Jean-Marie Dufour
March 7, 2014

TIME SERIES ANALYSIS
EXERCISES
STOCHASTIC PROCESSES 3

1. Let $T \subseteq \mathbb{R}$. Which condition must satisfy a family of joint distribution functions $F(x_1, , \ldots , x_n; t_1, , \ldots , t_n)$ defined for all finite subsets $\{t_1, \ldots , t_n\} \subseteq T$, where $n \geq 1$, in order for these distributions to be the distributions of a stochastic process? [See Brockwell and Davis (1991, Theorem 1.2.1).]

2. Show that any even and positive-definite function $\gamma : \mathbb{Z} \to \mathbb{R}$ is the autocovariance function of a Gaussian stationary process. [See Brockwell and Davis (1991, Theorem 1.5.1).]

3. Consider the function $\gamma : \mathbb{Z} \to \mathbb{R}$ defined by

$$\begin{aligned} \gamma(k) &= 1 \ , \text{ if } k = 0 \\ &= \rho \ , \text{ if } |k| = 1 \\ &= 0 \ , \text{ otherwise.} \end{aligned}$$

   Show this function is an autocovariance function if and only if $|\rho| \leq 0.5$. [See Brockwell and Davis (1991, Example 1.5.1).]

4. Let $(Z_t : t \in \mathbb{Z})$ be $i.i.d.$ $N[0,\ \sigma^2]$ random variables, and let a, b, and c be real constants. Determine which ones of the following processes are second-order stationary. For each stationary process, determine its mean and autocovariance function.

   (a) $X_t = a + b\, Z_t + c\, Z_{t-1}$
   (b) $X_t = a + b\, Z_0$
   (c) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$
   (d) $X_t = Z_0 \cos(ct)$
   (e) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$
   (f) $X_t = Z_t Z_{t-1}$

5. Let $(Y_t : t \in \mathbb{Z})$ be a second-order stationary process such that $E(Y_t) = 0$, $\forall t$, and let a and b be real constants.

(a) If $X_t = a + bt + s_t + Y_t$, where s_t is a periodic function with period 12, show that the process

$$Z_t = (1-B)(1-B^{12})X_t$$

is second-order stationary.

(b) If $X_t = (a+bt)s_t + Y_t$, where s_t is a periodic function with period 12, show that the process

$$Z_t = (1-B^{12})(1-B^{12})X_t$$

is second-order stationary.

6. Let $(X_t : t \in \mathbb{Z})$ and $(Y_t : t \in \mathbb{Z})$ be second-order stationary processes uncorrelated with each other, i.e. such that $Cov(X_s, Y_t) = 0,\ \forall s, t$.

   (a) Show that the process $X_t + Y_t$ is second-order stationary

   (b) Find the autocovariance function of $X_t + Y_t$.

7. Consider the process

$$S_t = \mu + S_{t-1} + u_t\ ,\ t = 1, 2, \ldots$$

   where $S_0 = 0$ and $u_1,\ u_2, \ldots$ are $i.i.d.$ random variables with mean zero and variance σ^2.

   (a) Determine the mean and covariance function of the process S_t. Is this process strictly stationary? second-order stationary?

   (b) Show that the process $Y_t = (1-B)S_t$, $t = 1, 2, \ldots$ is strictly stationary. Compute its mean and autocovariance function.

References

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods*. Springer-Verlag, New York, second edn.