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SOME GROUP PROPERTIES OF TRIPLE COUNTERPOINT AND THEIR INFLUENCE ON COMPOSITIONS BY J. S. BACH

Daniel Harrison

Invertible counterpoints in tonal music are a special and unique class of thematic repetition. Like unvaried reprises, recapitulations, and transpositional sequences, invertible counterpoints reuse thematic material in different parts of a musical form. Unlike these other procedures, which preserve interrelationships among the several melodic lines, invertible counterpoint alters these interrelationships so that the melodic components are rearranged with respect to each other upon repetition. For this reason invertible counterpoint can be a more subtle and covert means of thematic repetition than the other procedures mentioned above, and therefore can become a powerful way to unify musical structure. In the hands of J. S. Bach, who recognized new and unexplored possibilities for invertible counterpoints, it is used so often and with such imagination that the formal economies of his compositions are often extraordinarily simple in relation to the beautifully complex sound of the music.

In the pages following, I will examine both the general properties of triply invertible counterpoints (triple counterpoint) and the use of this compositional technique in some of Bach's music. The reader might well ask why a study of Bach's use of double counterpoint is not prerequisite. The answer lies in the different abstract structure of these two types of invertible counterpoint and the effect these differences have on compositional procedure.

For instance, in double counterpoint with two voices only two possible arrangements of melodic lines are possible: line A as the treble part and line B as the bass part, or line B as treble and line A as bass.[1] Therefore, a passage written in double counterpoint can be used only once more before a repetition of a previous section is heard.[2] Triple counterpoint, on the other hand, has six functionally distinct arrangements of three lines. The compositional implications of this difference between these two types can thus be summed up neatly: for every measure written in double counterpoint one other measure having a different musical foreground can be generated; but for every measure written in triple counterpoint, five other measures can be generated. This difference affects such compositional decisions as how long a passage of invertible counterpoint will be, what structural function will it perform, and how many times it will be used.

Another difference between these two types of invertible counterpoint is that inversions at intervals other than the octave and its equivalents (e.g., @12th, @10th) are common in double counterpoint but rare in triple counterpoint. The compositional possibilities of a double counterpoint invertible at both octave and non-octave intervals are correspondingly greater. In fact they are so interesting as to deserve a separate study. My discussion will not deal with the issues of non-octave inversions in triple counterpoint, intriguing as they are, and will concentrate on inversions at octave intervals.

Finally, Bach quite naturally approaches two-part composition differently from three-part. In keyboard works, for instance, these two types of writing are characteristic of different genres. Comparing double versus triple counterpoint would thus involve comparing not only these abstract theoretical entities, but also their role in compositions that are themselves distinguished by the respective demands of two- and three-part writing. Even in a comparison involving similar genres – Inventions and Sinfonias, for example – it would be difficult, not to say misleading, to atttempt a separation of genre and technique.

Examining the group structure of triple counterpoint discloses some special properties, properties that control the latent structural potentialities which Bach recognized and exploited in this procedure. After examining these group properties, I will discuss their influence on the formal organization of some of Bach's music, showing that triple counterpoint, instead of being some dry-as-dust and esoteric artifice, is a musically satisfying way of gaining structural coherence without sacrificing variety in the musical foreground.

TRIPLE COUNTERPOINT AS A PERMUTATION GROUP OF DEGREE 3. I shall attach the letter-tokens A, B, and C to three separate melodic lines. These tokens can be placed in different *arrangements* by operations called *permutations*. Musically speaking, arrangements reflect the placement of the musical lines in one of three voice parts, treble, middle, and bass. If I define permutation Z to replace token A by token C, token B by token A, and token C by token B, the operation Z can be written symbolically as in Figure 1a, where each letter in the left column is replaced by the letter in the same row in the right column.[3] The notation illustrates permutation Z by showing "before" and "after" arrangements of the tokens. In Figure 1a permutation Z produces the arrangement (C, A, B) from (A, B, C).

In Figure 1b permutation Z acts on a different "before" arrangement (A, C, B), producing a different "after" arrangement (C, B, A). Even though tokens C and B in the left column of Figure 1b have exchanged "registral" positions compared to their counterparts shown in the left column of Figure 1a, this difference does not affect the action of permutation Z; despite the rearrangement in the left column, token C still is replaced by B, and token B by token A, and token A by token C.

Permutations can be combined to produce other permutations. For instance, let us define permutation Y to replace tokens A with A, B with C, and C with B (Figure 2). The product of permutation Z followed by permutation Y is shown in Figure 3a. ZY replaces A by C through permutation Z, and then replaces C by B through permutation Y. (The rows of Y are vertically rearranged in the Figure.) Thus permutation ZY ultimately replaces A with B. By inspecting the other rows of Figure 3a, we can determine the effect of ZY on tokens B and C in the same fashion. Figure 3b shows that the compound permutation YZ produces an arrangement different than that produced by ZY, an important demonstration that multiple permutations can fail to commute.

The notation for permutations found in Figures 1 through 3 is unwieldy, especially when multiple permutations are considered. I will now use a shorthand which breaks permutations down into *cycles*. An illustration of this notation is shown in Figure 4a. Each token in the parentheses is replaced by its successor on the right, and the last token by the first. Permutation Z is thus notated as the cycling of token A into C, C into B, and B into A. Every permutation can be represented in cyclic notation. Permutation Y for instance, shown in Figure 2, is the cycle (A)(B C), where A is cycled into itself, B is cycled into C, and C into B. Adhering to mathematical convention, cycles of a single element, such as (A) in permutation Y, will be omitted, so that permutation Y is written as (B C). Figure 4b shows how the product of permutations can be represented as a cycle. The permutation Y = (B C) takes token A into token A; Z = (A C B) then takes token A into token C. So the net effect of YZ is to take A into C. We can start writing a cycle for YZ as YZ = (A C ?. Since we have considered only the effect of the two permutations on token A, we are unable to state what happens to token C or how token B is affected. To get rid of the question mark in the cycle, we must next compute the effect of YZ on token C. Y = (B C) takes C into B; Z = (A C B) takes B into A; YZ thus takes

$$Z = \begin{pmatrix} A & C \\ B & A \\ C & B \end{pmatrix}$$

Figure 1a

$$Z = \begin{pmatrix} A & C \\ C & B \\ B & A \end{pmatrix}$$

Figure 1b

$$Y = \begin{pmatrix} A & A \\ B & C \\ C & B \end{pmatrix}$$

Figure 2

$$ZY = \begin{pmatrix} A & C \\ B & A \\ C & B \end{pmatrix} \begin{pmatrix} C & B \\ A & A \\ B & C \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \\ C & C \end{pmatrix}$$

