A LABEL-FREE DEVELOPMENT

FOR 12-PITCH-CLASS SYSTEMS

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For some time, it has been customary in much of the important theoretical literature to label the twelve pitch-classes[1] by the numbers 0 through 11. For instance, Babbitt,[2] Perle,[3] and Forte[4] all do so. The numbers involved should not strictly be called "integers": the appropriate mathematical term is "integers modulo 12," or better, because it avoids all possible confusion, "residues modulo 12." An "integer" to a mathematician is an ordinary counting number: after counting to 11, one continues on to 12, 13, etc. In contrast, the residues modulo 12 (or "mod 12") have a circular structure: if one continues past 11, one returns to 0.

The residues mod 12 have a natural algebraic structure. It would be out of place to develop the structure rigorously here, but I shall outline the basic elements of its manipulation. One can add the residues m and n. Their residue sum is that residue corresponding to their integer sum, if the latter is less than 12. In case the integer sum is 12 or more, the residue sum is that residue corresponding to the integer sum minus 12. Under these rules, 3 + 4 = 7, 8 + 5 = 1, 6 + 6 = 0, etc. By analogy, "three hours clockwise and four more hours clockwise gives seven hours clockwise in all; eight hours clockwise and five more hours clockwise gives one hour clockwise in all," and so on. Residues also have negatives. Minus-zero is zero, and for a nonzero residue n, minus-n is the residue corresponding to the integer 12 minus n. Thus $-8 = 4$, $-6 = 6$, etc. Again, "If I go eight hours counterclockwise, the net result will be the same as if I had gone four hours clockwise," and so on. One can subtract residues. The residue difference "m minus n" is taken to be the residue sum of m and (the residue) minus-n.

Under these conventions, many of the familiar features of ordinary integer arithmetic carry over to residue arithmetic. Mathematicians say that the residues mod 12 form "a group under addition." The zero residue plays a special role in this structure. It is singled out as the unique residue m such that m plus n is always n itself, for any residue n. The zero residue is also involved in the concept of negative residues. Minus-n can be defined as precisely that residue which, together with a given n, sums to zero. These considerations involving the zero residue are essential and typical aspects of mathematical group structure, as that is manifested in the present case.

The zero residue thereby enjoys a certain priority over the others, which is formally somewhat in the nature of a "tonic" status. For instance, why is "three o'clock" labelled with the residue 3? Because it is three hours clockwise from "zero o'clock" (which is of course the same as "twelve o'clock" in this context). That is, we consider 12 o'clock a *referential* hour. Each hour is labelled as "n o'clock" according to its distance clockwise from that unique referential hour.

Now it is very useful for many computational purposes to label the twelve pc's with the residues mod 12. Nevertheless, the practice can lead to problems in conceptualizing the exact nature of the musical structure involved. By labelling any one pc with the zero residue, we are implicitly (if only formally) attributing a tonic status to that pc, in exactly the sense that twelve o'clock enjoys a tonic status among the hours.

Forte,[5] for example, labels the pitch-class C with the residue zero once and for all. This leads him to a "fixed-DO" system of pc labelling. E-flat is thereby labelled as "3" once and for all, because it is "three hours clockwise" from the referential C. So, even though I may hear E-flat as a musically "tonic" pitch-class for Webern's *Piano Variations,* I am bound by the system to conceptualize my musical tonic as a "3" rather than a "0." This is at least awkward and, as we shall see, it can too easily become more than just awkward. To be sure, one is accustomed to the specific awkwardness as typical of the problems involved in any fixed-DO musical theory. In France, for instance, the *Eroica* Symphony is conceptually organized not about a "DO" but rather about "*mi bémol,*" that is, the flatted third of some Platonic C in Heaven. But, even in the Paris Conservatory, they do not analyze the function of the opening chord as "bIII."

Babbitt and Perle[6] develop "moveable-DO" systems. The zero residue is used to label the first pitch-class of the most significant row-form in any specific musical context. But even moveable-DO systems of pc labelling are still conceptually problematic. First, does one really want to claim that in *any* pertinent musical context there must be a priori one of the pitch-classes which we shall want to analyze as functioning referentially over all the others? That is a difficult pill to swallow if one claims to be investigating "atonal" music, or music composed with "twelve tones related only to one another" and not to any one tone of reference. I pass over the theoretical problem involved in asserting a "most significant row-form" for any serial context. Let us suppose (what will frequently occur) that we agree on such a "tonic" row-form for some context; yet another problem arises. On what grounds are we then to attribute a tonic function a priori to the first pc of that row? Many subjects of tonal fugues do not begin on the tonic degree. (The analogy is of course not exact but is, I think, methodologically to the point.) If one wished to label pc's for the second movement of the *Piano Variations,* would not the pc A be the obvious candidate for the zero label? But no row-form used in the movement begins with A. (Some do begin with E-flat, but these do not have the thematic force of the opening row presentations.) The present topic will recur later in connection with a discussion of some aspects of Stravinsky's *In Memoriam.*

