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Revisiting da Costa logic

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A R T I C L E I N F O

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A B S T R A C T

In [25] Priest developed the da Costa logic (**daC**); this is a paraconsistent logic which is also a co-intuitionistic logic that contains the logic $\mathbf{C}_\omega$. Due to its interesting properties it has been studied by Castiglioni, Ertola and Ferguson, and some remarkable results about it and its extensions are shown in [8,11]. In the present article we continue the study of **daC**, we prove that a restricted Hilbert system for **daC**, named *DC*, satisfies certain properties that help us show that this logic is not a maximal paraconsistent system. We also study an extension of **daC** called $\mathbf{PH_1}$ and we give different characterizations of it. Finally we compare **daC** and $\mathbf{PH_1}$ with several paraconsistent logics.

1. Introduction

Paraconsistent logic is a well-known topic in the scientific community, and there is a lot of work done in this area. Today, paraconsistent logic touches various topics, such as ontology, the philosophy of science, applied science and technology. Thus, any advance in this area will be welcomed [3].

Briefly speaking, following Jean-Yves Béziau [5], a logic is paraconsistent if it has a negation ($\neg$) which is paraconsistent in the sense that the relation $\alpha, \neg\alpha \vdash \beta$ does not always hold for arbitrary formulas α and β, and at the same time it has strong properties that justify calling it a negation. Nevertheless, there is no paraconsistent logic that is unanimously recognized as a "good one" [5], and there are different proposals on what a paraconsistent logic should be [7].

Along this article we will focus on two paraconsistent logics, one is da Costa logic (**daC**) [25], also known as Priest–da Costa logic, and the other one is an extension of **daC** known as $\mathbf{PH_1}$ [11]. We will discuss

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some relevant properties that can be considered as "desirable" in such logics, (particularly when our aim is to apply them to intelligent agents) and the relation of **daC** and $\mathbf{PH_1}$ with respect to some paraconsistent logics. The present work is mainly a theoretical contribution to the area of paraconsistent logic and is also related to modal and constructive logics. To some extent, our work is related to the area of Non-Monotonic Reasoning (NMR): several authors have applied different logics in the modeling of non-monotonic reasoning by means of completions [10,14]. In fact, in [18] an interesting approach for Knowledge Representation (KR) was proposed and developed in [17,24]. This approach can be supported by any paraconsistent logic stronger than or equal to $\mathbf{C}_\omega$ [22,23], which is the case of **daC** and $\mathbf{PH_1}$.

The structure of this article is as follows. In Section 2, we present the necessary background in order to simplify the reading of the article, and a brief introduction to several of the logics that are relevant for the results of later sections. In Section 3, we show that **daC** does not satisfy a number of intuitive properties concerning the behavior of negation, it is not a maximal paraconsistent logic in the strong sense, and we compare it with some paraconsistent logics.

In Section 4 we obtain different characterizations of $\mathbf{PH_1}$ and we compare it with some paraconsistent logics as well.

In Section 5, we present our conclusions, some conjectures and ideas for future work.

2. Background

We first introduce a few basic definitions and clarify concepts such as maximality. Then we define some properties of a negation and, finally, we give a brief review of some paraconsistent logics. We assume that the reader has some familiarity with basic results from logic such as chapter one in [15].

2.1. Logic systems

We consider a formal (propositional) language $\mathcal{L}$ built from: an enumerable set of atoms (denoted by $p, q, r, \ldots$) and the set of connectives $\{\wedge, \vee, \rightarrow, \neg\}$.[1] *Formulas* are constructed as usual and will be denoted as lowercase Greek letters. *Theories* are sets of formulas and will be denoted as uppercase Greek letters.

A *logic* is simply a set of formulas that is closed under Modus Ponens (*MP*) and substitution. All the logics involved in this article are propositional logics and all of them have the same language, $\mathcal{L}$.[2] The elements of a logic are called *theorems*. Let X be a logic, the notation $\vdash_X \alpha$ is used to state that the formula α is a theorem of X (i.e. $\alpha \in X$).[3] Thought many different approaches have been used to define logics, we will use two of them, axiomatic systems (also known as Hilbert style proof systems) and many-valued systems.

We say that a logic X is *weaker than or equal to* a logic Y if $X \subseteq Y$. Sometimes we refer to this as Y *extends* X. Similarly, we say that X is *stronger than or equal to* Y if $Y \subseteq X$.

2.2. Maximality properties

Maximality is a desirable property of paraconsistent logics. However, the notion of maximality in the field of paraconsistent logic is far from being unique. One can define maximality of a logic with respect to some other logic, as in [6]. But in the literature one can find stronger notions of maximality. For example, a notion of absolute maximality, in the sense that it is not defined with respect to some other given logic, is the following:

[1] We use $\alpha \leftrightarrow \beta$ to abbreviate $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

[2] In the following sections we will mention some connectives and constants that do not appear in $\mathcal{L}$. All such cases will be abbreviations of formulas in the original language.

[3] We drop the subscript X in $\vdash_X$ when the given logic is understood from the context.

Table 1
Definition of some many-valued paraconsistent systems.

Logic	Domain	Designated values	Connectives	References
$\mathbf{P^3}$	$\{0,1,2\}$	$1,2$	Table A.3	[6]
$\mathbf{G}'_3$	$\{0,1,2\}$	2	Table A.4	[23,19]
$\mathbf{CG}'_3$	$\{0,1,2\}$	$1,2$	Table A.4	[20]
P-FOUR	$\{0,1,2,3\}$	3	Table A.5	[21,23]
$\mathbf{M4_p}$	$\{0,1,2,3\}$	$2,3$	Table A.6	[4]

Definition 2.1. A paraconsistent logic L_2 is *maximally paraconsistent in the strong sense* [2], if every logic L_1 in the language of L_2 that properly extends L_2 (i.e. $\vdash_{L_2}\subset\vdash_{L_1}$) is no longer paraconsistent.

In what follows, we will be specific when we refer to the term *maximal paraconsistent logic*.

2.3. Properties related to negation

In the study of paraconsistent logics the negation plays the main role. The negation connective's behavior in this case is different from the classical or the intuitionistic one. Then any principle in each one of these logics must not be taken as a theorem in paraconsistent logics by default. Some of the desirable properties of a negation operator are listed in the definition below; some others will be introduced in the article when necessary.

Definition 2.2. Given a logic X, we say that it satisfies the principle on the left column, if the corresponding condition on the right column holds.

DN (*Double negation*)	$\vdash_X \alpha \leftrightarrow \neg\neg\alpha$
NI (*Double negation of identity law*)	$\vdash_X \neg\neg(\alpha \to \alpha)$
DM1 (*De Morgan's law 1*)	$\vdash_X (\neg\alpha \wedge \neg\beta) \to \neg(\alpha \vee \beta)$
DM2 (*De Morgan's law 2*)	$\vdash_X \neg(\alpha \vee \beta) \to (\neg\alpha \wedge \neg\beta)$
DM3 (*De Morgan's law 3*)	$\vdash_X (\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$
DM4 (*De Morgan's law 4*)	$\vdash_X \neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$
DNC (*Distribution of* $\neg\neg$ *over* $\wedge$)	$\vdash_X \neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$
DNI (*Distribution of* $\neg\neg$ *over* $\to$)	$\vdash_X \neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$
$\neg\neg$-*Necessitation*	If $\vdash_X \alpha$ then $\vdash_X \neg\neg\alpha$
WE (*Weak explosion*)	$\vdash_X \neg\alpha \to (\neg\neg\alpha \to \beta)$
WC (*Weak contrapositive*)	$\vdash_X (\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$

2.4. Some paraconsistent logics

We now move to review some well known paraconsistent logics. We start with the definition of several many-valued systems in Table 1. The truth tables of these logics (Table A.3–Table A.6) can be found in Appendix A. Then we continue with Hilbert systems for $\mathbf{C}_\omega$ and $\mathbb{Z}$.