Figure 3a

$$YZ = \begin{pmatrix} A & A \\ B & C \\ C & B \end{pmatrix} \begin{pmatrix} A & C \\ C & B \\ B & A \end{pmatrix} = \begin{pmatrix} A & C \\ B & B \\ C & A \end{pmatrix}$$

Figure 3b

$$Z = \begin{pmatrix} A & C \\ B & A \\ C & B \end{pmatrix} = (A \quad C \quad B)$$

Figure 4a

$$\begin{aligned} Y &= (B\ C) \\ Z &= (A\ C\ B) \\ YZ &= (BC)(ACB) = (AC) \end{aligned}$$

Figure 4b

C into A. Because YZ takes A and C into each other, we can now close the parentheses around this cycle. Therefore YZ = (A C)(?. Since only one token is now left to replace the question mark, YZ must take B into itself, and we can deduce YZ = (A C)(B), which is simply (A C) in the simplified notation.[4]

The actions of these permutations within a musical context are illustrated in Example 1, which excerpts portions of Bach's f-minor Sinfonia, BWV 795. The three lines that make up mm. 7–9 are shown in Example 1a. For purposes of illustration, the three lines are separated by voice part (treble, middle, and bass) and are given separate staves. In Example 1b, taken from mm. 31–33, the material of mm. 7–9 is rearranged through the action of permutation Z. Lines A and B now occupy the voice parts formerly inhabited by lines B and C respectively, while line C occupies the former voice part of line A. Bach has rewritten line B slightly (at *) so that it can be a more appropriate bass line than the corresponding section of line B found in Example 1a. In Example 1c, taken from mm. 18–20, the lines are transposed to the key of the dominant and presented in an arrangement generated from the action of permutation Y on the music of Example 1a. Again line B is slightly rewritten, this time changing register at †.

As mentioned earlier, six functional arrangements of tokens A, B, and C are possible. There are also six permutations which can produce all these arrangements from any given one. Both permutations and arrangements are shown in Table 1. All six permutations are shown with the same "before" arrangement (A, B, C). This arrangement is taken into the six possible arrangements shown in the right column of each group; the permutations which effect these transformations are also written in cyclic notation. The permutation that takes the tokens A, B, and C into themselves is the *identity* permutation, labelled I in the Table.

The multiplication of any two of the six permutations produces one of those six permutations. These products are shown in Table 2. Each entry on the Table is the product of the cycle at the left of its row with the cycle at the top of its column. For example, the permutation (A B)—entered in the fifth row of the third column—is the product of (A C), which leads the fifth row, and (A C B), which tops the third column. Thus (A B) = (A C)(A C B). The Table reveals that three important properties of mathematical group structure have been satisfied: first, the product of any two permutations within the group belongs to the group; second, an identity element is found within the group; third, an inverse for any permutation is contained within the group. By an "inverse" for a generic permutation P, I mean a permutation Q satisfying PQ = QP = I. One writes $Q = P^{-1}$.[5]

The box in the top left corner of Table 2 encloses a proper subgroup *H* in *G*—that is, a subset of *G* which is itself a group. Besides the identity permutation, subgroup *H* contains the 3-cycle permutations (A B C) and (A C B). The 2-cycle permutations (A B), (A C), and (B C) do not form

Example 1

$$\begin{pmatrix} A & B \\ B & A \\ C & C \end{pmatrix} \quad \text{or} \quad (A\ B) \qquad\qquad \begin{pmatrix} A & B \\ B & C \\ C & A \end{pmatrix} \quad \text{or} \quad (A\ B\ C)$$

$$\begin{pmatrix} A & A \\ B & C \\ C & B \end{pmatrix} \quad \text{or} \quad B\ C) \qquad\qquad \begin{pmatrix} A & C \\ B & A \\ C & B \end{pmatrix} \quad \text{or} \quad (A\ C\ B)$$

$$\begin{pmatrix} A & C \\ B & B \\ C & A \end{pmatrix} \quad \text{or} \quad (A\ C) \qquad\qquad \begin{pmatrix} A & A \\ B & B \\ C & C \end{pmatrix} \quad \text{or} \quad (A)\ \ (B)\ \ (C) \ = \ \text{I (identity)}$$

Table 1

a subgroup since they do not satisfy the requirements for group structure under multiplication. Nonetheless, it will be instructive to consider the 2-cycle permutations as a subset K of G. The following properties about both K and H should be noted:

1. The elements of K combine into elements of H. (The elements of subgroup H, by definition, combine among themselves.)
2. Each permutation in K is its own inverse. The two 3-cycle permutations in H are inverses each of the other.

The group-theoretic distinction between 2-cycle and 3-cycle permutations has a strong effect on the analysis of their respective actions in musical composition. 3-cycle permutations, for instance, apply to genuine and proper triple counterpoints since all three lines are rearranged, as seen in the transformation of Example 1a into 1b under the 3-cycle permutation (A B C). The 2-cycle permutations, however, can be analyzed as double counterpoints of two out of three lines. For instance, the permutation (A C) which changes the arrangement of lines in Example 1b into that of Example 1c does not affect the relative position of line B. Hence, it is possible to describe Example 1c as a double counterpoint of the top two lines @8ve with respect to Example 1b.[6]

Focusing now on the 3-cycle permutations of subgroup H, we find that because of the different inverse relationships with respect to G, H satisfies the conditions of a *cyclic group,* a group which can be generated by repeated combinations of one element within the group.[7] H can be generated cyclically in two related ways. Using superscripts to indicate how many times a permutation is combined with itself, we write: H = (A B C), $(\text{A B C})^2$ = (A C B), $(\text{A B C})^3$ = I; or alternately: H = (A C B), $(\text{A C B})^2$ = (A B C), $(\text{A C B})^3$ = I.

The cyclic property of subgroup H is of particular interest because it can be used to generate two distinct arrangements of tokens from a starting arrangement. This type of generation is shown in Figure 5. In Figure 5a, permutation (A C B) is applied again and again, starting with arrangement #1; this produces arrangements #2 and #3 before cycling back to #1. In Figure 5b permutation (A B C) is used in the same way. Here, given the starting arrangement, only two other arrangements are possible no matter which 3-cycle permutation is used; the choice of permutation affects only the order in which the arrangements appear. Figure 5b retrogrades Figure 5a; that is because (A B C) = $(\text{A C B})^{-1}$.

