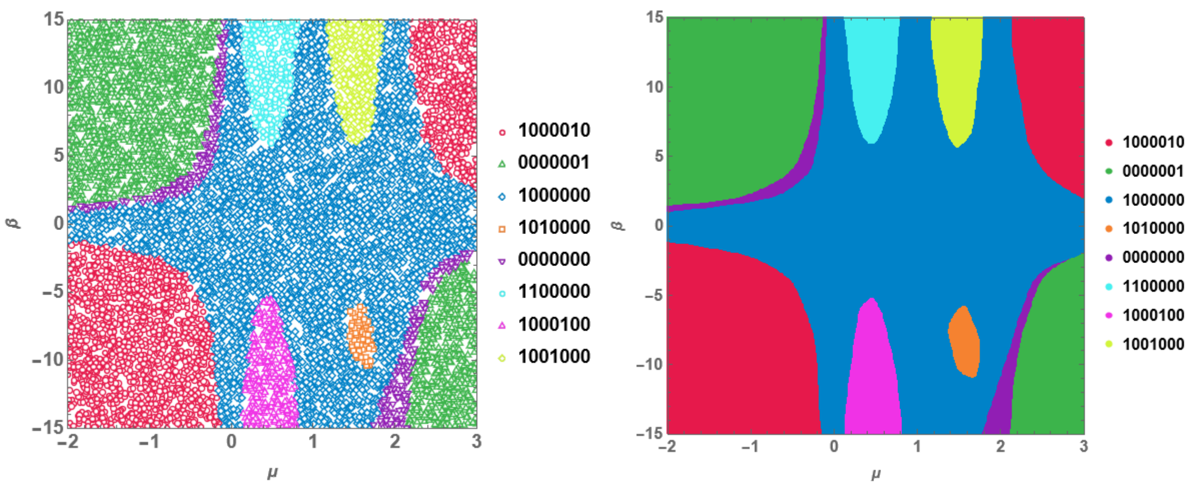

Figure 3: (Left)Random points in the $\beta - \mu$ plane colored according to which order parameters at them were found to be above the threshold. (Right)The result of applying a classifier to our data, giving a qualitative impression of the phases.

of having two 'outcrops' of phase 1010 (corresponding to 4-site percolation), we have 4 different 'outcrops', corresponding to 6,18,8,26-site percolation starting at the topleft and going clockwise. Therefore the opposite 'outcrops' are in a sense graph complements of eachother, since their characteristic degrees add up to the total of 26. The degenerate and disconnected phases are again very similar to what was observed in the square lattice case. Phase 1010000 is expected to take on a similar shape in the diagram in the thermodynamic limit of infinite lattices, since the size it occupies was seen to grow with lattice size.

4 Hexagonal lattice

4.1 Methods

We will now use a (non-periodic) hexagonal lattices (such as what is shown in Figure 4), that are otherwise treated identically to before. In this case there will be 14 possible order parameters, since the maximum vertex degree is 12. It will again be neccessary to see which of these parameters will be relevant.

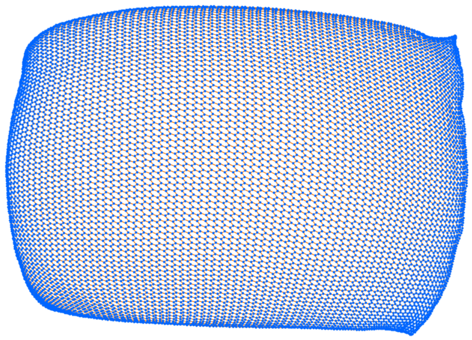

Figure 4: A 30×30 hexagonal lattice

4.2 Results

Again using a threshold of 0.45 gives stable results for the 70×70 hexagonal lattices we considered.We then find the relevant order parameters are those of bond percolation, 3,9,12,0-site percolation. We can then again name the phases by using a binary string formed in that order. The resultant classified data can be seen on the left of Figure 5, and the qualitative result of the neural net classifier can be seen on the right. We again see many resemblences with previous results, with 3-site and

