Properties of moments of random variables*

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Let X and Y be real random variables, and let r and s be real positive constants $(r > 0,\ s > 0)$. The distribution functions of X and Y are denoted $F_X(x) = \mathsf{P}[X \le x]$ and $F_Y(x) = \mathsf{P}[Y \le x]$.

1. Existence of moments

1.1 EXISTENCE OF ABSOLUTE AND ORDINARY MOMENTS. $\mathsf{E}(|X|)$ always exists in the extended real numbers $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ and $\mathsf{E}(|X|) \in [0, \infty]$; *i.e.*, either $\mathsf{E}(|X|)$ is a non-negative real number or $\mathsf{E}(|X|) = \infty$.

1.2 $\mathsf{E}(X)$ exists and is finite $\Leftrightarrow \mathsf{E}(|X|) < \infty$.

1.3 $\mathsf{E}(|X|) < \infty \Rightarrow |\mathsf{E}(X)| \le \mathsf{E}(|X|) < \infty$.

1.4 If $0 < r \le s$, then

$$\mathsf{E}(|X|^s) < \infty \Rightarrow \mathsf{E}(|X|^r) < \infty \,. \tag{1.1}$$

1.5 MONOTONICITY OF L_r. $L_s \subseteq L_r$ for $0 < r \le s$.

1.6 $\mathsf{E}(|X|^r) < \infty \Rightarrow \mathsf{E}(X^k)$ exists and is finite for all integers k such that $0 < k \le r$.

2. Moment inequalities

2.1 c_r-INEQUALITY.

$$\mathsf{E}(|X+Y|^r) \le c_r[\mathsf{E}(|X|^r) + \mathsf{E}(|Y|^r)] \tag{2.1}$$

where

$$\begin{aligned} c_r &= 1, & \text{if } 0 < r \le 1\,, \\ &= 2^{r-1}, & \text{if } r > 1\,. \end{aligned} \tag{2.2}$$

2.2 MEAN FORM OF c_r-INEQUALITY.

$$\begin{aligned} \mathsf{E}(|\tfrac{1}{2}(X+Y)|^r) &\le \left(\tfrac{1}{2}\right)^r [\mathsf{E}(|X|^r) + \mathsf{E}(|Y|^r)]\,, & \text{if } 0 < r \le 1, \\ &\le \tfrac{1}{2}[\mathsf{E}(|X|^r) + \mathsf{E}(|Y|^r)]\,, & \text{if } r > 1. \end{aligned} \tag{2.3}$$

2.3 CLOSURE OF L_r. Let a and b be real numbers. Then

$$X \in L_r \text{ and } Y \in L_r \Rightarrow aX + bY \in L_r\,. \tag{2.4}$$

2.4 HÖLDER INEQUALITY. If $r > 1$ and $\dfrac{1}{r} + \dfrac{1}{s} = 1$, then

$$\mathsf{E}(|XY|) \le [\mathsf{E}(|X|^r)]^{1/r}[\mathsf{E}(|Y|^s)]^{1/s}\,. \tag{2.5}$$

2.5 CAUCHY-SCHWARZ INEQUALITY.

$$\mathsf{E}(|XY|) \le [\mathsf{E}(X^2)]^{1/2}[\mathsf{E}(Y^2)]^{1/2}. \tag{2.6}$$

2.6 MINKOWSKI INEQUALITY. If $r \ge 1$, then

$$\mathsf{E}(|X+Y|^r)^{1/r} \le [\mathsf{E}(|X|^r)]^{1/r} + [\mathsf{E}(|Y|^r)]^{1/r}. \tag{2.7}$$

2.7 MOMENT MONOTONICITY. $[\mathsf{E}(|X|^r)]^{1/r}$ is a non-decreasing function of r, *i.e.*

$$0 < r \le s \Rightarrow [\mathsf{E}(|X|^r)]^{1/r} \le [\mathsf{E}(|X|^s)]^{1/s}. \tag{2.8}$$

