Distribution and quantile functions *

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1. Monotone functions

1.1 Definition MONOTONE FUNCTION. *Let D a non-empty subset of $\mathbb{R}$, $f : D \to E$, where E is a non-empty subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and let I be a non-empty subset of D.*

(a) *f is nondecreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2), \quad \forall x_1, x_2 \in I \ .$$

(b) *f is nonincreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2), \quad \forall x_1, x_2 \in I \ .$$

(c) *f is strictly increasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \forall x_1, x_2 \in I \ .$$

(d) *f is strictly decreasing on I iff*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \quad \forall x_1, x_2 \in I \ .$$

(e) *f is monotone on I iff f is nondecreasing, nonincreasing, increasing or decreasing.*

(f) *f is strictly monotone on I iff f is strictly increasing or decreasing.*

1.2 Remark It is clear that:

(a) an increasing function is also nondecreasing;

(b) a decreasing function is also nonincreasing;

(c) if f is nondecreasing (alt., strictly increasing), the function

$$g(x) = -f(x)$$

is nonincreasing (alt., strictly decreasing) on I, and the function

$$h(x) = -f(-x)$$

is nondecreasing on $I_1 = \{x : -x \in I\}$..

1.3 Theorem CONTINUITY OF MONOTONE FUNCTIONS. *Let $I = (a,\, b) \subseteq \mathbb{R}$, where $-\infty \leq a < b \leq \infty$, and $f : I \to \mathbb{R}$ be a nondecreasing function on I. Then the function f has the following properties.*

(a) *For each $x \in (a,\, b)$, set*

$$\begin{aligned} f(x_+) &= \lim_{\delta \downarrow 0} \left\{ \inf_{x<y<x+\delta} f(y) \right\} \ , \ f\left(x^+\right) = \lim_{\delta \downarrow 0} \left\{ \sup_{x<y<x+\delta} f(y) \right\} , \\ f(x_-) &= \lim_{\delta \downarrow 0} \left\{ \inf_{y-\delta<x<y} f(y) \right\} \ , \ f\left(x^-\right) = \lim_{\delta \downarrow 0} \left\{ \sup_{y-\delta<x<y} f(y) \right\} . \end{aligned}$$

Then, the four limits $f(x_+)$, $f(x^+)$, $f(x_-)$ and $f(x^-)$ are finite and, for any $\delta > 0$ such that $[x-\delta,\ x+\delta] \subseteq (a,\, b)$,

$$f(x-\delta) \leq f(x_-) \leq f\left(x^-\right) \leq f(x) \leq f(x_+) \leq f\left(x^+\right) \leq f(x+\delta) .$$

(b) *For each $x \in (a,\, b)$, we have*

$$f(x_+) = f\left(x^+\right) \ , \ f(x_-) = f\left(x^-\right) \ ,$$

and the function $f(x)$ has finite unilateral limits:

$$f(x+) \equiv \lim_{y \downarrow x} f(y) = f(x_+) = f\left(x^+\right) \ , \ f(x-) \equiv \lim_{y \uparrow x} f(y) = f(x_-) = f\left(x^-\right) \ .$$

(c) *For each $x \in (a,\, b)$,*

$$\sup_{a<y<x} f(y) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<y<b} f(y) \ .$$

(d) *If $a < x < y < b$, then*

$$f(x+) \leq f(y-) \ .$$

(e) *If $a = -\infty$, the function $f(x)$ has a limit in the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\, +\infty\}$ as $x \to -\infty$,*

$$-\infty \leq f(-\infty) \equiv \lim_{x \to -\infty} f(x) < \infty$$

and, if $b = \infty$, the function $f(x)$ has a limit in $\overline{\mathbb{R}}$ as $x \to \infty$:

$$-\infty < f(+\infty) \equiv \lim_{x \to \infty} f(x) \leq \infty .$$

(f) *For each* $x \in (a,\, b)$, f *is continuous at* x *iff*

$$f\,(x-) = f\,(x+) \ .$$

(g) *The only possible kind of discontinuity of* f *on* $(a,\, b)$ *is a jump.*

(h) *The set of points of* $(a,\, b)$ *at which* f *is discontinuous is countable (possibly empty).*

(i) *The function*

$$f_R\,(x) = f\,(x+) \ , \quad x \in (a,\, b)$$

is right continuous at every point of $(a,\, b)$, *i.e.,*

$$\lim_{y \downarrow x} f_R\,(y) = f_R\,(x) \ , \quad \forall x \in (a,\, b) \ .$$

