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Author(s): John Clough
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USE OF THE EXCLUSION RELATION TO PROFILE PITCH-CLASS SETS

John Clough

In the development of pitch-class set (pc set) theory, there is no concept more central than that of *inclusion*. It is the basis of Forte's set-complex and plays an important role in recent investigations by Lewin, Morris, and Rahn.[1] Taking a different (or complementary) approach to the problem of characterizing and comparing pc sets, I shall explore here the *exclusion* relation, which in many cases yields sharp profiling of pc sets and provides an alternative perspective for judging the similarity of pc sets of the same or different cardinality.[2]

My discussion uses the following notational conventions:

notation	*meaning*
$\{0,1,2\}$	The literal pc set including pc 0, pc 1 and pc 2, in the universe of 12 pcs.
[0, 1, 2]	The class of pc sets equivalent to $\{0,1,2\}$ under T_nI, called a set class (SC).
#A	The number of elements (cardinality) of A where A is the name of a pc set, or the number of elements of each set in the SC where A is the name of a SC.

$\{0, 1, 2\}_8$ $[0, 1, 2]_8$	Same as for $\{0, 1, 2\}$ and $[0, 1, 2]$ above, but in the universe of *eight* pcs.
$P \subset Q$	Pc set P is a subset of pc set Q.
$P \sqsubset Q$	For every pc set P_i in SC P, there is a pc set Q_i in SC Q such that $P_i \subset Q_i$. (Note that capital letters may serve as names for pc sets or SCs, depending on the context.)

The exclusion set X(A). If $P \sqsubset Q$, I will say that P is a *subclass* of Q, or, speaking loosely, that P is included in Q. (More precisely, P is *abstractly* included in Q.) Thus, for example, the subclasses of A = [0, 1, 3, 4] are all of those SCs represented among the subsets of $\{0, 1, 3, 4\}$. A SC is profiled by ennumerating its subclasses, more particularly so when the multiplicity of each subclass is also specified, as formalized by Lewin.[3]

All SCs which are not subclasses of A are said to be (abstractly) excluded by A. Taken collectively, these excluded SCs form a profile of A by negation, a kind of *ground.* In order to characterize usefully this collection of SCs, we can begin by ignoring all SCs of cardinality greater than or equal to #A, for all such SCs (except A itself) are excluded by A, and therefore their exclusion offers no insight to the particularity of A. Listed in the first two columns of Table 1 are the included SCs (or subclasses) of A (ignoring cardinalities zero and one) and the excluded SCs of A (ignoring cardinalities greater than 3) for A = [0, 1, 3, 4].

The collection of excluded SCs, as exhibited in the middle column of Table 1, does not yield a particularly sharp profile of A. Suppose now that we cull from the list of excluded SCs the following reduced list of SCs called the "excluded*" list. If a symbol of the form "[S] *" appears on the excluded* list, it represents S *plus all of the SCs of lower cardinality than A which include S.* So if we write "[0, 5] *" on the excluded* list, that symbol represents [0, 5] plus all SCs of cardinal 3 which include [0, 5]. Note that all such SCs will be on the original excluded list, since, for any three SCs A, B, and C, with $\#A > \#B > \#C$, if $C \not\sqsubset A$ and $C \sqsubset B$, then $B \not\sqsubset A$. By placing [0, 5] * and [0, 6] * on the excluded* list of [0, 1, 3, 4], we represent, in addition to SCs [0, 5] and [0, 6], most of the trichord SCs on the original exluded list, as shown by comparison of the second and third columns of Table 1. We can think of [0, 5] * and [0, 6] * as representing *foreign strains* of SCs with respect to A. The relatively small excluded* list is a compact description of "not in A," a good "negative" of A.

Once it is generated, the excluded* list provides an efficient test for inclusion. If A and B are SCs with $\#A > \#B$, and we wish to test whether

Table 1. Included and excluded SCs of [0, 1, 3, 4].

included	excluded	excluded*
[0, 1]	[0, 5]	[0, 5]*
[0, 2]	[0, 6]	[0, 6]*
[0, 3]		
[0, 4]		
[0, 1, 3]	[0, 1, 2]	[0, 1, 2]
[0, 1, 4]	[0, 1, 5]	
	[0, 1, 6]	
	[0, 2, 4]	[0, 2, 4]
	[0, 2, 5]	
	[0, 2, 6]	
	[0, 2, 7]	
	[0, 3, 6]	
	[0, 3, 7]	
	[0, 4, 8]	[0, 4, 8]

$B \sqsubset A$ or $B \not\sqsubset A$, we need only check whether or not B includes one or more members of the excluded* list for A. If so, $B \not\sqsubset A$; if not, $B \sqsubset A$. For example, if A = [0, 1, 3, 4] and B = [0, 1, 5], we observe that [0, 5] $\sqsubset$ B and conclude that $B \not\sqsubset A$ because A excludes *all* SCs which include [0, 5]. The *complete* structure of B is irrelevant to the question of its inclusion in A; the inclusion of [0, 5] in B is sufficient to decide the issue.

It may appear that the excluded* list does not yield greater efficiency in simple decisions like the above. Surely we can see immediately that [0, 1, 5] $\not\sqsubset$ [0, 1, 3, 4]. However, it is plain that we see this precisely because we see that [0, 1, 3, 4] cannot include any SC that includes [0, 5]. For a second example, suppose A = [0, 1, 3, 4, 6, 7, 9, 10]—the octatonic scale SC, and B = [0, 2, 4, 5, 7]—the "major pentachord". Aware that [0, 2, 4] (and all SCs including it) are foreign to the octatonic scale, we need go no further than the observation that [0, 2, 4] $\sqsubset$ B, to decide that $B \not\sqsubset A$. Here, in essence, we invoke the fact that [0, 2, 4]* is a member of the excluded* list for the octatonic scale SC.

Definition and construction of X(A).

