Journal of Applied Probability & Statistics Vol. **x**, No. x, xxx–xxx

Tests of Causality between two infinite-order autoregressive series

Chafik Bouhaddioui

Department of Statistics, United Arab Emirates University, Al Ain, UAE. Email: ChafikB@uaeu.ac.ae

Jean-Marie Dufour

Department of Economics, McGill University, Montreal, Canada. Email: jean-marie.dufour@mcgill.ca

Abstract

We propose a test for non-causality at various horizons for infinite-order vector autoregressive. We first introduce multiple horizon infinite-order vector autoregressions which can be approximated by a multiple horizon finite-order vector autoregressions. The order is assumed to increase with the sample size. Under some regularity conditions, we study the estimation of parameters obtained from the approximation of the infinite-order autoregression by a finite-order autoregression. The test can be implemented simply through linear regression methods and do not involve the use of artificial simulations. The asymptotic distribution of the new test statistic is derived under the hypothesis of non-causality at various horizons. An asymptotic power of the test will be studied. Bootstrap procedures are also considered. The methods are applied to a VAR model of the US economy.

Keywords: Causality tests; infinite-order vector autoregressive process; causality; consistency; asymptotic power; bootstrap.

2000 Mathematics Subject Classification: 62M10, 62M15.

1 Introduction

The concept of causality introduced by Wiener [22] and Granger [13] is now a basic notion for studying dynamic relationships between time series. The literature on this topic is considerable; see, for example, the reviews of Pierce and Haugh [19], Newbold [18], Geweke [10], Lütkepohl [15] and Gourieroux and Monfort [12, chapter 10]. The original definition of Granger [13], which is used or adapted by most authors on this topic, refers to the predictability of a variable $X(t)$, where t is an integer, from its own past, the one of another variable $Y(t)$ and possibly a vector $Z(t)$ of auxiliary variables, **one period ahead**: more precisely, we say that Y causes X in the sense of Granger if the observation of Y up to time t $(Y(\tau) : \tau \leq t)$ can help one to predict $X(t+1)$when the corresponding observations on X and Z are available $(X(\tau), Z(\tau) : \tau \leq t)$; a more formal definition will be given below.

Recently, however, Lütkepohl [16] and Dufour and Renault [9] have noted that, for multivariate models where a vector of auxiliary variables Z is used in addition to the variables of interest X and Y, it is possible that Y does not cause X in this sense, but can still help to predict X **several periods ahead**; on this issue, see also Sims [21], Renault, Sekkat and Szafarz [20] and Giles [11]. For example, the values $Y(\tau)$ up to time t may help to predict $X(t+2)$, even though they are useless to predict $X(t+1)$. This is due to the fact that Y may help to predict Z one period ahead, which in turn has an effect on X at a subsequent period. It is clear that studying such indirect effects can have a great interest for analyzing the relationships between time series. In particular, one can distinguish in this way properties of "short-run (non-)causality" and "long-run (non-)causality".

In that case, Dufour, Pelletier and Renault [8] studied the problem of testing non-causality at various horizons as defined in Dufour and Renault [9] for finite-order vector autoregressive (VAR) models. However, it is usually known that a finite-order VAR process is a rough approximation to the true data generation process (DGP) of a given multivariate time series. Different tests were developed to study causality and orthogonality of two infinite-order autoregressive vector series, see Bouhaddioui and Roy [3], Bouhaddioui and Dufour [4] and [5].

In this paper, we develop a test of non-causality at various horizons as defined in Dufour and Renault [9] for infinite-order autoregressive (VAR(∞)) series. For a given sample size N, the true VAR(∞) model is approximated by a finite order VAR(p) model where the order p is a function of the sample size N that tends, at some rate, to infinity as N increases.

In such models, the non-causality restriction at horizon one takes the form of the relatively simple zero restrictions on the coefficients of the VAR [see Boudjellaba, Dufour and Roy [2] and Dufour and Renault [9]]. However, as cited in Dufour, Pelletier and Renault [8], non-causality restrictions at higher horizons (greater than or equal to 2) are generally nonlinear, taking the form of zero restrictions on multilinear forms in the coefficients of the VAR. When applying standard test statistics such as Wald-type test criteria, such forms can easily lead to asymptotically singular covariance matrices, so that standard asymptotic theory would not apply to such statistics. Further, calculation of the relevant covariance matrices _ which involve the derivatives of potentially large numbers of restrictions _ can become quite awkward.

Consequently, we propose simple tests for non-causality restrictions at various horizons [as defined in Dufour and Renault [9]] which can be implemented only through linear regression methods and do not involve the use of artificial simulations [e.g., as in Lütkepohl-Burda [17]]. This will be done, in particular, by considering multiple horizon infinite-order vector autoregressions [called *(∞,h)-autoregressions*] which can be approximated by a multiple horizon finite-order vector autoregressions [called *(p,h)-autoregressions*]. The order p obeys to some constraints and is a function of the sample size. Under some regular conditions, the parameters of interest can be estimated by usual linear methods.

In section 2, we describe the model considered and introduce the notion of infinite-order autoregression at horizon h [or (∞,h)-autoregression] which will be the basis of our method. We study the estimation of parameters obtained from the approximation of the (∞,h)-autoregression by a finite (p,h)-autoregression. In section 3, we study the testing of non-causality at various horizons for infinite-order stationary processes. In section 4, we study the asymptotic power of the test under a specific local alternatives. Finally, in section 5, we conduct a small Monte Carlo experiment in order to study the exact level and power of the test for finite samples.

