REGIONS:

A THEORY OF TONAL SPACES IN

EARLY MEDIEVAL TREATISES

Norman Carey

David Clampitt

The discussion of harmonics in the ninth-century *Scolica enchiriadis* includes a diagram (reproduced as figure 1) that displays the relationships among the consonances and their correspondences with numerical proportions. This passage and diagram are included in Strunk's anthology (1950, 136).[1] The numbers in the diagram are the elements of the classical *tetractys*—6, 8, 9, 12—duplicated at the octave to form what we may refer to as the *heptactys,* the set {6,8,9,12,16,18,24}. The *heptactys* is our point of departure and the locus of intersection between ancient theoretical tradition and work of the last few decades that falls under the rubric of diatonic set theory, a relation already suggested in a general way in Gauldin (1983).

In this paper we generate the elements of the *Scolica* diagram in a manner that may well be ahistorical, although it uses no concept that could not have been available in the ancient or medieval worlds. The *heptactys* will then be shown to belong to a class of pitch collections called *regions,* generated in the same way. Several other medieval constructions, including Pseudo-Odo's gamut, Guido's hexachord, the tetrachord of the finals, and the diamond-shaped diagrams in Guido's *Micrologus* and elsewhere, may be construed as regions, as we will see.

Regions exist in pitch space. Taking a perspective that focuses exclu-

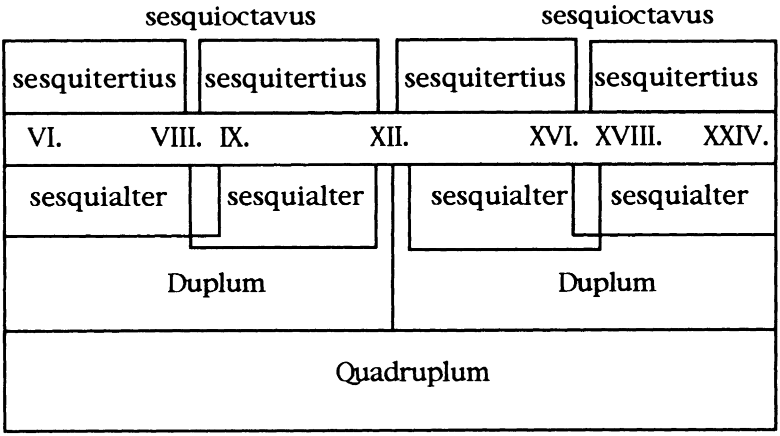


Figure 1. The *heptactys* {6, 8, 9, 12, 16, 18, 24} in *Scolica enchiriadis.*

sively on the pitch-class content of regions yields the *well-formed scales* defined in Carey and Clampitt (1989) and discussed below. The language of pitch class and interval class allows a much more tractable theory, but one recovers some partly veiled or completely lost information by adopting a point of view, akin to that of the medieval theorists, that treats pitch class more flexibly.

I

The *heptactys* includes the perfect eleventh, represented by the ratios 16:6 and 24:9 (that is, 8:3 in reduced terms). *Musica* and *Scolica enchiriadis* accept the perfect eleventh as a consonance in contradiction to strict Pythagorean tradition, 8:3 being neither a multiple nor a superparticular ratio. Barbera (1984) locates possible sources for what he calls this "expansion of the musical tetractys" in treatises by Theōn of Smyrna and Gaudentius, while Phillips (1984, 1990) identifies Cassiodorus and Calcidius as sources of *Scolica* and of its treatment of the eleventh as a consonance, though not of the *heptactys* itself.

The *heptactys* reappears in the eleventh-century treatises by Aribo Scholasticus and Hermannus Contractus, and in the *Quaestiones in musica.* Aribo's *De musica* contains a diagram similar to figure 1 (Smits van Waesberghe 1951, 62–63), while Hermannus identifies the *heptactys* with "the entire complex of consonances" (Ellinwood 1936, 21). One possible justification for the *heptactys* is that it presents the six consonances accepted by many medieval theorists: fourth (4:3), fifth (3:2), octave (2:1), eleventh (8:3), twelfth (3:1), and double octave (4:1). Moreover, the con-

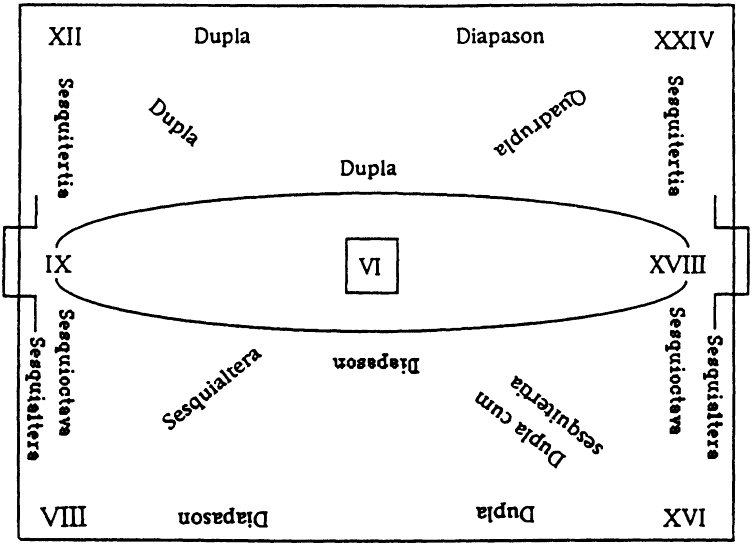


Figure 2. The *heptactys* in *Quæstiones in musica.*

sonances are represented in least integers such that they share a common point of origin, either 6 or 24.[2] The diagram of the *heptactys* in the *Quaestiones in musica,* shown in figure 2, presents the consonances in relation to 6.[3]

Recall that the proportion 6:8:9:12 has the property that 8 and 9 are the harmonic and arithmetic means, respectively, between 6 and 12. Another justification for the inclusion of the duplication at the octave of this proportion is to include the geometric mean as well: 6:12:24 reduces to 1:2:4, numbers in geometric proportion. All three means are discussed in the third part of *Scolica enchiriadis* (Schmid 1981, 139–41), beginning with a reconstruction of the *heptactys,* on the way to the formation of a diatonic scale.

The proper context for medieval discussions of harmonics, and the appropriate setting for the *heptactys* in our exposition of its generation, is the classical set in the Pythagorean tradition, the infinite sequence of integers that are products of powers of 2 and 3, a set we call $\hat{P}$ (pronounced, if you like, "P-hat"). The beginning of the sequence of $\hat{P}$ numbers is shown below the formal definition of $\hat{P}$:

$$\hat{P} = \left\{\hat{p} = 2^a 3^b \middle| a,b \in J = \{0,1,2,\ldots\}\right\}$$

$$1,2,3,4,6,8,9,12,16,18,24,27,32,36,48,54,64,72,\ldots$$

We can assign note names in ordinary musical notation to elements of $\hat{P}$ in the following way: we assign a tone F_0 to the number 1; the number 2^a3^b represents the note a octaves and b twelfths upwards from F_0, where the subscript on a note refers to the octave register.[4] This correspondence is shown below:

1	2	3	4	6	8	9	12	16	18	24	27
F_0	F_1	C_1	F_2	C_2	F_3	G_3	C_3	F_4	G_4	C_4	D_4
32	36	48	54	64	72	81	96	108	128	144	162
F_5	G_5	C_5	D_5	F_6	G_6	A_6	C_6	D_6	F_7	G_7	A_7
192	216	243	256	288	324	384	432	486	512	576	648
C_7	D_7	E_7	F_8	G_8	A_8	C_8	D_8	E_8	F_9	G_9	A_9

Since $\hat{P}$ consists exactly of all products of non-negative powers of 2 and 3, it has another representation as ordered pairs of non-negative integers, where the first element of the ordered pair represents the exponent on 2, and the second element the exponent on 3. For example, 12 equals $2^2 3^1$, or (2,1), and represents the note 2 octaves and 1 twelfth from F_0, or C_3. Stated formally, this one-to-one correspondence is $\hat{P} \leftrightarrow J \times J$: $2^a 3^b \leftrightarrow (a,b)$:

1	2	3	4	6	8	9	12	16	18	24
(0,0)	(1,0)	(0,1)	(2,0)	(1,1)	(3,0)	(0,2)	(2,1)	(4,0)	(1,2)	(3,1)
F_0	F_1	C_1	F_2	C_2	F_3	G_3	C_3	F_4	G_4	C_4
27	32	36	48	54	64	72	81	96	108	128
(0,3)	(5,0)	(2,2)	(4,1)	(1,3)	(6,0)	(3,2)	(0,4)	(5,1)	(2,3)	(7,0)
D_4	F_5	G_5	C_5	D_5	F_6	G_6	A_6	C_6	D_6	F_7

The representation of $\hat{P}$ by a set of ordered pairs suggests a coordinate system, and indeed $\hat{P}$ can be put in a two-dimensional setting, superimposed on the positive quadrant of the lattice of integers. Figure 3 shows a small portion of the set.

This version of $\hat{P}$ was known in antiquity and is referred to as the Nicomachus Triangle, or the *eschequier* [modern French, *échiquier*] of Nicole Oresme, after the fourteenth-century mathematician, cleric, advisor to Charles V, and associate of Philippe de Vitry. In his *Livre du ciel,* a glossed vernacular translation of Aristotle's *De cœlo,* Oresme asserts: "Truly, this figure is full of very great mysteries and one can make very

27	54	108	216	432	864
9	18	36	72	144	288
3	6	12	24	48	96
1	2	4	8	16	32

Figure 3. Two-dimensional setting of $\hat{P}$.

beautiful and marvelous observations in it, for it contains virtually all of speculative music."[5] Oresme's chessboard is displayed in figure 4.

II

In this section we define the notion of a region. Regions will be constructed by employing two properties, *connectedness* and *complementarity,* defined in the set $\hat{P}$. As we will see, these properties have some historical sanction. We have chosen them for their efficacy, however, not in an overt attempt at historical reconstruction.