2.8 Theorem LIAPUNOV THEOREM. $\log[\mathsf{E}(|X|^r)]$ *is a convex function of* r, i.e. *for any* $\lambda \in [0,1]$,

$$\log[\mathsf{E}(|X|^{\lambda r+(1-\lambda)s})] \le \lambda \log[\mathsf{E}(|X|^r)] + (1-\lambda)\log[\mathsf{E}(|X|^s)]. \tag{2.9}$$

2.9 LOWER BOUNDS ON THE MOMENTS OF A SUM. If $\mathsf{E}(|X|^r) < \infty$, $\mathsf{E}(|Y|^r) < \infty$ and $\mathsf{E}(Y \mid X) = 0$, then

$$\mathsf{E}(|X+Y|^r) \ge \mathsf{E}(|X|^r), \quad \text{for } r \ge 1. \tag{2.10}$$

2.10 JENSEN INEQUALITY. If $g(x)$ is a convex function on $\mathbb{R}$ and $\mathsf{E}(|X|) < \infty$, then, for any constant $c \in \mathbb{R}$,

$$g(c) \le \mathsf{E}[g(X - EX + c)] \tag{2.11}$$

and, in particular,

$$g(EX) \le \mathsf{E}[g(X)]. \tag{2.12}$$

3. Markov-type inequalities

3.1 MARKOV INEQUALITIES. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(X)$ is a real random variable, $\mathsf{E}(|g(X)|) < \infty$ and

$$\mathsf{P}[0 \le g(X) \le M] = 1 \tag{3.1}$$

where $M \in [0, \infty]$. If $g(x)$ is a non-decreasing function on $\mathbb{R}$, then, for all $a \in \mathbb{R}$,

$$\frac{\mathsf{E}[g(X)] - g(a)}{M} \le \mathsf{P}[X \ge a] \le \frac{\mathsf{E}[g(X)]}{g(a)}. \tag{3.2}$$

If $g(x)$ is a non-decreasing function on $[0,\infty)$ and $g(x) = g(-x)$ for any x, then, for all $a \ge 0$,

$$\frac{\mathsf{E}[g(X)] - g(a)}{M} \le \mathsf{P}[|X| \ge a] \le \frac{\mathsf{E}[g(X)]}{g(a)} \tag{3.3}$$

where $0/0 \equiv 1$.

3.2 CHEBYSHEV INEQUALITIES. If $\mathsf{P}[|X| \leq M] = 1$, where $M \in [0, \infty]$, then, for all $a \geq 0$,

$$\frac{\mathsf{E}\,(|X|^r) - a^r}{M^r} \leq \mathsf{P}[|X| \geq a] \leq \frac{\mathsf{E}\,(|X|^r)}{a^r}\,. \tag{3.4}$$

3.3 Theorem REFINED MARKOV INEQUALITIES. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(X)$ is a real random variable, $\mathsf{E}(|g(X)|) < \infty$ and*

$$0 \;\leq\; g(x) \leq M_U \text{ for } x \geq A_U\,, \tag{3.5}$$
$$0 \;\leq\; g(x) \leq M_L \text{ for } x \leq A_L\,, \tag{3.6}$$

where $0 \leq M_U \leq \infty$, $0 \leq M_L \leq \infty$, $0 \leq A_U \leq \infty$ and $0 \leq A_L \leq \infty$. Let also

$$C_U(g,\,a) = \int\limits_{[a,\infty)} g(x)\,dF_X(x)\,, \quad C_L(g,\,a) = \int\limits_{(-\infty,a]} g(x)\,dF_X(x)\,. \tag{3.7}$$

(a) *If $g(x)$ is nondecreasing on $[A_U,\infty)$, then, for $a \geq A_U$,*

$$\frac{C_U(g,\,a)}{M_U} \leq \mathsf{P}[X \geq a] \leq \frac{C_U(g,\,a)}{g\,(a)}\,. \tag{3.8}$$

(b) *If $g(x)$ is nonincreasing on $(-\infty, A_L]$, then, for $a \leq A_L$,*

$$\frac{C_L(g,\,a)}{M_L} \leq \mathsf{P}[X \leq a] \leq \frac{C_L(g,\,a)}{g\,(a)}\,. \tag{3.9}$$

