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DYNAMICAL SYSTEMS ON MONOIDS: TOWARD A GENERAL THEORY OF DETERMINISTIC SYSTEMS AND MOTION

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Dynamical systems are mathematical structures whose aim is to describe the evolution of an arbitrary deterministic system through time, which is typically modeled as (a subset of) the integers or the real numbers. We show that it is possible to generalize the standard notion of a dynamical system, so that its time dimension is only required to possess the algebraic structure of a monoid: first, we endow any dynamical system with an associated graph and, second, we prove that such a graph is a category if and only if the time model of the dynamical system is a monoid. In addition, we show that the general notion of a dynamical system allows us not only to define a family of meaningful dynamical concepts, but also to distinguish among a cluster of otherwise tangled notions of reversibility, whose logical relationships are finally analyzed.

Keywords: dynamical system, reversibility, irreversibility, category theory.

1 Introduction

A dynamical system is a kind of mathematical model that purports to formally capture the intuitive notion of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space (Arnold 1977 [1]; Szlenk 1984 [6]; Giunti 1997 [3]; Hirsch, Smale and Devaney, 2004 [2]). Let Z be the integers, Z^+ the non-negative integers, R the reals and R^+ the non-negative reals; below is a standard definition of a dynamical system.

Definition 0: *DS is a dynamical system* iff *DS* is a pair $(M, (g^t)_{t\in T})$ such that

(i) T is either Z, Z^+, R, or R^+. Any $t \in T$ is called a *duration* of the system, and T is called its *time set*;

(ii) M is a non-empty set. Any $x \in M$ is called a *state* of the system, and M is called its *state space*;

(iii) $(g^t)_{t\in T}$ is a family indexed by T of functions from M to M. For any $t \in T$, the function g^t is called the *state transition of duration t* (briefly, *t-transition*, or *t-advance*) of the system;

(iv) for any v, $t \in T$, for any $x \in M$,
  (a) $g^0(x) = x$;
  (b) $g^{v+t}(x) = g^v(g^t(x))$.

Examples of dynamical systems with discrete time and discrete state space are Turing machines and cellular automata; with discrete time and continuous state space: systems specified by difference equations (*e.g.* iterated mappings on R); with continuous time and continuous state space: systems specified by ordinary differential equations.

Definition 0 captures the intuitive notion of a deterministic system in the following sense. In the first place, condition (iii) should be interpreted as telling us the state of the system after an evolution of an arbitrary duration $t \in T$, provided that the state of the system at the present time $t_0 \in T$ is known; in other words, if at instant t_0 the system is in state $x \in M$, then at instant $t+t_0$ the system is in state $g^t(x)$. In addition, condition (iv.a) tells us that, whatever state the system is in, the evolution of duration 0 does not modify that state; and, finally, condition (iv.b) tells us that any evolution of duration $v+t$ can always be decomposed in two successive evolutions, the first one of duration t, and the second one of duration v.

However, Definition 0 is not fully explicit, for it does not make clear *exactly which structure* on the time set T is needed, in order to support appropriate dynamics for the system. By condition (i), T is either Z, Z^+, R, or R^+. With respect to the addition operation, these four models share the structure of a linearly ordered commutative monoid; but it is by no means obvious that *all* this structure on T is needed for a *general* definition of a dynamical system.

It is our contention that the minimal structure on the time set that underpins a materially adequate definition of a dynamical system is just that of a *monoid* (sec. 2, Definition 1). To substantiate our tenet we will focus first on the directed graph that any dynamical system induces on its state space, and on a revealing link between this graph and category theory. We will then prove that such a graph can be made into a category if, and only if, the algebraic structure of the time set is that of a monoid (sec. 3, Theorem 1). Finally, we will show how an algebraic time model as simple as a monoid is nevertheless sufficient to support a variety of significant dynamical concepts, as well as a rich web of relations among them (sec. 4).

2 Dynamical systems on monoids

To start with, we make the algebraic structure of the set T of durations explicit, by requiring that a binary operation + be given on T and the pair $L = (T, +)$ be a monoid. We then define the notion of a *dynamical system on L* by modifying Definition 0 as follows.

