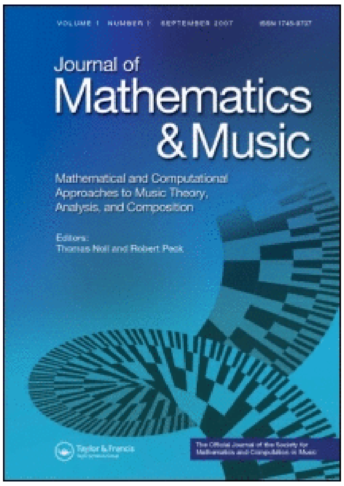

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Generalized diatonic scales

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The article studies the problem of generalizing the concept of 'diatonic scale' for a given ambient chromatic of N tones: 'Which subset $A \subset \mathbb{Z}_N$ shall be considered as a *generalized diatonic scale*?' Each generic type of well-formed scale has exactly two specific manifestations in chromatic universes, which are large ME^*-scales, i.e. which are maximally even non-degenerate well-formed scales, whose cardinality exceeds half of the chromatic cardinality. A qualitative distinction between these two large ME^*-scales of the same type can be comfortably made on the basis of the shuffled Stern–Brocot tree, which is introduced in Section 2. The shuffled Stern–Brocot tree represents the same abstract binary tree as the traditional Stern–Brocot tree, but has a different planar arrangement.

Candidates and final choices for generalized diatonic scales are studied in Section 3. Candidates are those large ME^*-scales $A \subset \mathbb{Z}_N$ which are *tightly generated* by a prime residue class m mod N. According to this property there is no non-degenerate m-generated well-formed scale W properly included between the translation $T_1(A^c)$ of the complement of A and A itself. This property is equivalent to the fact that the associated ratio m/N is a chromatic number, i.e. that its penultimate predecessor on the Stern–Brocot tree is a convergent for m/N, which again is equivalent to the fact that the ratio m/N corresponds to a right branching on the shuffled Stern–Brocot tree. Tight generatedness of a large ME^*-scale is also equivalent to the fact that the small-scale step of chromatic size 1 is at the same time also the rarer step, while the large step of size 2 is at the same time also the more frequent step. A generalized diatonic scale A minimizes the cardinality difference between $|A|$ and $|A^c|$ among all tightly generated large ME^*-scales in $\mathbb{Z}_N$. For N divisible by 4, this definition reproduces exactly the family of hyperdiatonic scales, as studied by Agmon (*Journal of Music Theory,* **33**(1), 1–25, 1989) and Clough and Douthett (*Journal of Music Theory* **35**, 93–173, 1991). For N odd we exclude the trivial case of 2-generated scales and obtain a rich inventory of generalized diatonic scales, such as the 7-generated 11-tone scale in $\mathbb{Z}_{19}$. An interesting point for N odd is also the generalization of the tritone as a minimal limited transposition subset.

We show that Ivan Wyschnegradsky has already done pioneering work on this subject (1916). His 11-generated 13-tone scale in the quarter-tone chromatic $\mathbb{Z}_{24}$ is a hyperdiatonic scale in the sense of Agmon (1989) and Clough and Douthett (1991) as above, and his argumentation in favour of this scale anticipates central points of the discussion in this article.

Keywords: generalized diatonic scales; diatonicism; well-formed scales; diatonicized chromaticism; N-Tone equal temperament; Stern-Brocot tree; Wyschnegradsky

1. Introduction

Is there an analogue to the standard diatonic scale in a given chromatic system with N notes rather than 12? The present article addresses this question on the basis of mathematical and

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music-theoretical arguments and explores cross-connections to related work by other authors. A historically and theoretically interesting source to this question is the preface of Ivan Wyschnegradsky's *Preludes, opus 22*, composed in 1916. In Section 4 we will recapitulate some of his ideas, which anticipate music-theoretical insights, which were realized and elaborated in academic discussion several decades later, especially in the work of Eytan Agmon [1] and in the joint article by John Clough and Jack Douthett [2]. The present article reviews these ideas against the background of the theory of well-formed scales [3] and recent transformational accounts.

In music theory, a (standard) diatonic scale is a seven-note scale comprising two types of interval: whole steps and half steps, with the half steps maximally separated. In addition diatonic scales are obtained from a chain of six successive fifths rescaled in an octave. In the last few decades, music theorists have studied these properties. John Clough and Gerald Meyerson [4] introduced the term *Myhill's property* to characterize the situation in a scale when each generic interval comes in exactly two sizes. They studied a class of scales which they called *generalized circles of fifths* (see also [5]), The term *maximally even* (ME) was coined by John Clough and Jack Douthett [2] to refer to scales that are subsets of a chromatic scale and in a well-defined sense are spread out as much as possible within that chromatic. The authors observe that not all maximally even sets have Myhill's property, and conversely, sets with Myhill's property are not always maximally even. Those ME scales that do have Myhill's Property will be denoted by the expression ME^*. The asterisk stands for a particular arithmetic relation between the cardinality d of the scale and the cardinality N of the ambient chromatic, namely when they are coprime, such as 7 and 12. Norman Carey and David Clampitt [3,6–11] studied the concept of *well-formedness*, which is one of the most important for investigations in diatonicity. They introduced *self-similarity* [6] and showed that any scale with real pitches modulo octave (*mod* 1) has Myhill's property if and only if it is a non-degenerate well-formed scale. A central insight of their theory is the existence of a linear automorphism of the generic scale which translates the counting of steps into the counting of generator intervals and vice versa.[1] Vittorio Cafagna [14] and Domenico Vicinanza [15] connected this translation with the concept of *winding numbers*. Thomas Noll [16] linked symplectic transformations of integer pairs with continued fractions and developed in [17] a transformational model for *scale states* based on the Stern–Brocot tree (see also [12,13]).

The initial context for the determination of a generalized diatonic in a given N-chromatic universe is provided in terms of the *large ME* scales* (see Section 3 for details). With the attribute 'large' we wish to select those ME scales A, whose cardinality is larger than half of the chromatic cardinality N: $|A| > N/2$. This implies that the scale steps have the specific chromatic sizes 1 and 2. These scales have a generalized circle of fifths, whose generator ("the generalized fifth") shall be denoted by m. Under this perspective the scales are non-degenerate well-formed scales in the broader sense of [3] with the extra property that they are maximally even and large. The small complements of such scales have no adjacent pitch classes. Conversely, if the complement A^c of any non-degenerate well-formed scale $A \subset \mathbb{Z}_N$ has no adjacent pitch classes, one may already conclude that A is maximally even with specific step sizes 1 and 2 (see Proposition 3.1). Thus, the task in searching for a unique generalized diatonic scale (up to transposition) in a given chromatic universe consists in making a choice among the large ME* scales. On the one hand, we recover Eytan Agmon's concept of a diatonic scale, for which Clough and Douthett ([2], p. 123) coined the term *hyperdiatonic scale*. Such a scale is defined, whenever N is divisible by 4. The odd diatonic cardinality $|A|$ and the chromatic cardinality N satisfy the equation $N = 2|A| - 2$. Ivan Wyschnegradsky's 13-tone diatonic in a 24-quarter-tone system is an instance of this particular type (see Section 4). On the other hand, the domain large ME* scales can be reduced by superimposing a *tightness condition* for the inclusion of a translation of A^c within A, which is equivalent to the condition for the ratio m/N, to be a *chromatic number*. The latter condition has been proposed by Thomas Noll in [16] as a component in his definition of *pseudo-diatonic system*. In this investigation Noll combines insights by Gerard Balzano [18] and Eytan

Agmon [19] and connects them with a generalized view on Jean Philippe Rameau's concept of double employment. It seems, however, that he was not aware about the tightness condition as a consequence of his proposal.

