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Simple exact bounds for distributions of linear signed rank statistics*

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Abstract: Three types of simple bounds on the tail areas of arbitrary linear signed rank statistics are proposed: exponential (Bernstein-type) bounds, Chebyshev bounds and Berry–Esséen–Zolotarev bounds. All of them are valid for arbitrary sample size, explicit and readily computable, hence fully operational in testing or confidence estimation problems. No restriction whatever is made on score functions, which is of particular interest when 'regression constants' are involved, or when dealing with ties or discrete data. Unlike signed rank tests relying on approximate critical values derived from asymptotic normal distributions or asymptotic expansions, the tests based on the bounds proposed here are exact, in the sense that they never reject too often under the null hypothesis.

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1. Introduction

A large number of testing procedures in nonparametric statistics are based on linear signed rank statistics of the form

$$\bar{S} = \sum_{t=1}^{n} s(X_t)\, a_n(R_t^+, t), \tag{1.1}$$

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where $X_1, X_2, \ldots, X_n$ is a sample of real random variables,

$$s(x) = \begin{cases} -1 & \text{if } x<0, \\ 0 & \text{if } x=0, \\ 1 & \text{if } x>0, \end{cases}$$

is the sign of x, $a_n(r,t)$ is a function of n, $r \in \{1,2,\ldots,n\}$ and t (score function), and, letting $u(x)=1$ or 0 according as $x \geqslant 0$ or $x<0$,

$$R_t^+ = \sum_{i=1}^{n} u(|X_t| - |X_i|) \tag{1.2}$$

is the rank of $|X_t|$ among $|X_1|, \ldots, |X_n|$. For example, if we set $X_t = Y_t - \mu_0$ and $a_n(r,t) = a_n(r)$, $\bar{S}$ can be used to test whether independent symmetrically distributed observations $Y_1, \ldots, Y_n$ have a given median μ_0: see Hájek and Šidák (1967, Ch. II.4.9), Hollander and Wolfe (1973) or Hušková (1984). This includes, in particular, comparisons of paired samples; see Hájek (1969, pp. 109–112) and Pratt and Gibbons (1981, Ch. 3). Other important problems include testing against simple regression alternatives, in which case $a_n(r,t) = c_t a_n(r)$ and $c_1, \ldots, c_n$ are arbitrary (known) regression constants (Puri and Sen (1985), Ch. 3), and testing against trend or serial dependence (Dufour (1981), Hallin, Laforet and Mélard (1989), Hallin and Puri (1991)).

Under the null hypothesis, the variables $X_1, \ldots, X_n$ are usually independent with continuous (not necessarily identical) distributions symmetric about zero. More generally, the distribution of $\boldsymbol{X} = (X_1, \ldots, X_n)'$ can be assumed to be *orthant symmetric* (Efron (1969)), i.e. $(\delta_1 X_1, \ldots, \delta_n X_n)'$ is distributed like $(X_1, \ldots, X_n)'$ for all $\delta_i = \pm 1$, $i = 1, \ldots, n$. When the distribution of the vector $\boldsymbol{X}$ is continuous and orthant symmetric, the distribution of a linear signed rank statistic is uniquely determined, irrespective of the other characteristics of the distribution, thus allowing for genuinely distribution-free testing procedures. However, except for a few special cases, such as the sign test $[a_n(r,t)=1]$ or the Wilcoxon test $[a_n(r,t)=r]$, for which relatively simple formulae are available, the null distribution of signed rank statistics has to be obtained either by relying on approximations (mostly, normal approximations) or by using computationally expensive algorithms.

The asymptotic distribution of linear signed rank statistics has been intensively studied; see Hájek and Šidák (1967), Hušková (1970, 1984), Koul and Staudte (1972), Albers, Bickel and van Zwet (1976), Puri and Ralescu (1982, 1984), and Puri and Sen (1985, Ch. 3). Convergence to normality requires restrictive regularity conditions on the scores (even under the null hypothesis) and may perform well in specific cases (like the Wilcoxon signed statistic with $n>50$). For general score functions and given value of n, however, it is clear that the normal approximation can be highly inaccurate. Further, even though rates of convergence to normality have been extensively investigated (see Puri and Seoh (1984a,b,c, 1985), Puri and Wu (1986), Ralescu and Puri (1985), Seoh and Puri (1985), Seoh, Ralescu and Puri (1985), Thompson, Govindarajulu and Doksum (1967), and Wu (1987)), no readily operational bound on the difference betweeen the distribution of a general linear signed rank statistic and the normal distribution is available in the literature. Similarly, for approximations based on asymptotic expansions, like Edgeworth expansions (see Albers, Bickel and van Zwet (1976), Fellingham and Stoker (1964), Puri and Seoh (1984a, b) and Thompson, Govindarajulu and Doksum (1967)), no such bound is available in the case of general linear signed rank statistics, and there is no guarantee that 'corrected' critical values obtained from such expansions do yield tests that do not reject too often. The exact null distribution of a general linear signed rank statistic (at least conditional on the vector of ranks) can be determined by enumeration or by using the inversion algorithm proposed by Pagano and Tritchler (1983). Since the number of cases to consider grows as fast as 2^n, it is clear that the first method rapidly becomes impractical, even with a high-speed computer. The second method is considerably more efficient than enumeration but nevertheless requires the use of finite Fourier transforms and remains computationally expensive even for samples of moderate size ($n>30$, say).

In this paper, we study an alternative strategy which consists in bounding the tail areas of linear signed rank statistics under the null hypothesis. For any such statistic, we derive simple exact bounds on the conditional tail areas $P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+]$, where $\boldsymbol{R}^+=(R_1^+,\dots,R_n^+)'$ and $\sigma_s^2=\sum_{t=1}^n [a_n(R_t^+,t)]^2$. By 'exact' bounds, we mean here bounds valid for any sample size. When $a_n(r,t)=a_n(r)$ and the observations have continuous distributions, this yields bounds on $P[\bar{S}/\sigma_s \geqslant x]$. These bounds provide conservative critical regions (or critical values) for tests based on linear signed rank statistics. No restriction is imposed on the form of the scores. Because of their generality and simplicity, the bounds we are proposing are easy to use for practical work and may advantageously replace or complement other methods that are either approximate or more difficult to implement. Further, they lead to tests that cannot reject too often under the null hypothesis that $(X_1,\dots,X_n)'$ follows a continuous orthant symmetric distribution. The latter property holds both conditionally on $\boldsymbol{R}^+$ and (consequently) unconditionally.

Three types of upper bounds on tail areas are considered below: exponential (Bernstein-type) bounds, Chebyshev bounds and Berry–Esséen–Zolotarev bounds. We give four bounds of the exponential type. Even though these four bounds are uniformly ordered, so that one of them is uniformly tightest, the four of them still are of practical interest, because the larger ones are simpler to compute. Exponential bounds are especially useful for large values of x. To obtain Chebyshev-type bounds, we consider in particular inequalities based on moments of order greater than 2. For this purpose, we derive the exact even moments (up to order 12) of arbitrary linear combinations of independent symmetric Bernoulli variables. We observe that Chebyshev bounds based on higher-order moments can be considerably tighter than those resulting from the familiar second-order inequality. We also show that normal and binomial moments yield valid (though looser) upper bounds in this context. Chebyshev bounds are especially useful for intermediate values of x ($1.5 \leqslant x \leqslant 3.0$, say). The Berry–Esséen–Zolotarev bounds are based on an estimate of the maximal difference between the distribution function of $\bar{S}/\sigma_s$ (conditional on $\boldsymbol{R}^+$) and the standard normal distribution. This estimate follows by combining results of van Beek (1972) and Zolotarev (1965) on the distance between the distribution of an arbitrary linear combination of independent symmetric Bernoulli variables and the N(0, 1) distribution. The resulting bounds are both exact and easy to compute. These features strongly contrast with previous work on Berry–Esséen bounds for linear signed rank statistics, where emphasis is put on *rates of convergence* (Puri and Seoh (1984a,b,c, 1985), Puri and Wu (1986), Ralescu and Puri (1985), Seoh (1990), Seoh and Puri (1985), Seoh, Ralescu and Puri (1985), Wu (1987)). The Berry–Esséen–Zolotarev upper bound on tail areas is especially useful for smaller values of x. The same results also yield lower bounds on tail areas of linear combinations of independent symmetric Bernoulli variables, which can be used to build liberal critical regions for tests based on linear signed rank statistics. Finally, numerical comparisons between the various bounds we are proposing are presented for five different scores, including the Wilcoxon, Cauchy and van der Waerden scores as well as scores for tests against simple regression alternatives. It is found that, by taking the best of the bounds suggested, one can get a remarkably tight and useful bound. Further, none of the three types of bounds considered (exponential, Chebyshev, Berry–Esséen–Zolotarev) uniformly dominates the others: depending on the value of x, any of these three types can yield the best bound.

The exponential, Chebyshev and Berry–Esséen–Zolotarev bounds are described in Sections 2, 3 and 4 respectively. Section 5 discusses how they can be used to build conservative and liberal critical regions for tests based on linear signed rank statistics. Some numerical comparisons between the bounds are presented in Section 6.

2. Exponential bounds

In this section, we give exponential upper bounds on the tail areas of linear signed rank statistics. They are based on the following proposition.