(j) *The function*

$$f_L\,(x) = f\,(x-)$$

is left continuous at every point of $(a,\, b)$, *i.e.,*

$$\lim_{y \uparrow x} f_L\,(y) = f_L\,(x) \ , \quad \forall x \in (a,\, b) \ .$$

1.4 Theorem CHARACTERIZATION OF THE CONTINUITY OF MONOTONE FUNCTIONS. *Let* $f : D \to \mathbb{R}$ *a monotone function, where* D *is a non-empty subset of* $\mathbb{R}$ *and* I *a non-empty subset of* D. *Then*

$$f \text{ is continuous on } I \text{ iff } f(I) \text{ is an interval.}$$

1.5 Theorem MONOTONE INVERSE FUNCTION THEOREM. *Let* I *be an interval in* $\mathbb{R}$, *and* $f : I \to \mathbb{R}$. *If* f *is continuous and strictly monotone, then* $J = f(I)$ *is an interval and the function* $f : I \to J$ *is an homeomorphism* (*i.e.,* $f : I \to J$ *is a bijection such that* f *and* f^{-1} *are continuous*).

1.6 Theorem STRICT MONOTONICITY AND HOMEOMORPHISMS BETWEEN INTERVALS. *Let* I *and* J *be intervals in* $\mathbb{R}$ *and* $f : I \to J$.

(a) *If* f *is an homeomorphism, then* f *is strictly monotone.*

(b) f *is an homeomorphism* $\Leftrightarrow$ f *is continuous and strictly monotone*
$\Leftrightarrow$ $f^{-1} : J \to I$ *exists and is an homeomorphism*
$\Leftrightarrow$ $f^{-1} : J \to I$ *exists, is continuous and strictly monotone.*

1.7 Lemma CHARACTERIZATION OF RIGHT (LEFT) CONTINUOUS FUNCTIONS BY DENSE SETS. *Let* f_1 *and* f_2 *be two real-valued functions defined on the interval* $(a,\, b)$ *such that the functions* f_1 *and* f_2 *are either both right continuous or both left continuous at each point* $x \in (a,\, b)$ *, and let* D *be a dense subset of* $(a,\, b)$ *. If*

$$f_1\,(x) = f_2\,(x) \ , \quad \forall x \in D\,,$$

then

$$f_1\,(x) = f_2\,(x) \ , \quad \forall x \in (a,\, b) \ .$$

1.8 Theorem CHARACTERIZATION OF MONOTONIC FUNCTIONS BY DENSE SETS. *Let* f_1 *and* f_2 *be two monotonic nondecreasing functions on* $(a,\, b)$ *, let* D *be a dense subset of* $(a,\, b)$ *, and suppose*

$$f_1\,(x) = f_2\,(x) \ , \quad \forall x \in D\,.$$

(a) *Then* f_1 *and* f_2 *have the same points of discontinuity, they coincide everywhere in* $(a,\, b)$ *, except possibly at points of discontinuity, and*

$$f_1\,(x+) - f_1\,(x-) = f_2\,(x+) - f_2\,(x-) \ , \quad \forall x \in (a,\, b) \ .$$

(b) *If furthermore* f_1 *and* f_2 *are both left continuous (or right continuous) at every point* $x \in (a,\, b)$*, they coincide everywhere on* $(a,\, b)$ *, i.e.,*

$$f_1\,(x) = f_2\,(x) \ , \quad \forall x \in (a,\, b) \ .$$

1.9 Theorem DIFFERENTIABILITY OF MONOTONE FUNCTIONS. *Let* $I = (a,\, b) \subseteq \mathbb{R}$*, where* $-\infty \le a < b \le \infty$*, and* $f : I \to \mathbb{R}$ *be a nondecreasing function on* I*. Then* f *is differentiable almost everywhere on* I*.*