2 Multiple horizons infinite-order autoregressions

Consider a m-dimensional infinite-order autoregressive process of the form

$$\boldsymbol{X}_t = \boldsymbol{\mu}_t + \sum_{k=1}^{\infty} \boldsymbol{\pi}_k \boldsymbol{X}_{t-k} + \boldsymbol{a}_t \ , \ \ t \in \mathbb{Z}, \tag{2.1}$$

where $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \ldots, X_{mt})'$ is a $m \times 1$ random vector, $\boldsymbol{\mu}_t$ is a deterministic trend, $\sum_{k=1}^{\infty} \|\boldsymbol{\pi}_k\| < \infty$ and $\|.\|$ is the Euclidean matrix norm defined by $\|\boldsymbol{A}\| = tr(\boldsymbol{A}'\boldsymbol{A})$. We suppose that $\mathbb{E}[\boldsymbol{a}_s \boldsymbol{a}_t'] = \delta_{ts}\boldsymbol{\Sigma}$, $\forall t, s \in \mathbb{Z}$ with $\delta_{ts} = 1$ if $t = s$ and 0 elsewhere and $\boldsymbol{\Sigma}$ is positive definite matrix. The innovation process $\{\boldsymbol{a}_t\}$ satisfies the following assumption

Assumption 2.1. The m-dimensional strong white noise $\boldsymbol{a} = \{\boldsymbol{a}_t = (a_{1t}, \ldots, a_{mt})'\}$ is such that $\mathbb{E}(\boldsymbol{a}) = 0$, its covariance regular matrix $\boldsymbol{\Sigma}$ and finite fourth moments.

This usual representation, 2.1, is an autoregression at horizon 1. At time $t + h$, where h is an integer, we can write

$$\boldsymbol{X}_{t+h} = \boldsymbol{\mu}_t^{(h)} + \sum_{k=1}^{\infty} \boldsymbol{\pi}_k^{(h)} \boldsymbol{X}_{t+h-k} + \sum_{j=0}^{h-1} \boldsymbol{\psi}_j \boldsymbol{a}_{t+h-j} \ , \ \ t \in \mathbb{Z},$$

where $\boldsymbol{\psi}_0 = \mathbb{I}_m$ is the $(m \times m)$ identity matrix. The appropriate formulas for the coefficients $\boldsymbol{\pi}_k^{(h)}$, $\boldsymbol{\mu}_t^{(h)}$ and $\boldsymbol{\psi}_j$ are given in Dufour and Renault [9] and Dufour, Pelletier and Renault [8]. We can also write that equation in the following way

$$\boldsymbol{X}_{t+h}' = \boldsymbol{\mu}_t^{(h)'} + \boldsymbol{X}(t)'\boldsymbol{\pi}^{(h)} + \boldsymbol{u}_{t+h}' \ , \ \ t \in \mathbb{Z}, \tag{2.2}$$

where $\boldsymbol{X}(t)' = [\boldsymbol{X}_t', \boldsymbol{X}_{t-1}', \ldots]$, $\boldsymbol{\pi}^{(h)} = [\boldsymbol{\pi}_1^{(h)}, \boldsymbol{\pi}_2^{(h)}, \ldots]'$ and $\boldsymbol{u}_{t+h}' = \sum_{j=0}^{h-1} \boldsymbol{\psi}_j \boldsymbol{a}_{t+h-j}$. We call 2.2 an "*infinite-order autoregression at horizon* h". We suppose that the deterministic part of such autoregression is a linear function of a finite-dimensional parameter vector, *i.e.*

$$\boldsymbol{\mu}_t^{(h)} = \gamma(h) D_t^{(h)} \tag{2.3}$$

where $\gamma(h)$ is a $m \times n$ coefficient vector and $D_t^{(h)}$ is a $n \times 1$ vector of deterministic regressors. If $\boldsymbol{\mu}_t$ is a constant vector, then $\boldsymbol{\mu}_t^{(h)}$ is also a constant vector which can be denoted by $\boldsymbol{\mu}_t^{(h)} = \mu_h$.

Based on realization $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_T$ of length T, we fit an autoregression model of order p, whose coefficients are denoted by $\boldsymbol{\pi}_{1,p}^{(h)}, \ldots, \boldsymbol{\pi}_{p,p}^{(h)}$ and we can write

$$\boldsymbol{\pi}^{(h)}(p) = \left[\boldsymbol{\pi}_{1,p}^{(h)}, \ldots, \boldsymbol{\pi}_{p,p}^{(h)}\right]'. \tag{2.4}$$

If we define by

$$\boldsymbol{\Pi}^{(h)}(p) \quad = \quad \begin{bmatrix} \gamma(h)' \\ \boldsymbol{\pi}^{(h)}(p) \end{bmatrix} = \left[\Pi_1^{(h)}(p), \Pi_2^{(h)}(p), \ldots, \Pi_p^{(h)}(p)\right]$$

the corresponding least square estimator $\hat{\boldsymbol{\Pi}}^{(h)}(p)$ is given by

$$\hat{\boldsymbol{\Pi}}^{(h)}(p) = \left[\overline{\boldsymbol{X}}_p(h)' \overline{\boldsymbol{X}}_p(h)\right]^{-1} \overline{\boldsymbol{X}}_p(h)' \boldsymbol{x}_h^{(h)} \tag{2.5}$$

where $\boldsymbol{x}_h^{(k)} \quad = \quad \begin{bmatrix} \boldsymbol{X}'_{0+h} \\ \boldsymbol{X}'_{1+h} \\ \vdots \\ \boldsymbol{X}'_{T-k+h} \end{bmatrix}, \quad \overline{\boldsymbol{X}}_p(k) \quad = \quad \begin{bmatrix} \boldsymbol{X}(0,p)' \\ \boldsymbol{X}(1,p)' \\ \vdots \\ \boldsymbol{X}(T-k,p)' \end{bmatrix}_{(T-k+1)\times(n+mp)}$,

$\boldsymbol{X}(t,p) = \begin{bmatrix} \boldsymbol{D}_t^{(h)'} \\ \boldsymbol{X}_t(p) \end{bmatrix}_{(n+mp)\times 1}$, $\boldsymbol{X}_t(p)' = \left[\boldsymbol{X}'_t, \boldsymbol{X}'_{t-1}, \ldots, \boldsymbol{X}'_{t-p+1}\right]$.

In the sequel, we state the following assumptions.