Definition 1: *DS_L is a dynamical system on L* iff DS_L is a pair $(M, (g^t)_{t\in T})$ and L is a pair $(T, +)$ such that

(i) $L = (T, +)$ is a monoid. Any $t \in T$ is called a *duration* of the system, T is called its *time set*, and L its *time model*;

(ii) M is a non-empty set. Any $x \in M$ is called a *state* of the system, and M is called its *state space*;

(iii) $(g^t)_{t\in T}$ is a family indexed by T of functions from M to M. For any $t \in T$, the function g^t is called the *state transition of duration t* (briefly, *t-transition*, or *t-advance*) of the system;

(iv) for any v, $t \in T$, for any $x \in M$,
   (a) $g^0(x) = x$, where 0 is the unity of L;
   (b) $g^{v+t}(x) = g^v(g^t(x))$.

Definition 2: *DS_L is a possible dynamical system on L* iff DS_L is a pair $(M, (g^t)_{t\in T})$ and L is a pair $(T, +)$ such that
T is a non-empty set and + is a binary operation on T;
M is a non-empty set and $(g^t)_{t\in T}$ is a family indexed by T of functions from M to M.

Definition 3: *f is a ρ-isomorphism of DS_{L_2} in DS_{L_1}* iff $DS_{L_2} = (N, (h^v)_{v\in V})$ is a possible dynamical system on $L_2 = (V, \oplus)$, $DS_{L_1} = (M, (g^t)_{t\in T})$ is a possible dynamical system on $L_1 = (T, +)$, ρ is an isomorphism of L_2 in L_1, and $f: N \rightarrow M$ is a bijection such that, for any $x \in N$ and $v \in V$, $f(h^v(x)) = g^{\rho(v)}(f(x))$.

Definition 4: *DS_{L_2} is isomorphic to DS_{L_1}* iff for some f and ρ, f is a ρ-isomorphism of DS_{L_2} in DS_{L_1}.

By Definition 4, it is easy to verify:

Proposition 1: *Being isomorphic to* is an equivalence relation on any given set of possible dynamical systems.

It is also not difficult to show that the relation of isomorphism is compatible with the property of being a dynamical system on a monoid, that is to say,

Proposition 2: If DS_{L_2} is isomorphic to DS_{L_1} and DS_{L_2} is a dynamical system on L_2, then DS_{L_1} is a dynamical system on L_1.

This allows us to speak of abstract dynamical systems on monoids in exactly the same sense we talk of abstract groups, fields, lattices, order structures, etc. We thus define:

Definition 5: *AS is an abstract dynamical system* iff *AS* is an equivalence class of isomorphic dynamical systems.

Definition 6: *P* is a *dynamical property* iff for any two possible dynamical systems DS_{L_2} and DS_{L_1},

(i) if DS_{L_2} has P, then DS_{L_2} is a dynamical system on L_2;
(ii) if DS_{L_2} has P, and DS_{L_2} is isomorphic to DS_{L_1}, then DS_{L_1} has P.

By Definition 6, a dynamical property is *proper* to dynamical systems and *preserved* by isomorphism. Dynamical properties can thus be regarded as the *specific structural* properties of dynamical systems. It is then easily shown:

Proposition 3: any two dynamical systems on monoids have exactly the same dynamical properties iff they are isomorphic.

Proof: If two dynamical systems on monoids are isomorphic, then by condition (ii) of Definition 6, they have exactly the same dynamical properties. Conversely, for any two non-isomorphic dynamical systems on monoids, DS_{L_1} and DS_{L_2}, there is a dynamical property they do not share; namely, the property of being isomorphic to DS_{L_1}. *Q.E.D.*

By *general dynamical systems theory* we mean the mathematical theory whose Suppes' style axiomatization (1957, ch. 12) is given by Definition 1. Since general dynamical systems theory is programmatically concerned with the study of dynamical properties, it regards any two isomorphic dynamical systems on monoids as identical.

3 Transition graphs and their relation to category theory

A *graph* in Lambek's sense (Marquis, 2010) is a quadruple (X, A, σ, τ), where X and A are non-empty sets, while both σ and τ are functions from A to X. An element of X is called an *object*, *node*, *point* or *vertex* of the graph, while a member of A is called an *arrow* or a *directed edge* of the graph. For any arrow $a \in A$, $\sigma(a)$ is called its *source*, and $\tau(a)$ its *target*. With the notation $x\ {}_a\!\!\longrightarrow y$ we mean that a is an arrow with source x and target y (briefly, a is an arrow from x to y).