The concept of *chromatic numbers* enters scale theory via the mathematical connections among well-formed scales, the Stern–Brocot tree and a Cayley graph of the matrix monoid $SL_2(\mathbb{N})$. Details follow in Section 2, where we depart from the classical Stern–Brocot tree in favour of a shuffled alternative. An inspection of the standard diatonic scale shall first illustrate the connection between the diatonic as a fifth-generated well-formed scale, an associated ratio 4/7 and a matrix

$$\mathcal{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \in SL_2(\mathbb{N}).$$

The numerator 4 represents the number of diatonic steps in the generating fifth, while the denominator 7 represents the number of diatonic steps in the octave. The rows (3, 1) and (5, 2) of the matrix $\mathcal{A}$ provide a refinement of information about the multiplicities of whole steps and half steps: The generating fifth consists of 3 whole steps and 1 half step, while the octave consists of 5 whole steps and 2 half steps.[2] If x and y represent the sizes of whole step and half step, respectively, we may express the above information more elegantly as a linear equation

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g \\ 1 \end{pmatrix} \quad \text{with solution} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \cdot \begin{pmatrix} g \\ 1 \end{pmatrix} = \begin{pmatrix} 2g-1 \\ -5g+3 \end{pmatrix}$$

The transformation of the initial vector $(g, 1)^{\mathrm{T}}$ into the step vector $(x, y)^{\mathrm{T}}$ entails an immediate music-theoretical question. The two step types are distinguished by two salient properties: The whole step is *larger* and *more frequent* (occurring 5 times), while the half step is *smaller* and *rarer* (occurring 2 times). How is this related to the above transformation, which formally characterizes the whole step as a substitution for the fifth and the half step as a substitution for the octave? Section 2 answers this question in terms of an alternative "shuffled" tree-structure of the Stern–Brocot tree and associated transformations. This modification guarantees that the solution coordinate x always represents the rare step, while y always represents the frequent step. Furthermore, the shuffled tree-structure puts the chromatic numbers in obvious positions.

2. The shuffled Stern–Brocot tree

The Cayley graph of the monoid $SL_2(\mathbb{N}) = \langle \mathcal{T}, \mathcal{S} \rangle$, freely generated by the two matrices

$$\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is classically associated with the Stern–Brocot tree (see [20, 21]), a binary tree of positive rational numbers obtained by iteratively inserting the mediant of two adjacent fractions, introduced by John Farey [22].

Each matrix

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{N})$$

is associated with the fraction

$$\frac{a+b}{c+d}.$$

This ratio is the value $\mu_{\mathcal{A}}(1)$ of the Moebius transform associated with $\mathcal{A}$, evaluated at the argument $z = 1$. Recall that $\mu_{\mathcal{A}} : \mathbb{C}^* \to \mathbb{C}^*$ is a fractional linear transformation, given by the formula

$$\mu_{\mathcal{A}}(z) := \frac{az + b}{cz + d}.$$

By the *shuffled Stern–Brocot tree* we mean a different planar arrangement of the same abstract binary tree. This means that the two successors n_1 and n_2 of a given node n are the same as in the original tree, but there is a different criterion for their placement to the left or to the right below n: the uppermost three nodes $1/1$, $1/2$ and $2/1$ are posed as in the original tree. For any node that is not the root $1/1$, we may then determine the direction of the edge leading to it: either from the left or from the right. The shuffling-principle is given as follows: if a successor n' of a node n in the original Stern–Brocot tree is reached from the same direction as n is reached from its own predecessor node $\mathrm{pre}(n)$, then in the shuffled tree n' is placed to the left lower side of n, otherwise it is posed to the right lower side of n.

There is an analogous Cayley graph of a freely generated matrix monoid $\langle \mathcal{R}, \mathcal{L} \rangle \subset GL(2, \mathbb{N})$, which corresponds to the shuffled Stern–Brocot tree in the same manner as the Cayley graph of $SL(2, \mathbb{N})$ corresponds to the original tree, namely via the values of the associated Moebius transforms at the argument $z = 1$. The two generating matrices associated with the shuffled Stern–Brocot tree are

$$\mathcal{R} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mathcal{T}.$$

It remains to be shown that these matrices represent the same abstract Stern–Brocot tree and that the Cayley graph of $\langle \mathcal{R}, \mathcal{L} \rangle$ realizes the above mentioned shuffling-principle for its planar arrangement. The matrix $\mathcal{R}$ is not an element of $SL_2(\mathbb{N})$, because $\det(\mathcal{R}) = -1$. We may distinguish between two cases.

(1) Each matrix $\mathcal{A} \in \langle \mathcal{R}, \mathcal{L} \rangle$ with an even number of occurrences of $\mathcal{R}$ in its generation has $\det(\mathcal{A}) = 1$ and belongs to $SL_2(\mathbb{N})$, i.e. it belongs also to the monoid $\langle \mathcal{T}, \mathcal{S} \rangle$ and consequently is associated with the same ratio $\mu_{\mathcal{A}}(1)$ on both trees: the original Stern–Brocot tree and the shuffled version.

(2) Each matrix

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \langle \mathcal{R}, \mathcal{L} \rangle$$

with an odd number of occurrences of $\mathcal{R}$ in its generation has $\det(\mathcal{A}) = -1$ and we find that the matrix

$$\tilde{\mathcal{A}} = \begin{pmatrix} b & a \\ d & c \end{pmatrix} = \mathcal{A} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with exchanged columns belongs to $SL_2(\mathbb{N}) = \langle \mathcal{T}, \mathcal{S} \rangle$. Although the two associated Moebius transforms

$$\mu_{\mathcal{A}}(z) := \frac{az + b}{cz + d} \quad \text{and} \quad \mu_{\tilde{\mathcal{A}}}(z) := \frac{bz + a}{dz + c}$$

differ from each other, we find nevertheless that their values at $z = 1$ coincide: $\mu_{\mathcal{A}}(1) = \mu_{\tilde{\mathcal{A}}}(1)$. Consequently, $\mathcal{A}$ is associated with the same ratio in the shuffled Stern–Brocot tree as is $\tilde{\mathcal{A}}$ on the original Stern–Brocot tree.