Assumption 2.2. p is a function of T such that

$$p \to \infty \quad , \quad \frac{p^3}{T} \longrightarrow 0 \text{ and } \sqrt{T} \sum_{j=p+1}^{\infty} \|\boldsymbol{\pi}_j\| < \infty \text{ as } T \to \infty. \tag{2.6}$$

To derive the asymptotic properties of $\hat{\boldsymbol{\pi}}^{(h)}(p)$, we can also express the OLS estimator as

$$\hat{\boldsymbol{\Pi}}^{(h)}(p) \quad = \quad \hat{\boldsymbol{\Gamma}}'_{1,p} \hat{\boldsymbol{\Gamma}}_p^{-1}, \tag{2.7}$$

where $\hat{\boldsymbol{\Gamma}}_{1,p} = (T-p)^{-1} \sum_{t=p+1}^{T-h} \boldsymbol{X}_t(p) \boldsymbol{X}'_t$ and $\hat{\boldsymbol{\Gamma}}_p = (T-p)^{-1} \sum_{t=p+1}^{T-h} \boldsymbol{X}_t(p) \boldsymbol{X}_t(p)'$. The consistency rate of $\hat{\boldsymbol{\Pi}}^{(h)}(p)$ is given in the following proposition.

Proposition 2.1. *Let* $\{\boldsymbol{X}_t\}$ *be a process given by* (2.2) *and satisfying the Assumption 2.1. Under the Assumption 2.2, the estimator* $\hat{\boldsymbol{\Pi}}^{(h)}(p)$ *defined by* (2.7) *is such that*

$$\|\hat{\boldsymbol{\Pi}}^{(h)}(p) - \boldsymbol{\Pi}^{(h)}(p)\| \quad = \quad O_p(\frac{p^{1/2}}{T^{1/2}}).$$

Now, if we suppose that the process $\boldsymbol{X_t}$ is second-order stationary, we define the autocovariance matrix at lag k by $\boldsymbol{\Gamma}(k) = \mathbb{E}(\boldsymbol{X}_t \boldsymbol{X}'_{t+k})$, $k \in \mathbb{Z}$.

3 Tests of causality for an infinite-order stationary h-autoregressions

Let us now consider the following hypothesis

$$\mathcal{H}_0(h) : \boldsymbol{R}\Pi_i^{(h)} \quad = \quad 0, \quad i = 1, 2, \ldots,$$

where $\boldsymbol{R}$ is a $q \times (n + mp)$ matrix of rank q. In particular, as mentioned in Dufour, Pelletier and Renault [8], if we wish to test the hypothesis that x_{jt} does not cause x_{it} at horizon h, the restriction would take the form

$$\mathcal{H}_{j \nrightarrow i}^{(h)} : \pi_{ijk}^{(h)} \quad = \quad 0, \quad k = 1, 2, \ldots,$$

where $\boldsymbol{\pi}_k^{(h)} = \left[\pi_{ijk}^{(h)}\right]_{i,j=1,\ldots,m}$, $k = 1, 2, \ldots$.

From the model (2.2) and from the equation (2.7), we have

$$vec(\hat{\boldsymbol{\Pi}}^{(h)}(p)) \quad = \quad vec\left\{\left[(T-p)^{-1}\sum_{t=p+1}^{T-h}\boldsymbol{X}_t(p)\boldsymbol{X}'_t\right]\left[(T-p)^{-1}\sum_{t=p+1}^{T-h}\boldsymbol{X}_t(p)\boldsymbol{X}_t(p)'\right]\right\}.$$

If we denote by

$$\boldsymbol{W}_{1T} \quad = \quad vec[(T-p)^{-1}\sum_{t=p+1}^{T-h}\boldsymbol{u}_{t+h}\boldsymbol{X}'_t(p)]$$

$$\boldsymbol{W}_{2T} \quad = \quad vec[(T-p)^{-1}\sum_{t=p+1}^{T-h}\{\boldsymbol{u}_{t+h}(p) - \boldsymbol{u}_{t+h}\}\boldsymbol{X}'_t(p)],$$

where $\boldsymbol{u}_{t+h}(p) = \boldsymbol{X}'_{t+h} - \boldsymbol{X}'_t(p)\boldsymbol{\Pi}^{(h)}(p)$, and under the hypothesis $\mathcal{H}_0$, we have

$$(\mathbb{I}_p \otimes \boldsymbol{R})vec(\hat{\boldsymbol{\Pi}}^{(h)}(p)) \quad = \quad (\mathbb{I}_p \otimes \boldsymbol{R})(\hat{\boldsymbol{\Gamma}}_p^{-1} \otimes \mathbb{I}_m)(\boldsymbol{W}_{1T} + \boldsymbol{W}_{2T}),$$

which can be also decomposed as

$$\begin{aligned}(\mathbb{I}_p \otimes \boldsymbol{R})vec(\hat{\boldsymbol{\Pi}}^{(h)}(p)) &= (\mathbb{I}_p \otimes \boldsymbol{R})(\boldsymbol{\Gamma}_p^{-1} \otimes \mathbb{I}_m)\boldsymbol{W}_{1T} + (\mathbb{I}_p \otimes \boldsymbol{R})(\boldsymbol{\Gamma}_p^{-1} \otimes \mathbb{I}_m)\boldsymbol{W}_{2T} \\ &\quad + (\mathbb{I}_p \otimes \boldsymbol{R})\{(\hat{\boldsymbol{\Gamma}}_p^{-1} - \boldsymbol{\Gamma}_p^{-1}) \otimes \mathbb{I}_m\}(\boldsymbol{W}_{1T} + \boldsymbol{W}_{2T}) \\ &= \boldsymbol{A}(p) + \boldsymbol{B}_1(p) + \boldsymbol{B}_2(p) \\ &= \boldsymbol{A}(p) + \boldsymbol{B}(p).\end{aligned} \tag{3.1}$$