Definition. Given a set class A, the exclusion set of A, or X-set of A, symbolized X(A), is the set of SCs $\{X_1, X_2, \ldots, X_n\}$ having the following properties:

(1) For all X_i in X(A), $\#X_i < \#A$.
(2) For any two distinct SCs in X(A), say X_i and X_j, $X_i \not\sqsubset X_j$.
(3) For any set class B with $\#B < \#A$, if there is some X_i in X(A) such that $X_i \sqsubseteq B$, then $B \not\sqsubset A$; otherwise $B \sqsubset A$.

The above three properties are sufficient to define X(A). Two additional properties follow logically:

(4) For all X_i in X(A), $X_i \not\sqsubset A$.
(5) For any set class B with $\#B < \#A$, if $B \not\sqsubset A$, then there is at least one member of X(A), say X_i, such that $X_i \sqsubseteq B$; on the contrary, if $B \sqsubset A$, then $X_i \not\sqsubset B$ for all X_i in X(A).

For any particular set class A, is there more than one collection which qualifies as X(A)? Or, perhaps, none at all? Either circumstance would seriously weaken the potential usefulness of X(A). Here is the answer.

For any SC A with $\#A \geqslant 3$, there exists a unique X(A).

The above is proved in the appendix. In the remainder of this section, I take a less formal, more graphic approach to the matter of constructing X(A) and understanding its properties.

Listed in Table 2 are the X-sets for a number of SCs in the 12-pc universe. In order to avoid unnecessary clutter, X-sets are displayed without enclosing curly brackets and without commas between SCs.

The notion of the X-set may be nicely visualized if the universe of SCs is pictured as a lattice where each point represents a particular SC and each line segment between two points represents an inclusion relation between two SCs of adjacent cardinality, as suggested by Regener.[4] This visualization also provides a convenient setting in which to describe the construction of the X-set for a particular SC. It is impractical to display here the very large lattice for the 12-pc universe. Instead, for purposes of illustration, we shall work with a lattice for the universe of eight pcs, as displayed in Figure 1. We can think of this universe as one which contains eight equal-tempered "steps" per octave; there are also other universes of eight pcs which are modelled by the lattice of Figure 1. All SC designations in Figures 1-3 are given in condensed form, without enclosing brackets, with no separation between pc numbers, and with the context "8-pc universe" understood. The lack of mirror symmetry around the SCs of cardinal four is due to the fact that $[0, 1, 3, 4]_8$ and $[0, 1, 2, 5]_8$ are a Z-related pair.[5]

We now proceed to construct X(A) for A = $[0, 1, 3, 4, 6]_8$. First, ignoring all SCs of cardinal five or greater, we divide the remaining SCs into two categories: those included by A (the subclasses of A) and those excluded by A, as indicated by circles and squares, respectively, in Figure 2. This "intermediate" set of excluded SCs is related to $[0, 1, 3, 4, 6]_8$, as the middle column of Table 1 is related to [0, 1, 3, 4] in the case examined earlier. It satisfies properties (1) and (3), but not property (2), of the definition of an X-set. From the excluded set of SCs, we now delete all SCs which include SCs of lower cardinality than themselves in the excluded set. The remaining SCs are those enclosed in thick squares in Figure 3. The reader may verify that this set of SCs has all three properties specified in the definition. It therefore constitutes X(A), which we write as follows:

For A = $[0, 1, 3, 4, 6]_8$,
X(A) = $[0, 1, 2]_8$, $[0, 1, 4, 5]_8$, $[0, 2, 4, 6]_8$

Note that A is a "deep scale" after Gamer in the 8-pc universe (generateable by interval class 3 or 5).[6] If we compare X(A) to the X-set of the major-scale SC in the 12-pc universe (also a deep scale whose X-set is given in Table 2), we find an interesting similarity. Both X-sets contain the chromatic triple—$[0, 1, 2]_8$ in 8 pcs and [0, 1, 2] in 12 pcs—and both contain all of the SCs (in their respective universes) that have two (disjunct) tritones. (The interval 4 is the "tritone" in 8 pcs.) Very loosely, X(A) is a "subset" of the X-set of the major-scale SC.

Using a notation adapted from Regener, we can describe the manner

Table 2. Illustrative X-sets in the universe of 12 pcs.

A	X(A)
[0, 1, 4]	[0, 2] [0, 5] [0, 6]
[0, 2, 5, 8]	[0, 1] [0, 2, 4] [0, 2, 7] [0, 4, 8]
[0, 2, 4, 7, 9] (pentatonic scale)	[0, 1] [0, 6] [0, 4, 8]
[0, 1, 2, 3, 4, 5]	[0, 6] [0, 2, 7] [0, 3, 7] [0, 4, 8]
[0, 1, 3, 4, 5, 8]	[0, 6] [0, 2, 4] [0, 1, 2, 3] [0, 2, 3, 5] [0, 2, 5, 7]
[0, 1, 4, 5, 8, 9]	[0, 2] [0, 6]
[0, 2, 4, 6, 8, 10] (whole-tone scale)	[0, 1] [0, 3] [0, 5]
[0, 1, 3, 5, 6, 8, 10] (major scale)	[0, 1, 2] [0, 1, 4] [0, 4, 8] [0, 1, 6, 7] [0, 2, 6, 8] [0, 3, 6, 9]
[0, 1, 3, 4, 6, 8, 9] (harm. minor scale)	[0, 1, 2] [0, 1, 6, 7] [0, 2, 6, 8] [0, 1, 4, 7] [0, 2, 4, 6]
[0, 1, 3, 4, 6, 7, 9, 10] (octatonic scale)	[0, 1, 2] [0, 2, 4] [0, 4, 8] [0, 2, 7] [0, 1, 5]

in which X(A) represents the complete collection of SCs of lower cardinality than A and excluded by A. Let the symbol SL(M,N) denote the sublattice from M to N, that is all of the points on the lattice between M and N, including M and N. Let ϕ and $\cup$ denote the points representing the null set and the universal set, respectively. Finally, let the symbol SL(M,N,c) denote the set of points on SL(M,N) of cardinality less than c. Now for a SC A, all of the subclasses of A are represented by SL(ϕ,A). The remaining SCs of lower cardinality than A, those excluded by A, are the union of SL(X_1, $\cup$, #A), SL(X_2, $\cup$, #A), ... , SL(X_n, $\cup$, #A), where $X(A) = \{X_1, X_2, \ldots, X_n\}$ as defined above. X(A) is not necessarily the only set of SCs which represent in this manner all of the SCs excluded by A; however it is the *minimal* set which does so.