For each matrix $\mathcal{A} \in \langle \mathcal{R}, \mathcal{L} \rangle$ let $\mathrm{card}_{\mathcal{R}}(\mathcal{A})$ denote number of occurrences of $\mathcal{R}$ in its generation. The bijection $\sigma : \langle \mathcal{R}, \mathcal{L} \rangle \to \langle \mathcal{T}, \mathcal{S} \rangle$ between both monoids has the formula

$$\sigma(\mathcal{A}) = \mathcal{A} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\mathrm{card}_{\mathcal{R}}(\mathcal{A})} = \begin{cases} \mathcal{A} & \text{for } \mathcal{A} \in \langle \mathcal{R}, \mathcal{L} \rangle \cap \langle \mathcal{T}, \mathcal{S} \rangle, \\ \tilde{\mathcal{A}} & \text{for } \mathcal{A} \in \langle \mathcal{R}, \mathcal{L} \rangle \backslash \langle \mathcal{T}, \mathcal{S} \rangle, \end{cases}$$

and σ sends $\mathcal{L} = \mathcal{T}$ to itself and $\mathcal{R}$ to $\tilde{\mathcal{R}} = \mathcal{S}$. Note that this bijection is not homomorphic. The matrices $\mathcal{L}$ and $\mathcal{R}$ verify the relation $\mathcal{L}^{-1}\mathcal{R} = \mathcal{R}^{-1}\mathcal{L}$, while the matrices $\mathcal{S}$ and $\mathcal{T}$ verify the relation $\mathcal{S}^{-1}\mathcal{T}\mathcal{S}^{-1} = \mathcal{T}\mathcal{S}^{-1}\mathcal{T}$. □

Instead of homomorphy we need to check the shuffling property. Each element in $\mathcal{A} \in \langle \mathcal{R}, \mathcal{L} \rangle$ is represented by its word $w_1(\mathcal{A}) \in \{R, L\}^*$ with letters 'R' and 'L', and likewise each element in $\mathcal{B} \in \langle \mathcal{T}, \mathcal{S} \rangle$ is represented by its word $w_2(\mathcal{B}) \in \{S, T\}^*$ with letters 'S' and 'T'. Let $w_\sigma : \{R, L\}^* \to \{S, T\}^*$ denote the expression of the bijection σ in terms of words, i.e. $w_\sigma(w_1(\mathcal{A})) := w_2(\sigma(\mathcal{A}))$. What needs to be shown is that whenever in a word $w_2(\sigma(\mathcal{A}))$ a letter at position k repeats its predecessor (no matter whether this letter is T or S), then the associated letter in the word $w_\sigma^{-1}(w_2(\sigma(\mathcal{A}))) = w_1(\mathcal{A})$ at position k is L. If, however, in $w_2(\sigma(\mathcal{A}))$ a letter at position k differs from its predecessor, then the associated letter in $w_1(\mathcal{A})$ at position k is R.

Our sketch of a proof starts with the observation that multiplying a matrix $\mathcal{A}$ by $\mathcal{T}$ from the right is the same as adding the left column of $\mathcal{A}$ to its right column, while leaving the left column unchanged, and multiplying a matrix $\mathcal{A}$ by $\mathcal{S}$ from the right is the same as adding the right column of $\mathcal{A}$ to its left column, while leaving its right column unchanged. This means that, in both cases, one of the two columns is replaced by the sum of the columns, while the other is still in its original place. However, multiplication of a matrix $\mathcal{A}$ by $\mathcal{R}$ from the right is like adding the right column of $\mathcal{A}$ to its left column, followed by an exchange of both columns. Thus, in both cases of multiplication by $\mathcal{L} = \mathcal{T}$ or by $\mathcal{R}$ from the right, the sum of the two columns ends up on the right side while one of the summands is 'stored' on the left side. As long as we repeatedly multiply a matrix $\mathcal{A}$ with $\mathcal{T}$, we repeatedly multiply $\sigma^{-1}(\mathcal{A})$ with $\mathcal{L}$, correspondingly. Once we change from T to S and thereby change the side of the 'sum column' in $\mathcal{A} \cdot \mathcal{S}$ from right to left we need to multiply $\sigma^{-1}(\mathcal{A})$ with $\mathcal{R}$ in order to keep the sum column on the right side. If we now go on to multiply $\mathcal{A} \cdot \mathcal{S}$ with $\mathcal{S}$, we need to multiply $\sigma^{-1}(\mathcal{A}) \cdot \mathcal{R}$ with $\mathcal{L}$, because we simply go on further to increase the 'wrong-sided' sum column. Only if we switch back to multiply with $\mathcal{T}$ again do we need to apply $\mathcal{R}$ in order to compensate for switching the sides again. ■

As a consequence of the shuffling property we obtain the following characterization of chromatic numbers. Noll defines chromatic numbers as positive ratios whose penultimate semi-convergent is a proper convergent. This means that the associated word on the Stern–Brocot tree has a suffix ST or TS. In both cases we have a change of the letter in the last position. And this means that the last letter of the associated word on the Stern–Brocot tree needs to be R. In other words, on the shuffled Stern–Brocot tree chromatic numbers are always right successors.

Scale generators, such as the fifth $g = \log_2(\omega)$ are usually chosen between 0 and 1. Thus, it is sufficient to inspect the half wing of the shuffled Stern–Brocot tree, which starts at $R(1) = 1/2$. It corresponds to the left wing of the ordinary Stern–Brocot tree (depicted by the grey rectangle in Figure 1) in which the rational numbers of each horizontal line are the same but in another order.

We return to our initial example from the introduction. The node $u = RRRL$ with matrix

$$\mathcal{A} = \mathcal{R}\mathcal{R}\mathcal{R}\mathcal{L} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

is associated with the Moebius transform

$$\mu_{\mathcal{A}}(z) = \frac{x+3}{2x+5},$$

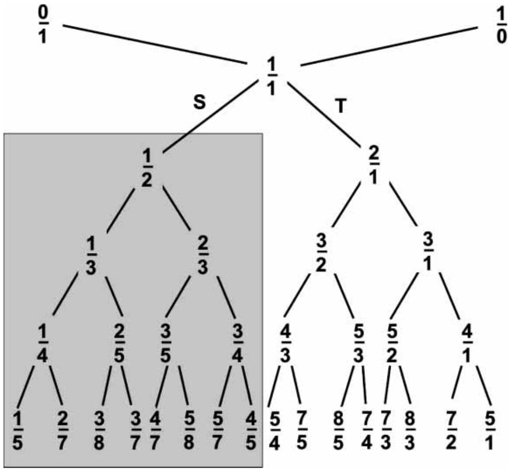

Figure 1. Stern–Brocot tree.

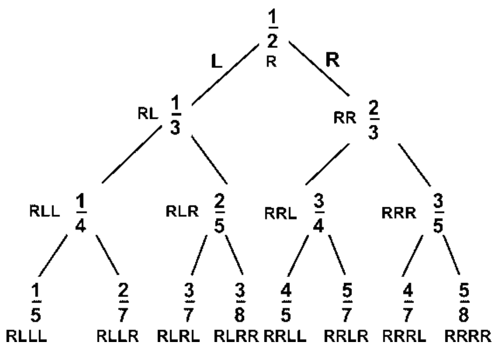

Figure 2. Shuffled Stern–Brocot tree.

with value $\mu_{\mathcal{A}}(1) = 4/7$. For the just fifth $\omega = 3/2$ with logarithm $g = \log_2(3/2)$, the infinite path in the shuffled Stern–Brocot tree starts with $R^3LRLRL^2R^2L^4RL\ldots$ The prefix R^3L represents the matrix $\mathcal{A}$ in the example. It represents the rational number $4/7$ and encodes the diatonic scale. This time the linear equation yields x as the 'rare step'-solution and y as the 'frequent step'-solution:

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \omega \\ 1 \end{pmatrix} \quad \text{with solution } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} \omega \\ 1 \end{pmatrix} = \begin{pmatrix} -5\omega + 3 \\ 2\omega - 1 \end{pmatrix}.$$

3. Generalized diatonic scales

In the light of our remarks in the introduction, we may characterize *large ME* scales* as follows.