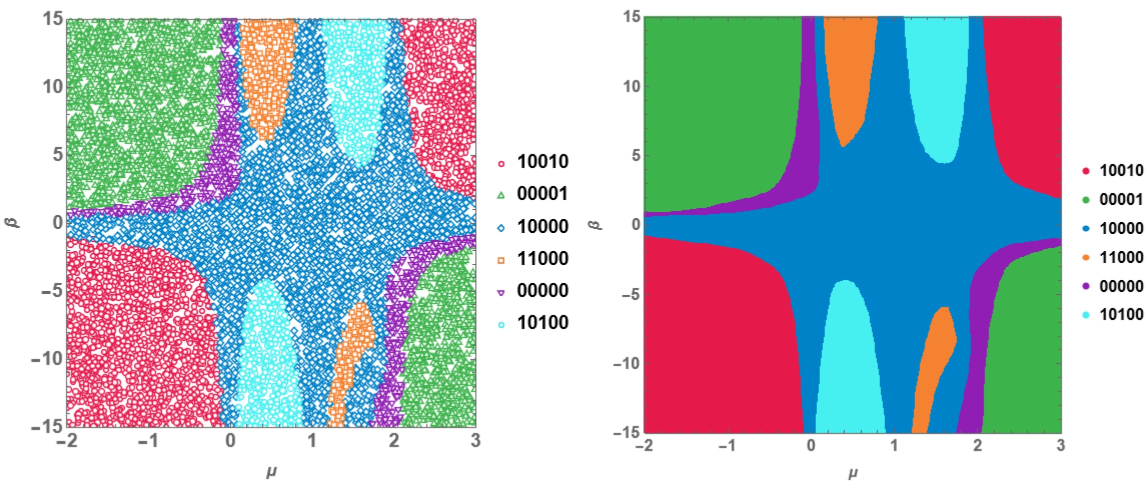

Figure 5: (Left)Random points in the $\beta - \mu$ plane colored according to which order parameters at them were found to be above the threshold. (Right)The result of applying a classifier to our data, giving a qualitative impression of the phases.

9-site percolation making up the orange and light blue 'outcrops' respectively. We see again that 'outcrops' that are across from eachother again have characteristic vertex degrees that sum to the total. The right bottom outcrop seems to again be falling victim to the finite systems we are considering.

5 Square lattice bond percolation scaling exponents

We are interristed in the critical exponents corresponding to the order parameters. We will limit our scope here to investigate how the order parameter of percolation I scales near transition as purely a function of either μ or β. We will specifically be interristed in if the apparent symmetry between the percolation I transition in quadrants 2 and 4 translates to symmetry of scaling exponents.

5.1 Methods

We will investigate the percolation I order parameter at 4 locations. At $\beta = \pm 20$ fixed and $\mu = \pm 5$ fixed. We will let the other parameter vary close to the theoretical transition, and then fit a powerlaw to the data 'above' the theoretical transition (using this theoretical transition value in our powerlaw fit). The theoretical transition lines are given by solving the equation $\frac{1}{2}\left(2 - \frac{1}{1+e^{\beta\mu}} - \frac{1}{1+e^{\beta(\mu-1)}} = 0.250368\right)$, where the left hand side is the edge occupancy (obtained from a straight forward but tedious probability calculation), and the right hand side is the percolation threshold of a 'Union Jack lattice', which is to first approximation accurate enough for our purposes. When we have constant β, we will therefore be fitting a powerlaw of the form $a|\mu - \mu_c|^\alpha$ on the side of the transition that has larger order parameter. Note that there is no intention for α to correspond to conventions of naming of scaling exponents. Similarly we will have at constant μ a powerlaw $a|\beta - \beta_c|^\gamma$. For both we will need to obtain a and α,γ respectively, although it is only the exponent that we are interrished in.

5.2 Results

We find for $\beta = 20$ a scaling exponent of $\alpha \approx 0.102$ and for $\beta = -20$ we find $\alpha \approx 0.098$. When we have $\mu = -5$ we find a scaling exponent of $\gamma \approx 0.$ and for $\mu = 5$ we have $\gamma \approx 0.107$. This seems quite close and suggests that the scaling exponent is independent of direction in the $\mu - \beta$ plane one approaches the transition from. Bigger lattices and more data will be neccessary to confirm such a claim.

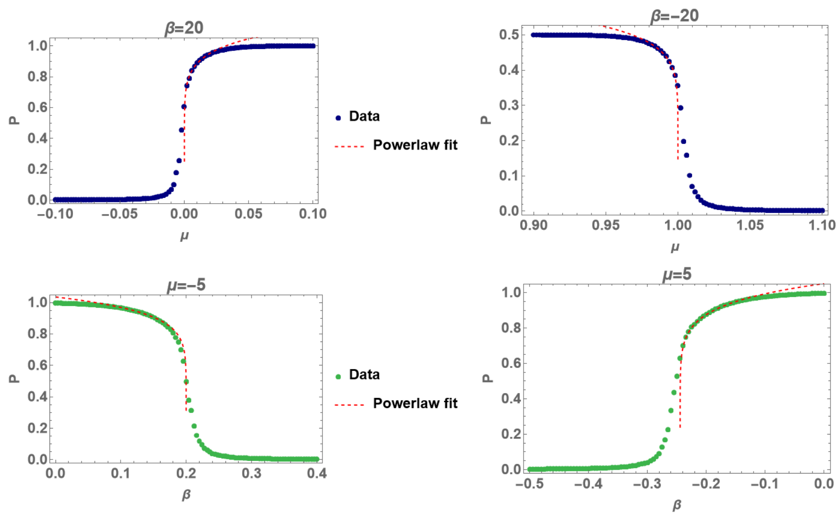

Figure 6: Fits of powerlaw scaling.

6 Conclusions

We have found that the phase diagrams of square, cubic and hexagonal lattices share some non-trivial qualitative features, prompting the question of which underlying graphs would support such structure. It seems like a safe conjecture that vertex transitive lattices in euclidean spaces would all support such structure. There is also then the question of where else we can expect to find similar behaviour. Different models of random graphs come to mind as interristing avenues to explore. By the nature of critical exponents, it seems like a natural conjecture that the exponent we found is universal, not caring about what the underlying graph is, where in the phase diagram the transition is or even which type of percolation transition is considered. This is a very wide conjecture with very little numerical data to back it up, making it a perfect continuation of this assignment. Finding a precise value of this (conjectured to be universal) exponent is then also an important question.