2. Generalized inverse of a monotonic function

2.1 Definition GENERALIZED INVERSE OF A NONDECREASING RIGHT-CONTINUOUS FUNCTION. *Let* f *be a real-valued, nondecreasing, right continuous function defined on the open interval* $(a,\, b)$ *where* $-\infty \le a < b \le \infty$*. Then the generalized inverse of* f *is defined by*

$$f^*(y) = \inf\{x \in (a,\, b) : f(x) \ge y\} \tag{2.1}$$

for $-\infty < y < \infty$ $($*with the convention* $\inf(\emptyset) = b)$*. Further, we define* f^{-1} *as the restriction of* f^* *to the interval* $(\inf(f)\,,\, \sup(f)) \equiv (\inf\{f(x) : x \in (a,\, b)\}, \sup\{f(x) : x \in (a,\, b)\})$ *:*

$$f^{-1}(y) = f^*(y) \quad \textit{for } \inf(f) < y < \sup(f)\ . \tag{2.2}$$

2.2 Definition GENERALIZED INVERSE OF A NONDECREASING LEFT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, left continuous function defined on the open interval $(a,\, b)$ where $-\infty \le a < b \le \infty$. Then the generalized inverse of f is defined by*

$$f^{**}(y) = \sup\{x \in (a,\, b) : f(x) \le y\} \tag{2.3}$$

for $-\infty < y < \infty$ (with the convention $\sup(\emptyset) = a$).

2.3 Proposition GENERALIZED INVERSE BASIC EQUIVALENCE (RIGHT-CONTINUOUS FUNCTION). *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval $(a,\, b)$ where $-\infty \le a < b \le \infty$. Then, for $x \in (a,\, b)$ and for every real y,*

$$y \le f(x) \Leftrightarrow f^*(y) \le x\,, \tag{2.4}$$

$$y > f(x) \Leftrightarrow f^*(y) > x\,, \tag{2.5}$$

$$f[f^*(y)] \ge y\,. \tag{2.6}$$

2.4 Proposition GENERALIZED INVERSE BASIC EQUIVALENCE (LEFT-CONTINUOUS FUNCTION). *Let f be a real-valued, nondecreasing, left continuous function defined on the open interval $(a,\, b)$ where $-\infty \le a < b \le \infty$. Then, for $x \in (a,\, b)$ and for every real y,*

$$y \le f(x) \Leftrightarrow f^{**}(y) \ge x\,. \tag{2.7}$$

2.5 Proposition CONTINUITY OF THE INVERSE OF A NONDECREASING RIGHT-CONTINUOUS FUNCTION. *Let f be a real-valued, nondecreasing, right continuous function defined on the open interval $(a,\, b)$ where $-\infty \le a < b \le \infty$, and set*

$$a(f) = \inf\{x \in (a,\, b) : f(x) > \inf(f)\}\ , \quad b(f) = \sup\{x \in (a,\, b) : f(x) < \sup(f)\}\ . \tag{2.8}$$

Then, f^* is nondecreasing and left continuous. Moreover

$$\lim_{y \to -\infty} f^*(y) = a\ , \quad \lim_{y \to \infty} f^*(y) = b \tag{2.9}$$

and

$$\lim_{y \to \inf(f)} f^{-1}(y) = a(f)\ , \quad \lim_{y \to \sup(f)} f^{-1}(y) = b(f)\ . \tag{2.10}$$

3. Distribution functions

3.1 Definition DISTRIBUTION AND SURVIVAL FUNCTIONS OF A RANDOM VARIABLE. *Let X be a real-valued random variable. The distribution function of X is the function*

$F(x)$ *defined by*

$$F(x) = \mathsf{P}[X \leq x] \tag{3.1}$$

and its survival function is the function $G(x)$ *defined by*

$$G(x) = \mathsf{P}[X \geq x] \,. \tag{3.2}$$

3.2 Proposition PROPERTIES OF DISTRIBUTION FUNCTIONS. *Let* X *be a real-valued random variable with distribution function* $F(x) = \mathsf{P}[X \leq x]$. *Then*

(a) $F(x)$ *is nondedreasing;*

(b) $F(x)$ *is right-continuous;*

(c) $F(x) \to 0$ *as* $x \to -\infty\,;$

(d) $F(x) \to 1$ *as* $x \to \infty\,;$

(e) $\mathsf{P}[X = x] = F(x) - F(x-)\,;$

(f) *for any* $x \in \mathbb{R}$ *and* $q \in (0,\, 1)$,

$$\{\, \mathsf{P}[X \leq x] \geq q \text{ and } \mathsf{P}[X \geq x] \geq 1{-}q \,\} \Longleftrightarrow \{\, \mathsf{P}[X < x] \leq q \text{ and } \mathsf{P}[X > x] \leq 1{-}q \,\}\,.$$