PROPOSITION 3.1 *If a non-degenerate well-formed scale A which is a subset of a chromatic universe $\mathbb{Z}_N$ has a complement A^c without adjacent pitch classes it is necessarily a large ME* scale.*

Proof In addition to the maximal evenness of the abstract step pattern of the scale A—which holds for every well-formed scale—we may further conclude from the lack of adjacent pitch classes in A^c that this ME pattern is entirely made up of 1's and 2's. From this fact we may conclude that the scale itself is maximally even and large. An ME scale can only be generated if the generator m and the chromatic cardinality N are coprime, and thus A has to be a large ME* scale. ■

If one thinks of a large ME* scale A as the white keys of a maximally even keyboard layout, the proposition focuses on the property that there are no black keys adjacent to each other. Informally, we may therefore alternatively speak of a *white key scale*.

Example 3.2 The set $A = \{0, 1, 2, 3, \ldots, N-2\}$ is a trivial 1-generated white-key scale in $\mathbb{Z}_N$ and the set $B = \{0, 2, 4, \ldots, 2k\}$ is a trivial 2-generated white-key scale in $\mathbb{Z}_{2k+1}$. Both sets are well-formed and non-degenerate. A^c reduces to one note (one black key). B^c is the set $\{1, 3, 5, \ldots, 2k-1\}$, which also has no adjacent pitch classes.

Proposition 3.3 shows that—in general—there are m-generated non-degenerate well-formed scales that are not m-generated large ME* scales and it shows how they are related with the large ME* scales.

PROPOSITION 3.3 *The cardinality of an m-generated large ME* scale in $\mathbb{Z}_N$ is maximal among all m-generated non-degenerate well-formed scales in $\mathbb{Z}_N$.*

Proof Different m-generated non-degenerate well-formed scales correspond to different specific step sizes (in the sense that different scales differ at least in one step size). Proposition 3.1 implies that the specific step sizes of a large ME* scale need to be 1 and 2, which implies its maximality. ■

Example 3.4 In the 14-chromatic universe $\mathbb{Z}_{14}$, the scale $C = \{0, 1, 2, 5, 6, 7, 10, 11\}$ is a 5-generated non-degenerate well-formed scale. But its complement $\{3, 4, 8, 9, 12, 13\}$ has adjacent pitch classes. Thus, the scale C is not a large ME* scale. However the scale $\{0, 1, 2, 3, 5, 6, 7, 8, 10, 11, 12\}$ is a large ME* scale.

The following proposition (Proposition 3.5) recapitulates a characterization of large ME* scales its connection with their small ME* complements.

PROPOSITION 3.5 *Any large ME* scale A in $\mathbb{Z}_N$ and its complement A^c are both maximally even subsets of $\mathbb{Z}_N$. A contains a translation of its complement. More precisely, the translation $T_{-1}(x) = x - 1$ of one degree down of the elements of A^c is included in A,*

$$T_{-1}(A^c) \subset A.$$

Proof As A is well-formed we can infer that its step interval pattern is maximally even. As the step sizes of A are furthermore 1 and 2 we may further infer that also step intervals of A^c are adjacent numbers and that A and A^c are both maximally even well-formed scales. They are both instances of generalized circles of fifths in the sense of [4]. From [24, Theorem 5.2], we know that a translate of A^c is contained in A. ■

DEFINITION 3.6 *An m-generated large ME* scale A ($m \geq 2$) is tightly generated if there is no non-degenerate well-formed m-generated scale W with*

$$T_{-1}(A^c) \subset W \subset A.$$

Before we further investigate this property we will use it in Definition 3.7 below, which provides an answer to the initial question implied by the title of this paper: 'Is there an analogue to the standard diatonic scale in a given chromatic system with N notes rather than 12?'

DEFINITION 3.7 *Within a fixed chromatic universe* $\mathbb{Z}_N$ ($N \geq 12$, $N \neq 15$) *the generalized diatonic scale (or simply the diatonic scale) is a tightly m-generated large ME* scale A (with generator* $m > 2$) *such that it minimizes the cardinality difference* $|A| - |A^c|$ *among all tightly m-generated large ME* scales (with varying generators* $m > 2$).

The condition $m > 2$ takes into account that for odd chromatic cardinality N there is always a 2-generated trivial generalized diatonic scale $A = \{0, 2, 4, \ldots, N-1\}$ of cardinality $|A| = (N+1)/2$, with the unbeatably small difference $|A| - |A^c| = 1$. Eytan Agmon [1] discards these scales, while John Clough and Jack Douthett [2] acknowledge their role within the construction of second-order ME sets.[3] Definition 3.7 thus presupposes the existence of tightly generated large ME* scales with generators $m > 2$. Above $N = 12$ there is one known exception, namely $N = 15$. Conjecture 3.12 claims that beyond this example there is always a non-trivial generalized diatonic scale.

Example 3.8 For $N = 12$, the traditional diatonic scale $A = \{0, 1, 3, 5, 6, 8, 10\}$ is a 5-generated (or 7-generated) large ME* scale. Its complementary set is the traditional pentatonic scale and is a translate of the 5-generated subscale $\{0, 3, 5, 8, 10\} \subset A$, which is a small ME* scale. The only 5-generated set $\{0, 1, 3, 5, 8, 10\}$ between A and A^c is not well-formed.

Example 3.9 In light of the prominence of the 19-tone chromatic as a tempered version of the *Cembalo Cromatico*, it is interesting to note that the tightly 8-generated large ME* 12-tone complement $\{0, 2, 4, 5, 7, 8, 10, 12, 13, 15, 16, 18\}$ of the small ME* scale $\{1, 3, 6, 9, 11, 14, 17\}$—which as a black key scale in $\mathbb{Z}_{19}$ corresponds to the traditional diatonic scale—is not the generalized diatonic scale in $\mathbb{Z}_{19}$. Instead it is the 7-generated scale $\{0, 2, 4, 6, 7, 9, 11, 13, 14, 16, 18\}$ of cardinality 11.[4]

The subsequent definitions and propositions demonstrate the benefit of the shuffled Stern–Brocot tree for the study of large ME* scales. In the introduction we associated the ratio 4/7 and the matrix

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \in SL_2(\mathbb{N})$$

with the standard diatonic scale. In Section 2 we replaced this matrix by its shuffled version

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \in \langle \mathcal{R}, \mathcal{L} \rangle.$$

Recall that the ratio 4/7 is a *generic* characterization of the diatonic scale: the numerator 4 represents the number of diatonic steps in the generating fifth, while the denominator 7 represents the number of diatonic steps in the octave; 3 and 1 are the numbers of occurrences of the frequent and the rare step within the generating fifth, respectively; 5 and 2 are the numbers of occurrences of the frequent and the rare step in the octave, respectively. This generic characterization does not tell whether the frequent step is smaller or larger. Later, in Proposition 3.13, we show that, for m-generated scales, tightness occurs exactly when the frequent step is also the larger one, namely of size 2.