Definition 1: Let S be a subset of $\hat{P}$ and let $\hat{p} \in \hat{P}$. S is a *connected subset of $\hat{P}$* if, whenever $s_a < \hat{p} < s_b$ where s_a and $s_b \in S$, it is true that $\hat{p} \in S$. The *connected* subsets of $\hat{P}$ may be regarded simply as uninterrupted sections of the sequence of $\hat{P}$ numbers—for example, {8,9,12}.

It appears that the property of connectedness in $\hat{P}$ was a desirable one for some of the ancient authorities. Timaios of Locri, for example, presents a long series of connected $\hat{P}$ numbers beginning with 384, or C_8, the point at which the diatonic scale first appears in the sequence of $\hat{P}$ numbers (Tobin 1985, 43). In *Scolica enchiriadis* itself, the one time the scale is associated with numbers it begins as a connected set from C_7, or 192 (Schmid 1981, 145). This yields the notes of one octave of the diatonic scale, except that the note B cannot be represented as an integer.

Elements that are adjacent in the ordered sequence will be called *connected elements.* For example, 9 and 12 are connected elements in $\hat{P}$. Of special importance will be connected pure powers of 2 and 3, which will be represented by 2^A and 3^B. For example 8 and 9, or 2^3 and 3^2 are con-

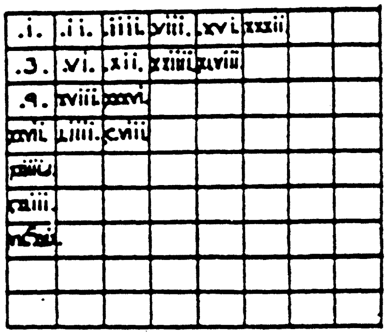


Figure 4. The *eschequier* [*échiquier*] of Oresme.

nected, as are 27 and 32, or 3^3 and 2^5. The first six pairs $(2^A, 3^B)$ of connected pure powers are shown below, together with the interval determined by each pair.[6]

A	B	2^A	3^B	interval
1	0	2	1	P8
1	1	2	3	P5
2	1	4	3	P4
3	2	8	9	M2
5	3	32	27	m3
8	5	256	243	m2

Definition 2: Let $S_{(A,B)}$ be the connected subset of $\hat{P}$ that contains all of $\hat{P}$ up to, but not including, 2^A or 3^B. For example, $S_{(5,3)}$ is the connected set of $\hat{P}$ elements that are less than 27; that is, the $\hat{P}$ elements from 1 to 24.

Definition 3: Let $G_{(A,B)}$ be the set of all $\hat{P}$ numbers such that the exponent on 2 is less than A, and the exponent on 3 is less than B; that is, the set of all factors of $2^{A-1}3^{B-1}$. Note that $S_{(A,B)}$ is a subset of $G_{(A,B)}$. As shown in figure 5, $G_{(5,3)}$ consists of all factors of $2^4 3^2 = 144$. Thus $G_{(5,3)} = \{1,2,3,4,6,8,9,12,16,18,24,36,48,72,144\}$.

Definition 4: We say that g and g′ are *complementary factors* (or *complements*) if their product equals $2^{A-1}3^{B-1}$. Note that g and g′ necessarily both belong to $G_{(A,B)}$, since this set contains all the factors of $2^{A-1}3^{B-1}$.

Definition 5: We define a *set of complements* in $G_{(A,B)}$ to be a subset of $G_{(A,B)}$ such that, given any element of the subset, its complement is also an element of the subset. $G_{(A,B)}$ itself, of course, satisfies this definition.

9	18	36	72	144
3	6	12	24	48
1	2	4	8	16

Figure 5. $S_{(5,3)} \subset G_{(5,3)}$.

Definition 6: Finally, we define the *region* $R_{(A,B)}$ to be the largest connected set of complements in $G_{(A,B)}$. $S_{(A,B)}$ is the largest connected subset of $G_{(A,B)}$, so the region is a subset of $S_{(A,B)}$: $R_{(A,B)} \subset S_{(A,B)} \subset G_{(A,B)}$.

As figure 5 shows, $G_{(5,3)}$ consists of the fifteen factors of 144 and contains the eleven elements of $S_{(5,3)}$ plus the four elements 144, 72, 48, and 36, the elements in the upper right hand corner of $G_{(5,3)}$. The complements of these four elements are the first four elements of $S_{(5,3)}$: 1, 2, 3, and 4. The remaining seven elements—6, 8, 9, 12, 16, 18, 24—form the region $R_{(5,3)}$, the largest connected subset of complements in $G_{(5,3)}$. Thus, the *heptactys* in *Scolica enchiriadis* is a region as defined.

This set is shown in figure 6 as a symmetrical subset of $G_{(5,3)}$ on the Nicomachus Triangle. It should not be surprising that a symmetrical figure results from this definition, since the relation of complementarity is a symmetrical one: if x is the complement of y, then y is the complement of x. The formation of symmetrical sets of complements is not unprecedented. An example is in the treatise on monochord division, *Katagraphetai toinun hō kanōn,* probably from the fourth century of the Christian era. In this scale, (of which the *heptactys* is a subset), pairs of rational numbers whose product is 144 form a symmetrical scale spanning two octaves (Markovits 1977, 37–38).

One should keep in mind that this definition of $R_{(A,B)}$ starts from the premise that 2^A and 3^B are connected. Since A–1 and B–1 play roles in the construction, we require that A and B be positive integers. The first three regions in $\hat{P}$ are shown below:[7]

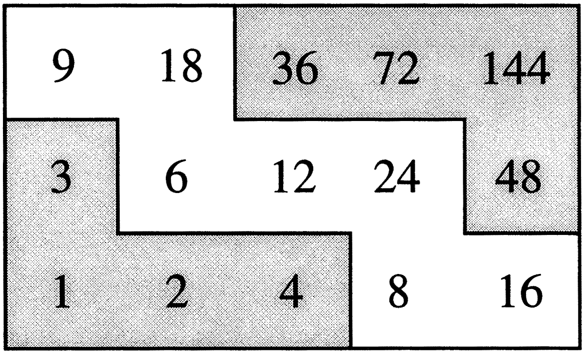


Figure 6. The region $R_{(5,3)} \subset G_{(5,3)}$.

2^A	3^B	$G_{(A,B)}$	$S_{(A,B)}$	$R_{(A,B)}$
2	3	$G_{(1,1)}=\{1\}$	$S_{(1,1)}=\{1\}$	$R_{(1,1)}=\{1\}$
4	3	$G_{(2,1)}=\{1,2\}$	$S_{(2,1)}=\{1,2\}$	$R_{(2,1)}=\{1,2\}$
8	9	$G_{(3,2)}=\{1,2,3,4,6,12\}$	$S_{(3,2)}=\{1,2,3,4,6\}$	$R_{(3,2)}=\{2,3,4,6\}$

Note that in each region there are A+B−1 elements. For example, there are 7 elements in $R_{(5,3)}$, and 7=5+3−1. We will show that in general a region with parameters A and B will contain A+B−1 elements.

The next connected pair of pure powers of 2 and 3 after 27 and 32 is 256 and 243, or 2^8 and 3^5, and 256/243 represents the interval known to the ancients as *limma,* that is, the Pythagorean diatonic half-step. This, by the way, is the example Boethius uses for his notion of *primi numeri,* by which he means, not prime numbers, but pairs of numbers in reduced terms. Our connected pairs of pure powers would be included among his *primi numeri,* and it is interesting that the example Boethius gives is a connected pair (Bower 1989, 26–27). Figure 7 shows the twelve elements of the region $R_{(8,5)}$ as a subset of $G_{(8,5)}$. The region $R_{(8,5)}$ contains all modal varieties of the usual (anhemitonic) pentatonic scale. The cardinality of the region is again A+B−1, that is, 8+5−1, or 12.

The elements 2^{11} and 3^7 are the next connected pair of pure powers in $\hat{P}$. The ratio $3^7/2^{11}$ represents the interval from F to F♯, the chromatic half-step, or *apotomē.* This interval determines the diatonic region $R_{(11,7)}$, in which there are 11+7−1 or 17 elements. The region begins on 384, or C_8, and ranges through two octaves and a major third to conclude on 1944, or E_{10}. *Scolica enchiriadis* also presents the C scale, in an alphabetic notation, spanning two octaves. The two-octave scale is the final stage in the discussion of harmonics in *Scolica.* The two-octave span is customary,

81	162	324	648	1296	2592	5184	10368
27	54	108	216	432	864	1728	3456
9	18	36	72	144	288	576	1152
3	6	12	24	48	96	192	384
1	2	4	8	16	32	64	128

$R_{(8,5)} \subset G_{(8,5)}$

THE 12 ELEMENTS OF

THE REGION $R_{(8,5)}$

48	54	64	72	81	96	108	128	144	162	192	216
(4,1)	(1,3)	(6,0)	(3,2)	(0,4)	(5,1)	(2,3)	(7,0)	(4,2)	(1,4)	(6,1)	(3,3)
C_5	D_5	F_6	G_6	A_6	C_6	D_6	F_7	G_7	A_7	C_7	D_7

Figure 7. The pentatonic region

but the choice of C is unusual at this time.[8] At an intermediate stage, one octave of the scale is associated with numbers beginning, as we observed above, an octave too low to find a complete scale in whole numbers.

Even more tantalizing is the remark by the anonymous author of *De modorum formulis* a few centuries later, cited by Handschin: "We have surely recognized that the letter C, which the ancients considered to be third, is first according to numerical proportions."[9] Handschin (1948, 214) was aware that the *tetraktys,* pentatonic, and diatonic could be realized as connected subsets of Pythagorean numbers beginning on 6, 48, and 384, respectively. Furthermore, he noted that 384 must be interpreted as C to be consistent with the medieval gamut. In relation to this fact he found intriguing (as do we) the remark by the *De modorum* anonymous.[10]

The next example is the chromatic region $R_{(19,12)}$. We have, however, just about reached the limit of the usefulness of our ad hoc approach to the determination of regions, since the numbers are becoming rather large. In this case 2^{19} and 3^{12} are connected in $\hat{P}$, and their ratio corresponds to the Pythagorean comma. In turn, the intervals between adjacent elements within this region are the diatonic and chromatic half-steps, *limma* and *apotomē.*[11]

Following this is the region $R_{(27,17)}$, which contains the division of the octave in 17 parts, described by the Arabic theorist Al-Fārābī in the late ninth or early tenth century (Sachs 1943, 278–80). The 17-note system also plays a role in Renaissance theories of chromaticism, discussed, for example, in the *Libellus monochordi* of Prosdocimo de' Beldomandi (Herlinger 1987, 139–47). Yet another region, $R_{(84,53)}$, contains a 53-note division of the octave discussed by Chinese theorists (Daniélou 1943, 177).