Also affected by the choice of permutation, in Figures 5a and 5b, is the overall registral effect of a musical realization. In Figure 5a permutation (A C B) continually overlaps the top two tokens with a token which had previously been on the bottom. When realized musically, this characteristic of the Figure will give the illusion that the corresponding musical lines are "sinking" to the bass with each subsequent arrangement. In Figure 5b the

	I	(A B C)	(A C B)	(A B)	(A C)	(B C)
I	I	(A B C)	(A C B)	(A B)	(A C)	(B C)
(A B C)	(A B C)	(A C B)	I	(B C)	(A B)	(A C)
(A C B)	(A C B)	I	(A B C)	(A C)	(B C)	(A B)
(A B)	(A B)	(A C)	(B C)	I	(A B C)	(A C B)
(A C)	(A C)	(B C)	(A B)	(A C B)	I	(A B C)
(B C)	(B C)	(A B)	(A C)	(A B C)	(A C B)	I

Table 2

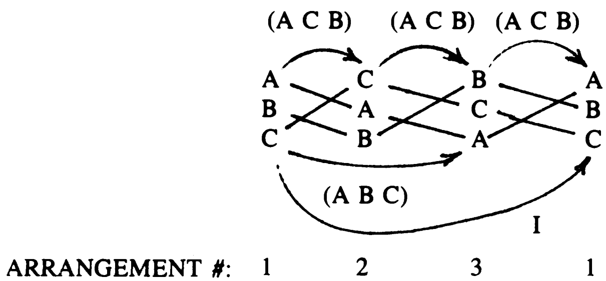


Figure 5a

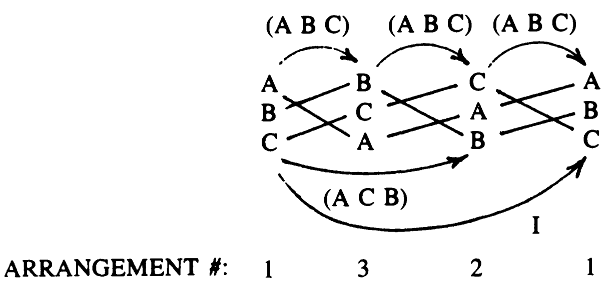


Figure 5b

situation is reversed. The apparent motion of musical lines will be in the opposite direction; they will "rise" to the treble with each subsequent arrangement. I will thus characterize the cyclic application of a 3-cycle permutation as either a TB progression ("treble to bass") as in Figure 5a, or as a BT progression ("bass to treble,"), as in Figure 5b.

The remaining three arrangements not shown in Figure 5 can also be generated cyclically if a starting arrangement different from any of the first three is used (Figure 6). An arrangement of Figure 6 can be generated from an arrangement of Figure 5 only by means of some 2-cycle permutation from subset *K*. Figure 7 shows how the various arrangements of Figure 6 are related to those of Figure 5 by 2-cycle permutations. I will henceforth refer to these two collections of three arrangements each as the two *conjugations* of subgroup *H*. I intend the word "conjugation" here as a grammatical metaphor. Later on, I shall show how Bach "runs through" such conjugations in segments of his musical discourse.

Some interesting properties of the two conjugations are these:

1. When a 3-cycle permutation from *H* is used to generate both conjugations, one conjugation will have a TB progression and the other a BT progression.
2. If each arrangement of lines in a conjugation makes correct three-part counterpoint, then each arrangement in the other conjugation will also make correct counterpoint.

The first property can be verified by examining Figures 5 and 6. The second property results from the generational principle of the conjugations, a principle that causes every line to be cycled through the bass once in each conjugation. Since the integrity of an invertible counterpoint depends upon the formation of permissible intervals above the bass, the success of line A, for instance, as the bass of an arrangement in one conjugation guarantees its success as a bass in the other conjugation, the inversion of the other two parts notwithstanding.[8]

BACH'S USE OF CONJUGATIONS IN COMPOSITION. Bach recognized new opportunities for structural control afforded by grouping triple counterpoints into conjugations. In two compositions in which all six arrangements of a triple counterpoint are used, Bach deployed the conjugations in such a way that they become important structural pillars in their own right, transcending their apparent function as an organizational convenience.

In the D-major Sinfonia, BWV 789, the two conjugations appear at the beginning and end of the piece to create an unusual type of exposition and recapitulation (Figure 8). This conjugational structure is given additional, thematic importance by the role one of the melodic lines plays as the main subject of the Sinfonia. The three lines are shown in Example 2 as they