The problems we have seen arising in conceptualizing a "zero pc," fixed or moveable, give rise to problems in conceptualizing the nature of and relations among pc inversions. The residues mod 12 do possess one "most natural" inversion: inversion about the residue 0. The twelve pitch-classes, however, do *not* possess, in the raw, any one such "most natural" inversion. There are twelve ways of inverting the total pc chromatic into itself, and any one of these twelve inversions may assume priority in a given musical context. In the *Piano Variations*, inversion about A, the same pc operation as inversion about E-flat, enjoys clear priority, among the twelve inversions, throughout the second movement, and at the opening and close of the third. In Schoenberg's *String Trio*, inversion about B and B-flat, the same pc operation as inversion about E and F, enjoys clear overall structural priority, considering the opening, reprise, and close of the piece.[7] If I am going to label inversions for these pieces, then, I will have a conceptually good model by labelling inversion about E-flat as "I_0" for the Webern, and inversion about E and F as "I_0" for the Schoenberg. In the former case, I can still hold onto a moveable-DO pc labelling system, labelling the pc E-flat with the residue zero, to make the mathematics of the labels model the musical structure. But in the latter case, I cannot salvage the model with any moveable-DO. There is no pc which reflects into itself under inversion about E and F, which I might use as a zero pitch-class around which the inversion reflected things. In particular, the first pc of a row will not work in this connection.

Going beyond such specific contexts, it seems problematical to claim that there must always be one pc inversion that assumes priority in any pertinent musical context, which therefore "ought" to be labelled as "I_0" for that context. The algebra of the pc labels, though, will always supply us with exactly such an "I_0," whenever we use the pc labels.

In what follows, then, I shall show that the system of transpositions and inversions on the total pc chromatic can be developed without having to label pitch-classes. This will leave us free to ignore labelling completely, or, given a specific analytic context, we may adopt any one of a number of possible conventions for labelling pc's and/or inversions, depending on that context. We will examine various possible conventions.

We will start with the unlabelled total chromatic, an unordered collection of the twelve pitch-classes, to which we shall refer simply as C, F-sharp (= G-flat), etc. We shall also presume that we are convinced of the fundamental importance of a construct called "the interval from s to t," where s and t are pitch-classes. This is a function which assigns to the ordered pair s and t a value int(s,t) which is one of the residues modulo 12. The assignment of this value will be made following the most common convention. Thus, int(C,F-sharp) = 6, int(B-flat,D) = 4, int(D,B-flat), the interval *from* D *to* B-flat = 8, int(A,G-sharp) = 11, etc. If we imagine the pitch-classes arranged in the intuitively obvious way around the face of a clock, then int(s,t) will be the number of hours clockwise from s to t.[8]

Now while the pc's themselves do not exhibit algebraic behavior, the intervals do. Specifically, the intervals take on values in the additive group of residues modulo 12, and the intervals compose with one another, and have complements, following the arithmetic of that group. The following two formulas are all-important.

(1) For any pitch-classes s, t, and u, int(s,u) = int(s,t) + int(t,u).

(2) For any pitch-classes s and t, int(t,s) = - int(s,t).

The plus and minus signs are to be considered modulo 12. Formula (1) says that if you measure the interval from s to t, and add to it, mod 12, the interval from t to u, you will end up having measured the interval from s to u. Formula (2) says that the interval from t to s is the mod 12 complement of the interval from s to t. We can also think of the minus sign here as meaning "counterclockwise": if s to t is n hours clockwise, then t to s is n hours *counter*clockwise, which is –n hours clockwise. We shall take the two formulas as established. We shall also take as established a proposition which asserts that, if I give you a pitch-class s and an interval n, you can find a unique pitch-class t which is n hours clockwise from s, i.e., which solves the equation int(s,t) = n. For instance, if I give you s = F-sharp and n = 9, you can produce a unique t satisfying int(F-sharp,t) = 9, namely, t = E-flat.

(3) Given any pitch-class s and any interval n, there is a unique pitch-class t satisfying int(s,t) = n.

We can now define pc transpositions without recourse to pc labelling. The operation "T_n," for n, a residue mod 12, transforms a sample pc s into that unique pc "$T_n(s)$" satisfying

(4) $\mathrm{int}(s,T_n(s)) = n$.

By virtue of proposition (3), given the sample s and the desired interval of transposition n, the value of "$T_n(s)$" is uniquely determined by the relation (4). Notice that the subscript "n" on "T_n" is completely independent of any pc labelling system.

We can also define inversions independent of pc labelling. We shall do so by stipulating a pair (u,v) of pitch-classes such that the inversion is to reflect u into v, and we shall denote the inversion operation that does so by, "I^{uv}." Let us consider such an inversion: given a sample s, let us set $n = \mathrm{int}(u,s)$. Since our sample s is n hours clockwise from u, the inverted image of s will appear reflected as n hours *counter*clockwise from v, the reflected image of u. That $I^{uv}(s)$ is n hours counterclockwise from v can be expressed in our formalism by writing $\mathrm{int}(v,I^{uv}(s)) = -n$. Theorem (3) assures us that one unique pitch-class "$I^{uv}(s)$" is indeed determined by the latter equation. Substituting $n = \mathrm{int}(u,s)$ in the equation, we arrive at the following characterizing equation for I^{uv}, given pitch-classes u and v.

(5) $\mathrm{int}(v,I^{uv}(s)) = -\mathrm{int}(u,s)$ for any pitch-class s.

Via (2), we can write "int(s,u)" for "– int(u,s)" in the above formula, obtaining an alternate characterizing equation for I^{uv}.