A less formal approach to the construction of X-sets involves the following puzzle. Suppose that for a particular SC we have reached the situation depicted in Figure 2. We know, for all SCs of cardinality less than #A, whether they are included or excluded by A. Now let us imagine that the inclusion lattice represents an unusual communications network in which a message may be introduced at any point whereupon it will travel automatically *from left to right* over all possible paths to the rightmost node of the lattice. ("From left to right" here means to a node of the next vertical rank to the right.) We wish now to introduce a message into the network of Figure 2 in such a way that it reaches all of the "squared" nodes but none of the "circled" nodes. How can this be accomplished *most efficiently*? That is, what is the minimal set of nodes to which we should initially send our message? A little experimentation will show that it is just exactly the set defined as X(A).

For SCs of cardinal less than 5 in the 12-pc universe, X-sets are easily constructed by hand. This is also true for SCs of cardinal greater than 7, because of the relationship between complementation and X-sets (to be treated later in this article). Unfortunately, the construction of X-sets for SCs of cardinal 5-7 is tedious in most cases, and the determination of these X-sets would ideally be accomplished by referring to a table. (By the time this article appears, I hope to be able to distribute a computer-generated complete roster of X-sets in the 12-pc universe.)

X-set in the universe of 6 and 7 pcs. Figures 4A and 4B display the inclusion lattices for the universes of six and seven pcs, respectively (with SC designations in condensed form as in Figures 1-3). These small universes provide simple illustrations of the differentiations afforded by analysis of X-sets. In six pc (modeled, of course, by the whole-tone scale):

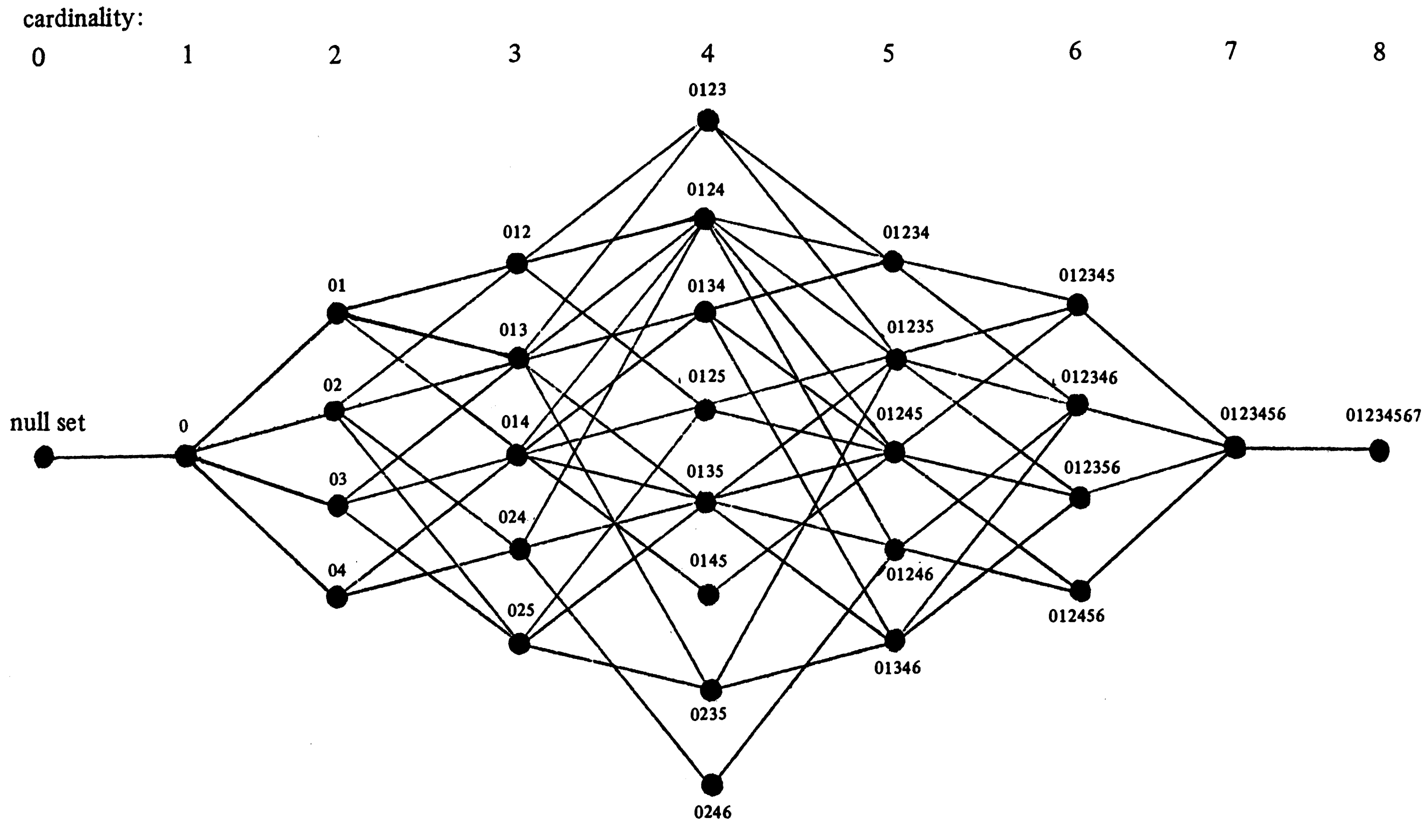

Figure 1. Lattice of SC inclusion in the universe of 8 pcs.

