Hilbert spaces*

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References: Brockwell and Davis (1991), Chapter 2.

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1. Inner-product spaces

1.1 Definition INNER PRODUCT SPACE. *Let $\mathscr{H}$ be a vector space on a set scalars S, where $S = \mathbb{R}$ or $S = \mathbb{C}$. An* inner product *on H is an application which associates to each pair of elements x and y in $\mathscr{H}$ a complex number $\langle x, y\rangle$ such that, for all x, y, $z \in \mathscr{H}$,*

(a) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(b) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

(c) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$, *for all* $\alpha \in S$;

(d) $\langle x, x\rangle \geq 0$;

(e) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

If $\langle\cdot,\cdot\rangle$ is an inner product on $\mathscr{H}$, the pair $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is called an inner product space*, and the elements of $\mathscr{H}$ are also called elements of the inner-product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. When there is no ambiguity on the definition of the inner product, the inner-product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ may simply be denoted $\mathscr{H}$.*

1.2 Remark If $S = \mathbb{R}$, condition (a) reduces to

(a)$'$ $\langle x, y\rangle = \langle y, x\rangle$.

1.3 Definition NORM ASSOCIATED WITH AN INNER PRODUCT. *The* norm *of an element x of an inner-product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is defined by*

$$\|x\| = \sqrt{\langle x, x\rangle} . \tag{1.1}$$

1.4 Example EUCLIDEAN SPACE. $\mathbb{R}^n$ with the usual scalar product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{1.2}$$

where $x = (x_1, \ldots, x_x)'$ $y = (y_1, \ldots, y_x)'$, where $x_i \in \mathbb{R}$ and $y_j \in \mathbb{R}$ for all i and j, is an inner-product space whose norm is the usual Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} . \tag{1.3}$$

1.5 Example COMPLEX EUCLIDEAN SPACE. $\mathbb{C}^n$ with the scalar product

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i \tag{1.4}$$

where $x = (x_1, \ldots, x_n)'$ $y = (y_1, \ldots, y_n)'$, where $x_i \in \mathbb{C}$ and $y_j \in \mathbb{C}$ for all i and j, is an inner-product space. The associated norm is:

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}. \tag{1.5}$$

1.6 Example REAL-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2 = L^2(\Omega, \mathscr{A}, P)$ of all the random variables $X : \Omega \to \mathbb{R}$ such that $E(X^2) < \infty$ with, for any $X, Y \in L^2$,

$$\langle X,Y\rangle = E(XY) \tag{1.6}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(X^2)]^{1/2}. \tag{1.7}$$

1.7 Example COMPLEX-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$ of all the random variables $X : \Omega \to \mathbb{C}$ such that $E(|X|^2) < \infty$ with

$$\langle X,Y\rangle = E(X\overline{Y}) \tag{1.8}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(|X|^2)]^{1/2}. \tag{1.9}$$

1.8 Proposition CAUCHY-SCHWARZ INEQUALITY. *Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be an inner-product space. Then, for all $x, y \in \mathscr{H}$,*

$$|\langle x,y\rangle| \leq \|x\|\,\|y\| \tag{1.10}$$

and

$$|\langle x,y\rangle| = \|x\|\,\|y\| \;\Leftrightarrow\; x = \left(\frac{\langle x,y\rangle}{\langle y,y\rangle}\right) y\,. \tag{1.11}$$

1.9 Definition ANGLE AND ORTHOGONALITY. *Let x and y be two elements of an inner product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. The* angle *between the vectors x and y is defined by*

$$\theta = \cos^{-1}[\langle x,y\rangle/(\|x\|\,\|y\|)]\,. \tag{1.12}$$

x and y are said to be orthogonal *(denoted $x \perp y$) if and only if*

$$\langle x,y\rangle = 0\,. \tag{1.13}$$

1.10 Proposition PROPERTIES OF THE NORM. *Let* $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ *be an inner-product space with the norm* $\|x\| = \sqrt{\langle x,x\rangle}$. *Then the following properties hold for all* $x, y \in \mathscr{H}$:

(a) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$;

(b) $\|x+y\| \leq \|x\| + \|y\|$; (*Triangle inequality*)

(c) $\|\alpha x\| \leq |\alpha|\,\|x\|$, *for all* $\alpha \in \mathbb{C}$;

(d) $\|x\| \geq 0$;

(e) $\|x\| = 0$ *if and only if* $x = 0$;

(f) $\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$. (*Parallelogram law*)