A *deductive system* is a graph together with a family $(id_x)_{x\in X}$ of arrows and a partial binary operation $\circ$ on A that satisfy (i) for any $x \in X$, $\sigma(id_x) = x$ and $\tau(id_x) = x$; (ii) for any $x, y, z \in X$, for any $a, b \in A$, if $x\ {}_a\!\!\longrightarrow y$ and $y\ {}_b\!\!\longrightarrow z$, then $x\ {}_{b\circ a}\!\!\longrightarrow z$. For any $x \in X$, id_x is called the *x-identity arrow*, and the partial operation $\circ$ is called *arrow composition*. (For ease of understanding, we suggest reading "$b\circ a$" as "b following a".)

Finally, a *category* is a deductive system that also satisfies (iii) for any $w, x, y, z \in X$, for any $a, b, c \in A$, if $w\ {}_a\!\!\longrightarrow x$, $x\ {}_b\!\!\longrightarrow y$ and $y\ {}_c\!\!\longrightarrow z$, then $c\circ(b\circ a) = (c\circ b)\circ a$; (iv) for any $a \in A$, $id_{\tau(a)}\circ a = a$ and $a\circ id_{\sigma(a)} = a$.

There is an interesting link between general dynamical systems theory and category theory. Let us first of all notice that the family of t-transitions of a possible dynamical system DS_L naturally gives rise to a particular graph in Lambek's sense. We call this graph the *transition graph* of a possible dynamical system; the definition is below.

Definition 7: *$G(DS_L)$ is the transition graph of DS_L* iff $DS_L = (M, (g^t)_{t\in T})$ is a possible dynamical system on $L = (T, +)$ and $G(DS_L)$ is the quadruple (M, A, σ, τ) such that

(i) $A = \{a$: for some $t \in T$, $x \in M$, $a = (x, t, g^t(x))\}$;

(ii) $\sigma: A \to M$ such that, for any triple $a \in A$, $\sigma(a)$ is its first element;

(iii) $\tau: A \to M$ such that, for any triple $a \in A$, $\tau(a)$ is its last element.

From now on we indicate an arbitrary arrow $a = (x, t, g^t(x))$ of a transition graph with the notation $x \xrightarrow{t} g^t(x)$. We remark that this notation *exclusively* applies to arrows of transition graphs. It should not be confused with the more general notation $y \,{}_b\!\!\longrightarrow z$, which instead applies to an arbitrary arrow b of *any* graph. Also note that the first notation is just a different way of writing the triple $(x, t, g^t(x)) = a$, while the general notation does not stand for an arrow, but it is rather an abbreviation for the *statement* "b is an arrow with source y and target z".

A *quasi-dynamical system DS_L* on $L = (T, +)$ differs from a dynamical system just for the fact that the binary operation + not necessarily is associative; in other words, the time model L is not assumed to be a monoid, but a magma with unity. Below is the precise definition.

Definition 8: *DS_L is a quasi-dynamical system on L* iff DS_L is a pair $(M, (g^t)_{t\in T})$ and L is a pair $(T, +)$ such that

(i) $L = (T, +)$ is a magma with unity;

(ii) M is a non-empty set;

(iii) $(g^t)_{t\in T}$ is a family indexed by T of functions from M to M;

(iv) for any $v, t \in T$, for any $x \in M$,

  (a) $g^0(x) = x$, where 0 is the unity of L;

  (b) $g^{v+t}(x) = g^v(g^t(x))$.

Let DS_L be a quasi-dynamical system on L and $G(DS_L) = (M, A, \sigma, \tau)$ be its transition graph. We now equip $G(DS_L)$ with a family of x-identity arrows $(id_x)_{x\in M}$, where, for any $x \in M$, $id_x = x \xrightarrow{0} g^0(x)$.

As for the operation of arrow composition, we define it as follows. For any two arrows $a = x \xrightarrow{t} g^t(x)$ and $b = g^t(x) \xrightarrow{u} g^u(g^t(x))$, $b \circ a = x \xrightarrow{u+t} g^{u+t}(x)$.

We show below that the transition graph $G(DS_L)$ of a quasi-dynamical system DS_L, together with the family of x-identity arrows $(id_x)_{x\in M}$ and the operation of arrow composition $\circ$, is a category if and only if L is a monoid.