(c) *If $g(x)$ is nondecreasing on $[A_U,\infty)$ and nonincreasing on $(-\infty, A_L]$, then, for $a \geq \max\{|A_U|,\,|A_L|\}$,*

$$\begin{aligned} \mathsf{P}[|X| \geq a] \;&\leq\; \frac{C_U(g,\,a)}{g\,(a)} + \frac{C_L(g,\,a)}{g\,(-a)} \\ &\leq\; \frac{C_U(g,\,a) + C_L(g,\,a)}{\min\{g\,(a)\,,\,g\,(-a)\}}\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathsf{P}[|X| \geq a] \;&\geq\; \frac{C_U(g,\,a)}{M_U} + \frac{C_L(g,\,a)}{M_L} \\ &\geq\; \frac{C_U(g,\,a) + C_L(g,\,a)}{\max\{M_U\,,\,M_L\}}\,. \end{aligned} \tag{3.11}$$

4. Moments and behavior of tail areas

4.1 Lemma RIEMANN-STIELTJES INTEGRATION BY PARTS. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ two real-valued functions and $-\infty < a \leq b < +\infty$. If the (Riemann-Stieltjes) integral $\int_a^b g(x)\,d\,f(x)$*

exists, then the integrals $\int_a^b f(x)\,dg(x)$ *and* $\int_a^b [A - f(x)]\,dg(x)$ *also exist and*

$$\begin{aligned}\int_a^b g(x)\,d f(x) &= g(b)f(b) - g(a)f(a) - \int_a^b f(x)\,dg(x) \\ &= [A - f(a)]g(a) - g(b)[A - f(b)] + \int_a^b [A - f(x)]\,dg(x)\,, \end{aligned} \tag{4.1}$$

for any real constant A, *with*

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,g'(x)dx \tag{4.2}$$

and

$$\int_a^b [A - f(x)]\,dg(x) = \int_a^b [A - f(x)]\,g'(x)dx \tag{4.3}$$

if g *is continuous on* $[a,b]$ *as well as differentiable on* (a,b) *and the Riemann integral* $\int_a^b f(x)\,g'(x)dx$ *exists* $($*where* g' *can take arbitrary real values at* a *and* $b)$.

4.2 Lemma CENTERED RIEMANN-STIELTJES INTEGRATION BY PARTS. *Let* $f:\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}\to\mathbb{R}$ *two real-valued functions and* $-\infty < a \le c \le b < +\infty$. *If the integrals* $\int_a^b g(x)\,d f(x)$, $\int_a^c g(x)\,d f(x)$ *and* $\int_c^b g(x)\,d f(x)$ *exist, then the integrals* $\int_a^c f(x)\,dg(x)$ *and* $\int_c^b f(x)\,dg(x)$ *also exist, and*

$$\begin{aligned}\int_a^b g(x)\,d f(x) &= Ag(c) - \{g(b)[A - f(b)] + g(a)f(a)\} \\ &\quad + \int_c^b [A - f(x)]\,dg(x) - \int_a^c f(x)\,dg(x) \end{aligned} \tag{4.4}$$

for any real constant A, *with*

$$\int_a^c f(x)\,dg(x) = \int_a^c f(x)\,g'(x)\,dx \tag{4.5}$$

if g *is continuous on* $[a,c]$ *as well as differentiable on* (a,c) *and the Riemann integral* $\int_a^c f(x)\,g'(x)dx$ *exists* $($*where* g' *can take arbitrary real values at* a *and* $c)$, *and*

$$\int_c^b [A - f(x)]\,dg(x) = \int_c^b [A - f(x)]\,g'(x)\,dx \tag{4.6}$$

if g *is continuous on* $[c,\ b]$ *as well as differentiable on* (c,b) *and the Riemann integral* $\int_c^b [A - f(x)]\,g'(x)dx$ *exists* $($*where* g' *can take arbitrary real values at* c *and* $b)$.