1.11 Proposition CONVERGENCE IN NORM. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of an iner product space* $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. *We say that* x_n *converges to* $x \in \mathscr{H}$ *if and only if*

$$\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 . \tag{1.14}$$

1.12 Proposition CONTINUITY OF INNER PRODUCT. *Let* $\{x_n : n = 1, 2, ...\}$ *and* $\{y_n : n = 1, 2, ...\}$ *be two sequences of elements of an inner-product space* $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ *such that* $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0$ *and* $\|y_n - y\| \underset{n\to\infty}{\longrightarrow} 0$, *where* $x, y \in \mathscr{H}$ *.Then*

$$\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\| \tag{1.15}$$

and

$$\langle x_n, y_n\rangle \underset{n\to\infty}{\longrightarrow} \langle x, y\rangle \ . \tag{1.16}$$

1.13 Example MEAN SQUARE CONVERGENCE. Let $\{X_n : n = 1, 2, ...\}$ a sequence of real random variables in $L^2 \equiv L^2(\Omega, \mathscr{A}, P)$. If we define $\langle X, Y\rangle = E(XY)$, then the convergence in norm represents mean-square convrgence:

$$\|X_n - X\| \underset{n\to\infty}{\longrightarrow} 0 \ \Leftrightarrow \ E[(X_n - X)^2] \underset{n\to\infty}{\longrightarrow} 0 . \tag{1.17}$$

2. Hilbert spaces

2.1 Definition CAUCHY SEQUENCE. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of an inner-product space* $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. *Then* x_n *is a* Cauchy sequence *of* $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ *if and only if*

$$\|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0 . \tag{2.1}$$

2.2 Definition HILBERT SPACE. *Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ an inner-product space. Then $(\mathscr{H}, \langle\cdot,\cdot\rangle)$is a* Hilbert space *if and only if every Cauchy sequence of $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ converges in norm to an element of $\mathscr{H}$,*

$$\|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0 \Rightarrow \exists x \in \mathscr{H} \text{ such that } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0\,. \tag{2.2}$$

2.3 Proposition NORM CONVERGENCE AND CAUCHY CRITERION. *Let $\{x_n : n = 1, 2, ...\}$ be a sequence of elements of a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. Then x_n converges in norm if and only if x_n is a Cauchy sequence.*

2.4 Theorem COMPLETENESS OF $L^2(\Omega, \mathscr{A}, P)$. *The inner-product space $L^2(\Omega, \mathscr{A}, P)$ with $\langle X,Y\rangle = E(XY)$ is complete and thus constitutes a Hilbert space.*

PROOF. See Brockwell and Davis (1991, Section 2.10). □

3. Projection theorems

3.1 Definition CLOSED SUBSPACE. *A linear subspace $\mathscr{M}$ of a Hilbert space $\mathscr{H}$ is a* closed subspace *of $\mathscr{H}$ if and only if $\mathscr{M}$ contains all its limit points, i.e.*

$$x_n \in \mathscr{M}\,,\ \forall n\,,\ \text{and } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow x \in \mathscr{M}\,. \tag{3.1}$$

3.2 Definition ORTHOGONAL COMPLEMENT. *The orthogonal complement $\mathscr{M}^\perp$ of a subset $\mathscr{M}$ of $\mathscr{H}$ is the set of all $x \in \mathscr{H}$ which are orthogonal to all the elements of $\mathscr{M}$, i.e.*

$$x \in \mathscr{M}^\perp \Leftrightarrow \langle x,y\rangle = 0\,,\ \forall y \in \mathscr{M}\,. \tag{3.2}$$

3.3 Proposition CLOSURE OF ORTHOGONAL COMPLEMENT. *If $\mathscr{M}$ is any subset of a Hilbert space $\mathscr{H}$, then $\mathscr{M}^\perp$ is a closed subspace of $\mathscr{H}$.*

3.4 Theorem PROJECTION THEOREM. *Let $\mathscr{M}$ be a closed subspace of a Hilbert space $\mathscr{H}$. Then, for any $x \in \mathscr{H}$,*

(*a*) *there is a unique element $\hat{x} \in \mathscr{M}$ such that*

$$\|x - \hat{x}\| = \inf_{y\in\mathscr{M}} \|x - y\| \tag{3.3}$$

and

(*b*)

$$\begin{aligned} &\hat{x} \in \mathscr{M} \text{ and } \|x - \hat{x}\| = \inf_{y\in\mathscr{M}} \|x - y\| \\ &\Leftrightarrow \hat{x} \in \mathscr{M} \text{ and } x - \hat{x} \in \mathscr{M}^\perp\,. \end{aligned} \tag{3.4}$$