In extrapolating this concrete situation of the traditional diatonic we may investigate the hierarchy of generic descriptions of well-formed scales with the help of one-to-one correspondences among ratios on the shuffled Stern–Brocot tree, matrices in $\langle \mathcal{R}, \mathcal{L} \rangle$ and generic scale descriptions.

DEFINITION 3.10 *Let $d = |A|$ denote the cardinality of a well-formed scale A and let $1 \leq b < |A|$ a number coprime with d. Let*

$$\mathcal{G}_{b/d} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} b - b_0 & b_0 \\ d - d_0 & d_0 \end{pmatrix} \in \langle \mathcal{R}, \mathcal{L} \rangle$$

denote the matrix on the shuffled Stern–Brocot tree associated with the ratio b/d. On the basis of these coordinates we obtain a generic scale description: b represents the number of scale steps in the generating interval, and d represents the number of scale steps in the octave. b_0 and a_0 are the numbers of occurrences of the frequent and the rare step in the generating interval, respectively. d_0 and c_0 are the numbers of occurrences of the frequent and the rare step within the octave, respectively.

Definition 3.10 is merely a shuffled version of the hierarchy of well-formed scales according to Carey and Clampitt [3]. Now, in addition to this generic interpretation, we introduce specific one-to-one correspondences among ratios, matrices and certain m-generated scales.

DEFINITION 3.11 *Let N denote the cardinality of a chromatic system $\mathbb{Z}_N$ and $1 \leq m < N$ a number coprime with N. Let*

$$\mathcal{A}_{m/N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} m - b & b \\ N - d & d \end{pmatrix} \in \langle \mathcal{R}, \mathcal{L} \rangle$$

denote the matrix on the shuffled Stern–Brocot tree associated with the ratio m/N and let $A_{m/N} = \{km \mid k = 0, \ldots, d-1\}$ denote the associated m-generated scale of cardinality d.

The following proposition (Proposition 3.12) confirms that this association provides in fact one-to-one correspondences among ratios, matrices and large ME* scales. Furthermore, it provides a connection between the generic interpretation of ratios in the sense of Definition 3.10 and the specific interpretation of ratios in the sense of Definition 3.11.

PROPOSITION 3.12 *Each m-generated scale of the form $A_{m/N}$ in $\mathbb{Z}_N$ is a large ME* scale and every large ME* scale in $\mathbb{Z}_N$ is of the form $A_{m/N}$, i.e. corresponds to one node m/N in the shuffled Stern–Brocot tree.*[5] *The generic description of any large ME* scale $A_{m/N}$ corresponds to the immediate predecessor b/d of the node m/N on the shuffled Stern–Brocot tree.*

Proof The predecessor of ratio m/N on the shuffled Stern–Brocot tree is the ratio b/d with the associated matrix

$$\mathcal{G}_{b/d} = \begin{pmatrix} b - b_0 & b_0 \\ d - d_0 & d_0 \end{pmatrix}$$

Depending whether $\mathcal{A}_{m/N}$ is a left or right successor of $\mathcal{G}_{b/d}$, we have to distinguish the following two possibilities:

$$\mathcal{G}_{b/d} \cdot \mathcal{L} = \begin{pmatrix} b - b_0 & b_0 \\ d - d_0 & d_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b - b_0 & b \\ d - d_0 & d \end{pmatrix} = \mathcal{A}_{(2b-b_0)/(2d-d_0)}$$

$$\mathcal{G}_{b/d} \cdot \mathcal{R} = \begin{pmatrix} b - b_0 & b_0 \\ d - d_0 & d_0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b_0 & b \\ d_0 & d \end{pmatrix} = \mathcal{A}_{(b+b_0)/(d+d_0)}.$$

For both cases we calculate the specific step sizes of the scale $A_{m/N}$ by applying the inverse matrix $\mathcal{G}_{b/d}^{-1}$ to the vector $(m \quad N)^{\mathrm{T}}$ (see the introduction):

$$\begin{aligned}\mathcal{G}_{b/d}^{-1}\cdot\begin{pmatrix}2b-b_0\\2d-d_0\end{pmatrix} &= (-1)^r\begin{pmatrix}d_0 & -b_0\\-d+d_0 & b-b_0\end{pmatrix}\cdot\begin{pmatrix}2b-b_0\\2d-d_0\end{pmatrix}\\ &= (-1)^r(bd_0-b_0d)\cdot\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}\\ \mathcal{G}_{b/d}^{-1}\cdot\begin{pmatrix}b+b_0\\d+d_0\end{pmatrix} &= (-1)^r\begin{pmatrix}d_0 & -b_0\\-d+d_0 & b-b_0\end{pmatrix}\cdot\begin{pmatrix}b+b_0\\d+d_0\end{pmatrix}\\ &= (-1)^r(bd_0-b_0d)\cdot\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}.\end{aligned}$$

This shows that all scales of the form $A_{m/N}$ are degenerate well-formed scales with specific step sizes 1 and 2, i.e. they are large ME* scales. Conversely, if we are given a large ME* scale, whose generic description is given by the matrix $\mathcal{G}_{b/d}$, we may apply this matrix to its step size vector $(2 \quad 1)^{\mathrm{T}}$ or $(1 \quad 2)^{\mathrm{T}}$ in order to obtain

$$\begin{aligned}\mathcal{G}_{b/d}\cdot\begin{pmatrix}2\\1\end{pmatrix} &= \begin{pmatrix}b-b_0 & b_0\\d-d_0 & d_0\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2b-b_0\\2d-d_0\end{pmatrix}\\ \mathcal{G}_{b/d}\cdot\begin{pmatrix}1\\2\end{pmatrix} &= \begin{pmatrix}b-b_0 & b_0\\d-d_0 & d_0\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}b+b_0\\d+d_0\end{pmatrix}.\end{aligned}$$

The resulting vectors describe exactly the chromatic sizes of the generating interval and of the octave for the scales $\mathcal{A}_{(2b-b_0)/(2d-d_0)}$ and $\mathcal{A}_{(b+b_0)/(d+d_0)}$, which completes the proof. ■

In the case of the standard diatonic which is generically characterized by

$$\mathcal{G}_{4/7}=\begin{pmatrix}1 & 3\\2 & 5\end{pmatrix},$$

one finds $\mathcal{G}_{4/5}\cdot\mathcal{L}=\mathcal{A}_{5/9}$ and $\mathcal{G}_{4/5}\cdot\mathcal{R}=\mathcal{A}_{7/12}$. The 7-diatonic scale $A_{7/12}=\{0,2,4,6,7,9,11\}$ in $\mathbb{Z}_{12}$ is tightly generated, while the 5-generated large ME* scale $A_{5/9}=\{0,1,2,3,5,6,7\}$ in $\mathbb{Z}_9$ is not. $A_{5/9}$ contains the translate $\{0,5\}$ of its non-degenerate well-formed complement $\{8,4\}$ with step sizes 4 and 5. And between $\{0,5\}$ and $\{0,1,2,3,5,6,7\}$ there is the 5-generated non-degenerate well-formed scale $\{0,1,2,5,6\}$ with step sizes 1 and 3. Proposition 3.13 below shows that this binary division in a tightly generated case and one non-tightly generated case exemplifies the general case, namely the binary distribution of large ME* scales (for varying m) along the binary tree structure of the shuffled Stern–Brocot tree.

