The Geometry of Musical Structure: A Brief Introduction and History

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The purpose of this article is to provide a general introduction to the idea that musical structure can be represented geometrically. It begins by defining those elements that appear in the models in the three papers that follow. A brief review is given of psychological results demonstrating that these music-theoretic definitions describe implicit knowledge that listeners have of tonal-harmonic music. This is followed by a brief historical summary of maps that were proposed previously of keys, tones, and chords, and a hybrid tone-chord map. The article then summarizes a number of issues that arise in connection with geometric representations. Finally, the innovations in the three proposed models are highlighted, particularly their contributions to visualizing the dynamics and temporal relationships that exist in extended musical pieces.

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WHAT IS TONALITY (OR KEY)?

As background, let us start with some definitions from music theory, a highly developed field that provides a technical language for describing the structure of music. It is most developed in connection with traditional tonal-harmonic music of the 18th and 19th centuries, but has continued to evolve to describe compositional practice in more recent Western art music, jazz, and other popular styles, as well as music of non-Western cultures. Here we concentrate on compositions in tonal-harmonic music, and only the most essential aspects will be presented. In this context, the theory was developed with two basic objectives: to teach composition and to analyze the works of major composers.

According to Hyer [2001], tonality is "One of the main conceptual categories in Western musical thought, the term most often refers to the orientation of melodies and harmonies toward a referential (or tonic) pitch class." Thus, tonality is an important theoretical construct. Despite this, it is somewhat elusive and difficult to define. In Western tonal-harmonic music, tonality is equivalent to key, which is notated by a key signature. Let us establish some serviceable definitions that will be important for the tonality models presented in the following articles. To anticipate, the models in various ways and to varying degrees describe musical relationships for three kinds of elements: tones, chords, and keys.

The most basic element of music is the *tone* (or *pitch*). This is what you hear when you press down on a piano key, for example. It is a sound with a fixed *frequency*

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(expressed in cycles per second, called Hz – short for Hertz). If you are reading music, the tone is notated by a symbol, a *note.* It specifies the tone (as well as its duration, and likely also its loudness or dynamics). Other aspects that might be shown in the notation are how the tone is to be played by a particular instrument (e.g., whether it is to be plucked on a stringed instrument, called *pizzicato*), the phrasing (the degree to which tones should be played as connected or separated), and so on.

Each tone (and note) has a name, say, C, D, E, … , which are repeated in each octave. For example, an octave above the middle C on the piano is another tone that is also called C. A *pitch class* contains all the tones with the same name in different octaves. Two tones form an *interval*, and the distance between the two tones is measured in semitones (which is the smallest interval in the Western chromatic scale). Each interval has a name: 1 semitone = minor second; 2 semitones = major second; 3 semitones = minor third; 4 semitones = major third, and so on. The two tones can be played either at the same time (simultaneous interval) or one after the other (successive interval).

Chords play a central role in tonal-harmonic music. A *chord* is formed when three (or more) tones are sounded (together or in close temporal proximity). When there are three tones (or pitch classes), which happens frequently, the chord is called a *triad*. Different kinds of chords are formed depending on the intervals between the tones. A major chord has a major third between the lowest tone and the middle tone and a minor third between the middle tone and the highest tone (the tones C E G form a major chord). A minor chord has a minor third between the lowest tone and the middle tone and a major third between the middle tone and the highest tone (the tones A C E form a minor chord).

A piece of music can be described as being in a particular *tonality* or *key*, such as the key of C major or A minor. This has a variety of implications for how the music is structured. The key specifies the most stable tone, Hyer's referential tone or *tonic*. Often the piece will end on that tone and it will appear prominently throughout the piece. Implicit in all three models is the idea that by tracking the sounded tones, noting those that appear frequently and in special relationships to one another (such as those in certain chords), we can determine the local harmony (chords) and – on a somewhat longer timescale – the key.

The tonality or key also specifies the *scale*, the tones that will appear most frequently (although non-scale or chromatic tones can also appear). The notes of the C major scale are C D E F G A B, whereas the notes of the G major scale are G A B C D E F#. The key signature specifies how many sharps (#) or flats (*b*) will appear in the scale. These two scales overlap in all but one tone (F vs. F#), and consequently they are described as closely related; their tonics are separated by an interval of a perfect fifth (7 semitones).

From this develops the notion that keys are more or less related to one another. One important representation for key distance is the circle of fifths (shown here for major keys): C – G – D – A – E – B – F#/G*b* – D*b* – A*b* – E*b* – B*b* – F – C. Each tone is separated from its neighbors by the interval of a fifth. There is also a circle of fifths for minor keys. It is interesting to note that the tones of any (major or natural minor) scale are neighbors on the circle. For example, the tones of G major scale appear as the subset C – G – D – A – E – B – F#. All three models use the circle of fifths in one way or another, as described below.