2.4.1. The logic $\mathbf{C}_\omega$

The logic $\mathbf{C}_\omega$ is defined by the following set of axiom schemata and the rule of inference *MP* [9]:

Pos1	$\alpha \to (\beta \to \alpha)$	**Pos2**	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
Pos3	$(\alpha \wedge \beta) \to \alpha$	**Pos4**	$(\alpha \wedge \beta) \to \beta$
Pos5	$\alpha \to (\beta \to (\alpha \wedge \beta))$	**Pos6**	$\alpha \to (\alpha \vee \beta)$
Pos7	$\beta \to (\alpha \vee \beta)$	**Pos8**	$(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
$\mathbf{C}_\omega\mathbf{1}$	$\alpha \vee \neg\alpha$	$\mathbf{C}_\omega\mathbf{2}$	$\neg\neg\alpha \to \alpha$

Note that **Pos1**–**Pos8** and *MP* defines *Positive Intuitionistic Logic* (**Pos**).

2.4.2. The logic $\mathbb{Z}$

Béziau offers an axiomatization for $\mathbb{Z}$, the restricted[4] Hilbert system *HZ* (see [5], def 3.1). The system *HZ* contains all of the axioms of $\mathbf{C}_\omega$ and *MP* plus the following schemata and rule:

Peirce $((\alpha \to \beta) \to \alpha) \to \alpha$

AZ2 $((\alpha \wedge \neg\beta) \wedge \neg(\alpha \wedge \neg\beta)) \to (\alpha \wedge \neg\alpha)$

AZ3 $\neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$

RZ: $\dfrac{\alpha \to \beta}{\neg(\alpha \wedge \neg\beta)}$

Many interesting facts about $\mathbb{Z}$ and $\mathbb{S}5$ can be found in [5], e.g. *HZ* can be interpreted as an axiomatization of the modal logic $\mathbb{S}5$ that uses a negation operator as a primitive connective rather than a necessity or possibility operator.

More recently, Omori and Waragai in [16] gave an alternative axiomatization of $\mathbb{Z}$ using *MP* and *RZ* as inference rules and the axioms **Pos1**–**Pos8**, **Peirce**, $\mathbf{C}_\omega\mathbf{2}$, plus the schema:

AZ2′ $\neg(\alpha \wedge \neg\beta) \to (\neg\beta \to \neg\alpha)$

A deeper study of the properties of the several logics we have cited in this section and many others can be found in [20].

3. Da Costa logic

The attempt to relate Intuitionistic Logic and paraconsistent logics is not new, as Béziau notes it in [3]. According to him, intuitionistic logic appears as a dual of a particular paraconsistent logic, but in fact, there is more than one way to dualize Intuitionistic Logic. All depends on how one sees it, or how one defines it. For example using algebraic, relational, axiomatic or sequent approaches. In each case, one gets different results, and the result also depends on the connectives dualized; some of these results can be found in [13,26,1,27].

If we accept Kripke frames as the standard semantics of Intuitionisitic Logic (**Int**), then what logic do we obtain when we dualize the negation? According to Graham Priest [25], one of the reasons that pushed da Costa to build $\mathbf{C}_\omega$ was to find a dual for intuitionistic negation. To support his assertion Priest created a system called da Costa logic, based on the Kripke semantics for **Int** and dualizing the conditions for intuitionistic negation, he shows the close relationship between his system and $\mathbf{C}_\omega$ proving that it contains $\mathbf{C}_\omega$ as well as **Int**, that the $\{\vee, \wedge, \neg\}$-fragment of **daC** coincides with the classical one, it is finitely trivializable and finally, it is algebrizable by da Costa algebras.

Three years after the introduction of **daC**, Castiglioni and Ertola [8] proved that the Deduction Theorem holds true, as well as the finite model property and related properties such as the finite satisfiability property. Since **daC** is finitely axiomatizable and has the finite model property, then it is decidable.

Here we present a Hilbert style axiomatization of da Costa logic, some results about the behavior of the negation connective of this logic and a brief comparison with the logics presented in Section 2.4.

[4] In a Hilbert style system $\Gamma \vdash_X \varphi$ denotes that φ can be derived from the axioms of X and the formulas in Γ by a sequence of applications of the inference rules. In a *restricted Hilbert style proof system* $\Gamma \vdash_X \varphi$ denotes that $\vdash_X \alpha_1 \to (\ldots(\alpha_n \to \varphi)\ldots)$ where $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$ and $n \in \mathbb{N}$. It is important to note that both notions agree in case X is a finitary logic with deduction theorem.

3.1. Kripke semantics for da Costa logic

This subsection is taken from [25]. A Kripke frame is a structure $\mathcal{F} = \langle W, R\rangle$ where W is a set of worlds, R is a binary relation on worlds which is reflexive and transitive. A Kripke model for da Costa logic is a structure $\mathcal{W} = \langle W, R, v\rangle$, where $\langle W, R\rangle$ is a Kripke frame and v assigns a truth value, 1 or 0, to every atomic formula p, at each world, subject to the following constraint: for each atomic formula p: if wRk and $v_w(p) = 1$, then $v_k(p) = 1$ (Heredity Constraint or monotonicity). The truth conditions for the logical operators $\rightarrow, \vee, \wedge$ correspond to Intuitionistic Logic. However the negation is dualized (with respect to **Int**), namely we have that:

$$v_w(\neg\alpha) = 1 \text{ if and only if for some } t \text{ such that } tRw,\ v_t(\alpha) = 0.$$

It is shown in [25] that monotonicity generalizes to all formulas. The notions of soundness and completeness respect to frames or classes of frames are defined as usual. An easy way of representing Kripke models is to depict them as graphs where worlds in W are the nodes, the relation R corresponds to the graph's edges and v is specified by placing at the left of a node the atomic formulas that are true and at the right side those that are false.

3.2. Hilbert style axiomatization of **daC**

In [25], Priest performs a deep study of **daC** and he presents tableaux and natural deduction systems for it and in addition to the Kripke semantics he also gives algebraic and topological semantics for **daC**. In [8], one can find a Frege–Hilbert style presentation of **daC** used to prove that the logic has the finite model property and related properties such as the finite satisfiability property. Here we present the same system but using a restricted Hilbert style axiomatization in order to make explicit the restriction over the rule corresponding to the disjunctive syllogism.

Definition 3.1. The restricted Hilbert system DC is defined by including the axiom schemata of **Pos** plus $\mathbf{C}_\omega\mathbf{1}$. We have two rules of inference MP and

$$MG : \frac{\alpha \vee \beta}{\neg\alpha \rightarrow \beta}.$$

Theorem 3.2. *The set of theorems of the system DC coincides with the theorems of* **daC**.

Proof. With respect to soundness, it is easy to verify that the rule of inference MG is logically valid in the Kripke semantics of **daC**. Suppose that for every world w, it holds $v_w(\alpha \vee \beta) = 1$. Let t be any particular world. We need to show that $v_t(\neg\alpha\rightarrow\beta) = 1$. Let s be any world such that $tRs, v_s(\neg\alpha) = 1$. We have to prove that $v_s(\beta) = 1$. There exists a world k such that $kRs, v_k(\alpha) = 0$. Since $v_k(\alpha \vee \beta) = 1$ and $v_k(\alpha) = 0$, then $v_k(\beta) = 1$. By monotonicity $v_s(\beta) = 1$, as desired. Regarding completeness, one can check that the same proof provided by Priest in section 4 of [25] holds also here. In fact, only the first paragraph of Lemma 2 in [25] requires the use of MG. □

3.3. Properties of da Costa logic

We start with a basic result of a given logic that satisfies certain properties. After that we present some results about $\mathbf{C}_\omega$ and we conclude the present section with a long list of interesting results about **daC**.