3.5 Corollary EXISTENCE OF PROJECTION MAPPING OF $\mathscr{H}$ ONTO $\mathscr{M}$. *Let $\mathscr{M}$ be a closed subspace of a Hilbert space $\mathscr{H}$, and let I be the identity mapping on $\mathscr{H}$. Then there is a unique mapping $P_{\mathscr{M}}$ of $\mathscr{H}$ onto $\mathscr{M}$ such that $I - P_{\mathscr{M}}$ maps $\mathscr{H}$ onto $\mathscr{M}^{\perp}$. $P_{\mathscr{M}}$ is called the projection mapping of $\mathscr{H}$ onto $\mathscr{M}$.*

3.6 Theorem PROPERTIES OF PROJECTION MAPPINGS. *Let $\mathscr{H}$ be a Hilbert space and $P_{\mathscr{M}}$ the projection mapping onto the closed subspace $\mathscr{M}$. Then the following properties hold for all $x, y \in \mathscr{H}$:*

(a) *each element $x \in \mathscr{H}$ has a unique representation as the sum of an element of $\mathscr{M}$ and an element of $\mathscr{M}^{\perp}$,*

$$x = P_{\mathscr{M}}x + (I - P_{\mathscr{M}})x \; ; \tag{3.5}$$

(b) $\|x\|^2 = \|P_{\mathscr{M}}x\|^2 + \|(I - P_{\mathscr{M}})x\|^2$;

(c) $P_{\mathscr{M}}(\alpha x + \beta y) = \alpha P_{\mathscr{M}}x + \beta P_{\mathscr{M}}y$, *for all* α , $\beta \in S$;

(d) $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow P_{\mathscr{M}}x_n \underset{n\to\infty}{\longrightarrow} P_{\mathscr{M}}x$;

(e) $x \in \mathscr{M} \Leftrightarrow P_{\mathscr{M}}x = x$;

(f) $x \in \mathscr{M}^{\perp} \Leftrightarrow P_{\mathscr{M}}x = 0$;

(g) *if $\mathscr{M}_1$ and $\mathscr{M}_2$ are two subspaces of $\mathscr{H}$ such that $\mathscr{M}_1 \subseteq \mathscr{M}_2$, then*

$$P_{\mathscr{M}_1}(P_{\mathscr{M}_2}x) = P_{\mathscr{M}_1}x \, . \tag{3.6}$$

4. Orthonormal sets

4.1 Definition CLOSED SPAN. *Let $\mathscr{M}$ be a subset of a Hilbert space $\mathscr{H}$. The closed span of $\mathscr{M}$, denoted $\overline{sp}(\mathscr{M})$, is the smallest closed subspace of $\mathscr{H}$ that contains $\mathscr{M}$. If $\mathscr{M} = \{e_t : t \in T\}$, we can also denote $\overline{sp}(\mathscr{M})$ by $\overline{sp}\{e_t : t \in T\}$.*

4.2 Proposition CLOSED SPAN OF A FINITE SET. *The closed span of a finite set $\mathscr{M} = \{x_1, ..., x_n\}$ is the set of all linear combinations*

$$y = \alpha_1 x_1 + ... + \alpha_n x_n \; , \; \alpha_1 \, , \, ... \, , \, \alpha_n \in S \tag{4.1}$$

where $S = \mathbb{R}$ or $S = \mathbb{C}$.

4.3 Definition ORTHOGONAL AND ORTHONORMAL SETS. *Let $\{e_t : t \in T\}$ be a subset of an inner-product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. Then $\{e_t : t \in T\}$ is said to be* orthogonal *if and only if*

$$\langle e_s, e_t\rangle = 0\,,\ \text{for } s \neq t\ . \tag{4.2}$$

Further, $\{e_t : t \in T\}$ is said to be orthonormal *if and only it is orthogonal and $\langle e_t, e_t\rangle = 1$ for all $t \in T$.*

4.4 Theorem PROJECTIONS ON FINITE ORTHONORMAL SETS. *If $\{e_1, \ldots, e_n\}$ is an orthonormal subset of a Hilbert space $\mathscr{H}$ and $\mathscr{M} = \overline{sp}\{e_1, \ldots, e_n\}$, then the following properties hold for all $x \in \mathscr{H}$:*