Proposition 1. *Let $T=\sum_{t=1}^{n} w_t S_t$, where $S_1,\ldots,S_n$ are independent random variables such that $P[S_t=-1]=P[S_t=1]=\frac{1}{2}$, $t=1,\ldots,n$, and where $w_1,\ldots,w_n$ denote fixed real numbers such that $\sum_{t=1}^{n} w_t^2=1$. Then, for all $x>0$,*

$$P[T\geqslant x]\leqslant \inf_{z\geqslant 0}\left[\mathrm{e}^{-zx}\prod_{t=1}^{n}\cosh(w_t z)\right]$$

$$\leqslant \mathrm{e}^{-x^2}\prod_{t=1}^{n}\cosh(w_t x)\leqslant \mathrm{e}^{-x^2}[\cosh(x/\sqrt{n})]^n<\mathrm{e}^{-x^2/2}, \tag{2.1}$$

where $\cosh(x)=(\mathrm{e}^x+\mathrm{e}^{-x})/2$. *Further, for $x>0$,*

$$\inf_{z\geqslant 0}\left[\mathrm{e}^{-zx}\prod_{t=1}^{n}\cosh(w_t z)\right] = \begin{cases} 0 & \textit{if } \sum_{t=1}^{n}|w_t|<x, \\ (\tfrac{1}{2})^{n^*} & \textit{if } \sum_{t=1}^{n}|w_t|=x, \\ \mathrm{e}^{-z^*x}\prod_{t=1}^{n}\cosh(w_t z^*) & \textit{if } \sum_{t=1}^{n}|w_t|>x, \end{cases} \tag{2.2}$$

where $n^*=\mathrm{card}(\{w_t\colon w_t\neq 0,\ 1\leqslant t\leqslant n\})$ *and* z^* *is the unique positive value of* z *such that*

$$\sum_{t=1}^{n}|w_t|\,[(1-\mathrm{e}^{-2|w_t|z})/(1+\mathrm{e}^{-2|w_t|z})] = x. \tag{2.3}$$

Proof. By the independence of $S_1,\ldots,S_n$, the moment generating function of T is

$$E[\mathrm{e}^{zT}] = \prod_{t=1}^{n}E[\mathrm{e}^{zw_tS_t}] = \prod_{t=1}^{n}\cosh(w_t z).$$

Then, by Markov's inequality, we have for all $z\geqslant 0$,

$$P[T\geqslant x]\leqslant P[zT\geqslant zx] = P[\mathrm{e}^{zT}\geqslant \mathrm{e}^{zx}]\leqslant E[\mathrm{e}^{zT}]/\mathrm{e}^{zx} = \left[\prod_{t=1}^{n}\cosh(w_t z)\right]\Big/\mathrm{e}^{zx}.$$

Hence

$$P[T\geqslant x]\leqslant \inf_{z\geqslant 0}\left[\mathrm{e}^{-zx}\prod_{t=1}^{n}\cosh(w_t z)\right]\leqslant \mathrm{e}^{-x^2}\left[\prod_{t=1}^{n}\cosh(w_t x)\right],$$

where the last inequality follows by taking $z=x$. Further, from Corollary 1 and Example 2 of Eaton (1970),

$$E[\mathrm{e}^{xT}] = \prod_{t=1}^{n}\cosh(w_t x)\leqslant \prod_{t=1}^{n}\cosh(x/\sqrt{n}) = [\cosh(x/\sqrt{n})]^n,$$

for all $x>0$. Since

$$\cosh(x) = \sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!} < \sum_{j=0}^{\infty}\frac{x^{2j}}{j!\,2^j} = \mathrm{e}^{x^2/2},$$

for $x>0$, we have

$$\mathrm{e}^{-x^2}\left[\prod_{t=1}^{n}\cosh(w_t x)\right]\leqslant \mathrm{e}^{-x^2}[\cosh(x/\sqrt{n})]^n < \mathrm{e}^{-x^2}[\mathrm{e}^{x^2/(2n)}]^n = \mathrm{e}^{-x^2/2},$$

from which (2.1) follows.

In order to obtain (2.2), let $H(z)\equiv \mathrm{e}^{-zx}\prod_{t=1}^{n}\cosh(w_t z)$ and $h(z)\equiv\log[H(z)]$. Noting that $H(z)=\mathrm{e}^{-zx}\prod_{t=1}^{n}\cosh(|w_t|\,z)$, the derivative $h'(z)$ of $h(z)$ takes the form

$$h'(z) = \left\{\sum_{t=1}^{n}|w_t|\,[(1-\mathrm{e}^{-2|w_t|z})/(1+\mathrm{e}^{-2|w_t|z})]\right\}-x, \tag{2.4}$$

where $x>0$ and $w_t \neq 0$ for at least one value of t. It is easy to see that the function $h'(z)$ is everywhere continuous, strictly increasing for $z>0$, with $h'(0)=-x<0$ and $h'(z)\to\sum_{t=1}^{n}|w_t|-x$ as $z\to\infty$. Consequently, if $\sum_{t=1}^{n}|w_t|>x$, the equation $h'(z)=0$ has a unique solution $z^*>0$, with $h'(z)<0$ for $0\leqslant z<z^*$ and $h'(z)>0$ for $z>z^*$. Thus $h(z^*)$ is the minimum of $h(z)$ over the interval $z\geqslant 0$ or, equivalently, $H(z^*)=\inf_{z\geqslant 0}H(z)$ where z^* is the solution of equation (2.3). If $\sum_{t=1}^{n}|w_t|\leqslant x$, we have $h'(z)<0$ for all $z>0$, so that $h(z)$ and $H(z)$ have no minimum. In this case, it is easy to check directly that

$$\inf_{z\geqslant 0} H(z) = \lim_{z\to\infty} H(z) = \begin{cases} 0 & \text{if } \sum_{t=1}^{n}|w_t|<x, \\ (\tfrac{1}{2})^{n^*} & \text{if } \sum_{t=1}^{n}|w_t|=x. \end{cases} \qquad \square$$

By the symmetry of the distribution of T, $P[|T|\geqslant x]=2P[T\geqslant x]$ for $x>0$, so that the four bounds in (2.1) also apply to $\frac{1}{2}P[|T|\geqslant x]$. Denote them by $E_1(x)\leqslant E_2(x)\leqslant E_3(x)\leqslant E_4(x)$, in ascending order. Starting from the largest one, $E_4(x)\equiv e^{-x^2/2}$, the bounds become increasingly tighter but also increasingly difficult to compute. The inequality $P[T>x]<e^{-x^2/2}$ (for $x>0$) was given by Edelman (1986) and, in a somewhat different form, by Efron (1969). It can also be derived from Hoeffding's (1963) inequality and improves the classical Bernstein (1924, 1927) inequality by taking into account the special form of the T statistic. It yields a rapidly decreasing bound on tail areas for statistics of the form $T=\sum_{t=1}^{n} w_t S_t$ with $\sum_{t=1}^{n} w_t^2=1$. (Note that the condition $x>0$ is not stated explicitly, neither by Edelman (1986) nor by Efron (1969), though it is straightforward to see that the inequality does not generally hold for $x<0$.) Unlike the simple bound $E_4(x)$, which only depends on x, $E_3(x)\equiv e^{-x^2}[\cosh(x/\sqrt{n})]^n$ also takes into account the sample size n, and provides a uniform improvement over $E_4(x)$. For example, letting $n=10$ and $x=3$, we have $E_3(x)=0.0064$ while $E_4(x)=0.0111$. Neither of these two bounds depends on the weights $w_1,\ldots,w_n$. If, however, the set of weights $\{w_1,\ldots,w_n\}$ is fixed with probability one (which is the case for linear signed rank statistics whenever $a_n(r,t)=a_n(r)$ and $X_1,\ldots,X_n$ have continuous distributions), we can also use $E_2(x)\equiv e^{-x^2}[\prod_{t=1}^{n}\cosh(w_t,x)]$, which is uniformly smaller than $E_3(x)$ and $E_4(x)$ (for $x>0$) and remains easy to compute. Equality between $E_2(x)$ and $E_3(x)$ occurs when all the weights are equal ($w_t=1/\sqrt{n}$, $t=1,\ldots,n$). In all other cases, $E_2(x)$ takes into account the structure of the weights and can lead to substantial improvement when the weights are dispersed. For example, when $w_1\to 1$ (hence $w_j\to 0$, $j=2,\ldots,n$), $n=10$ and $x=3$, the ratio $E_2(x)/E_3(x)$ converges to 0.1933. Finally, $E_1(x)$ is obtained by searching for the infimum of the function $H(z)=e^{-zx}\prod_{t=1}^{n}\cosh(w_t z)$ for $z\geqslant 0$, and can lead to sizeable improvements over the three other bounds. Section 6 provides numerical illustrations of this fact. Note also that any number $\bar{z}>0$ such that $H(\bar{z})<H(x)$ yields a valid upper bound $P[T\geqslant x]\leqslant H(\bar{z})$ which is better than $E_2(x)$.

Though a general closed-form expression for $\inf\{H(z)\colon z \geqslant 0\}$ is apparently impossible, this infimum can be obtained fairly easily by applying numerical techniques. An especially simple method is based on expression (2.2). When $\sum_{t=1}^{n} |w_t| > x > 0$, the latter requires finding $z^* > 0$ such that $h'(z^*) = 0$, where $h'(z)$ is given by (2.4). From Proposition 1, we know that z^* exists and is unique. Further, $h'(z)$ is continuous and strictly increasing for $z \geqslant 0$. Thus, given any $\bar{z}$ such that $h'(\bar{z}) > 0$, one can use a bisection algorithm to locate the value of z^* inside the interval $(0, \bar{z})$. This procedure typically converges very quickly. Note that we also can minimize $H(z)$ directly by using various optimization algorithms, e.g., the Newton–Raphson or the Davidson–Fletcher–Powell algorithm; see Avriel (1976, Chapters 8–11) or Judge et al. (1985, Appendix B).

Proposition 1 can be applied to linear signed rank statistics by noting that $s(\boldsymbol{X}) = [s(X_1), \dots, s(X_n)]'$ and $|\boldsymbol{X}| = (|X_1|, \dots, |X_n|)'$ are independent when $X_1, \dots, X_n$ are independent with distributions symmetric about zero and $P(X_t = 0) = 0$, $t = 1, \dots, n$. This yields the following corollary.

