

Probabilistic \mathcal{EL}

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Motivation

- ▶ Description Logics are accepted formalism for KR
- ▶ polynomial time reasoning for \mathcal{EL}
- ▶ desired: ability to represent uncertain knowledge

Example

- ▶ a certain indication has a probable cause
- ▶ a finding is not certain
- ▶ somebody with a flew has fever with at least 75%

Description Logics

- ▶ family of logics to describe terminological knowledge
- ▶ decidable fragments of FOL
- ▶ Concepts: $C ::= \top \mid A \mid \neg C \mid C_1 \sqcap C_2 \mid \exists r.C \mid \dots$
- ▶ terminological knowledge (TBox): set of GCIs $C \sqsubseteq D$
- ▶ instance knowledge (ABox): set of assertions $C(a), r(a, b)$

EXAMPLE for DL

Semantics

given by interpretations $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$

- ▶ domain Δ , set of individuals
- ▶ concept names: $A^{\mathcal{I}} \subseteq \Delta$
- ▶ conjunction: $(C_1 \sqcap C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
- ▶ negation: $(\neg C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}$
- ▶ roles: $r^{\mathcal{I}} \subseteq \Delta \times \Delta$
- ▶ existential restrictions:
 $(\exists r.C)^{\mathcal{I}} = \{d \in \Delta \mid \exists e \in C^{\mathcal{I}}.(d, e) \in r^{\mathcal{I}}\}$

Semantics for Probability

- ▶ important problem in philosophy: mathematical, classical, subjective, frequency, ...
- ▶ desired: formal semantics for probabilities
- ▶ FOL: statistical vs. subjective interpretation [Halpern, 1990]
- ▶ probability distribution on the domain vs. possible world semantics

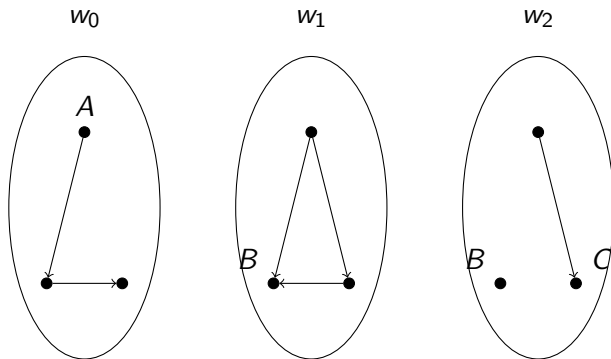
Possible World Semantics

- ▶ additional concept constructors $P_{\geq p}C$, $\exists P_{\geq p}r.C$
- ▶ extend DL-interpretation with a set of worlds
- ▶ probability distribution on the worlds
- ▶ concept & role names are interpreted locally: $A^{\mathcal{I},w}$, $r^{\mathcal{I},w}$
- ▶ interpretation of new constructors:

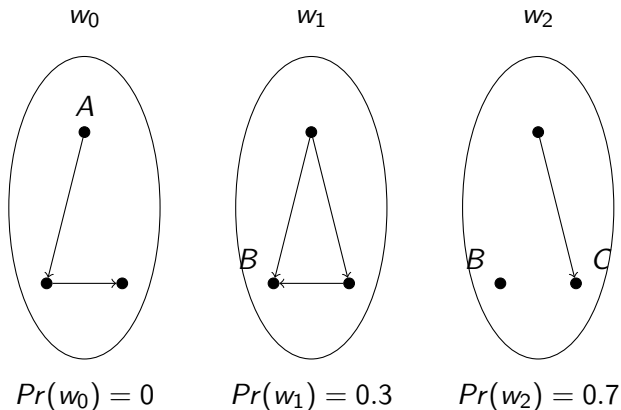
$$(P_{\geq p}C)^{\mathcal{I},w} = \{d \in \Delta \mid \sum_{v:d \in C^{\mathcal{I},v}} Pr(v) \geq p\}$$

$$(\exists P_{\geq p}C)^{\mathcal{I},w} = \{d \in \Delta \mid \exists e \in C^{\mathcal{I},w}. \sum_{v:(d,e) \in r^{\mathcal{I},v}} Pr(v) \geq p\}$$