The tonality or key specifies certain chords (harmonies) that will be important structural elements, specifically the chords built on the tonic (the I chord), the fourth tone of the scale (the IV chord), and the fifth tone of the scale (the V chord). In a major key, all these will be major chords (a very consonant sound). Ends of major phrases (and especially the very end of the piece) are marked by conventional sequences of chords, called cadences. These often conclude with the V I chords. Finally, within a piece changes of key may occur; these are called *modulations*. These can be temporary, and are indicated in the notation by extra sharps or flats, or they can be indicated explicitly by a change in key signature.

THE COGNITION OF TONALITY

Starting in the late 1970s cognitive psychologists started investigating the idea that music theory might be more than an aid to composition and music analysis. The idea being tested was that music theory might also describe concepts listeners have about structure in music. Of particular interest for psychology was implicit knowledge in listeners who have not undergone explicit training in music theory. It is learned from experience and cannot be easily verbalized, somewhat like knowledge of grammar in one's native language.

Experimental studies have demonstrated that listeners have implicit knowledge of scale structure, harmony, and key distance, as described by music theorists. The studies used techniques that bypassed the technical language of music theory. For example, listeners might judge whether a tone fits with a melody, or whether two chords are similar, or detect changes in sequences that, given the key, may or may not violate expectations. The degree to which the results conformed to music-theoretic descriptions is reflected in Hyer's [2001] statement: "A further complication (and recurrent tension) has to do with whether the term [tonality] refers to the objective properties of the music – its fixed, internal structure – or the cognitive experience of listeners."

A few findings are noted here (for reviews, see Krumhansl [1990; 2004]). As evidence for listeners' sensitivity to tonality, experiments show that they can detect tones that are not in the scale. In fact, listeners have trouble remembering them and confuse them with other, more expected tones. They judge the tonic to be the tone that fits best with a key-defining context such as a scale or chord cadence. Below the tonic in the hierarchy are the fifth and third scale degrees. There is also a hierarchy of chords, headed by the tonic (I) chord, then the V and IV chords. Listeners have more difficulty assimilating shifts between distantly related keys than between closely related keys. Many other related experimental results have found evidence for implicit knowledge of tonality – and not just in trained musicians but also those without musical training, children, and listeners from other musical cultures.

Stepping away from these specific findings, tonality is important for our experience of music. It provides a framework within which some elements are unstable and engenders expectations for how the music might resolve to more stable elements. This gives the music a sense of variation in tension over time. Patterns of tension-release-tension-release, and so on, serve to organize the music into segments, provide a sense of the music moving forward in time, and links the cognition of music to emotional responses.

These developments in cognitive psychology were coincident with an increased interest in more explicitly cognitive approaches in music theory (for example, Lerdahl and Jackendoff [1983]; further developed in Lerdahl [2001]) and in an expanded interest in using computer models to aid in composing and analyzing music. The articles in this issue represent three of the most recent and most innovative approaches to modeling tonality in geometric terms. The major innovation that unites them is a concern with modeling the real-time experience of tonality, the fact that the sense of key changes dynamically in time. Before turning to the issues and problems addressed by the three approaches, let us consider some of their historical precedents.

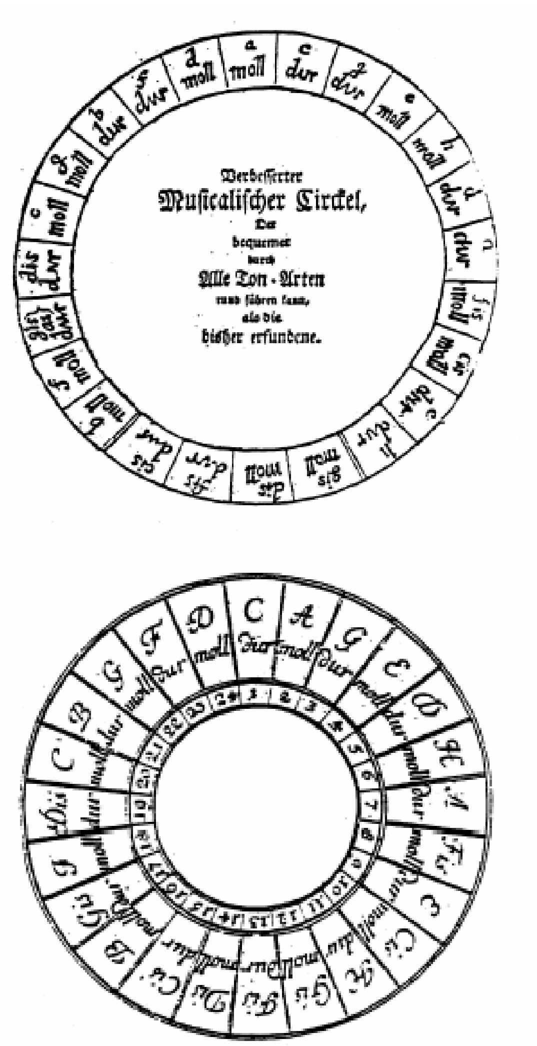

Fig. 1. The musical circles showing key relationships proposed by (top) Mattheson [1980], originally published in 1735; and (bottom) Heinichen [1969], originally published in 1728.