Corollary 1.1. *Let $X_1, \dots, X_n$ be independent random variables with distributions symmetric about zero and $P[X_t = 0] = 0$, $t = 1, \dots, n$, and let $\bar{S} = \sum_{t=1}^{n} s(X_t) a_n(R_t^+, t)$ be any linear signed rank statistic such that $\sum_{t=1}^{n} a_n(R_t^+, t)^2 > 0$ with probability* 1. *Then, for any $x > 0$,*

$$P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+] \leqslant \inf_{z \geqslant 0} \left[\mathrm{e}^{-zx} \prod_{t=1}^{n} \cosh(w_t z) \right] \leqslant \mathrm{e}^{-x^2} \prod_{t=1}^{n} \cosh(w_t x)$$

$$\leqslant \mathrm{e}^{-x^2} \left[\cosh \frac{x}{\sqrt{n}} \right]^n < \mathrm{e}^{-x^2/2}, \tag{2.5}$$

where $\boldsymbol{R}^+ = (R_1^+, \dots, R_n^+)'$, $\sigma_s^2 = \sum_{t=1}^{n} a_n(R_t^+, t)^2$ and $w_t = a_n(R_t^+, t)/\sigma_s$, $t = 1, \dots, n$.

When $a_n(r, t) = a_n(r)$ and $X_1, \dots, X_n$ have continuous distributions, the standard error σ_s is fixed, i.e. $\sigma_s = [\sum_{t=1}^{n} a_n(t)^2]^{1/2}$ with probability 1, and the bounds in (2.5) are constants. We can thus replace $P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+]$ by $P[\bar{S}/\sigma_s \geqslant x]$ and use any of the bounds in (2.5) to compute conservative critical values as well as p-values for tests based on $\bar{S}/\sigma_s$.

For general scores $a_n(r, t)$, it is easier to compute p-values. Let $T = \bar{S}/\sigma_s$, $G(x \mid \boldsymbol{R}^+) = 1 - F(x \mid \boldsymbol{R}^+)$, where $F(x \mid \boldsymbol{R}^+)$ is the cumulative distribution function of $\bar{S}/\sigma_s$ conditional on $\boldsymbol{R}^+$, and let $U(x, \boldsymbol{R}^+)$ be any one of the four bounds in (2.5). Clearly, $G(T \mid \boldsymbol{R}^+) \leqslant \alpha$ is a critical region (for a right-one-sided test) with size not larger than α conditional on $\boldsymbol{R}^+$, and thus also unconditionally $(0 < \alpha < 1)$. Since $G(T \mid \boldsymbol{R}^+) \leqslant U(T \mid \boldsymbol{R}^+)$, the critical region $U(T \mid \boldsymbol{R}^+) \leqslant \alpha$ has size not larger than α both conditional on $\boldsymbol{R}^+$ and unconditionally. Left-one-sided and two-sided critical regions can be defined in a similar way.

3. Chebyshev-type bounds

When some or all the moments of a given statistic are known, it is also possible to bound their tail areas with Chebyshev-type inequalities. We will see that the moments of an arbitrary linear signed rank statistic can be computed fairly easily up to a high order. It is worthwhile considering higher-order moments (greater than 2) because the bound based on the absolute moment of order p declines like $1/x^p$. For example, consider the case where $X \sim \mathrm{N}(0,1)$. The Chebyshev inequality of order p is $P[|X| \geqslant x] \leqslant E(|X|^p)/x^p$ for $x>0$. In this case, the bounds on $P(|X| \geqslant 3.0)$ with $p=2$, 4, 6, 8, 10, 12 are respectively 0.1111, 0.0370, 0.0206, 0.0160, 0.0160, 0.0196. For $P(|X| \geqslant 2.5)$, the corresponding bounds are 0.1600, 0.0768, 0.0614, 0.0688, 0.0991, 0.1744. The tightest bound on $P[|X| \geqslant 3.0]$ is thus obtained by taking $p=8$ or 10, while for $P[|X| \geqslant 2.5]$ it is obtained with $p=6$. Thus, by considering higher-order moments, one can get considerable improvement over the usual Chebyshev inequality of order 2. In practice, of course, such bounds are useful when the moments of X are known but the distribution function of X is unknown or difficult to compute. We will now see that this is precisely the case with linear signed rank statistics.

Consider again a statistic of the form $T=\sum_{t=1}^{n} w_t S_t$, where $w_1, \ldots, w_n$ are fixed numbers with $\sum_{t=1}^{n} w_t^2 = 1$ and $S_1, \ldots, S_n$ are independent random variables such that $P[S_t=1]=P[S_t=-1]=\frac{1}{2}$, $t=1,\ldots,n$. Clearly $E(T)=0$ and $E(T^2)=1$. Since $E(S_t^p)=1$ and $E(S_t^{p-1})=0$, for any even integer p, the fourth moment of T is

$$E(T^4) = \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \sum_{t_3=1}^{n} \sum_{t_4=1}^{n} w_{t_1} w_{t_2} w_{t_3} w_{t_4} E(S_{t_1} S_{t_2} S_{t_3} S_{t_4})$$
$$= \sum_{t=1}^{n} w_t^4 + 3 \sum_{t_1 \neq t_2}^{n} w_{t_1}^2 w_{t_2}^2.$$

Denote the power sums of the weights by

$$W_r = \sum_{t=1}^{n} w_t^r \tag{3.1}$$

and use the development of the symmetric function $\sum_{t_1 \neq t_2} w_{t_1}^2 w_{t_2}^2$ in terms of power sums:

$$\sum_{t_1 \neq t_2} w_{t_1}^2 w_{t_2}^2 = W_2^2 - W_4;$$

see Kendall, Stuart and Ord (1983, p. 708). Since $W_2 = 1$, this yields

$$E(T^4) = 3 - 2W_4, \tag{3.2}$$

which is easy to compute. Though the calculations are increasingly tedious, the other even moments of T can be derived in a similar way. The results up to $p=12$ are:

$$E(T^6) = 15 - 30W_4 + 16W_6, \tag{3.3}$$

$$E(T^8) = 105 - 420 W_4 + 140 W_4^2 + 448 W_6 - 272 W_8, \tag{3.4}$$

$$E(T^{10}) = 945 - 6300 W_4 + 6300 W_4^2 + 10080 W_6 - 6720 W_6 W_4 - 12240 W_8 + 7936 W_{10}, \tag{3.5}$$

$$E(T^{12}) = 10395 - 103950 W_4 + 207900 W_4^2 - 46200 W_4^3 + 221760 W_6 - 443520 W_6 W_4 + 118272 W_6^2 - 403920 W_8 + 269280 W_8 W_4 + 523776 W_{10} - 353792 W_{12}. \tag{3.6}$$

To the best of our knowledge, the latter expressions have not been reported anywhere else. Formulae (3.2) to (3.6) are simple to compute. The first term in each of them is the corresponding moment of a N(0,1) variable. If W_4 approaches zero, then W_6, W_8, W_{10} and W_{12} also approach zero and the moments approach those of a normal variable. This will hold, for example, if $n \to \infty$ and the asymptotic distribution of T is normal. Further, it is of interest to note that all the even moments of T are bounded by the moments of the N(0,1) distribution as well as by those of a centered binomial distribution. For any even positive integer p, the following inequality holds:

$$E(T^p) \leqslant E(Y^p) \leqslant E(Z^p) = (p-1)(p-3)\cdots 3 \cdot 1 = \frac{p!}{2^{p/2}(p/2)!}, \tag{3.7}$$

where $Y = \sum_{t=1}^{n} S_t/\sqrt{n}$ and $Z \sim \mathrm{N}(0,1)$; see Efron (1969) and Eaton (1970). Up to order 12, the moments $E(Y^p)$ follow from formulae (3.2) to (3.6) with W_r replaced by $n(1/n)^{r/2}$. More generally, the moments $E(Y^p)$ can be computed by noting that

$$E(Y^p) = \left(\frac{2}{\sqrt{n}}\right)^p \nu_p, \tag{3.8}$$

where ν_p is the moment of order p about the mean of the $\mathrm{Bin}(n, \frac{1}{2})$ binomial distribution. Further, ν_p can be obtained easily from the recurrence relation

$$\nu_p = \frac{n}{4} \sum_{j=0}^{p-2} \binom{p-1}{j} \nu_j - \frac{1}{2} \sum_{j=0}^{p-2} \binom{p-1}{j} \nu_{j+1}, \quad p = 2, 4, 6, \ldots, \tag{3.9}$$

where $\nu_0 = 1$ and $\nu_j = 0$ for $j = 1, 3, 5, \ldots$, and $\binom{n}{r} = n!/[r!(n-r)!]$; see Romanovsky (1923) or Kendall and Stuart (1977, Vol. 1, p. 129).

For any even positive integer p and for any $x > 0$, we thus have the inequalities

$$P[T \geqslant x] \leqslant \frac{E(T^p)}{2x^p} \leqslant \frac{E(Y^p)}{2x^p} \leqslant \frac{(p-1)(p-3)\cdots 3 \cdot 1}{2x^p}, \tag{3.10}$$

where $Y = \sum_{t=1}^{n} S_t/\sqrt{n}$ and the symmetry of the distribution of T has been taken into account. The three upper bounds in (3.10) are uniformly ordered but the larger bounds again are simpler to compute. On the other hand, the tightest bound, which takes the weights w_t into account, can be *much* smaller than the two other ones. For example, for $n = 10$ and $p = 8$, $E(Z^p) = 105$, $E(Y^p) = 64.8$ and $E(T^p)$ can be as low as 1 (e.g., when $w \to 1$).

