Distribution-free bounds for serial correlation coefficients *

Jean-Marie Dufour † — Université de Montréal
Abdeljelil Farhat ‡ — Université de Montréal
Marc Hallin § — Université Libre de Bruxelles

First version: August 1993
Revised: June 2003
This version: August 2003
Compiled: December 3, 2003, 6:04pm

*The authors thank two anonymous referees and the Editor, Ronald Gallant, for several useful comments. This work was supported by the the Belgian Federal Government (P.A.I. contract), the Canada Research Chair Program (Chair in Econometrics, Université de Montréal), the Institut de Finance mathématique de Montréal (IFM2), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems*, (MITACS)], the Communauté françcaise de Belgique (A.R.C. contract), the Intercollegiate Center for Management Sciences (Bruxelles), the Fonds de recherche sur la société et la culture (Québec), the Fonds de recherche sur la nature et les technologies (Québec), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, the Canada Council for the Arts (Killam Fellowship), and the Alexander-von-Humboldt Foundation (Germany).

† Canada Research Chair Holder (Econometrics). Centre interuniversitaire de recherche en analyse des organisations (CIRANO), Centre interuniversitaire de recherche en économie quantitative (CIREQ), and Département de sciences économiques, Université de Montréal. Mailing address: Département de sciences économiques, Université de Montréal, C.P. 6128 succursale Centre-ville, Montréal, Québec, Canada H3C 3J7. TEL: 1 514 343 2400; FAX: 1 514 343 5831; e-mail: jean.marie.dufour@umontreal.ca. Web page: http://www.fas.umontreal.ca/SCECO/Dufour

‡ Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Centre interuniversitaire de recherche en économie quantitative (CIREQ). Adresse postale: CIREQ, Université de Montréal, C.P. 6128 succursale Centre-ville, Montréal, Québec, Canada H3C 3J7. TEL: 1 514 343 6557; FAX: 1 514 343 5831; courriel: abdeljelil.farhat@umontreal.ca.

§ Institut de Statistique, Université Libre de Bruxelles, Campus Plaine, C.P. 210, B-1050 Bruxelles, Belgium

ABSTRACT

We consider the problem of testing whether the observations $X_1, \ldots, X_n$ of a time series are independent with unspecified (possibly nonidentical) distributions symmetric about a common known median. Various bounds on the distributions of serial correlation coefficients are proposed : exponential bounds, Eaton-type bounds, Chebyshev bounds and Berry-Esséen-Zolotarev bounds. The bounds are exact in finite samples, distribution-free and easy to compute. The results are applied to U.S. data on interest rates.

Key words : autocorrelation; nonparametric test; distribution-free test; heteroskedasticity; interest rates; robustness.
JEL classification code : C14, C22, C12, C32, E4

Contents

List of Propositions and Theorems **ii**

1. Introduction **1**

2. Exponential bounds **3**

3. Bounds based on moments **4**

4. Berry-Esséen-Zolotarev bounds **6**

5. Simulation experiment **7**

6. Empirical illustration **12**

A. Appendix: Proofs **15**

B. Appendix: Conditional moments of the autocorrelations **16**

List of Propositions and Theorems

2.1 Proposition : Exponential bounds . . . 3
3.1 Proposition : Improved Eaton-Pinelis bounds . . . 5
3.2 Proposition : Generalized Chebyshev bounds . . . 5
4.1 Proposition : Berry-Esséen-Zolotarev bounds . . . 6
Proof of Proposition **2.1** . . . 15
Proof of Proposition **3.1** . . . 16
Proof of Proposition **3.2** . . . 16
Proof of Proposition **4.1** . . . 16

List of Tables

1 Heterokedasticity patterns studied . . . 8
2 Empirical levels of alternative tests for serial dependence at nominal level 0.05 . . 9
3 Empirical levels of alternative tests for serial dependence at nominal level 0.05 . . 10
4 Empirical levels of alternative tests for serial dependence at nominal level 0.05 . . 11
5 Commercial Paper Interest Rate (logarithm, first differences): Autocorrelations and bounds on p-values for two-sided tests . . . 13
6 Commercial Paper Interest Rate (logarithm, second differences) : Autocorrelations and bounds on p-values for two-sided tests . . . 14

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a time series of length n. In many situations, it is of interest to test whether the $X_t^\prime s$ are independent against an alternative of serial dependence, say, at lag $k\ (k \geq 1)$. If under the null hypothesis the observations are assumed to be identically distributed with known means μ_t, a natural test consists in rejecting the null hypothesis for large or small values of the autocorrelation coefficient

$$r_k = \sum_{t=1}^{n-k} (X_t - \mu_t) \left(X_{t+k} - \mu_{t+k}\right) \Big/ \sum_{t=1}^{n} (X_t - \mu_t)^2 \,, \tag{1.1}$$

where $1 \leq k \leq n-1$. Under general regularity conditions, the distribution of r_k is approximately normal with mean zero and variance n^{-1}; see Anderson (1971, chapter 8) or Brockwell and Davis (1991, chapter 7).

When the observations are not identically distributed, such a procedure can clearly be inappropriate. In this paper, we study the null hypothesis H_0 under which the observations $X_1, \ldots, X_n$ are independent but possibly nonidentically distributed, with distributions symmetric about known medians μ_t. No assumption about the existence of the moments of $X_1, \ldots, X_n$ is made, and the distribution of the observations can be discrete. Without loss of generality, we can assume that $\mu_1 = \cdots = \mu_n = 0$, and we henceforth set $\mu_t = 0,\ t = 1, \ldots, n$.

The hypothesis H_0 is "nonparametric" in the sense that no finite-dimensional parameter vector can determine entirely the probability distribution of the observations $X_1, X_2, \ldots, X_n$. Following standard terminology [see Lehmann (1986, sections 3.1 and 3.5)], a test of H_0 has *level* α if the probability of rejecting H_0 is *not greater than* α under any distribution of $X = (X_1, \ldots, X_n)^\prime$ included in H_0 $(0 < \alpha < 1)$. If moreover the supremum of the rejection probabilities over H_0 is equal to α, one says that the test has *size* α. Since H_0 covers a wide variety of probability distributions and because of the "parametric origin" of the coefficient r_k, the distribution of r_k under H_0 depends on the *form* of the distribution of the observations. Without additional assumptions, it is unknown. Consequently, no *similar* critical region of the type $|r_k| > c$ (where c is a nonstochastic critical point which depends on the level of the test) does exist: *i.e.*, for $0 < c < 1$, the probability of the event $|r_k| > c$ is not constant over the set of data generating processes (DGP) in H_0, and finding a valid critical value involves bounding the distribution of r_k over H_0 or considering data-dependent critical regions for r_k. In particular, there is strictly no guarantee that the actual sizes of tests based on the asymptotic (normal) distribution of r_k will be less than or equal to their nominal level (as tests of H_0) in finite samples. The same will hold *a fortiori* for critical values obtained under parametric assumptions, *e.g.*, the assumption that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables according to $N\left(0,\, \sigma^2\right)$ distribution [in which case exact critical values may be computed using Imhof's algorithm]: such critical values – though they belong to daily practice – simply do not yield valid tests of the nonparametric hypothesis H_0.

The objective of this paper is to develop *finite-sample* (α-level) tests based on r_k for the *nonparametric null hypothesis* H_0. In other words, we need to ensure that the probability of rejecting H_0 is not greater than α under any DGP in H_0. This problem is quite distinct from the one where one tries to approximate the distribution of r_k under some *specific* distribution included in H_0 (like the i.i.d. Gaussian model). Following a classic nonparametric technique, we shall do this here by using an appropriate *conditioning*. When $X_1, X_2, \ldots, X_n$ are absolutely continuous, the vector of absolute values $|X| = (|X_1|, \ldots, |X_n|)'$ is a complete sufficient statistic for H_0. Further, classical arguments of similarity and Neyman structure lead one to consider tests that are conditional with respect to the complete sufficient statistic $|X|$; see Lehmann (1986, chapter 4). The conditional distribution of $X = (X_1, X_2, \ldots, X_n)'$ given $|X|$ is then determined by the distribution of the signs of $X_1, \ldots, X_n$. Since, under H_0, the signs are independent symmetric Bernoulli variables, the conditional distribution of r_k (given the vector of absolute values $|X|$) may in principle be computed, *e.g.*, by enumeration. In practice, however, the conditional distribution of r_k depends on each specific sample, because it is a function of $|X|$, and so finding critical values may be difficult. This problem is also met in the well-known case of permutation t-tests; see Pratt and Gibbons (1981, chapter 4).