(5′) $\mathrm{int}(v,I^{uv}(s)) = \mathrm{int}(s,u)$ for any pitch-class s.

This method defines "inversion taking u to v" for any possible choice of u and v. It works in particular for u = v. In this case we will write "I^{u}" for "I^{uu}" and refer to "inversion about u." While we have a great many names for inversions, there are only twelve distinct operations so defined, each with many names. For instance, inversion taking C to F is the same operation as inversion taking C-sharp to E, which is the same as inversion taking E to C-sharp, which is the same as inversion taking D-sharp to D, etc. That is, if we pick any sample s and compute $I^{CF}(s)$ via formula (5), we will obtain the same result as we would had we computed $I^{\text{C-sharp}\,E}(s)$ via formula (5):

we cannot distinguish the effect of the operations on the total chromatic, so we write $I^{CF} = I^{C\text{-sharp}E}$. Similarly, this operation = $I^{EC\text{-sharp}} = I^{D\text{-sharp}D}$, etc.[9]

In spite of the superfluity of names for inversion operations, it is not implausible to have available the resource of conceptualizing each in so many different ways. Very frequently, the interchange of a certain u with a certain v will be a highly prominent feature of the aural effect of an inversion in a specific musical context. This accords with the (by no means exact) analogy of pitch-contour inversion in tonal counterpoint. Very often one wants to answer $\hat{1}$ and $\hat{5}$, in the inverted form, by $\hat{5}$ and $\hat{1}$ respectively, the two degrees being analogs of our present "u and v." The notion is strongly operative in measure 7 of Schoenberg's opus 23 number 3. One hears the mezzo forte entry of the subject, inverted to begin D–B-flat, very much in connection with the prime form that opened the piece with B-flat–D. (There are to be sure other significant features relating the forms.) Webern, in his serial practice, frequently elides the last two pc's u and v of one row form together with the first two pc's u and v of the suitably retrograde-inverted form: RI^{uv} of the first form.

We shall defer for the moment consideration of any formal proof that there are only twelve inversion operations; such a proof will be indicated later. What will concern us now is a very important and fundamental theoretical question which is obscured when one develops a pc labelled system, but which is presently highlighted. Among the many and various conceivable operations one might perform on pitch-classes, why are transpositions and inversions ones which we particularly want to investigate? The question is obscured when one starts off from the beginning with pc labelling because the logic of the arithmetic forces the operations, as it were, onto the musical-theoretic structure which the arithmetic labels. That is, we have to consider the pc operations "natural" in that context precisely because they reflect "natural" algebraic manipulations of the labels. But if we call into question the conceptual validity of the labelling model, as we have, does there remain any formal (as contrasted with historical-analytic) ground for attributing any particular conceptual significance to transpositions and inversions, as among other possible operations on the total pc chromatic? (N.B.: we are not as yet talking formally about rows; we are simply considering the unordered collection of twelve pitch-classes as a *Gestalt* structured by the interval concept.) The answer is yes, and it can be expressed in the form of a theorem.

(6) Let us say that an operation X on the total chromatic "preserves intervals" if int(X(s),X(t)) = int(s,t) for every s and t. Let us say that an operation Y "reverses intervals" if int(Y(s),Y(t)) = int(t,s) (= –int(s,t)) for every s and t. Then an operation X preserves intervals if and only if X is a transposition; and an operation Y reverses intervals if and only if Y is an inversion.

The theorem is proved in Appendix A. Given that we are interested in "intervals" as structuring the total chromatic, it is certainly natural that we will be particularly interested in those operations which preserve or reverse intervals in the sense of (6), and the theorem tells us that these operations are precisely the transpositions and the inversions. In light of the "if and only if," we could in fact have *defined* transpositions and inversions by those properties. The notion is suggestive in developing analogs for the system in other musical dimensions (particularly various temporal systems). Many finesses would be involved, however, which would lead us beyond the scope of this paper.

Still eschewing pc labelling, one can develop formulas to express the compositions of the operations as presently defined and notated.

(7) (a) $T_mT_n = T_{m+n}$ (the sum modulo 12).
(b) $T_nI^{uv} = I^{uw}$ where $w = T_n(v)$.
(c) $I^{uv}T_n = I^{ux}$ where $x = T_{-n}(v)$.
(d) $I^{uv}I^{wx} = T_n$ where n = int(w,v) – int(u,x).

The proofs are presented in Appendix B. Thus, for example, we have $T_5I^{\text{B-flat}\,E} = I^{\text{B-flat}\,A}$, a special case of (b). This signifies that given any s, the transposition by 5 of the inversion of s via the inversion that reflects B-flat to E is the same pc as is the inversion of s via the operation that inverts B-flat to A. Let us study another example. Since int(F,F-sharp) – int(D,A) = 1 – 7 = 6, we have $I^{D\,\text{F-sharp}}\,I^{FA} = T_6$ as a special case of (d). This signifies that if you invert any s via the inversion that reflects F to A, and then invert the result via the inversion that reflects D to F-sharp, the net result will be to have transposed s by a tritone. Symbolically, $I^{D\,F\text{-sharp}}(I^{FA}(s)) = T_6(s)$.