Maps of Keys

Geometric representations of tonality have a long history. This section considers some of the many representations of key relationships that have been proposed. Apart from historical interest, they demonstrate some of the strengths and weaknesses of different approaches, and raise issues that will be relevant to the three models presented later in this issue.

Figure 1 shows two early geometric representations of musical keys devised by composer/theorists to show the relationships between major and minor keys. Both take the form of circles. The circle on the left comes from Mattheson [1980], originally

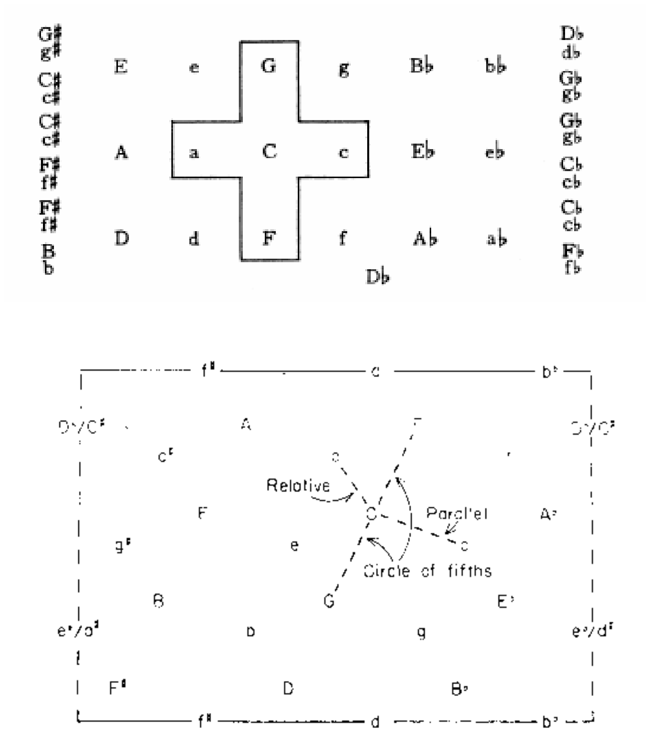

Fig. 2. Representations of key relationships from (top) Schoenberg [1969;1954] and (bottom) Krumhansl and Kessler [1982].

published 1735. The circle on the right comes from Heinichen [1969], originally published 1728, who wrote a piece of music that progresses systematically around the circle. (This and two other such pieces can be found in *Three Musical Circles for Keyboard* [Rasch 1983].)

Both circles are trying to grapple with the problem that every major key is closely associated with four different keys: the keys with tonics a perfect fifth above and below its tonic, and the relative and parallel minor key. Consider the reference key C major (C dur, at the top of each circle). As noted above, it should be close to G major (a fifth above C on the circle of fifths) because their scales differ only in one tone (F vs. F#). It is also close to F major (a fifth below C on the circle of fifths) whose scale also differs from C major only in one tone (B vs. B*b*). C major is also close to A minor (A moll), its relative minor, because the C major scale (C D E F G A B) has the same tones as the A minor natural scale (A B C D E F G), only they begin on different notes and, thus, have different tonics. Finally, C major is also close to C minor, its parallel minor, which has the same tonic (and also the same fifth scale tone and V chord, structurally important elements in these keys.)

The first circle attempts a solution by grouping together two neighboring major keys on the circle of fifths (for example, C major and G major) and flanking them on either side with their relative minor keys (a minor on the left of C major and E minor on the right of G major). This solution satisfies the relative major/minor relationship and keeps the ordering on the circle of fifths intact, but loses the parallel major/minor relationship. The second circle intersperses major and minor keys, which separately form one circle of fifths for major keys and one circle of fifths for minor keys. Again, however the parallel major/minor relationship is sacrificed. The general point that arises from these circles is that a low-dimensional space may not be sufficient to represent all of the desirable musical relations.

