Derivation of optical flow formulas for course AE4317

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March 21, 2016

Abstract

Reproduction of the derivation of optical flow formulas as explained in the lecture.

1 Deriving the optical flow formula

Each point in an image (x, y) has an associated optical flow (u, v). Here we will derive an expression for the optical flow in terms of the movement of the camera, translational velocities (U, V, W) and rotation rates (A, B, C), and the coordinate of the projected point in the real world, (X, Y, Z). This derivation is based on [2], from which we adopt the notation.

We also use the coordinate system as introduced in [2], which is shown in Figure 1. In the system, O represents the center of the camera, X is positive toward the left, Y positive up, and Z is positive in the direction of the camera's principal axis.

We start with the formulas for the image coordinate of a world point P at (X, Y, Z):

$$x = \frac{X}{Z}, \tag{1}$$

$$y = \frac{Y}{Z}, \tag{2}$$

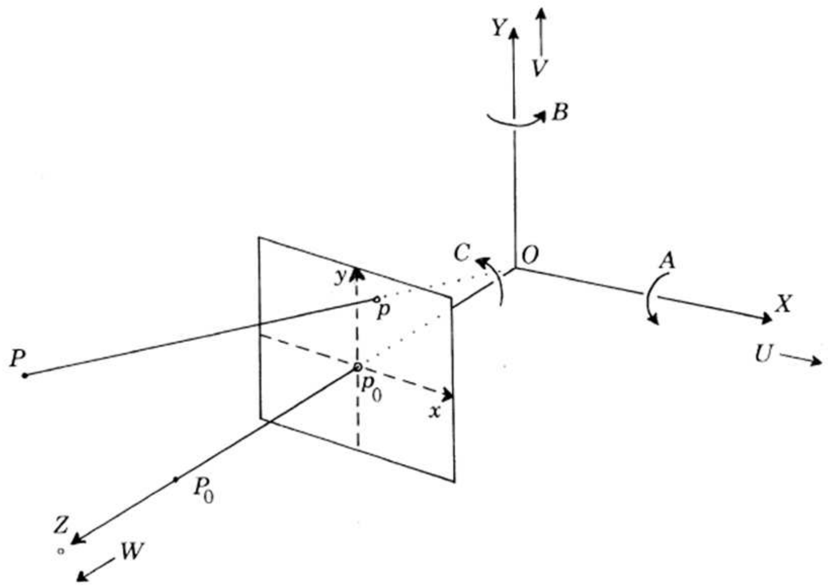

Figure 1: Coordinate system from [2].

where the reader should note that x and y are normalized image coordinates. The optical flow then is:

$$u = \frac{\partial}{\partial t} x = \dot{x} \tag{3}$$

$$v = \frac{\partial}{\partial t} y = \dot{y} \tag{4}$$

Here, we first express $\dot{x}$ and $\dot{y}$ in the variables representing the camera movement and P's coordinate.

$$\dot{x} = \frac{\dot{X}}{Z} - \frac{X\dot{Z}}{Z^2}. \tag{5}$$

$$\dot{y} = \frac{\dot{Y}}{Z} - \frac{Y\dot{Z}}{Z^2}. \tag{6}$$

Which shows that we should find the expressions for $\dot{X}$, $\dot{Y}$ and $\dot{Z}$.

We will determine the time derivative of x, as the derivation of $\dot{y}$ is similar. Let us start with $\dot{X}$, which captures the change in coordinate X as measured from the camera's origin over time. Obviously, if the camera has a positive

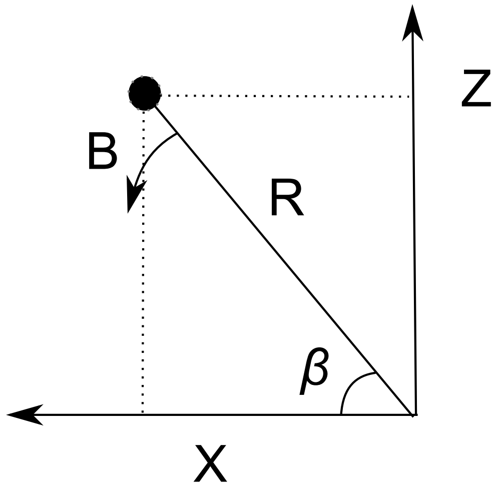

Figure 2: Rotation around Y-axis, with rotation rate B.