Theorem 3.3. *Let X be any logic that extends* **Pos** *and satisfies the substitution property.*[5] *Assume in addition that there exists a formula β such that $\vdash_X \beta$ and $\vdash_X \neg\neg\beta$. Then for every formula α, $\vdash_X \alpha$ implies $\vdash_X \neg\neg\alpha$.*

Proof. Suppose $\vdash_X \alpha$ and $\vdash_X \beta$. Then using **Pos** we have $\vdash_X \alpha \leftrightarrow \beta$. Hence by the substitution property $\vdash_X \neg\neg\alpha \leftrightarrow \neg\neg\beta$. So, $\vdash_X \neg\neg\beta \to \neg\neg\alpha$. Since $\vdash_X \neg\neg\beta$, by *MP* we obtain $\vdash_X \neg\neg\alpha$ as desired. □

Theorem 3.4. *Let X be any logic that extends* **Pos** *and satisfies* $\mathbf{C}_\omega\mathbf{1}$ *and* **AZ2**.
Then it also satisfies **AZ2′**.

Proof. In **Pos** the following theorems and rule hold:

Identity law	**IL**	$\alpha \to \alpha$
Import and export premises	**IEP**	$(\alpha \to (\beta \to \gamma)) \leftrightarrow ((\alpha \wedge \beta) \to \gamma)$
Permutation law	**PL**	$(\alpha \to (\beta \to \gamma)) \leftrightarrow (\beta \to (\alpha \to \gamma))$
Commutativity of $\wedge$	**COM**	$(\alpha \wedge \beta) \leftrightarrow (\beta \wedge \alpha)$
Contraction of $\wedge$	**CON**	$(\alpha \to (\alpha \wedge \beta)) \to (\alpha \to \beta)$
Hypothetical syllogism	*HS*:	$\dfrac{\alpha \to \beta, \beta \to \gamma}{\alpha \to \gamma}$

Using the above properties of **Pos** and the hypotheses we have that:

By **IL** we have $\vdash (\alpha \to \neg\alpha) \to (\alpha \to \neg\alpha)$. By **PL** this is equivalent to $\vdash \alpha \to ((\alpha \to \neg\alpha) \to \neg\alpha)$. We also have the following instance of **Pos1** $\vdash \neg\alpha \to ((\alpha \to \neg\alpha) \to \neg\alpha)$. Applying *MP* twice to the appropriate instance of **Pos8** and the last two steps we get $\vdash (\alpha \vee \neg\alpha) \to ((\alpha \to \neg\alpha) \to \neg\alpha)$. Now by using $\mathbf{C}_\omega\mathbf{1}$ and *MP* $\vdash (\alpha \to \neg\alpha) \to \neg\alpha$. By **CON** it holds that $\vdash (\alpha \to (\alpha \wedge \neg\alpha)) \to (\alpha \to \neg\alpha)$. Now we apply *HS* to the previous two steps and obtain

$$\vdash (\alpha \to (\alpha \wedge \neg\alpha)) \to \neg\alpha. \tag{1}$$

Since **AZ2** also holds, then $\vdash ((\alpha \wedge \neg\beta) \wedge \neg(\alpha \wedge \neg\beta)) \to (\alpha \wedge \neg\alpha)$. By Commutativity and associativity of $\wedge$ from the previous step we get $\vdash ((\neg\beta \wedge \neg(\alpha \wedge \neg\beta)) \wedge \alpha) \to (\alpha \wedge \neg\alpha)$. Equivalently by **IEP** $\vdash (\neg\beta \wedge \neg(\alpha \wedge \neg\beta)) \to (\alpha \to (\alpha \wedge \neg\alpha))$. By *HS* applied to the appropriate instance of **COM** and the previous step

$$\vdash (\neg(\alpha \wedge \neg\beta) \wedge \neg\beta) \to (\alpha \to (\alpha \wedge \neg\alpha)). \tag{2}$$

By *HS* applied to (1) and (2) we obtain $\vdash (\neg(\alpha \wedge \neg\beta) \wedge \neg\beta) \to \neg\alpha$. Equivalently by **IEP** $\vdash \neg(\alpha \wedge \neg\beta) \to (\neg\beta \to \neg\alpha)$. □

Theorem 3.5. *Let X be any logic that extends* $\mathbf{C}_\omega$ *and satisfies De Morgan's laws* **DM3** *and* **DM4**, *i.e.* $\vdash \neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$. *Then* $X + \mathbf{AZ2'} = X + \mathbf{WE}$.

Proof. Assume that $\vdash$ corresponds to $X + \mathbf{AZ2'}$. By $\mathbf{C}_\omega\mathbf{1}$ and **DM3** we have that $\vdash \neg\neg\alpha \vee \neg\neg\neg\alpha$ and $\vdash (\neg\neg\alpha \vee \neg\neg\neg\alpha) \to \neg(\neg\alpha \wedge \neg\neg\alpha)$. Applying *MP* we obtain $\vdash \neg(\neg\alpha \wedge \neg\neg\alpha)$ and equivalently $\vdash \neg(\neg\neg\alpha \wedge \neg\alpha)$. By *MP* applied to this last result and the **AZ2′** instance $\vdash \neg(\neg\neg\alpha \wedge \neg\alpha) \to (\neg\alpha \to \neg\neg\neg\alpha)$ we obtain:

$$\vdash \neg\alpha \to \neg\neg\neg\alpha. \tag{3}$$

Using properties of **Pos**, the instance **Pos7** given by $\vdash \neg\alpha \to (\neg\neg\beta \vee \neg\alpha)$ and (3) we obtain that $\vdash \neg\alpha \to (\neg\neg\beta \vee \neg\neg\neg\alpha)$. By **DM3** we have $\vdash (\neg\neg\beta \vee \neg\neg\neg\alpha) \to \neg(\neg\beta \wedge \neg\neg\alpha)$. By *HS* applied to the last two formulas,

[5] Let X be a logic, we say X satisfies the substitution property if $\vdash_X \alpha \leftrightarrow \beta$ implies $\vdash_X \varphi[\alpha/b] \leftrightarrow \varphi[\beta/b]$ for any formulas α, β and φ and any atom b that appears in φ.

we get $\vdash \neg\alpha \to \neg(\neg\beta \wedge \neg\neg\alpha)$. Take the instance of **AZ2′** given by $\vdash \neg(\neg\beta \wedge \neg\neg\alpha) \to (\neg\neg\alpha \to \neg\neg\beta)$. Using *HS* we get $\vdash \neg\alpha \to (\neg\neg\alpha \to \neg\neg\beta)$. By **IEP** from the last result we get:

$$\vdash (\neg\alpha \wedge \neg\neg\alpha) \to \neg\neg\beta. \tag{4}$$

Now by *HS* applied to the $\mathbf{C}_\omega\mathbf{2}$ instance $\vdash \neg\neg\beta \to \beta$ and (4) we get $\vdash (\neg\alpha \wedge \neg\neg\alpha) \to \beta$ (*Weak explosion*), as desired.

Assume now that $\vdash$ corresponds to $X+\mathbf{WE}$. Using the instance of **Pos1** $\vdash \neg\alpha \to (\neg\beta \to \neg\alpha)$, the equivalent version of *Weak explosion* $\vdash \neg\neg\beta \to (\neg\beta \to \neg\alpha)$, and the appropriate instance of **Pos8**, by *MP* applied twice, we obtain $\vdash (\neg\alpha \vee \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$. Finally, we also have the instance of **DM4** $\vdash \neg(\alpha \wedge \neg\beta) \to (\neg\alpha \vee \neg\neg\beta)$, applying *HS* with the previous step we get $\vdash \neg(\alpha \wedge \neg\beta) \to (\neg\beta \to \neg\alpha)$, as desired. □

The following results are simple facts about $\mathbf{C}_\omega$.

Theorem 3.6. *In $\mathbf{C}_\omega$, if $\vdash \alpha \to \beta$ then $\vdash \neg\alpha \vee \beta$.*

Proof. Straightforward from $\mathbf{C}_\omega\mathbf{1}$ and some properties of **Pos**. □

Corollary 3.7. *In $\mathbf{C}_\omega$, it holds that $\vdash \alpha \vee \neg\neg\neg\alpha$.*

Proof. Straightforward from $\mathbf{C}_\omega\mathbf{2}$, Theorem 3.6 and some properties of **Pos**. □

Now we move to **daC** and we present several results about it. In the following theorem we summarize some results already known about **daC** from [25].