Since the inequalities (3.10) are valid for any even integer p, this suggests minimizing the bounds with respect to p. Using the formula for the p-th moment of the N(0, 1) distribution in (3.7), it is easy to see that

$$\min\left\{\frac{E(Z^p)}{x^p} : p=2,4,\dots\right\} = \frac{(p_*-1)(p_*-3)\cdots 3\cdot 1}{x^{p_*}}, \tag{3.11}$$

where $p_* = \max\{2, \bar{p}\}$ and $\bar{p}$ is the largest even integer (including zero) such that $\bar{p} < x^2+1$. Hence, for all $x>0$,

$$P[T \geqslant x] \leqslant \frac{(p_*-1)(p_*-3)\cdots 3\cdot 1}{2x^{p_*}}. \tag{3.12}$$

We could not find a similar closed-form expression for the bounds based on $E(T^p)$ and $E(Y^p)$. However, using (3.8) and (3.9), it is easy to consider bounds based on the moments $E(Y^p)$ up to arbitrarily large values of p.

It will be convenient to summarize the above results in the following proposition.

Proposition 2. *Let the assumptions of Proposition* 1 *hold. Then, for any positive even integer p and for any* $x>0$,

$$P[T \geqslant x] \leqslant \frac{E(T^p)}{2x^p} \leqslant \frac{E(Y^p)}{2x^p} \leqslant \frac{(p-1)(p-3)\cdots 3\cdot 1}{2x^p},$$

where $Y = \sum_{t=1}^n S_t/\sqrt{n}$, *and*

$$P[T \geqslant x] \leqslant \frac{(p_*-1)(p_*-3)\cdots 3\cdot 1}{2x^{p_*}},$$

where $p_* = \max\{2, \bar{p}\}$ *and* $\bar{p}$ *is the largest even integer such that* $\bar{p} < 1+x^2$. *Further, for* $p=2,4,\dots,12$, *the moments* $E(T^p)$ *are given by Equations* (3.2) *to* (3.6).

Inequalities (3.10) and (3.12) can be applied to linear signed rank statistics in the same way as the exponential bounds of Section 2. This yields the following corollary.

Corollary 2.1. *Let the assumptions of Corollary* 1.1 *hold. Then, for any positive even integer p and for all* $x>0$,

$$P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+] \leqslant \frac{\mu_p(\boldsymbol{R}^+)}{2x^p} \leqslant \frac{E(Y^p)}{2x^p} \leqslant \frac{(p-1)(p-3)\cdots 3\cdot 1}{2x^p}, \tag{3.13}$$

and

$$P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+] \leqslant \frac{(p_*-1)(p_*-3)\cdots 3\cdot 1}{2x^{p_*}}, \tag{3.14}$$

where $\boldsymbol{R}^+ = (R_1^+, \dots, R_n^+)'$,

$$\sigma_s^2 = \sum_{t=1}^n a_n(R_t^+, t)^2, \qquad \mu_p(\boldsymbol{R}^+) = E(T^p \mid \boldsymbol{R}^+),$$

$$T=\sum_{t=1}^{n} w_t s(X_t), \qquad w_t=a_n(R_t^+,t)/\sigma_s, \quad t=1,\dots,n,$$

$p_* = \max\{2, \bar{p}\}$ *and* $\bar{p}$ *is the largest even integer such that* $\bar{p}<x^2+1$.

When $a_n(r,t)=a_n(r)$ and $X_1,\dots,X_n$ are continuous random variables, σ_s and $\mu_p(\boldsymbol{R}^+)$ do not depend on the rank vector $\boldsymbol{R}^+$ (with probability 1) and $P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+]$ can be replaced by the unconditional probability $P[\bar{S}/\sigma_s \geqslant x]$. In all cases, conservative p-values can be computed as in the case of exponential bounds.

4. Berry–Esséen–Zolotarev bounds

The exponential and Chebyshev bounds proposed in the previous sections appear to be helpful mainly in producing upper bounds on the tail areas of linear signed rank statistics when x is not too small ($x>1.5$, say). For smaller values of x, these bounds are usually too conservative for practical purposes, and may even be greater than 1. In this section, we derive bounds of the Berry–Esséen type whose behavior is more satisfactory for smaller values of x. In addition, these results yield *lower* bounds on the tail areas of linear signed rank statistics.

To get these bounds, we combine inequalities due to van Beek (1972) and Zolotarev (1965) and obtain a bound on the distance between the cumulative distribution function of an arbitrary linear combination of independent symmetric Bernoulli variables and that of a N(0, 1) variable. The basic result is given by the following proposition.

Proposition 3. *Let* $T=\sum_{t=1}^{n} w_t S_t$, *where* $S_1,\dots,S_n$ *are independent random variables such that* $P[S_t=1]=P[S_t=-1]=\frac{1}{2}$, *and where* $w_1,\dots,w_n$ *are fixed real numbers such that* $\sum_{t=1}^{n} w_t^2=1$. *Then*

$$|P(T<x)-\Phi(x)| \leqslant \min\left\{0.7975\sum_{t=1}^{n}|w_t|^3,\ 0.366145\left(\sum_{t=1}^{n}|w_t|^3\right)^{1/4}\right\} \leqslant 0.366145, \tag{4.1}$$

for all x, *where* $\Phi(x)$ *denotes the* N(0, 1) *cumulative distribution function.*

Proof. The statistic T is a sum of independent random variables $X_t=w_tS_t$ such that $E(X_t)=0$, $E(X_t^2)=w_t^2$ and $E(|X_t|^3)=|w_t|^3$. By the Berry–Esséen inequality there exists a universal constant C_1 such that

$$|P(T<x)-\Phi(x)| \leqslant C_1\sum_{t=1}^{n} E(|X_t|^3) = C_1\sum_{t=1}^{n}|w_t|^3, \quad \text{for all } x, \tag{4.2}$$

see Petrov (1975, p. 111, Theorem 3). Further, it was shown by van Beek (1972) that $C_1 \leqslant 0.7975$. Note also that $C_1 \geqslant 0.4097$ (Esséen (1956)).

Let us now suppose (without loss of generality) that $w_t \neq 0$, $t=1,\dots,n$, and denote by F_t the cumulative distribution function of X_t. Then, by Theorem 1 of Zolotarev (1965) with $m=2$ and $r=3$, we have

$$|P(T<x)-\Phi(x)| \leqslant C_2(2\pi)^{-3/8}\nu(3)^{1/4}, \quad \text{for all } x,$$

where $C_2=0.57465$ (Zolotarev (1965), p. 478) and

$$\nu(3)=\sum_{t=1}^{n}\int |x|^3 \left| \mathrm{d}\left(F_t(x)-\Phi\left(\frac{x}{|w_t|}\right)\right)\right|.$$

Further,

$$\nu(3)\leqslant \sum_{t=1}^{n}\left\{\int |x|^3\,\mathrm{d}F_t(x)+|w_t|^3\int\left|\frac{x}{w_t}\right|^3 \mathrm{d}\Phi\left(\frac{x}{|w_t|}\right)\right\}$$

$$=\left[1+\frac{4}{\sqrt{2\pi}}\right]\sum_{t=1}^{n}|w_t|^3;$$

hence

$$|P(T<x)-\Phi(x)|\leqslant C_2(2\pi)^{-3/8}\left[1+\frac{4}{\sqrt{2\pi}}\right]^{1/4}\left(\sum_{t=1}^{n}|w_t|^3\right)^{1/4}$$

$$\leqslant 0.366145\left(\sum_{t=1}^{n}|w_t|^3\right)^{1/4}\leqslant 0.366145, \quad \text{for all } x, \tag{4.3}$$

where the last inequality follows by noting that $\sum_{t=1}^{n}|w_t|^3\leqslant 1$. Inequality (4.1) follows by combining (4.2) and (4.3). □

Corollary 3.1. *Let the assumptions of Corollary* 1.1 *hold. Then*

$$|P[\bar{S}/\sigma_s<x\mid \boldsymbol{R}^+]-\Phi(x)|\leqslant \min\left\{0.7975\sum_{t=1}^{n}|w_t|^3,\ 0.366145\left(\sum_{t=1}^{n}|w_t|^3\right)^{1/4}\right\}\equiv\Delta$$

$$\leqslant 0.366145, \tag{4.4}$$

for all x, *where* $\boldsymbol{R}^+=(R_1^+,\dots,R_n^+)'$, $\sigma_s^2=\sum_{t=1}^{n}a_n(R_t^+,t)^2$ *and* $w_t=a_n(R_t^+,t)/\sigma_s$, $t=1,\dots,n$.

Inequality (4.4) yields both upper and lower bounds on the distribution function of $\bar{S}/\sigma_s$ (conditional on $\boldsymbol{R}^+$) or, equivalently, on the tail areas of $\bar{S}/\sigma_s$:

$$[1-\Phi(x)]-\Delta\leqslant P[\bar{S}/\sigma_s\geqslant x\mid \boldsymbol{R}^+]\leqslant[1-\Phi(x)]+\Delta. \tag{4.5}$$

When $a_n(r,t)=a_n(r)$ and $X_1,\dots,X_n$ have (absolutely) continuous distributions, the quantity Δ is a fixed number (with probability 1) and

$$[1-\Phi(x)]-\Delta\leqslant P[\bar{S}/\sigma_s\geqslant x]\leqslant[1-\Phi(x)]+\Delta. \tag{4.6}$$

The lower bound is informative whenever $[1-\Phi(x)]-\Delta>0$. The upper bound $[1-\Phi(x)]+\Delta$ is usually dominated by the exponential and Chebyshev bounds for values of x not too close to zero ($x>1$, say) but still may yield the best upper bound for x sufficiently small. This will be illustrated numerically in Section 6.

The interesting feature of the bounds in (4.5) and (4.6) is that they are sufficiently explicit to yield upper and lower bounds for p-values of tests based on linear signed rank statistics. Further, no assumption on the behavior of the generalized score $a_n(r, t)$ is required, whether in small samples or asymptotically. These characteristics can be contrasted with previous work on Berry–Esséen or Cramér-type bounds for linear signed rank statistics, where only rates of convergence are studied and various restrictions are imposed on the scores; see Puri and Seoh (1984a, b, c, 1985), Puri and Wu (1986), Ralescu and Puri (1985), Seoh (1990), Seoh and Puri (1985), Seoh, Ralescu and Puri (1985) and Wu (1987). The Berry–Esséen inequality is used here as an operational finite-sample approximation theorem, not as a theoretical result on rates of convergence to asymptotic normal distributions.

