Probabilistic $\mathcal{EL}$

Jean Christoph Jung

AG Theoretische Grundlagen der Künstlichen Intelligenz
Universität Bremen

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Motivation

- Description Logics are accepted formalism for KR
- polynomial time reasoning for $\mathcal{EL}$
- desired: ability to represent uncertain knowledge

Example

- a certain indication has a probable cause
- a finding is not certain
- somebody with a flew has fever with at least 75%

Description Logics

- family of logics to describe terminological knowledge
- decidable fragments of FOL
- Concepts: $C ::= \top \mid A \mid \neg C \mid C_1 \sqcap C_2 \mid \exists r.C \mid \ldots$
- terminological knowledge (TBox): set of GCIs $C \sqsubseteq D$
- instance knowledge (ABox): set of assertions $C(a)$, $r(a, b)$

Example

$$Mother \sqsubseteq Female \sqcap \exists child.\top$$
$$Parent \equiv Mother \sqcup Father$$
$$GrandMother \sqsubseteq Mother \sqcap \exists child.Parent$$

Semantics

given by interpretations $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$

- domain Δ, set of individuals
- concept names: $A^{\mathcal{I}} \subseteq \Delta$
- conjunction: $(C_1 \sqcap C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
- negation: $(\neg C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}$
- roles: $r^{\mathcal{I}} \subseteq \Delta \times \Delta$
- existential restrictions:
  $(\exists r.C)^{\mathcal{I}} = \{d \in \Delta \mid \exists e \in C^{\mathcal{I}}.(d, e) \in r^{\mathcal{I}}\}$

Semantics for Probability

- important problem in philosophy: mathematical, classical, subjective, frequency, ...
- desired: formal semantics for probabilities
- FOL: statistical vs. subjective interpretation [Halpern, 1990]
- probability distribution on the domain vs. possible world semantics

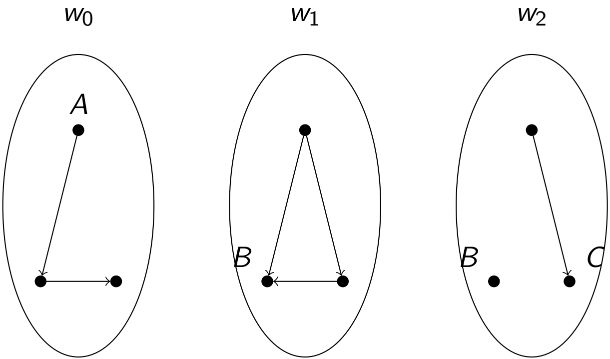
Possible World Semantics

- additional concept constructors $P_{\geq p}C$, $\exists P_{\geq p}r.C$
- extend DL-interpretation with a set of worlds
- probability distribution on the worlds
- concept & role names are interpreted locally: $A^{\mathcal{I},w}$, $r^{\mathcal{I},w}$
- interpretation of new constructors:

$$(P_{\geq p}C)^{\mathcal{I},w} = \{d \in \Delta \mid \sum_{v:d \in C^{\mathcal{I},v}} Pr(v) \geq p\}$$

$$(\exists P_{\geq p}C)^{\mathcal{I},w} = \{d \in \Delta \mid \exists e \in C^{\mathcal{I},w}. \sum_{v:(d,e) \in r^{\mathcal{I},v}} Pr(v) \geq p\}$$

Possible World Semantics – Example



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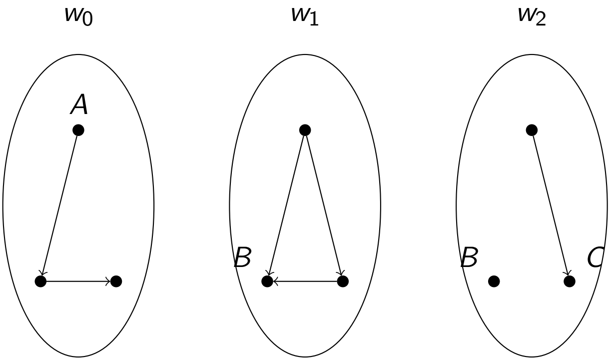