Theorem 1: Let $DS_L = (M, (g^t)_{t\in T})$ be a quasi-dynamical system on $L = (T, +)$, and $G(DS_L) = (M, A, \sigma, \tau)$ be the transition graph of DS_L; let $(id_x)_{x\in M}$ be the family of x-identity arrows, and $\circ$ be the operation of arrow composition, as defined above. $G^*(DS_L) = (M, A, \sigma, \tau, (id_x)_{x\in M}, \circ)$ is a category iff L is a monoid.

Proof: We show first that $G^*(DS_L)$ is a deductive system. Condition (i) of the definition of a deductive system follows from the definition of the family $(id_x)_{x\in X}$ and from condition (iv.a) of Definition 8; as for condition (ii), it follows from the definition of the operation $\circ$ and from condition (iv.b) of Definition 8.

Next, we show that, if L is a monoid, then $G^*(DS_L)$ is a category as well. For any three arrows $a = x \xrightarrow{t} g^t(x)$, $b = g^t(x) \xrightarrow{u} g^u(g^t(x))$, $c = g^u(g^t(x)) \xrightarrow{v} g^v(g^u(g^t(x)))$, we have:

$$c\circ(b\circ a) = c\circ(x \xrightarrow{u+t} g^{u+t}(x)) = \\ c\circ(x \xrightarrow{u+t} g^u(g^t(x)) = x \xrightarrow{v+(u+t)} g^v(g^u(g^t(x))) \tag{1}$$

$$(c\circ b)\circ a = (g^t(x) \xrightarrow{v+u} g^{v+u}(g^t(x))))\circ a = \\ (g^t(x) \xrightarrow{v+u} g^v(g^u(g^t(x))))\circ a = x \xrightarrow{(v+u)+t} g^v(g^u(g^t(x))) \tag{2}$$

As L is a monoid, from (1) and (2), and by associativity of +, $c\circ(b\circ a) = (c\circ b)\circ a$; thus, condition (iii) of the definition of a category holds.

Finally, condition (iv) of the definition of a category holds as well, since for any three arrows $id_{\sigma(a)} = x \xrightarrow{0} g^0(x)$, $a = x \xrightarrow{t} g^t(x)$, $id_{\tau(a)} = g^t(x) \xrightarrow{0} g^0(g^t(x))$, we have:

$$id_{\tau(a)}\circ a = (g^t(x) \xrightarrow{0} g^0(g^t(x)))\circ a = \\ (g^t(x) \xrightarrow{0} g^t(x))\circ a = x \xrightarrow{0+t} g^t(x) = x \xrightarrow{t} g^t(x) = a \tag{3}$$

$$a\circ id_{\sigma(a)} = a\circ(x \xrightarrow{0} g^0(x)) = a\circ(x \xrightarrow{0} x)) = \\ x \xrightarrow{t+0} g^t(x) = x \xrightarrow{t} g^t(x) = a \tag{4}$$

Therefore, if L is a monoid, then $G^*(DS_L)$ is a category.

Conversely, we show that, if L is not a monoid, then $G^*(DS_L)$ is not a category, for condition (iii) of the definition of a category fails. Suppose L is not a monoid. Then, as L is a magma with unity, + is not associative. Hence, for some $v, u, t \in T$, $v+(u+t) \neq (v+u)+t$, and thus $x \xrightarrow{v+(u+t)} g^v(g^u(g^t(x))) \neq x \xrightarrow{(v+u)+t} g^v(g^u(g^t(x)))$. On the other hand, for any three arrows $a = x \xrightarrow{t} g^t(x)$, $b = g^t(x) \xrightarrow{u} g^u(g^t(x))$, $c = g^u(g^t(x)) \xrightarrow{v} g^v(g^u(g^t(x)))$, equations (1) and (2) above hold. Therefore, $c\circ(b\circ a) \neq (c\circ b)\circ a$, so that condition (iii) of the definition of a category fails. *Q.E.D.*

Theorem 1 provides us with a justification for our claim that the *minimal* structure on the time set that supports a materially adequate definition of a dynamical system is *at least* that of a monoid. For, if it is not, the transition graph of the system cannot even be made into a category.

4 Motion, orbits, and types of reversibility/irreversibility in dynamical systems on monoids

In this section we show that general dynamical systems theory, though based on a time model as simple as a monoid, is nevertheless sufficient to define a variety of genuine dynamical concepts, as well as to prove about them significant and sometimes even surprising results. Due to space limits, we will often skip details of the proofs, or we will even omit some proofs.