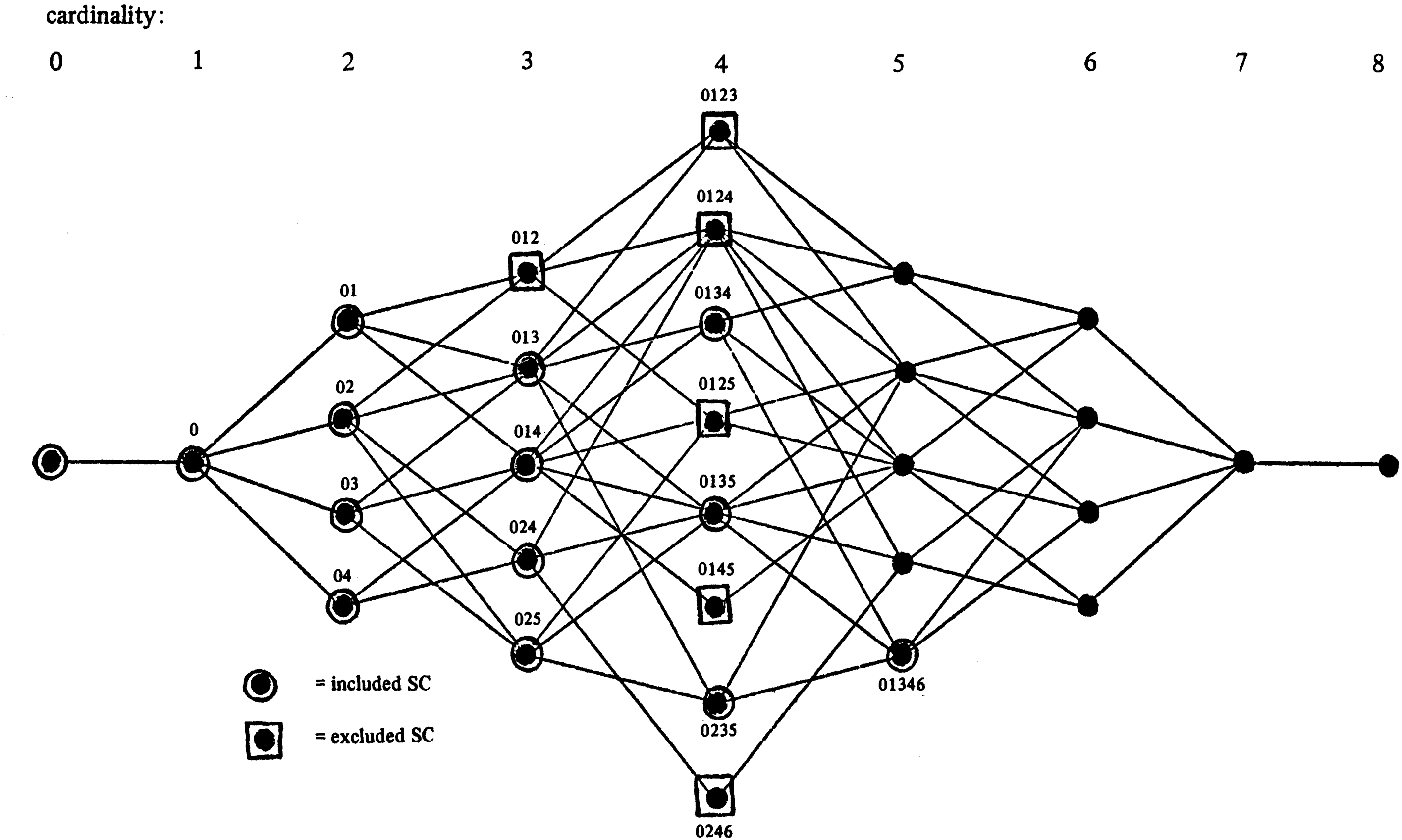

Figure 2. Included and excluded SCs of $[0, 1, 3, 4, 6]_8$.