For the problem of testing H_0 against location-shift alternatives, simple bounds for the conditional and unconditional distributions of the t-statistic were provided in Edelman (1986, 1990) and Dufour and Hallin (1991, 1993); similar bounds for general linear signed rank statistics have also been proposed in Dufour and Hallin (1992). Beyond the important advantage of exactness for any sample size, extensive comparisons in Dufour and Hallin (1991, 1992, 1993) indicate that the bounds studied (exponential, Chebyshev-type, Eaton-type, Berry-Esséen) can be surprisingly tight, especially if one takes the minimum of the various bounds.

In this paper, we give analogous results for tests of H_0 based on r_k against serial dependence alternatives. Four types of bounds are presented: (1) exponential bounds (Proposition **2.1**); (2) improved Eaton bounds (Proposition **3.1**); (3) Chebyshev-type bounds (Proposition **3.2**); (4) Berry-Esséen-Zolotarev bounds (Proposition **4.1**). The exponential bounds are based on the conditional moment generating function of r_k (given $|X|$), the improved Eaton and Chebyshev-type bounds on conditional moments of r_k (a truncated third moment in the case of the Eaton bound), while the Berry-Esséen-Zolotarev bound is based on the normal distribution function. The exponential, Eaton, Chebyshev and Berry-Esséen bounds extend to the case of autocorrelation coefficients the bounds proposed in Dufour and Hallin (1991, 1992, 1993).

All these bounds are exact in finite samples and simple to compute. They are applicable despite the presence of general forms of nonnormality and heteroskedasticity (provided the symmetry hypothesis holds). In particular, no assumption about the existence of moments is required, and the variables considered may have continuous or discrete distributions. None of the bounds given uniformly dominates the others. While the three first classes of bounds are especially useful to obtain upper bounds for small tail areas, the Berry-Esséen bounds can be tighter for larger tail areas (*i.e.*, tails associated with points that are closer to the center of the distribution) and yield lower bounds on tail areas as well. Conservative conditional (given $|X|$) as well as unconditional conservative p-values, or critical points, for tests based on r_k can be obtained from any one of these bounds. Since all the bounds are simple to compute, the obvious strategy here is to take the smallest p-value yielded by the different bounds (or, equivalently, the tightest critical point). Such p-values provide a useful nonparametric check on the significance of tests based on autocorrelation coefficients.

The exponential bounds are described in section 2, the Eaton and Chebyshev bounds are given in section 3, while the Berry-Esséen bounds are derived in section 4. In section 5, simulation results on the performance of the bounds are presented. In section 6, we illustrate the use of the bounds by applying them to data on commercial paper interest rates in the U.S.

2. Exponential bounds

In the following proposition, we derive exponential bounds for the tail areas of the conditional distributions of r_k given $|X|$ under the null hypothesis that $X_1, \ldots, X_n$ are independent with distributions symmetric about zero. The proofs of the propositions appear in Appendix A.

Proposition 2.1 EXPONENTIAL BOUNDS. *Let $X_1, \ldots, X_n$ be independent random variables with distributions symmetric about zero, $|X| = (|X_1|, \ldots, |X_n|)'$, and*

$$r_k = \sum_{t=1}^{n-k} X_t X_{t+k} \Big/ \sum_{t=1}^{n} X_t^2, \quad 1 \leq k \leq n-1, \tag{2.1}$$

$$w_{kt} = |X_t X_{t+k}| \Big/ \left(\sum_{\tau=1}^{n-k} X_\tau^2 X_{\tau+k}^2 \right)^{\frac{1}{2}}, \quad t = 1, \ldots, n-k, \tag{2.2}$$

where we use the convention $0/0 = 0$. Then the conditional distribution of r_k given $|X|$ is symmetric about zero and, for all $y > 0$ and $1 \leq k \leq n-1$,

$$\begin{aligned} \mathsf{P}[r_k \geq y \mid |X|] &\leq B_k(y_k, |X|) \leq \exp\left(-y_k^2\right) \prod_{t=1}^{n-k} \cosh(w_{kt} y_k) \\ &\leq \exp\left(-y_k^2\right) \left[\cosh\left(y_k / \sqrt{n_k^*}\right)\right]^{n_k^*} \leq \exp\left(-y_k^2 / 2\right), \end{aligned} \tag{2.3}$$

where $y_k = y/D_k(|X|)$, $\cosh(x) = (e^x + e^{-x})/2$, $n_k^* = card(\{t : |X_t X_{t+k}| \neq 0, \ 1 \leq t \leq n-k\})$ is the number of products $X_t X_{t+k}$ different from zero,

$$\begin{aligned} D_k(|X|) &= \left(\sum_{t=1}^{n-k} X_t^2 X_{t+k}^2 \right)^{\frac{1}{2}} \Big/ \left(\sum_{t=1}^{n} X_t^2 \right), \\ B_k(y, |X|) &= \inf_{z \geq 0} \left\{ \exp(-zy) \prod_{t=1}^{n-k} \cosh(w_{kt} z) \right\} \end{aligned}$$

and the four bounds in (2.3) *are set equal to zero when $D_k(|X|) = 0$.*

From the symmetry of the conditional distribution of r_k, it is clear that $\mathsf{P}[|r_k| \geq y \mid |X|] = 2\mathsf{P}[r_k \geq y \mid |X|] = 2\mathsf{P}[r_k \leq -y \mid |X|]$, so that (2.3) can also be used to bound $\mathsf{P}[r_k \leq -y]$ and $\mathsf{P}[|r_k| \geq y \mid |X|]$ for any $y > 0$. In (2.3), four bounds on the tail areas $\mathsf{P}[r_k \geq y \mid |X|]$ are given. Denote them by $E_{1k} \leq E_{2k} \leq E_{3k} \leq E_{4k}$ in ascending order. These bounds are increasingly looser, but the larger ones are easier to compute. In particular, E_{2k}, E_{3k} and E_{4k} only require information about the second empirical moments of the sample (r_k and $\sum X_t^2$), which may be useful when the complete observation vector $X = (X_1, \ldots, X_n)'$ is not available to an investigator. The exponential bound $E_{4k} = \exp\left(-y_k^2/2\right)$ is similar to a bound given by Edelman (1986) and Efron (1969) for the case of t-statistics; for an earlier related result, see also Hoeffding (1963). In contrast with the case of t-statistics, however, this bound now explicitly depends on $|X|$ through $D_k(|X|)$. The second largest bound $E_{3k} = \exp\left(-y_k^2\right)\left[\cosh\left(y_k \,/\, \sqrt{n_k^*}\right)\right]^{n_k^*}$ uniformly improves the latter by explicitly taking into account the sample size and the lag. It is based on a result given by Eaton (1970) for linear combinations of independent Bernoulli variables. For example, for $n-k=10$ and $y_k=3$, we have $E_{3k}=0.0064$ while $E_{4k}=0.0111$. Similarly, the bound $E_{2k} = \exp\left(-y_k^2\right) \mathit{\Pi} \cosh\left(w_{kt} y_k\right)$ improves the two previous ones by explicitly taking into account the weights $w_{kt},\ t=1, \ldots, n-k$. When the weights are equal, *i.e.*, $w_{kt} = 1/\sqrt{n_k^*},\ t=1, \ldots, n-k$, the bounds E_{2k} and E_{3k} coincide. In the other cases, E_{2k} can yield substantial improvements over E_{3k}, especially when the data contain a large outlier. For example, if $w_{kt} \to 0$, $n_k^* = 10$ and $y_k = 3$, the ratio E_{2k}/E_{3k} converges to 0.1933. Finally, the smallest bound $E_{1k} \equiv B_k(y_k, |X|)$ is obtained by finding the infimum of the function $M_k(z) = \exp(-zy_k)\prod_{t=1}^{n-k}\cosh(w_{kt}z)$ for $z \geq 0$. E_{4k} in turn can yield substantial improvement over the previous bounds. The function $B_k(y, |X|)$ has the following more explicit expression :

$$
\begin{aligned}
B_k(y, |X|) &= 0\,, && \text{if } \sum_{t=1}^{n-k} w_{kt} < y, \\
&= \left(\tfrac{1}{2}\right)^{n_k^*}\,, && \text{if } \sum_{t=1}^{n-k} w_{kt} = y, \\
&= \exp\left(-z_k^* y\right)\prod_{t=1}^{n-k}\cosh(w_{kt} z_k^*)\,, && \text{if } \sum_{t=1}^{n-k} w_{kt} > y,
\end{aligned}
\tag{2.4}
$$

and z_k^* is the unique positive value of z that solves the equation

$$
\sum_{t=1}^{n-k} w_{kt}\left[\left(1-e^{-2w_{kt}z}\right) \,/\, \left(1+e^{-2w_{kt}z}\right)\right] = y\,.
$$

It is fairly easy to compute $B_k(y, |X|)$ by numerical methods; for further discussion, see Dufour and Hallin (1992, pp. 315-317).