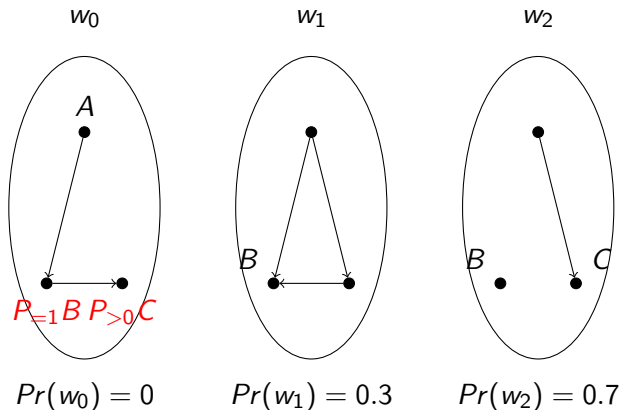
Possible World Semantics – Example



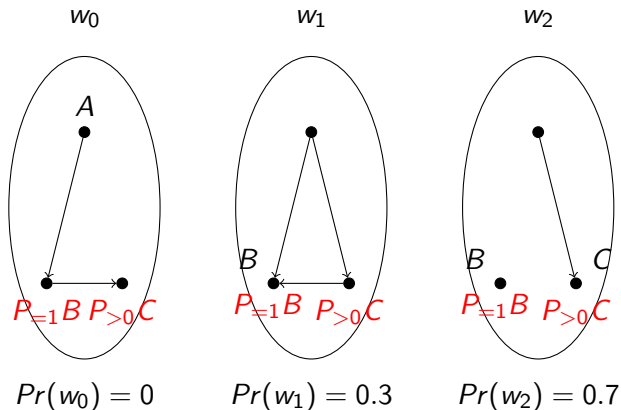
Possible World Semantics – Example



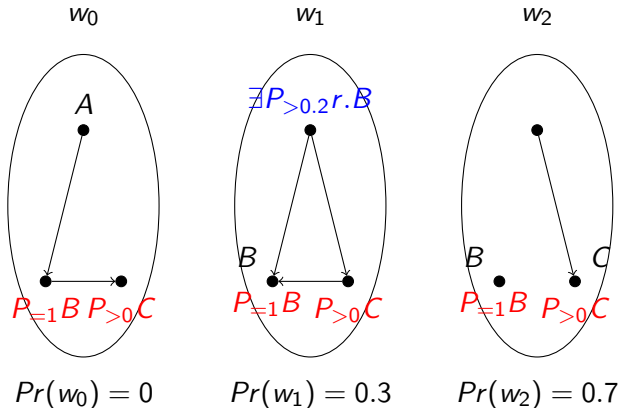
Possible World Semantics – Example



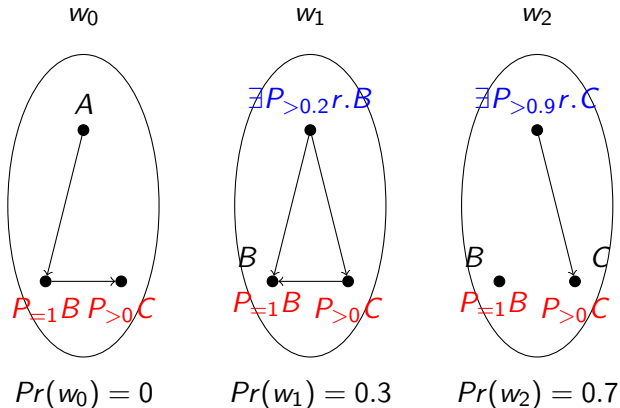
Possible World Semantics – Example



Possible World Semantics – Example



Possible World Semantics – Example



Worlds with probability zero

- ▶ takes time getting used to it
- ▶ philosophical justification:
logically contradictory \iff infinitely improbable
- ▶ restriction to one world with probability 0 possible
- ▶ difference between $\Box C$ and $P_{=1}C$, e.g., no direct reduction from $S5_{\mathcal{ALC}}$ to $\text{Prob-}\mathcal{ALC}^{01}$ possible

Reasoning in \mathcal{EL}

\mathcal{EL} : restriction to constructors $\exists r.A$ and \sqcap

- ▶ turns out to be expressive enough in many cases (e.g. medical knowledge bases)
- ▶ extensions: concrete domains, fixpoint operators, \perp , ...
- ▶ polynomial reasoning
- ▶ finite model property
- ▶ **convex logic**: $C \sqsubseteq D_1 \sqcup D_2$ implies $C \sqsubseteq D_1$ or $C \sqsubseteq D_2$

important observation: **non-convex** extensions of \mathcal{EL} are as hard as the corresponding variant of \mathcal{ALC} .