3.3 Proposition PROPERTIES OF SURVIVAL FUNCTIONS. *Let* X *be a real-valued random variable with distribution function* $G(x) = \mathsf{P}[X \leq x]$. *Then*

(a) $G(x)$ *is nonincreasing;*

(b) $G(x)$ *is left-continuous;*

(c) $G(x) \to 1$ *as* $x \to -\infty\,;$

(d) $G(x) \to 0$ *as* $x \to \infty\,;$

(e) $\mathsf{P}[X = x] = G(x) - G(x+)\,;$

(f) $G(x) = 1 - F(x) + \mathsf{P}\left[S = x\right]\,.$

4. Quantile functions

4.1 Definition QUANTILE FUNCTION. *Let* $F(x)$ *be a distribution function. The quantile function associated with* F *is the generalized inverse of* F*, i.e.*

$$F^{-1}(q) \equiv F^{-}(q) = \inf\{x : F(x) \geq q\}\ ,\ 0 < q < 1\ . \tag{4.1}$$

4.2 Theorem PROPERTIES OF QUANTILE FUNCTIONS. *Let $F(x)$ be a distribution function. Then the following properties hold:*

(*a*) $F^{-1}(q) = \sup\{x : F(x) < q\}\ ,\ 0 < q < 1\ ;$

(*b*) $F^{-1}(q)$ *is nondecreasing and left continuous;*

(*c*) $F(x) \geq q \Leftrightarrow x \geq F^{-1}(q)$, *for all* $x \in \mathbb{R}$ *and* $q \in (0,\, 1)$;

(*d*) $F(x) < q \Leftrightarrow x < F^{-1}(q)$, *for all* $x \in \mathbb{R}$ *and* $q \in (0,\, 1)$;

(*e*) $F[F^{-1}(q)-] \leq q \leq F[F^{-1}(q)]$, *for all* $q \in (0,\, 1)$;

(*f*) $F^{-1}[F(x)] \leq x \leq F^{-1}[F(x)+]$, *for all* $x \in \mathbb{R}$;

(*g*) *if* F *is continuous at* $x = F^{-1}(q)$, *then* $F[F^{-1}(q)] = q$;

(*h*) *if* F^{-1} *is continuous at* $q = F(x)$, *then* $F^{-1}[F(x)] = x$;

(*i*) *for* $q \in (0,\, 1)$, $F[F^{-1}(q)] = q \Leftrightarrow q \in F[\mathbb{R}]$;

(*j*) $F[F^{-1}(q)] = q$ *for all* $q \in (0,\, 1)$ $\Leftrightarrow (0,\, 1) \subseteq F[\mathbb{R}]$
$\Leftrightarrow F$ *is continuous*
$\Leftrightarrow F^{-1}$ *is strictly increasing* ;

(*k*) *for any* $x \in \mathbb{R}$, $F^{-1}[F(x)] = x \Leftrightarrow F(x - \varepsilon) < F(x)$ *for all* $\varepsilon > 0$;

(*l*) $F^{-1}[F(x)] = x$ *for all* $x \in \mathbb{R}$ $\Leftrightarrow F$ *is strictly increasing*
$\Leftrightarrow F^{-1}$ *is continuous* ;

(*m*) F *is continuous and strictly increasing* $\Leftrightarrow F^{-1}$ *is continuous and strictly increasing;*

(*n*) $F^{-1} \circ F \circ F^{-1} = F^{-1}$ *or, equivalently,*

$$F^{-1}\left(F\left[F^{-1}(q)\right]\right) = F^{-1}(q)\,,\ \text{for all } q \in (0,\, 1)\ ;$$

(*o*) $F \circ F^{-1} \circ F = F$ *or, equivalently,*

$$F\left(F^{-1}\left[F(x)\right]\right) = F(x)\,,\ \text{for all } x \in \mathbb{R}\ .$$

4.3 Theorem CHARACTERIZATION OF DISTRIBUTIONS BY QUANTILE FUNCTIONS. *If $G(x)$ is a real-valued nondecreasing left continuous function with domain $(0,\, 1)$, there is a unique distribution function F such that $G = F^{-1}$.*

4.4 Theorem DIFFERENTIATION OF QUANTILE FUNCTIONS. *Let $F(x)$ be a distribution function. If F has a positive continuous $f(x)$ density f in a neighborhood of $F^{-1}(q_0)$, $where$ $0 < q_0 < 1$, then the derivative $dF^{-1}(q)/dq$ exists at $q = q_0$ and*

$$\left.\frac{dF^{-1}(q)}{dq}\right|_{q_0} = \frac{1}{f\left(F^{-1}(q_0)\right)} \,. \tag{4.2}$$