$Pr(w_0) = 0$ $Pr(w_1) = 0.3$ $Pr(w_2) = 0.7$

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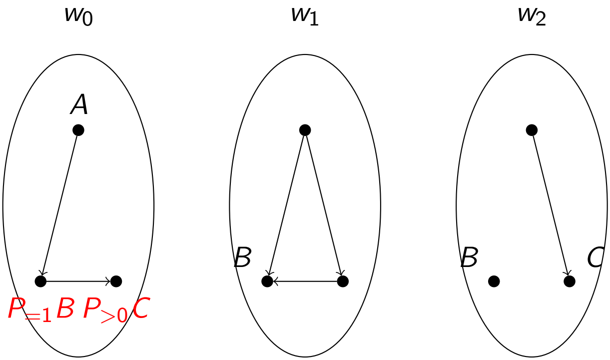

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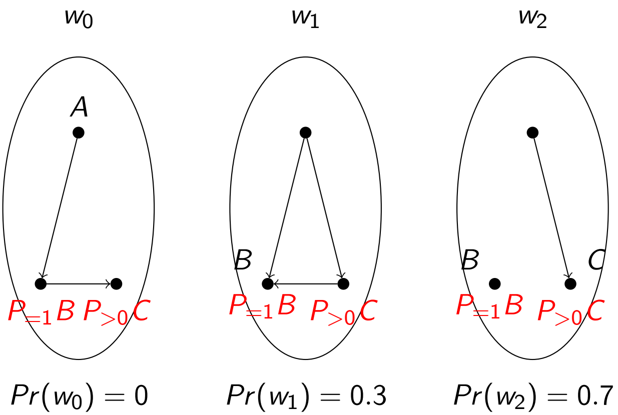

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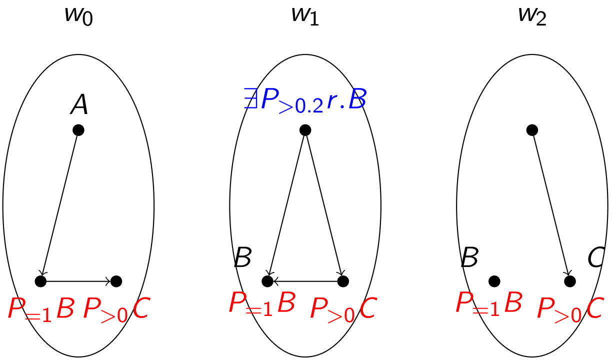

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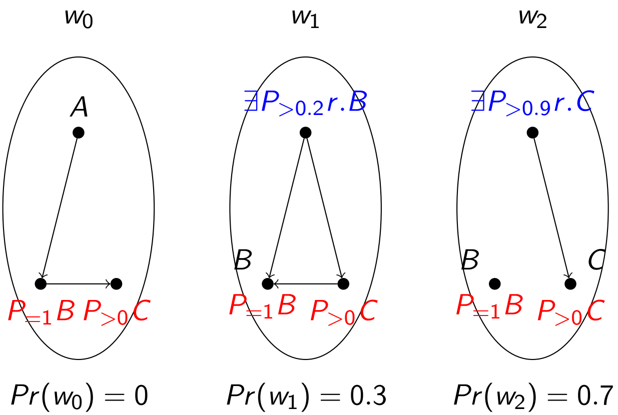

Worlds with probability zero

- takes time getting used to it
- philosophical justification:
  logically contradictory $\Longleftrightarrow$ infinitely improbable
- special behaviour, since probabilistic operators range only over worlds w with $Pr(w) > 0$
- often: restriction to one world with probability 0 possible
- difference between $\Box C$ and $P_{=1}C$, e.g., no direct reduction from $S5_{\mathcal{ALC}}$ to Prob-$\mathcal{ALC}^{01}$ possible

Reasoning in $\mathcal{EL}$

$\mathcal{EL}$: restriction to constructors $\exists r.A$ and $\sqcap$

- turns out to be expressive enough in many cases (e.g. medical knowledge bases)
- extensions: role hierarchies, fixpoint operators, $\bot$, ...
- polynomial reasoning
- finite model property
- convex logic: $C \sqsubseteq D_1 \sqcup D_2$ implies $C \sqsubseteq D_1$ or $C \sqsubseteq D_2$

important observation: non-convex extensions of $\mathcal{EL}$ are as hard as the corresponding variant of $\mathcal{ALC}$.