Theorem 3.8. *In the system* **daC** *the following holds:*

1. $\nvdash \alpha \to \neg\neg\alpha$.
2. $\vdash \neg\neg\alpha \to \alpha$.
3. $\nvdash (\alpha \wedge \neg\alpha) \to \beta$.
4. $\vdash \beta \to (\alpha \vee \neg\alpha)$.
5. $\vdash \neg(\alpha \wedge \neg\alpha)$.
6. $\vdash \neg(\alpha \vee \neg\alpha) \to \beta$.
7. *Weak substitution property:*
   if $\vdash \alpha \leftrightarrow \beta$ then $\vdash \neg\beta \leftrightarrow \neg\alpha$.

Theorem 3.9. *In* **daC**, *it holds that $\vdash \neg\alpha \leftrightarrow \neg\neg\neg\alpha$.*

Proof. One implication is direct from item 2 of Theorem 3.8 and the other follows by Corollary 3.7 and *MG*. □

Corollary 3.10. *In* **daC** *there exists a formula β such that $\vdash \beta$ and $\vdash \neg\neg\beta$.*

Proof. Let β be the formula $\neg(\alpha \wedge \neg\alpha)$. By item 5 in Theorem 3.8 we have that $\vdash \neg(\alpha \wedge \neg\alpha)$ i.e. $\vdash \beta$. Using this and Theorem 3.9 we have that $\vdash \neg\neg\neg(\alpha \wedge \neg\alpha)$ i.e. $\vdash \neg\neg\beta$. □

Theorem 3.11. *System DC has the substitution property.*

Proof. The proof is by induction over the length of the formula. The base case and the induction step for conjunction, disjunction and implication is pretty straightforward. Finally, the induction step for negation is direct from the weak substitution property (item 7 in Theorem 3.8). □

Another important property of **daC** is that it is finitely trivializable[6] [25] even when $\mathbf{C}_\omega$ is not. In da Costa logic, it is possible to define the logical constants $\top$ and $\bot$. The first one as $\alpha \vee \neg\alpha$ and the second one as $\neg(\alpha \vee \neg\alpha)$, for any α we choose.

The presence of a bottom particle and all axioms of **Pos** produce full Intuitionistic Logic. In particular, we may define the intuitionistic negation of α in **daC** as $\sim \alpha := \alpha \to \bot$. It is not difficult to check that $\sim \alpha \to \neg\alpha$ is a theorem in **daC**, but not the other way around. Then, intuitionistic negation ($\sim$) is stronger than the native negation of da Costa logic ($\neg$). Hence, da Costa logic is an extension of **Int**, in fact we can see **daC** as a conservative expansion of **Int** by adding $\neg$ as a new negation operator. However, **Int** does not contain da Costa logic.

Another important consequence of the definability of the intuitionistic negation ($\sim$) in **daC** is that the consistency connective $\circ$ is also definable by

$$\circ\alpha :=\sim (\alpha \wedge \neg\alpha).$$

Once a logic has a consistency connective $\circ$, it is natural to ask if it is a Logic of Formal Inconsistency (**LFI**). Ferguson proved not only that **daC** is an **LFI** (Observation 1. in [11]) but that it is a **C**-system and since the connective $\circ$ is definable then it is a **dC**-system (Corollary 5.1. in [11]).

Let us start with some results about **daC**; first we will see if some interesting rules hold up. From [16] it is known that rules such as $\neg\neg$-Necessitation, also known as rule $R2$: $\dfrac{\alpha}{\neg\neg\alpha}$, and rule $R3$: $\dfrac{\alpha \to \beta}{\neg\beta \to \neg\alpha}$ hold up in $\mathbb{Z}$, this is also the case for **daC**.

Theorem 3.12. *The following rules hold up in* **daC**.

1. *If* $\vdash \neg\alpha \vee \beta$ *then* $\vdash \neg\beta \to \neg\alpha$.
2. $R3$.
3. $\neg\neg$*-Necessitation.*
4. *If* $\vdash \alpha \to \beta$ *then* $\vdash \neg\alpha \vee \neg\neg\beta$.

Proof.

1. Suppose $\vdash \neg\alpha \vee \beta$. Then by using **Pos** we obtain $\vdash \beta \vee \neg\alpha$. Finally, by MG we obtain the desired result, namely $\vdash \neg\beta \to \neg\alpha$.
2. Direct from Theorem 3.6 and previous item.
3. The substitution property holds up in **daC** and by Corollary 3.10 there exists a formula β such that $\vdash \beta$ and $\vdash \neg\neg\beta$. Hence, by Theorem 3.3 the result follows immediately.
4. By rule $R3$ we have that $\vdash \alpha \to \beta$ implies $\vdash \neg\beta \to \neg\alpha$. Thus, by Theorem 3.6 we get $\vdash \neg\neg\beta \vee \neg\alpha$. Hence, by **Pos** we get $\vdash \neg\alpha \vee \neg\neg\beta$ as desired. $\square$

Theorem 3.13. *In the system* **daC** *the following holds:*

1. **DM2**.
2. **DM3**.
3. **DM4**.
4. **NI**.

[6] A logic L is said to be *finitely trivializable* when it has finite trivial theories.

Proof.

1. By **Pos** we have that $\vdash \alpha \rightarrow (\alpha \vee \beta)$ and $\vdash \beta \rightarrow (\alpha \vee \beta)$. By rule $R3$ (item 2 Theorem 3.12) we have that $\vdash \neg(\alpha \vee \beta) \rightarrow \neg\alpha$ and $\vdash \neg(\alpha \vee \beta) \rightarrow \neg\beta$. Also by **Pos** $\vdash (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$. Using the appropriate instance of it and after the application of MP twice we obtain **DM2**.
2. We have **Pos3**, then $\vdash (\alpha \wedge \beta) \rightarrow \alpha$. Using rule $R3$ we get $\vdash \neg\alpha \rightarrow \neg(\alpha \wedge \beta)$. Similarly, from **Pos4** we obtain $\vdash \neg\beta \rightarrow \neg(\alpha \wedge \beta)$. Since **Pos8** holds, by using MP twice we obtain **DM3**.
3. By axiom $\mathbf{C_\omega 1}$ we have $\vdash \alpha \vee \neg\alpha$ and $\vdash \beta \vee \neg\beta$. Using distributivity properties in **Pos**, we obtain $\vdash (\alpha \wedge \beta) \vee (\neg\alpha \vee \neg\beta)$. Then by MG we obtain **DM4**.
4. Identity law holds up in **Pos** so $\vdash \alpha \rightarrow \alpha$ and by $\neg\neg$-Necessitation rule (item 3 of Theorem 3.12) $\vdash \neg\neg(\alpha \rightarrow \alpha)$ follows immediately. □

Theorem 3.14. *Logic* **daC** *does not satisfy:*

1. **WE**.
2. *Extended law of excluded middle,* **EE**: $\alpha \vee (\alpha \rightarrow \beta)$.
3. **DM1**.
4. **DNC**.
5. **DNI**.
6. **WC**.
7. **AZ2′**.
8. **AZ2**.

Proof. See Appendix B. □

The previous results about **daC** will help us know more about the paraconsistency of **daC** and to elucidate the relation between this logic and some other paraconsistent logics such as $\mathbf{G'_3}$, $\mathbb{Z}$, $\mathbf{P^3}$ and $\mathbf{M4_p}$. Let us start with $\mathbf{G'_3}$. Apart from the many-valued characterization given in the Section 2, there is a Hilbert system for it in [19]. We can also find at the end of section 2 of [12] a brief study of extensions of fragments of Heyting Brouwer logic. This is the case of the family of logics $\mathbf{daCG_n}$, each one an extension of **daC** characterized by a Kripke frame for **daC** which is linearly ordered and has $n - 1$ points. We have that $\mathbf{G'_3}$ corresponds to $\mathbf{daCG_3}$ and then it is characterized by a Kripke frame for **daC** with two linearly ordered points.