(a) $P_{\mathscr{M}}x = \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i\,;$

(b) $\|P_{\mathscr{M}}x\|^2 = \sum_{i=1}^{n} |\,\langle x, e_i\rangle\,|^2;$

(c) $\left\|x - \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i\right\| \leq \left\|x - \sum_{i=1}^{n} c_i e_i\right\|$, *for all* $c_1, \ldots, c_n \in S$;

(d)
$$\left\|x - \sum_{i=1}^{n} \langle x, e_i\rangle\, e_i\right\| = \left\|x - \sum_{i=1}^{n} c_i e_i\right\| \iff c_i = \langle x, e_i\rangle\ ,\ i = 1, \ldots, n; \tag{4.3}$$

(e) $\sum_{i=1}^{n} |\,\langle x, e_i\rangle\,|^2 \leq \|x\|^2\ .$ (*Bessel's inequality*)

4.5 Remark The coefficients $c_i = \langle x, e_i\rangle\ ,\ i = 1, \ldots, n$, are called the *Fourier coefficients* relative to the set $\{e_1, \ldots, e_n\}$.

4.6 Corollary PROJECTION ON A UNIDIMENSIONAL SUBSPACE. *Suppose $\mathscr{M} = \overline{sp}\{e_1\}$ where $e_1 \in \mathscr{H}$ and $e_1 \neq 0$. Then*

$$P_{\mathscr{M}}x = \frac{\langle x, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = \frac{\langle x, e_1\rangle}{\|e_1\|^2} e_1\ . \tag{4.4}$$

4.7 Corollary PROJECTIONS ON FINITE ORTHOGONAL SETS. *Let $\{e_1, \ldots, e_n\}$ be a finite set of non-zero orthogonal vectors in a Hilbert space $\mathscr{H}$, let $\mathscr{M} = \overline{sp}\{e_1, \ldots, e_n\}$, and let $\beta_i(x) = \langle x, e_i\rangle / \langle e_i, e_i\rangle$, $i = 1, \ldots, n$. Then the following properties hold for all $x \in \mathscr{H}$:*

(a) $P_{\mathscr{M}}x = \sum_{i=1}^{n} \beta_i(x) e_i\,;$

(b) $\|P_{\mathscr{M}}x\|^2 = \sum_{i=1}^{n} |\,\langle x, e_i\rangle\,|^2 / \|e_i\|^2 = \sum_{i=1}^{n} |\,\langle x, e_i\rangle\,|^2 / \langle e_i, e_i\rangle = \sum_{i=1}^{n} |\beta_i(x)|^2\, \|e_i\|^2\ ;$

(c) $\left\| x - \sum_{i=1}^{n} \beta_i(x) e_i \right\| \le \left\| x - \sum_{i=1}^{n} c_i e_i \right\|$, *for all* $c_1, \dots, c_n \in S$;

(d)
$$\left\| x - \sum_{i=1}^{n} \beta_i(x) e_i \right\| = \left\| x - \sum_{i=1}^{n} c_i e_i \right\| \Leftrightarrow c_i = \beta_i ,\ i = 1, \dots, n; \tag{4.5}$$

(e) $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 / \|e_i\|^2 = \sum_{i=1}^{n} |\beta_i(x)|^2 \|e_i\|^2 \le \|x\|^2$. (*Bessel's inequality*)

4.8 Definition SEPARABILITY. *The Hilbert space* $\mathscr{H}$ *is* separable *if and only if* $\mathscr{H} = \overline{sp}\{e_t : t \in T\}$ *where* T *is a finite or countable infinite set.*

4.9 Theorem ORTHONORMAL REPRESENTATIONS IN SEPARABLE HILBERT SPACES. *Let* $\mathscr{H} = \overline{sp}\{e_1, e_2, \dots\}$ *be a separable Hilbert space where* $\{e_1, e_2, \dots\}$ *is an orthonormal set. Then the following properties hold for all* $x \in \mathscr{H}$:

(a) *the set of all finite linear combinations of* $\{e_1, e_2, \dots\}$ *is dense in* $\mathscr{H}$ *, i.e., for any* $\varepsilon > 0$, *there is an* $n > 0$ *and constants* $c_1, \dots, c_n \in S$ *such that*
$$\left\| x - \sum_{i=1}^{n} c_i e_i \right\| < \varepsilon ; \tag{4.6}$$

(b) $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$, *i.e.*
$$\left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\| \underset{n\to\infty}{\longrightarrow} 0 ; \tag{4.7}$$

(c) $\|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$;

(d) $\langle x, y \rangle = \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle$, $\forall y \in \mathscr{H}$; (*Parseval's identity*)

(e) $x = 0 \Leftrightarrow \langle x, e_i \rangle = 0$, , $i = 1, 2, \dots$.