PROPOSITION 3.13 *Let $\mathcal{A}_{m_1/N_1}=\mathcal{G}_{b/d}\cdot\mathcal{L}$ and $\mathcal{A}_{m_2/N_2}=\mathcal{G}_{b/d}\cdot\mathcal{R}$ denote the left and right successors of the matrix $\mathcal{G}_{b/d}$ on the binary Cayley graph of the monoid $\langle\mathcal{R},\mathcal{L}\rangle$. The m_2-generated large ME* scale A_{m_2/N_2} in $\mathbb{Z}_{N_2}$ is tightly generated, while the m_1-generated large ME* scale A_{m_1/N_1} in $\mathbb{Z}_{N_1}$ is not. In the tightly generated case, the larger step of size* 2 *occurs more frequent than the smaller step of size* 1.

Proof As

$$\mathcal{G}_{b/d}=\begin{pmatrix}b-b_0 & b_0\\d-d_0 & d_0\end{pmatrix}$$

is an element of the monoid $\langle\mathcal{R},\mathcal{L}\rangle$, we know that $b-b_0<b_0$ and $d-d_0<d_0$. This implies that the chromatic cardinality $N_1=(d-d_0)+d$ is smaller than the chromatic cardinality

$N_2 = d_0 + d$. It also implies that the number $d - d_0$ of steps of size 2 in A_{m_1/N_1} is smaller than the number d_0 of steps of size 2 in A_{m_2/N_2}. Thus, to complete the proof, it remains to be shown that the right successor corresponds to a tightly generated scale, while the left successor does not. The crucial point is that the cardinalities of the complements $A^c_{m_1/N_1} = \mathbb{Z}_{N_1} \backslash A_{m_1/N_1}$ and $A^c_{m_2/N_2} = \mathbb{Z}_{N_2} \backslash A_{m_2/N_2}$ coincide with the frequencies $d - d_0$ and d_0 of the steps of size 2 in A_{m_1/N_1} and A_{m_2/N_2}, respectively. This is simply because in each size 2 gap there sits a black key—an element of the complement.

The predecessor of the ratio b/d on the shuffled Stern–Brocot tree is b_0/d_0. Consequently, b_0/d_0 is a semi-convergent of both ratios m_1/N_1 and m_2/N_2. In accordance with the theory of well-formed scales [3] we may conclude that the m_1-generated scale $B_1 = \{km_1 \mid k = 0, \ldots, d_0 - 1\}$ of cardinality d_0 within $\mathbb{Z}_{N_1}$ is a non-degenerate well-formed scale. Likewise, the m_2-generated scale $B_2 = \{km_2 \mid k = 0, \ldots, d_0 - 1\}$ of cardinality d_0 within $\mathbb{Z}_{N_2}$ is a non-degenerate well-formed scale. The complement of A_{m_2/N_2} also has cardinality d_0 and therefore B_2 is a translate of $A^c_{m_2/N_2}$. But d_0 is larger than the cardinality $d - d_0$ of the complement of A_{m_1/N_1}. Thus, on the one hand, we have a proper inclusion $T_{-1}(A^c_{m_1/N_1}) \subset B_1 \subset A_{m_1/N_1}$, which shows that the left successor $\mathcal{A}_{m_1/N_1} = \mathcal{G}_{b/d} \cdot \mathcal{L}$ is associated with a non-tightly generated scale. On the other hand, a proper inclusion $T_{-1}(A^c_{m_2/N_2}) = B_2 \subset B'_2 \subset A_{m_2/N_2}$ of well-formed scales would imply that there is an intermediate semi-convergent of the ratio m_2/N_2 between b_0/d_0 and d/b. But there is no such intermediate semi-convergent and therefore the right successor $\mathcal{A}_{m_2/N_2} = \mathcal{G}_{b/d} \cdot \mathcal{R}$ is associated with a tightly generated scale. ■

Example 3.14 For $N = 13$, the number $m/N = 5/13$ is associated with the word RLR^3 and the matrix $\mathcal{U}$. The predecessor on the shuffled Stern–Brocot tree is 3/8 and is associated with the word RLR^2 and the matrix $\mathcal{V}$

$$\mathcal{U} = \mathcal{RLR}^3 = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}, \qquad \mathcal{V} = \mathcal{RLR}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.$$

The generalized diatonic scale A is the 5-generated large ME* scale $A = \{0, 2, 4, 5, 7, 9, 10, 12\}$. Its complement is, up to translation, the scale $B = \{0, 2, 5, 7, 10\}$. There is no well-formed scale between A and B. The scale A is tightly generated. The word RLR^3 associated with m/N ends by R. The first column of the matrix $\mathcal{U}$ is equal to the second column of $\mathcal{V}$. Instead consider now the number $m/N = 3/13$, whose associated word is RL^3RL. This is associated with the matrix

$$\mathcal{U} = \mathcal{RL}^3\mathcal{RL} = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}, \qquad \mathcal{V} = \mathcal{RL}^3\mathcal{R} = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}.$$

The 3-generated large ME* scale $A = \{0, 2, 3, 5, 6, 8, 9, 11, 12\}$ is not tightly generated. Its complement is, up to a translation, $A^c = \{0, 3, 6, 9\}$. The scale $B = \{0, 3, 6, 9, 12\}$ is a 3-generated well-formed scale between A and A^c. The associated word RL^3RL does not end with R. The first column of the matrix $\mathcal{U}$ is not equal to the second column of $\mathcal{V}$.

Generalized diatonic scales $D \subset \mathbb{Z}_N$ for $N = 12, \ldots, 24$ are presented in the Table 1. The scale generators m are listed in the third column and the cardinalities $|D|$ are listed in the last column, accordingly.

Table 2 shows the same data for quarter-tone ($N = 24$) to sixth-tone systems ($N = 36$).

As already mentioned in connection with the exceptional case $N = 15$, the existence of non-trivial generalized diatonic scales is not guaranteed for odd cardinalities N. However, on the basis of computer experiments we formulate the following conjecture.

Table 1. Generalized diatonic scales $D \subset \mathbb{Z}_N$ for $N = 12, \ldots, 24$.