4.3 Lemma BOUNDED MONOTONICITY CONDITION FOR TAIL CONVERGENCE OF AN INTEGRABLE FUNCTION. *Let* $f:\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}\to\mathbb{R}$ *two real-valued functions, and let* m, M *be two real constants.*

(a) *If $f(x)$ is monotonic nondecreasing on the interval $(-\infty, m)$ with finite limit as $x \to -\infty$, and if g satisfies the inequality*

$$|g(a)| \le B_L(x)\,, \text{ for } x \le a < m \tag{4.7}$$

where $B_L(x)$ is a real-valued function such that $\int_{-\infty}^{m} B_L(x)\,d\,f(x)$ exists, then

$$0 \le |g(a)|\,[f(a) - f(-\infty)] \le \int_{-\infty}^{a} B_L(x)\,d\,f(x)\,, \text{ for } a < m, \tag{4.8}$$

where $f(-\infty) = \lim_{x \to -\infty} f(x) > -\infty$, and

$$\lim_{a \to \infty} g(a)[\,f(a) - f(-\infty)] = 0\,. \tag{4.9}$$

(b) *If $f(x)$ is monotonic nondecreasing on the interval $[M, \infty)$ with finite limit as $x \to \infty$, and if g satisfies the inequality*

$$|g(b)| \le |g(x)| + B_U(x)\,, \text{ for } x \ge b > M \tag{4.10}$$

where $B_U(x)$ is a real-valued function such that $\int_{M}^{\infty} B_U(x)\,d\,f(x)$ exists, then

$$0 \le |g(b)|\;[f(\infty) - f(b)] \le \int_{b}^{\infty} B_U(x)\,d\,f(x)\,, \text{ for } b > M\,, \tag{4.11}$$

where $f(\infty) = \lim_{x \to \infty} f(x) < \infty$, and

$$\lim_{b \to \infty} g(b)\,[f(\infty) - f(b)] = 0\,. \tag{4.12}$$

It is easy to see that (4.7) holds whenever $\int_{-\infty}^{m} |g(x)|\,d\,f(x)$ exists and one of the following conditions holds: for some real constant B,

$$|g(a)| \le |g(x)| + B\,, \text{ for } x \le a < m\,; \tag{4.13}$$

$$|g(x)| \text{ is nondecreasing on the interval } (M, \infty)\,; \tag{4.14}$$

$$g(x) \text{ is bounded on the interval } (M, \infty)\,. \tag{4.15}$$

Further, in case (4.13), we have:

$$0 \le |g(a)|\,[f(a) - f(-\infty)] \le \int_{-\infty}^{a} |g(x)|\,d\,f(x) + B\,[f(a) - f(-\infty)]\,. \tag{4.16}$$

Similarly, (4.10) holds whenever one of the following conditions holds: for some real constant B,

$$|g(a)| \le |g(x)| + B\,, \text{ for } x > b > M\,; \tag{4.17}$$

$$|g(x)| \text{ is nonincreasing on the interval } (-\infty, m)\,; \tag{4.18}$$

$$g(x) \text{ is bounded on the interval } (-\infty, m)\,. \tag{4.19}$$

Further, in case (4.17), we have:

$$0 \le |g(b)|\,[f(\infty) - f(b)] \le \int_b^\infty |g(x)|\, df(x) + B\,[f(\infty) - f(b)] \tag{4.20}$$

4.4 Proposition MOMENT EXISTENCE AND TAIL AREA DECAY. *Let $r > 0$. If $\mathsf{E}(|X|^r) < \infty$, then*

$$\begin{aligned} \lim_{x\to\infty} \{x^r\, \mathsf{P}[X \ge x]\} &= \lim_{x\to-\infty} \{|x|^r\, \mathsf{P}[X \le x]\} \\ &= \lim_{x\to\infty} \{x^r\, \mathsf{P}[|X| \ge x]\} = 0\,. \end{aligned} \tag{4.21}$$

In particular, if $\mathsf{E}(|X|) < \infty$, then

$$\begin{aligned} \lim_{x\to\infty} \{x\, \mathsf{P}[X \ge x]\} &= \lim_{x\to-\infty} \{|x|\, \mathsf{P}[X \le x]\} \\ &= \lim_{x\to\infty} \{x\, \mathsf{P}[|X| \ge x]\} = 0\,. \end{aligned} \tag{4.22}$$