4.10 Corollary ORTHOGONAL REPRESENTATIONS IN SEPARABLE HILBERT SPACES. *Let* $\mathscr{H} = \overline{sp}\{e_1, e_2, \dots\}$ *be a separable Hilbert space where* $\{e_1, e_2, \dots\}$ *is a set of non-zero orthogonal vectors, and let* $\beta_i(x) = \langle x, e_i \rangle / \langle e_i, e_i \rangle$, $i = 1, \dots, n$. *Then the following properties hold for all* $x \in \mathscr{H}$:

(a) *the set of all finite linear combinations of* $\{e_1, e_2, \dots\}$ *is dense in* $\mathscr{H}$ *, i.e., for any* $\varepsilon > 0$, *there is an* $n > 0$ *and constants* $c_1, \dots, c_n \in S$ *such that*

$$\left\| x - \sum_{i=1}^{n} c_i e_i \right\| < \varepsilon \; ; \tag{4.8}$$

(b) $x = \sum_{i=1}^{\infty} \beta_i(x) e_i$, *i.e.*

$$\left\| x - \sum_{i=1}^{n} \beta_i e_i \right\| \underset{n\to\infty}{\longrightarrow} 0 \; ; \tag{4.9}$$

(c) $\|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 / \|e_i\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 / \langle e_i, e_i \rangle = \sum_{i=1}^{n} |\beta_i(x)|^2 \|e_i\|^2$;

(d)
$$\begin{aligned} \langle x, y \rangle &= \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle / \|e_i\|^2 = \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle / \langle e_i, e_i \rangle \\ &= \sum_{i=1}^{\infty} \beta_i(x) \overline{\beta_i(y)} \|e_i\|^2 \ , \ \forall y \in \mathscr{H} \; ; \end{aligned} \qquad \text{(Parseval's identity)}$$

(e) $x = 0 \Leftrightarrow \langle x, e_i \rangle = 0$, $i = 1, 2, \dots$.

4.11 Corollary PROJECTIONS IN SEPARABLE HILBERT SPACES. *Under the assumptions of Theorem* **4.9***, we have for each* $x \in \mathscr{H}$:

(a) *for any* $\varepsilon > 0$, *there is an* $n > 0$ *such that*

$$\|x - P_{E_n} x\| < \varepsilon \tag{4.10}$$

where $E_n = \overline{sp}\{e_1, \dots, e_n\}$;

(b) $P_{E_n} x \underset{n\to\infty}{\longrightarrow} x$.

5. Conditional expectations

5.1 Definition CONDITIONAL EXPECTATION WITH RESPECT TO A CLOSED SUBSPACE. *Let* $\mathscr{M}$ *be a closed subspace of* $L^2 \equiv L^2(\Omega, \mathscr{A}, P)$ *containing the constant functions and let* $Y \in L^2$. *Then the conditional expectation of* Y *given* $\mathscr{M}$ *is the projection of* Y *on* $\mathscr{M}$:

$$E_{\mathscr{M}} Y = P_{\mathscr{M}} Y \; . \tag{5.1}$$

5.2 Definition CONDITIONAL EXPECTATION WITH RESPECT TO A RANDOM VECTOR. *Let* $X = (X_1, \dots, X_k)'$ *be a vector of random variables in* $L^2 \equiv L^2(\Omega, \mathscr{A}, P)$ *and let* $L^2(X)$ *be the closed* *subspace of* $L^2(\Omega, \mathscr{A}, P)$ *consisting of all the random variables in* L^2 *of the form* $\phi(X)$ *for some function* ϕ*. Then the conditional expectation of* Y *given* X *is the projection of* Y *on* $L^2(X)$ *:*

$$E(Y|X) = E(Y|X_1, \dots, X_k) = P_{L^2(X)}Y \; . \tag{5.2}$$

6. Sources and additional references

A good summary of Hilbert space theory aimed at applications in time series analysis may be found in Brockwell and Davis (1991, Chapter 2). Other good reviews appear in: Debnath and Mikusiński (1990) for general applications, Small and McLeish (1994) for applications in statistical theory, and Young (1988) for a more mathematically oriented presentation.

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