N	Diatonic scale D	m	$\|D\|$
12	{0, 1, 3, 5, 6, 8, 10}	5	7
13	{0, 2, 4, 5, 7, 9, 10, 12}	5	8
14	{0, 1, 3, 4, 6, 7, 9, 10, 12}	3	9
15	{0, 2, 4, 6, 8, 10, 12, 14} = trivial	2	8
16	{0, 1, 3, 5, 7, 8, 10, 12, 14}	7	9
17	{0, 1, 3, 5, 6, 8, 10, 11, 13, 15}	5	10
18	{0, 2, 4, 5, 7, 9, 10, 12, 14, 15, 17}	5	11
19	{0, 2, 4, 6, 7, 9, 11, 13, 14, 16, 18}	7	11
20	{0, 1, 3, 5, 7, 9, 10, 12, 14, 16, 18}	9	11
21	{0, 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19}	8	13
22	{0, 1, 3, 5, 6, 8, 10, 11, 13, 15, 16, 18, 20}	5	13
23	{0, 1, 3, 5, 7, 8, 10, 12, 14, 15, 17, 19, 21}	7	13
24	{0, 1, 3, 5, 7, 9, 11, 12, 14, 16, 18, 20, 22}	11	13

Table 2. The same data as in Table 1 for quarter-tone to sixth-tone systems.

N	Diatonic scale D	m	$\|D\|$
24	{0, 1, 3, 5, 7, 9, 11, 12, 14, 16, 18, 20, 22}	11	13
25	{0, 2, 4, 6, 8, 9, 11, 13, 15, 17, 18, 20, 22, 24}	9	14
26	{0, 2, 4, 6, 7, 9, 11, 13, 14, 16, 18, 20, 21, 23, 25}	7	15
27	{0, 1, 3, 5, 6, 8, 10, 11, 13, 15, 16, 18, 20, 21, 23, 25}	5	16
28	{0, 1, 3, 5, 7, 9, 11, 13, 14, 16, 18, 20, 22, 24, 26}	13	15
29	{0, 1, 3, 5, 7, 9, 10, 12, 14, 16, 18, 19, 21, 23, 25, 27}	9	16
30	{0, 1, 3, 5, 7, 8, 10, 12, 14, 15, 17, 19, 21, 22, 24, 26, 28}	7	17
31	{0, 2, 4, 6, 8, 10, 11, 13, 15, 17, 19, 21, 22, 24, 26, 28, 320}	11	17
32	{0, 1, 3, 5, 7, 9, 11, 13, 15, 16, 18, 20, 22, 24, 26, 28, 30}	15	17
33	{0, 2, 4, 6, 7, 9, 11, 13, 14, 16, 18, 20, 21, 23, 25, 27, 28, 30, 32}	7	19
34	{0, 2, 4, 6, 8, 9, 11, 13, 15, 17, 18, 20, 22, 24, 26, 27, 29, 31, 33}	9	19
35	{0, 1, 3, 5, 7, 9, 11, 12, 14, 16, 18, 20, 22, 23, 25, 27, 29, 31, 33}	11	19
36	{0, 1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 20, 22, 24, 26, 28, 30, 32, 34}	17	19

CONJECTURE 3.15 In each N-chromatic universe $\mathbb{Z}_N$, $N \geq 12$, $N \neq 15$, there is exactly one generalized diatonic scale. In other words, every natural number $N \geq 12$, $N \neq 15$ is the denominator of a chromatic number m/N with $m > 2$.

There is another interesting structural aspect of the traditional diatonic scale, which has also been investigated in the context generalizing it. In the 12-tone chromatic the diatonic scale contains a tritone, which embodies an ambiguity between augmented fourth and diminished fifth. The tritone as a two-tone set is invariant under the tritone transposition T_6 and is therefore an instance of a *limited transposition set* in $\mathbb{Z}_{12}$. Limited transposition sets play an important role in the music theory of the 12-tone system, including the study of scales. Olivier Messiaen dedicated widely recognized investigations into limited transposition sets after he explored them compositionally in his *Preludes* for piano of 1929. In tonal harmony it is well known that the diminished seventh chord links different major and major keys and offers certain means for modulation. This chord contains two tritones and is a limited transposition set. However, the smallest limited transposition set is the tritone, which is a subset of the diatonic scale.[6] Enharmonic modulations emphasize the ambiguity of the tritone.

In the N-chromatic universe, it is therefore interesting to see whether or not the generalized diatonic scale has a limited transposition subset. The concept of the *hyperdiatonic scale* as introduced by Clough and Douthett [2] after the works of Agmon [1] involves the tritone as a central constituent. Therefore, we propose the following generalization of this concept.

DEFINITION 3.16 *A generalized diatonic scale $A \subset \mathbb{Z}_N$ is a hyperdiatonic scale if it contains a (non-trivial) limited transposition subset $L \subset \mathbb{Z}_N$ of smallest possible cardinality in the N-chromatic universe.*

If $N = p_1^{r_1} \cdots p_s^{r_s}$ is the prime power decomposition of N with prime numbers $p_1 < p_2 < \cdots < p_s$, the limited transposition subsets of smallest possible cardinality in the N-chromatic universe are exactly the translates of the set $(N/p_1)\mathbb{Z}_N$. Whenever $N = p_1$ is prime, the generalized diatonic scale is not a hyperdiatonic scale in the sense of Definition 3.16, because the only limited transposition set is the entire chromatic system $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. There are also chromatic cardinalities N, which are not prime numbers, whose generalized diatonic scales are not hyperdiatonic. For $N = 21$, the generalized diatonic scale $\{0, 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19\}$ is a diatonic scale of generator 8 without any limited transposition set.

Definition 3.16 does not postulate uniqueness of the limited transposition subset, such that hyperdiatonic scales may also contain certain larger limited transposition subsets. For $N = 14$, the generalized diatonic scale $\{0, 1, 3, 4, 6, 7, 9, 10, 12\}$ is 3-diatonic and contains the limited transposition set $\{0, 3, 7, 10\}$, which includes the smallest limited transposition set $\{0, 7\}$.

4. Wyschnegradsky's diatonicized chromaticism

The final section of this article is dedicated to a historically interesting instance of a hyperdiatonic scale. In the *Preface* of his *Preludes, opus 22*, composed in 1916, Ivan Wyschnegradsky[7] [25] defined what is meant by a diatonic scale in the quarter-tone system (see Figure 3). He wrote:

> Structurally the work is based on a non-symmetrical scale comprising 13 tones, which is similar to the traditional diatonic scale. The latter comprises two tetrachords separated by two intervallic units (i.e. two semitones). In the same way, the 13-tone scale consists of two heptachords (same structure as in the tetrachord), also separated by two interval units. But since this unit in this case is the quarter-tone, the distance between the two heptachords will be that of a semitone.

He compares the two scales and discusses 'the analogy between the relationship of the 7-tone diatonic scale to the whole-tone system on one hand, and the relationship of the 13-tone scale to the chromatic scale on the other hand'. He continues:

> Another analogy must be mentioned. That is the capacity of both scales to be expressed in terms of a circle of fourths or fifths. In the old scale we have six perfect fourths or perfect fifths embracing half of the complete circle of 12 fourths or 12 fifths. In the new scale, we have 12 major fourths or 12 minor fifths, also embracing half of the complete circle, not of 12 but of 24 major fourths or 24 minor fifths. It is important to bear in mind that both of these intervals, the major fourth (11 quarter-tones = perfect fourth plus one quarter-tone) and the minor fifth (13 quarter-tones = perfect fifth minus one quarter-tone), which are the structural basis of the 13-tone diatonicism, are not mere accidental results of an unequal division of the octave, but natural intervals. Their frequency ratio is: for the major fourth 11/8, for the minor fifth 16/11. Notice that the number 11, absent in all the tonal relationship known in music up to now, is present in both intervals, as well as in some other quarter-tone intervals.