We start with the observation that the minimal framework of general dynamical systems theory provides us with a quite general concept of *motion*. Let Y and Z be any two sets; let $eval$: $Z^Y \times Y \to Z$ such that, for any function $f \in Z^Y$, for any $y \in Y$, $eval(f, y) = f(y)$; we then define:

Definition 9: *g_x is the motion with initial state x of DS_L* iff $DS_L = (M, (g^t)_{t\in T})$ is a dynamical system on $L = (T, +)$, for any $x \in M$, for any $t \in T$, g_x: $T \to M$ and $g_x(t) = eval(g^t, x)$. (g_x is also called *the x-motion of DS_L* or *the x-evolution of DS_L*.)

Intuitively, for any $x \in M$, the x-evolution g_x represents the motion of the system when the state at the initial time t_0 is x; more precisely, if the state at t_0 is x, then, for any $t \in T$, $g_x(t)$ is the state at time $t + t_0$.

Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$, and $x \in M$; we then define:

Definition 10:
(i) *The orbit of* $x = orb(x) = \{z: z = g^t(x)$, for some $t \in T\}$;
(ii) *r is an orbit* iff for some $x \in M$, $r = orb(x)$;
(iii) *the phase portrait* $= \{r$: r is an orbit$\}$.

Note that, by Definition 10.i and Definition 9, for any $x \in M$, $orb(x)$ = the image of g_x.

From an intuitive point of view, if a system is deterministic and two orbits have a state in common, then, from that state on, they must coincide; that is to say, deterministic systems do not admit crossing orbits. The following proposition expresses exactly this fact.

Proposition 4: Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$. For any $x, y, z \in M$, if $z \in orb(x)$ and $z \in orb(y)$, then $orb(z) \subseteq orb(x)$ and $orb(z) \subseteq orb(y)$.

Proof: By Definition 10.i and condition (iv.b) of Definition 1. *Q.E.D.*

As the time model $L = (T, +)$ of a dynamical system DS_L is a monoid, for any $t \in T$, there is at most one inverse; when the inverse of t exists, we indicate it by $-t$. The next proposition highlights three interesting consequences of the existence of the inverse $-t$ of a duration $t \in T$.

Proposition 5: Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$. For any $t \in T$, if the inverse $-t$ of t exists, then
(i) g^t is a bijection;

(ii) g^{-t} is the inverse of g^t with respect to the operation of function composition $\circ$;
(iii) $g^{-t} = (g^t)^{-1}$, where $(g^t)^{-1}$ is the inverse function of g^t.

Proof: both (i) and (ii) follow from conditions (iv.a) and (iv.b) of Definition 1; (iii) follows from (ii) and the fact that $(g^t)^{-1}$ is the inverse of g^t with respect to $\circ$. *Q.E.D.*

Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$, $x \in M$, and $t \in T$; we then define:

Definition 11:
(i) *x is a fixed point* iff $orb(x) = \{x\}$;
(ii) *x is a periodic point* iff for some $t \in T$, $t \neq 0$ and $g^t(x) = x$;
(iii) *t is a period of x* iff $t \neq 0$ and $g^t(x) = x$;
(iv) *t is the period of x* iff t is a period of x and, for any $t^* \in T$, if t^* is a period of x, $t^* \neq t$ and $t^* \neq -t$, then there is $k \geq 2$ such that $t^* = \underline{t+t\ldots}_{k\text{ times}}$ or $t^* = \underline{-t+-t\ \ldots}_{k\text{ times}}$.

Note that, by Definitions 11.iii and 11.ii, if t is a period of x, x is a periodic point; conversely, if x is a periodic point, there is t such that t is a period of x, but not necessarily is such a t unique. Also, by Definition 11.iii and Proposition 5, if t is a period of x and the inverse $-t$ of t exists, then $-t$ is a period of x as well.

By Definition 11.iv, if t is the period of x, then (*i*) t is unique iff the inverse of t does not exist, or $-t = t$; (*ii*) if t is not unique, then $-t$ is the period of x as well, but nothing else is the period of x.

Proposition 6: If x is a fixed point, then x is a periodic point and, for any $t \in T$, if $t \neq 0$, t is a period of x.