Since they depend on $|X|$ only through $D_k(|X|)$, the two largest bounds E_{3k} and E_{4k} in (2.3) also yield simple unconditional bounds : for all $y > 0$,

$$
\mathsf{P}\left[r_k \geq y D_k(|X|)\right] \leq \exp\left(-y^2\right)\left[\cosh\left(y/\sqrt{n-k}\right)\right]^{n-k} \leq \exp\left(-y^2/2\right)\,. \tag{2.5}
$$

However, in most practical cases, the weights w_{kt} are known so that the better bounds E_{1k} and E_{2k} are available: conditional critical values based on the latter always yield less conservative tests (both conditionally and unconditionally).

3. Bounds based on moments

The exponential bounds described in Proposition **2.1** are based on the conditional moment generating function of r_k. In this section, we give two sets of bounds based on considering appropriate moments of r_k. The first one applies results from Eaton (1970), Pinelis (1991) and Dufour and Hallin (1993), and is based on minimizing a truncated third order moment. We denote by $\varphi(y) = (2\pi)^{-1/2} \exp\left(-y^2/2\right)$ the $N(0,\,1)$ density function, and by $(y)_+$ the positive part of any real number y, *i.e.*, $(y)_+ = \max(0,\,y)$.

Proposition 3.1 IMPROVED EATON-PINELIS BOUNDS. *Under the assumptions and notations of Proposition* **2.1***, we have :*

$$\begin{aligned} \mathsf{P}\left[r_k \geq y \mid |X|\right] &\leq \min\left\{B_E\left(y_k;\, n_k^*\right),\, 0.5\, y_k^{-2},\, 0.5\right\} = B_{EP}^*\left(y_k;\, n_k^*\right) \\ &\leq \min\left\{B_E\left(y_k\right),\, 0.5\, y_k^{-2},\, 0.5\right\} = B_{EP}\left(y_k\right) \end{aligned} \quad (3.1)$$

for all $y > 0$, where

$$B_E(y;\, m) = (0.5) \inf_{0 \leq c < y} \left\{ (0.5)^m \sum_{j=0}^{m} \binom{m}{j} f_c\left[(m/4)^{-1/2}\left(j - (m/2)\right)\right] / (y-c)^3 \right\}, \quad (3.2)$$

$f_c(x) = \left[(|x| - c)_+\right]^3$, $\binom{m}{j} = m!/\left[j!\,(m-j)!\right]$, *and*

$$\begin{aligned} B_E(y) &= \inf_{0 \leq c < y} \int_c^\infty \left(\frac{z-c}{y-c}\right)^3 \varphi(z)\, dz \\ &= \inf_{0 \leq c < y} \left\{\left[\varphi(c)\left(2 + c^2\right) - \left(1 - \Phi(c)\right)\left(c^3 + 3c\right)\right] / (y-c)^3\right\}. \end{aligned} \quad (3.3)$$

Calculation of the bounds, $B_{EP}^*(y;\, m)$ and $B_{EP}(y)$ is discussed in Dufour and Hallin (1993), where the associated (conservative) critical values for standard significance levels are also reported. It is of interest to note that the bound B_{EP} enjoys an optimality property in the sense that it is tightest among all bounds based on expectations of convex functions of a standard normal variable; see Pinelis (1991) and Dufour and Hallin (1993). Note also that the function $B_E(y;\, m)$ is monotonic increasing in m, *i.e.*, $B_E(y;\, m) \leq B_E(y;\, m+1)$ for $y > 0$.

Another related method consists in bounding the tail areas of r_k with Chebyshev-type inequalities. As observed in Dufour and Hallin (1992), such bounds can be quite tight, especially if they are based on higher-order moments (i.e., moments of order greater than 2). We summarize these in the following proposition.

Proposition 3.2 GENERALIZED CHEBYSHEV BOUNDS. *Let the assumptions and notations of Proposition* **2.1** *hold. Then, for any positive even integer p and for any $y > 0$,*

$$\mathsf{P}\left[r_k \geq y \mid |X|\right] \leq \frac{E\left[r_k^p \mid |X|\right]}{2y^p} \leq \frac{D_k\left(|X|\right)^p E\left[Y\left(n_k^*\right)^p\right]}{2y^p}$$

$$\leq \left[\frac{(p-1)(p-3)\cdots 3\cdot 1}{2y^{p}}\right] D_k(|X|)^p \,, \tag{3.4}$$

and

$$\mathsf{P}\left[r_k \geq y \mid |X|\right] \leq \left[\frac{(p_k^*-1)(p_k^*-3)\cdots 3\cdot 1}{2y^{p_k^*}}\right] D_k(|X|)^{p_k^*} \,, \tag{3.5}$$

where $Y(m)$ refers to a $Bin(m,\,0.5)$ random variable, $p_k^* = \max\{2,\,\overline{p}_k\}$ and $\overline{p}_k$ is the largest even integer such that $\overline{p}_k < 1 + y_k^2$.

To implement the first bound in (3.4), we need the conditional moments of r_k given $|X|$. These can be established easily from (A.1), (A.2) and (A.3) in the proof of Proposition **2.1** and equations (3.2) to (3.6) in Dufour and Hallin (1992); the appropriate expressions are given in Appendix B. Even moments $E\left(r_k^p \mid |X|\right)$ of order greater than 12 can be established by analogous methods, but the algebra is correspondingly more involved. The standardized binomial moments can be computed up to any desired order from formulae (3.8) and (3.9) in Dufour and Hallin (1992), and so the two larger bounds in (3.4) above can be obtained easily for any value of p. Clearly, the bounds in (3.4) can be computed for several values of p and the minimum of these bounds again provides a valid bound. The bound (3.5) is the explicit solution of this minimization process (over all even values of $p \geq 2$) based on the third bound in (3.4), which is based on the moments of a $N(0,\,1)$ distribution.

4. Berry-Esséen-Zolotarev bounds

The results of the two previous sections yield *upper* bounds on the tail areas of autocorrelation coefficients under the null hypothesis of independence, and they can therefore be used to check whether we can safely reject the null hypothesis at a given level under relatively weak nonparametric assumptions. Further, these bounds are reasonably tight only when y is not too small (say, $y > 1.5$). In many cases, it would also be helpful to have a *lower* bound which could be used to decide whether an autocorrelation coefficient unambiguously lies in the acceptance region of the (conditional) test based on r_k.

Unfortunately, it appears much more difficult to obtain lower bounds similar to the upper bounds previously given. In order to obtain such lower bounds as well as upper bounds whose behavior may be more satisfactory for lower values of y, we will consider bounds of the Berry-Esséen type. More precisely, in the following proposition, we combine results of van Beek (1972) and Zolotarev (1965) to bound the distance between the conditional distribution of r_k and a normal distribution.

Proposition 4.1 BERRY-ESSÉEN-ZOLOTAREV BOUNDS. *Under the assumptions and notations of Proposition* **2.1** *and provided $X_t X_{t+k} \neq 0$ for at least one $t\,(1 \leq t \leq n-k)$, we have*

$$\begin{aligned}&\left|\mathsf{P}\left[r_k < y \mid |X|\right] - \Phi\left[y \,/\, D_k(|X|)\right]\right| \\ &\qquad \leq \Delta \equiv \min\left\{0.7975 \sum_{t=1}^{n} |w_{kt}|^3 \,,\; 0.366145 \left(\sum_{t=1}^{n} |w_{kt}|^3\right)^{1/4}\right\}\end{aligned}$$

$$\leq 0.366145 \tag{4.1}$$

for all y, where $\Phi(y)$ denotes the $N(0, 1)$ distribution function.