5. Bounds tests

Denote by $E_i(x, w)$, $i=1, \ldots, 4$, the four upper bounds in (2.1) in increasing order (i.e., $E_1 \leqslant \cdots \leqslant E_4$), and by $C_j(x, w, p)$, $j=1,2,3$, the three Chebyshev bounds in (3.10), also in increasing order, where $w=(w_1, \ldots, w_n)'$. Let

$$\bar{C}_j(x, w, P) = \min\{C_j(x, w, p)\colon\ p=2,4,\ldots,P\}, \tag{5.1}$$

where P is some positive even integer. There is no simple ordering between the exponential, Chebyshev and Berry–Esséen–Zolotarev bounds. Further, for Chebyshev bounds, we can search over several values of p. Since all are easy to compute, this suggests combining the different bounds to get an even tighter one. Moreover, the Berry–Esséen–Zolotarev inequality yields a lower bound on $P[T \geqslant x]$.

Let

$$U_{ij}(x, w, P) = \min\{E_i(x, w),\ \bar{C}_j(x, w, P),\ 1-\Phi(x)+\Delta\}, \tag{5.2}$$

where Δ is the smallest of the two upper bounds in (4.1). Then for any statistic of the form $T=\sum_{t=1}^{n} w_t S_t$, as defined in Proposition 1, we have

$$1-\Phi(x)-\Delta \leqslant P[T \geqslant x] \leqslant U_{11}(x, w, P) \leqslant U_{ij}(x, w, P), \tag{5.3}$$

for all $x>0$ and all positive even integers P, $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 3$. The last inequality follows by observing that

$$E_1(x, w) \equiv \inf_{z \geqslant 0} \left[\mathrm{e}^{-zx} \prod_{t=1}^{n} \cosh(w_t z) \right] \leqslant E_i(x, w), \quad i=2,3,4,$$

and

$$C_1(x, w, p) = \frac{E(T^p)}{2x^p} \leqslant C_j(x, w, p), \quad j=2,3.$$

The bound $U_{11}(x, w, P)$ is thus the best upper bound available from the inequalities developed in the preceding sections (with moments up to order P). However, in some situations, one of the larger bounds may be more convenient from a computational point of view.

For any linear signed rank statistic $\bar{S}=\sum_{t=1}^{n} s(X_t)a_n(R_t^+,t)$, as defined in Corollary 1.1, inequality (5.3) yields

$$1-\Phi(x)-\Delta \leqslant P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+] \leqslant U_{11}(x,w,P) \leqslant U_{ij}(x,w,P), \tag{5.4}$$

for all $x>0$ and all positive even integers P, $1\leqslant i\leqslant 4$, $1\leqslant j\leqslant 3$, where $w_t=a_n(R_t^+,t)/\sigma_s$, $t=1,\dots,n$. For general scores $a_n(r,t)$, the bounds on $P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+]$ depend on $\boldsymbol{R}^+$. When $a_n(r,t)=a_n(r)$ and $X_1,\dots,X_n$ are independent with continuous distributions symmetric about zero, the bounds do not depend on $\boldsymbol{R}^+$ (with probability 1). In this case, we can replace $P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+]$ by the unconditional probability $P[\bar{S}/\sigma_s \geqslant x]$.

Inequality (5.4) may be used to compute conservative and liberal critical values for tests based on $\bar{S}/\sigma_s$. For a test that rejects the null hypothesis when $\bar{T}\equiv\bar{S}/\sigma_s$ is large (right one-sided test), the critical region $U_{ij}(\bar{T},w,P)\leqslant\alpha$ has size *not greater* than α, both conditional on $\boldsymbol{R}^+$ and unconditionally. In contrast, the critical region $1-\Phi(\bar{T})-\Delta\leqslant\alpha$ has size *not smaller* than α and contains the conservative critical region. When $U_{ij}(\bar{T},w,P)\leqslant\alpha$, we can be sure that the test rejects the null hypothesis at level α while, if $1-\Phi(\bar{T})-\Delta>\alpha$, we can be sure that the test does not reject at level α. For further discussion of bounds tests, see Dufour (1989, 1990).

6. Numerical comparisons

In this section, we compare the bounds of the preceding sections for linear signed rank statistics of the form $\bar{S}=\sum_{t=1}^{n} s(X_t)a_n(R_t^+,t)$ as defined in Corollary 1.1. Three sample sizes ($n=25$, 50 and 1000) and five different scores are considered:

$$a_{1n}(r,t) = r, \tag{6.1}$$

$$a_{2n}(r,t) = \Phi^{-1}\left[\frac{1}{2}\left(1+\frac{r}{n+1}\right)\right], \tag{6.2}$$

$$a_{3n}(r,t) = \cos\left[\pi\left(1+\frac{r}{n+1}\right)\right], \tag{6.3}$$

$$a_{4n}(r,t) = t^2, \tag{6.4}$$

$$a_{5n}(r,t) = \begin{cases} 1 & \text{if } t=1,\dots,n-1, \\ \gamma_n & \text{if } t=n, \end{cases} \tag{6.5}$$

where $\gamma_n=15$, 20, 100 for $n=25$, 50 and 1000 respectively, and Φ^{-1} denotes the inverse of the cumulative standard normal distribution function. a_{1n} is the score function of the Wilcoxon signed rank test, a_{2n} is the van der Waerden score (van Eeden (1963)) and a_{3n} is the optimal score associated with the Cauchy density (Philippou (1984)). a_{4n} corresponds to a test against a regression alternative with double-exponential errors and regression constants proportional to t^2 (quadratic trend), i.e. $a_{4n}(r,t)=c_t a_n(r)$ with $c_t=t^2$ and $a_n(r)=1$. Finally, a_{5n} may correspond to a situation where the last value of the regressor differs substantially from the others; a_{5n} yields a signed rank statistic for which the normal approximation is particularly poor. Among these five signed rank statistics, only the distribution of the Wilcoxon statistic has been well tabulated (Wilcoxon, Katti and Wilcox (1973)).

For each of these statistics, we present in Tables 1 to 5 the distance Δ as defined by Corollary 3.1, the upper bounds for $P[\bar{S}/\sigma_s \geqslant x \mid \boldsymbol{R}^+]$ and $x=0.5\,(0.5)\,4.0$ and the lower bound $1-\Phi(x)-\Delta$. For all cases studied here, we assume that $|X_1|,\dots,|X_n|$ are all different, with no zeros, so that $\boldsymbol{R}^+$ is a permutation of $(1,2,\dots,n)$ and $P[\bar{S}/\sigma_s \geqslant x|\boldsymbol{R}^+]=P[\bar{S}/\sigma_s \geqslant x]$. This, of course, holds with probability one if $X_1,\dots,X_n$ have continuous distributions. The upper bounds E_1 to E_4 are the four upper bounds given in (2.5):

$$E_1=\inf_{z\geqslant 0}\left[\mathrm{e}^{-zx}\prod_{t=1}^{n}\cosh(w_t z)\right],\qquad E_2=\mathrm{e}^{-x^2}\prod_{t=1}^{n}\cosh(w_t x),$$

$$E_3=\mathrm{e}^{-x^2}\left[\cosh\left(\frac{x}{\sqrt{n}}\right)\right]^n,\qquad E_4=\mathrm{e}^{-x^2/2}.$$

The Chebyshev upper bounds are $C_p=\mu_p(\boldsymbol{R}^+)/(2x^p)$, with $p=2,4,6,8,10,12$, where the exact moments are obtained from (3.2) to (3.6). The bounds CB and CN were obtained by minimizing $E(Y^p)/(2x^p)$ and $E(Z^p)/(2x^p)$ with respect to even values of p, where $E(Y^p)$ and $E(Z^p)$ are the binomial and normal moments. CN was computed with the closed-form expression on the right-hand side of (3.12), while CB was obtained by considering the bounds $E(Y^p)/(2x^p)$, $p=2,4,\dots,30$. For each x, the value p^* of p that yields the minimum is reported. $1-\Phi(x)+\Delta$ is the Berry–Esséen–Zolotarev upper bound based on Corollary 3.1. The lowest of all the upper bounds is then given. When a bound yields a value greater than one or less than zero, which is possible for Chebyshev and Berry–Esséen–Zolotarev bounds, it is replaced by 1 or 0. Finally, we report estimates of the actual tail areas $P[\bar{S}/\sigma_s \geqslant x]$ and approximations based on Edgeworth expansions. For the Wilcoxon signed rank statistic with $n=25$ and $n=50$, these probabilities are taken from Wilcoxon, Katti and Wilcox (1973). In the other cases, they were evaluated by Monte-Carlo simulations, with 100000 replications for $n=25$ and 50, and 10000 replications for $n=1000$. The Edgeworth approximations are based on the standard formula

$$1-F_E(x)=1-\left\{\Phi(x)-\frac{k_4}{24}(x^3-3x)\phi(x)\right\},$$

where $k_4=\mu_4(\boldsymbol{R}^+)-3$, $\phi(x)=(2\pi)^{-1/2}\exp(-x^2/2)$ and $\Phi(x)$ is the cumulative $\mathrm{N}(0,1)$ distribution function; see Johnson and Kotz (1970, Ch. 12, p. 17). We take into account here the fact that $\mu_3(\boldsymbol{R}^+)=0$.

Six major observations emerge from the results presented in the tables.

(1) Each of the three smallest exponential bounds may yield a substantial improvement with respect to the higher ones, especially for large x. The improvement yielded by E_1 is most striking for the last score (a_5). But, of course, the larger bounds are easier to compute.