From formulas (7) one can derive various standard consequences. For instance, via (b) and (c) one infers that $I^{uv}T_n = T_{-n}I^{uv}$. From (d), one can infer that $I^{uv}I^{uv} = T_0$, the identity operation. One can then, via (d), infer that $I^{wx} = I^{uv}$ if and only if int(w,v) = int(u,x); that in turn is the case, via (2), if and only if int(v,w) = int(x,u) which, via (5′), is the case if and only if $I^{uv}(x) = w$. From this, it can be verified that there are exactly twelve distinct inversion operations. Their effects on the "clock" of pitch-classes are easily visualized.[10]

These computations are somewhat cumbersome, compared to the speed and efficiency of the same work using a pc labelling model. But I feel it is very much worth while noting that the work *can* be carried through without recourse to such a model. In particular, the homogeneous group structure of the twenty-four operations is very much highlighted in this development. Rather than singling out one "tonic" inversion operation, which we imagine as being followed (or preceded) by various transpositions, to "derive" the other inversions, we have here what is in fact the conceptual case: twelve a priori equally "natural" inversion operations. The face of a clock without numbers can be flipped over onto itself in twelve equally "good" ways, just as a square can be flipped over onto itself in four equally "good" ways—horizontally, vertically, or about either diagonal. It is true that we may "derive" inversion J as inversion I followed by transposition T, writing J = TI. But we can equally well "derive" I here as J followed by T′ and write I = T′J. Or for that matter, we can "derive" T as I followed by J and write T = JI. Or we can "derive" J as T′ followed by I and write J = IT′. None of these ways of regarding the manifold interrelationships of the operations is a priori any more or any less "natural" than any of the others. Which of these various operations we may decide to regard as "primary," and which as "derived," in a given context, is a matter to be determined, if at all, by that context.

The latter observation is particularly cogent when it comes to devising suggestive notation for the analysis of a particular passage or piece. We can, for instance, label $I^{B\,B\text{-flat}}$ as "I_0" and thence label the twelve inversions as "I_n" = T_nI_0. This makes conceptual sense in Schoenberg's *String Trio* where, if we symbolize the "tonic hexachord" of the piece by "H," $T_n(H)$ and $I_n(H)$, in the above notation, produce respectively the hexachord and its complement transposed by n. Or we can label I^A as "I_0" and then label the twelve inversions as "I_n" = I_0T_n: (N.B. this is *not* the same as T_nI_0!) That notation makes conceptual sense for contexts in which whatever happens in the pitch structure gets mirrored by inversion about A. (See Webern, *Piano Variations* 2nd movement, the opening section of the *Symphony,* or the opening section of Boulez's *Structures I.*) In this notational convention, whenever T_n of a row is *dux,* I_n of that row will be *comes,* and vice versa. For each of these notational conventions, we can derive efficient algebraic formulas for combining the subscripts of the labelled inversion operations with each other and with the transposition subscripts. The formulas will be different, depending on the notational convention selected.

We can, in certain contexts, switch our "I_0" in midstream, letting our notation reflect an analytic assertion of "modulation" in that sense.[11] We could, should a context warrant, stake out an "I_0," a "J_0," etc., to reflect the structuring influence of various inversions during a piece, using formulas (7) to work out efficient algebraic translation formulas from left or right I_n notation (as per the preceding paragraph) to left or right J_n notation, etc. In short, we are free to label (or not to label) inversions as seems most directly suited to our analytic conceptions in a given analytic context,[12] and this without necessarily having labelled the pitch-classes themselves by any fixed-DO or moveable-DO method.

Beyond such analytic considerations, the freedom afforded by the absence of pc labelling in our model can be put to advantage in certain purely theoretical situations. For instance, Babbitt[13] ponders the "sums of set numbers of a collection," in the sense that the differences of such set numbers (my "pc labels") yield the intervallic content of that collection. Pointing out that the label sums also reflect important structural features–common-note relations between the collection and its various inverted forms, he is still concerned that the label differences are "observational" while the label sums are not. That is, loosely speaking, we can say that we "hear" the differences of the labels, as intervals within the collection, while we cannot say that we "hear" the sums of the labels in any similar sense, significant though they are systematically.

The methodological difficulty here can be removed by dispensing with the pc labelling that underlies the formulation of the theoretical constructs under consideration as "sums" and "differences" of such labels. Let us denote by "w" the assumed moveable-DO of Babbitt's labelling system, and by "P" the collection of pitch-classes under examination. The numbers given by his tabulation of label sums are then, as he recognizes, the various values of my "interval function between $I^w(P)$ and P,"[14] or equivalently, Regener's "common-note function between P and $I^w(P)$."[15] Put this way, the question quickly arises, Why, of the various possible inverted forms of P, is $I^w(P)$ to be singled out here a priori? Why not consider instead the interval function between J(P) and P, where J(P) is some other, perhaps more "natural" inverted form of P? The latter interval function will be some cyclic rotation of the former. Suppose, for instance, that P were an invertible hexachord. Would not its complement be the most "natural" inverted form to consider in this connection? But its complement could not possibly be $I^w(P)$: the combinatorial inversion J could not leave w fixed.