It is clear that inequality (4.1) can provide both upper and lower bounds on the tail areas of r_k :

$$BE_L \equiv 1 - \Phi[y \,/\, D_k(|X|)] - \Delta \leq \mathsf{P}[r_k \geq y \mid |X|] \leq 1 - \Phi[y \,/\, D_k(|X|)] + \Delta \equiv BE_U. \tag{4.2}$$

The normal approximation is seen to be good when $\sum |w_{kt}|^3$ is small. It also follows from (4.1) that the conditional distribution of r_k given $|X|$ (hence also its unconditional distribution) converges to a normal distribution when $\sum |w_{kt}|^3$ goes to zero. But, of course, the main interest of (4.1) lies in the fact that it is an operational finite-sample approximation result, not a convergence theorem.

5. Simulation experiment

In order to provide some evidence on the size and power of the proposed bounds, we considered an AR(1) process of the form:

$$X_t = \rho X_{t-1} + u_t\,, \quad t = 1, \ldots, n, \tag{5.1}$$
$$u_t = d_t\, v_t\,, \quad t = 1, \ldots, n, \tag{5.2}$$

where the variables $v_t\,,\ \ t = 1, \ldots, n$, are i.i.d., the d_t's are scale parameters which determine the form of the heteroskedasticity, and $X_t = 0$ (fixed). Two types of distributions for v_t were considered:

$$\text{(G)} \quad v_t \overset{i.i.d.}{\sim} N(0, 1), \quad t = 1, \ldots, n, \tag{5.3}$$
$$\text{(C)} \quad v_t \overset{i.i.d.}{\sim} Cauchy, \quad t = 1, \ldots, n. \tag{5.4}$$

For the error heterogeneity, the patterns described in Table 1 were. studied.

The results of the simulations are reported in Tables 2 - 4. In these tables, the statistics $|t(r_1)|$, $|\tilde{t}(r_1)|$, $|t(\hat{\rho}_k)|$ and $|\bar{t}(\hat{\rho}_k)|$ represent four alternative ways of standardizing standard (parametric) autocorrelation coefficients while E_{11} represents the best exponential bound. The autocorrelation statistics are:

$$|t(r_k)| = |\sqrt{n}\, r_k|\,, \quad |\tilde{t}(r_k)| = |\, r_k/\sigma_k|\,, \tag{5.5}$$
$$|t(\hat{\rho}_k)| = |\sqrt{n}\, \hat{\rho}_k|\,, \quad |\bar{t}(\hat{\rho}_k)| = |(\hat{\rho}_k - \mu_k)/\sigma_k|\,, \tag{5.6}$$

where r_k is defined in (2.1),

$$\hat{\rho}_k = \sum_{t=1}^{n-k} (X_t - \bar{X})(X_{t+k} - \bar{X}) \,/\, \sum_{t=1}^{n} (X_t - \bar{X})^2 \tag{5.7}$$

is the usual "centered" autocorrelation coefficient, while $\mu_k = -(n-k)/[n(n+1)]$ and $\sigma_k^2 =$

Table 1. Heteroskedasticity patterns studied

	Type			
M1	Homoskedasticity	$d_t =$	$1\,,$	$t = 1, ..., n$
M2	One outlier I	$d_t =$	$10\,,$	if $t = n/2$
		$=$	$1\,,$	otherwise
M3	One outlier II	$d_t =$	$100\,,$	if $t = n/2$
		$=$	$1\,,$	otherwise
M4	Exponential I	$d_t =$	$e^{t/10}\,,$	$t = 1, ..., n$
M5	Exponential II	$d_t =$	$e^{t/2}\,,$	$t = 1, ..., n$
M6	Two outliers I	$d_t =$	$10\,,$	if $t = \frac{n}{2}$ or $\frac{n}{2} + 1$
			$1\,,$	otherwise
M7	Two outliers II	$d_t =$	$100\,,$	if $t = \frac{n}{2}$ or $\frac{n}{2} + 1$
			$1\,,$	otherwise
M8	Two outliers III	$d_t =$	$10^6\,,$	if $t = \frac{n}{2}$ or $\frac{n}{2} + 1$
			$1\,,$	otherwise

$(n-k)/[n(n+2)]$ are the adjusted mean and variance suggested in Dufour and Roy (1985) for the case of a sequence of i.i.d. observations.

For each of the above parametric statistics, we also report the results of tests based on four ways of computing critical values: (1) standard asymptotic normal critical values; (2) critical points based on the Imhof algorithm assuming the observations are i.i.d. Gaussian (the Imhof critical values were also cross-checked by simulation); (3) critical values obtained by simulation under each specific distribution and heteroskedasticity pattern considered; (4) critical values based on the largest critical that we could find over the different null hypotheses considered. Of course, the second method is best choice under the assumption made by the Imhof algorithm, but will not correct the level under the other independence models (e.g., with heteroskedasticity). The third method provides a theoretical benchmark that cannot be achieved in practice, because the heteroskedasticity pattern is typically unknown in practice. The fourth method is the one closest to what one would like to do to obtain a distribution-free tests that is robust to non-normality and heteroskedasticity of unknown form, based on these statistics. It is not clear, however, that the (marginal) distributions of these statistics under the null hypothesis can be bounded in a useful way under the (very wide) hypotheses considered by the conditional bound procedures we propose [for further of discussion of this point, see Pratt and Gibbons (1981, chapter 4), Dufour and Hallin (1991, section 1), Dufour (2003, section 4.2)]. We do not have a way of producing provably valid critical values for these tests. So the "size-corrected" critical values used for the unconditional tests remain too , "small" and the powers presented overestimate the true power of these procedures for the nonparametric null hypothesis studied.

All the tests are performed at the 0.05 nominal level. Sample sizes $n = 30$, 60 were considered.

Table 2. Empirical levels of alternative tests for serial dependence at nominal level $\alpha = 0.05$