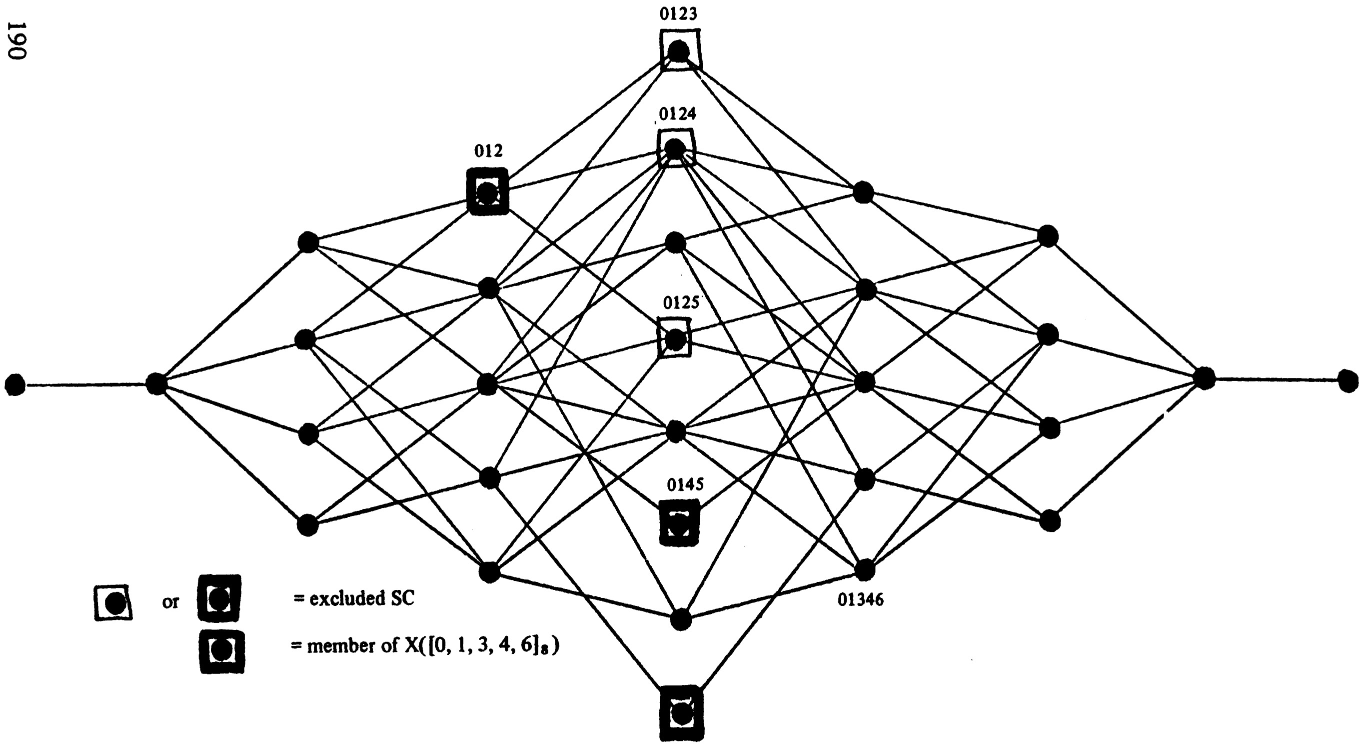

Figure 3. X-set of $[0, 1, 3, 4, 6]_8$.

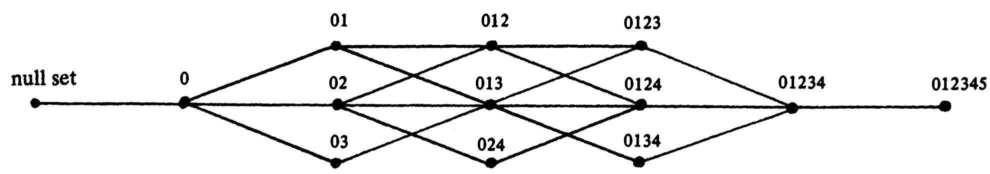

Figure 4A. Lattice of SC inclusion
in the universe of 6 pcs.

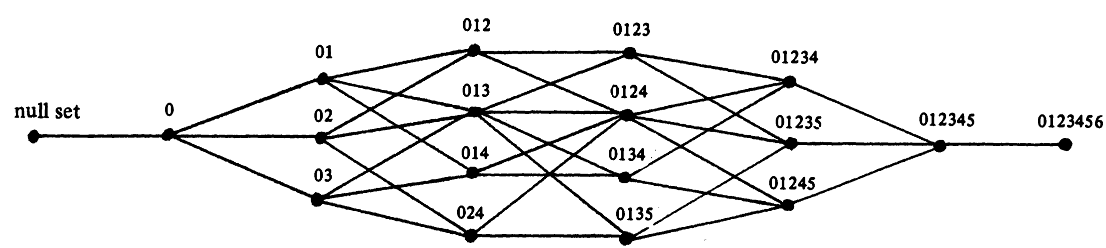

Figure 4B. Lattice of SC inclusion
in the universe of 7 pcs.

$X([0,1,2]_6) = \{[0,3]_6\}$
$X([0,1,3]_6) = \text{null set}$
$X([0,2,4]_6) = \{[0,1]_6, [0,3]_6\}$

And in seven pcs (modelled by the diatonic scale under a particular equivalence relati6n):[7]

$X([0,1,2,3]_7) = \{[0,1,4]_7, [0,2,4]_7\}$
$X([0,1,2,4]_7) = \text{null set}$
$X([0,1,3,4]_7) = \{[0,1,2]_7, [0,2,4]_7\}$
$X([0,1,3,5]_7) = \{[0,1,2]_7, [0,1,4]_7\}$

It is interesting to note that the four tetrachord SCs in the seven-pc universe all include the all-interval trichord $[0,1,3]_7$, which therefore does not appear anywhere in the X-sets of the tetrachords. Of the remaining three trichords, each appears exactly twice, with a different associate, among the X-sets of the tetrachords, exhausting all the combinations of these trichords into pairs. These facts are in keeping with the idea I have argued elsewhere, that distinctions among tetrachords (and trichords), under the diatonic equivalence relation, are quite palpable.[8]

Some additional properties of X-sets.

1. M5 operation. For any SC A and its $X(A) = \{X_1, X_2, \ldots, X_n\}$, $X(M5(A)) = \{M5(X_1), M5(X_2), \ldots, M5(X_n)\}$. This follows immediately from the fact that inclusion and exclusion relations are preserved under M5.

2. Complementation. For any SC A and its X(A), let X(A)com denote the set of complements of all SCs in X(A). So $X(A) = \{X_1, X_2, \ldots, X_n\}$ and $X(A)\text{com} = \{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n\}$. Since, for any two SCs M and N, if $M \not\sqsubset N$, then $\bar{N} \not\sqsubset \bar{M}$, it follows that for any member of X(A)com, say $\bar{X}_i$, $\bar{A} \not\sqsubset \bar{X}_i$. With reference to $\bar{A}$, X(A)com will have the following properties, corresponding to the properties of X(A):

   (i) For all $\bar{X}_i$ in X(A)com, $\#\bar{X}_i > \#\bar{A}$.
   (ii) For any two distinct SCs in X(A)com, say $\bar{X}_i$ and $\bar{X}_j$, $\bar{X}_i \not\sqsubset \bar{X}_j$.
   (iii) For any SC B, with $\#B > \#\bar{A}$, if there is some $\bar{X}_i$ in X(A)com such that $B \sqsubset \bar{X}_i$, then $\bar{A} \not\sqsubset B$; otherwise $\bar{A} \sqsubset B$.
   (iv) For all X_i in X(A)com, $\bar{A} \not\sqsubset \bar{X}_i$.
   (v) For any SC B, with $\#B > \#\bar{A}$, if $\bar{A} \not\sqsubset B$, then there is some $\bar{X}_i$ in X(A)com such that $B \sqsubset \bar{X}_i$; on the contrary, if $\bar{A} \sqsubset B$, then $B \not\sqsubset \bar{X}_i$ for all $\bar{X}_i$ in X(A)com.