Also, we denote by $\boldsymbol{\Sigma}_u^{(h)}$ the covariance matrix of $\{\boldsymbol{u}_t^{(h)}\}$ and $\hat{\boldsymbol{\Sigma}}_u^{(h)}$ is its estimator such that

$$\sqrt{T}(\hat{\boldsymbol{\Sigma}}_u^{(h)} - \boldsymbol{\Sigma}_u^{(h)}) \quad \xrightarrow{L} \quad N(0, \boldsymbol{\Omega}_h) \tag{3.2}$$

where $\xrightarrow{L}$ denotes convergence in law and $\boldsymbol{\Omega}_h$ is a given matrix. The test statistic is based on the following quadratic form

$$\mathcal{T}(\hat{\boldsymbol{u}}^{(h)}, \hat{\boldsymbol{\Sigma}}_u^{(h)}) \quad = \quad T\hat{\boldsymbol{\pi}}^{(h)}(p)'(\mathbb{I}_p \otimes \boldsymbol{R})'\hat{\boldsymbol{\Delta}}_{p,h}^{-1}(\mathbb{I}_p \otimes \boldsymbol{R})\hat{\boldsymbol{\pi}}^{(h)}(p) \tag{3.3}$$

where $\hat{\boldsymbol{\pi}}^{(h)}(p) = vec(\hat{\boldsymbol{\Pi}}^{(h)}(p))$ and $\hat{\boldsymbol{\Delta}}_{p,h} = (\mathbb{I}_p \otimes \boldsymbol{R})(\hat{\boldsymbol{\Gamma}}_p^{-1} \otimes \hat{\boldsymbol{\Sigma}}_u^{(h)})(\mathbb{I}_p \otimes \boldsymbol{R}')$. Now, we consider the following statistic test which is a standardized version of $\mathcal{T}(\hat{\boldsymbol{u}}^{(h)}, \hat{\boldsymbol{\Sigma}}_u^{(h)})$ given by

$$\mathcal{Q}_T^{(h)} \quad = \quad \frac{\mathcal{T}(\hat{\boldsymbol{u}}^{(h)}, \hat{\boldsymbol{\Sigma}}_u^{(h)}) - pq}{\sqrt{2pq}} \tag{3.4}$$

The main result to test the non-causality is stated in the following theorem.

Theorem 3.1. *Let $\{\boldsymbol{X}_t\}$ be a process given by* (2.2) *and satisfying the Assumption 2.1. If the order p satisfies the Assumption 2.2 and under any hypothesis of the form $\mathcal{H}_0(h)$, we have*

$$\mathcal{Q}_T^{(h)} \quad \xrightarrow{L} \quad N(0,1).$$

Proof. Using the decomposition (3.1), we start by defining the pseudo-statistic

$$\mathcal{T}_T^{(h)} \quad = \quad T\boldsymbol{A}(p)'\boldsymbol{\Delta}_{p,h}^{-1\,\prime}\boldsymbol{A}(p)$$

where $\boldsymbol{\Delta}_{p,h} = (\mathbb{I}_p \otimes \boldsymbol{R})(\boldsymbol{\Gamma}_p^{-1} \otimes \boldsymbol{\Sigma}_u^{(h)})(\mathbb{I}_p \otimes \boldsymbol{R}')$. Thus, we can write

$$\mathcal{Q}_T^{(h)} \quad = \quad \frac{\mathcal{T}_T^{(h)} - pq}{\sqrt{2pq}} + \frac{T\mathcal{Z}_p^{(h)}}{\sqrt{2pq}}, \tag{3.5}$$

where

$$\mathcal{Z}_p^{(h)} \quad = \quad \boldsymbol{A}(p)'\{\hat{\boldsymbol{\Delta}}_{p,h}^{-1} - \boldsymbol{\Delta}_{p,h}^{-1}\}\boldsymbol{A}(p) + 2\boldsymbol{A}(p)'\hat{\boldsymbol{\Delta}}_{p,h}^{-1}\boldsymbol{B}(p) + \boldsymbol{B}(p)'\hat{\boldsymbol{\Delta}}_{p,h}^{-1}\boldsymbol{B}(p).$$

The asymptotic distribution of $\mathcal{Q}_T^{(h)}$ follows from the next two propositions. The proof is given in the appendix of a technical report available from the author (Bouhaddioui and Dufour [6]).

Proposition 3.1. *Under the assumptions of Theorem 3.1, we have that*

$$\mathcal{Z}_p^{(h)} \quad = \quad o_p(\frac{p^{1/2}}{T}).$$

Proposition 3.2. *Under the assumptions of Theorem 3.1, we have that*

$$\frac{\mathcal{T}_T^{(h)} - pq}{\sqrt{2pq}} \quad \xrightarrow{L} \quad N(0,1).$$

□

4 Local asymptotic power

Let consider the series of local alternative hypothesis

$$\mathcal{H}_{\boldsymbol{\varphi}_p(T)}(h) : \boldsymbol{R}\Pi_i^{(h)} \quad = \quad \varphi_i(T), \ \ i = 1, 2, \dots,$$

where $\boldsymbol{\varphi}_p(T) = vec(\varphi_1(T), \dots, \varphi_p(T)) = T^{\kappa} vec(\varphi_1, \dots, \varphi_p) = \boldsymbol{\varphi}_p =$ and $\|\varphi_i(T)\| \to 0$ as $T \to \infty$. We can give the following theorem which characterizes the local power behavior of the test for the particular class of local alternatives designed by $\mathcal{H}_{\boldsymbol{\varphi}_p(T)}(h)$. The proof is given in the appendix of a technical report available from the author (Bouhaddioui and Dufour [6]).

Theorem 4.1. *Let $\{\boldsymbol{X}_t\}$ be a process given by* (2.2) *and satisfying the Assumption 2.1. If the order p satisfies the Assumption 2.2 and under the local alternative hypothesis of the form $\mathcal{H}_{\boldsymbol{\varphi}_p(T)}(h)$ with $\boldsymbol{\varphi}_p(T) = T^{-1/2}\boldsymbol{\varphi}_p$, we have*

$$\frac{\mathcal{Q}_T^{(h)} - \boldsymbol{\mu}_{\boldsymbol{\varphi}_p}^{(h)}}{\boldsymbol{\sigma}_p^{(h)}} \quad \xrightarrow{L} \quad N(0,1)$$

where $\boldsymbol{\mu}_{\boldsymbol{\varphi}_p}^{(h)} = \frac{\boldsymbol{\varphi}_p' \boldsymbol{\Delta}_{p,h}^{-1} \boldsymbol{\varphi}_p}{\sqrt{2pq}}$ and $\boldsymbol{\sigma}_p^{(h)} = \sqrt{1 + \boldsymbol{\mu}_{\boldsymbol{\varphi}_p}^{(h)}}$.