Theorem 3.15. *Logic* $\mathbf{G'_3}$ *properly extends* **daC**.

Proof. The Hilbert system for $\mathbf{G'_3}$ in [19] includes all axioms of $\mathbf{C_\omega}$. All that is left is to see that rule MG holds in $\mathbf{G'_3}$. However, if $\alpha \vee \beta$ is a tautology in $\mathbf{G'_3}$, then $\neg\alpha \rightarrow \beta$ is also a tautology, so, the inclusion is immediate. The *Weak explosion* principle holds in $\mathbf{G'_3}$, it is nevertheless false in **daC**. □

Since $\mathbf{G'_3}$ is a paraconsistent logic that extends da Costa logic, it is immediate that:

Corollary 3.16. *Da Costa logic is not maximally paraconsistent in the strong sense (Definition 2.1).*

Theorem 3.17. *If we add to* **daC** *the Extended law of excluded middle (i.e.* $\alpha \vee (\alpha \rightarrow \beta)$*), then the resulting system is no longer paraconsistent.*

Proof. If we apply MG to $\vdash \alpha \vee (\alpha \rightarrow \beta)$, then we obtain $\vdash \neg\alpha \rightarrow (\alpha \rightarrow \beta)$, or equivalently $\vdash (\alpha \wedge \neg\alpha) \rightarrow \beta$. □

If we try to compare $\mathbb{Z}$ and **daC**, then we can get the following two theorems about the rules RZ and MG.

Theorem 3.18. *In* **daC** *the rule RZ (i.e. if* $\vdash \alpha \to \beta$ *then* $\vdash \neg(\alpha \wedge \neg\beta)$*) holds.*

Proof. By item 4 of Theorem 3.12 $\vdash \alpha \to \beta$ implies $\vdash \neg\alpha \vee \neg\neg\beta$. By item 2 of Theorem 3.13 $\vdash (\neg\alpha \vee \neg\neg\beta) \to \neg(\alpha \wedge \neg\beta)$. Hence, by *MP* we obtain that $\vdash \neg(\alpha \wedge \neg\beta)$, as desired. □

Theorem 3.19. *Let X be the logic obtained by the addition of the rule MG to logic* $\mathbb{Z}$*. The system X is no longer paraconsistent. In fact* $X =$ **CL***.*

Proof. We have from Fact 2.1 in [5] that the $\{\vee, \wedge, \to\}$-fragments of $\mathbb{Z}$ and **CL** agree; hence, $\vdash \alpha \vee (\alpha \to \beta)$. Applying *MG* we obtain $\vdash \neg\alpha \to (\alpha \to \beta)$, or equivalently by **IEP** $\vdash (\alpha \wedge \neg\alpha) \to \beta$. In other words $\mathbb{Z} + MG$ is no longer paraconsistent. For the second result, comparing the axiomatization of **Pos** and one of the Łukasiewicz axiomatizations of **CL** (see exercise 1.58 in [15]), we have that **Pos**$+\{\neg\alpha \to (\alpha \to \beta), (\neg\alpha \to \alpha) \to \alpha\} =$ **CL**. We have already proven that $\vdash \neg\alpha \to (\alpha \to \beta)$. From Fact 2.1 in [5], we have that $(\neg\alpha \to \alpha) \to \alpha$ is in $\mathbb{Z}$. Since $\mathbb{Z}$ extends **Pos** then the result follows immediately. □

Theorem 3.20. *Logic* **daC** *is neither comparable with* $\mathbf{P^3}$ *nor with* $\mathbf{M4_p}$*.*

Proof. For the first case we have that **Peirce** law is a theorem in $\mathbf{P^3}$ but it is not the case for **daC**. On the other hand **DM4** is a theorem of **daC** (item 3 of Theorem 3.13) but it is not a theorem of $\mathbf{P^3}$. For the second case we have that **WE** is a theorem of $\mathbf{M4_p}$ but it is not a theorem of **daC** (Theorem 3.14). In **daC** the schema **NI** is a theorem but it is not the case for $\mathbf{M4_p}$. □

Theorem 3.21. *The addition of* **AZ2′** *or* **WE** *to* **daC** *leads us to the same logic.*

Proof. We have that **daC** extends $\mathbf{C}_\omega$, by Theorem 3.13 it satisfies **DM3** and **DM4**, then the result follows directly from Theorem 3.5. □

Due to the importance of $\mathbb{Z}$ we would like to compare it with **daC**. With the information we have so far, we might think that logic $\mathbb{Z}$ extends **daC**. However, this does not happen as we can see in the following theorem.

Theorem 3.22. *The logics* **daC** *and* $\mathbb{Z}$ *are not comparable*

Proof. On one hand, we have that **WE** (Weak Explosion) is valid in $\mathbb{Z}$ but by Theorem 3.14 we know that it does not hold in **daC**. On the other hand $\neg(\neg(\alpha \vee \beta) \vee \alpha) \to \beta$ is a theorem of **daC** but it does not hold for $\mathbb{Z}$. To prove that $\neg(\neg(\alpha \vee \beta) \vee \alpha) \to \beta$ is a theorem of **daC**, let v be a valuation and w_0 an arbitrary world. Let us suppose that w is a world such that $w_0 R w$ and $v_w(\neg(\neg(\alpha \vee \beta) \vee \alpha)) = 1$. Then there must be a world s such that sRw and $v_s(\neg(\alpha \vee \beta) \vee \alpha) = 0$, then $v_s(\alpha) = 0$ and $v(\neg(\alpha \vee \beta)) = 0$. This means that for all t such that tRs $v_t(\alpha \vee \beta) = 1$. Since R is reflexive particularly $v_s(\alpha \vee \beta) = 1$ but $v_s(\alpha) = 0$, then $v_s(\beta) = 1$. By the Heredity Constraint $v_w(\beta) = 1$ and therefore $v_{w_0}(\neg(\neg(\alpha \vee \beta) \vee \alpha) \to \beta) = 1$. To check that $\neg(\neg(\alpha \vee \beta) \vee \alpha) \to \beta$ does not hold in $\mathbb{Z}$, since it is sound with respect to **P-FOUR**, it is enough to find a valuation v in **P-FOUR** such that the value of the formula is not designated. This is the case for $v(\alpha) = 2$ and $v(\beta) = 1$ where $v(\neg(\neg(\alpha \vee \beta) \vee \alpha) \to \beta) = 1$. □

4. Extending daC

There are many ways in which a logic can be extended. In [11], the authors present some ideas for extending da Costa logic using the techniques from analyzing superintuitionistic logic. In [11], we also find a study about maximality and its relation to Logics of Formal Inconsistency. The set of extensions of **daC** is denoted by *Ext***daC** and the members of this collection are called *super Priest-da Costa logics* (sdc-logics). A member L of the class *Ext***daC** is a proper sdc-logic if $\mathbf{daC} \subsetneq L \subsetneq \mathbf{CL}$.

Here we extend **daC** by adding an axiom that is a well known theorem of the paraconsistent logics $\mathbb{Z}$ and $\mathbf{G'_3}$, the *Weak explosion*. In [11], this proper super Priest-da Costa logic is presented as **WECQ** or $\mathbf{PH_1} = \mathbf{daC} + \mathbf{WE}$. Since our intention is to compare $\mathbf{PH_1}$ with other paraconsistent logics we present some results.

Theorem 4.1. *In* $\mathbf{PH_1}$ *the following schemata hold.*

1. *Weak contrapositive* $\mathbf{WC}: (\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$.
2. $((\neg\alpha \vee \beta) \wedge \neg\neg\alpha) \to \beta$.
3. $\neg(\neg\alpha \vee \neg\neg\beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\beta)$.
4. **DNI**.
5. **DNC**.