3. Distinctness. The following is a conjecture based in large part on the lack of a counterexample. For any two distinct SCs A and B with #A = #B, X(A) and X(B) are distinct. They may, of course, intersect in one or more SCs. (In some cases, two or more SCs of *different* cardinality have the same X-set. For example, [0, 2, 6], [0, 2, 4, 8], [0, 2, 4, 6, 8] and [0, 2, 4, 6, 8, 10] all have the following X-set: {[0, 1], [0, 3], [0, 5]}.)

X-sets and lattice pathways. If we walk to the right along the lattice of the 12-pc universe, following the line segments (which signify inclusion) and arriving at SCs of ever increasing cardinality, what can we observe about the X-sets of these SCs? As the SCs in such a chain are all related by inclusion, it will not be surprising if their X-sets exhibit similarities. Tables 3A and 3B display the situations for the chains of SCs generated by interval class 1 and interval class 5, respectively. Because these chains are M5-related, so are the X-sets for SCs of the same cardinality in them.

The X-sets of each chain are arranged in Tables 3A and 3B so that similarities and differences between SCs of adjacent cardinality in the same chain may be readily observed. As these examples show, given SCs A and A′, with #A + 1 = #A′, and $A \sqsubset A'$, the intersection of X(A) and X(A′) is generally extensive between these two X-sets, a fact which permits simple comparison in most cases. For example, comparing A = the pentatonic scale SC [0, 2, 4, 7, 9] and A′ = the diatonic hexachord [0, 2, 4, 5, 7, 9] on the basis of their X-sets, we find that both X(A) and X(A′) include [0, 6] and [0, 4, 8], whereas X(A) includes [0, 1] (which X(A′) doesn't) and X(A′) includes [0, 1, 2] and [0, 1, 4] (which X(A) doesn't). Interpreted, these data indicate that all SCs with a tritone or an augmented triad are inimical to both A and A′, whereas A and A′ are differentiated by the fact that A excludes *all* SCs that have an interval class 1 and, of such SCs, A′ excludes only those which contain [0, 1, 2] or [0, 1, 4].

Exclusion and pc accretion. As noted previously, given two SCs A and B, with #A>#B, if we wish to determine whether $B \sqsubset A$, and X(A) is known, it will be easier in many cases to make this determination indirectly by invoking property no. 3 of the definition of X(A), than to test for $B \sqsubset A$ directly. Given A=the diatonic SC [0, 1, 3, 5, 6, 8, 10] and B = [0, 3, 6, 7], we observe that [3, 6, 7] (= [0, 1, 4]) is in B and also in X(A); therefore $B \not\sqsubset A$. I suspect that this is how we (subconsciously) decide the question (a ubiquitous one in listening to tonal music) of whether there exists a major scale from which a given set of pitches can be extracted.

In cases where a pc set is built by *accretion* and at some stage be-

Table 3A. SCs generated by ic 1, and their X-sets.

A	X(A)
[0, 1]	null
[0, 1, 2]	[0, 3] [0, 4] [0, 5] [0, 6]
[0, 1, 2, 3]	[0, 4] [0, 5] [0, 6]
[0, 1, 2, 3, 4]	[0, 5] [0, 6] [0, 4, 8]
[0, 1, 2, 3, 4, 5]	[0, 6] [0, 4, 8] [0, 2, 7] [0, 3, 7]
[0, 1, 2, 3, 4, 5, 6]	[0, 4, 8] [0, 2, 7] [0, 3, 7] [0, 1, 6, 7] [0, 2, 6, 8] [0, 3, 6, 9]

Table 3B. SCs generated by ic 5, and their X-sets.

A	X(A)
[0, 5]	null
[0, 2, 7]	[0, 1] [0, 3] [0, 4] [0, 6]
[0, 2, 5, 7]	[0, 1] [0, 4] [0, 6]
[0, 2, 4, 7, 9]	[0, 1] [0, 6] [0, 4, 8]
[0, 2, 4, 5, 7, 9]	[0, 6] [0, 4, 8] [0, 1, 2] [0, 1, 4]
[0, 1, 3, 5, 6, 8, 10]	[0, 4, 8] [0, 1, 2] [0, 1, 4] [0, 1, 6, 7] [0, 2, 6, 8] [0, 3, 6, 9]

comes incompatible with a referential SC, there is often an incremental reduction in the degree of inclusion. By "degree of inclusion" I mean what Lewin calls the *covering number,* that is the number of pc sets of a specified SC which include a given pc set.[9] Suppose that the pc set {0, 3, 6, 7, 10} is constructed note-by-note, with pc 6 or pc 7 as the last member to be added. Under these conditions, the enlarging set becomes incompatible with the diatonic set only when the last pc is added. The extent to which the reduction in degree of inclusion is gradual depends on the order in which pcs are added. Table 4 shows two contrasting cases within the above situation.

I suspect that, in practice, if we are given {0, 3, 6, 7, 10} all at once, and we wish to know whether it can be extracted from some diatonic set, the X-set of the diatonic SC (subconciously invoked) plays a role in our judgement. (Detection of {3, 6, 7} ($\subset$ [0, 1, 4]) rules out a diatonic SC parent for {0, 3, 6, 7, 10}.) I suspect that it does so also in case 1 of Table 4, when pc 6 is added, finally forming {0, 3, 6, 7, 10}. However in case 2 of Table 4, the penultimate pc set {0, 3, 6, 10} suggests only *one* diatonic set, a pc set which includes also pcs 1, 5 and 9. In this context, pc 7 will be detected immediately as foreign to the one potential diatonic set (if we are attuned to this) and the X-set reference is not necessary.

Exclusion and chord polarity. Lewin, Morris, and Rahn have proposed various metrics pertaining to similarity between and among pc sets of the same or different cardinality.[10] Comparison of X-sets offers a different perspective on this matter, although it remains to be seen whether such comparison is susceptible to meaningful quantification, as in the work cited above. The following example serves as an illustration.

Suppose A = the diatonic SC [0, 1, 3, 5, 6, 8, 10] and B = the "octatonic heptachord" [0, 1, 3, 4, 6, 7, 9],
Then X(A) = {[0, 1, 2], [0, 1, 4], [0, 4, 8], [0, 1, 6, 7], [0, 2, 6, 8], [0, 3, 6, 9]}
and X(B) = {[0, 1, 2], [0, 2, 4], [0, 4, 8], [0, 2, 7], [0, 1, 5]}.