Proof: By Definition 11.i, 11.ii and 11.iii. *Q.E.D.*

For an arbitrary dynamical system DS_L, we are now able to distinguish three mutually exclusive and jointly exhaustive types of orbit: *periodic*, *eventually periodic*, and *aperiodic*. Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$; we define:

Definition 12:
(i) *r is a periodic orbit* iff for some $x \in M$, $r = orb(x)$ and x is a periodic point;
(ii) *r is an eventually periodic orbit* iff r is an orbit, r is not a periodic orbit and, for some $x \in r$, $orb(x)$ is a periodic orbit;
(iii) *r is an aperiodic orbit* iff r is an orbit, r is not a periodic orbit, and r is not an eventually periodic orbit.

Intuitively an orbit is *merging* when it shares a state with a different orbit and, from that state on, the two orbits coincide; this idea is expressed by the next definition.

Definition 13: *r is a merging orbit* iff for some $x, y \in M$, $r = orb(x)$, $orb(x) \cap orb(y) \neq \varnothing$, $\neg(orb(x) \subseteq orb(y))$ and $\neg(orb(y) \subseteq orb(x))$.

By combining Definitions 12 and 13 we obtain a partition of the phase portrait of an arbitrary dynamical system DS_L into six orbit types. It is thus interesting to study how the instantiation pattern of the six orbit types varies with specific characters of the systems considered.

For example, if we only consider dynamical systems on Z or Z^+, and we also take into account whether they have finite or infinite state space M, we get the instantiation patterns shown in Table 1.

Table 1. Instantiation of orbit types in four different types of dynamical systems.

	Dynamical System Types			
Orbit Types	DS_Z and M finite	DS_{Z^+} and M finite	DS_Z and M infinite	DS_{Z^+} and M infinite
(i.a)	0	0	0	0
(i.b)	1	1	1	1
(ii.a)	0	1	0	1
(ii.b)	0	1	0	1
(iii.a)	0	0	0	1
(iii.b)	0	0	1	1

Note: M = state space; (i.a) = periodic and merging, (i.b) = periodic and not merging, (ii.a) = eventually periodic and merging, (ii.b) = eventually periodic and not merging, (iii.a) = aperiodic and merging, (iii.b) = aperiodic and not merging; 1 = orbit type is instantiated, 0 = orbit type is not instantiated.

Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$; the family of t-transitions $(g^t)_{t\in T}$ allows us to introduce purely dynamical concepts of *past* and *future*, as follows. Let 0 be the unity of L, and $x \in M$:

Definition 14:

(i) $P^t(x)$ *is the t-past of* x iff $t \in T-\{0\}$ and $P^t(x) = \{y: g^t(y) = x\}$;

(ii) $F^t(x)$ *is the t-future of* x iff $t \in T-\{0\}$ and $F^t(x) = \{y: g^t(x) = y\}$;

(iii) $P(x)$ *is the past of* x iff $P(x) = \cup_{t\in T-\{0\}} P^t(x)$;

(iv) $F(x)$ *is the future of* x iff $F(x) = \cup_{t\in T-\{0\}} F^t(x)$.

Note that, by Definition 14.ii, for any $x \in M$ and $t \in T-\{0\}$, $F^t(x)$ is a singleton. Analogous definitions can be given for a set of states $X \subseteq M$.

It is quite unexpected that general dynamical systems theory might provide us with fine distinctions among many different concepts of reversibility/irreversibility. However, within this theory we can in fact distinguish at least *eight* notions of reversibility. Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$; we define:

Definition 15:

(i) DS_L *is reversible* iff for any $x \in M$, for any $t \in T$, for some $w \in T$, $g^w(g^t(x)) = x$;

(ii) DS_L *is strictly reversible* iff for any $t \in T$, for some $w \in T$, for any $x \in M$, $g^w(g^t(x)) = x$;

(iii) DS_L *is logically reversible* iff for any $t \in T$, g^t is injective;

(iv) DS_L *has complete past* iff for any $t \in T$, g^t is surjective;

(v) DS_L *is completely logically reversible* iff for any $t \in T$, g^t is bijective;

(vi) DS_L *is time symmetric* iff DS_L is completely logically reversible and there is $\sim: M \to M$ such that, for any $x \in M$, for any $t \in T$, $\sim(g^t(\sim(x))) = (g^t)^{-1}(x)$;

(vii) DS_L *is space invertible* iff there is $\sim: M \to M$ such that, for any $x \in M$, for any $t \in T$, $g^t(\sim(g^t(\sim(x)))) = x$;

(viii) DS_L *is time invertible* iff $L = (T, +)$ is a group.