To address this, Schoenberg [1969], originally published 1954, proposed a two-dimensional grid centered around a particular key. This representation, shown at the left of Figure 2, has C major as the central key. It is flanked above and below by its neighbors on the circle of fifths, on the left with its relative minor and on the right with its parallel minor. This satisfies the local relationships. More distant keys are shown in the fringes, and their relationships to one another are not made explicit.

Krumhansl and Kessler [1982] showed that the entire configuration of 24 keys can be mapped with a geometrically regular representation on a torus in four-dimensions. This is shown at the right of Figure 2, where it is understood that the top and bottom edges connect with each other and the left and right edges also connect with each other. Unlike the other representations shown so far, this is based on psychological data, namely, the hierarchies of tones in the 24 major and minor keys (the method is described in more detail in the article by Toiviainen, this issue). The tonal hierarchies of each pair of keys were correlated, and these correlations were analyzed using multidimensional scaling, which constructs a geometric map from similarity data (in this case the correlations between tonal hierarchies [Krumhansl and Kessler 1982]).

All three representations described in this issue rely on the circle of fifths. It appears as one of the two organizing principles of the torus representation (in addition to the major/minor key relationships; Toiviainen, this issue). The spiral array (see Chew and François, this issue) orders tones according to the circle of fifths, with the result that there is also a circle of fifths ordering for keys. And, Sapp (this issue) capitalizes on the circle of fifths by mapping keys that are close on the circle onto similar colors.

The torus representation was used [Krumhansl and Kessler, 1982] as a way of showing how listeners' sense of key developed and changed over time as chord sequences were sounded. The method, collecting perceptual judgments on a chord-by-chord basis, was inordinately time-consuming and tedious for participants in the experiment. Also importantly, the resulting representation (points indicating the strongest key on the torus) had no way of showing the relative strengths of keys. In other words, at some points in time listeners may have had a strong sense of key, at other points only a weak sense; but this wasn't represented on the key map. This is another consideration that is important for representing tonality: to have some way of showing the strength of a key, not just the locus. This was one of the main motivations for the representation devised by Toiviainen (this issue); [Krumhansl and Toiviainen 2001; Toiviainen and Krumhansl 2003], which uses color to show key strengths, and one of the features contained in the visual representation by Chew and François (this issue, where key strength is shown by the size of the marker for key).

Maps of Tones

The representations so far have all dealt with musical keys, an abstract concept with implications for both scale and harmony. Other representations have been devised to show the relationships between tones. Some of these grew from an interest in tuning, that is, the precise adjustment of tone frequencies. It has long been known that some intervals, those with frequencies that can be expressed by simple ratios of numbers, have a pleasant, smooth percept, called consonance. For example, the perfect fifth is made up of tones with a 3:2 ratio of frequencies. Another musical interval that is important for forming chords is the major third, ideally with a frequency ratio of 5:4. Unfortunately, it is not possible to make all intervals simultaneously fit exactly these simple ratios. This has led to a variety of schemes to minimize the discrepancies from ideal ratios.

One such model is shown in Figure 3. Ellis added the diagram to his translation of Helmholtz's [1954, p. 463] treatise *On the Sensations of Tone as a Physiological Basis for the Theory of Music* (originally published 1885). It shows the tones in a grid in which the vertical axis corresponds to the interval of a perfect fifth, and the horizontal axis corresponds to the interval of a major third. The reference pitch is C and the numbers beside each other tone show how distant they are from the C in cents (100 cents = 1 semitone).

Ideally, the G should be 700 cents (7 semitones) away from C, and the value 702 comes close. In this scheme, the E should be 400 cents, but the value of 386 cents is quite far off. As can be seen, the discrepancies tend to increase with distance from the reference tone. Most fixed-pitch instruments, such as piano, now compromise with what is called equal-tempered tuning in which tones are equally separated on a log-frequency scale. This has proved serviceable because it allows the instrument to play music written in any key. To celebrate this advance, Bach composed the Well-tempered Clavier preludes and fugues; favorite examples for testing key-finding algorithms. This diagram,however, is important for emphasizing the relationship of the major third in addition to the fifth. This thirds relationship is used by Sapp (this issue) to determine the