Now constructing the intersection and the difference sets of X(A) and X(B),

Let XX = X(A)$\cap$X(B) = {[0, 1, 2], [0, 4, 8]}
OCT $\sim$ MAJ = X(A)$\sim$X(B) = {[0, 1, 4], [0, 1, 6, 7], [0, 2, 6, 8], [0, 3, 6, 9]}
and MAJ $\sim$ OCT = X(B)$\sim$X(A) = {[0, 2, 4], [0, 2, 7], [0, 1, 5]}.

Sets XX, OCT$\sim$MAJ (read "octatonic, not major") and MAJ$\sim$OCT (read "major, not octatonic") partition X(A)$\cup$X(B) into three disjoint subsets, providing a test for allegiance to the diatonic SC versus the

Table 4. Reduction in covering number as pc set is built by accretion.

	n	B_n	$COV(B_n, A)$, $A = [0, 1, 3, 5, 6, 8, 10]$
case 1	1	{10}	7
	2	{0, 10}	5
	3	{0, 7, 10}	4
	4	{0, 3, 7, 10}	3
	5	{0, 3, 6, 7, 10}	0
case 2	1	{0}	7
	2	{0, 6}	2
	3	{0, 3, 6}	1
	4	{0, 3, 6, 10}	1
	5	{0, 3, 6, 7, 10}	0

octatonic-heptachord SC. Set XX represents those strains of SCs (of cardinality less than seven) which are included by neither the diatonic nor the octatonic heptachord SCs; set OCT~MAJ represents those strains of SCs which are included at least partly by the octatonic-heptachord SC but not by the diatonic SC; and set MAJ~OCT represents those strains of SCs which are included at least partly by the diatonic SC but not at all by the octatonic-heptachord SC.[11] From these observations, we acquire some sense of the polarity between the two SCs in question.

Now suppose we are given a third SC C, and we wish to determine whether it is more similar to A than B or *vice versa*. In general, if $C \sqsubset A$ and $C \not\sqsubset B$, then C is more similar to A; on the other hand if $C \not\sqsubset A$ and $C \sqsubset B$, then C is more similar to B. (Counterexamples can be constructed in the presence of unorthodox transformations.) Suppose that C = [0, 4, 7, 8, 10]. Here $C \not\sqsubset A$ and $C \not\sqsubset B$; so a test for simple inclusion is of no help. We now proceed to examine, then, the *particularities* of the exclusion relations $C \not\sqsubset A$ and $C \not\sqsubset B$, with reference to the partitioning of $X(A) \cup X(B)$ established above.

C contains one of the SCs in XX, namely [0, 4, 8]. Since XX contains only sets excluded by both A and B, the fact that $C \sqsupset [0, 4, 8]$ is sufficient to decide its exclusion by both A and B. While it may be of some interest, the duality of these exclusion relationships is not helpful in differentiating between C's relationships with A and B. Looking further, we find that C contains exactly one of the sets in MAJ~OCT, namely [0, 2, 4]. On this basis we recognize a particular likeness between A and C which is not evident between B and C. There is a strain of SCs represented by (and including) [0, 2, 4] which is totally excluded by B but at least partially included by A and C. However, C also contains exactly one of the SCs in OCT~MAJ, namely [0, 1, 4], and on this basis seems "closer" to B. In summary, C seems to be balanced midway between A and B, with comparable allegiances and disallegiances to both.

For a second example, suppose that A and B are defined as above and C = [0, 1, 2, 5, 7]. Here again, C includes one of the SCs in XX, namely [0, 1, 2]. But in contrast to the previous case, C includes two of the SCs in MAJ~OCT, namely [0, 1, 5] and [0, 2, 7] (twice), but only one of the SCs in OCT~MAJ, namely [0, 1, 4], and for this reason C seems closer to A than to B.

In order to quantify these comparisons, we would need to devise some means of dealing with the complexities of intersecting sublattices. In the last case presented above, for example, [0, 1, 5] and [0, 2, 7], both members of MAJ~OCT, represents strains which intersect at the tetrachord level in C = [0, 1, 2, 5, 7]:

$[0,1,5] \sqsubset [0,1,5,7]$
$[0,2,7] \sqsubset [0,1,5,7]$ $([0,2,7] = [0,5,7])$
$[0,1,5,7] \sqsubset [0,1,2,5,7]$.

This intersection tends to limit the number of tetrachords in C which are included by A but excluded by B (in addition to [0, 1, 5, 7], only [0, 2, 5, 7] falls into this category), thus weakening the case that C is more similar to A than to B. There are other obstacles to constructing a good metric for SC comparsion based on the exclusion relation. However, it would seem that any such metric must be logically equivalent to one based on the *inclusion* relation. In any event, the notion of the X-set should prove to be useful as a different perspective, an additional means of acquiring what Rahn calls "feel" for pc sets and their relationships.

APPENDIX

To prove the existence of X(A) for any SC A with $\#A \geqslant 3$, we construct a set of SCs T and show that it has properties (1), (2) and (3) on page 180.

Let FEWER (A) = {all SCs S: $\#S < \#A$}
EXCL (A) = {all SCs S: $S \not\sqsubset A$}
FEWEX (A) = FEWER (A) $\cap$ EXCL(A).

Construct T by the following algorithm:

1. Assign all S in FEWEX(A) with $\#S=2$ to membership in T. Proceed to step 2.
2. Give n the initial value $n=\#A-1$. Proceed to step 3.
3. If $n=2$, the construction of T is now complete.
   If $n>2$, proceed to step 4.
4. Examine each of the SCs of cardinality n in FEWEX(A). If, for any such SC S_j, there is no SC S_i of cardinatliy n–1 in FEWEX(A) such that $S_i \sqsubset S_j$, assign S_j to membership in T. Proceed to step 5.
5. Decrease the value of n by one. Return to step 3.

We now show that T qualifies as X(A).