III

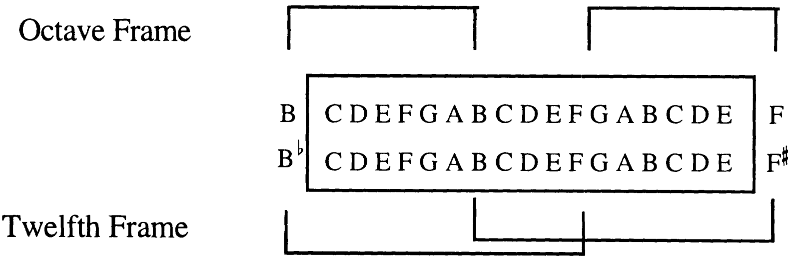
What is perhaps most striking about the regions we have constructed is that they contain various notable scales, especially the pentatonic, diatonic, and chromatic scales. These scales are worth investigating on their own account—as we will see, they all share the property we have called well-formedness. But to identify the scales within regions is to focus on the pitch classes they contain, and it is worth questioning one's almost reflexive move to abstract pitch-class information from a pitch structure.

Going back to first principles, we recall that regions were defined in the context of $\hat{P}$, which was generated entirely by two numbers, 2 and 3. As mathematical entities, these numbers play perfectly symmetrical roles in $\hat{P}$ and in its regions. There is no reason to assign a privileged status to either of them, prior to interpreting them as representing the intervals octave and twelfth. Even when that interpretation is made, there is no necessity to favor the octave over the twelfth by imposing octave equivalence. The intervals play symmetrical roles in the construction of regions, and either (or neither) may be chosen as a modular frame. Within the region, the intervals remain in a kind of balance.

To begin to understand the nature of this balance, consider the diatonic region $R_{(11,7)}$. The region allows for 10 perfect octaves from C—c to e—e′ and 6 perfect twelfths from C—g to A—e′. This space is maximal in this respect, as figure 8 shows. Either interval is extended outside the region only at the expense of the other.

Another lesson to be drawn is that there is nothing in the formal construction that requires that the generating numbers be 2 and 3. Certainly a medieval musician would only deal with integers, in particular the powers of 2 and 3, but if one thinks carefully about the definitions, there is nothing in the notions of connectedness and complementarity that precludes the generating numbers from being rational or even irrational. Regions are not necessarily defined only in Pythagorean tonal spaces.

If we construct P* from the values 2 and 3/2 representing, respectively, the octave and perfect fifth, the regions in P* appear to be truncated versions of the regions in $\hat{P}$. The beginning of the sequence of P* elements is shown below, together with the sequence of ordered pairs



Perfect octaves and twelfths are preserved within $R_{(11,7)}$

Figure 8. The diatonic octave-twelfth region.

(a,b) where a and b represent the exponents on 2 and $\frac{3}{2}$, respectively, and of interpretation in pitches:

1	$\frac{3}{2}$	2	$\frac{9}{4}$	3	$\frac{27}{8}$	4	$\frac{9}{2}$	$\frac{81}{16}$	6	$\frac{27}{4}$
(0,0)	(0,1)	(1,0)	(0,2)	(1,1)	(0,3)	(2,0)	(1,2)	(0,4)	(2,1)	(1,3)
F_0	C_0	F_1	G_1	C_1	D_1	F_2	G_2	A_2	C_2	D_2
$\frac{243}{32}$	8	9	$\frac{81}{8}$	$\frac{729}{64}$	12	$\frac{27}{2}$	$\frac{243}{16}$	16	$\frac{2187}{128}$	
(0,5)	(3,0)	(2,2)	(1,4)	(0,6)	(3,1)	(2,3)	(1,5)	(4,0)	(0,7)	
E_2	F_3	G_3	A_3	B_3	C_3	D_3	E_3	F_4	$F\sharp_4$	

For example, the diatonic region generated by octaves and fifths could be represented by the notes C_2 to E_3. Since 2^4 and $\left(\frac{3}{2}\right)^7$ are the connected elements that determine this region, we refer to it as $R_{(4,7)}$ in P*. There are 7+4−1=10 elements in this region, and it represents the largest diatonic span in which octaves and fifths may be maintained.[12] As we have stated, the generating intervals themselves are balanced in the region so that either one (or neither) may be chosen as the interval establishing a modular equivalence relation.

This brings us to a reconsideration of what is surely the most notorious aspect of the *Enchiriadis* treatises: the *dasian* notation and the strange scale associated with it, first elucidated by Spitta (1889). As figure 9 shows, this scale comprises a series of four disjunct Dorian tetrachords, plus two *residual* or *remnant* notes. Although the emphasis in the treatises is on the Dorian tetrachord, with its tone-semitone-tone symmetry, the dasian scale can also be regarded as a series of identical conjunct pentachords, that is, periodic at the fifth.[13]

Such a scale lends itself ideally to organum at the fifth. In organum at the octave or in composite organum, on the other hand, augmented

Dasian Scale

Residui (Remanentes)	c♯''	Deuteros
	b'	Archoos
Excellentes	a'	Tetrardos
	g'	Tritos
	f♯'	Deuteros
	e'	Archoos
Superiores	d'	Tetrardos
	c'	Tritos
	b	Deuteros
	a	Archoos
Finales	g	Tetrardos
	f	Tritos
	e	Deuteros
	d	Archoos
Graves	c	Tetrardos
	B♭	Tritos
	A	Deuteros
	G	Archoos

Figure 9. The scale associated with the dasian notation.

octaves would occur if all the voices moved according to the dasian scale, an issue the student interlocutor raises near the end of *Scolica enchiriadis.* Just as the periodicity at the octave in the usual diatonic pitch-space allows two voices to move perpetually in parallel octaves, while voices moving in parallel fifths will encounter a diminished fifth, in the dasian scale the reverse is the case: parallel fifths may be maintained perpetually, while motion in parallel octaves will eventually result in an augmented octave. See figure 10. Within the region, where the dasian and diatonic scales coincide, no conflicts arise.[14]

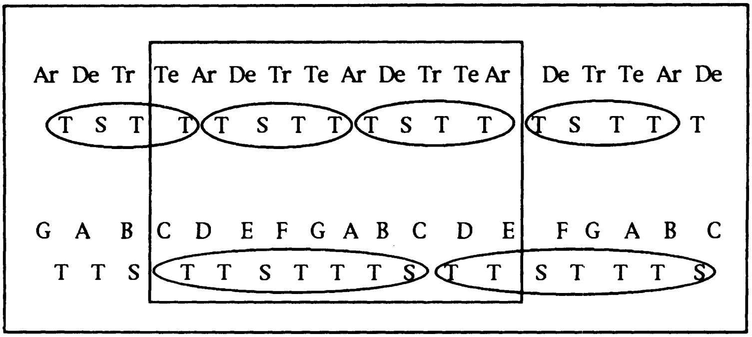


Figure 10. The octave-fifth region as the space in which the dasian and diatonic scales coincide.

IV

The dasian system was perhaps not only motivated by the practice of organum. It also addressed a fundamental concern of early medieval theory. In the ninth century, the classification of chant was being accomplished simultaneously with the establishment of the gamut and a suitable notation. One of the features of the diatonic system that was evident to those engaged in these tasks was that similar tonal patterns existed in the neighborhoods of pitches a fifth or a fourth apart. Thus, four finals—D,E,F,G—sufficed to classify chants according to their endings.[15] Put another way, given the system of four *maneriae,* each embracing two modes, some chants (or portions of them) might be placed equally well a fifth (more rarely, a fourth) above one of the finals.[16]

This observation of similarity developed into the doctrine of the affinities.[17] The radical approach of the *Enchiriadis* treatises was to make affinity at the fifth perfect by treating the tetrachord of the finals as a modular unit, with the fifth as interval of periodicity. The choice of the fifth rather than the octave yielded a more manageable basic unit, for both the voice and the mind.[18] Position finding, that is, determining modal identity, thus became a function of the placement of a single semitone. As Phillips (1984, 161) points out, in *Scolica enchiriadis* the semitone is considered to be the heart and soul of melody: "cor atque animus . . . cantilenae."

Theorists of the eleventh century rejected the dasian scale: Guido of Arezzo (implicitly) and Hermannus Contractus (explicitly) were both critical.[19] Thus Guido: "Likeness is not complete except at the octave." Handschin, however, emphasizes that Guido, despite his critical stance, was truly a follower of the *Enchiriadis* tradition in the great attention he paid the affinities (1948, 323). Guido's problem was to reconcile the four basic "qualities" of the dasian system with "the seven different notes."[20]

Guido's principal statement on the affinities is chapter 7 of *Micrologus.* He describes the *modus vocum,* or characteristic motion through tonal space, that determines affinity. The first *modus vocum,* for example, is "when from a note one descends by a tone and ascends by a tone, a semitone, and two tones, as from A and D," (Babb 1978, 63); similarly for the finals E and F and the notes a fifth above and a fourth below them. Guido treated the final G as an exception, a note without an affinity. For Guido, essentially the same tonal space exists within the sixths bounded by Γ and E, and C and a.