5 Empirical illustration and Simulation

In this section, we present an application of these causality tests at various horizons to macroeconomic time series. The data set considered is the one used Bernanke and Mihov [1] and Dufour, Pelletier and Renault [8] in order to study United States monetary policy. The data set considered consists of monthly observations on nonborrowed reserves (NBR, also denoted ω_1), the federal funds rate (r, ω_2), the GDP deflator (P, ω_3) and real GDP (*GDP*, ω_4). The monthly data on GDP and GDP deflator were constructed by state space methods from quarterly observations [for more details, see Bernanke and Mihov [1]]. The sample goes from January 1965 to December 1996 for a total of 384 observations. In what follows, all variables were first transformed by a logarithmic transformation.

Before performing the causality tests, we must specify the order of the VAR model. First, in order to get apparently stationary time series, all variables were transformed by taking first differences of their logarithms. In particular, for the federal funds rate, this helped to mitigate the effects of possible break in the series in the years 1979-1981. Once the data is made stationary, we use a nonparametric approach for the estimation and Akaike's information criterion to specify the orders of the long VAR(k) models. Using Akaike's criterion for the unconstrained VAR model, which corresponds to four variables, we observe that it is minimized at $k = 16$.

Vector autoregressions of order p at horizon h were estimated as described in section 2 and the matrix $\hat{V}_{ip}^{NW}$, required to obtain covariance matrices, were computed. On looking at the values of the test statistics $\mathcal{Q}_T^{(h)}$ and their corresponding p-values at various horizons it quickly becomes evident that the $N(0,1)$ asymptotic approximation of the statistic $\mathcal{Q}_T^{(h)}$ is very poor. Now, let consider the GDP as the VAR(16) estimated with our data in first difference but we impose that some coefficients are zero such that the federal funds rate does not cause GDP up to horizon h and then we test the $r \overset{h}{\nrightarrow} GDP$ hypothesis. The constraints of non-causality from j to i up to horizon h that we impose are

$$\begin{aligned} \hat{\pi}_{ijl} &= 0, \ \ 1 \leq l \leq p, && (5.1)\\ \hat{\pi}_{ikl} &= 0, \ \ 1 \leq l \leq h, \ 1 \leq k \leq m. && (5.2) \end{aligned}$$

Rejection frequencies for this case are given in Table 5.1.
In light of these results we computed the p-values by doing a parametric bootstrap, i.e. doing an asymptotic Monte Carlo test based on a consistent point estimate [see Dufour(2006) [7]]. The procedure to test the hypothesis $\omega_j \overset{h}{\nrightarrow} \omega_i | I_{(j)}$ is the following:

1. An unrestricted VAR(p) model is fitted for the horizon one, yielding the estimates $\hat{\Pi}^{(1)}$ and $\hat{\Sigma}$ for $\Pi^{(1)}$ and Σ. The autoregressive order was obtained by minimizing the AIC criterion for $p \leq P$, where P was fixed to 24.

2. An unrestricted (p,h)-autoregression is fitted by least squares, yielding the estimate $\hat{\Pi}^{(h)}$ of $\Pi^{(h)}$.

3. The test statistic $\mathcal{Q}_T^{(h)}$ for testing non-causality at the horizon h from ω_j to ω_i is computed. We denoted by $\mathcal{Q}_{T,j\nrightarrow i}^{(h)}(0)$ the test statistic based on the actual data.

4. N simulated samples from (2.2) are drawn by Monte Carlo methods, using $\Pi^{(h)} = \hat{\Pi}^{(h)}$ and $\Sigma = \hat{\Sigma}$. We impose the constraints of non-causality $\hat{\pi}_{ijk} = 0, k = 1, \ldots, p$. Estimates of the impulse response coefficients are obtained from $\hat{\Pi}^{(1)}$ through the relations described in Eq. (2.4). We denote by $\mathcal{Q}_{T,j\nrightarrow i}^{(h)}(n)$ the test statistic for $\mathcal{H}_{j\nrightarrow i}^{(h)}$ based on nth simulated sample ($1 \leq n \leq N$).

5. The simulated p-value $\hat{p}[\mathcal{Q}_{T,j\nrightarrow i}^{(h)}(0)]$ is obtained, where
$$\hat{p}[x] = \{1 + \sum_{n=1}^{N} \mathcal{I}[\mathcal{Q}_{T,j\nrightarrow i}^{(h)}(n) - x]\}/(n+1),$$
where $\mathcal{I}[x] = 1$ if $x \geq 0$ and $\mathcal{I}[x] = 0$ if $x < 0$.

6. The null hypothesis $\mathcal{H}_{j\nrightarrow i}^{(h)}$ is rejected at level α if $\hat{p}[\mathcal{Q}_{T,j\nrightarrow i}^{(h)}(0)] \leq \alpha$.

7. Finally, for each nominal level $\alpha = 1\%, 5\%$ and 10%, we obtained from the 2000 realizations (with $N = 999$), the empirical frequencies of rejection of the null hypothesis of non-correlation. The power analysis was conducted in the similar way using the model VAR$_\delta$ for different values of δ.

From looking at the results in Table 5.1, we see that we get a much better size control by using this bootstrap procedure. The rejection frequencies over 1000 replications (with $N = 999$) are very close to the nominal size. Although the coefficients ψ_j's are functions of π_j's we do not constrain them in the bootstrap procedure because there is no direct mapping from $\pi_k^{(h)}$ to π_k and ψ_j. This certainly produces a power loss but the procedure remains valid because the ψ_j's are computed with the $\hat{\pi}_k$, which are consistent estimates of the true π_k both under the null and alternative hypothesis. To illustrate that our procedure has power for detecting departure from the null hypothesis of non-causality at a given horizon we ran the following Monte Carlo experiment. We again took a VAR(16) fitted on our data in first differences and we imposed the constraints (5.1)-(5.2) so that there was no causality from r to GDP up to horizon 12. Next the value of one coefficient previously set to zero was changed to induce causality from r to GDP at horizon 4 and higher: $\pi_3(1, 3) = \theta$. As θ increases from zero to one the strength of the causality from r to GDP is higher. Under this setup, we could compute the power of our simulated test procedure to reject the null hypothesis of non-causality at a given horizon. The power curves are plotted as a function of θ for the various horizons. The level of the tests was controlled through the bootstrap procedure. In this experiment we took again $N = 999$ and we did 1000 simulations. These curves are plotted in Bouhaddioui-Dufour (2009) [6]. As expected, the power curves are flat at around 5% for horizons one to three since the null is true for these horizons. For horizons four and up we get the expected result that power goes up as θ moves from zero to one, and the power curves gets flatter as we increase the horizon.