1. Since $T \subseteq$ FEWER(A), T has property (1). [Read "T" for "X(A)" in property (1).]
2. Choose any two SCs in FEWEX(A), S_a and S_z with $\#S_a < \#S_z$. If $S_a \sqsubset S_z$, then by definition of the set-class inclusion relation, there exists some S_y with $\#S_y + 1 = \#S_z$, such that $S_a \sqsubseteq S_y \sqsubset S_z$. Because of the adjacent cardinalities of S_y and S_z, the operation of step 4 in the construction of T ensures that S_z is not assignable to T. In summary, $S_a \sqsubset S_z$ leads to $S_z \notin T$. It follows that $S_z \in T$ implies $S_a \not\sqsubset S_z$. Thus T has property (2).
3. Since $T \subseteq$ EXCL(A), for any SC t in T, $t \not\sqsubset A$. Therefore, for any SC S, $t \sqsubseteq S$ implies $S \not\sqsubset A$. This covers the first part of property (3). Now suppose that $S \in$ FEWEX(A), which is to say $S \not\sqsubset A$. By the operation of step 4 in the construction of T, if there is some S' in FEWEX(A) with $\#S' + 1 = \#S$ and $S' \sqsubset S$, then S is not in T. At the next iteration of step 4, if there is some S'' in FEWEX(A) with $\#S'' + 1 = \#S'$ and $S'' \sqsubset S'$, then S' is not in T, etc., etc. Since FEWEX(A) is finite and contains no SCs of cardinality less than 2, the test for an included subclass of adjacent cardinality eventually fails, and one of S, S', S'', and so forth, is assigned to T. Because of the transitivity of subclass inclusion, there is some t (one of S, S', S'', and so forth) in T such that $t \sqsubseteq S$. In summary, $S \not\sqsubset A$ implies that there is some t in T such that $t \sqsubseteq S$. It follows that if there is no t in T such that $t \sqsubseteq S$, then $S \sqsubset A$. Thus T has property (3).
   Since T has properties (1), (2) and (3), the existence of X(A) is proved for any A with $\#A \geqslant 3$. [The empty X(A) is not ruled out, but its occurrence would appear to be limited to universes of 7 or fewer pc's. For example, $X([0, 1, 3]_7)$ is empty.]

To prove the uniqueness of X(A), suppose that X(A) is not unique and that X and Y are both X-sets of A, with $X \neq Y$. Since $X \neq Y$, one or both of X and Y must contain a SC not in the other. Say $X_i \in X$ and $X_i \notin Y$. By property (4), $X_i \not\sqsubseteq A$. Therefore by property (5), there is a SC in Y, say Y_i, such that $Y_i \sqsubseteq X_i$. But $X_i \notin Y$; therefore $Y_i \neq X_i$; so $Y_i \sqsubset X_i$. Likewise, by (4) and (5), $Y_i \not\sqsubseteq A$, and there is a SC in S, say X_j, such that $X_j \sqsubseteq Y_i$. Combining the two abstract inclusions, we get $X_j \sqsubseteq Y_i \sqsubset X_i$. But this implies $X_j \sqsubset X_i$ which contradicts property (2). Therefore X(A) must be unique.

NOTES

1. Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973), pp. 93-100. David Lewin, "Forte's Interval Vector, My Interval Function, and Regener's Common-Note Function," *Journal of Music Theory* 21 (1977): 194-237; also Lewin, "Some New Constructs Involving Abstract PC Sets and Probabilistic Implications," *Perspectives of New Music* 18/2 (1980): 433-444. Robert Morris, "Set Groups, Complementation, and Mappings among Pitch-Class Sets," *Journal of Music Theory* 26 (1982): 101-144. John Rahn, "Relating Sets," *Perspectives of New Music* 18/2 (1980): 483-498.
2. Howard Boatwright, in his *Introduction to the Theory of Music* (New York: W. W. Norton & Co., 1956), lists on p. 149 "chromatic formulae" which correspond to what is called the X-set of the diatonic SC in the present article. To my knowledge, this is the earliest published expression of the idea of characterizing a set of pitches by means of an exclusion relation. As an afterthought in my "Pitch-Set Equivalence and Inclusion (A Comment on Forte's Theory of Set-Complexes)," *Journal of Music Theory* 9 (1965): 163-171, I generalize Boatwright's idea to the arbitrary set of pcs.
3. Lewin, "Forte's Interval Vector," pp. 197-201.
4. Eric Regener, "On Allen Forte's Theory of Chords," *Perspectives of New Music* 13/1 (1974): 191-212. See especially pp. 208-209.
5. Adapting Forte's notation to the 8-pc universe, the interval class vector of these SCs is [2121]. For an explanation of the Z-relation, see Forte, *The Structure of Atonal Music,* pp. 21-24.
6. Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11 (1967): 32-59.
7. Under the diatonic equivalence relation, all 2nds, 7ths, 9ths, 14ths, etc., form a single interval class, one of three such classes. See my "Aspects of Diatonic Sets," *Journal of Music Theory* 23 (1979): 45-61.
8. I argue this claim in "Aspects of Diatonic Sets" and in "Diatonic Interval Sets and Transformational Structures," *Perspectives of New Music* 18/2 (1980): 461-482.
9. Lewin, "Some Constructs," p. 434.
10. Lewin, "A Response to a Response on PCSet Relatedness," *Perspectives of New Music* 18/2 (1980): 498-502; Morris, "A Similarity Index for Pitch-Class Sets," *Perspectives of New Music* 18/2 (1980): 483-498; and Rahn, "Relating Sets."
11. The statements in this sentence generalize to two arbitrarily chosen SCs A and B, of the same cardinality, if and only if there is no X_i in X(A) and no X_j in X(B), with $\#X_i \neq \#X_j$, such that $X_i \sqsubset X_j$ or $X_j \sqsubset X_i$. If, for example $X_i \sqsubset X_j$, then X_i represents a strain of SCs which is partly included in B, but the parallel statement does not hold—X_j does not represent a strain of SCs which is partly included in A. As a substrain of X_i, the X_j strain is totally excluded by A.