Sample size: $n = 30$	Asymptotic tests and bounds									
Error distribution (v_t)	N(0, 1)					Cauchy				
Heteroskedasticity type	M1	M2	M5	M7	M8	M1	M2	M5	M7	M8
$\lvert t(r_1)\rvert$	3.90	1.67	33.33	49.96	50.49	2.47	2.43	17.57	24.15	33.96
$\lvert \tilde{t}(r_1)\rvert$	4.96	2.20	35.82	52.78	53.31	2.95	2.88	19.21	26.34	36.06
$\lvert t(\hat{\rho}_k)\rvert$	4.22	1.91	31.71	47.21	47.76	2.43	2.32	16.68	22.52	32.00
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	4.65	2.17	34.22	50.50	51.09	2.88	2.62	18.28	24.44	34.35
E_{11}	0.95	0.87	2.10	0.77	0.00	1.10	1.33	2.94	1.70	0.00
Best bound	1.11	1.00	2.10	0.78	0.00	1.16	1.36	2.94	1.70	0.00
	Tests based on Imhof critical values									
$\lvert t(r_1)\rvert$	5.09	2.30	36.25	53.20	53.88	3.04	2.96	19.50	26.68	36.48
$\lvert \tilde{t}(r_1)\rvert$	5.45	2.49	36.99	54.05	54.77	3.21	3.17	19.92	27.17	37.20
$\lvert t(\hat{\rho}_k)\rvert$	5.57	2.53	34.60	50.82	51.68	2.95	3.10	18.50	25.35	35.06
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	5.20	2.47	35.49	51.98	52.53	3.08	2.82	19.14	25.78	35.39
	Tests based on global size correction									
$\lvert t(r_1)\rvert$	0.01	0.00	4.67	0.00	0.00	0.03	0.00	1.42	0.13	0.00
$\lvert \tilde{t}(r_1)\rvert$	0.01	0.00	4.67	0.00	0.00	0.03	0.00	1.42	0.13	0.00
$\lvert t(\hat{\rho}_k)\rvert$	0.02	0.00	4.69	0.00	0.00	0.02	0.00	1.26	0.11	0.00
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	0.03	0.00	4.70	0.00	0.00	0.02	0.01	1.45	0.16	0.00
	Tests based on model-specific size correction									
$\lvert t(r_1)\rvert$	4.76	4.86	4.67	4.71	5.12	5.07	5.05	4.67	5.12	4.87
$\lvert \tilde{t}(r_1)\rvert$	4.76	4.86	4.67	4.71	5.12	5.07	5.05	4.67	5.12	4.87
$\lvert t(\hat{\rho}_k)\rvert$	4.68	4.62	4.69	5.01	5.25	5.08	5.10	4.67	5.14	4.99
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	4.53	4.85	4.70	4.73	4.92	4.95	4.93	4.69	4.87	4.74
Sample size: $n = 60$	Asymptotic tests and bounds									
$\lvert t(r_1)\rvert$	4.20	2.83	51.06	64.44	66.24	2.88	2.92	30.43	30.64	48.45
$\lvert \tilde{t}(r_1)\rvert$	4.89	3.22	52.25	65.24	67.27	3.07	3.09	31.20	31.39	49.41
$\lvert t(\hat{\rho}_k)\rvert$	4.24	2.81	50.60	63.79	65.73	2.84	2.88	29.91	30.26	47.99
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	4.77	3.05	51.26	64.61	66.54	2.95	3.13	30.81	30.81	48.65
E_{11}	0.72	0.91	2.24	0.76	0.00	1.15	1.08	2.77	1.29	0.01
Best bound	1.01	1.21	2.24	0.76	0.00	1.24	1.10	2.77	1.29	0.01
	Tests based on Imhof critical values									
$\lvert t(r_1)\rvert$	4.99	3.29	52.34	65.41	67.41	3.12	3.12	31.42	31.44	49.51
$\lvert \tilde{t}(r_1)\rvert$	5.20	3.48	52.65	65.70	67.62	3.21	3.19	31.69	31.78	49.84
$\lvert t(\hat{\rho}_k)\rvert$	5.12	3.34	51.84	64.89	66.66	3.09	3.09	31.01	31.21	48.97
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	4.95	3.16	51.83	65.03	66.99	3.07	3.20	31.24	31.35	49.10
	Tests based on global size correction									
$\lvert t(r_1)\rvert$	0.00	0.00	5.03	0.00	0.00	0.00	0.00	1.81	0.08	0.00
$\lvert \tilde{t}(r_1)\rvert$	0.00	0.00	5.03	0.00	0.00	0.00	0.00	1.81	0.08	0.00
$\lvert t(\hat{\rho}_k)\rvert$	0.00	0.00	4.98	0.00	0.00	0.00	0.00	1.86	0.08	0.00
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	0.00	0.00	4.92	0.00	0.00	0.00	0.00	1.83	0.07	0.00
	Tests based on model-specific size correction									
$\lvert t(r_1)\rvert$	4.61	5.20	5.03	5.01	5.28	4.99	4.93	4.93	5.08	5.13
$\lvert \tilde{t}(r_1)\rvert$	4.61	5.20	5.03	5.01	5.28	4.99	4.93	4.93	5.08	5.13
$\lvert t(\hat{\rho}_k)\rvert$	4.45	5.04	4.98	5.07	5.07	5.00	4.81	4.93	5.20	5.06
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	4.56	5.18	4.92	4.68	4.74	4.90	4.90	4.98	5.07	5.02

Table 3. Empirical powers of alternative tests for serial dependence at nominal level $\alpha = 0.05$
$X_t = 0.2X_t + u_t$

Sample size: $n = 30$	Asymptotic tests and bounds									
Error distribution (v_t)	N(0,1)					Cauchy				
Test \ Model	M1	M2	M5	M7	M8	M1	M2	M5	M7	M8
E_{11}	4.67	6.83	3.38	19.71	47.23	14.19	14.50	7.13	19.67	44.30
Best bound	5.44	7.27	3.38	19.71	47.23	14.40	14.66	7.13	19.67	44.30
	Tests based on global size correction									
$\lvert t(r_1)\rvert$	0.11	0.02	7.47	0.15	0.00	0.10	0.13	3.15	0.51	0.00
$\lvert \tilde{t}(r_1)\rvert$	0.11	0.02	7.47	0.15	0.00	0.10	0.13	3.15	0.51	0.00
$\lvert t(\hat{\rho}_k)\rvert$	0.08	0.02	6.50	0.02	0.00	0.06	0.07	2.45	0.37	0.00
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	0.19	0.07	8.52	0.52	0.00	0.15	0.20	3.56	0.73	0.00
	Tests based on model-specific size correction									
$\lvert t(r_1)\rvert$	18.47	22.02	7.47	24.52	24.70	16.43	15.35	8.69	13.24	16.69
$\lvert \tilde{t}(r_1)\rvert$	18.47	22.02	7.47	24.52	24.70	16.43	15.35	8.69	13.24	16.69
$\lvert t(\hat{\rho}_k)\rvert$	12.00	13.43	6.50	19.59	19.16	11.22	10.45	7.41	11.08	13.53
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	17.55	20.60	8.52	24.79	24.63	16.09	15.37	9.08	13.87	17.30
Sample size: $n = 60$	Asymptotic tests and bounds									
E_{11}	11.16	14.20	3.23	22.35	47.89	24.65	25.42	7.70	24.62	45.18
Best bound	13.03	15.56	3.23	22.35	47.89	24.95	25.71	7.70	24.62	45.18
	Tests based on global size correction									
$\lvert t(r_1)\rvert$	0.00	0.00	7.69	0.07	0.00	0.01	0.01	3.01	0.28	0.00
$\lvert \tilde{t}(r_1)\rvert$	0.00	0.00	7.69	0.07	0.00	0.01	0.01	3.01	0.28	0.00
$\lvert t(\hat{\rho}_k)\rvert$	0.00	0.00	7.36	0.02	0.00	0.01	0.01	2.75	0.22	0.00
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	0.00	0.01	8.27	0.12	0.00	0.01	0.01	3.19	0.29	0.00
	Tests based on model-specific size correction									
$\lvert t(r_1)\rvert$	32.79	41.35	7.69	24.71	24.91	40.82	40.66	8.94	11.64	17.72
$\lvert \tilde{t}(r_1)\rvert$	32.79	41.35	7.69	24.71	24.91	40.82	40.66	8.94	11.64	17.72
$\lvert t(\hat{\rho}_k)\rvert$	26.36	33.25	7.36	23.09	22.74	27.10	27.18	8.22	10.88	16.64
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	31.39	40.17	8.27	25.14	25.14	39.30	39.85	9.22	12.05	18.45

Table 4. Empirical powers of alternative tests for serial dependence at nominal level $\alpha = 0.05$
$X_t = 0.9X_t + u_t$

Sample size: $n = 30$	Asymptotic tests and bounds; $n = 30$									
Error distribution (v_t)	N(0,1)					Cauchy				
Test \ Model	M1	M2	M5	M7	M8	M1	M2	M5	M7	M8
E_{11}	98.22	97.89	18.77	83.89	84.49	94.54	94.97	35.42	90.24	89.73
Best bound	98.51	98.08	18.77	84.23	84.88	94.72	95.13	35.45	90.58	89.97
	Tests based on global size correction									
$\lvert t(r_1)\rvert$	93.19	96.38	39.32	79.25	78.94	93.70	94.52	56.05	88.42	86.34
$\lvert \tilde{t}(r_1)\rvert$	93.19	96.38	39.32	79.25	78.94	93.70	94.52	56.05	88.42	86.34
$\lvert t(\hat{\rho}_k)\rvert$	80.50	90.34	39.57	71.47	70.86	86.57	87.84	55.08	81.75	80.00
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	85.73	93.08	46.32	73.47	73.24	89.90	91.05	60.90	84.02	81.64
	Tests based on model-specific size correction									
$\lvert t(r_1)\rvert$	99.66	99.96	39.32	83.81	83.48	98.97	99.20	70.81	92.79	89.47
$\lvert \tilde{t}(r_1)\rvert$	99.66	99.96	39.32	83.81	83.48	98.97	99.20	70.81	92.79	89.47
$\lvert t(\hat{\rho}_k)\rvert$	98.44	99.69	39.57	77.33	76.97	98.36	98.60	70.71	89.14	85.12
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	98.92	99.86	46.32	78.88	78.62	98.59	98.77	76.99	90.30	86.37
Sample size: $n = 60$	Asymptotic tests and bounds									
E_{11}	99.99	100	18.98	86.00	85.48	98.91	99.02	34.67	95.63	90.61
Best bound	99.99	100	18.98	86.28	85.83	98.98	99.04	34.76	95.77	90.84
	Tests based on global size correction									
$\lvert t(r_1)\rvert$	99.05	99.52	39.25	80.41	79.77	97.88	98.15	53.85	92.80	86.90
$\lvert \tilde{t}(r_1)\rvert$	99.05	99.52	39.25	80.41	79.77	97.88	98.15	53.85	92.80	86.90
$\lvert t(\hat{\rho}_k)\rvert$	97.08	98.63	39.96	76.35	75.52	96.89	97.04	54.04	90.16	83.86
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	97.75	99.00	42.80	77.01	76.42	97.22	97.45	56.56	90.77	84.40
	Tests based on model-specific size correction									
$\lvert t(r_1)\rvert$	100	100	39.25	84.79	84.39	99.71	99.67	70.18	96.05	90.04
$\lvert \tilde{t}(r_1)\rvert$	100	100	39.25	84.79	84.39	99.71	99.67	70.18	96.05	90.04
$\lvert t(\hat{\rho}_k)\rvert$	100	100	39.96	81.49	80.79	99.67	99.58	70.06	94.66	87.73
$\lvert \bar{t}(\hat{\rho}_k)\rvert$	100	100	42.80	82.13	81.45	99.68	99.59	73.25	94.96	88.31