Obviously, by negating each of the previous concepts of reversibility, we obtain eight corresponding concepts of irreversibility.

The order of the eight reversibility concepts *does not* correspond to their logical strength; in particular, (i) and (viii) are not, respectively, the weakest and the strongest concept; rather, (i) and (viii) are the weakest and the strongest *proper* concept of reversibility – there is only one more proper concept of reversibility, *i.e.* (ii). The following proposition provides details about the exact logical relations among the eight reversibility concepts of Definition 15.

Proposition 7:

(i) (15.viii) → (15.ii);

(ii) (15.ii) → (15.i);

(iii) (15.v) → (15.iv);

(iv) (15.v) → (15.iii);

(v) (15.vii) → (15.v);

(vi) (15.vii) ↔ (15.vi)

(vii) (15.ii) → (15.v).

Proof:

(i) Follows from Definitions 15.viii and 15.ii, and Proposition 5;

(ii) follows from Definitions 15.ii and 15.i;

(iii) follows from Definitions 15.v and 15.iv;

(iv) follows from Definitions 15.v and 15.iii;

(v) suppose DS_L is space invertible; hence, by Definition 15.vii, there is $\sim: M \to M$ such that, for any $x \in M$, for any $t \in T$,

$$g^t(\sim(g^t(\sim(x)))) = x \tag{5}$$

thus, by applying $\sim$ to both members of (5),

$$\sim(g^t(\sim(g^t(\sim(x))))) = \sim x \tag{6}$$

consequently, by substituting $\sim(x)$ for x in (6),

$$\sim(g^t(\sim(g^t(\sim(\sim(x)))))) = \sim(\sim(x)) \tag{7}$$

on the other hand, by setting $t = 0$ in (5),

$$\sim(\sim(x)) = x \tag{8}$$

so that, by (8), $\sim$ is an involution on M and, by (8) and (7), for any $t \in T$, the composed function $\sim\circ g^t$ is also an involution on M. Therefore, both $\sim\circ g^t$ and $\sim$ are bijections. Injectivity of $\sim\circ g^t$ and the fact that $\sim$ is an involution entail injectivity of g^t; also, surjectivity of $\sim\circ g^t$, the fact that $\sim$ is an involution, and surjectivity of $\sim$ entail surjectivity of g^t. Therefore, by Definition 15.v, DS_L is completely logically reversible;

(vi) the implication from right to left follows from Definitions 15.vi and 15.vii; the implication fron left to right follows from Definition 15.vii, thesis (v), and Definition 15.vi;

(vii) logical reversibility follows easily from Definitions 15.ii and 15.iii by reduction; in turn, also by reductio, complete past follows from Definitions 15.ii, 15.iv, and 15.iii.

Q.E.D.

Proposition 7 describes *all* the implications among the eight reversibility concepts of Definition 15; in other words, no other implication holds, except for those licensed by transitivity. To prove this claim, however, we need several counterexamples, which go beyond the scope of this paper. We only show below some of the most significant counterexamples.

Example 1: $\neg((15.i) \rightarrow (15.iii))$ and $\neg((15.i) \rightarrow (15.iv))$. This is shown by the dynamical system DS_L defined below. DS_L is in fact reversible, logically irreversible, and with incomplete past. Also note that L is a non-commutative monoid.

Let $L = (T, +)$, where $T = \{0, 1, 2, 3\}$ and the sum operation + is defined by table 2.

Table 2. The sum operation +.

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	1	2	3
3	3	1	2	3

Note: Read column+row, as reading order matters.

Let $DS_L = (M, (g^t)_{t\in T})$, where $M = \{x_1, x_2, x_3\}$ and, for any $t \in T$, the t-transition g^t is defined by table 3.