Table 1
Bounds for the Wilcoxon statistic; $a_n(r,t) = a_{1n}(r,t) = r$

	x ($n=25$, $\Delta=0.2051$)							
	0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds E_1	0.8822	0.6029	0.3146	0.1219	0.0334	0.0059	0.0006	0.0000
E_2	0.8822	0.6031	0.3156	0.1241	0.0359	0.0075	0.0011	0.0001
E_3	0.8823	0.6045	0.3193	0.1286	0.0389	0.0087	0.0014	0.0002
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p) $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
4	1	1	0.2824	0.0893	0.0366	0.0175	0.0095	0.0056
6	1	1	0.5697	0.1014	0.0266	0.0089	0.0035	0.0016
8	1	1	1	0.1535	0.0257	0.0060	0.0017	0.0006
10	1	1	1	0.2845	0.0305	0.0049	0.0011	0.0003
12	1	1	1	0.6139	0.0422	0.0047	0.0007	0.0001
CB	1	0.5000	0.2222	0.0913	0.0283	0.0061	0.0009	0.0001
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(22)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
BE $= 1-\Phi(x)+\Delta$	0.5137	0.3538	0.2719	0.2279	0.2113	0.2065	0.2053	0.2051
Best upper bound	0.5137	0.3638	0.2222	0.0893	0.0257	0.0047	0.0006	0.0000
Type[a]	BE	BE	C2	C4	C8	C12	E1	E1
$1-\Phi(x)-\Delta$ (lower bound)	0.1034	0	0	0	0	0	0	0
Tail area[b]	0.3075	0.1627	0.0668	0.0226	0.0053	0.0009	0.0001	0.0000
Edgeworth	0.3114	0.1615	0.0677	0.0221	0.0054	0.0009	0.0001	0.0000

	x ($n=50$, $\Delta=0.1458$)							
	0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds E_1	0.8823	0.6047	0.3197	0.1287	0.0387	0.0085	0.0013	0.0001
E_2	0.8823	0.6048	0.3199	0.1293	0.0394	0.0089	0.0015	0.0002
E_3	0.8824	0.6055	0.3220	0.1318	0.0413	0.0098	0.0017	0.0002
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p) $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
4	1	1	0.2893	0.0915	0.0375	0.0181	0.0098	0.0057
6	1	1	0.6126	0.1090	0.0286	0.0096	0.0038	0.0017
8	1	1	1	0.1775	0.0298	0.0069	0.0020	0.0007
10	1	1	1	0.3526	0.0389	0.0063	0.0013	0.0004
12	1	1	1	0.8536	0.0607	0.0068	0.0011	0.0002
CB	1	0.5000	0.2222	0.0925	0.0295	0.0070	0.0012	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(18)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
BE $= 1-\Phi(x)+\Delta$	0.4543	0.3044	0.2126	0.1685	0.1520	0.1471	0.1460	0.1458
Best upper bound	0.4543	0.3044	0.2126	0.0915	0.0286	0.0063	0.0011	0.0001
Type[a]	BE	BE	BE	C4	C6	C10	C12	E1
$1-\Phi(x)-\Delta$ (lower bound)	0.1628	0.0129	0	0	0	0	0	0
Tail area[b]	0.3093	0.1592	0.0677	0.0225	0.0058	0.0011	0.0001	0.0000
Edgeworth	0.3100	0.1601	0.0672	0.0224	0.0058	0.0011	0.0001	0.0000

	x ($n=1000$, $\Delta=0.0327$)							
	0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds E_1	0.8825	0.6064	0.3244	0.1350	0.0437	0.0110	0.0021	0.0003
E_2	0.8825	0.6064	0.3244	0.1350	0.0437	0.0110	0.0021	0.0003
E_3	0.8825	0.6065	0.3245	0.1352	0.0438	0.0110	0.0022	0.0003
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p) $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
4	1	1	0.2959	0.0936	0.0384	0.0185	0.0100	0.0069
6	1	1	0.6561	0.1168	0.0306	0.0103	0.0041	0.0018
8	1	1	1	0.2036	0.0342	0.0079	0.0023	0.0008
10	1	1	1	0.4559	0.0490	0.0079	0.0017	0.0004
12	1	1	1	1	0.0856	0.0096	0.0015	0.0003
CB	1	0.5000	0.2222	0.0937	0.0307	0.0079	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(12)	(16)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
BE $= 1-\Phi(x)+\Delta$	0.3413	0.1914	0.0996	0.0555	0.0390	0.0341	0.0330	0.0328
Best upper bound	0.3413	0.1914	0.0996	0.0555	0.0306	0.0079	0.0015	0.0002
Type[a]	BE	BE	BE	BE	C6	C8	C12	C8
$1-\Phi(x)-\Delta$ (lower bound)	0.2753	0.1259	0.0341	0	0	0	0	0
Tail area[b]	0.3127	0.1580	0.0656	0.0224	0.0072	0.0022	0.0002	0.0000
Edgeworth	0.3086	0.1587	0.0668	0.0227	0.0062	0.0013	0.0002	0.0000

[a] The lowest upper bound is identified by one of the following symbols: E_1 for the exponential bound, C_p for the Chebyshev bound based on the p-th moment, CB and CN for the best Chebyshev bounds based on binomial and normal moments respectively, and BE for the Berry-Esséen-Zolotarev bound $1-\Phi(x)+\Delta$.

[b] For $n=25$ and 50, the actual tail areas were obtained from the table of Wilcoxon, Katti and Wilcox (1973). For $n=1000$, they were evaluated by Monte-Carlo simulation (10000 replications).

Table 2

Bounds for the Van der Waerden statistic; $a_n(r,t)=a_{2n}(r,t)=\Phi^{-1}\left[\frac{1}{2}\left(1+\frac{r}{n+1}\right)\right]$

		x ($n=25$, $\Delta=0.2360$)							
		0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds	E_1	0.8820	0.6014	0.3105	0.1166	0.0295	0.0044	0.0003	0.0000
	E_2	0.8820	0.6018	0.3124	0.1206	0.0337	0.0067	0.0009	0.0001
	E_3	0.8823	0.6045	0.3193	0.1286	0.0389	0.0087	0.0014	0.0002
	E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p)	$p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
	4	1	1	0.2769	0.0876	0.0359	0.0173	0.0093	0.0055
	6	1	1	0.5383	0.0958	0.0251	0.0084	0.0033	0.0015
	8	1	1	1	0.1376	0.0231	0.0054	0.0016	0.0005
	10	1	1	1	0.2386	0.0256	0.0041	0.0009	0.0002
	12	1	1	1	0.4762	0.0327	0.0037	0.0006	0.0001
CB		1	0.5000	0.2222	0.0913	0.0283	0.0061	0.0009	0.0001
(p^*)		(2)	(2)	(2)	(4)	(6)	(10)	(14)	(22)
CN		1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$BE=1-\Phi(x)+\Delta$		0.5446	0.3947	0.3029	0.2588	0.2423	0.2374	0.2363	0.2361
Best upper bound		0.5446	0.3947	0.2222	0.0876	0.0231	0.0037	0.0003	0.0000
Type[a]		BE	BE	C2	C4	C8	C12	E1	E1
$1-\Phi(x)-\Delta$ (lower bound)		0.0725	0	0	0	0	0	0	0
Tail area[a]		0.3121	0.1622	0.0678	0.0224	0.0047	0.0005	0.0000	0.0000
Edgeworth		0.3125	0.1626	0.0680	0.0219	0.0050	0.0007	0.0000	0.0000

		x ($n=50$, $\Delta=0.1712$)							
		0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds	E_1	0.8823	0.6039	0.3173	0.1257	0.0364	0.0074	0.0010	0.0001
	E_2	0.8823	0.6039	0.3179	0.1269	0.0377	0.0082	0.0013	0.0001
	E_3	0.8824	0.6055	0.3220	0.1318	0.0413	0.0098	0.0017	0.0002
	E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p)	$p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
	4	1	1	0.2860	0.0905	0.0371	0.0179	0.0096	0.0057
	6	1	1	0.5923	0.1054	0.0276	0.0093	0.0037	0.0016
	8	1	1	1	0.1662	0.0279	0.0065	0.0019	0.0006
	10	1	1	1	0.3256	0.0350	0.0056	0.0012	0.0003
	12	1	1	1	0.7542	0.0518	0.0058	0.0009	0.0002
CB		1	0.5000	0.2222	0.0925	0.0295	0.0070	0.0012	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(10)	(14)	(18)
CN		1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$BE=1-\Phi(x)+\Delta$		0.4797	0.3298	0.2380	0.1939	0.1774	0.1725	0.1714	0.1712
Best upper bound		0.4797	0.3298	0.2222	0.0905	0.0276	0.0056	0.0009	0.0001
Type[a]		BE	BE	C2	C4	C6	C10	C12	E1
$1-\Phi(x)-\Delta$ (lower bound)		0.1374	0	0	0	0	0	0	0
Tail area[a]		0.3087	0.1584	0.0662	0.0213	0.0051	0.0009	0.0001	0.0000
Edgeworth		0.3106	0.1608	0.0674	0.0223	0.0056	0.0010	0.0001	0.0000

		x ($n=1000$, $\Delta=0.0400$)							
		0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds	E_1	0.8825	0.6064	0.3242	0.1348	0.0435	0.0109	0.0021	0.0003
	E_2	0.8825	0.6064	0.3243	0.1348	0.0435	0.0109	0.0021	0.0003
	E_3	0.8825	0.6065	0.3245	0.1352	0.0438	0.0110	0.0022	0.0003
	E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p)	$p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
	4	1	1	0.2957	0.0936	0.0383	0.0185	0.0100	0.0058
	6	1	1	0.6546	0.1165	0.0305	0.0102	0.0041	0.0018
	8	1	1	1	0.2027	0.0340	0.0079	0.0023	0.0008
	10	1	1	1	0.4525	0.0486	0.0078	0.0017	0.0004
	12	1	1	1	1	0.0847	0.0095	0.0015	0.0003
CB		1	0.5000	0.2222	0.0937	0.0307	0.0079	0.0015	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(10)	(12)	(16)
CN		1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$BE=1-\Phi(x)+\Delta$		0.3485	0.1986	0.1068	0.0627	0.0462	0.0413	0.0402	0.0400
Best upper bound		0.3485	0.1986	0.1068	0.0627	0.0305	0.0078	0.0015	0.0002
Type[a]		BE	BE	BE	BE	C6	C10	C12	C8
$1-\Phi(x)-\Delta$ (lower bound)		0.2686	0.1187	0.0268	0	0	0	0	0
Tail area[a]		0.3097	0.1575	0.0676	0.0232	0.0076	0.0022	0.0003	0.0000
Edgeworth		0.3087	0.1588	0.0668	0.0227	0.0062	0.0013	0.0002	0.0000

[a] The actual tail areas for Tables 2, 3, 4 and 5 were evaluated by Monte-Carlo simulations. For $n=25$ and 50, 100000 replications were used, while, for $n=1000$, only 10000 replications were used.