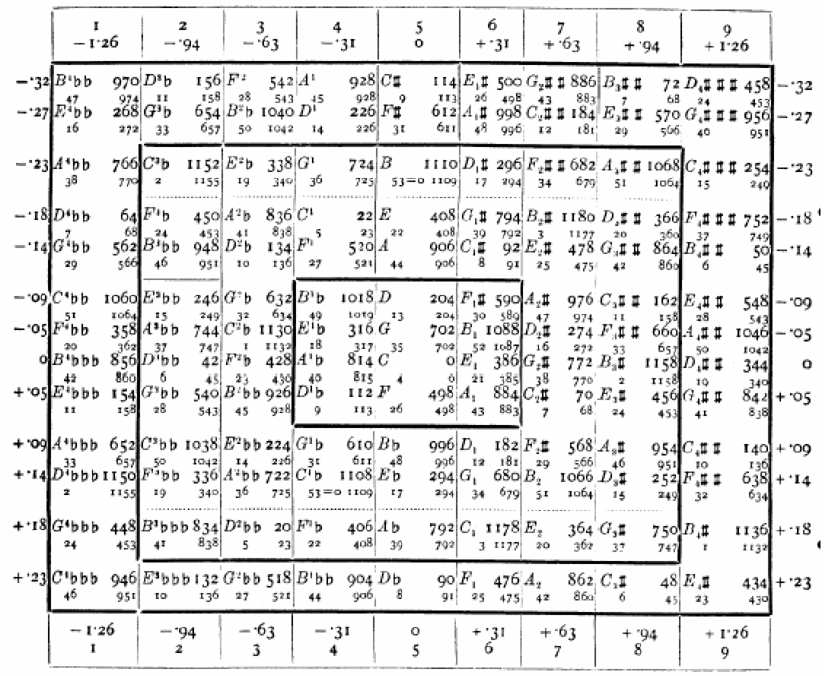

Fig. 3. The map of musical tones added by Ellis to his translation of Helmholtz [1954], originally published in 1885.

most likely triad for a set of tones. It is explicit in Chew and François' (this issue) in the visual representation of chords, which appear as triangles in the spiral array. And the keys with tonics separated by thirds appear on horizontal lines in the torus of keys (Toiviainen, this issue).

These two intervals, the perfect fifth and major third, also appear in a grid proposed by Longuet-Higgins and Steedman [1971]. The grid was used in this early attempt to automate key-finding. Here the fifths are on the horizontal dimension and the thirds are on the vertical dimension, although otherwise it is essentially equivalent to the representation proposed by Ellis. They note that the scale tones of C major fall in a compact region (shown on the left) and the scale tones of C minor fall in a compact region (shown on the right). They used this map of tones as the basis for finding the keys in Bach's Well-tempered Clavier (see Krumhansl [1990], for more details of their approach).

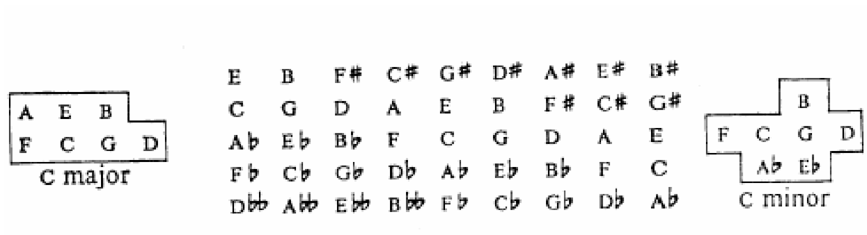

Fig. 4. The map of musical tones used in Longuet-Higgins and Steedman's [1971] key-finding algorithm.

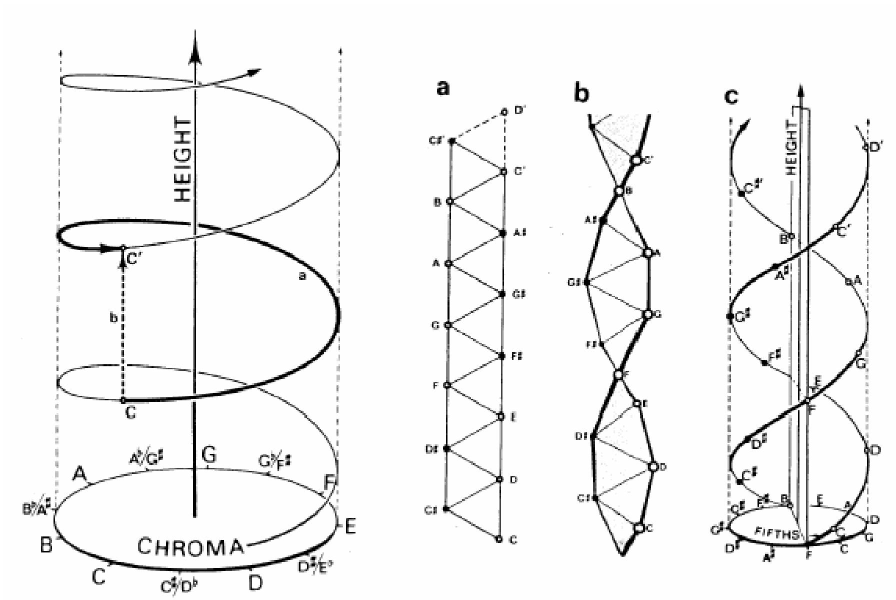

Fig. 5. Two helical representations of musical pitch proposed by Shepard [1982], the single helix (left) based on pitch proximity on the chroma circle and octave equivalence, and the double helix (right) based on two whole tone scales where the projection onto the horizontal is the circle of fifths.