4.5 Proposition *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \le x]$ and survival function $G(x) = \mathsf{P}[X \ge x]$. Then, for any $q \in (0,\, 1)$,*

(*a*) $\mathsf{P}[X \le F^{-1}(q)] \ge q$ *and* $\mathsf{P}[X \ge F^{-1}(q)] \ge 1 - q$;

(*b*) $\mathsf{P}[X < F^{-1}(q)] \le q$ *and* $\mathsf{P}[X > F^{-1}(q)] \le 1 - q$.

5. Distributions, quantiles and transformations of random variables

5.1 Notation *X is a random variable with distribution function $F_X(x) = \mathsf{P}[X \le x]$. $U(0,\, 1)$ a uniform random variable on the interval $(0,\, 1)$.*

5.2 Definition QUANTILE OF RANDOM VARIABLE. *A* quantile *of order q of the random variable X is any number m_q such that $\mathsf{P}\left(X \le m_q\right) \ge q$ and $\mathsf{P}\left(X \ge m_q\right) \ge 1 - q$, where $0 \le q \le 1$. In particular, $m_{0.5}$ is a* median *of X, while $m_{0.25}$ and $m_{0.75}$ are lower and upper* quartiles *of X.*

5.3 Theorem PROPERTIES OF QUANTILE TRANSFORMATION. *Let $F(x)$ be a distribution function, and U a random variable with distribution $D(x)$ such that $D(0) = 0$ and $D(1) = 1$. If $X = F^{-1}(U)$, then, for all $x \in \mathbb{R}$,*

$$X \le x \Leftrightarrow F^{-1}(U) \le x \Leftrightarrow U \le F(x) \tag{5.1}$$

or, equivalently,

$$\mathbf{1}\{X \le x\} = 1\{F^{-1}(U) \le x\} = \mathbf{1}\{U \le F(x)\} \,, \tag{5.2}$$

and

$$\mathsf{P}[X \le x] = \mathsf{P}[F^{-1}(U) \le x] = \mathsf{P}[U \le F(x)] = D\left(F(x)\right) \;; \tag{5.3}$$

further,

$$\mathbf{1}\{X < x\} = 1\{F^{-1}(U) < x\} = \mathbf{1}\{U \le F(x-)\} \text{ with probability } 1 \tag{5.4}$$

and
$$\mathsf{P}[X < x] = \mathsf{P}[F^{-1}(U) < x] = \mathsf{P}[U \le F(x-)]\ . \tag{5.5}$$
In particular, if U follows a uniform distribution on the interval $(0,\, 1)$, i.e. $U \sim U(0,\, 1)$, the distribution function of X is F :
$$\mathsf{P}[X \le x] = \mathsf{P}[U \le F(x)] = F(x)\ . \tag{5.6}$$

5.4 Theorem PROPERTIES OF DISTRIBUTION TRANSFORMATION. *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \le x]$. Then the following properties hold:*

(*a*) $\mathsf{P}[F(X) \le u] \le u$, *for all* $u \in [0,\, 1]$;

(*b*) $\mathsf{P}[F(X) \le u] = u \Leftrightarrow u \in \mathrm{cl}\{F(\mathbb{R})\}$,
where $\mathrm{cl}\{F(\mathbb{R})\}$ *is the closure of the range of* F;

(*c*) $\mathsf{P}[F(X) \le F(x)] = \mathsf{P}[X \le x] = F(x)$, *for all* $x \in \mathbb{R}$;

(*d*) $F(X) \sim U(0,\, 1) \Leftrightarrow F$ *is continuous;*

(*e*) *for all* x, $\mathbf{1}\{F(X) \le F(x)\} = 1\{X \le x\}$ *with probability* 1 ;

(*f*) $F^{-1}\left(F\left(X\right)\right) = X$ *with probability* 1.