The 13-tone scale can be transposed on each of the 24 degrees of the quarter-tone scale. The first transposition is called 'Position Do'.

Each prelude of opus 22 is written in a different position. The first prelude uses position C, and the third prelude uses position B (Figure 4). Note that this scale cannot be transposed in the 12-tone system.

Figure 3. Diatonicized chromatism scale.

Figure 4. Wyschnegradsky, *Preludes op. 22, no. 3.*

Wyschnegradsky's diatonicized chromatic scale is an instance of a hyperdiatonic scale. In particular it belongs to the family of generalized diatonic scales in the N-tone chromatic universes for N divisible by 4. Wyschnegrasky's studies, which were published in 1979, but whose beginnings date back to 1916, should be acknowledged as predecessors of the later investigations by Agmon [1] and Clough and Douthett [2].[8] We summarize some basic facts in the following theorem.

THEOREM 4.1 *Let k be a non-negative integer $k \geq 2$. The generalized diatonic scale in the $N = 4k + 4$-tone chromatic is*

$$W_k = \{0, 1, 3, 5, \ldots, 2k+1, 2k+2, 2k+4, \ldots, 4k+2\}.$$

This scale has $|W_k| = 2k + 3$ tones (scale degrees), and is generated by $m = 2k + 1$. The step pattern contains exactly two semitones of chromatic size 1 (between 0 and 1 as well as between $2k + 1$ and $2k + 2$). All other steps are whole steps of chromatic size 2. The three numbers N, m and $|W_k|$ satisfy the equation

$$N = m + |W_k| = 4k + 4.$$

The scale W_k is a hyperdiatonic scale which contains the limited transposition set $\{0, 2k + 2\}$.

Proof We reduce the first $k + 2$ even and the first $k + 1$ odd multiples of $2k + 1$ modulo $4k + 4$ and obtain:

$$0(2k+1) = 0 \qquad 1(2k+1) = (2k+2) - 1$$
$$2(2k+1) = -2 \qquad 3(2k+1) = (2k+2) - 3$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

$$2k(2k+1) = 2k+4 \qquad (2k+1)(2k+1) = 1.$$
$$(2k+2)(2k+1) = 2k+2$$

This calculation shows that the $2k + 3$-element $2k + 1$-generated scale is exactly W_k. Close inspection shows also that the step pattern is exactly as described in the proposition. The balance $2 \times 1 + (2k + 1) \times 2 = 4k + 4$ yields $4k + 4$ chromatic steps in total. The semitones are maximally evenly distributed, because they are interrupted by k whole steps between 1 and $2k + 1$ and $k + 1$ whole steps between $2k + 2$ and $4k + 4 = 0$. Thus W_k is a large ME^*-scale. It is tightly generated because the small step 1 is also the rare step. The difference between the cardinalities of W_k and its complement is $|W_k| - |W_k^c| = 2k + 3 - [4k + 4 - (2k - 3)] = 2$, which cannot be surpassed. Hence W_k is the generalized diatonic scale in Z_{4k+4} with the asserted properties. In particular it is a hyperdiatonic scale. ■

Without proof we mention that the ratio $(2k+1)/(4k+4)$ is associated with the word $RLRL^{k-1}R$ on the shuffled Stern–Brocot tree. The penultimate and ultimate predecessors of $(2k+1)/(4k+4)$ on the Stern–Brocot tree are $k/(2k+1)$ and $(k+1)/(2k+3)$, respectively. $k/(2k+1)$ is a convergent for $(2k+1)/(4k+4)$, which implies that W_k is tightly generated.

Similarly there are two families of hyperdiatonic scales in chromatic universes with $N \equiv 2 \mod 4$ which split in two subcases $\mod 8$:

(1) generators $m/N = (2k+1)/(8k+6)$ for hyperdiatonic scales in chromatic universes with $N \equiv 6 \mod 8$ are associated with the words $RLLLRL^{k-1}R$ of the shuffled Stern–Brocot tree, while
(2) generators $m/N = (2k+1)/(8k+2)$ for hyperdiatonic scales in chromatic universes with $N \equiv 2 \mod 8$ are associated with the words $RLLRRL^{k-1}R$ of the shuffled Stern–Brocot tree.[9]

The family of trivial generalized diatonic scales in chromatic cardinalities with odd cardinality $N = 2k+1$ and generators $m/N = 2/(2k+1)$ is associated with the simple words $RL^{k-1}R$ on the shuffled Stern–Brocot tree. For the non-trivial generalized diatonic scales in chromatic cardinalities with odd cardinality, i.e. for generators $m > 2$, much more diversity can be observed in the associated pathways along the tree. It remains an open question whether this reflects a potential musical diversity in the character of these scales.

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Notes

1. The generic level of description in [4], [3] is algebraically linked to what I call *cyclic tone systems* as a special cases of *tuning groups* (see [12] and [13]). The generic interval system controls the counting of steps and—in the case of generated scales—of generator intervals within the generalized circle of “fifths”. Well-formed scale theory—as mentioned above—is essentially built upon a linear automorphism of the generic interval system which exchanges the circle of steps with the generalized circle of fifths. In contrast to the external definition of the generic interval system (like a metalanguage for counting in a scale, so to say), cyclic tone systems are defined as factor groups of the free group, generated by intervals such as octave and fifth. This is an intrinsic definition where the cyclic tone system is constructed within the specific level. The group isomorphism between the generic interval system and the cyclic tuning group provides a music-theoretically interesting connection between these levels.
2. The general theory of well-formed scales does not require the choice of an equally tempered 12-chromatic universe from the outset. The diatonic scale is generated by a fifth (e.g. with frequency ratio $\omega = 3/2$ and spanning the linear pitch interval $g = log_2(\omega)$ within the octave with frequency ratio 2/1 and linear pitch interval $log_2(2) = 1$. But any generator “close enough” to $g = log_2(\omega)$, such as $g = 7/12$ yields the same structural results. The mathematical condition for “close enough” is that the ratio 4/7 needs to be a semi-convergent of the number g.
3. Seventh chords in the generic diatonic $\mathbb{Z}_7$ are the white keys of such a trivial generalized diatonic scale, while the complementary triad forms the black keys. The same observation drives Noll’s definition of a pseudo-diatonic system [16].
4. The microtonal composer Mark Gould (see [23]) confirmed in e-mail conversations that he favours this scale.
5. Recall that we restrict our considerations to the left half of the entire tree below the node $1/2$.
6. The harmonic minor scale also contains two tritones.
7. According to his son Dimitri, Ivan Wyschnegradsky—who lived in Paris—wanted the transliteration of his name written without accents. I follow his preference (see also [26]).
8. Clough and Douthett [2], pp. 168–69 dedicate particular investigations to the study of interval circles in the quarter-tone diatonic set $M_{24,13}$, i.e. in ‘Wyschnegradsky’s’ hyperdiatonic scale.
9. This is even true for $k = 1$ with the word $RLLRRL^{-1}R = RLLRL$.

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