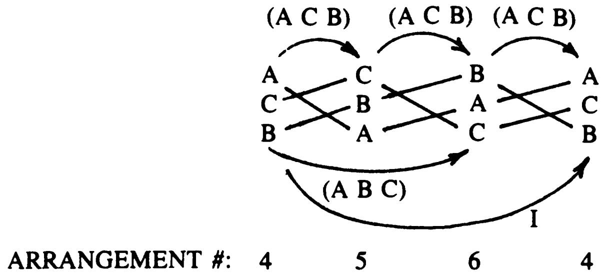


Figure 6a

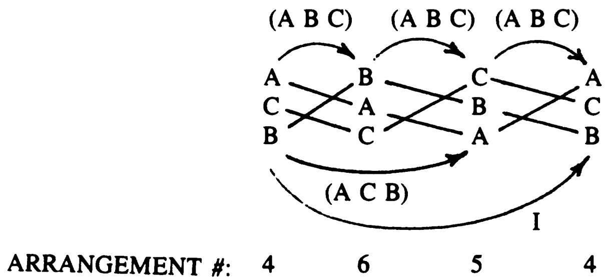


Figure 6b

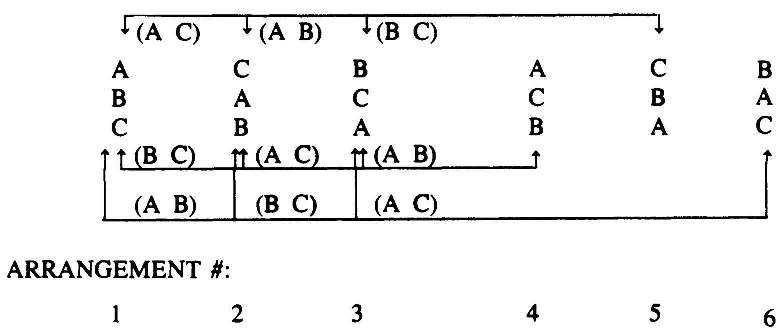


Figure 7

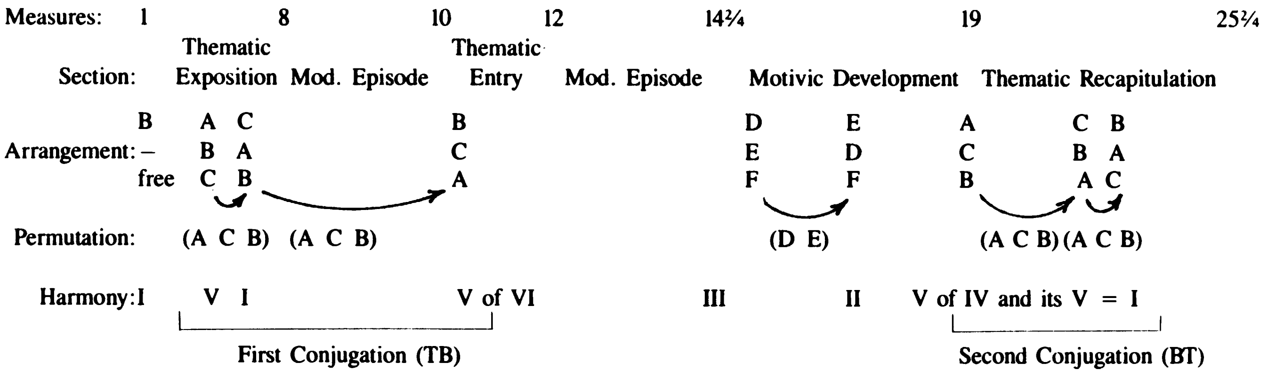

Measures: 1 8 10 12 14¾ 19 25¾
Section: Thematic Exposition Mod. Episode Thematic Entry Mod. Episode Motivic Development Thematic Recapitulation
Arrangement: B – free
A B C
C A B
B C A
D E F
E D F
A C B
C B A
B A C
Permutation: (A C B) (A C B) (D E) (A C B) (A C B)
Harmony: I V I V of VI III II V of IV and its V = I
First Conjugation (TB)
Second Conjugation (BT)

Figure 8

appear in the second half of measure 3. Again, the lines are separated by voice part and presented on their own staves. The brackets enclose those portions of the lines which are common to five of the six contrapuntal arrangements; the remaining arrangement uses shortened versions of lines A and C, an abridgement which will be discussed below. Thematically, the lines used in this triple counterpoint correspond to the subject of the Sinfonia (line B of Example 2), and two countersubjects (lines A and C).

A fugue-like thematic exposition comprises the first section, mm. 1–8. The subject first appears in the top voice, accompanied by a bass line while the middle voice is silent. The answer follows in the middle voice (mm. 3–4) while the outer voices have the two countersubjects (Example 2). This is the first appearance of all three lines to be used in the triple counterpoint. The thematic exposition ends after the completion of the bass subject in mm. 6–8. The arrangement of voices accompanying the bass subject in mm. 6–8 is generated by the permutation (A C B) acting on the arrangement of mm. 3–5. The two three-voice thematic statements in the exposition thus belong to the same conjugation, as shown on Figure 8.

Although the thematic exposition ends at m. 8, the first conjugation of triple counterpoint is not yet completed. A short modulating episode in mm. 8–10 separates the first two arrangements of the conjugation from the final arrangement, which begins in m. 10. This episode, by isolating the thematic exposition, emphasizes fugue-like aspects of the structure. Yet, the thematic statement in m. 10, which would be appropriately located in the form of a fugal structure, asserts an importance beyond that of a typical thematic entry in traditional theories pertaining to fugue. The entry completes the first conjugation, thus closing as it were a "*conjugational* exposition." The sense of closure is projected by the strong authentic cadence on VI in m. 12, the first authentic cadence since the thematic exposition. The importance of the arrival on VI in m. 12 is indicated by the length of the dominant which supports the thematic entry—the only time the theme enters in a dominant rather than a tonic context. The arrival at m. 12 is also emphasized by the immobile bass on the long tonic B following the downbeat of that measure. This strong rhetoric, making the cadence an appropriate ending for the conjugational section, causes lines A and C to be altered in the triple counterpoint arrangement at the third beat of m. 11, an alteration that accommodates the unusual harmonic setting of the thematic statement.[9] These features are all associated with the completion of the first conjugation, and they all delineate a strong closure. The term "conjugational exposition," which I introduced above, is appropriate to reference the formal section.

Just as the thematic exposition was isolated by an episode, so too is the conjugational exposition. An episode modulating to III in mm. 12–14 follows the final arrangement of the first conjugation. This episode leads not to another thematic statement, as did the first episode, but to yet another episode (mm. 14–18), one in which motivic development is the primary feature instead of harmonic modulation. The developmental episode of mm. 14–18, together with the modulating epsiode of mm. 12–14 which precedes it, comprises the longest section of the piece during which the theme is not heard in its entirety.

This large middle section of the Sinfonia, mm. 12–18, separates the two conjugations of triple counterpoint. As Figure 8 shows, the second conjugation begins to unfold at m. 19, where the subject simultaneously returns. The section preceding m. 19 ends with a pedal point supporting a local V^7, a gesture which in the context suggests an upcoming structural boundary. And indeed we get a full thematic recapitulation over mm. 19–25. The recapitulation is coupled to a harmonic structure typical of a thematic exposition – that is, a subject-answer relationship in the first two thematic statements. However, Bach skews the harmonic relationship by beginning the recapitulation on the subdominant. The placement of the answer on the required "dominant" of the subdominant allows Bach to exploit the double function of this harmony deftly, sneaking into the tonic, as it were, by the back door.

Unlike the treatment given to the first conjugation, the second is presented as an uninterrupted block of music. No arrangement is separated from any other by an episode. The identification of conjugation with subject is thus made complete; the re-exposition of the subject in mm. 19–25 coincides with the exposition of the second conjugation. The complementary relationship of the two conjugations is given further strength by the use of the same permutation (A C B) to generate the arrangements which, as was mentioned above, results in a TB progression in the first conjugation and a BT in the second.

While the second conjugation varies the first both in the arrangements heard and in the rising sense of the BT progression, there is still a sense in which we can say that mm. 19–25 give a conjugational "recapitulation" of mm. 1-12. The sense is that which David Lewin calls transformational "isography."[10] The diagram of Figure 9 shows a graph illustrating this idea. Each small circle (node) on the diagram represents a formal "place" where a contrapuntal arrangement can occur. The nodes can be filled with "contents" in a variety of ways. If we input the arrangement (A, B, C) as the contents of the left-hand node on the graph, the indicated transformational chain will produce arrangement (C, A, B) as the contents of the middle node, and arrangement (B, C, A) as the contents of the right-hand node; this gives us the overall content of a network modeling the conjugational exposition of the Sinfonia. If, on the other hand, we input the arrangement (A, C, B) into the left-hand node, the chain will produce (C, B, A) as the contents of the middle node, and (B, A, C) as the contents of the right-hand node; this models what I have called the conjugational "recapitulation." What is "recapitulated" here is the action represented in the transformational graph.