5.5 Theorem QUANTILES AND P-VALUES. *Let X be a real-valued random variable with distribution function $F(x) = \mathsf{P}[X \le x]$ and survival function $G(x) = \mathsf{P}[X \ge x]$. Then, for any $x \in \mathbb{R}$,*
$$\begin{aligned} G(x) &= \mathsf{P}[G(X) \ge G(x)] \\ &= \mathsf{P}\big[X \ge F^{-1}\big((F\left(x\right) - p_F(x))^+\big)\big] \\ &= \mathsf{P}\big[X \ge F^{-1}\big((1 - G\left(x\right))^+\big)\big] \end{aligned} \tag{5.7}$$
where $p_F(x) = \mathsf{P}[X = x] = F(x) - F(x-)$.

5.6 Theorem QUANTILES OF TRANSFORMED RANDOM VARIABLES. *Let X be a real-valued random variable with distribution function $F_X(x) = \mathsf{P}[X \le x]$. If $g(x)$, $x \in \mathbb{R}$, is a nondecreasing left continuous function, then*
$$F_{g(X)}^{-1} = g\left(F_X^{-1}\right) \tag{5.8}$$
where $F_{g(X)}(x) = \mathsf{P}[g(X) \le x]$ and $F_{g(X)}^{-1}(q) = \inf\{x : F_{g(X)}(x) \ge q\}\ ,\ 0 < q < 1$.

6. Multivariate generalizations

6.1 Notation CONDITIONAL DISTRIBUTION FUNCTIONS. *Let* $X = (X_1, \ldots, X_k)'$ *a* $k \times 1$ *random vector in* $\mathbb{R}^k$. *Then we denote as follows the following set of conditional distribution functions:*

$$\begin{aligned} F_{1|\cdot}(x_1) &= F_1(x_1) = \mathsf{P}[X_1 \le x_1] \, , \\ F_{2|\cdot}(x_2|x_1) &= \mathsf{P}[X_2 \le x_2 \,|\, X_1 = x_1] \, , \\ &\vdots \\ F_{k|\cdot}(x_k \,|\, x_1, \ldots, x_{k-1}) &= \mathsf{P}[X_k \le x_k \,|\, X_1 = x_1, \ldots, X_{k-1} = x_{k-1}] \, . \end{aligned} \tag{6.1}$$

Further, we define the following transformations of $X_1, \ldots, X_k$:

$$\begin{aligned} Z_1 &= F_1(X_1) \, , \\ Z_2 &= F_{2|\cdot}(X_2 \,|\, X_1) \, , \\ &\vdots \\ Z_k &= F_{k|\cdot}(X_k \,|\, X_1, \ldots, X_{k-1}) \, . \end{aligned} \tag{6.2}$$

6.2 Theorem TRANSFORMATION TO $i.i.d.$ $U(0,1)$ VARIABLES (ROSENBLATT). *Let* $X = (X_1, \ldots, X_k)'$ *be a* $k \times 1$ *random vector in* $\mathbb{R}^k$ *with an absolutely continuous distribution function* $F(x_1, \ldots, x_k) = \mathsf{P}[X_1 \le x_1, \ldots, X_k \le x_k]$. *Then the random variables* $Z_1, \ldots, Z_k$ *are independent and identically distributed according to a* $U(0, 1)$ *distribution.*

7. Proofs and additional references

1.3 Rudin (1976), Chapter 4, pp. 95-97, and Chung (1974), Section 1.1. For (a)-(b), see Phillips (1984), Sections 9.1 (p. 243) and 9.3 (p. 253).

1.4 - 1.6 Ramis, Deschamps, and Odoux (1982), Section 4.3.2, p.121.

1.7 Chung (1974), Section 1.1, p. 4.

1.9 Haaser and Sullivan (1991), Section 9.3; Riesz and Sz.-Nagy (1955/1990), Chapter 1.

?? (**??**) and (**??**) are given by Gleser (1985, Lemma 1, p. 957).

2.3 (2.4) is proved by Reiss (1989, Appendix 1, Lemma A.1.1). (2.5) and (2.6) are also given by Gleser (1985, Lemma 1, p. 957).

2.4 Reiss (1989), Appendix 1, Lemma A.1.3.

2.5 Reiss (1989), Appendix 1, Lemma A.1.2.

3.2 (f) Lehmann and Casella (1998), Problem 1.7 (for the case $q = 1/2)$.

4.2 (a) Williams (1991), Section 3.12 (p. 34).

5.4 (a)-(b) Shorack and Wellner (1986), Chapter 1, Proposition 2.

5.6 See Parzen (1980) and Shorack and Wellner (1986, page 9, Exercise 3).

6.2 See Rosenblatt (1952).

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