In this connection, $I^w(P)$ is in fact being singled out a priori from among the various inverted forms of P only because of the labelling model, which arbitrarily introjects into the general situation a "tonic" pitch-class w, whose tonicity is, in general, extraneous to the intervallic relations obtaining between P and its various inverted forms. The "sums" that appear so methodologically problematic appear so precisely because they *are* extraneous to those intervallic relations, in general: the "sums" are in the picture as such only insofar as the various pitch classes involved are all being labelled by their intervals from the arbitrary w. (Later work will, I hope, clarify this perhaps too cryptic remark.)

To pursue the matter farther, let us suppose that P is an invertible hexachord with combinatorial inversion J. If "f" denotes my "interval function between J(P) and P," then the equation "f(0) = 0" expresses the fact the J(P) and P have no common pitch-classes. If g is the interval function between P and itself (my "intervallic content of P," which differs slightly from Forte's "interval vector"), f and g are related by the simple formula $f(n) = 6 - g(n)$. Thus, the more ways the interval n can appear within P, the fewer ways can it appear in "counterpoint" between J(P) and P. Conversely, the more ways n can appear in the "counterpoint," the fewer ways can it appear within P. All this works strictly as per the formula. In that the intervallic content g of P might reflect, for example, the "overall sound" of the piano part in Schoenberg's *Violin Fantasy,* while the interval function f would reflect the "overall sound" of the counterpoint between piano and violin in that piece, the formula reflects a certain complementary aspect of those two "overall sounds" in the piece, a matter which I find amply observational.

We can find the residue k such that $I^wJ = T_k$. It will then be the case that $I^w = T_kJ$. If f* is the interval function from $I^w(P)$ to P, to which Babbitt's formulation leads us, it can (and later will) be shown that f* is related to f by the formula $f^*(n) = f(n + k)$ for each argument n. Using this relation, we can pass from the observational statement "J(P) and P have no common pitch-class" to the equivalent arithmetic statement "$f(0) = 0$," which now translates to a more complicated arithmetic statement "$f^*(-k) = 0$." The latter, in Babbitt's development, reflects the fact that "there is no way of summing up a pair of w labels of pitch-classes within P so as to add up to -k." It will be noted to what extent the w labels, their sums, and the mysterious "k" here introject extraneous and distracting factors into the theoretical situation. These factors are not actually involved with the behavior of P and its inverted forms at all; they are functions of the relation of that behavior to the extraneous w labelling system. Divested of the distracting labels, what "$f^*(-k) = 0$" actually tells us is that there are no intervals of -k between J(P) transposed k semitones and P. The reason for this does not involve k nor the "tonic" w and its labels in any essential way, but only the relation between J(P) and P. In like manner, the straightforward formula $f(n) = 6 - g(n)$ translates to $f^*(n) = 6 - g(n + k)$. The "k" is again an extraneous feature to the essential situation, and the w labels pile Pelion on Ossa should we interpret the latter equation as signifying "the number of ways we can sum up a pair of w labels within P to a sum of n is equal to 6 minus the number of ways we can subtract a pair of w labels within P to a difference of n + k." This statement is true enough, but it is not a statement about the essential structure of P. Rather it is a statement about the relation of that structure to the extraneous w labelling system, the mysterious "k" arising here from the relation of the extraneous "tonic" inversion I^w to the actual structuring inversion J in the situation. The lack of "observational" character about the statement is due precisely to the fact that it mixes in the notationally fortuitous along with the structurally essential in its arithmetic.

For another example, let P be the collection (C, C-sharp, D, D-sharp, E), and suppose we wish to consider relations between P and its various inverted forms. Leaving aside any specific compositional context in which P might appear, if one wanted to attribute abstract priority to one of the twelve inversions in connection with P, the obvious candidate is I^D, since that operation inverts P into itself. If f is the interval function between $I^D(P)$ and P, then f is simply the intervallic content of P (in my sense). "f(0) = 5" expresses the fact that $I^D(P)$ has 5 common pitch-classes with P. Now suppose one adopts the pitch-class E as a systematic DO, focussing on I^E as a notationally "tonic" inversion. If f* is the interval function between $I^E(P)$ and P, then f*(n) = f(n + 4) for each interval n. As will be shown later, this relation obtains because $I^E = T_4I^D$. Our "f(0) = 5" accordingly translates into "f*(–4) = 5," attributing mysterious significance to the "4" here. The "4," as we have just noted, actually arises from the relation of the structurally natural inversion I^D in this context to the notationally "tonic" inversion $I^E = T_4I^D$.

This, I think,[16] is the "interval of 4" to which Babbitt refers in connection with P as a prime collection for Stravinsky's *In Memoriam.* That is, it arises because he attaches priority to I^E rather than I^D in this context. The priority of I^E in turn arises from his attaching pitch-class priority to E as a DO for the piece, using pc labels accordingly. That choice in turn arises from the fact that P appears in the piece in a thematic linear ordering of which E is the first pitch-class. This does not seem sufficient ground to me for considering E, and I^E, to have such analytic priority in the piece. For instance, the retrograde ordering is also prominent, and begins with D. Furthermore, there are as good (even better, to my ear) grounds for hearing I^D as a "tonic inversion" in the music as for so hearing I^E.

Let me recapitulate somewhat. On abstract grounds, the "interval of 4" here is only confusing. "$f^*(-4) = 5$," divested of the E labels, merely asserts that there are 5 intervals of -4 between $I^D(P)$ transposed by 4 and P itself. The reason for this has nothing essential to do with "4": there are 5 intervals of minus-anything between $I^D(P)$ transposed by that thing and P itself. The reason has everything to do with the fact that $I^D(P) = P$, a fact which indicates that we ought, other things being equal, to attribute priority to I^D, not I^E, in considering P.