Table 3

Bounds for the signed rank statistic with Cauchy scores; $a_n(r,t) = a_{3n}(r,t) = \cos\left[\pi\left(1 + \dfrac{r}{n+1}\right)\right]$

	x ($n=25$, $\Delta=0.1925$) 0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds E_1	0.8822	0.6034	0.3160	0.1238	0.0348	0.0066	0.0007	0.0000
E_2	0.8822	0.6035	0.3168	0.1255	0.0368	0.0078	0.0012	0.0001
E_3	0.8823	0.6045	0.3193	0.1286	0.0389	0.0087	0.0014	0.0002
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p) $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
4	1	1	0.2843	0.0900	0.0368	0.0178	0.0096	0.0056
6	1	1	0.5813	0.1035	0.0271	0.0091	0.0036	0.0016
8	1	1	1	0.1597	0.0268	0.0062	0.0018	0.0006
10	1	1	1	0.3037	0.0326	0.0053	0.0011	0.0003
12	1	1	1	0.6760	0.0465	0.0052	0.0008	0.0002
CB	1	0.5000	0.2222	0.0913	0.0283	0.0061	0.0009	0.0001
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(22)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
BE = 1 $\Phi(x)+\Delta$	0.5011	0.3512	0.2593	0.2153	0.1987	0.1939	0.1927	0.1925
Best upper bound	0.5011	0.3512	0.2222	0.0900	0.0268	0.0052	0.0007	0.0000
Type	BE	BE	C2	C4	C8	C12	E1	E1
$1-\Phi(x)-\Delta$ (lower bound)	0.1160	0	0	0	0	0	0	0
Tail area	0.3087	0.1604	0.0658	0.0220	0.0051	0.0008	0.0001	0.0000
Edgeworth	0.3110	0.1611	0.0675	0.0222	0.0055	0.0009	0.0001	0.0000

	x ($n=50$, $\Delta=0.1358$) 0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds E_1	0.8824	0.6050	0.3204	0.1297	0.0395	0.0088	0.0014	0.0002
E_2	0.8824	0.6050	0.3206	0.1302	0.0400	0.0092	0.0016	0.0002
E_3	0.8824	0.6055	0.3220	0.1318	0.0413	0.0098	0.0017	0.0002
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p) $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
4	1	1	0.2903	0.0919	0.0376	0.0181	0.0098	0.0057
6	1	1	0.6194	0.1102	0.0289	0.0097	0.0038	0.0017
8	1	1	1	0.1814	0.0304	0.0071	0.0021	0.0007
10	1	1	1	0.3760	0.0404	0.0065	0.0014	0.0004
12	1	1	1	0.9329	0.0641	0.0072	0.0011	0.0002
CB	1	0.5000	0.2222	0.0925	0.0295	0.0070	0.0012	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(18)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
BE = 1 $\Phi(x)+\Delta$	0.4443	0.2944	0.2026	0.1585	0.1420	0.1371	0.1360	0.1358
Best upper bound	0.4443	0.2944	0.2026	0.0919	0.0289	0.0065	0.0011	0.0002
Type	BE	BE	BE	C4	C6	C10	C12	E1
$1-\Phi(x)-\Delta$ (lower bound)	0.1728	0.0229	0	0	0	0	0	0
Tail area	0.3071	0.1579	0.0655	0.0214	0.0058	0.0011	0.0002	0.0000
Edgeworth	0.3098	0.1599	0.0672	0.0225	0.0059	0.0011	0.0002	0.0000

	x ($n=1000$, $\Delta=0.0303$) 0.5	1	1.5	2	2.5	3	3.5	4
Exponential bounds E_1	0.8825	0.6065	0.3244	0.1351	0.0437	0.0110	0.0021	0.0003
E_2	0.8825	0.6065	0.3244	0.1351	0.0437	0.0110	0.0021	0.0003
E_3	0.8825	0.6065	0.3245	0.1352	0.0438	0.0110	0.0022	0.0003
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev bounds (C_p) $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
4	1	1	0.2960	0.0937	0.0384	0.0185	0.0100	0.0059
6	1	1	0.6565	0.1168	0.0306	0.0103	0.0041	0.0018
8	1	1	1	0.2039	0.0342	0.0080	0.0023	0.0008
10	1	1	1	0.4568	0.0491	0.0079	0.0017	0.0004
12	1	1	1	1	0.0859	0.0096	0.0015	0.0003
CB	1	0.5000	0.2222	0.0937	0.0307	0.0079	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(12)	(16)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
BE = 1 $\Phi(x)+\Delta$	0.3388	0.1889	0.0971	0.0530	0.0365	0.0316	0.0305	0.0303
Best upper bound	0.3388	0.1889	0.0971	0.0530	0.0306	0.0079	0.0015	0.0002
Type	BE	BE	BE	BE	C6	C10	C12	C8
$1-\Phi(x)-\Delta$ (lower bound)	0.2783	0.1284	0.0365	0	0	0	0	0
Tail area	0.3211	0.1612	0.0708	0.0252	0.0063	0.0013	0.0002	0.0000
Edgeworth	0.3086	0.1587	0.0668	0.0227	0.0062	0.0013	0.0002	0.0000

Table 4
Bounds for the signed rank statistic scores; $a_n(r,t)=a_{4n}(r,t)=t^2$

	x ($n=25$, $\Delta=0.2520$)							
	0.5	1	1.5	2	2.5	3	3.5	4
Exponential E_1	0.8820	0.6009	0.3088	0.1141	0.0275	0.0035	0.0001	0.0000
bounds E_2	0.8820	0.6012	0.3111	0.1192	0.0328	0.0063	0.0008	0.0001
E_3	0.8823	0.6045	0.3193	0.1286	0.0389	0.0087	0.0014	0.0002
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
bounds (C_p) 4	1	1	0.2748	0.0870	0.0356	0.0172	0.0093	0.0054
6	1	1	0.5256	0.0935	0.0245	0.0082	0.0033	0.0015
8	1	1	1	0.1308	0.0219	0.0051	0.0015	0.0005
10	1	1	1	0.2187	0.0235	0.0038	0.0008	0.0002
12	1	1	1	0.4161	0.0286	0.0032	0.0005	0.0001
CB	1	0.5000	0.2222	0.0913	0.0283	0.0061	0.0009	0.0001
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(22)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$\mathrm{BE}=1-\Phi(x)+\Delta$	0.5606	0.4107	0.3189	0.2748	0.2583	0.2534	0.2523	0.2521
Best upper bound	0.5606	0.4107	0.2222	0.0870	0.0219	0.0032	0.0001	0.0000
Type	BE	BE	C2	C4	C8	C12	E1	E1
$1-\Phi(x)-\Delta$ (lower bound)	0.0565	0	0	0	0	0	0	0
Tail area	0.3127	0.1625	0.0691	0.0219	0.0045	0.0005	0.0000	0.0000
Edgeworth	0.3129	0.1630	0.0681	0.0218	0.0049	0.0006	0.0000	0.0000

	x ($n=50$, $\Delta=0.1792$)							
	0.5	1	1.5	2	2.5	3	3.5	4
Exponential E_1	0.8822	0.6037	0.3169	0.1250	0.0359	0.0071	0.0009	0.0001
bounds E_2	0.8822	0.6038	0.3175	0.1264	0.0374	0.0081	0.0012	0.0001
E_3	0.8824	0.6055	0.3220	0.1318	0.0413	0.0098	0.0017	0.0002
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
bounds (C_p) 4	1	1	0.2854	0.0903	0.0370	0.0178	0.0096	0.0056
6	1	1	0.5887	0.1048	0.0275	0.0092	0.0036	0.0016
8	1	1	1	0.1639	0.0275	0.0064	0.0019	0.0006
10	1	1	1	0.3178	0.0341	0.0055	0.0012	0.0003
12	1	1	1	0.7258	0.0499	0.0056	0.0009	0.0002
CB	1	0.5000	0.2222	0.0925	0.0295	0.0070	0.0012	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(18)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$\mathrm{BE}=1-\Phi(x)+\Delta$	0.4877	0.3379	0.2460	0.2020	0.1854	0.1806	0.1794	0.1792
Best upper bound	0.4877	0.3379	0.2222	0.0903	0.0275	0.0055	0.0009	0.0001
Type	BE	BE	C2	C4	C6	C10	E1	E1
$1-\Phi(x)-\Delta$ (lower bound)	0.1293	0	0	0	0	0	0	0
Tail area	0.3090	0.1595	0.0655	0.0216	0.0050	0.0009	0.0001	0.0000
Edgeworth	0.3107	0.1609	0.0675	0.0222	0.0056	0.0010	0.0001	0.0000