Now that we have shown that bootstrap procedure does have power we present causality tests at horizon one to 24 for every pair of variables in Tables 5.2 and 5.3. For every horizon we have 12 causality tests and we group them by pairs. When we say that a given variable cause or does not cause another, it should be understood that we mean the growth rate of the variables. The p-values are computed by taking $N = 999$. Table 5.4 summarize the results by presenting the significant results at the 5% and 10% levels.

The first thing to notice is that we have significant causality results at short horizons for some pairs of variables while we have it at longer horizons for other pairs. This is an interesting illustration of the concept of causality at horizon h of Dufour and Renault [9].

The instrument of the central bank, the nonborrowed reserves, cause the federal funds rate at horizon one, the prices at horizons 1, 2, 3 and 9 (10% level). It does not cause the other two variables at any horizon and except the GDP at horizon 12 and 16 (10% level) nothing is causing it. We see that the impact of variations in the nonborrowed reserves is over a very short term. Another variable, the GDP, is also causing the federal funds rates over short horizons (one to five months).

An interesting result is the causality from the federal funds rate to the GDP. Over the first few months the funds rate does not cause GDP, but from horizon 3 (up to 20) we do find

Table 5.1: Rejection using the asymptotic distribution N(0,1) with VAR(16)

	Asymptotic					Bootstrap					
h	$\alpha=5\%$	$\alpha=10\%$	h	$\alpha=5\%$	$\alpha=10\%$	h	$\alpha=5\%$	$\alpha=10\%$	h	$\alpha=5\%$	$\alpha=10\%$
1	22.2	32.2	7	36.9	52.3	1	5.2	9.6	7	4.3	10.8
2	23.4	38.6	8	39.4	55.6	2	4.8	9.3	8	4.1	9.6
3	27.6	40.5	9	42.8	58.4	3	4.9	10.5	9	4.8	9.1
4	31.2	44.2	10	46.6	61.5	4	5.8	11.0	10	5.6	10.7
5	32.7	46.4	11	49.5	63.4	5	6.0	9.8	11	5.4	10.4
6	34.6	48.8	12	53.7	66.8	6	5.3	10.2	12	4.1	9.1

significant causality. This result can easily be explained by, e.g. the theory of investment. Notice that we have following indirect causality. Nonborrowed reserves do not cause GDP directly over any horizon, but they cause the federal funds rate which in turn causes GDP. Concerning the observation that there are very few causality results for long horizons, this may reflect the fact that, for stationary processes, the coefficients of prediction formulas converge to zero as the forecast horizon increases.
Using the results of Proposition 4.5 in Dufour and Renault [9], we know that for the highest horizon that we have to consider is 33 since we have a VAR(16) with four time series. Causality tests for horizons 25 through 33 were also computed but are not reported. Some p-values smaller or equal to 10% are scattered over horizons 30 to 33 but no discernible pattern emerges.

Acknowledgements

The authors wish to thank a referee for helpful comments that led to a more concise and improved presentation. This work was supported by the William Dow Chair in Economics (McGill University), the Bank of Canada (Research Fellowship), the Canada Research Chair Program (Chair in Econometrics, Université de Montréal, Canada), the Alexander-von-Humboldt Foundation(Germany), the Institut de finance mathématique de Montréal (IFM2), the Canadian Network of Centres of Excellence [program on Mathematics of Information Technology and Complex Systems (MITACS)], the Canada Council for the Arts (Killam Fellowship), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, the Fonds de recherche sur la société et la culture (Québec), and the Fonds de recherche sur la nature et les technologies (Québec), and the Fonds de recherche sur la nature et les technologies (Québec).

Table 5.2: Causality tests and simulated p-values for series in first differences (of logarithm) for horizon 1 to 12.