"Other things being equal" is an important qualification. If one wanted to assert analytic grounds for I^E potency in the piece, the relation of I^E to I^D *would* be important, the two inversions entering into musical contention or alternation. In that case the concomitant "interval of 4" relating the two inversions would be analytically significant. That is a different matter, though, from the emergence of an E "tonic" and I^E "priority" as a result of notational convention, rather than analytic assertion.

Let me voice another caution, just to touch an obvious base. All this is not to say that the interval of 4 has no *other* special significance in the piece. It obviously does. For one thing, it "spans" P in the obvious sense. But that makes it all the more desirable not to confuse "real" structural 4's with notational will-o'-the-wisp 4's.

In the Stravinsky case, the label-free structure of the situation could still be reflected by appropriate labelling, that is, if one took D as a DO and attached notational priority thereby to what is in fact the natural contextual inversion I^D. But no such expedient could work in the preceding case where P was an invertible hexachord: the structurally privileged combinatorial inversion J could never be an I^w, for any w one tried as a DO.

The mathematical formalities invoked in the foregoing discussion are worked out in Appendix C.

To sum up, I have tried to show that the practice of pitch-class labelling is

(A) theoretically suspect, in that it implicitly asserts that there is always one a priori "tonic" pitch-class, whether fixed or moveable, from which we are to measure intervals to the other pitch-classes in affixing their labels.
(B) theoretically suspect, in that it leads to an implicit assertion that there is systematically one "most natural" inversion I^w, rather than twelve equally plausible inversions.
(C) analytically complicating and confusing, in that even when one wants to assert analytic contextual priority for one of the twelve inversions J, it may be impossible to express J in the form I^w for any w. This must be the case, for example, for any serial piece exploiting a "tonic" level for a combinatorially invertible hexachord.
(D) analytically restrictive, in that it can artificially exclude a priori what may be the "best notation" in a given musical context to reflect the structure one's analysis wants to assert. This obtains even beyond (C) as a special case. The practice more or less imposes one notation in advance of analysis, rather than reflecting the richness of the group structure of the family of transpositions and inversions, leading to many possible contextual notations.
(E) theoretically confusing, in that the mechanics of pitch-class labelling can introduce will-o'-the-wisp intervals and quantities (such as "sums of labels") into theoretical investigations, distracting attention from the structurally essential to the notationally fortuitous.

and perhaps most important, (F) systematically unnecessary, in that all the theoretical constructs one needs can be developed without the aid of pitch-class labelling along the lines of (1) through (7) in the preceding text, together with such adjunct constructs as "interval function," which can be developed label-free. I would particularly like to recall theorem (6) in this connection, which shows there is a formal rationale other than the algebra of pitch-class labels to attribute particular significance to the twelve transpositions and the twelve inversions among the many conceivable possible operations on the total chromatic.

In distinction to points (A) through (F), it is undeniable that pitch-class labelling can be highly useful as a computational and notational tool in many theoretical and analytic contexts. I would only urge that it be recognized as just that, so as to minimize the difficulties and fortuities to which it can lead if allowed to assume a more central theoretical role.

APPENDIX A: proof of theorem (6)

We show that $T = T_n$ preserves intervals: given s and t,

$$\begin{aligned} \mathrm{int}(T(s),T(t)) &= \mathrm{int}(T(s),s) + \mathrm{int}(s,t) + \mathrm{int}(t,T(t)) && \text{via (1)} \\ &= -\,\mathrm{int}(s,T(s)) + \mathrm{int}(s,t) + \mathrm{int}(t,T(t)) && \text{via (2)} \\ &= -\,n + \mathrm{int}(s,t) + n && \text{via (4)} \\ &= \mathrm{int}(s,t) \ldots \text{ as desired.} \end{aligned}$$

We show that $I = I^{uv}$ reverses intervals: given s and t,

$$\begin{aligned} \mathrm{int}(I(s),I(t)) &= \mathrm{int}(I(s),v) + \mathrm{int}(v,I(t)) && \text{via (1)} \\ &= -\,\mathrm{int}(v,I(s)) + \mathrm{int}(v,I(t)) && \text{via (2)} \\ &= \mathrm{int}(u,s) + \mathrm{int}(t,u) && \text{via (5) and (5}'\text{)} \\ &= \mathrm{int}(t,u) + \mathrm{int}(u,s) && \text{which, via (1)} \\ &= \mathrm{int}(t,s) \ldots \text{ as desired.} \end{aligned}$$

Now suppose that X preserves intervals. Fix an arbitrary w and set $n = \mathrm{int}(w,X(w))$. We show that $X = T_n$. For, given any sample s, we have

$$\begin{aligned} \mathrm{int}(s,X(s)) &= \mathrm{int}(s,w) + \mathrm{int}(w,X(w)) + \mathrm{int}(X(w),X(s)) && \text{via (1)} \\ &= \mathrm{int}(s,w) + n + \mathrm{int}(X(w),X(s)) \\ &= \mathrm{int}(s,w) + n + \mathrm{int}(w,s) && \text{since X preserves ints} \\ &= \mathrm{int}(s,w) + n - \mathrm{int}(s,w) && \text{via (2)} \\ &= n. \end{aligned}$$

So X(s) is the pc satisfying $\mathrm{int}(s,X(s)) = n$. Via (4), we conclude that $X(s) = T_n(s)$. This being the case for any sample s, $X = T_n$.