Proof. See Appendix B. □

At the moment we have that $\mathbf{PH_1}$, $\mathbf{G'_3}$ and $\mathbb{Z}$ seem to be pretty close. For example, schemata **WE**, **DNI** and **DNC** hold up in all of them. However, there are significant differences between them as we will see. We start this task by finding different characterizations of $\mathbf{PH_1}$. In [11] we can find two semantical characterization of $\mathbf{PH_1}$ using Kripke style models. The first one is the following.

Definition 4.2. Let $\mathcal{W} = \langle W, R\rangle$ be a frame. It is *backwards convergent* if for all $t, u, v \in W$:

$$\text{If } tRv \text{ and } uRv\text{, then there exists } w \in W \text{ such that } wRt \text{ and } wRu.$$

Theorem 4.3. *(See Theorem 2.1 in [11].) Logic* $\mathbf{PH_1}$ *is characterized by the backwards convergent frames.*

Using this characterization we can see that $((\alpha \wedge \neg\beta) \wedge \neg(\alpha \wedge \neg\beta)) \to (\alpha \wedge \neg\alpha)$ is a theorem of $\mathbf{PH_1}$.

Theorem 4.4. *In* $\mathbf{PH_1}$ *the schema* **AZ2** *holds.*

Proof. Let $\mathcal{W} = \langle W, R, v\rangle$ be a da Costa interpretation based on a backwards convergent frame. Let $w \in W$ and suppose that $v_w((\alpha\wedge\neg\beta)\wedge\neg(\alpha\wedge\neg\beta)) = 1$. Then $v_w(\alpha) = 1$ and there are $s, t \in W$ such that sRw, tRw, $v_s(\beta) = 0$ and $v_t(\alpha \wedge \neg\beta) = 0$. Let us see that it is not possible that $v_t(\alpha) = 1$. Suppose that $v_t(\alpha) = 1$ then $v_t(\neg\beta) = 0$. From the last condition we have that $v_t(\beta) = 1$ and for all $u \in W$ such that uRt, $v_u(\beta) = 1$. Since $\mathcal{W}$ is backwards convergent, there exists $w' \in W$ such that $w'Rs$ and $w'Rt$. As $w'Rt$, $v_{w'}(\beta) = 1$, which contradicts the monotonicity of the interpretation because $w'Rs$ and $v_s(\beta) = 0$. Since $v_t(\alpha) \neq 1$ we have that $v_t(\alpha) = 0$, then $v_w((\alpha \wedge \neg\alpha) = 1$. Therefore $(\alpha \wedge \neg\beta) \wedge \neg(\alpha \wedge \neg\beta) \to (\alpha \wedge \neg\alpha)$ is a theorem of $\mathbf{PH_1}$. □

The following is a result of $\mathbf{PH_1}$ already known from [11].

Theorem 4.5. *In* $\mathbf{PH_1}$ *schema* **DM1** *holds.*

In fact, by adding **DM1** or **WE** to **daC** we obtain the same logic. Even more:

Theorem 4.6. *The addition of* **DM1**, **WE** *or* $\mathbf{AZ2'}$ *to* **daC** *leads to the same logic, in fact:*

$$\mathbf{PH_1} = \mathbf{daC} + \mathbf{WE} = \mathbf{daC} + \mathbf{AZ2'} = \mathbf{daC} + \mathbf{DM1}.$$

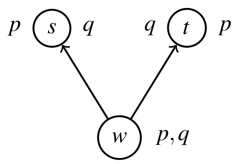

Fig. 1. Kripke countermodel for Theorem 4.8.

Proof. The first equality is by the definition of $\mathbf{PH_1}$, the second one is by Theorem 3.21 and the third one is by Theorem 2.4 in [11]. □

This last equality led Ferguson to the second semantical characterization of $\mathbf{PH_1}$.

Theorem 4.7. *Logic* $\mathbf{PH_1}$ *is characterized by the frames* $\mathcal{F} = \langle W, R\rangle$ *such that the pair* (W, R) *is a tree.*[7]

Proof. By Theorem 4.6 and Theorem 2.3 in [11] **daC+DM1** is characterized by the class of *1-rooted* frames i.e. by trees. □

Now that we have several characterizations of $\mathbf{PH_1}$ we go back to the task of comparing it with $\mathbf{G'_3}$ and $\mathbb{Z}$.

Theorem 4.8. *Logic* $\mathbf{G'_3}$ *extends* $\mathbf{PH_1}$, *but not the other way around.*

Proof. It is known that $\mathbf{G'_3}$ extends **daC** and it satisfies *Weak explosion*. So, the inclusion is immediate. Dummett's axiom, **DA** :$(\alpha \to \beta) \vee (\beta \to \alpha)$, holds up in $\mathbf{G'_3}$. However, $\mathbf{PH_1}$ is characterized by trees and in the tree depicted in Fig. 1 the instance $(p \to q) \vee (q \to p)$ of **DA** does not hold up in the point w. □

Corollary 4.9. *Logic* $\mathbf{PH_1}$ *is not comparable with* $\mathbb{Z}$.

Proof. Straightforward from Theorem 3.22. □

Theorem 4.10. *Logic* $\mathbf{PH_1}$ *is not comparable with* **P-FOUR**.

Proof. From Theorem 4.5 **DM1** holds up in $\mathbf{PH_1}$. However, **DM1** is not a theorem of **P-FOUR**. On the other hand **Peirce** holds up in **P-FOUR** but it is not a theorem of $\mathbf{G'_3}$ and since $\mathbf{G'_3}$ extends $\mathbf{PH_1}$ (Theorem 4.8) then it does not hold up in $\mathbf{PH_1}$. □

We can find many of the previous results about $\mathbf{PH_1}$ and some other paraconsistent logics in Table 2. As we may suspect from Table 2:

Theorem 4.11. *The relation between* **daC**, $\mathbf{PH_1}$, $\mathbf{G'_3}$ *and* $\mathbf{CG'_3}$ *is the following:*

$$\mathbf{daC} \subsetneq \mathbf{PH_1} \subsetneq \mathbf{G'_3} \subsetneq \mathbf{CG'_3}$$

Proof. The first inclusion is by definition of $\mathbf{PH_1}$ and it is proper since **WE** is not a theorem of **daC** (Theorem 3.14). The second proper inclusion is Theorem 4.8 and the final one is direct from the definition of $\mathbf{G'_3}$ and the fact that **Peirce** holds up in $\mathbf{CG'_3}$ but not in $\mathbf{G'_3}$. □

[7] A *tree* is a partially ordered set (poset) $(T, <)$ such that for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$.

Table 2
Comparison of **daC** and $\mathbf{PH_1}$ and other paraconsistent logics.

	Formula	**P-FOUR**	$\mathbf{M4_p}$	$\mathbb{Z}$	**daC**	$\mathbf{PH_1}$	$\mathbf{G'_3}$	$\mathbf{CG'_3}$
WE	$\neg\alpha \to (\neg\neg\alpha \to \beta)$	✓	✓	✓	X	✓	✓	✓
NI	$\neg\neg(\alpha \to \alpha)$	✓	X	✓	✓	✓	✓	✓
WC	$(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$	✓	✓	✓	X	✓	✓	✓
DNI	$\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$	✓	✓	✓	X	✓	✓	✓
DNC	$\neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$	✓	✓	✓	X	✓	✓	✓
DM1	$(\neg\alpha \wedge \neg\beta) \to \neg(\alpha \vee \beta)$	X	✓	X	X	✓	✓	✓
DM2	$\neg(\alpha \vee \beta) \to (\neg\alpha \wedge \neg\beta)$	✓	✓	✓	✓	✓	✓	✓
DM3	$(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$	✓	✓	✓	✓	✓	✓	✓
DM4	$\neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$	✓	✓	✓	✓	✓	✓	✓
Peirce	$((\alpha \to \beta) \to \alpha) \to \alpha$	✓	✓	✓	X	X	X	✓
AZ2	$(\alpha \wedge \neg\beta) \wedge \neg(\alpha \wedge \neg\beta) \to (\alpha \wedge \neg\alpha)$	✓	✓	✓	X	✓	✓	✓
$\neg\neg-$Necessitation	If $\vdash \alpha$ then $\vdash \neg\neg\alpha$	✓	X	✓	✓	✓	✓	X

5. Conclusions

We studied the behavior of the negation in a large family of paraconsistent logics. We focused on the relation between five interesting logics namely: **daC**, $\mathbf{PH_1}$, $\mathbf{G'_3}$, $\mathbf{CG'_3}$ and $\mathbb{Z}$.