Example 2

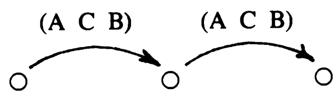


Figure 9

Example 3

As we know, the relationship between the two conjugations is determined by the 2-cycle permutations of subset *K*. As if to underscore this trans-conjugational role, Bach uses one of the 2-cycle permutations on three new lines in the section of motivic development which separates the two conjugations, mm. 14–18. In Figure 8, I have marked these three new lines as D, E, and F in order to illustrate the permutation. As one sees by comparing Example 3 to Example 2, lines D and E are made up of head and tail motives from the subject of the Sinfonia, and line F is derived from parts of the countersubject of line C, as well as from parts of the subject. Despite their close relationship to the lines used in the conjugations, lines D, E, and F participate in an invertible counterpoint separate from that of lines A, B, and C. As Figure 8 shows, the arrangement (D, E, F), which starts in the second half of m. 14, is permuted by (D E) to produce the arrangement (E, D, F) starting in m. 16. The use of the 2-cycle permutation (D E) on motivically related lines subtly adumbrates the upcoming change of conjugation while still maintaining the separation of the two structures.

The F♯-major Fugue from the Well Tempered Clavier, Book 2 also uses two conjugations of triple counterpoint to develop a structural parallelism. In Figure 10 two formal conjugations are bracketed involving tokens A, B and C. At first glance, the widely separated, block-like deployment of the conjugations seems to be the same structural idea used in the D-major Sinfonia, the only difference being that each conjugation makes a complete cycle under permutation (A B C); the first and last arrangements of each conjugation are thus identical.[11] However, Bach invests this technique of deployment with a more sophisticated construction that governs connections between arrangements in both conjugations. This innovation allows him to connect the conjugations to the other structural parallelisms shown in Figure 10, creating a highly unified and economical composition.

The three lines organized by conjugation are shown in Example 4 as they appear in m. 13. Although the Example shows the vertical fit of the three lines, it cannot reveal the remarkable horizontal organization which controls the unfolding of the individual conjugations. As Figure 10 shows, the arrangements in each conjugation appear in uninterrupted sequence. It is thus possible to examine a particular voice part in each conjugation to discover how each line of the triple counterpoint is connected to its successor in that voice. The results of such an examination are shown in Example 5. The treble line of mm. 13–20 is used as a reference; each line of the triple counterpoint is marked off by brackets—including the two appearances of line A. Between the brackets are "connectors" which bridge two lines of the triple counterpoint. Below the connectors found in the treble of mm. 13–20 are shown all the remaining connectors used in each voice part in both conjugations, transposed to the key of the Example. Connectors having variant forms are grouped together, such as connectors α, β, and ε. With respect to the connectors used in the treble line of mm. 13–20, the task of connectors

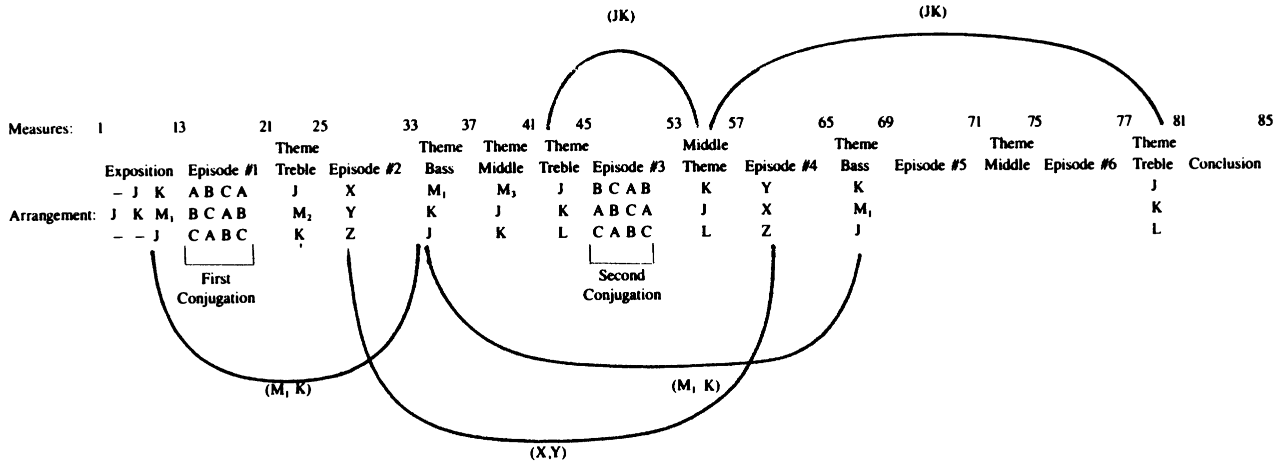
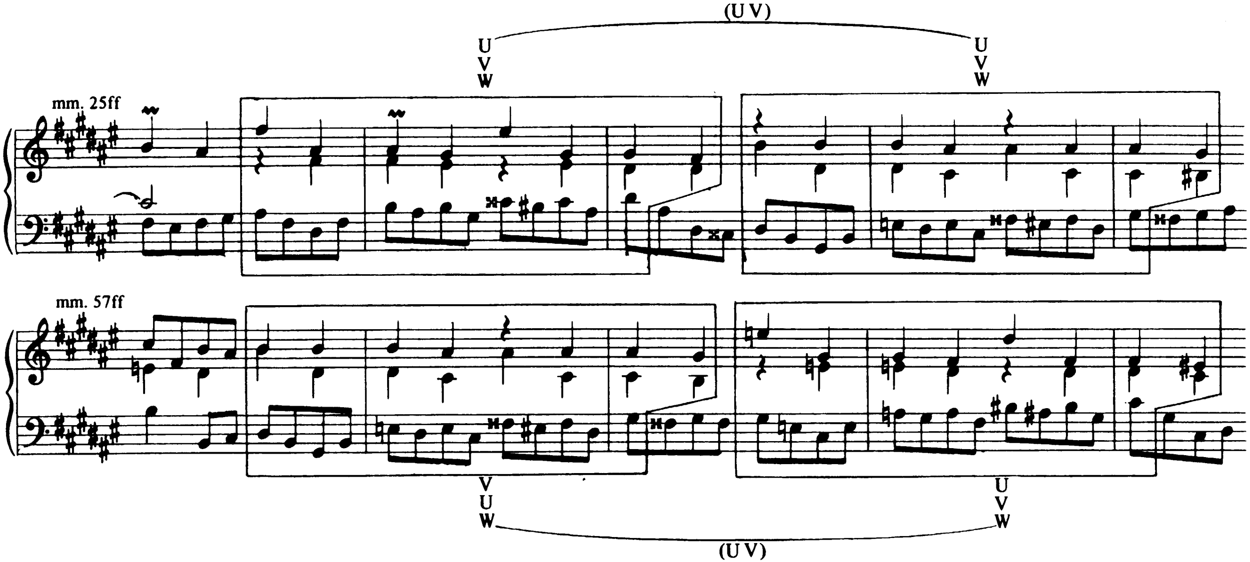