Shepard [1982] also made contributions that stimulated efforts to devise geometric representations of music. Two different proposals are shown in Figure 5. The figure on the left shows the representation that takes the form of a single helix. Neighboring tones on the helix form the chromatic scale. The helix wraps around so that tones separated by octaves (with frequency ratios of 2:1) are located above one another. By splitting the chromatic scale into two whole-tone scales and twisting them into a double helix, one arrives at the representation on the far right. The feature of greatest interest is that when projected back down onto the circle the tones fall along the circle of fifths. As noted above, the circle of fifths plays a role in all three models described in this issue.

One final representation of music tones is presented that comes from a psychological experiment [Krumhansl 1979]. In the experiment, each trial began with a C major context that was followed by two tones. Listeners were asked to rate how similar the first tone was to the second tone in the context of the key. (Variations of these instructions, as well as minor key contexts, were used in later experiments [Krumhansl 1990].) All possible pairs of tones were sounded and the resulting multidimensional scaling solution is shown in Figure 6.

Note that tones high in the hierarchy of the key, C, E, and G (and C', which denotes the C an octave above C) are drawn close to one another in the C major context. The context enhances the degree to which they are heard as similar. In other key contexts, the results would be different. For example, in D major, D F#, and A would be close together. This raises the issue that, at least on the level of tones, geometric models may have to shift tone positions in order to express how they are perceived in relationship to one another in that context. To date, this idea has not been pursued extensively.

Another important finding from this, and other related studies, is that perceived similarities between tones are not symmetric. For example, in C major the tone B is judged as much more similar to the tonic C, than the tonic C is to the tone B. These findings are musically interpretable; certain tones are unstable and part of listeners' knowledge of tonality includes how unstable tones typically resolve to more stable

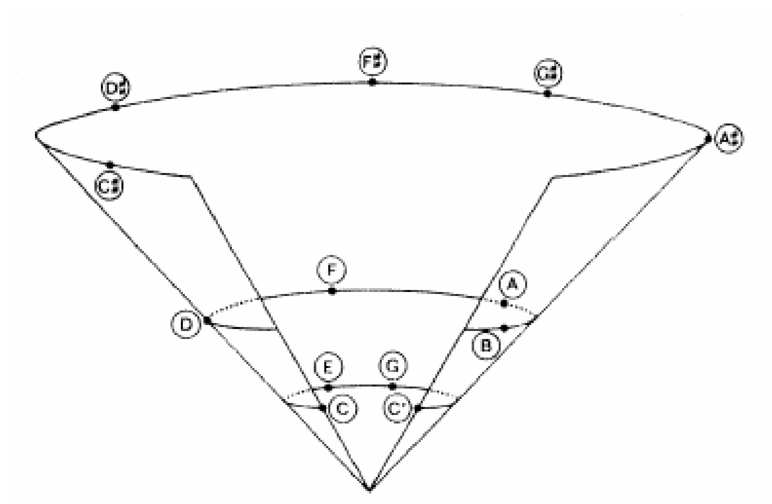

Fig. 6. A multidimensional scaling solution of tones presented in a C major context [Krumhansl 1979].

tones. (Incidentally, asymmetries have also been found in key distances; a modulation to the key a fifth above, say C major to G major, is perceived as less distant than a modulation to the key a fifth below, say from G major to C major [Thompson and Cuddy 1989; Cuddy and Thompson 1992].) To my knowledge, no solution to the problem of representing such asymmetries has yet been proposed.

MAPS OF CHORDS

Next we turn to maps of the relationships between musical chords. Music theory would predict that there is a stable core containing the most structurally significant chords: I, V, and IV, that is, the triads built on the first, fifth, and fourth tones of the scale. One study, with results shown in Figure 7, validates this prediction [Krumhansl et al. 1982]. This is a study in which pairs of chords were presented for similarity judgments. As can be seen, these chords were located in a compact cluster in the center of the multidimensional scaling solution (on the left) and joined together first in the clustering, or tree, solution (on the right).