4.5 Theorem DISTRIBUTION DECOMPOSITION OF r-MOMENTS. *For any $r > 0$,*

$$\int_0^\infty x^r dF_X(x) = r\int_0^\infty x^{r-1}[1 - F_X(x)]dx\,, \tag{4.23}$$

$$\begin{aligned} \mathsf{E}(|X|^r) &= r\int_0^\infty x^{r-1}\mathsf{P}(|X| \ge x)dx \\ &= r\int_0^\infty x^{r-1}[1 - F_X(x) + F_X(-x)]dx\,, \end{aligned} \tag{4.24}$$

and

$$\begin{gathered} \mathsf{E}(|X|^r) < \infty \Leftrightarrow x^{r-1}\mathsf{P}(|X| \ge x) \textit{ is integrable on } (0, +\infty) \\ \Leftrightarrow |x|^{r-1}\,[1 - F_X(x) + F_X(-x)] \textit{ is integrable on } (0, +\infty) \\ \Leftrightarrow \int_0^\infty x^{r-1}[1 - F_X(x)]dx < \infty \textit{ and } \int_{-\infty}^0 |x|^{r-1} F_X(x)dx < \infty\,. \end{gathered} \tag{4.25}$$

4.6 Proposition DISTRIBUTION DECOMPOSITION OF THE FIRST ABSOLUTE MOMENT.

$$\int_0^\infty x dF_X(x) = \int_0^\infty [1 - F_X(x)]dx\,, \tag{4.26}$$

$$\mathsf{E}|X| = \int_0^\infty \mathsf{P}(|X| \ge x)dx\,, \tag{4.27}$$

and

$$\mathsf{E}(|X|) < \infty \Leftrightarrow \mathsf{P}(|X| \ge x) \textit{ is integrable on } (0, +\infty)$$

$$\Leftrightarrow [1-F_X(x)+F_X(-x)] \text{ is integrable on } (0,+\infty)$$

$$\Leftrightarrow \int_0^{\infty}[1-F_X(x)]dx < \infty \text{ and } \int_{-\infty}^{0} F_X(x)dx < \infty. \tag{4.28}$$

4.7 Proposition MOMENT-TAIL AREA INEQUALITIES. *Let $g(x)$ be a nonnegative strictly increasing function on $[0,\infty)$ and let $g^{-1}(x)$ be the inverse function of g. Then,*

$$\sum_{n=1}^{\infty} \mathsf{P}\left[|X| \geq g^{-1}(n)\right] \leq \mathsf{E}[g(X)] \leq \sum_{n=0}^{\infty} \mathsf{P}\left[|X| > g^{-1}(n)\right]. \tag{4.29}$$

In particular, for any $r>0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathsf{P}(|X| \geq n^{1/r}) &\leq \mathsf{E}(|X|^r) \leq \sum_{n=0}^{\infty} \mathsf{P}(|X| > n^{1/r}) \\ &\leq 1+\sum_{n=1}^{\infty} \mathsf{P}(X > n^{1/r}). \end{aligned} \tag{4.30}$$

4.8 Corollary MEAN-TAIL AREA INEQUALITIES. *If X is a positive random variable,*

$$\sum_{n=1}^{\infty} \mathsf{P}(X \geq n) \leq \mathsf{E}(X) \leq 1+\sum_{n=1}^{\infty} \mathsf{P}(X > n). \tag{4.31}$$

5. Moments of sums of random variables

In this section, we consider a sequence $X_1,\ldots,X_n$ of random variables, and study the moments of the corresponding sum and average:

$$S_n = \sum_{i=1}^{n} X_i, \quad \bar{X}_n = S_n/n. \tag{5.1}$$

5.1 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES.