(JK)
(JK)
Measures: 1 13 21 25 33 37 41 45 53 57 65 69 71 75 77 81 85
Exposition
Episode #1
Theme Treble
Episode #2
Theme Bass
Theme Middle
Theme Treble
Episode #3
Middle Theme
Episode #4
Theme Bass
Episode #5
Theme Middle
Episode #6
Theme Treble
Conclusion
Arrangement:
– J K
J K M₁
– – J
A B C A
B C A B
C A B C
J M₂ K
X Y Z
M₁ K J
M₃ J K
J K L
B C A B
A B C A
C A B C
K J L
Y X Z
K M₁ J
J K L
First Conjugation
Second Conjugation
(M₁ K)
(M₁ K)
(X,Y)

Figure 10

Example 4

Example 5



Example 6

β, γ, and ε_2 is to bridge a change in register. Connector ζ also effects a change of register, but it also slightly alters the coupling between line C and A in some arrangements.

A complete summary of connectors and arrangements in both conjugations is shown in Figure 11. Where no special connector is indicated, the bridge between two lines is the same connector used in the treble passage, mm. 13–20, shown in Example 5. Discounting the use of variants and of connector ζ, the progression of lines in both conjugations is identical. For instance, the connectors in the bass line of both conjugations are varieties of ε, α, and δ. In fact, correspondences between the connectors and arrangements is so close that it is possible to conceive of each voice part in a conjugation as a concatenation of the lines of triple counterpoint, as illustrated in Figure 12. For instance, the bass part, made up of lines C, A, B, C, is defined as a composite line F; B, C, A, B is defined as a composite line E; and A, B, C, A is defined as line D. The relationship between the two conjugations can then be roughly described as the action of permutation (D E) on the composite lines. The result of this organization is that mm. 13–21 and mm. 45–53 can be analyzed as having two levels of triple counterpoint, a small-scale relationship by conjugation-shift, with transformational isography, and a large-scale, somewhat freer relationship by the 2-cycle permutation (D E).

Multiple levels of triple counterpoint are also found in another pair of episodes, mm. 25-33 (episode #2) and mm. 57-65 (episode #4). Example 6 shows the beginnings of both episodes and also marks off the triple counterpoints within the excerpt. Both episodes are made up of two-measure units which are repeated using a 2-cycle permutation that affects only the top two lines. As seen in the example, the lines are labelled U, V, and W and are permuted by (U V). This 2-cycle permutation is used throughout both episodes. However, the ordering of arrangements of the two-measure units of episode #2 is reversed in episode #4—that is, the arrangement (U, V, W) begins episode #2 at m. 25, but arrangement (V, U, W) begins episode #4 at m. 57. If the same process of linear concatenation used above with the conjugations is employed here, the relationship between these two episodes is defined by a large-scale, 2-cycle permutation, labelled (X Y) in Figure 13. The similarities in the structures of triple counterpoints between these two episodes and the episodes containing the conjugations is striking; the significant difference is that a single type of permutation—one that works on only the top two lines—is used to create the double level of triple counterpoint in episodes #2 and #4.[12]

As Figure 10 shows, other structural parallelisms resulting from the application of the same type of 2-cycle permutation are found throughout the fugue. The triple counterpoints associated with the fugue theme are perhaps the most striking. Example 7 shows the openings of the fugue subject (J), principal countersubject (K), and four additional secondary countersubjects

	FIRST CONJUGATION							SECOND CONJUGATION						
mm.	13		15		17		19	45		47		49		51
	A		B		C		A	B	γ	C	ζ	A	β_2	B
	B	γ	C		A	β_1	B	A		B		C	ζ	A
	C	ε_1	A	α_1	B	δ	C	C	ε_2	A	α_2	B	δ	C

CONNECTORS

Figure 11

$$\begin{array}{l} A + B + C + A = D \\ B + C + A + B = E \\ C + A + B + C = F \end{array} \qquad \begin{array}{l} B + C + A + B = E \\ A + B + C + A = D \\ C + A + B + C = F \end{array}$$

(D E)

Figure 12

mm. 25–33	mm. 57–65
U + U + U + V = X	V + U + V + U = Y
V + U + V + U = Y	U + V + U + V = X
W + W + W + W = Z	W + W + W + W = Z

(X Y)

Figure 13

used throughout the fugue (L, M_1, M_2, and M_3).[13] It is readily apparent from the Example that countersubjects M all work on the same voice-leading principles: a span of an ascending fourth sequenced down a step. With the exception of M_2 and M_3, which appear only once in the fugue, the secondary countersubjects particpate in permutations of triple counterpoint, as seen in Figure 10. All these permutations are the type of 2-cycle permutation affecting the top two lines.

Because of the use of this permutation throughout the fugue, extending even to the conjugational relationship, structural units can be analyzed based on the same principles of linear concatenation used in connection with other episodes. For instance, mm. 21–33, containing episode #2 and a thematic statement in the bass, can be related to mm. 57–65 by means of a large-scale permutation (N O), illustrated in Figure 14.