Table 3. The t-transitions family $(g^t)_{t\in T}$.

	g^0	g^1	g^2	g^3
x_1	x_1	x_2	x_1	x_3
x_2	x_2	x_2	x_1	x_3
x_3	x_3	x_2	x_1	x_3

Example 2: $\neg((15.v) \rightarrow (15.i))$. This is shown by the dynamical system $DS_L = (Z, (s^k)_{k\in Z^+})$ on $L = (Z^+, +)$, where s^0 is the identity function on Z and, for any $k > 0$, s^k is the k-th iteration of the successor function on Z. Then, obviously, DS_L is both completely logically reversible and irreversible.

Example 3: $\neg((15.v) \rightarrow (15.vi))$. This is shown by the dynamical system $DS_L = (Z, (g^t)_{t\in T})$ on $L = (T, \circ)$, where T is the set of all bijective functions on Z, $\circ$ is the operation of function composition and, for any $t \in T$ and any $x \in Z$, $g^t(x) = t(x)$. On the one hand, DS_L is completely logically reversible, as all its state transitions are bijective by definition.
On the other hand, suppose for reductio DS_L is time-symmetric; then, by Definition 15.vi, there is $\sim: Z \rightarrow Z$ such that, for any $x \in Z$, for any $t \in T$,

$$\sim(g^t(\sim(x))) = (g^t)^{-1}(x) \tag{9}$$

hence, by setting in (9) $t = 0$ = the identity function on Z, $\sim$ turns out to be an involution on Z, and thus a bijection on Z, so that $\sim \in T$. Also, by applying $\sim$ to both sides of (9),

$$\sim(\sim(g^t(\sim(x)))) = \sim((g^t)^{-1}(x)) \tag{10}$$

by (10), as $\sim$ is an involution,

$$g^t(\sim(x)) = \sim((g^t)^{-1}(x)) \tag{11}$$

since $\sim$ is an involution, $\sim = (\sim)^{-1}$, so that by (11),

$$g^t(\sim(x)) = (\sim)^{-1}((g^t)^{-1}(x)) \tag{12}$$

for any $u, v \in T$, $(u)^{-1}\circ(v)^{-1} = (v\circ u)^{-1}$, thus by (12),

$$g^t(\sim(x)) = (g^t\circ\sim)^{-1}(x) \tag{13}$$

by (13), (9), and being $\sim$ an involution,

$$g^t(\sim(x)) = \sim(g^t\circ\sim(\sim(x))) = \sim(g^t(\sim(\sim(x)))) = \sim(g^t(x)) \tag{14}$$

by applying $\sim$ to both sides of (14), and being $\sim$ an involution,

$$\sim(g^t(\sim(x))) = \sim(\sim(g^t(x))) = g^t(x) \tag{15}$$

by (15) and (9),

$$(g^t)^{-1}(x) = g^t(x) \tag{16}$$

that is to say, all bijective functions on Z would be involutions, which is plainly false. Hence, DS_L is not time-symmetric.

Example 4: $\neg((15.vii) \rightarrow (15.i))$. Let $DS_L = (Z, (g^k)_{k\in Z^+})$ be the dynamical system on $L = (Z^+, +)$ such that, for any $z \in Z$ and any $t \in Z^+$, $g^t(x) = t + x$. DS_L is space-invertible, for consider the function $\sim: Z \rightarrow Z$, $\sim(x) = -x$; then, for any $x \in Z$ and any $k \in Z^+$,

$$g^k(\sim(g^k(\sim(x)))) = k + (-(k + (-x))) = k - k + x = x \tag{17}$$

on the other hand, DS_L is not reversible, for let x and $k > 0$; then, $g^k(x) = k + x > x$, and thus, for any $n \in Z^+$, $g^n(g^k(x)) = n + k + x \geq k + x > x$.

A concept related to the notions of reversibility/irreversibility is that of a *garden of Eden*, that is, a state without past. Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$, and $x \in M$; we define:

Definition 16: *x is a garden of Eden* iff $P(x) = \varnothing$.

It is then easy to show:

Proposition 8: Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on $L = (T, +)$. If DS_L is reversible, then for any $x \in M$, x is not a garden of Eden; the converse implication is false.

Proof: If DS_L is reversible, then for any $x \in M$, and any $t \in T - \{0\}$, there is $w \in T$ such that $g^w(g^t(x)) = x$; but then, if $w \neq 0$, x is not a garden of Eden and, if $w = 0$, $g^t(x) = x$, so that x is not a garden of Eden either. Hence, DS_L has no garden of Eden. Falsity of the converse is shown by Example 2 above. *Q.E.D.*

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