	h		1	2	3	4	5	6	7	8	9	10	11	12
NBR	$\nrightarrow$	r	43.584	39.375	28.332	24.268	18.314	20.895	22.745	19.843	21.226	23.054	20.667	18.445
			(0.035)	(0.064)	(0.281)	(0.325)	(0.412)	(0.364)	(0.401)	(0.267)	(0.358)	(0.285)	(0.257)	(0.349)
r	$\nrightarrow$	NBR	27.895	25.853	26.933	21.562	22.572	18.942	22.193	49.773	30.381	47.923	36.859	33.287
			(0.267)	(0.346)	(0.389)	(0.420)	(0.369)	(0.572)	(0.357)	(0.031)	(0.139)	(0.073)	(0.146)	(0.201)
NBR	$\nrightarrow$	P	52.368	47.329	45.835	40.376	37.289	48.104	42.864	39.249	40.982	35.849	33.457	29.045
			(0.003)	(0.042)	(0.063)	(0.168)	(0.368)	(0.043)	(0.082)	(0.235)	(0.189)	(0.390)	(0.471)	(0.569)
P	$\nrightarrow$	NBR	20.347	23.458	25.926	21.285	24.338	26.371	28.912	31.026	32.902	33.954	36.789	35.496
			(0.690)	(0.562)	(0.483)	(0.439)	(0.523)	(0.379)	(0.428)	(0.284)	(0.301)	(0.358)	(0.274)	(0.305)
NBR	$\nrightarrow$	GDP	29.267	27.834	22.863	26.891	23.897	20.915	17.983	15.789	20.951	18.937	23.179	25.951
			(0.091)	(0.285)	(0.589)	(0.459)	(0.672)	(0.579)	(0.459)	(0.631)	(0.584)	(0.479)	(0.395)	(0.421
GDP	$\nrightarrow$	NBR	21.894	24.671	26.740	23.934	27.891	36.956	39.034	41.361	20.941	40.921	37.812	42.846
			(0.734)	(0.657)	(0.620)	(0.572)	(0.495)	(0.406)	(0.329)	(0.279)	(0.396)	(0.258)	(0.185)	(0.162)
r	$\nrightarrow$	P	36.189	37.228	32.963	30.951	28.954	24.518	27.913	26.841	31.852	38.752	41.954	31.953
			(0.085)	(0.082)	(0.201)	(0.174)	(0.273)	(0.246)	(0.310)	(0.356)	(0.274)	(0.184)	(0.243)	(0.361)
P	$\nrightarrow$	r	25.983	18.653	23.841	19.846	17.439	15.742	14.952	12.762	16.529	17.930	19.203	22.391
			(0.571)	(0.730)	(0.562)	(0.638)	(0.749)	(0.809)	(0.761)	(0.893)	(0.751)	(0.650)	(0.704)	(0.659)
r	$\nrightarrow$	GDP	30.781	33.610	36.074	42.358	40.227	45.332	63.228	79.225	92.556	98.265	102.347	109.104
			(0.301)	(0.192)	(0.072)	(0.041)	(0.053)	(0.017)	(0.005)	(0.003)	(0.001)	(0.002)	(0.001)	(0.002)
GDP	$\nrightarrow$	r	45.274	48.739	43.996	42.337	39.058	37.072	32.916	29.045	30.517	33.928	30.841	34.968
			(0.028)	(0.031)	(0.043)	(0.032)	(0.045)	(0.078)	(0.169)	(0.291)	(0.206)	(0.265)	(0.178)	(0.342)
P	$\nrightarrow$	GDP	25.865	26.285	22.324	17.653	14.327	18.462	20.846	24.739	22.894	27.541	23.218	25.431
			(0.468)	(0.263)	(0.361)	(0.635)	(0.792)	(0.671)	(0.530)	(0.368)	(0.429)	(0.404)	(0.367)	(0.470)
GDP	$\nrightarrow$	P	27.834	25.719	30.260	37.984	46.228	38.539	36.226	40.338	33.941	28.946	30.860	26.701
			(0.420)	(0.321)	(0.165)	(0.073)	(0.032)	(0.082)	(0.076)	(0.041)	(0.234)	(0.320)	(0.286)	(0.346)

Table 5.3: Causality tests and simulated p-values for series in first differences (of logarithm) for horizon 13 to 24.

	h		13	14	15	16	17	18	19	20	21	22	23	24
NBR	$\nrightarrow$	r	22.561	20.981	24.376	19.867	23.196	30.458	36.719	42.951	50.983	37.942	32.901	28.937
			(0.658)	(0.571)	(0.682)	(0.561)	(0.621)	(0.439)	(0.429)	(0.359)	(0.286)	(0.461)	(0.671)	(0.727)
r	$\nrightarrow$	NBR	53.289	50.921	46.871	42.571	37.682	39.872	33.718	30.791	38.963	48.931	36.916	30.992
			(0.168)	(0.184)	(0.238)	(0.439)	(0.561)	(0.459)	(0.574)	(0.621)	(0.566)	(0.359)	(0.466)	(0.554)
NBR	$\nrightarrow$	P	43.985	45.647	40.368	42.389	50.356	41.882	38.649	58.902	60.035	55.341	48.467	45.392
			(0.451)	(0.362)	(0.450)	(0.423)	(0.278)	(0.367)	(0.562)	(0.239)	(0.178)	(0.190)	(0.254)	(0.321)
P	$\nrightarrow$	NBR	39.045	36.891	37.841	30.956	25.874	38.904	44.396	42.951	48.903	52.314	47.825	43.671
			(0.376)	(0.257)	(0.351)	(0.567)	(0.623)	(0.284)	(0.326)	(0.287)	(0.241)	(0.189)	(0.257)	(0.348)
NBR	$\nrightarrow$	GDP	33.454	28.904	23.456	25.671	20.903	18.793	28.583	23.674	31.083	33.865	39.865	42.347
			(0.561)	(0.346)	(0.589)	(0.420)	(0.782)	(0.803)	(0.634)	(0.746)	(0.639)	(0.549)	(0.476)	(0.386
GDP	$\nrightarrow$	NBR	47.890	53.763	68.256	64.963	52.930	49.045	43.960	40.819	32.489	28.793	33.478	35.672
			(0.345)	(0.231)	(0.074)	(0.081)	(0.189)	(0.278)	(0.348)	(0.459)	(0.672)	(0.758)	(0.720)	(0.546)
r	$\nrightarrow$	P	23.856	38.561	37.428	39.045	28.983	30.163	37.439	46.842	50.263	53.980	59.125	55.471
			(0.673)	(0.385)	(0.479)	(0.376)	(0.654)	(0.457)	(0.402)	(0.376)	(0.228)	(0.268)	(0.128)	(0.208)
P	$\nrightarrow$	r	24.561	22.174	28.940	30.984	22.895	32.054	20.981	21.267	24.618	34.901	37.279	38.235
			(0.748)	(0.704)	(0.629)	(0.673)	(0.707)	(0.579)	(0.658)	(0.745)	(0.569)	(0.583)	(0.607)	(0.564)
r	$\nrightarrow$	GDP	130.742	110.226	95.786	94.893	78.654	85.849	75.356	68.925	63.952	56.476	52.845	55.671
			(0.001)	(0.001)	(0.003)	(0.002)	(0.031)	(0.024)	(0.042)	(0.063)	(0.088)	(0.138)	(0.204)	(0.249)
GDP	$\nrightarrow$	r	40.652	37.658	42.398	31.278	26.874	20.210	18.934	20.313	31.682	39.028	41.204	32.571
			(0.367)	(0.472)	(0.320)	(0.539)	(0.662)	(0.827)	(0.869)	(0.758)	(0.528)	(0.458)	(0.431)	(0.548)
P	$\nrightarrow$	GDP	11.274	13.256	17.823	21.034	18.920	23.561	27.351	38.419	49.735	55.301	58.247	67.208
			(0.962)	(0.943)	(0.895)	(0.845)	(0.873)	(0.742)	(0.784)	(0.534)	(0.356)	(0.284)	(0.236)	(0.174)
GDP	$\nrightarrow$	P	27.361	36.712	32.901	43.561	58.903	48.752	41.328	61.045	63.451	68.201	58.762	54.612
			(0.562)	(0.358)	(0.428)	(0.385)	(0.273)	(0.329)	(0.374)	(0.268)	(0.304)	(0.224)	(0.287)	(0.359)