However, just as perceived tone distances depend on context, so do chord distances. Figure 8 shows the multidimensional scaling results for the chords of two distantly related keys, C major and F# major [Krumhansl et al. 1982]. On the left are the results when the chords are presented in the context of C major; on the right are the results when chords are presented in the context of F# major; and in the center are the results when the

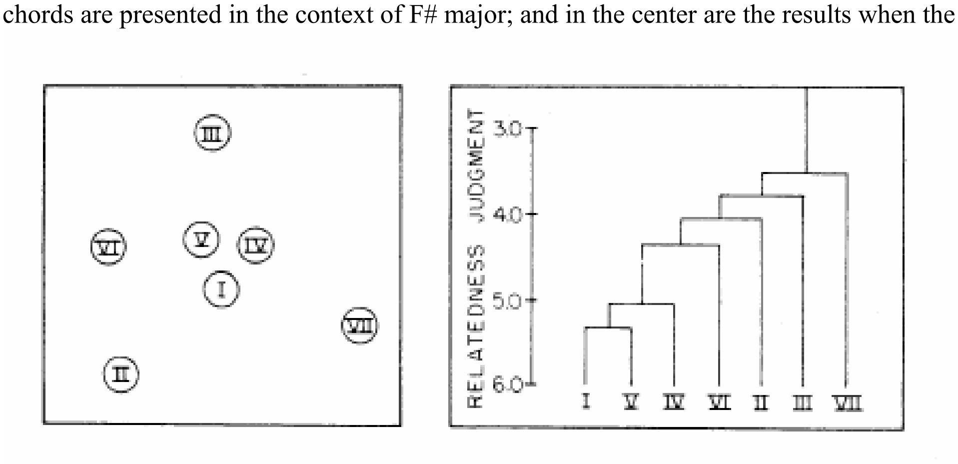

Fig. 7. Two ways of representing the perceived similarity of chords [Krumhansl et al. 1982]: multidimensional scaling (left) and hierarchical clustering (right).

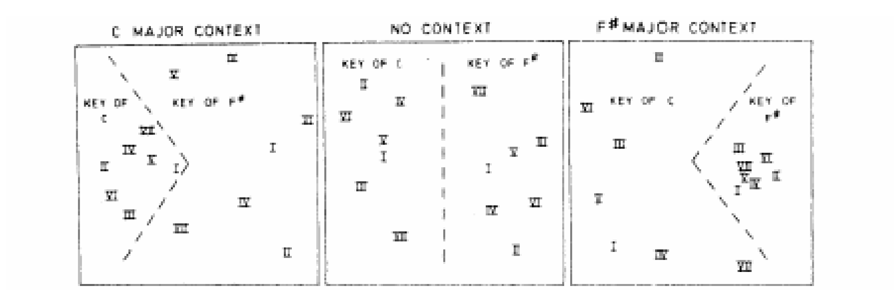

Fig. 8. Shows that the key context in which chord pairs are presented strengthens the relationships between some chords and weakens others (from Krumhansl et al.[1982]; Krumhansl [1990]).

chords are presented in isolation. The context clearly had an effect on the perceived relatedness of the chords, drawing together those that belong to the key. In addition, these studies show asymmetries such that less stable chords are judged as more related to more stable chords than the reverse. As with tones and keys, these finding pose problems for geometric representations.

A HYBRID MAP OF TONES AND CHORDS

Finally, we take a look at a hybrid model that includes both tones and chords. It comes from a strand of music theory that has developed from the theorist Hugo Riemann (see for example the special issue of *Journal of Music Theory*, 1998, on *Neo-Riemannian Theory*). Figure 9 shows the central representation, which is called the *Tonnetz*. It consists of a grid of tones connected by lines to form triangles. The tones of each triangle form a chord. One triangle contains the tones C E G, the tonic chord of C major. Shifting to the left and up one step gives the chord A C E, the tonic triad of the relative minor, A minor. Shifting down gives the chord C E*b* G, the tonic triad of the parallel minor, C minor. Shifting to the upper right gives the tones E G B, which is the triad of the relative minor of G major (C major's neighbor on the circle of fifths, with tonic triad reached by shifting directly right from the C major triad). The shift to E G B is labeled L for *Leittonwechsel,* meaning exchanging the C with the leading tone B. In fact, one can see this as a close key to C major in the torus representation of musical keys (Figure 2).

This representation makes the important suggestion that tone relations and key relations are interdependent. On the level of tones, those found to be closely related (by intervals of fifths and thirds) form triads that are tonic triads in closely related keys. This suggests that it may be possible to combine tones, chords, and perhaps keys in a single representation, as has been done by Chew [2000] and shown visually by Chew and François (this issue). The model proposed by Sapp (this issue) draws on the *Tonnetz* to determine probable chords from the sounded tones. Other related evidence for hybrid representations comes from the findings that chord hierarchies can be derived from tone hierarchies, and that both tone and chord hierarchies generate the same model of key distances [Krumhansl 1990]. At the same time, some caution should be observed. Various results differ at the level of tones, chords, and keys. For example, for tones the most central are the first, third, and fifth scale degrees, whereas for chords the most central are the triads built on the first, fourth, and fifth scale degrees. Also, the distinction