$$\begin{aligned} \mathsf{E}(|S_n|^r) &\leq \sum_{i=1}^{n} \mathsf{E}(|X_i|^r), && \text{if } 0<r\leq 1, \\ &\leq n^{r-1}\sum_{i=1}^{n} \mathsf{E}(|X_i|^r), && \text{if } r>1, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \mathsf{E}(|\bar{X}_n|^r) &\leq \left(\tfrac{1}{n}\right)^r \sum_{i=1}^{n} \mathsf{E}(|X_i|^r), && \text{if } 0<r\leq 1, \\ &\leq \tfrac{1}{n}\sum_{i=1}^{n} \mathsf{E}(|X_i|^r), && \text{if } r>1. \end{aligned} \tag{5.3}$$

5.2 Proposition MINKOWSKI INEQUALITY FOR n VARIABLES. *If $r \geq 1$, then*

$$[\mathsf{E}(|S_n|^r)]^{1/r} \leq \sum_{i=1}^{n} [\mathsf{E}(|X_i|^r)]^{1/r} \tag{5.4}$$

and

$$\begin{aligned} [\mathsf{E}(|\bar{X}_n|^r)]^{1/r} &\leq \frac{1}{n}\sum_{i=1}^{n} [\mathsf{E}(|X_i|^r)]^{1/r} \\ &\leq \left\{\frac{1}{n}\sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\right\}^{1/r}. \end{aligned} \tag{5.5}$$

5.3 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER CONDITIONAL SYMMETRY. *If the distribution of X_{k+1} given S_i is symmetric about zero for $k = 1, \ldots, n-1$, and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, then*

$$\mathsf{E}(|S_n|^r) \leq \sum_{i=1}^{n} \mathsf{E}(|X_i|^r) \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.6}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \leq \left(\frac{1}{n}\right)^r \sum_{i=1}^{n} \mathsf{E}(|X_i|^r) \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.7}$$

with equality holding when $r = 2$.

5.4 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES UNDER MARTINGALE CONDITION. *If*

$$\mathsf{E}(X_{k+1} \,|\, S_k) = 0 \quad \textit{a.s.,} \quad k = 1, \ldots, n-1, \tag{5.8}$$

and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \ldots, n$, then

$$\mathsf{E}(|S_n|^r) \leq 2\sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \leq r \leq 2\,, \tag{5.9}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \leq 2\left(\frac{1}{n}\right)^r \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \leq r \leq 2\,. \tag{5.10}$$

Furthermore, for $r = 2$,

$$\mathsf{E}({S_n}^2) = \sum_{i=1}^{n} \mathsf{E}(X_i^2)\,. \tag{5.11}$$

5.5 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF RANDOM VARIABLES

UNDER TWO-SIDED MARTINGALE CONDITION. *Let*

$$S_{m(k)} = \sum_{i=1,\ i\neq k}^{m+1} X_i\,, \quad 1 \le k \le m+1 \le n. \tag{5.12}$$

If

$$\mathsf{E}(X_k \,|\, S_{m(k)}) = 0 \quad \textit{a.s.,} \quad \textit{for } 1 \le k \le m+1 \le n, \tag{5.13}$$

and $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \dots, n$, *then*

$$\mathsf{E}(|S_n|^r) \le \left(2 - \frac{1}{n}\right) \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \le r \le 2\,, \tag{5.14}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \le \left(\frac{1}{n}\right)^r \left(2 - \frac{1}{n}\right) \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \le r \le 2\,. \tag{5.15}$$

5.6 Proposition BOUNDS ON THE ABSOLUTE MOMENTS OF A SUM OF INDEPENDENT RANDOM VARIABLES. *Let the random variables* $X_1, \dots, X_n$ *be independent with* $\mathsf{E}(X_i) = 0$ *and* $\mathsf{E}(|X_i|^r) < \infty$, $i = 1, \dots, n$, *and let*

$$D(r) = [13.52/(2.6\pi)^r]\, \Gamma(r) \sin(r\pi/2)\,. \tag{5.16}$$

If $D(r) < 1$ *and* $1 \le r \le 2$, *then*

$$\mathsf{E}(|S_n|^r) \le [1 - D(r)]^{-1} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \tag{5.17}$$

and

$$\mathsf{E}(|\bar{X}_n|^r) \le \left(\frac{1}{n}\right)^r [1 - D(r)]^{-1} \sum_{i=1}^{n} \mathsf{E}(|X_i|^r)\,, \quad \textit{for } 1 \le r \le 2\,. \tag{5.18}$$