Probabilistic $\mathcal{EL}$

aims for the design of "Prob-$\mathcal{EL}$":

- add probabilistic constructors
- retain polynomial reasoning

Attempt 1 – $P_{>0}C$ and $P_{>0.4}C$

Consider the KB $\mathcal{K} = (\mathcal{T}, \{C(a)\})$ with $\mathcal{T}$:

$$
\begin{aligned}
C &\equiv P_{>0.4}A_1 \sqcap P_{>0.4}A_2 \sqcap P_{>0.4}A_3 \\
D_1 &\equiv P_{>0}(A_1 \sqcap A_2) \\
D_2 &\equiv P_{>0}(A_1 \sqcap A_3) \\
D_3 &\equiv P_{>0}(A_2 \sqcap A_3)
\end{aligned}
$$

Attempt 1 – $P_{>0}C$ and $P_{>0.4}C$

Consider the KB $\mathcal{K} = (\mathcal{T}, \{C(a)\})$ with $\mathcal{T}$:

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Now we have

$$\mathcal{K} \models D_1(a) \sqcup D_2(a) \sqcup D_3(a)$$

but for all $i = 1, 2, 3$

$$\mathcal{K} \not\models D_i(a)$$

Attempt 2 – $P_{<1/3}$ and $\bot$

Consider TBox:

$$A_1 \sqcap A_2 \sqsubseteq \bot$$
$$A_2 \sqcap A_3 \sqsubseteq \bot$$
$$A_1 \sqcap A_3 \sqsubseteq \bot$$

Attempt 2 – $P_{<1/3}$ and $\perp$

Consider TBox:

$$A_1 \sqcap A_2 \sqsubseteq \perp$$
$$A_2 \sqcap A_3 \sqsubseteq \perp$$
$$A_1 \sqcap A_3 \sqsubseteq \perp$$

We obtain

$$\mathcal{K} \models P_{<1/3}A_1(a) \sqcup P_{<1/3}A_2(a) \sqcup P_{<1/3}A_3(a)$$

but for all $i = 1, 2, 3$

$$\mathcal{K} \not\models P_{<1/3}A_i(a)$$

Attempt 3 – $P_{>p}C$

Assume $p = 1/k$ for some positive integer k.
Consider now the knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with

$$\mathcal{T} = \{A_i \sqcap A_j \sqsubseteq P_{>p}C_{ij} \mid 1 \leq i < j \leq k\}$$

and

$$\mathcal{A} = \{P_{>p}A_i(a) \mid 1 \leq i \leq k\}$$

Attempt 3 – $P_{>p}C$

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Then,

$$\mathcal{K} \models \bigsqcup_{i<j} P_{>p}C_{ij}$$

Result

Many interesting combinations of $\mathcal{EL}$ with probabilistic operators lead to intractable reasoning.
Three possible settings:

1. restriction to "normal" TBoxes + $P_{>p}C$ for one probabilty p
2. TBoxes without probabilities, but still only one probability $P_{>p}C$
3. restriction to $P_{>0}C$ and $P_{=1}C$

Probabilities on the roles

- adding arbitrary $\exists P_{>p} r.C$ leads to similar problems
- So again, try $\exists P_{>0} r.C$ or $\exists P_{=1} r.C$.
- convex extension
- From [Lutz & Schröder, 2010] it is known that $\mathcal{EL}$ extended with $P_{>0} C$ and one of the two constructors is PSPACE-hard
- best known upper bound: 2-EXPTIME.

The logic $S5_{\mathcal{EL}}$

- two-dimensional logic with possible world semantics
- $\Diamond C$ roughly $P_{>0}C$ and $\exists\Diamond r.C$ roughly $\exists P_{>0}r.C$
- "roughly": worlds with probability zero
- PSPACE algorithm for instance checking in $S5_{\mathcal{EL}}$
- we believe that with a similar technique we get an algorithm for the corresponding $\mathcal{EL}$-variant

Conclusions and future directions

- more careful analysis of statistical semantics
- how to convert from statistical to subjective semantics, i.e. how to get degree of belief from statistical knowledge

Relevant References

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An analysis of First-Order Logics of Probability
Artificial Intelligence, Vol 46, 1990

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Probabilistic Description Logics for Subjective Uncertainty
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