First we did a detailed study of the properties of logic **daC**. We saw that it satisfies the substitution property. We proved that it is not a maximal paraconsistent logic in the strong sense, that **daC** is weaker than $\mathbf{G'_3}$, and it is not comparable with any of the logics $\mathbf{P^3}$, $\mathbf{M4_p}$ and $\mathbb{Z}$. After that, we concentrate in $\mathbf{PH_1}$ and we obtained different characterizations of it. At first glance $\mathbf{PH_1}$, $\mathbf{G'_3}$ and $\mathbb{Z}$ seem to be pretty close. For example, all of them satisfy many interesting properties related to negation, among them we can find: weak explosion, three De Morgan's laws, distribution of double negation and double negation of identity law. Later we found that the remaining De Morgan's law, the Peirce axiom, the Dummett's axiom and the $\neg\neg$-necessitation rule made the difference between these logics. From these differences we have that $\mathbf{PH_1}$ is not comparable with $\mathbb{Z}$ nor with **P-FOUR** and that $\mathbf{daC} \subsetneq \mathbf{PH_1} \subsetneq \mathbf{G'_3} \subsetneq \mathbf{CG'_3}$.

We saw some relations between $\mathbf{CG'_3}$ and some other logics with Hilbert style axiomatizations. Nevertheless, an axiomatization for $\mathbf{CG'_3}$ is still missing.

It is necessary to study the role of $\mathbf{PH_1}$ in *Ext***daC**. In general it is important to know if it is possible to find a logic in *Ext***daC** that enjoys properties such as definability of intuitionistic negation or the substitution property. If it is not the case, then prove that **daC** is maximal in these senses. Logics **daC** and $\mathbb{Z}$ are not comparable, then an interesting question arises. Is there a logic weaker than **daC** and $\mathbb{Z}$ such that it contains the positive fragment of **Int**, it preserves the substitution property and it has a reasonable semantics (such as Kripke semantics)? If the answer is positive, then which is the maximal logic in these senses.

From the results of the present article we can realize some relations between **daC** and **LFIs** logics. Ferguson presented some other relations in [11], but it is important to do a detailed study of these relations. Furthermore, the exploration of the relationship between the lattices *Ext***Int** (the class of extensions of **Int**) and *Ext***daC** is still missing.

Finally, we would like to thank Professor Graham Priest for his helpful answers to some of our questions and the helpful suggestions of the reviewers to improve this text.

Appendix A. Truth tables

Table A.3
Truth tables of $\mathbf{P^3}$.

$\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	1	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

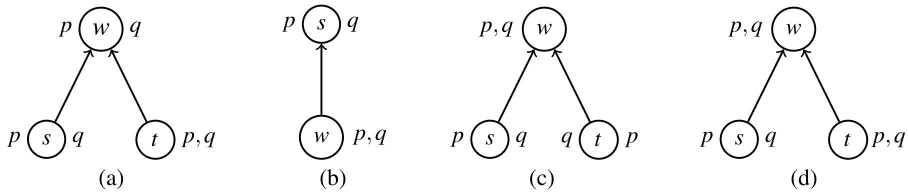

Fig. B.2. Kripke countermodels for Theorem 3.14.

Table A.4
Truth tables of $\mathbf{G}'_3$.

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

x	$\neg x$
0	2
1	2
2	0

Table A.5
Truth tables of **P-FOUR**.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

$\to$	0	1	2	3
0	3	3	3	3
1	2	3	2	3
2	1	1	3	3
3	0	1	2	3

x	$\neg x$
0	3
1	3
2	3
3	0

Table A.6
Truth tables of $\mathbf{M4_p}$.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

$\to$	0	1	2	3
0	3	3	3	3
1	2	2	2	3
2	1	1	2	3
3	0	1	2	3

x	$\neg x$
0	3
1	3
2	3
3	0

Appendix B. Proofs

Proof Theorem 3.14.

1. We prove that $\nvdash (\neg\alpha \wedge \neg\neg\alpha) \to \beta$ which is equivalent to check that **WE** does not hold.
   Let $\mathcal{W}$ be a Kripke interpretation for **daC** where $W = \{w, s, t\}$ is the set of worlds, the relation R is reflexive and transitive, and it is depicted in the Fig. B.2(a), $v(p) = \{w, s\}$ and $v(q) = \emptyset$. Then $v_t(p) = 0$ and $v_s(\neg p) = 0$ so $v_w(\neg p) = 1$ and $v_w(\neg\neg p) = 1$. Finally we have that $v_w(\neg p \wedge \neg\neg p) = 1$ but $v_w(q) = 0$ so $v_w((\neg p \wedge \neg\neg p) \to q) = 0$.
2. Let $\mathcal{W}$ be a Kripke interpretation for **daC** where $W = \{w, s\}$ is the set of worlds, the model is depicted in Fig. B.2(b). In this case $v_s(p) = 1$ and for any other variable and point the valuation is 0. $v_w(p) = 0$ and $v_w(p \to q) = 0$, since wRs, $v_s(p) = 1$ and $v_s(q) = 0$ therefore $v_w(p \vee (p \to q)) = 0$.
3. Let $\mathcal{W}$ be a Kripke interpretation for **daC** where $W = \{w, s, t\}$ is the set of worlds, which is represented graphically in Fig. B.2(c), $v(p) = \{w, s\}$ and $v(q) = \{w, t\}$. Since $v_s(q) = v_t(p) = 0$ then $v_w(\neg p \wedge \neg q) = 1$. On the other hand $v_x(p \vee q) = 1$ for all $x \in \mathcal{W}$ then $v_w(\neg(p \vee q)) = 0$. Finally $v_w((\neg p \wedge \neg q) \to \neg(p \vee q)) = 0$ as desired.
4. We check particularly that $\nvdash (\neg\neg\alpha \wedge \neg\neg\beta) \to \neg\neg(\alpha \wedge \beta)$.
   Let $\mathcal{W}$ be the same Kripke interpretation as in the previous case, represented graphically in Fig. B.2(c). Since $v_s(p) = v_t(q) = 1$ and s, t do not have predecessors, then $v_s(\neg p) = v_t(\neg q) = 0$ and $v_w(\neg\neg p \wedge \neg\neg q) = 1$. In this case $v_s(p \wedge q) = v_t(p \wedge q) = 0$ and $v_x(\neg(p \wedge q)) = 1$ for all $x \in \mathcal{W}$, then $v_w(\neg\neg(p \wedge q)) = 0$. So we obtain that $v_w((\neg\neg p \wedge \neg\neg q) \to \neg\neg(p \wedge q)) = 0$
5. Let $\mathcal{W}$ be a Kripke interpretation for **daC** where $W = \{w, s, t\}$ is the set of worlds, which is represented graphically in Fig. B.2(d). Here we see that $v(p) = \{w, s\}$ and $v(q) = \{w\}$, then $v_s(\neg p) = 0$, $v_s(\neg q) = v_t(\neg q) = v_w(\neg q) = 1$. As a result $v_w(\neg\neg p) = 1$, $v_w(\neg\neg q) = 0$ and $v_w(\neg\neg p \to \neg\neg q) = 0$. Also we have

that $v_t(p \to q) = 1$, then $v_t(\neg(p \to q)) = 0$ and we obtain that $v_w(\neg\neg(p \to q)) = 1$ and $v_w(\neg\neg(p \to q) \to (\neg\neg p \to \neg\neg q)) = 0$, as desired.