Hermannus, writing a few decades after *Micrologus,* treated the affinities in a more logically satisfying (if less practical) way than Guido in chapter 7. His rule: "Take any tetrachord you wish—say, that of the *graves* [A-D]; add a tone at both ends; you then have the limits of the tonal patterns which form the basis of the modes [*sedes troporum*]. There are four modes and as many tonal patterns [*modi vocum*]." (Ellinwood 1936, 57.) Thus his *modi vocum* agree with Guido's except that G also has affinities at the fifth above and fourth below, d and D.[21] His more logical arrangement of the affinities is apparent if represented on the diatonic circle of fifths thus: F↔C G↔D↔A E↔B. The note D is both protus in the *finales* and tetrardus in the *graves.* The fifths and fourths that are privileged under this arrangement connect tones that have the same position within their tetrachords and are called by Hermannus natural (*naturalis*); the others, connecting C to G and A to E are formal (*formalis*).

If C and G are taken as lower boundaries, and a and e as upper, they mark the limits of affinity, since C and G are only similar in ascent and a and e are only similar in descent. The boundary pitches within both theorists' conceptions are those of the natural and hard hexachords, as they were later called. While neither used the term, the hexachord is described by Hermannus's rule. As we will see, the hexachord mediates among what we call the various diatonic regions.

Guido gave further consideration to the affinities in chapter 8 of *Micrologus,* in an account more theoretically consistent than that of the previous chapter. Here he considers the general affinity of all notes a fifth or a fourth apart.

Figure 11 is Guido's representation of $R_{(4,7)}$ in P*, the diatonic region generated by octave and fifth.[22] The same region appeared before in the context of the dasian scale, where four qualities are determined by fifth equivalence. Here, seven pitch classes (*septem discrimina vocum*) are determined by octave equivalence. There are ten different notes in the diagram (F G a and ♮ appear twice). The three octaves in the region are shown vertically, while the six fifths appear on the right side. The equivalence at the octave assumed in this context means that fifths are not to

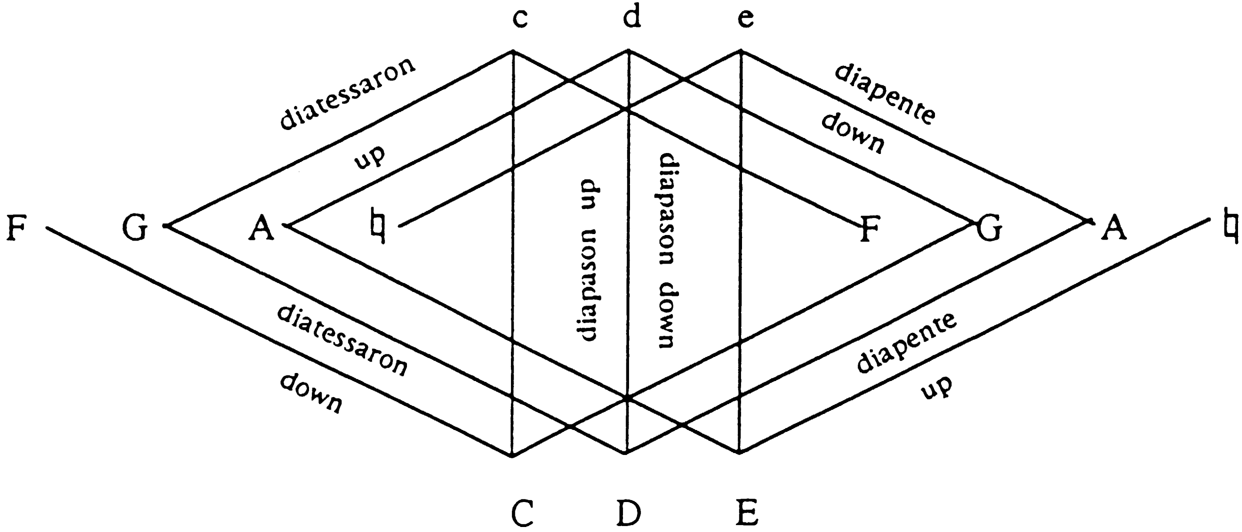


Figure 11. Diagram from Chapter 8 of Guido of Arezzo's *Micrologus*.

be found below F and above ♮, which the diagram shows by isolating them as end points only.

The six fifths imply a structure of overlapping natural and hard hexachords, C D E F G a and G a ♮ c d e. Guido in this chapter is also concerned with what he considers a lesser affinity, affinity at the fourth. The six perfect fourths are shown in a complementary fashion on the left side of the diagram. A possible reinterpretation of the diagram brings out the lesser affinity: The hexachord ordering natural-hard was suggested by three upward fifths from C D E and three downward fifths from c d e. In our reinterpretation, projecting downward and upward fourths from a central trichord C D E leads to the other hexachord order, hard-natural:[23]

Γ	A	B	C	D	E			
			C	D	E	F	g	a

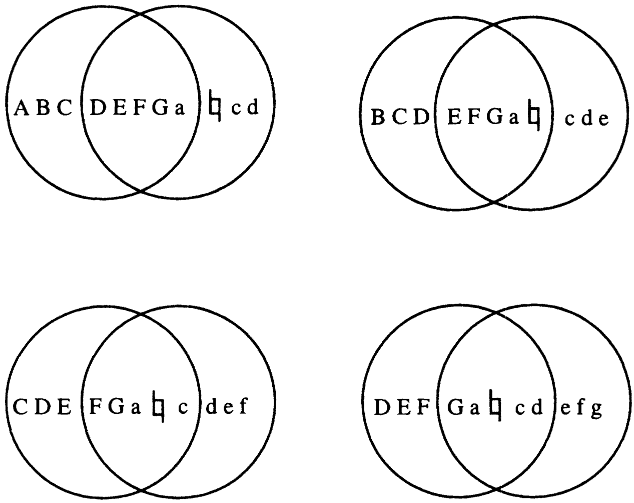
This is the diatonic region generated by octaves and fourths. This region is the maximal space in which perfect octaves and fourths may be maintained; that is, it is the maximal space within which octave equivalence and fourth equivalence are equally valid. Taking the fourth as a modular interval there are three classes or qualities. This interpretation is consistent with Guido's text, which asserts that *three* notes—C, D, E—capture the possible tonal characters.

> All the modes and the "distinctions" of the modes are connected with these three notes [C, D, E]. . . . All other notes have some concordance with these three, either below or above.[24]

The idea that there were three basic qualities was only taken up again in the fourteenth-century *Quatuor principalia.*[25] In this treatise the hexachord is divided into three pairs, ut-fa, re-sol, and mi-la, but if we treat the pairs as three equivalences, we have the presentation of the diatonic octave-fourth region below. Continuation outside the region using the fourth as modular interval leads to soft b, making this presentation highly suggestive of the three hexachords of the mature solmization system.

ut re mi	ut re mi	ut re mi
Γ A B	C D E	F g a

The diatonic region generated by octaves and fourths is closely related to the corresponding octave-eleventh region. This region coincides with the gamut of Pseudo-Odo (leaving out the alternate ninth step, the soft b), spanning two octaves and a tone, from Γ to aa. The elevenths reflect the combined ranges of authentic and plagal modes, as displayed in the treatises of first Aribo (Smits van Waesberghe 1951, 18–20), then John of Affllighem (Smits van Waesberghe 1950, 94–95; Babb 1978, 124) and



The ranges of authentic and plagal modes as a symmetrical subset of the diatonic octave-eleventh region

Γ A B C D E F G a ♮ c d e f g ${}^{a}_{a}$

Figure 12. Aribo's circle diagrams.

Jacques of Liège (Bragard 1973, 155–56) (figure 12). Their diagrams are of the elevenths formed by the *naturalis* fourths and fifths of Hermannus. The addition of the elevenths composed of the *formalis* fourths and fifths covers all six elevenths in the region. In the mathematical determination of the region, 2^{10} and $(8/3)^7$ are connected elements in the appropriate $\hat{P}$-type space. There are 9 octaves and 6 elevenths in the region, and 16 elements in all.

Strictly parallel four-voice organum must take place within the confines of either the octave-eleventh region or the octave-twelfth region. An example that moves within an octave-eleventh region is found in Chapter 15 of the *Musica enchiriadis.*

The hexachord figures prominently in all of the diatonic regions we have seen so far. The octave-fifth region is composed of two hexachords in the order natural-hard, while hexachords in the order hard-natural form the octave-fourth region. The octave-twelfth region is made up of two pairs of natural-hard hexachords an octave apart. The octave-eleventh

$\frac{81}{16}$	$\frac{27}{4}$	$\frac{9}{1}$	$\frac{12}{1}$	$\frac{16}{1}$
$\frac{27}{8}$	$\frac{9}{2}$	$\frac{6}{1}$	$\frac{8}{1}$	$\frac{32}{3}$
$\frac{9}{4}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{16}{3}$	$\frac{64}{9}$
$\frac{3}{2}$	$\frac{2}{1}$	$\frac{8}{3}$	$\frac{32}{9}$	$\frac{128}{27}$
$\frac{1}{1}$	$\frac{4}{3}$	$\frac{16}{9}$	$\frac{64}{27}$	$\frac{256}{81}$

Figure 13. $P' = \left\{ \left(\frac{4}{3}\right)^a \left(\frac{3}{2}\right)^b \middle| a,b \in J \right\}$.

region is formed by two hard-natural pairs an octave apart. In the chapter called "De sedibus modorum" of the late eleventh-century treatise *Musica* of Wilhelm of Hirsau, two pairs of hexachords cover in this way the gamut of Pseudo-Odo, that is, the octave-eleventh region (Harbinson 1975, 59–60).

The hexachord itself is a region in the tonal space generated by fourths ($\frac{4}{3}$) and fifths ($\frac{3}{2}$). Letting P′ represent this space, we construct the hexachord in a manner parallel to our earlier presentation of the *heptactys* as a region in $\hat{P}$. Figure 13 is a two-dimensional view of P′.