On a scale even larger and—it must be admitted—more inexact than those examined so far, the first sixty-nine measures of the fugue can be roughly divided into formally comparable and parallel sections (Figure 15). Bracket 1 encloses the sections discussed in the previous paragraph related by the permutation (N O). Bracket 2, in addition to the sections encompassed in bracket 1, encloses sections containing a conjugation followed by some thematic statement. Bracket 3 increases the area covered to include the thematic statements of the exposition and the "expositional" thematic statements of mm. 33–45—so named because of the progressive placement of the fugue subject in each voice without the interruption of an episode.[14] These parallelisms are, of course, not exact. Short figures connecting the individual sections are often different in the parallel sections, as are the location of subjects in the various thematic statements and choice of their accompanying countersubjects. Yet the many contributions of triple counterpoint to the construction of parallelisms in the fugue are undeniable and impressive. Bach's integration of structures generated from the different properties of 2- and 3-cycle permutations shows a profound understanding of the compositional opportunities afforded by triple counterpoint and opens up new and fruitful areas of inquiry for the analyst.

The majority of Bach's works which use triple counterpoint do not employ all six possible arrangements. Rather, triple counterpoint is generally a secondary means of organization in the service of other structures—harmonic, thematic, and so forth. When five or fewer arrangements are used in a composition, the conjugations cannot be explicitly used in their entirety. Yet even then Bach does not completely abandon the conjugational principle, for in many cases one complete conjugation is used while members of the other are smoothly incorporated into the structure.

In a few cases, Bach makes a point of using only one complete conjugation. This practice, to one familiar with his techniques, defines the complementary relationship of the two conjugations almost as clearly as is possible by the use of both conjugations. Because the contrapuntal integrity

Example 7

mm. 25–37	mm. 57–69
X + M_1 = N	Y + K = O
Y + K = O	X + M_1 = N
Z + J = P	Z + J = P

(N O)

Figure 14

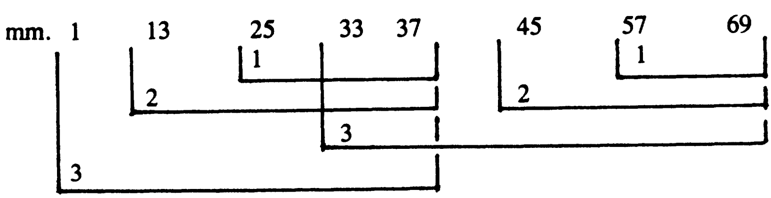


Figure 15

of one conjugation guarantees the integrity of the other, the remaining three arrangements are, in a sense, present by implication. A good example of a single conjugation used in this way is found in the middle section of the organ Tocatta in F major, BWV 540 (mm. 218–238, 270–290, 332–352).[15] Instead of grouping the arrangements of the conjugation together in a single block, Bach separates each arrangement by lengthy sequential passages. Since the passages in triple counterpoint are not part of the surrounding sequences, they are a welcome relief from the otherwise unrelenting linear motion, standing out as islands of structural stability in the sea of sequences. The roughly equidistant placement of the arrangements thus creates a three-part division within the particular section of the tocatta. By expanding the structural role of each arrangement, Bach changes the structural role of the conjugation itself; he emphasizes here the division of a single conjugation into three arrangements, rather than the articulation of an entire triple counterpoint into two conjugations.

When in addition to a complete conjugation, an arrangement or two belonging to the other conjugation is used, Bach often integrates the arrangements from two conjugations into a composite structure by taking advantage of the properties of 2-cycle permutations under multiplication. Figure 16 shows the progression of arrangements in Contrapunctus 8 from the *Art of Fugue,* BWV 1080; the lines of this triple counterpoint are the themes of the triple fugue. As the Figure shows, during the unfolding of the conjugation by the action of permutation (A C B), an arrangement from the other conjugation is intercalated between the third and fourth (= first) arrangements. By using two successive permutations from subset *K* which produce the permutation (A C B) under combination, the arrangement from the other conjugation is gracefully introduced into the scheme. The use of permutations (B C) and (A C) actually reinforces the governance of permutation (A C B) as the overall generator of the conjugation by diminuting it, so to speak, into two permutational factors; the intercalated permutations (B C) and (A C) are as it were "passing permutations" filling in the overall (A C B).

An example of conjugational intercalation in a more complex structure is found in the f-minor Sinfonia, BWV 795. The three lines of its triple counterpoint were already displayed in Example 1. Figure 17 shows the progression of the various arrangements. The lines on Figure 17 are labelled according to the scheme of Example 1, which accounts for the unusual labelling of their first complete arrangement (m. 3) as (C, A, B). Although nine arrangements are found in the Sinfonia, only four different possibilities of the triple counterpoint are used: a complete conjugation containing three arrangements and an arrangement from the other conjugation. The complete conjugation appears twice, unfolded first through a BT progression as part of the thematic exposition, and then through a TB progression in the final section, a progression interrupted only by a temporarily immobilizing

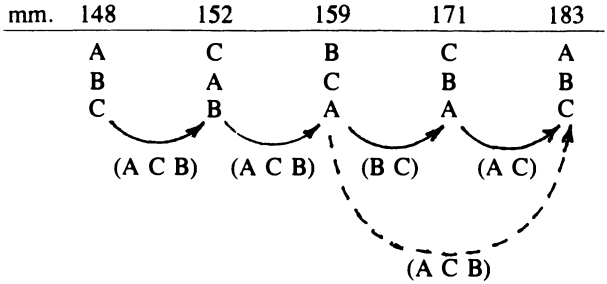


Figure 16

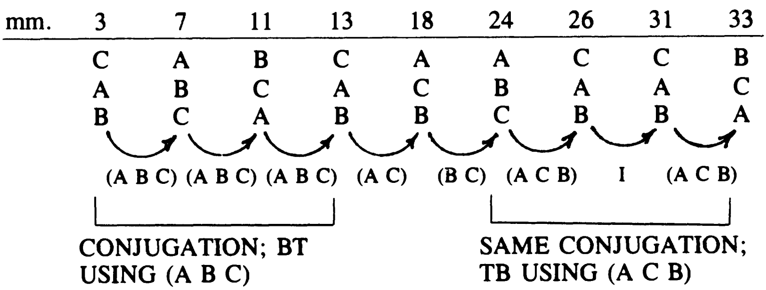


Figure 17

identity permutation. The two unfoldings are separated by an intercalated arrangement from the other conjugation which not only provides a buffer between the two unfoldings, but also reinforces the status of line B as the favored bass. The more natural bass line A—more natural because of its widespread use as a ground bass—is saved for structurally climactic points: the first harmonic turn away from the tonic-dominant relationship of the exposition in mm. 11–13, and the final appearance of the three lines in mm. 33–35.