Probabilistic \mathcal{EL}

aims for the design of “Prob- \mathcal{EL} ”:

- ▶ add probabilistic constructors
- ▶ retain polynomial reasoning

Attempt 1 – $P_{>0}C$ and $P_{>0.4}C$

We add the constructors $P_{>0}C$ and $P_{>0.4}A$.

Consider now the KB $\mathcal{K} = (\mathcal{T}, \{C(a)\})$ with \mathcal{T} :

$$C \equiv P_{>0.4}A_1 \sqcap P_{>0.4}A_2 \sqcap P_{>0.4}A_3$$

$$D_1 \equiv P_{>0}(A_1 \sqcap A_2)$$

$$D_2 \equiv P_{>0}(A_1 \sqcap A_3)$$

$$D_3 \equiv P_{>0}(A_2 \sqcap A_3)$$

Attempt 1 – $P_{>0}C$ and $P_{>0.4}C$

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Consider now the KB $\mathcal{K} = (\mathcal{T}, \{C(a)\})$ with \mathcal{T} :

$$\begin{aligned}C &\equiv P_{>0.4}A_1 \sqcap P_{>0.4}A_2 \sqcap P_{>0.4}A_3 \\D_1 &\equiv P_{>0}(A_1 \sqcap A_2) \\D_2 &\equiv P_{>0}(A_1 \sqcap A_3) \\D_3 &\equiv P_{>0}(A_2 \sqcap A_3)\end{aligned}$$

Now we have

$$\mathcal{K} \models D_1(a) \sqcup D_2(a) \sqcup D_3(a)$$

but for all $i = 1, 2, 3$

$$\mathcal{K} \not\models D_i(a)$$

Attempt 2 – $P_{<1/3}$ and \perp

Consider TBox:

$$A_1 \sqcap A_2 \sqsubseteq \perp$$

$$A_2 \sqcap A_3 \sqsubseteq \perp$$

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Attempt 2 – $P_{<1/3}$ and \perp

Consider TBox:

$$A_1 \sqcap A_2 \sqsubseteq \perp$$

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$$A_1 \sqcap A_3 \sqsubseteq \perp$$

We obtain

$$\mathcal{K} \models P_{<1/3}A_1(a) \sqcup P_{<1/3}A_2(a) \sqcup P_{<1/3}A_3(a)$$

but for all $i = 1, 2, 3$

$$\mathcal{K} \not\models P_{<1/3}A_i(a)$$

Attempt 3 – $P_{>p}C$

Assume $p = 1/k$ for some positive integer k .

Consider now the knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with

$$\mathcal{T} = \{A_i \sqcap A_j \sqsubseteq P_{>p}C_{ij} \mid 1 \leq i < j \leq k\}$$

and

$$\mathcal{A} = \{P_{>p}A_i(a) \mid 1 \leq i \leq k\}$$

Attempt 3 – $P_{>p}C$

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Then,

$$\mathcal{K} \models \bigsqcup_{i < j} P_{>p}C_{ij}$$

Result

Many interesting combinations of \mathcal{EL} with probabilistic operators lead to intractable reasoning.

We found only three possible settings:

1. restriction to “normal” TBoxes + $P_{>p}C$ for one probability p
2. TBoxes without probabilities, but still only one probability $P_{>p}C$
3. restriction to $P_{>0}C$ and $P_{=1}C$

Probabilities on the roles

- ▶ adding arbitrary $\exists P_{>p}r.C$ leads to similar problems
- ▶ So again, try $\exists P_{>0}r.C$ or $\exists P_{=1}r.C$.
- ▶ From [Lutz & Schröder, 2010] it is known that \mathcal{EL} extended with $P_{>0}C$ and one of the two constructors is PSPACE-hard
- ▶ best known upper bound: 2-EXPTIME.

The logic $S5_{\mathcal{EL}}$

- ▶ two-dimensional logic with possible world semantics
- ▶ $\Diamond C$ roughly $P_{>0}C$
- ▶ $\exists\Diamond r.C$ roughly $\exists P_{>0}r.C$
- ▶ “roughly”: worlds with probability zero
- ▶ PSPACE algorithm for reasoning in $S5_{\mathcal{EL}}$
- ▶ we believe that with a similar technique we get an algorithm for the corresponding \mathcal{EL} -variant

Relevant References



Joseph Halpern

An analysis of First-Order Logics of Probability
Artificial Intelligence, Vol 46, 1990



Carsten Lutz, Lutz Schröder

Probabilistic Description Logics for Subjective Uncertainty
Proceedings of KR 2010