The beginning of the sequence of P′ is shown below. Connected pairs of pure powers of $\frac{4}{3}$ and $\frac{3}{2}$ are indicated by asterisks:

	*	*	*		*	*			*	*	
1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{16}{9}$	$\frac{2}{1}$	$\frac{9}{4}$	$\frac{64}{27}$	$\frac{8}{3}$	$\frac{3}{1}$	$\frac{256}{81}$	$\frac{27}{8}$	
(0,0)	(1,0)	(0,1)	(2,0)	(1,1)	(0,2)	(3,0)	(2,1)	(1,2)	(4,0)	(0,3)	

The connected pair $(\frac{4}{3})^A$, $(\frac{3}{2})^B$ that forms the limit for the hexachord region is $(\frac{4}{3})^4$, $(\frac{3}{2})^3$, or $\frac{256}{81}$ and $\frac{27}{8}$. This pair determines the set $G_{(4,3)}$, con-

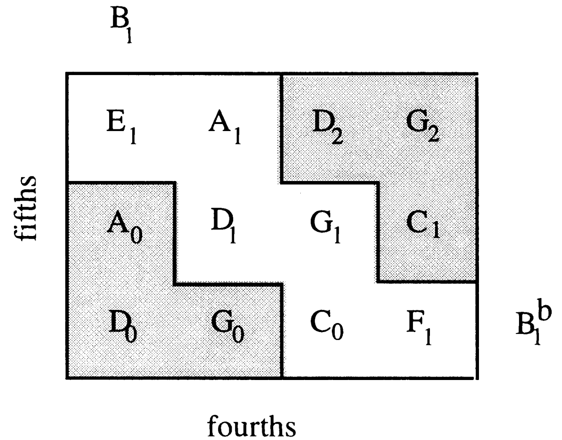


Figure 14. The hexachord as region $R_{(4,3)}$ in P′.

sisting of the factors of $(\frac{4}{3})^3(\frac{3}{2})^2 = \frac{16}{3}$. The connected part of $G_{(4,3)}$ is $S_{(4,3)}$, which contains the P′ elements from 1 to $\frac{3}{1}$.

The largest subset of complements in $S_{(4,3)}$ is $R_{(4,3)}$. This set is:

$$\left\{\frac{16}{9}, \frac{2}{1}, \frac{9}{4}, \frac{64}{27}, \frac{8}{3}, \frac{3}{1}\right\}$$

Interpreting $\frac{16}{9}$ as the note C (equivalently, interpreting 1 as D), the region is composed of the notes C D E F G A, the natural hexachord. The hexachord represents the largest diatonic span in which all fifths and fourths are perfect. Figure 14 shows the 3 horizontal perfect fourths and the 2 vertical perfect fifths, and suggests the partitioning of the hexachord into two *ut re mi* trichords at C D E and F G A.

It is also interesting to note that the determining pair $\frac{256}{81}$ and $\frac{27}{8}$ would be assigned the notes B♭ and B in this notation, suggesting the mutation into the soft hexachord through the B♭, and into the hard hexachord through the B. The determining ratio is the *apotomē,* which we met with above as the determining ratio for the diatonic octave-twelfth region in $\hat{P}$. All of the regions that we have called diatonic belong to a definable class, and it is the case that the *apotomē* is the determining ratio for all diatonic regions.

Finally, there is yet another diatonic region that is fundamental in medieval theory. This is the TST tetrachord, that is, the tetrachord of the finals, a symmetrical subset of the hexachord. This region is $R_{(3,2)}$ in the space $P^{\sim}$ generated by the minor third $\left(\frac{32}{27}\right)$ and perfect fourth $\left(\frac{4}{3}\right)$. The

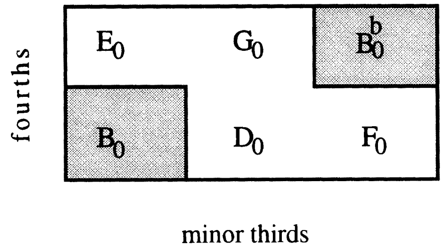


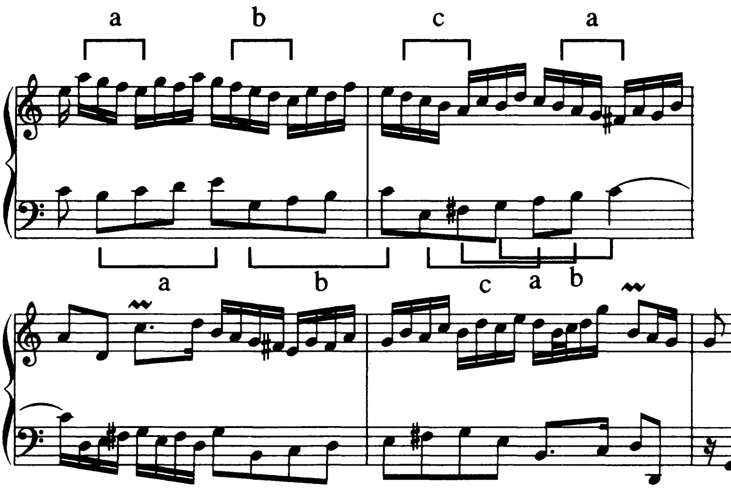
Figure 15. The tetrachord of the finals, $R_{(3,2)}$ in $P^{\sim}$.

cube of the minor third ratio and the square of the perfect fourth ratio are connected in this space. Again, the connected pair resolves to the *apotomē* as determining ratio, since $\left(\frac{4}{3}\right)^2 / \left(\frac{32}{27}\right)^3 = \frac{3^7}{2^{11}}$. Figure 15 shows the region $R_{(3,2)}$ in $P^{\sim}$, interpreted as the tetrachord of the finals. Hucbald's unerring choice of the TST tetrachord as the modular unit in his reconfiguration of the Greater Perfect System appears all the more prescient.[26]

Some of the tonal spaces we have been discussing arise in two examples by J. S. Bach. In example 1, the subject and answer from the C-Major Fugue from Book I of the *Well-Tempered Clavier,* Bach uses the maximum diatonic space that permits a real answer. The notes of the subject are all of the notes of the natural hexachord, while those of the real answer comprise the hard hexachord, at the fifth above. Thus, the entire space of subject and answer is the diatonic region determined by octave and fifth. This is a common enough procedure in fugue, with several more examples to be found in WTC alone. Here, the structure of the hexachord region itself is revealed by the shape of the subject, which emphasizes the three perfect fourths within the hexachord.

The second example involves the octave-fourth region, and is taken from the C-Major Invention. Bach has made use of the largest diatonic span in which both the octave frame and fourth frame may be preserved (example 2). The sequence beginning in m. 3 presents the three different species of filled-in descending perfect fourths: half-step on the bottom, on top, and in the middle, designated by *a, b,* and *c,* respectively. Considering the potential role of the fourth in this region, the modulation takes on a new significance. The descending perfect fourth usurps the octave as interval of periodicity, through sequence gaining inertial or conservative force to preserve its frame past the confines of C major. Once all of the species of fourths have been exhausted, the pattern begins again.

Example 1. J. S. Bach, subject of Fugue 1 from *WTC* I, BWV 846.



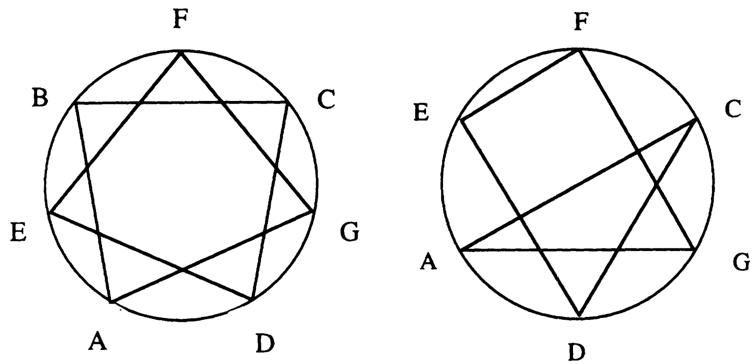
Example 2. J. S. Bach, Invention in C major, BWV 772.

V

Regions are tonal spaces generated by two coequal intervals. The perspective shifts from the bounded space of regions to the infinitely extendible space of scales, if the intervals are assigned fundamentally dissimilar roles. One interval is chosen as the modular frame, and the other interval becomes the now single generator of the notes of the scale. The scales that arise in this way are *well-formed.*

Definition: A well-formed scale is generated by a single interval, where the generating interval is spanned by a constant number of step intervals (see p. 139).

In the particular case of the diatonic set, the perfect fifth generates the scale, and all perfect fifths are indeed spanned by a constant number of



a. The diatonic set (well-formed)

b. The hexachord (not well-formed)

Figure 16. Well-formedness.

steps, to wit, 4. Also, as figure 16a makes clear, diatonic steps are always the sum of two fifths (modulo the octave). These two statements are equivalent.

Intervals may be measured generically or specifically. Generic measure counts the number of scale steps that the interval spans. In the context of well-formed scales, specific measure is expressed as an integer multiple of the generating interval—the formal fifth—reduced modulo the formal octave. For example, in the usual diatonic set, 4 fifths modulo the octave is the major third, and -3 fifths is the minor third.

The musical significance of well-formedness is principally this: in a well-formed set, the interval determined by *n* formal fifths always retains its generic identity, no matter where in the set it may be located. In particular, this means that if *n* formal fifths is a step interval somewhere in the set, it is always a step, never a skip. Contrast this with the situation in the hexachord, considered now as a scale rather than as a region. It has some of the characteristics of the diatonic set. It is generated by perfect fifths, and it is a deep scale (as defined in Gamer (1967)), but it is not a well-formed scale. In the scale C D E F G A (C), the minor third (-3 fifths) is sometimes a step (at A to C) and sometimes a skip (at D to F and E to G). This distinction was drawn by Dahlhaus (1990, 172): "The pentatonic and heptatonic scales are *systems*. In comparison, the hexachord is a mere auxiliary construction. As a system it would be self-contradictory . . . would . . . lead to the absurd consequence that the listener would have to alternate between the idea of the minor third as a 'step' and as a 'leap.'" In figure 16b, the lack of well-formedness in the hexachord is shown as a failure of symmetry preservation, as opposed to the case of the diatonic set.