Table 5.4: Summary of Causality relations at various horizons for series in first difference.

h	1	2	3	4	5	6	7	8	9	10	11	12
$NBR \nrightarrow r$	$\star\star$	$\star$										
$r \nrightarrow NBR$								$\star\star$		$\star$		
$NBR \nrightarrow P$	$\star\star$	$\star\star$	$\star$			$\star\star$	$\star$					
$P \nrightarrow NBR$												
$NBR \nrightarrow GDP$	$\star$											
$GDP \nrightarrow NBR$												
$r \nrightarrow P$	$\star$	$\star$										
$P \nrightarrow r$												
$r \nrightarrow GDP$			$\star$	$\star\star$	$\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$
$GDP \nrightarrow r$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star$						
$P \nrightarrow GDP$												
$GDP \nrightarrow P$					$\star$	$\star\star$	$\star$	$\star$	$\star\star$			
h	13	14	15	16	17	18	19	20	21	22	23	24
$NBR \nrightarrow r$												
$r \nrightarrow NBR$												
$NBR \nrightarrow P$												
$P \nrightarrow NBR$												
$NBR \nrightarrow GDP$												
$GDP \nrightarrow NBR$			$\star$	$\star$								
$r \nrightarrow P$												
$P \nrightarrow r$												
$r \nrightarrow GDP$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star\star$	$\star$	$\star$			
$GDP \nrightarrow r$												
$P \nrightarrow GDP$												
$GDP \nrightarrow P$												

Note: The symbols $\star$ and $\star\star$ indicate rejection of the non-causality hypothesis at the 10% and 5% levels, respectively.

References

[1] B. Bernanke, I. Mihov, (1998). Measuring monetary policy, *The Quarterly Journal of Economics* **113**(3),869–902.

[2] H. Boudjellaba, J.-M. Dufour, R. Roy (1992). Testing causality between two vectors in multivariate arma models, *Journal of the American Statistical Association* **87**, 1082–1090.

[3] C. Bouhaddioui, R. Roy (2006). A generalized portmanteau test for independence of

two infinite-order vector autoregressive series, *Journal of Time Series Analysis* **27**(4), 505–544.

[4] C. Bouhaddioui, J.-M. Dufour (2008). Tests for Non-Correlation of two infinite-order cointegrated vector autoregressive series, *Journal of Applied Probability and Statistics* **3 (1)**, 77–94.

[5] C. Bouhaddioui, J.-M. Dufour (2009). Semiparametric tests of orthogonality and causality between two cointegrated series with application to Canada/US monetary interactions, *Discussion Paper*, CIRANO and CIREQ, 35 pages.

[6] C. Bouhaddioui, J.-M. Dufour (2009). Short run and long run causality in an infinite-order autoregressive vector, *Technical Report*, CIRANO and CIREQ, 27 pages.

[7] J.-M. Dufour (2006). Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics in econometrics, *Journal of Econometrics* **133**(2) 443–477.

[8] J.-M. Dufour, D. Pelletier, É. Renault (2006). Short run and long run causality in time series: Inference, *Journal of Econometrics* **132**(2), 337–362.

[9] J. -M. Dufour, É. Renault (1998). Short run and long run causality in time series: Theory, *Econometrica* **66**, 1099–1125.

[10] J. Geweke (1984). Inference and causality in economic time series, *in* Z. Griliches, M. D. Intrilligator, eds, 'Handbook of Econometrics, Volume 2', North-Holland, Amsterdam, pp. 1102–1144.

[11] J. A. Giles (2002). Time series analysis testing for two-step Granger noncausality in trivariate VAR models, *in* A. Ullah, A. K. Wan, A. Chaturvedi, eds, 'Handbook of Applied Econometrics and Statistical Inference', Marcel Dekker, New York, chapter 18, pp. 371–399.

[12] C. Gouriéroux, A. Monfort (1997). *Time Series and Dynamic Models*, Cambridge University Press, Cambridge, U.K.

[13] C. Granger (1969). Investigating causal relations by econometric models and cross-spectral methods, *Econometrica* **37**, 424–459.

[14] R. Lewis, G. C. Reinsel (1985). Prediction of multivariate time series by autoregressive model fitting, *Journal of Multivariate Analysis* **16**, 393–411.

[15] H. Lütkepohl (2005). *New Introduction to Multiple Time Series Analysis*, Springer-Verlag, Berlin.

[16] H. Lütkepohl (1993). Testing for causation between two variables in higher dimensional VAR models, *in* H. Schneeweiss, K. Zimmermann, eds, 'Studies in Applied Econometrics', Springer-Verlag, Heidelberg.

[17] H. Lütkepohl, M. M. Burda (1997). Modified Wald tests under nonregular conditions, *Journal of Econometrics* **78**, 315–332.

[18] P. Newbold (1982). Causality testing in economics, *in* O. D. Anderson, ed., Time Series Analysis: Theory and Practice 1, North-Holland, Amsterdam.

[19] D. A. Pierce, L. D. Haugh (1977). Causality in temporal systems: Characterizations and survey, *Journal of Econometrics* **10**, 257–259.

[20] E. Renault, K. Sekkat, A. Szafarz (1998). Testing for spurious causality in exchange rates, *Journal of Empirical Finance* **5**,47–66.
[21] C. Sims (1980). Macroeconomic and reality, *Econometrica* **48**, 1–48.
[22] N. Wiener (1956). *The Theory of Prediction*, New York, chapter 8.