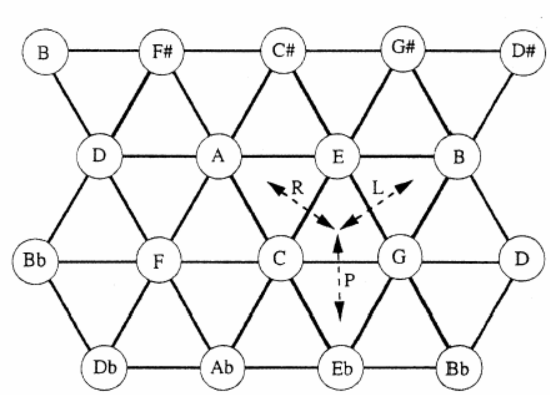

Fig. 9. The hybrid map of tones and chords (shown as points of triangles) in Riemann's theory from Gollin [1998].

between major and minor only applies to the levels of chords and keys; no such distinction applies to tones.

ISSUES ARISING FROM PRIOR MAPS AND INNOVATIONS IN THREE MODELS

Various issues arise from the previously proposed geometric representations. The first is the issue of dimensionality, that is, how many dimensions are needed to describe the musical relationships of interest? This is a recurrent tension because it is only possible to visualize easily models in a small number of dimensions. One of the new developments in the proposed models is introducing color to add dimensionality. In the case of Sapp (this issue), the keys are represented by color such that closely related keys are shown in similar colors, and Chew and François (this issue) use color to distinguish between the various elements, tones, chords, and keys.

Another issue that emerges is how to represent how strongly a key is established. This could be a property of the music, that is, some musical styles or passages within a piece in a particular style might establish a key far more clearly than other musical styles or passages. This could also be an aspect of the psychological response, such that listeners are able to perceive a tonality much more clearly given some styles or passages. The present contributions focus primarily on the former, because none of them explicitly involves perceptual data. In the representation of Toiviainen (this issue), strength of key is indicated by color. In the representation of Chew and François (this issue), strength of key is represented by the size of the pulsating orb.

A third issue is the influence of context on the relationships between musical elements. One manifestation of this might be that the basic geometric representation is deformed systematically depending on the tonal context. None of the present models engages this possible complexity, although it is supported by various psychological results showing that the perceived similarity between tones and chords depends on the context in which they are embedded.

A related issue also comes from psychological findings, which is that there are asymmetries in the distances between elements. That is, the distance from one element to a second is not the same as the distance from the second to the first. This has been found on all levels of tones, chords, and keys. These results are intrinsically incompatible with geometrical models unless the representation dynamically adjusts in response to each sounded element.

A final issue concerns which elements are represented geometrically. Many previous models propose separate representations for tones, chords, and keys. In part motivated by psychological data as well as theoretical considerations (see for example, Lerdahl, 2001), it has been advantageous to distinguish between these different elements because somewhat different relationships hold for each. The present proposals take different stances on the issue. Toiviainen (this issue) uses a representation that includes just musical keys. Sapp (this issue) identifies chords at the lowest time scale and key or tonality at the higher time scales. Chew and François (this issue) proposes a visual representation that simultaneous contains all tones, chords, and keys.

A common theme of all three models proposed here is the importance of representing how the tonality or key changes in time. In addressing this problem, they make major advances over previous proposals. In traditional music theory and analysis, a section of a piece (or an entire piece) is analyzed with respect to a reference key. Chords are analyzed with respect to their function in the key (or keys). This provides a relatively static notion of tonality. In both Toiviainen and Chew and François (this issue) models, the representation is updated in real time, so that it is possible to track easily the changes of key as they occur. The latter model provides tracking of both short-term and long-term pitch contexts in order to determine keys and chords. The Sapp (this issue) model approaches the problem by presenting the results on different time scales, with lower levels representing more local chord information and higher levels representing longer-term key centers. The self-similarity maps of Toiviainen (this issue) show provide a representation of how rapidly key changes occur and how different sections of a piece relate to one another.

As we will see by examining each of the three models, music is highly dynamic, constantly evolving over time. The strength of key varies, with times of great uncertainty and ambiguity alternating with times when the tonality is strong and unambiguous. These innovative models bring out vividly this characteristic of music. The models also represent both short-term and long-term relationships that exist within the music. They provide a visual representation that complements and does not require knowing the technical language of music theory. Rather, they show tonal, harmonic, and key relations in geometric representations that are visually accessible. Thus, they serve as a way to intuitively access musical structures that are implicitly understood even by untrained music listeners.

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