Another important property of well-formed sets is a consequence of this coordination of generic and specific intervals. All non-trivial well-formed scales have Myhill's Property (MP), identified by Clough and Myerson (1985). A set has MP if every non-zero generic interval comes in two specific varieties. For example, in the usual diatonic, 2nds, 3rds, 4ths and so on come in two sizes. If MP is defined appropriately it is actually equivalent to well-formedness.[27]

We have seen how to generate, in principle, all regions. Each pair of intervals gives rise to a $\hat{P}$-type space and to a hierarchy of regions. We have asserted that each region determines two essentially different well-formed scales, with the two intervals exchanging the roles of formal octave and formal fifth. We have also asserted that from a well-formed scale (given its formal fifth and formal octave) one can recover its region. But how do we know that this correspondence holds? Part VI develops the mathematical theory of regions, and reveals the relationship between regions and well-formed scales.

VI

The mathematics required to develop a theory of regions depends upon what has been called the most enduring problem of number theory: that of diophantine equations, which the Greek mathematicians were adept at solving. Figure 17 demonstrates the application of linear diophantine equations to the region $R_{(5,3)}$. The fact that there are always A+B−1 elements in the region $R_{(A,B)}$ depends upon the solution of the equation Ay+Bx=z in non-negative integers, assuming A and B are relatively prime. The equation $Ay+Bx \equiv z_{\mathrm{mod}AB}$ in modular arithmetic always has solutions, but Ay+Bx=z in ordinary arithmetic does not always have solutions in non-negative integers x and y. If a given z does have such solutions we will say that z is *solvable*. In the set $Z_{AB}=\{0, 1, \ldots AB-1\}$, the last A+B−1 consecutive integers are solvable. As figure 17 shows, for all of the values from 8 to 14, the equation 5y+3x=z has solutions in non-negative numbers x and y. The solvable elements 8 to 14 map onto the largest connected set of complements in $G_{(5,3)}$ in $\hat{P}$, that is, the region $R_{(5,3)}$. Of the numbers less than (A−1)(B−1) exactly half are solvable. The entire set of solvable elements among the numbers 0 to 14 maps onto the set of connected elements in $G_{(5,3)}$, that is, $S_{(5,3)}$.

In the remainder of this section we will operate in the most general setting possible: $P_{\nu\mu} = \{\nu^a\mu^b | a,b \in J = \{0,1,2,\ldots\}\}$ subject only to the proviso that neither ν nor μ be an integral power of the other.

The sets $G_{(A,B)}$, $Z_{AB} = \{0, 1, \ldots AB-1\}$ and $Z_A \times Z_B = \{(x,y) | 0 \le x <$

Z_{15}	$\overset{\sigma}{\leftarrow}$	$Z_5 \times Z_3$	$\overset{\rho}{\rightarrow}$	$G_{(5,3)}$
0*		(0,0)		1
1		(2,2)		36
2		(4,1)		48
3*		(1,0)		2
4		(3,2)		72
5*		(0,1)		3
6*		(2,0)		4
7		(4,2)		144
8*		(1,1)		6
9*		(3,0)		8
10*		(0,2)		9
11*		(2,1)		12
12*		(4,0)		16
13*		(1,2)		18
14*		(3,1)		24

$\rho: Z_5 \times Z_3 \rightarrow G_{(5,3)}: (x,y) \rightarrow 2^x 3^y$ *Solvable elements

$\sigma: Z_5 \times Z_3 \rightarrow Z_{15}: (x,y) \rightarrow (5y+3x)_{\text{mod } 15}$ $z=5y+3x$ for $z=8, 9, 10, 11, 12, 13, 14$ (also for $z=0, 3, 5, 6$)

$\tau = \rho\sigma^{-1}: Z_{15} \rightarrow G_{(5,3)}$

Figure 17. The mappings underlying $R_{(5,3)}$ in $\hat{P}$.

$A, 0 \leq y < B\}$ may all be considered to be isomorphic groups, when provided with the appropriate operations. For example, $G_{(A,B)}$ is a group with the operation defined as multiplication, where the exponent on ν is reduced modulo A, and the exponent on μ is reduced modulo B. The other two sets become additive groups with the additions defined in the natural way: addition modulo AB for Z_{AB} and component-wise addition modulo A and modulo B for $Z_A \times Z_B$. It is easy to show that the connectedness condition on ν^A and μ^B implies that A and B are relatively prime. It follows that both Z_{AB} and $Z_A \times Z_B$, with the appropriate additions, represent the cyclic group of order AB.[28] The isomorphisms among these groups are ρ: $Z_A \times Z_B \rightarrow G_{(A,B)}$:$(a,b) \rightarrow \nu^a\mu^b$ and σ: $Z_A \times Z_B \rightarrow Z_{AB}$:$(x,y) \rightarrow (Ay + Bx)_{\text{mod}AB}$. The one-to-one correspondence set up by σ is crucial: every element of Z_{AB} may be expressed uniquely in the form $(Ay+Bx)_{\text{mod}AB}$ for some (x,y) in $Z_A \times Z_B$, and we can define an inverse mapping σ^{-1}:$Z_{AB} \rightarrow Z_A \times Z_B$. Finally, we can define the isomorphism $\tau = \rho\sigma^{-1}$ which maps Z_{AB} to $G_{(A,B)}$.

We are now in a position to give a proof of what we will call the theorem of Lucas (1891, 3:480–83). We give our own proof, because it affords more insight into the question at hand than does that by Lucas.

Theorem of Lucas: Assume A and B are relatively prime with A>B>1. Then the A+B–1 integers from AB–(A+B)+1 to AB–1 are all solvable in positive integers x,y where $0 \leq x < A$ and $0 \leq y < B$. Of the remaining elements, half of them, numbering (A–1)(B–1)/2, are solvable, and the other (A–1)(B–1)/2 are not solvable. The element AB–(A+B) is not solvable, so AB–(A+B) is the smallest integer above which every integer is solvable. (It is easy to show that $z \geq AB$ is solvable. We omit this demonstration, and confine ourselves to the set Z_{AB}.)

If Ay+Bx=z with strict equality for z in Z_{AB} then z is solvable, by definition. If $z \in Z_{AB}$ is not solvable, then it must be that AB+z = Ay+Bx for some pair (x,y) in $Z_A \times Z_B$, and conversely, if Ay+Bx = AB+z for $z \in Z_{AB}$, then z is not solvable, since σ is one-to-one and onto Z_{AB}. Then consider the ordered pair (A–1, B–1). A(B–1)+B(A–1) = 2AB–(A+B) = AB+(AB–(A+B)), and since A>B>1, AB–(A+B)>0, so AB–(A+B) is not solvable. But this is the highest value the form Ay + Bx can take on (in terms of strict equality) when $0 \leq x < A$ and $0 \leq y < B$, i.e., when $(x,y) \in Z_A \times Z_B$, and since σ is onto Z_{AB}, the elements AB–(A+B)+1 to AB–1 are solvable for (x,y) in $Z_A \times Z_B$. That leaves the elements from 0 to AB–(A+B). There are (A–1)(B–1) of them since AB–(A+B)+1 = (A–1)(B–1).

Suppose an element (x′,y′) of $Z_A \times Z_B$ is a solution for n′, where $0 \leq n' <$ (A–1)(B–1). Then the $Z_A \times Z_B$ element ((A–1)–x′, (B–1)–y′) is not a solution for any z in Z_{AB}: A(B–1–y′)+B(A–1–x′) = 2AB–(A+B)–(Ay′+Bx′) = 2AB–(A+B)–n′ ≥ AB, since n′ ≤ AB–(A+B). Conversely, suppose (x″,y″) is not a solution for any z in Z_{AB}. Then Ay″+Bx″ ≥ AB. Then ((A–1)–x″, (B–1)–y″) is a solution for some element n″ such that $0 \leq n'' <$ (A–1)(B–1) = AB–(A+B)+1: A((B–1)–y″)+B((A–1)–x″) = 2AB–(A+B)–(Ay″+Bx″) ≤ AB–(A+B), since Ay″+Bx″ ≥ AB. Therefore (A–1)(B–1)/2 of the elements of Z_{AB} from 0 to AB–(A+B) are solvable, (A–1)(B–1)/2 are not.

The element of the form ((A–1)–x, (B–1)–y) used in the above proof is reminiscent of the notion of complementarity that was defined in $G_{(A,B)}$. The property of complementarity is a group property; that is, it is defined in terms of the group operation. In $G_{(A,B)}$, g and g′ were defined to be complements of each other if $g'g=\nu^{A-1}\mu^{B-1}$. Although the group operation in $G_{(A,B)}$ is slightly different from ordinary multiplication, clearly it remains true that if $g=\nu^a\mu^b$, $g'=\nu^{(A-1)-a}\mu^{(B-1)-b}$ is the complement of g. Then we can define complements in $Z_A \times Z_B$ and Z_{AB} so that complementarity is preserved by the various isomorphisms: In $Z_A \times Z_B$, given x=(a,b), let the complement of x be x′=((A–1)–a,(B–1)–b). Then it is true that $\rho(x') = (\rho(x))'$. Since $\sigma((A-1,B-1)) = AB-(A+B)$, as we saw above, if $z \in Z_{AB}$ we must have $z' \equiv (AB-(A+B)-z)_{\mathrm{mod}AB}$. Thus if $z \le AB-(A+B)$, $z' \le AB-(A+B)$, and if $z > AB-(A+B)$, then $z' > AB-(A+B)$, ordering the elements of Z_{AB} as ordinary integers.

Looking back at the proof of the theorem, evidently half of the elements of Z_{AB} from 0 to AB–(A+B) are solvable, and their complements are not. Furthermore, if, and only if, an element and its complement are both solvable, they both belong to the *abstract region* in Z_{AB} from AB–(A+B)+1 to AB–1. It will turn out that τ maps this set onto the region $R_{(A,B)}$.

Recalling our definition of a set of complements, we can see that both of the subsets {0, 1, . . . AB–(A+B)} and {AB–(A+B)+1, . . . AB–1} are sets of complements in Z_{AB}. Also, both subsets are connected, if we carry over the definition of connectedness to Z_{AB} and consider elements 0 to AB–1 to be ordered as ordinary integers. However, the abstract region has the additional property that all of its elements are solvable. Thus it will turn out that the region, the largest connected set of complements in $G_{(A,B)}$, corresponds to the largest connected set of *solvable* complements in Z_{AB}.