Now suppose that Y reverses intervals. Fix an arbitrary u and let $v = Y(u)$. We show that $Y = I^{uv}$. For, given any sample s, we have

$$\begin{aligned} \mathrm{int}(v,Y(s)) &= \mathrm{int}(Y(u),Y(s)) && \text{substituting } v = Y(u) \\ &= \mathrm{int}(s,u) && \text{since Y reverses intervals.} \end{aligned}$$

But that relation, via (5′), tells us that $Y(s) = I^{uv}(s)$. This being the case for any sample s, $Y = I^{uv}$ as an operation.

APPENDIX B: proofs of formulas (7)

(a) $int(T_{m+n}(s),T_mT_n(s)) = int(T_{m+n}(s),s) + int(s,T_mT_n(s))$ via (1)

$= - int(s,T_{m+n}(s)) + int(s,T_mT_n(s))$ via (2)

$= - (m + n) + int(s,T_mT_n(s))$ via (4)

$= - (m+n) + int(s,T_n(s)) + int(T_n(s),T_mT_n(s))$ via (1)

$= - (m+n) + (n) + (m)$ via (4)

$= 0.$

Since the interval from $T_{m+n}(s)$ to $T_mT_n(s)$ is zero, we conclude that $T_{m+n}(s) = T_mT_n(s)$. This being the case for any sample s, $T_{m+n} = T_mT_n$.

(b) $int(I^{uw}(s),T_nI^{uv}(s)) = int(I^{uw}(s),w) + int(w,T_nI^{uv}(s))$ via (1)

$= - int(w,I^{uw}(s)) + int(w,T_nI^{uv}(s))$ via (2)

$= int(u,s) + int(w,T_nI^{uv}(s))$ via (5)

$= int(u,s) + int(T_n(v),T_nI^{uv}(s)$ substituting for $w = T_n(v)$

$= int(u,s) + int(v,I^{uv}(s))$ via (6)

$= int(u,s) - int(u,s)$ via (5)

$= 0.$

The conclusion of the argument follows as in (a) above.

(c) $int(I^{uv}T_n(s),I^{ux}(s)) = int(I^{uv}T_n(s),v) + int(v,x) + int(x,I^{ux}(s))$

$= int(u,T_n(s)) + int(v,x) + int(s,u)$

$= int(s,u) + int(u,T_n(s)) + int(v,x)$

$= int(s,T_n(s)) + int(v,x)$

$= n + int(v,T_{-n}(v)) = n - n = 0.$

(d) $int(T_n(s),I^{uv}I^{wx}(s)) = int(T_n(s),s) + int(s,v) + int(v,I^{uv}I^{wx}(s))$

$= - n + int(s,v) - int(u,I^{wx}(s))$

$= - n + int(s,v) - int(u,x) - int(x,I^{wx}(s))$

$$\begin{aligned}&= -n + \mathrm{int}(s,v) - \mathrm{int}(u,x) + \mathrm{int}(w,s)\\ &= -n - \mathrm{int}(u,x) + \mathrm{int}(w,s) + \mathrm{int}(s,v)\\ &= -n - \mathrm{int}(u,x) + \mathrm{int}(w,v)\\ &= 0.\end{aligned}$$

APPENDIX C: on the "sums of set numbers" of a collection; mathematical formalities.

Let P be a collection of pitch-classes and w a "tonic" pc. We shall write "wlabel(s)" for int(w,s). (The labelling system labels the pitch-class s according to its intervallic distance from the "tonic" w.) For any pair of pitch-classes s and t, we compute

$$\begin{aligned}\mathrm{wlabel}(s) + \mathrm{wlabel}(t) &= \mathrm{int}(w,s) + \mathrm{int}(w,t) && \text{via definition}\\ &= \mathrm{int}(I^w(s),w) + \mathrm{int}(w,t) && \text{via (5) and (2)}\\ &= \mathrm{int}(I^w(s),t) && \text{via (1).}\end{aligned}$$

Now if f* is the interval function from $I^w(P)$ to P, f*(n) is the number of ways in which the interval n can be spanned from $I^w(P)$ to P, which is the number of pairs (s,t) of members of P such that $\mathrm{int}(I^w(s)t) = n$. Via the preceding computation, this is the number of such pairs whose wlabels sum to n. The notion of "summing" appears only because, in the above computation, the actual interval of interest, $\mathrm{int}(I^w(s),t)$, is broken down by the labelling convention into the sum int(w,s) + int(w,t). That is an aspect of the structure of the labelling convention itself, not of P and its various inversions.