	x ($n=1000$, $\Delta=0.0402$)							
	0.5	1	1.5	2	2.5	3	3.5	4
Exponential E_1	0.8825	0.6064	0.3243	0.1348	0.0435	0.0109	0.0021	0.0003
bounds E_2	0.8825	0.6064	0.3243	0.1348	0.0435	0.0109	0.0021	0.0003
E_3	0.8825	0.6065	0.3245	0.1352	0.0438	0.0110	0.0022	0.0003
E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev $p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
bounds (C_p) 4	1	1	0.2957	0.0936	0.0383	0.0185	0.0100	0.0058
6	1	1	0.6548	0.1165	0.0305	0.0102	0.0041	0.0018
8	1	1	1	0.2028	0.0340	0.0079	0.0023	0.0008
10	1	1	1	0.4530	0.0486	0.0079	0.0017	0.0004
12	1	1	1	1	0.0848	0.0095	0.0015	0.0003
CB	1	0.5000	0.2222	0.0937	0.0307	0.0079	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(10)	(12)	(16)
CN	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$\mathrm{BE}=1-\Phi(x)+\Delta$	0.3488	0.1989	0.1071	0.0630	0.0465	0.0416	0.0405	0.0403
Best upper bound	0.3488	0.1989	0.1071	0.0630	0.0305	0.0079	0.0015	0.0002
Type	BE	BE	BE	BE	C6	C8	C12	C8
$1-\Phi(x)-\Delta$ (lower bound)	0.2683	0.1184	0.0265	0	0	0	0	0
Tail area	0.3110	0.1577	0.0666	0.0231	0.0080	0.0023	0.0003	0.0000
Edgeworth	0.3086	0.1588	0.0668	0.0227	0.0062	0.0013	0.0002	0.0000

Table 5
Bounds for the signed rank statistic scores a_{5n}; $a_n(r,t)=a_{5n}(r,t)$

		x ($n=25$, $\Delta=0.3531$)								x ($n=50$, $\Delta=0.3511$)								x ($n=1000$, $\Delta=0.3534$)							
		0.5	1	1.5	2	2.5	3	3.5	4	0.5	1	1.5	2	2.5	3	3.5	4	0.5	1	1.5	2	2.5	3	3.5	4
Exponential	E_1	0.8784	0.5472	0.1008	0.0009	0.0000	0.0000	0.0000	0.0000	0.8786	0.5502	0.1195	0.0025	0.0000	0.0000	0.0000	0.0000	0.8784	0.5459	0.0965	0.0012	0.0000	0.0000	0.0000	0.0000
bounds	E_2	0.8790	0.5740	0.2585	0.0760	0.0141	0.0017	0.0001	0.0000	0.8791	0.5748	0.2599	0.0769	0.0145	0.0017	0.0001	0.0016	0.8789	0.5736	0.2579	0.0756	0.0140	0.0016	0.0001	0.0000
	E_3	0.8823	0.6045	0.3193	0.1286	0.0389	0.0087	0.0014	0.0002	0.8824	0.6055	0.3220	0.1318	0.0413	0.0098	0.0017	0.0017	0.8825	0.6065	0.3245	0.1352	0.0438	0.0110	0.0022	0.0003
	E_4	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0022	0.8825	0.6065	0.3247	0.1353	0.0439	0.0111	0.0022	0.0003
Chebyshev	$p=2$	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313	1	0.5000	0.2222	0.1250	0.0800	0.0556	0.0408	0.0313
bounds (C_p)	4	1	1	0.1349	0.0427	0.0175	0.0084	0.0046	0.0027	1	0.7061	0.1395	0.0441	0.0181	0.0087	0.0047	0.0028	1	0.6734	0.1330	0.0421	0.0172	0.0083	0.0045	0.0026
	6	1	1	0.1009	0.0180	0.0047	0.0016	0.0006	0.0003	1	1	0.1096	0.0195	0.0051	0.0017	0.0007	0.0003	1	1	0.0977	0.0174	0.0046	0.0015	0.0006	0.0003
	8	1	1	0.0883	0.0088	0.0015	0.0003	0.0001	0.0000	1	1	0.1020	0.0102	0.0017	0.0004	0.0001	0.0000	1	1	0.0842	0.0084	0.0014	0.0003	0.0001	0.0000
	10	1	1	0.0878	0.0049	0.0005	0.0001	0.0000	0.0000	1	1	0.1090	0.0061	0.0007	0.0001	0.0000	0.0000	1	1	0.0829	0.0047	0.0005	0.0001	0.0000	0.0000
	12	1	1	0.0970	0.0031	0.0002	0.0000	0.0000	0.0000	1	1	0.1306	0.0041	0.0003	0.0000	0.0000	0.0000	1	1	0.0914	0.0029	0.0002	0.0000	0.0000	0.0000
CB		1	0.5000	0.2222	0.0913	0.0283	0.0061	0.0009	0.0001	1	0.5000	0.2222	0.0925	0.0295	0.0070	0.0012	0.0002	1	0.5000	0.2222	0.0937	0.0307	0.0079	0.0015	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(10)	(14)	(22)	(2)	(2)	(2)	(4)	(6)	(10)	(14)	(18)	(2)	(2)	(2)	(4)	(6)	(10)	(12)	(16)
CN		1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002	1	0.5000	0.2222	0.0938	0.0307	0.0080	0.0015	0.0002
(p^*)		(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)	(2)	(2)	(2)	(4)	(6)	(8)	(12)	(16)
$BE=1-\Phi(x)+\Delta$		0.6617	0.5118	0.4199	0.3759	0.3593	0.3545	0.3533	0.3531	0.6597	0.5098	0.4180	0.3739	0.3574	0.3525	0.3514	0.3512	0.6619	0.5120	0.4202	0.3761	0.3596	0.3547	0.3536	0.3534
Best upper bound		0.6617	0.5000	0.0878	0.0009	0.0000	0.0000	0.0000	0.0000	0.6597	0.5000	0.1020	0.0025	0.0000	0.0000	0.0000	0.0000	0.6619	0.5000	0.0829	0.0012	0.0000	0.0000	0.0000	0.0000
Type		BE	C2	C10	E1	E1	E1	E1	E1	BE	C2	C8	E1	E1	E1	E1	E1	BE	C2	C10	E1	E1	E1	E1	E1
$1-\Phi(x)-\Delta$ (lower bound)		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Tail area		0.4634	0.2090	0.0157	0.0000	0.0000	0.0000	0.0000	0.0000	0.4585	0.1945	0.0206	0.0003	0.0000	0.0000	0.0000	0.0000	0.4656	0.2284	0.0174	0.0002	0.0000	0.0000	0.0000	0.0000
Edgeworth		0.3415	0.1916	0.0767	0.0154	−0.0035	−0.0041	−0.0017	−0.0004	0.3406	0.1907	0.0764	0.0156	−0.0032	−0.0039	−0.0016	−0.0004	0.3419	0.1920	0.0768	0.0153	−0.0036	−0.0041	−0.0017	−0.0004

(2) For Chebyshev bounds, using higher-order moments can yield very substantial improvements over the usual second-order inequality. The greater the value of x, the greater the order of the moment providing the best Chebyshev bound. As expected, the bound based on binomial moments is always smaller than the bound based on normal moments, but the latter is slightly easier to compute.

(3) There is no uniform ordering between the best exponential bound, the best Chebyshev bound and the Berry–Esséen–Zolotarev upper bound: (a) the exponential bound is the best upper bound for large x; (b) the Berry–Esséen–Zolotarev bound is best for low values of x; (c) the Chebyshev bounds are best for intermediate values of x (e.g., $1.0 \leqslant x \leqslant 3.0$). The three types of bounds are thus useful but the ranges over which each of them is best, differ from one statistic to the other.

(4) The Berry–Esséen–Zolotarev lower bound $1-\Phi(x)-\Delta$ is clearly less informative than its upper counterpart (often yielding values smaller than 0) but still can be useful for low values of x, especially when the sample size is large.

(5) When compared with the actual tail probability, the best upper bound is usually tighter for larger values of x (for $x \geqslant 2$, say), which is the range of practical concern. In the case of the fifth score (Table 5), it is interesting to note that, for $x \geqslant 1.5$, the best upper bound gives tail areas smaller than those resulting from a simple normal approximation.

(6) Edgeworth approximations at first sight perform quite well for the first four scores, where sufficient smoothness is present. However,

(6a) they behave extremely badly for the fifth score: for small values of x ($x=0.5$, 1.0), they severely underestimate the actual tail areas; for intermediate values ($x=2.0$), they overestimate them to the point of being more conservative than the best upper bound; for large values of x ($x>2.5$), they yield negative tail areas;

(6b) though fairly close, for the first four scores, to the actual or simulated tail areas, they cannot be considered perfectly safe: when actual tail areas are available (Wilcoxon scores, $n=25$, 50), it is clear again that the Edgeworth approximation can underestimate the true value ($n=25$, $x=1.0$, 2.0; $n=50$, $x=1.5$, 2.0); further, for $n=1000$, the Edgeworth approximation is smaller than the tail area (obtained by simulation) in 19 cases out of 32.

The fifth score is an example of a 'non-smooth' score-generating function, where both the normal approximation and Edgeworth expansions are quite unreliable. Such a poor performance is to be feared in all situations where, due to ill-balanced regression constants, the a_n scores present irregular features. The bounds we are proposing here may then be especially useful. Even in the case of 'smooth' score-generating functions, Edgeworth expansions nevertheless appear to provide estimates rather than upper bounds for tail areas, so that their use to obtain tests that do not reject too often under the null hypothesis is questionable.

Globally, these few numerical results clearly show that the bounds we are proposing in this paper provide a simple way of bounding null distributions of linear signed rank statistics without any assumption on the form of the scores. Of course, when the bounds do not yield a conclusive test, other exact methods, e.g., the

Pagano and Tritchler (1983) algorithm (if feasible), or an approximation may be employed to reach a conclusion.

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