To carry this project out and to evaluate the region requires that the solutions $(x,y) \in Z_A \times Z_B$ to the congruences $z \equiv Ay+Bx$ be determined. Then it must be shown that if the mapping $(x,y) \rightarrow Ay+Bx$ is increasing (as we know it is on the abstract region), so is the mapping $(x,y) \rightarrow \nu^x\mu^y$. Both these determinations depend upon the properties of continued fractions, and we omit the details. It is not difficult to show that the values $(a_k(-1)^k(n-B))_{\mathrm{mod}A}$ and $(b_k(-1)^k(A-n))_{\mathrm{mod}B}$, where a_k/b_k is the penultimate convergent in the continued fraction representing A/B, are solutions x and y, respectively, in non-negative integers to the equation Ay+Bx= AB–(A+B)+n for n from 1 to A+B–1. Again omitting the details, if and only if ν^A and μ^B are connected, then A/B belongs to the series of rational approximations of the number $\log_\nu\mu$ generated by its continued fraction, its convergents and semi-convergents.[29] One can verify on this basis, for example, that the monstrously large $\hat{P}$ numbers 2^{50508} and 3^{31867} are connected elements. The number of regions in $\hat{P}$ is thus seen to be infinite, since the value approximated is irrational.

The means are now at hand to characterize regions otherwise than by the initial definition, and a way of computing them can be given. The formula which produces the $P_{\nu\mu}$ numbers for the region associated with the ordered pair (A,B) is shown below:

$R_{(A,B)} = \left\{ \nu^{a_k(-1)^k(n-B)_{\mathrm{mod}A}} \mu^{b_k(-1)^k(A-n)_{\mathrm{mod}B}} \mid 1 \le n \le A + B - 1 \right\}$, where $\frac{a_k}{b_k}$ is the full convergent immediately preceding $\frac{A}{B}$ in the continued fraction for $\log_\nu \mu$.

The above formula tells us that the region is an ordered set: its elements increase as n increase, from 1 to A+B–1. Hence there are A+B–1 elements in the region. The derivation of well-formed scales from this space is achieved by first treating one of the generators as the formal octave. Without loss of generality, we assume that ν is the formal octave and μ is the formal fifth.[30] The interval μ within the region defines B pitch classes, which we can represent as $\mu^0, \mu^1, \mu^2, \ldots \mu^{B-1}$. We are labeling the pitch classes in the order in which they are generated. As one is only interested in pitch class in this context, it makes sense to disregard the exponents on ν and let those on μ "hop down," yielding the set of pitch classes {0, 1, . . . B–1}. We know that for n ranging from 1 to A+B–1, elements of the form $\left(b_k(-1)^k(A-n)\right)_{\mathrm{mod}B}$ are pitch classes in scale order. Let n range from A to A+B–1. Then the value (A–n) ranges from 0 to B–1. Resetting this expression to n, the mapping $n \to [b_k(-1)^k n]_{\mathrm{mod}B}$ arranges the pitch classes of the scale in order.[31]

Thus the inverse mapping $n \to [b_k^{-1}(-1)^k n]_{\mathrm{mod}B}$ returns pitch classes from scale order to formal fifths order; that is, to the order in which the pitch classes are generated. It follows that every formal fifth is spanned by $[b_k^{-1}(-1)^k]_{\mathrm{mod}B}$ scale steps. This recaptures the definition of well-formed scales which was given on p. 133.

Each pair, ν and μ as defined, gives rise to a distinct hierarchy of regions, ruled by the continued fraction of $\log_\nu \mu$. Each pair also gives rise to a hierarchy of well-formed scales in which ν is taken as the equivalence relation and μ determines pitch classes. Assuming $\mu' > \mu$, the spaces $P_{\nu\mu}$ and $P_{\nu\mu'}$ produce the same hierarchy of well-formed scales if $\mu\nu^k = \mu'$ for some positive integer k. Thus, assuming octave equivalence on both, the octave-fifth region $R_{(4,7)}$ in P* and the octave-twelfth region $R_{(11,7)}$ in $\hat{P}$ both produce the seven-note diatonic scales.

VII

The theory of regions provides a perspective from which to consider various issues in medieval theory. It prompts a reconsideration of the often-discussed relationships among tetrachords, hexachords, and gamuts. As we have seen, many of the tonal spaces employed by medieval theorists are regions, which have a structure nicely suited to a number of concerns. While there is no evidence that writers of the ninth to eleventh centuries possessed the general notion of a region, they were better situated in some respects than we are today to intuit the importance of the various diatonic regions. Our twentieth-century assumption of an embedding of the diatonic in an equal-tempered chromatic sometimes blinds us to transformational features of the system. For example, the notion of affinity has lost its significance for us: if we want the same tonal environment around a different note, we simply transpose. The writers of the early medieval treatises were sensitive to the affinities in part because they were inclined to treat pitch class, as we call it today, more flexibly. Depending upon the theoretical context, medieval authors might invoke various intervals of periodicity and consequent notions of pitch class. For example they might employ the usual seven letter names, A–G; or the four dasian qualities, Archoos, Deuteros, Tritos, Tetrardos; or even the three syllables, ut, re, mi; or they might choose not to consider pitch class at all, as in some of the alphabetic gamuts (e.g., A–P) found in Boethius and in the *Enchiriadis* treatises.

In practical terms, the problem of maintaining the perfection of two intervals simultaneously in parallel organum is solved within the all-embracing symmetry of the region. The finitude of regions was congenial to the medieval mind as well, consistent with the notions of gamut and ambitus. The identification of the Limited with orderliness and the Good has a philosophical basis in a tradition that leads from Pythagoras through Aristotle:

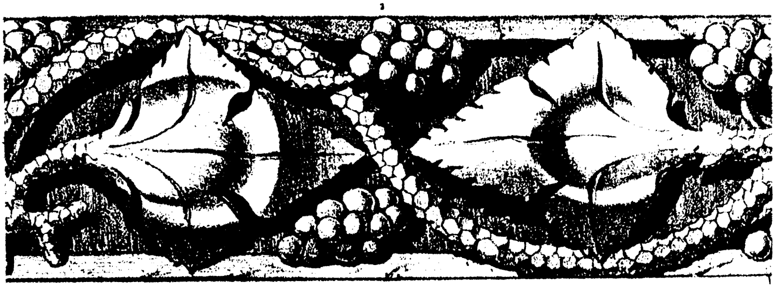
> Evil partakes of the nature of the Unlimited, Good of the Limited, as the Pythagoreans conjectured.[32]

Regions and scales are fundamentally different in conception. One thinks of scales as being, at least potentially, infinitely repeating in unbounded pitch space; hence the information they bear may be encapsulated in a set of pitch classes. Regions, on the other hand, are necessarily bounded.

Consider, for example, the hexachord, ubiquitous among the several diatonic regions we have identified. While its status as a scale is questionable, as a region it is the tonal space in which the perfection of fourth and fifth is maintained. Dahlhaus astutely pointed out its failure as a scale, but while Dahlhaus, Handschin and others have recognized the usefulness of hexachords in combination, they have disparaged it, in Dahlhaus's terms, as "a mere auxiliary construction." In itself, it is more properly viewed as a self-contained tonal matrix, containing as it does the three species of fourths, as John of Afflighem points out (Babb 1978, 111), as well as the two species of fifths that do not enclose the tritone.

This study has also shed light upon the nature of diatonicism in a broad sense. Most of the regions that we have discussed have belonged to a family of diatonic regions. The interval of the *apotomē* is the constant here, acting as the limit for all of them, including the tetrachord of the finals and the hexachord. Those who call the Guidonian hexachord *the* diatonic hexachord are, then, theoretically justified. (The notion of the *apotomē* itself has been generalized by Regener (1973), Blackwood (1985), and Carey and Clampitt (1989).)

Every region contains within itself two possible ways of spreading out into pitch space with one or the other of its generators chosen to be the interval of periodicity. The result of either of these choices is a well-formed scale. Just as the hexachords were for Wilhelm of Hirsau the *sedes troporum,* the seats of the modes, so are the well-formed scales enthroned in the regions.



NOTES

1. The diagram and notes on its transmission are found in Schmid (1981, 111). *Musica* and *Scolica enchiriadis* are usually treated as a pair of treatises. Whether they are by the same author and their relative and absolute chronology are open questions. Phillips, in her 1984 dissertation, discusses these matters, as well as the curious titles of these works, and places them in their historical context. Erickson (1995) is the first translation based on Schmid's edition.
2. The Pythagorean tradition transmitted by Boethius accepted five consonances: *diatessaron* (fourth), *diapente* (fifth), *diapason* (octave), *diapason cum diapente* (twelfth), and *disdiapason* (double octave). Boethius acknowledged that Ptolemy accepted *six* consonances, with the addition of *diapason cum diatessaron* (eleventh). Boethius's citation of Ptolemy is noted explicitly in *Musica enchiriadis* (Schmid 1981, 44). Boethius's contemporary Cassiodorus followed Ptolemy in the matter of the consonances, with his dictum "Symphoniae autem sex." (Mynors 1937, 145; also Strunk 1950, 89.) Other ninth-century treatises—notably, *Musica disciplina* of Aurelian of Réôme and *De institutione musica* of Hucbald—follow Cassiodorus and use similar language. See Phillips 1990, 108–18, and 1984, chapter 6, especially pp. 240–56.
3. The diagram is found in Steglich's edition (1911, 33). The *Quaestiones in musica* is probably from the late eleventh or early twelfth century. It is attributed by Steglich to Rudolph of St. Trond (1070–1138). Smits van Waesberghe (1949, 98–101), following a suggestion by P. Cölestin Vivell, argues that the author was Franco of Liège (1047–1083?).
4. The choice of pitch class F for the origin is simply for notational convenience. In this system, new octaves begin on F.
5. "Et verité est que ceste figure est plainne de tres grans misteres et peust l'en prendre en elle consideracions tres excellentement belles et merveilleuses, et contient en vertu toute musique speculative." Oresme, *Livre du ciel,* translation of Aristotle's *De cælo,* book 2, chapter 18, translation by Michael Long, cited in Long 1990. We are grateful to Professor Long for sharing with us a copy of his paper, and for information about Oresme.
6. Either $2^A > 3^B$ or $3^B > 2^A$. Whichever of $\frac{2^A}{3^B}$ or $\frac{3^B}{2^A}$ is greater than unity will be referred to as the determining ratio.
7. A version of $R_{(3,2)}$ appears in Zarlino's *Istitutioni Harmoniche* (1558, 17) in a passage concerning *musica humana.* Although the values are tripled, (to 6, 9, 12, 18), the accompanying diagram clearly shows the symmetry characteristic of any region.
8. According to Markovits (1977, 30–31, 44) this is the only scale in early medieval literature formed according to a monochord division based on C. Hucbald discusses a scale associated with water organs that begins on C. (Babb 1978, 24–25.)
9. "Quoniam autem C litteram, que apud veteres tertia habetur, pro certo comperimus ex numerorum ratione fore primam." (Coussemaker [1867] 1963, 2:85.) *De modorum formulis,* treated as a work of Guido of Arezzo by Coussemaker, is listed as *opus dubium* by Smits van Waesberghe (1953, 143). Handschin considers it definitely not by Guido, but by someone closely associated with him (1948, 337).
10. The possibility exists that the author of *De modorum formulis* is making a reference to a Middle Platonic tradition found in Timaios of Locri and Eudorus (Tobin