6. We check particularly that $\nvdash (\neg\alpha \to \neg\beta) \to (\neg\neg\beta \to \neg\neg\alpha)$.
   Let $\mathcal{W}$ be the same Kripke interpretation as in the previous case, represented graphically in Fig. B.2(d). As we have already seen $v_w(\neg\neg p \to \neg\neg q) = 0$. In this model also $v_w(\neg p) = v_w(\neg q) = 1$ then $v_w(\neg q \to \neg p) = 1$. Finally $v_w((\neg q \to \neg p) \to (\neg\neg p \to \neg\neg q)) = 0$, as desired.
7. Suppose $\mathbf{AZ2'}$ is a theorem in $\mathbf{daC}$. Since $\mathbf{daC}$ satisfies the hypotheses of Theorem 3.5 then $\mathbf{daC} = \mathbf{daC} + \mathbf{AZ2'} = \mathbf{daC} + \mathbf{WE}$, which contradicts the first item of this theorem.
8. Suppose $\mathbf{AZ2}$ is a theorem in $\mathbf{daC}$. Since $\mathbf{daC}$ satisfies the hypotheses of Theorem 3.4, then $\mathbf{AZ2'}$ is a theorem in $\mathbf{daC}$, which contradicts the previous item of this theorem. $\square$

Proof Theorem 4.1.

1. From [20] it is known that any logic that extends $\mathbf{C}_\omega$ with *Weak explosion* proves $(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$.
2. Consider the *Weak explosion* axiom version $\neg\alpha \to (\neg\neg\alpha \to \beta)$ and the instance of **Pos1** $\beta \to (\neg\neg\alpha \to \beta)$. Using the appropriate instance of **Pos8** and *MP* with the previous two formulas it follows that $\neg\alpha \vee \beta \to (\neg\neg\alpha \to \beta)$.
3. From **Pos6** and **Pos7**, $\vdash \neg\alpha \to (\neg\alpha \vee \neg\neg\beta)$ and $\vdash \neg\neg\beta \to (\neg\alpha \vee \neg\neg\beta)$. If we apply rule $R3$ to both formulas, then we obtain

$$\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to \neg\neg\alpha, \tag{B.1}$$

   and $\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to \neg\neg\neg\beta$. The application of *HS* to the last formula and $\vdash \neg\neg\neg\beta \to \neg\beta$ (Theorem 3.9) leads to

$$\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to \neg\beta. \tag{B.2}$$

   Also by **Pos** $\vdash (\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$, using the appropriate instance of it and after the application of *MP* with (B.1) and (B.2) we get $\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to (\neg\neg\alpha \wedge \neg\beta)$.
   For the other implication observe that, due to the weak explosion principle from $(\neg\alpha \vee \neg\neg\beta) \wedge (\neg\neg\alpha \wedge \neg\beta)$ anything follows, so that we have

$$(\neg\alpha \vee \neg\neg\beta) \to ((\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta)). \tag{B.3}$$

   If we use *MP* three times applied to the appropriate instance of **Pos8**, (B.3), the instance of **Pos1** $\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to ((\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta))$ and the instance of $\mathbf{C}_\omega\mathbf{1}$ $\vdash (\neg\alpha \vee \neg\neg\beta) \vee \neg(\neg\alpha \vee \neg\neg\beta)$, then we obtain $\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta)$.
4. By item 2 of this Theorem, $\vdash ((\neg\alpha \vee \beta) \wedge \neg\neg\alpha) \to \beta$. From **Pos** we have that $\vdash ((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)) \to ((\neg\alpha \vee \beta) \wedge \neg\neg\alpha)$ and by *HS*, $\vdash ((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)) \to \beta$. We can easily see that $\vdash ((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)) \to \neg\beta$. By **Pos** $\vdash (\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$, using the appropriate instance of it and after the applications of *MP* twice we obtain that:

$$\vdash ((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)) \to (\beta \wedge \neg\beta). \tag{B.4}$$

   Now by item 3 of this Theorem, $\vdash \neg(\neg\alpha \vee \neg\neg\beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\beta)$. Then (B.4) is equivalent to $\vdash ((\neg\alpha \vee \beta) \wedge \neg(\neg\alpha \vee \neg\neg\beta)) \to (\beta \wedge \neg\beta)$. By rule $R3$ we have that $\vdash \neg(\beta \wedge \neg\beta) \to \neg((\neg\alpha \vee \beta) \wedge \neg(\neg\alpha \vee \neg\neg\beta))$. But $\vdash \neg(\beta \wedge \neg\beta)$ (item 5 of Theorem 3.8) then by *MP* we get that:

$$\vdash \neg((\neg\alpha \vee \beta) \wedge \neg(\neg\alpha \vee \neg\neg\beta)). \tag{B.5}$$

Now by *MP* applied to the appropriate instance of **AZ2′** and (B.5) we have that $\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to \neg(\neg\alpha \vee \beta)$. By *HS* Applied to this result and item 2 of this Theorem $\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta)$ we obtain that:

$$\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \beta). \tag{B.6}$$

From $\vdash \alpha \wedge (\alpha \to \beta) \to \neg\alpha \vee \beta$ and $\vdash \neg\alpha \wedge (\alpha \to \beta) \to \neg\alpha \vee \beta$ we get that $\vdash (\alpha \to \beta) \to (\neg\alpha \vee \beta)$ and by rule $R3$ that $\vdash \neg(\neg\alpha \vee \beta) \to \neg(\alpha \to \beta)$. By *HS* applied to this result and (B.6) we have that $\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\alpha \to \beta)$ which by **IEP** is equivalent to $\vdash \neg\neg\alpha \to (\neg\beta \to \neg(\alpha \to \beta))$. Finally by rule $R3$ and properties of **Pos** we have that $\vdash \neg\neg\alpha \to (\neg\neg(\alpha \to \beta) \to \neg\neg\beta)$ and by **PL** $\vdash \neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$.

5. Applying rule $R3$ twice from **Pos3** we obtain $\vdash \neg\neg(\alpha \wedge \beta) \to \neg\neg\alpha$, analogously from **Pos4** we obtain $\vdash \neg\neg(\alpha \wedge \beta) \to \neg\neg\beta$. By **Pos** $\vdash (\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$, using the appropriate instance of it and after the applications of *MP* twice we obtain $\vdash \neg\neg(\alpha \wedge \beta) \to (\neg\neg\alpha \wedge \neg\neg\beta)$.
   For the other implication we use the following instance of **Pos8**, $\vdash \neg\alpha \to (\neg\neg\alpha \to \neg\beta) \to (\neg\beta \to (\neg\neg\alpha \to \neg\beta) \to (\neg\alpha \vee \neg\beta) \to (\neg\neg\alpha \to \neg\beta))$. By **WE** and **Pos1** we have that $\vdash \neg\alpha \to (\neg\neg\alpha \to \neg\beta)$ and $\vdash \neg\beta \to (\neg\neg\alpha \to \neg\beta)$. applying *MP* twice, we obtain $(\neg\alpha \vee \neg\beta) \to (\neg\neg\alpha \to \neg\beta)$. From here by **IEP** we get $\vdash \neg\neg\alpha \to ((\neg\alpha \vee \neg\beta) \to \neg\beta)$. Next, we have that by **DM3** and **DM4**, this is equivalent to $\neg\neg\alpha \to (\neg(\alpha \wedge \beta) \to \neg\beta)$. Now by using **WC** this is equivalent to $\vdash \neg\neg\alpha \to (\neg\neg\beta \to \neg\neg(\alpha \wedge \beta))$, and finally by **IEP** this is equivalent to $\vdash (\neg\neg\alpha \wedge \neg\neg\beta) \to \neg\neg(\alpha \wedge \beta)$. □

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