If I^w is the inversion in which we are most interested, contemplating P, well and good. If there is some other inversion J to which we wish to assign systematic or contextual analytic priority, then we will be interested in f(n), the number of ways in which the interval of n can be spanned between J(P) and P; this more than in f*. Setting $I^wJ = T_k$, via (7(d)), we will have $I^w = T_kJ$; then, continuing the earlier computation,

$$\begin{aligned}\mathrm{int}(I^w(s),t) &= \mathrm{int}(T_kJ(s),t)\\ &= \mathrm{int}(T_kJ(s),J(s)) + \mathrm{int}(J(s),t) && \text{via (1)}\\ &= \mathrm{int}(J(s),t) + \mathrm{int}(T_kJ(s),J(s))\\ &= \mathrm{int}(J(s),t) - k && \text{via (4) and (2).}\end{aligned}$$

Thus, whenever f* tabulates an interval of $n = \mathrm{int}(I^w(s),t)$, f will tabulate an interval of $n + k = \mathrm{int}(J(s),t)$, and vice versa.

So $f^*(n) = f(n+k)$. Or, equivalently, $f(n) = f^*(n - k)$. The two interval functions are cyclic rotations, each of the other. And it is f(n) in which we are really interested.

Without involving w or k at all, we can describe f(n) simply as the number of ways of spanning an interval of n from J(P) to P. Working the computations backwards, we may, if we wish, also express f(n) as the number of ways one can add wlabels from P in pairs to sum up to $(n-k)$. The latter formulation, however, involves not just the "natural" structure of P itself as we wish to consider it in the given context, but that structure in relation to an extraneously presumed "tonic" w, and the "tonic" I^w in relation to the actually significant J.

NOTES

1. Hereafter I shall abbreviate "pitch-class" by "pc" ad libitum.
2. Milton Babbitt, "Set Structure as a Compositional Determinant," *Journal of Music Theory* 5(1961): 72–94. The article is reprinted in *Perspectives on Contemporary Music Theory,* edited by Benjamin Boretz and Edward T. Cone (New York: W. W. Norton, 1972).
3. George Perle, *Serial Composition and Atonality* (Berkeley: University of California Press, 1962).
4. Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973).
5. Ibid.
6. Babbitt, "Set Structure," and Perle, *Serial Composition.*
7. Such matters are expounded at length in an earlier article of mine, "Inversional Balance as an Organizing Force in Schoenberg's Music and Thought," *Perspectives of New Music* 6(1968): 1–21.
8. The functionality of this concept arises from the following fact: if p and q are any pitches which belong to pitch-classes s and t respectively, and if N is the number of semitones up from p to q (= 0 if p = q; negative if q is lower than p), then the residue of N mod 12 will always be int(s,t) no matter in what registers the pitches p and q lie. Thus, "int(C,F) = 5" reflects the fact that the interval, in semitones up, between any pitch whose name is C and any pitch whose name is F will always be some multiple of 12 semitones, plus 5. For instance, middle C to Queen of the Night F is 29 semitones; $29 = (2.12) + 5$. High C to contra F is –55 semitones; $-55 = ((-5)\cdot 12) + 5$.

   It should be stressed that this notion of pitch interval, and hence the pc interval it induces, is completely independent of any fixed or moveable DO one might consider, whether on metaphysical grounds, analytic-contextual, or for purposes of tuning.

9. The second movement of the *Piano Variations* illustrates clearly to the ear that $I^{A} = I^{\text{B-flat G-sharp}} = I^{BG} = I^{\text{C F-sharp}} = I^{DE} = I^{\text{E-flat}} = I^{ED} = I^{\text{F C-sharp}} = I^{\text{F-sharp C}} = I^{GB} = I^{\text{G-sharp B-flat}}$ The *String Trio,* particularly measures 1, 41–44, 107½–109½, and the analogous portions of the reprise, does as much for $I^{\text{B B-flat}} = I^{CA} = I^{\text{C-sharp G-sharp}} = I^{DG} = I^{\text{D-sharp F-sharp}} = I^{EF} = I^{FE} = I^{\text{F-sharp D-sharp}} = I^{GD} = I^{\text{G-sharp C-sharp}} = I^{AC} = I^{\text{B-flat B}}$
10. See the early diagrams of Lewin, "Inversional Balance."
11. See my observations on the two powerful inversions in *Die Kreuze*, ibid.
12. Each such labelling system corresponds mathematically to a partition of the group of pc operations under discussion: into the family of left or right cosets of a certain contextually privileged subgroup.
13. Milton Babbitt, "The Structure and Function of Musical Theory: I," *College Music Symposium* 5(1965): 49–60. The article is reprinted in *Perspectives on Contemporary Music Theory.*
14. David Lewin, "Intervallic Relations between two Collections of Notes," *Journal of Music Theory* 3(1959): 298–301.
15. Eric Regener, "On Allen Forte's Theory of Chords," *Perspectives of Music Theory* 13(1974): 191–212.
16. The words "I think" arise from incertitude on my part over Babbitt's exact procedure, since he does not actually cite any form of P, ordered or unordered, nor does he indicate specifically which pitch-class he is assuming as a contextual "tonic" zero. My guess is that he followed the procedure I am about to outline. In that case, his "interval of 4" is a mistake for "interval of 8" (which is –4). Taking E as a zero pitch-class, and considering my cited form of P as E labelled, there will be 5 ways of summing pairs of E labels within P to a sum of 8. It is also possible (but I think not as likely) that, given the cited form of P, he is taking C as the "tonic" zero pitch-class rather than E. In that case, the "interval of 4" comes out right as such. Whatever his procedure, the sense of my following critique applies.