6. Proofs

1.1 to **3.2**. See Loève (1977, Volume I, Sections 9.1 and 9.3, pp. 151-162). For Jensen inequality, see also Chow and Teicher (1988, Section 4.3, pp. 103-106). Hannan (1985), Lehmann and Shaffer (1988), Piegorsch and Casella (1988) and Khuri and Casella (2002) discussed conditions for the existence of the moments of $1/X$.
2.9. See von Bahr and Esseen (1965, Lemma 3).
PROOF OF THEOREM **3.3** (a) For $x \geq a \geq A_U$, we have $g(x) \geq g(a)$ and $g(x) \leq M_U$, hence

$$C_U(g, a) = \int_{[a,\infty)} g(x)\, dF_X(x) \geq g(a) \int_{[a,\infty)} dF_X(x) = g(a) P[X \geq a]$$

and

$$\int_{[a,\infty)} g(x)\, dF_X(x) \leq M_U P[X \geq a] \,,$$

from which we get the inequality

$$\frac{C_U(g, a)}{M_U} \leq P[X \geq a] \leq \frac{C_U(g, a)}{g(a)} \,.$$

(b) For $x \leq a \leq A_L$, we have $g(x) \geq g(a)$ and $g(x) \leq M_L$, hence

$$C_L(g, a) = \int_{[-\infty,a)} g(x)\, dF_X(x) \geq g(a) \int_{[-\infty,a)} dF_X(x) = g(a) P[X \leq a]$$

and

$$\int_{[-\infty,a)} g(x)\, dF_X(x) \leq M_L P[X \leq a]$$

from which we get the inequality

$$\frac{C_L(g, a)}{M_L} \leq P[X \leq a] \leq \frac{C_L(g, a)}{g(a)} \,.$$

(c) For $a \geq \max(|A_U|, |A_L|)$, we have $a \geq A_U$ and $-a \leq A_L$, hence

$$\begin{aligned} P[|X| \geq a] &= P[X \geq a] + P[X \leq -a] \\ &\leq \frac{C_U(g, a)}{g(a)} + \frac{C_L(g, a)}{g(a)} \\ &\leq \frac{C_U(g, a) + C_L(g, a)}{\min\{g(a), g(-a)\}} \end{aligned}$$

and

$$
\begin{aligned}
P\left[|X| \geq a\right] &\geq \frac{C_U\left(g,a\right)}{M_U} + \frac{C_L\left(g,a\right)}{M_L} \\
&\geq \frac{C_U\left(g,a\right) + C_L\left(g,a\right)}{\max\left(M_U, M_L\right)}.
\end{aligned}
$$

□

PROOF OF LEMMA **4.1** The first identity in (4.1) part is given by Devinatz (1968, Theorem 5.4.8, page 213) and Protter and Morrey (1991, Theorem 12.12, page 320), while the second follows from the latter on observing that

$$
\begin{aligned}
-\int_a^b f(x)\, dg(x) &= \int_a^b [A - f(x) - A]\, dg(x) = \int_a^b [A - f(x)]\, dg(x) - A\int_a^b dg(x) \\
&= \int_a^b [A - f(x)]\, dg(x) - A\left[g(b) - g(a)\right] \qquad (6.1)
\end{aligned}
$$

and rearranging the terms of the sum. Equation (6.1) also entails the existence of the integral $\int_a^b [1 - f(x)]\, dg(x)$. The identities (4.2)-(4.3) follow on observing that we can write $dg(x) = g'(x)dx$ when g is differentiable [see Devinatz (1968, Theorem 5.4.7, page 213)]. □