velocity U, in its own reference frame the (static) point P will move with the same velocity in the other direction. So:

$$\dot{X} = -U + \dots \tag{7}$$

Furthermore, a rotation around the Y-axis (B) also has an effect on the change in coordinate X. Please look at Figure 2, depicting a point P that is rotated with a rotation rate B. First, we can express X and Z in terms of the angle β and distance of P to the origin, R:

$$X = R\cos(\beta), \tag{8}$$

$$Z = R\sin(\beta). \tag{9}$$

Then we can determine $\dot{X}$ as:

$$\dot{X} = \dot{R}\cos(\beta) - R\sin(\beta)\dot{\beta}, \tag{10}$$

where $\dot{R} = 0$ for a rotation, and $\dot{\beta} = B$:

$$\dot{X} = -R\sin(\beta)B = -BZ, \tag{11}$$

where the last equality uses Eq. 9. So this brings us to:

$$\dot{X} = -U - BZ + \dots \tag{12}$$

A similar reasoning (left as exercise for the student) leads to the addition of $+CY$. The derivations for $\dot{Y}$ are also similar, with as result:

$$\dot{X} = -U - BZ + CY, \tag{13}$$

$$\dot{Y} = -V - CX + AZ, \tag{14}$$

$$\dot{Z} = -W - AY + BX. \tag{15}$$

We can fill in these expressions into Eq. 5 and 6:

$$u = \dot{x} = \frac{-U - BZ + CY}{Z} - \frac{X(-W - AY + BX)}{Z^2}, \tag{16}$$

which we can write differently as:

$$u = \dot{x} = -\frac{U}{Z} - B + Cy + x\frac{W}{Z} + Axy - Bx^2, \tag{17}$$

where we have used $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$.

In the article [2], the optical flow formulas are finally written as follows:

$$u = (-\frac{U}{Z} - B + Cy) - x(-\frac{W}{Z} - Ay + Bx), \tag{18}$$

$$v = (-\frac{V}{Z} - Cx + A) - y(-\frac{W}{Z} - Ay + Bx) \tag{19}$$

2 Information available from optical flow

The first thing to notice about Eqs. 18 and 21 is that the flow has a separate translational and rotational component:

$$u = -\frac{U}{Z} + x\frac{W}{Z} + Axy - Bx^2 - B + Cy = u_T + u_R, \tag{20}$$

$$v = -\frac{V}{Z} + y\frac{W}{Z} - Cx + A + Ay^2 - Bxy = v_T + v_R. \tag{21}$$

Let us for now assume that we know the rotations, for instance by measuring the rotation rates with a gyro, and that we can cancel the rotational component of the optical flow. Then, we are left only with the translational component, which can provide us with interesting information. For instance, let's see what the physical meaning of the Focus of Expansion (FoE) is, the point in which the optical flow is zero:

$$u_T = 0 = -\frac{U}{Z} + x_{\text{FoE}}\frac{W}{Z} \tag{22}$$

$$x_{\text{FoE}} = \frac{U}{W} \tag{23}$$

Similarly:

$$v_T = 0 = -\frac{V}{Z} + y_{\text{FoE}}\frac{W}{Z} \tag{24}$$

$$y_{\text{FoE}} = \frac{V}{W} \tag{25}$$

, so that:

$$\frac{x_{\text{FoE}}}{y_{\text{FoE}}} = \frac{U}{V} \tag{26}$$

indicates the direction in which the camera is travelling. So, if a robot can find the point where the optical flow is zero, it can find out in what direction it is moving. Also, we can show that all the flow will move away from this point. We first re-express u_T, v_T as follows:

$$u_T = (-U + xW)/Z = (-\frac{U}{W} + x)\frac{W}{Z} = (x - x_{\text{FoE}})\frac{W}{Z} \tag{27}$$

$$v_T = (-\frac{V}{W} + y)\frac{W}{Z} = (y - y_{\text{FoE}})\frac{W}{Z} \tag{28}$$

, which means that:

$$u_T/v_T = (x - x_{\text{FoE}})/(y - y_{\text{FoE}}), \tag{29}$$

so that the optical flow always points straight away from the FoE.

Equations 27-29 also show how to determine the relative velocity in the direction of the camera's principal axis:

$$\frac{W}{Z} = \frac{u_T}{(x - x_{\text{FoE}})} = \frac{v_T}{(y - y_{\text{FoE}})}, \tag{30}$$

which is inversely related to the 'time-to-contact' $\frac{Z}{W}$ and the divergence $2\frac{W}{Z}$ (under a flat plane assumption). In the literature on optical flow landing, the 'ventral flow' indicates $\frac{U}{Z}$ (in the X-direction) or $\frac{V}{Z}$ (in the Y-direction).

This document does not discuss *how* to go from the optical flow vectors $(u, v)_i$ to knowing the ventral flow, time-to-contact, etc. For this, you can have a look at the article [1], part of the literature for the course.

References

[1] G C H E de Croon, H W Ho, C De Wagter, E Van Kampen, B Remes, and QP Chu. Optic-flow based slope estimation for autonomous landing. *International Journal of Micro Air Vehicles*, 5(4):287–298, 2013.

[2] H Christopher Longuet-Higgins and Kvetoslav Prazdny. The interpretation of a moving retinal image. *Proceedings of the Royal Society of London B: Biological Sciences*, 208(1173):385–397, 1980.