Results on empirical frequencies of type I errors (empirical level) appear in Table 2, while powers for $\rho = 0.2,\, 0.9$ appear in tables 3 - 4.[1]

We see from these results that the bounds constitute the only methods that allow one to control size for all the patterns considered. Further, once standard tests are corrected for size, the bounds can lead to substantial power gains. This holds even if the powers of the non-bound procedures are indeed overestimated. The adjustments required to correct the size of the alternative tests are simply too "large" to yield useful tests of the nonparametric hypothesis considered. This shows clearly that the distribution-free bounds presented in this paper can at least provide a useful check on the reliability of serial dependence tests that are not provably distribution-free. We also observed that the tightest exponential $E_{11} = B_1\left(y_1,\, |X|\right)$ yields the best results in terms of power (for a level of 0.05), with a performance that is very close to one one of the minimum value over all the bounds (which may come supplied by a different bound, depending on the sample).

6. Empirical illustration

In this section, we illustrate how the bounds derived above can be used by applying them to U.S. data on interest rates. We will study the autocorrelation structure of the first and second differences of the logarithm of the commercial paper rate [denoted by $\ln\left(r_t\right)$] from 1951 to 1983 (quarterly, 132 observations). The source of the data is Balke and Gordon (1986, pp. 789-808).

For these two series, we report in Tables 1 and 2 the usual centered autocorrelations $\hat{\rho}_k = \sum_{t=1}^{n-k}\left(X_t - \overline{X}\right)\left(X_{t+k} - \overline{X}\right) / \sum_{t=1}^{n}\left(X_t - \overline{X}\right)^2$, where $\overline{X} = \sum_{t=1}^{n} X_t/n$, and the uncentered autocorrelations r_k defined in (A.2) with $\mu_t = 0$, for $k = 1,\, \ldots\,,\; 20$. Since both series have means very close to zero, there is very little difference between the two sets of autocorrelation coefficients (see also the t-statistics reported in the tables). Even though we are mostly interested by the minimal upper bound on the p-value for testing independence, we also report the individual bounds for the sake of comparison (but one would not normally report all this information). The bounds reported are for two-sided tests, *i.e.*, we compute bounds on $\mathsf{P}\left[|r_k| \geq y \mid |X|\right] = 2\mathsf{P}\left[r_k \geq |y| \mid |X|\right]$ at $y = \hat{r}_k$ (observed value of r_k). The upper bounds on $\mathsf{P}\left[r_k \geq |y| \mid |X|\right]$ computed are based on the four exponential bounds $E_{1k} \leq E_{2k} \leq E_{3k} \leq E_{4k}$ from (2.3), the improved Eaton-Pinelis bound B_{EP}^{*} and the Eaton-Pinelis bound B_{EP} from (3.1), Chebyshev bounds based on the exact conditional even moments of r_k, binomial moments and normal moments as given in (3.4)-(3.5), and the Berry-Esséen-Zolotarev type bound BE_U given by (4.2). The Chebyshev bound (C) based on the exact moments of r_k is the tightest obtained from the first six even moments ($p = 2,\, 4,\, \ldots\,,\, 12$), the one based on the binomial moments is the best over the first 15 even moments ($p = 2,\, 4,\, \ldots\,,\, 30$), while the normal moment bound is based on (3.5). We also report the Berry-Esséen lower bound obtained from (4.2). When an upper bound exceeds one (which can occur for all bounds considered except B_{EP}^{*} and B_{EP}), we report 1.0 in the table (since a probability cannot be greater than 1.0). Similarly, when the lower bound is less than zero, we report 0.0 in the table.

From the results in Table 5, we see that the series $X_t = (1 - B)\ln\left(r_t\right) = \ln\left(r_t\right) - \ln\left(r_{t-1}\right)$ exhibits four autocorrelations r_k (at lags $k = 1,\, 2,\, 6,\, 7$) whose absolute values exceed two asymptotic

[1]More complete results are available from the authors upon request.

Table 5: U.S. Commercial Paper Interest Rate (logarithm, first differences) : $X_t = (1 - B)\ln(r_t)$
Autocorrelations and bounds on p-values for two-sided tests
Sample : 1951:2-1983:4 (quarterly, $n = 131$ observations)[a]

			Exponential bounds				Eaton-type bounds		Chebyshev-type bounds					Berry-Esséen	Best upper bound	Type	Lower bound
k	$\hat{\rho}_k$	r_k	E_1	E_2	E_3	E_4	B^*_{EP}	B_{EP}	$C(p^*)$		$CB(p^*)$		CN	BE_U			BE_U
1	0.3339	0.3385	0.0124	0.0200	0.0332	0.0347	0.0169	0.0175	0.0100	(12)	0.0235	(8)	0.0243	0.5457	0.0100*	C	0.0000
2	-0.2105	-0.2023	0.4277	0.4516	0.5053	0.5079	0.3494	0.3502	0.3353	(4)	0.3648	(2)	0.3648	0.6884	0.3353	C	0.0000
3	-0.0393	-0.0326	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2144
4	0.0911	0.0969	1.0000	1.0000	1.0000	1.0000	0.9179	0.9179	0.9179	(2)	0.9179	(2)	0.9179	0.6903	0.6903	BEU	0.0000
5	-0.0499	-0.0415	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9371	0.9371	BEU	0.2565
6	-0.2490	-0.2383	0.0314	0.0357	0.0460	0.0478	0.0240	0.0248	0.0246	(10)	0.0327	(8)	0.0338	0.4110	0.0240*	BEP*	0.0000
7	-0.2132	-0.2027	0.0467	0.0499	0.0598	0.0618	0.0320	0.0328	0.0350	(8)	0.0435	(8)	0.0446	0.3674	0.0320*	BEP*	0.0000
8	-0.0174	-0.0086	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4774
9	-0.0230	-0.0136	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4522
10	-0.1787	-0.1686	0.3146	0.3208	0.3458	0.3489	0.2255	0.2267	0.2307	(4)	0.2446	(4)	0.2460	0.5000	0.2255	BEP*, C	0.0000
11	-0.1190	-0.1088	0.7822	0.7837	0.7973	0.7993	0.5452	0.5452	0.5452	(2)	0.5452	(2)	0.5452	0.5684	0.5452	BEP*, C	0.0000
12	-0.0179	-0.0077	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4990
13	0.0118	0.0210	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2967
14	0.0296	0.0400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9983	0.9983	BEU	0.1485
15	0.0788	0.0905	1.0000	1.0000	1.0000	1.0000	0.9427	0.9427	0.9427	(2)	0.9427	(2)	0.9427	0.7598	0.7598	BEU	0.0000
16	0.1127	0.1228	0.7597	0.7636	0.7867	0.7888	0.5374	0.5374	0.5374	(2)	0.5374	(2)	0.5374	0.6671	0.5374	BEP*, C	0.0000
17	0.1005	0.1088	0.7276	0.7296	0.7459	0.7481	0.5085	0.5085	0.5085	(2)	0.5085	(2)	0.5085	0.5762	0.5085	BEP*, C	0.0000
18	-0.0055	0.0025	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4941
19	-0.1109	-0.1029	0.6813	0.6830	0.6973	0.6997	0.4761	0.4761	0.4761	(2)	0.4761	(2)	0.4761	0.5297	0.4761	BEP*, C	0.0000
20	-0.0640	-0.0567	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.7679	0.7679	BEU	0.0312

t-statistic for zero mean = 0.9875.