PROOF OF LEMMA **4.2** Using Lemma **4.2**, we get:

$$
\begin{aligned}
\int_a^b g(x)\, df(x) &= \int_a^c g(x)\, df(x) + \int_c^b g(x)\, df(x) \\
&= g(c)f(c) - g(a)f(a) - \int_a^c f(x)\, dg(x) \\
&\quad + [A - f(c)]g(c) - g(b)[A - f(b)] + \int_c^b [A - f(x)]\, dg(x) \\
&= Ag(c) - \{g(b)[A - f(b)] + g(a)f(a)\} \\
&\quad + \int_c^b [A - f(x)]\, dg(x) - \int_a^c f(x)\, dg(x). \qquad (6.2)
\end{aligned}
$$

□

PROOF OF LEMMA **4.3** (a) The existence of the limit $\lim_{x \to -\infty} f(x)$ entails that the integral $\int_{-\infty}^a df(x) = f(a) - f(-\infty)$ also exists. Since $f(x)$ is monotonic nondecreasing on the interval $(-\infty, m)$ and $\int_{-\infty}^m B_L(x)\, df(x)$ exists, we get from (4.7): for $a < m$,

$$
0 \leq \int_{-\infty}^a |g(a)|\, df(x) \leq \int_{-\infty}^a B_L(x)\, df(x) \qquad (6.3)
$$

hence

$$0 \leq |g(a)|\,[f(a) - f(-\infty)] \leq \int_{-\infty}^{a} B_L(x)\,d\,f(x). \tag{6.4}$$

Letting $a \to -\infty$, this yields

$$0 \leq \lim_{a\to-\infty} |g(a)|\,[f(a) - f(-\infty)] \leq \lim_{a\to-\infty} \int_{-\infty}^{a} B_L(x)\,d\,f(x) = 0 \tag{6.5}$$

and

$$\lim_{a\to\infty} g(a)[\,f(a) - f(-\infty)] = 0\,. \tag{6.6}$$

(b) The existence of the limit $\lim_{x\to\infty} f(x)$ entails that the integral $\int_b^\infty d\,f(x) = f(\infty) - f(b)$ also exists. Since $f(x)$ is monotonic nondecreasing on the interval (M, ∞) and $\int_M^\infty B_U(x)\,d\,f(x)$ exists, we get from (4.10): for $b > M$,

$$0 \leq \int_b^\infty |g(b)|\,d\,f(x) \leq \int_b^\infty B_U(x)\,d\,f(x) \tag{6.7}$$

hence

$$0 \leq |g(b)|\,[\,f(\infty) - f(b)] \leq \int_b^\infty B_U(x)\,d\,f(x)\,. \tag{6.8}$$

Letting $b \to \infty$, this yields

$$0 \leq \lim_{b\to\infty} |g(b)|\,[\,f(\infty) - f(b)] \leq \lim_{b\to\infty} \int_b^\infty B_U(x)\,d\,f(x) = 0 \tag{6.9}$$

and

$$\lim_{b\to\infty} g(b)\,[\,f(\infty) - f(b)] = 0\,. \tag{6.10}$$

□

4.4 to **4.6**. See Feller (1966, Section V.6, Lemma 1), Chung (1974, Section 3.2, Exercises 17-18), Serfling (1980, Section 1.14, pp. 46-47) and Chow and Teicher (1988, Section 4.3, pp. 103-106). For other inequalities involving absolute moments, the reader may consult Beesack (1984).

4.7. See Chow and Teicher (1988, Section 4.1, Corollary 3, p. 90).

4.8. The inequality (4.31) is given by Chung (1974, Theorem 3.2.1) and Serfling (1980, Section 1.3, p. 12).

5.1. See von Bahr and Esseen (1965), Chung (1974, p. 48) and Chow and Teicher (1988, p. 108).

5.2. See Chung (1974, p. 48).

PROOF OF PROPOSITION **5.2** The first inequality follows by recursion on applying the Minkowski inequality for two variables. The first part of the second inequality is obtained by multiplying both sides of the first one by $(1/n)$. The second part follows on observing that the function $x^{1/r}$ is concave in x for $x > 0$ when $r > 1$. □

5.3. See von Bahr and Esseen (1965, Theorem 1).

5.4. See von Bahr and Esseen (1965, Theorem 2).

5.5. See von Bahr and Esseen (1965, Theorem 3).

5.6. See von Bahr and Esseen (1965, Theorem 4).

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