1985, 6) where the proportions of the world soul are established upon the number 384.

11. The determining ratios gradually approach 1, i.e., unison, and from $3^{12}/2^{19}$ on would be called "commas." Crocker (1963, 197) drew attention to the sense in which the tone, the Pythagorean minor third, *limma,* and *apotomē* may all be considered commas.
12. One can describe all regions by a phrase like "the largest diatonic span in which octaves and fifths may be maintained." This is an easy test for a region, but it does not allow one to construct regions. As we will see below, each hierarchy of regions gives rise to two hierarchies of well-formed scales; given a well-formed scale and its generator (e.g., the diatonic scale and fifth), one can recover a region.
13. Our conception of periodicity at the fifth is supported by Phillips's (1984, 167–68) interpretation of the relevant passages. "[T]he interval of a fifth is the basis of this system. A diapason system includes seven discrete pitches with correspondence at the octave; the dasia pitch content contains four discrete pitches with correspondence at the fifth. To identify the tetrachord as the fundamental interval of the system would be equivalent to describing the seventh as the fundamental interval of the diapason system." On the theoretically extensible nature of the system: "The total pitch content is described by the author in *Musica,* chapter 1, as infinite and moving four by four through musical space: 'et intendendo et remittendo naturaliter ordo continuatur, ut semper quatuor et quatuor eiusdem conditionis sese consequantur.'"
14. Augmented octaves do not occur in the organum practice described in the *Enchiriadis* treatises. In *Musica enchiriadis* this is attributed to a "marvelous change" in the upper voice or voices ("mirabili mutatione," Schmid 1981, 34), and in the concluding pages of *Scolica enchiriadis,* the explanation is presented that a voice eight steps from a given voice is a doubling. As such, it respects the octave as the greatest consonance and follows the pattern of tones and semitones of the given voice, rather than following the path of an independent voice in its location on the dasian scale. "The voice . . . does not follow the order of its own place, but the order of the voice to which it consonantly responds." Schmid, 155: "vox . . . non sequitur sui loci ordinem, sed eius, cui consonanter respondet."
15. Throughout section IV we will refer to notes by the letters used in most medieval treatises from about 1000 on. Pseudo-Odo in *Dialogus de musica* used Roman letters for the two octaves of the Greater Perfect System which Hucbald had reorganized around the TST tetrachord of the finals. The *Dialogus* author also added the low Γ. Guido of Arezzo extended the gamut with the notes above *aa* to form another tetrachord. With the advent of the solmization system, *e la* or *ee* came into use. Thus the completed gamut:

$$\Gamma\ A\ B\ C\ D\ E\ F\ G\ a\ b\natural\ c\ d\ e\ f\ g\,{}^{a\ b\natural\ c\ d\ e}_{a\ b\natural\ c\ d\ e}$$

16. The availability of B♭ (on the authority of the Greek *synemmenon* tradition) meant that sometimes portions of chant could fit precisely at these other locations with the use of this note; occasionally they could *only* fit at these locations, if E♭s or F♯s were to be avoided.
17. Pesce (1987) surveys the treatment of the affinities from Hucbald and the *Enchiriadis* treatises to Glarean. The term *affinitas* was introduced by Guido of Arezzo.

18. See Crocker (1972) for a discussion of the advantages of a small scalar module, and particularly the importance of the tetrachord of the finals. The use of the tetrachord of the finals as the basic unit of construction is first found in the West in the *Enchiriadis* treatises and in Hucbald.
19. Guido does not name the treatise at this point, but he is clearly criticizing the dasian system (Babb 1978, 62). Hermannus mentions *Musica enchiriadis* in his somewhat differently formulated criticism (Ellinwood 1936, 23–24), and his scathing comments later (56) refer explicitly to the dasian scale.
20. Guido acknowledges seven pitch classes by quoting Vergil approvingly: ". . . septem discrimina vocum." (Babb 1978, 62.)
21. Crocker (1972) presents a pertinent treatment of Hermannus's theory. Both influenced by and critical of Handschin, Crocker places Hermannus in the *Enchiriadis* tradition, much as Handschin did Guido.
22. Different versions of this diagram are found in the work of later theorists, John of Afflighem (Smits van Waesberghe 1950, 75 and Babb 1978, 114) and Jacques of Liège (Bragard 1973, 77). John's diagram places the G a ♮ trichord on the left side and the F G a trichord on the right, and eliminates the indications of direction on the fourths and fifths. Jacques provides an extended commentary. His diagram improves upon Guido's by taking John's arrangement of its elements, and annotating it with Hermannus's enumeration of the species of the fifths and fourths, and his categories *naturalis* and *formalis.* The late eleventh-century *Commentarius anonymous* also treats the diagram and the accompanying passage at length (Smits van Waesberghe 1957, 126–30). The diagram is discussed in Crocker (1972, 24–25), Handschin (1948, 324–25), and Oesch (see note 23).
23. Oesch, in his *MGG* article on the hexachord, transforms the diagram in just this way. As we show, this presentation is closer to Guido's text. Oesch, col. 351: "The trichord c-d-e thereby assumes a central position, since above it f-g-a is adjoined and below it G-A-B, such that both hexachords, on G and on c, complement each other within the heptatonic." ("Das trichord c-d-e nimmt deshalb eine zentrale Stellung ein, weil sich ihm nach oben f-g-a und nach unten G-A-H anschließt, so daß sich die beiden Hexachorde auf G und c zur Heptatonik ergänzen.")
24. Trans. Babb (1978, 65). "Omnes itaque modi distinctionesque modorum his tribus aptantur vocibus. . . . Ergo omnes aliae voces cum his aliquam habent concordiam, seu in depositione seu in elevatione." (Smits van Waesberghe 1955, 127–28.) As Smits van Waesberghe's notes on the transmission of this text show, .C.D.E. is placed after the word "vocibus" in sixteen of the sources, and appears as a superscription over *vocibus* or *his* in three other sources. (1955, 127.) The distinctions of the modes are the plagal forms, as his anonymous commentator of the late-eleventh century explains. The commentator also explicates *his tribus vocibus* by .C.D.E. (Smits van Waesberghe 1957, 126.)
25. Coussemaker ([1867] 1963, 4:225). See Dahlhaus (1955, 290), and Dahlhaus (1990, 170–71).
26. An analogous complex of pentatonic regions exists, where the *limma* $\left(\frac{256}{243}\right)$ is the common determining ratio. The region corresponding to the hexachord is the tetrachord (e.g., CDFG, GACD), the region $R_{(3,2)}$ in P'. Two overlapping tetrachords cover either the pentatonic octave-fifth region or octave-fourth region, depending on their order. The pentatonic octave-fourth region itself spans an octave. The region in $P^{\sim}$ comparable to the tetrachord of the finals is $R_{(2,1)}$, which

reduces to the minor third, again a symmetrical subset of the pentatonic tetrachordal region. See Gauldin (1983, 41–46) for a discussion of pentatonic systems and their tetrachordal organization.

27. See Carey and Clampitt (1989, 202). Strictly speaking, this equivalence holds for what we called non-degenerate well-formed scales, that is, well-formed scales with two different step sizes. All equal-interval scales trivially satisfy the definition of well-formedness. These are the degenerate well-formed scales (1989, 200). See also Chalmers (1975) for a discussion of Moments of Symmetry (MOS), a class of scales first defined by Ervin Wilson in 1964 in a private communication to Dr. Chalmers. MOS, MP scales, and well-formed scales are all essentially equivalent concepts.
28. Balzano (1980) appeals to this isomorphism in the special case A=3, B=4 (more generally, A=k, B=k+1) to show how the usual chromatic may be considered to be the direct product of a diminished seventh chord and an augmented triad. He remarks that the usual diatonic is a compact region on the lattice corresponding to this direct product space. This is not a region in our sense, but it is the reflected image of a chromatic cluster, as Balzano remarks, and the chromatic cluster (more precisely, the chromatic hexachord) is a region in the space generated by $2^{1/3}$ and $2^{1/4}$, equal-tempered major and minor thirds.
29. For more information on continued fractions, see Carey and Clampitt 1989, 197–200.
30. The expression μ is more general here than the analogous expression in our 1989 article, and ν, replacing 2, completely generalizes the notion of formal octave.
31. See Carey and Clampitt (1989, 199), and "Errata," *Music Theory Spectrum* 12/1 (1990): 171.
32. Aristotle, Nicomachean Ethics. i–6; 1096 b 5. Cited in Guthrie (1988, 304).

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