The deployment and participation of the triple counterpoint in the f-minor Sinfonia is reminiscent of the technique in the D-major Sinfonia, as befits two compositions of the same genre and collection. In both works, the triple counterpoint is organized into two conjugational blocks having different directional progressions and separated by a section rearranged by one of the 2-cycle permutations. It is ironic that the well-known f-minor Sinfonia is often cited as a *tour de force* of triple counterpoint even though it possesses only four of the six combinations, while the relatively unheralded yet conjugationally complete D-major has rarely been acknowledged in theoretical literature as a significant technical achievement.[16]

* * *

I hope that the preceding and necessarily incomplete survey of triple counterpoint in Bach's compositions has shown some of the ways in which this most disciplined of composers handled materials that themselves were disciplined by the laws of both music and mathematics. I would hardly intimate that Bach, despite his wide learning, anticipated by almost a century the development of mathematical group theory. I believe it is nonetheless clear that Bach, the consummate contrapuntalist, understood that the arrangements of triple counterpoint could be neatly and suggestively collected into what I have termed conjugations.[17] It is reasonable to believe that the rotational property of the two conjugations—a property which enabled all lines to appear in every voice in only three arrangements—attracted Bach's attention and interest, as it did my own when I first started my researches into triple counterpoint. However, this property, as well as others having compositional implications, is but one part of the entire group structure of triple counterpoint, a structure which nourishes it as a compositional artifice. The grafting of group-theoretic ideas onto the relatively simple property of rotation does not then join two separate conceptual varietals, but rather attaches a vine to a branch of its own kind. The result endows invertible counterpoint with more powerful and suggestive theoretical support as well as wider ranging analytic implications. If we have long enjoyed the fruits of invertible counterpoint, we can now appreciate its cultivation.

NOTES

1. This apparent tautology ("double counterpoint in two voices") is actually a necessary specification; in the course of this article I will touch upon some double counterpoints which occur in three-voiced music.
2. It is, of course, possible to place a "double-counterpoint-of-a-double-counterpoint" (A over B changed to B over A changed back to A over B) at a different pitch level than the original A over B arrangement. This corresponds to a transposed repetition of A over B and is more often heard as such rather than as a double counterpoint of B over A. A more complicated situation arises when the "transposed repetition" is inverted at an interval that makes the registral distance between the two parts different from that found in the original passage. Besides the difference in registral space, such a change can also create voice-crossings not found in the original or, alternately, eliminate such voice-crossings, thereby changing slightly the shape of the musical foreground.
3. This notation is different that that used in standard texts on group theory, where A, B, and C are displayed horizontally and the permutation is written below it. My notation better reflects the standard notational arrangement of contrapuntal lines in registral space and will be useful when discussing examples of triple counterpoints in music. A useful reference work that discusses permutation groups is Walter Ledermann, *Introduction to the Theory of Finite Groups* (New York: Interscience Publishers, 1957), pp. 62–95.
4. Figure 3b shows the multiplication of the same two permutations Y and Z in non-cyclic notation.
5. The other requirement for group structure, that the combination of permutations be associative, is also satisfied, but cannot be seen at once by inspecting the Table.
6. It is sometimes more profitable analytically to describe invertible counterpoints of three-part music in just this way. For instance, Bach's organ sonatas—like others of his trio sonatas—are conceived as two treble voices and a bass. The upper two parts are often inverted under permutation (A B) while the continuo-like line C in the bass is left untouched. The technique is used, for example, in Sonata I BWV 525, I, mm. 1-4 and mm. 22-25; Sonata VI BWV 530, I, mm. 21-28 and mm. 29-36.
7. "Cyclic notation" and "cyclic group" are completely different ideas; the first is a notational convenience used with permutations, the second is a special property of certain groups which might or might not be permutation groups. Because subgroup *H* is both a cyclic group and a permutation group capable of being expressed in cyclic notation, the use of the same term in two different senses might create confusion. *Caveat lector.*
8. This property actually makes a permutation of triple counterpoint in which the top two lines exchange positions extraordinarily easy to compose. Although in an invertible counterpoint @8ve, for example, a fifth will still invert into a troublesome fourth, a bass part undernearth this inversion eliminates the problem since the interval in question is found between two upper parts and is therefore permissible.
9. In the arrangement of mm. 10–12 line A is in the bass; line C is in the middle.
10. David Lewin, *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1986), p. 183. Lewin develops a more formal definition of isography in pp. 193–201.
11. This situation was presented as an abstraction in Figure 5 and 6.
12. It is important to remind the reader that our permutations as defined operate on *thematic* tokens (line) A, B, C, etc., not on *registral* tokens like "top," "middle," and

"bottom." Thus (A B) changes line A to line B and line B to line A, in whatever registers the two lines appear; in contrast a registrally defined permutation, such as (top middle), would exchange the melodic lines of the top and middle voices, whatever those lines happened to be. (Top middle) is thus not a member of our permutational system as we have developed the system. Nevertheless the pertinence of the registral permutations is clear in connection with the present text. In addition to this important distinction between thematic and registral permutations, I should also point out here that other musical dimensions can be represented by tokens A, B, and C. For instance, the tokens could represent three instruments, or two manuals and pedal on an organ. All that is needed is some musical dimension which "separates" the three lines.

13. The only time outside the exposition that the theme appears without any of the secondary countersubjects—and indeed, without a straightforward version of the principal countersubject—is in mm. 71-75.
14. Although the thematic impression of this section is like that of an exposition, the section does not conform to the harmonic requirement of an exposition (the entries are on V, I, and VI), and the use of different secondary countersubjects for each thematic entrance somewhat obscures the relationship of this section to the exposition.
15. The triple counterpoint is not exact throughout each section. The conjugational relationship can thus best be seen in a few, relatively exact portions, viz. mm. 223-233, mm. 275-285, mm. 337-347.
16. The D-major Sinfonia is the only composition in the collection that uses all six possible arrangements of triple counterpoint.
17. Friedrich Wilhelm Marpurg recognized the basic idea of conjugational grouping in *Abhandlung von der Fuge.* He divided the six arrangements into two conjugational groups of three, but he considered one conjugation as the primary and basic structure, calling the three arrangements therein *die Hauptversetzungen.* Each arrangement in the other conjugation (*die Nebenversetzungen*) was defined as a double counterpoint of *one* arrangement in the *Hauptversetzungen.* Marpurg did not note that *any Haptversetzung* could produce *any Nebenversetzung* by means of some double counterpoint permutation. F. W. Marpurg, *Abhandlung von der Fuge,* 2 vols. (Berlin: A Haude and J. C. Spener, 1753-54), 2:6-7. There is an error in Marpurg's illustration of the *Nebenversetzungen* on p. 7. The contrapuntal arrangements in (2) and (3) should be exchanged each with the other to match the description in the text.