[a] The p-values reported one for two-sided tests. Each one is obtained by bounding $\mathsf{P}\left[|r_k| \geq y \mid |X|\right] = 2\mathsf{P}\left[r_k \geq y \mid |X|\right]$ where we take $y = |\hat{r}_k|$ (the observed value of $|r_k|$). The exponential bounds $E_1 \leq E_2 \leq E_3 \leq E_4$ are based on Proposition **2.1**, the Eaton-type bounds on $B^*_{EP} \leq B_{EP}$ on Proposition **3.1**, the Chebyshev bounds on Proposition **3.2**, and the Berry-Esséen bounds on Proposition **4.1**. C is the best Chebyshev bound based on the exact even moments of order $p = 2, 4, \ldots, 12$, CB the best Chebyshev bound based on the centered binomial moments of order $p = 2, 4, \ldots, 30$, and CN is based on (3.5); the moment yielding the best bound (p^*) is given in parenthesis. Best upper bounds lower than 0.05 have been starred $(*)$.

Table 6: U.S. Commercial Paper Interest Rate (logarithm, second differences) : $X_t = (1-B)^2 \ln(r_t)$
Autocorrelations and bounds on p-values for two-sided tests
Sample : 1951:3-1983:4 (quarterly, $n = 130$ observations)

			Exponential bounds				Eaton-type bounds		Chebyshev-type bounds					Berry-Esséen	Best upper bound	Type	Lower bound
k	$\hat{\rho}_k$	r_k	E_1	E_2	E_3	E_4	B^*_{EP}	B_{EP}	$C(p^*)$		$CB(p^*)$		CN	BE_U			BE_U
1	-0.0874	-0.0874	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9820	0.9820	BEU	0.0000
2	-0.5392	-0.5392	0.00002	0.0013	0.0042	0.0046	0.0019	0.0020	0.0006	(12)	0.0030	(12)	0.0033	0.5790	0.00002*	E1	0.0000
3	0.0282	0.0282	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2340
4	0.2043	0.2044	0.3403	0.3474	0.3749	0.3776	0.2473	0.2483	0.2518	(4)	0.2685	(4)	0.2699	0.5397	0.2473	BEP*	0.0000
5	0.0444	0.0445	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.0592
6	-0.1754	-0.1754	0.2373	0.2449	0.2710	0.2739	0.1708	0.1719	0.1774	(4)	0.1887	(4)	0.1897	0.5024	0.1708	BEP*	0.0000
7	-0.1201	-0.1201	0.5735	0.5773	0.5997	0.6020	0.4165	0.4165	0.4165	(2)	0.4165	(2)	0.4165	0.5697	0.4165	BEP*, C	0.0000
8	0.1448	0.1449	0.3858	0.3899	0.4111	0.4139	0.2750	0.2760	0.2863	(4)	0.3005	(4)	0.3022	0.4902	0.2750	BEP*	0.0000
9	0.1187	0.1187	0.4409	0.4484	0.4790	0.4817	0.3283	0.3292	0.3399	(4)	0.3512	(2)	0.3512	0.5774	0.3283	BEP*	0.0000
10	-0.1560	-0.1560	0.3192	0.3337	0.3729	0.3757	0.2458	0.2469	0.2410	(4)	0.2668	(4)	0.2683	0.6288	0.2410	C	0.0000
11	-0.0318	-0.0318	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9564	0.9564	BEU	0.2093
12	0.0612	0.0610	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	0.9182	0.9182	BEU	0.0000
13	0.0017	0.0017	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.6052
14	-0.0308	-0.0307	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.1379
15	0.0082	0.0082	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4532
16	0.0364	0.0364	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.1480
17	0.0699	0.0700	1.0000	1.0000	1.0000	1.0000	0.8318	0.8318	0.8318	(2)	0.8318	(2)	0.8318	0.6949	0.6949	BEU	0.0000
18	-0.0022	-0.0022	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.4939
19	-0.1100	-0.1100	0.6430	0.6499	0.6806	0.6830	0.4654	0.4654	0.4654	(2)	0.4654	(2)	0.4654	0.6795	0.4654	BEP*, C	0.0000
20	-0.0295	-0.0295	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(2)	1.0000	(2)	1.0000	1.0000	1.0000		0.2008

t-statistic for zero mean = -0.0790.

standard errors $\left(|r_k| \geq 2\ /\ \sqrt{n} = 2/\sqrt{131} = 0.175\right)$. Among these, three $(k = 1,\ 6,\ 7)$ are clearly significant at level 0.05 under the assumption that $X_1,\ \ldots\ ,\ X_n$ have distributions symmetric about zero. It is also of interest to note that the autocorrelations at lags $k = 3,\ 5,\ 8,\ 9,\ 12,\ 13,\ 14,\ 18$ are clearly not significant at level 0.05. Depending on cases, the best upper bound is obtained by using either a Chebyshev (C), Eaton-type (B_{EP}^{*}) or Berry-Esséen bound.

The autocorrelations for the second differences $X_t = (1-B)^2 \ln(r_t)$ in Table **??** exhibit only one autocorrelation (at $k = 2$) whose absolute value is greater than two asymptotic standard errors $\left(|r_k| \geq 2\ /\ \sqrt{n} = 2\ /\ \sqrt{130} = 0.175\right)$. The nonparametric upper bound on the p-value for $|r_2|$ indicates that this is significant even for a level as low as 0.00002; the best upper bound is given here by the exponential bound E_1. In this case, all the upper bounds (except the Berry-Esséen one) indicate that this is significant at level 0.005. The Berry-Esséen lower bound indicates that the autocorrelations at lags $k = 5,\ 12,\ 13$ are clearly not significant at level 0.05. Overall the second differences of $\ln(r_t)$ seem to have a simpler autocorrelation structure than the first differences $(1-B)\ln(r_t)$.

A. Appendix: Proofs

PROOF OF PROPOSITION **2.1** Let $Z_{kt} = X_t X_{t+k},\ t = 1,\ \ldots\ ,\ n-k$, and let sgn$(x)$ be the sign function : sgn$(x) = -1$ if $x < 0$, 0 if $x = 0$, and 1 if $x > 0$. Then we can write

$$r_k = D_k\left(|X|\right) \sum_{t=1}^{n-k} w_{kt} S_{kt} = D_k\left(|X|\right) R_k \tag{A.1}$$

where $S_{kt} = \mathrm{sgn}(Z_{kt}),\ t = 1,\ \ldots\ ,\ n-k$, $R_k = \sum_{t=1}^{n-k} w_{kt} S_{kt}$, and $\sum w_{kt}^2 = 1$. When $Z_{k1} = \cdots = Z_{k,n-k} = 0$, we have $r_k = D_k\left(|X|\right) = 0$, so that $\mathsf{P}\left[r_k \geq y \mid |X|\right] = 0$, and the result holds trivially. We now suppose that $Z_{kt} \neq 0$ for at least one t. Let $A_k\left(|X|\right) = \{t : |X_t| \neq 0,\ 1 \leq t \leq n-k\}$ and $B_k\left(|X|\right) = \{t : |X_t X_{t+k}| \neq 0,\ 1 \leq t \leq n-k\}$. Clearly, $t \in B_k\left(|X|\right)$ if and only if $t \in A_k\left(|X|\right)$ and $t + k \in A_k\left(|X|\right)$, and

$$R_k = \sum_{t=1}^{n-k} w_{kt} S_{kt} = \sum_{t \in B_k(|X|)} w_{kt} S_{kt}\ . \tag{A.2}$$

By the independence of $X_1,\ \ldots\ ,\ X_n$ and by the symmetry assumption, the variables in the set $\{\mathrm{sgn}\left(X_t\right) : t \in A_k\left(|X|\right)\}$ are independent conditional on $|X|$, with $\mathsf{P}\left[\mathrm{sgn}\left(X_t\right) = -1 \mid |X|\right] = \mathsf{P}\left[\mathrm{sgn}\left(X_t\right) = 1 \mid |X|\right] = 0.5$. Further, since $\mathrm{sgn}(Z_{kt}) = \mathrm{sgn}(X_t)\mathrm{sgn}(X_{t+k})$, it is easy to see that the variables in the set $\{S_{kt} : t \in B_k\left(|X|\right)\}$ are independent conditional on $|X|$ with

$$\mathsf{P}\left[S_{kt} = -1 \mid |X|\right] = \mathsf{P}\left[S_{kt} = 1 \mid |X|\right] = 0.5\ ; \tag{A.3}$$

see Dufour (1981). It is clear from (A.1) - (A.3) that the conditional distribution of r_k given $|X|$ is symmetric about zero. Further, using Markov's inequality and observing that $\cosh\left(w_{kt} z\right) =$

$\cosh(0) = 1$ for $t \notin B_k(|X|)$, we have:

$$\mathsf{P}\left[R_k \geq y \mid |X|\right] \leq E\left[\exp\left(z\, R_k \mid |X|\right)\right] \;/\; \exp(zy) = \prod_{t=1}^{n-k} \cosh(w_{kt} z) \;/\; \exp(zy) \qquad \text{(A.4)}$$

for all $z \geq 0$ and for all y. Consequently, for all $y > 0$,

$$\begin{aligned}
\mathsf{P}\left[R_k \geq y \mid |X|\right] &\leq \operatorname*{Inf}_{z \geq 0} \left\{ \exp(-zy) \prod_{t=1}^{n-k} \cosh(w_{kt} z) \right\} \leq \exp\left(-y^2\right) \prod_{t=1}^{n-k} \cosh(w_{kt} y) \\
&= \exp\left(-y^2\right) \prod_{t \in B_k(|X|)} \cosh(w_{kt} y) \leq \exp\left(-y^2\right) \left\{\cosh\left(y/\sqrt{n_k^*}\right)\right\}^{n_k^*} \\
&< \exp\left(-y^2\right) \left\{\exp\left[\left(y/\sqrt{n_k^*}\right)^2 /2\right]\right\}^{n_k^*} = \exp\left(-y^2/2\right), \qquad \text{(A.4)}
\end{aligned}$$

where the second inequality is obtained by taking $z = y$ in (A.4), the third one follows from Corollary 1 and Example 2 of Eaton (1970), and the last one is obtained by noting that $\cosh(x) < \exp\left(x^2 \,/\, 2\right)$ for $x > 0$ [Edelman (1986)]. Inequality (2.3) follows from (A.4) on observing that $r_k = D_k(|X|)\, R_k$. □

PROOF OF PROPOSITION **3.1** When $X_t X_{t+k} = 0$, for $t = 1, \ldots, n-k$, we have $r_k = 0$ and (3.1) clearly holds. When $X_t X_{t+k} \neq 0$ for some t, the result follows from (A.1) to (A.3), and then by applying Proposition 1 from Dufour and Hallin (1993) to R_k in (A.2). □

PROOF OF PROPOSITION **3.2** When $X_t X_{t+k} = 0$, for $t = 1, \ldots, n-k$, we have $r_k = 0$, and (3.4)–(3.5) clearly hold. Otherwise, the result follows from (A.1) to (A.3), and Proposition 2 in Dufour and Hallin (1992). □

PROOF OF PROPOSITION **4.1** The result is immediate from (A.1) to (A.3) and Proposition 3 from Dufour and Hallin (1992). □

B. Appendix: Conditional moments of the autocorrelations

The conditional moments $\left[E r_k^P \mid |X|\right]$ in (3.4) can be computed by noting that

$$E\left(r_k^p \mid |X|\right) = D_k(|X|)^p\, E\left(R_k^p \mid |X|\right) \qquad \text{(B.1)}$$

where, provided $D_k(|X|) \neq 0$ (otherwise, $r_k = 0$), $E\left(R_k^2 \mid |X|\right) = 1$ and, for $p = 4, 6, \ldots, 12$, $E\left(R_k^p \mid |X|\right)$ is given by the following formulae : setting $W_{kp} = \sum_{t=1}^{n-k} w_{kt}^p$,

$$\begin{aligned}
E\left(R_k^4 \mid |X|\right) &= 3 - 2\, W_{k4}\,, && \text{(B.2)} \\
E\left(R_k^6 \mid |X|\right) &= 15 - 30\, W_{k4} + 16\, W_{k6}\,, && \text{(B.3)} \\
E\left(R_k^8 \mid |X|\right) &= 105 - 420\, W_{k4} + 140\, W_{k4}^2 + 448\, W_{k6} - 272\, W_{k8}\,, && \text{(B.4)} \\
E\left(R_k^{10} \mid |X|\right) &= 945 - 6300\, W_{k4} + 6300\, W_{k4}^2 + 10080\, W_{k6} - 6720\, W_{k6} W_{k4}
\end{aligned}$$

$$
\begin{aligned}
& - 12240\ W_{k8} + 7936\ W_{k,10}\ , && \text{(B.5)} \\
E\left(R_k^{12} \mid |X|\right) \quad = \quad & 10395 - 103950\ W_{k4} + 207900\ W_{k4}^2 - 46200\ W_{k4}^3 + 221760\ W_{k6} \\
& - 443520\ W_{k6} W_{k4} + 118272\ W_{k6}^2 - 403920\ W_{k8} \\
& + 269280\ W_{k8} W_{k4} + 523776\ W_{k,10} - 353792\ W_{k,12}\ . && \text{(B.6)}
\end{aligned}
$$

References

Anderson, T. W. (1971), *The Statistical Analysis of Time Series*, John Wiley & Sons, New York.

Balke, N. S. and Gordon, R. J. (1986), Historical data, *in* R. J. Gordon, ed., 'The American Business Cycle: Continuity and Change', University of Chicago Press, Chicago, Illinois, pp. 781–850.

Brockwell, P. J. and Davis, R. A. (1991), *Time Series: Theory and Methods*, second edn, Springer-Verlag, New York.

Dufour, J.-M. (1981), 'Rank tests for serial dependence', *Journal of Time Series Analysis* **2**, 117–128.

Dufour, J.-M. (2003), 'Identification, weak instruments and statistical inference in econometrics', *Canadian Journal of Economics* **36**(4), 767–808.

Dufour, J.-M. and Hallin, M. (1991), 'Nonuniform bounds for nonparametric t tests', *Econometric Theory* **7**, 253–263.

Dufour, J.-M. and Hallin, M. (1992), 'Simple exact bounds for distributions of linear signed rank statistics', *Journal of Statistical Planning and Inference* **31**, 311–333.

Dufour, J.-M. and Hallin, M. (1993), 'Improved Eaton bounds for linear combinations of bounded random variables, with statistical applications', *Journal of the American Statistical Association* **88**, 1026–1033.

Dufour, J.-M. and Roy, R. (1985), 'Some exact results on sample autocorrelations and tests of randomness', *Journal of Econometrics* **29**, 257–273. Corrigendum 41(1989), 279-281.

Eaton, M. L. (1970), 'A note on symmetric Bernoulli random variables', *The Annals of Statistics* **41**, 1223–1226.

Edelman, D. (1986), 'Bounds for a nonparametric t table', *Biometrika* **73**, 242–243.

Edelman, D. (1990), 'An inequality of optimal order for the tail probabilities of the t-statistic under symmetry', *Journal of the American Statistical Association* **85**, 120–122.

Efron, B. (1969), 'Student's t-test under symmetry conditions', *Journal of the American Statistical Association* **81**, 461–470.

Hoeffding, W. (1963), 'Probability inequalities for sums of bounded random variables', *Journal of the American Statistical Association* **58**, 13–20.

Lehmann, E. L. (1986), *Testing Statistical Hypotheses, 2nd edition*, John Wiley & Sons, New York.

Pinelis, I. (1991), Extremal probabilistic problems and Hotelling's T^2 test statistic under symmetry conditions, Technical report, Statistics Department, University of Illinois (Champaign).

Pratt, J. W. and Gibbons, J. D. (1981), *Concepts of Nonparametric Theory*, Springer-Verlag, New York.

van Beek, P. (1972), 'An application of Fourier methods to the problem of sharpening the Berry-Esséen inequality', *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **23**, 187–196.

Zolotarev, V. M. (1965), 'On the closeness of the distributions of two sums of independent random variables', *Theory of Probability and Its Applications* **10**, 472–479.