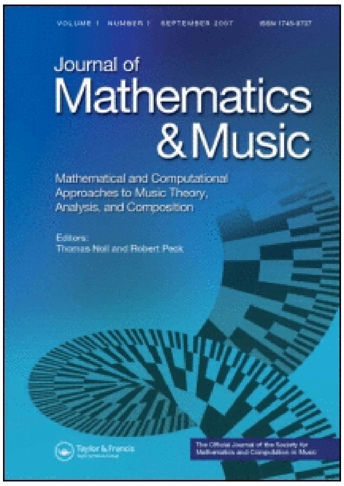

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Tonal trends and α-motif in the first movement of Brahms' String Quartet op. 51 nr. 1

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Tonal trends and α-motif in the first movement of Brahms' String Quartet op. 51 nr. 1

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In this paper, we study the first movement of Brahms' op. 51 nr. 1 in two musical dimensions. On the one hand, we investigate tonal trends and tonal extremes. On the other, we study the positions of Forte's (and Huron's) α-motif in relation to the musical form. We find that α-motives are especially prevalent in the development section. We also combine these two musical dimensions to see whether the α-motives have some relation to tonal trends. They proved to occur more prominently in remote tonal regions than in those near the main tonality.

Keywords: comparison structure analysis; tonal distance; motivic analysis; tonal principal component

1. Introduction

The aim of this article is to propose a computational study of the first movement of Brahms' String Quartet op. 51 nr. 1 in relation to two analytical aspects. We investigate the tonal evolution in the piece. Instead of focusing on key identifications that is often the aim in computer-assisted tonal analyses [1,2], the purpose of what we call *comparison structure analysis* (CSA) [3] is to find (i) tonal trends (or tonal stability) according to some reference pitch-class (pc) set and (ii) tonal extremes, i.e. musical segments that represent tonally the farthest points in the composition as far as their tonal stability values, compared with all musical segments, are concerned. We also study, statistically, the positions of Forte's and (Huron's) α-motif in relation to musical form and tonality (based on our CSA).

The aim of our automated approach to tonality analysis is first and foremost practical: to produce visual representations of dynamic musical properties by evaluating the prevalence of a chosen musical data structure evolving through a musical piece. CSA uses concepts and techniques drawn from traditional music analysis and *music information retrieval* (MIR). Theoretically defined musical entities, such as set classes, pc sets or particular rhythm patterns are detected in compositions using pattern extraction and pattern comparison algorithms that are typical within the field of MIR. Our CSA approach can be applied to monophonic as well as polyphonic music.

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In CSA, *continuous* musical data are automatically segmented into segments of the same cardinality and thereafter these segments are compared against a *comparison structure* (CS) using a similarity function in relation to a specific music dimension. In other words, the latter procedure measures, according to the selected music dimension, the prevalence of the CS at each point of the composition. Since the musical dimension under investigation is tonality, we consider pc information of the piece. Such tonality-based analyses with CSA have been demonstrated in [3,4]. The data consist of pc that are clustered into (non-ordered) *pc segments*, representing pc sets of equal cardinality.

Since 1982, when Krumhansl and Kessler [5] introduced their probe tone weightings for major and minor key profiles, researchers in the field have presented several successful applications for tonal analysis (e.g. Krumhansl–Schmuckler's [6] algorithm and Sapp's [2] application). In a way, our proposed CSA resembles their approaches that involve a key finding using key profiles and correlation. Technically speaking, their key profiles can be paralleled to our concept of CS. However, our approach is more flexible, since the CS can be any possible representative of all pc sets, except from the empty set and the total chroma (not to mention the CS of dimensions other than tonality; more examples of the usage of CSA can be found in [7,8]), whereas the key profiles are fixed.[1]

The analysis procedure occurs in multiple phases and produces as its end product a (discrete) curve, called the tonality CSA curve according to the selected CS, that describes how tonality evolves away from CS within the analysed piece. Since other dimensions of music such as rhythm and melody could be treated as well using CSA, it would have been possible to compare different musical dimensions, for example, by measuring the correlation between these different CSA curves. However, due to the focus of the present Special Issue, we decided in this study to consider instead the occurrences of Forte's [9] α-motif according to their placement in the musical form and in their tonal environment with the resulting tonality CSA curve by using various appropriate statistical tools.

2. CSA for tonal analysis

CSA consists of two main parts. First the segmentation procedure yields pc sets of a fixed cardinality extracted from the piece, called pc segments. The second part consists of the comparison procedures in which all pc segments are compared, using a tonal similarity distance function, first with the so-called CS, a selected pc set, second with each other. For finding the CS, we use the principal component analysis (PCA) for all pc segments.

2.1. *Segmentation procedures*

CSA depends on an automatic fixed-length segmentation that allows for a statistical or mathematical survey of the data. The problem is that there are, in theory, a combinatorial number of possible segmentations. For example, Hasty [10] has shown that multiple segmentations of the same material may be meaningfully developed from different perspectives. In the present analysis, we apply a segmentation method which aims at creating unordered pc set clusters, presented in [3]. The idea in the clustering segmentation is to associate at least one pc set of a selected fixed segmentation cardinality n_{seg} with each unique note onset time by collecting (i.e. clustering) the most proximate pc from its nearest environment. We denote each onset time as u_b^n, where b refers to a bar number and n refers to an order number of the onset time in the bar. We associate each onset time u_b^n with a cluster of k pc segments, the latter being denoted as $s_b^{n_k}$, where $k = 1, 2, \ldots$. For example, if $n_{\text{seg}} = 5$ and there are two different note onset times in a bar, the segmentation

Table 1. Pentachordal pc segments of bar 147.

Segment	pc
$s_{147}^{1_1}$	1,$\underline{2}$,4,6,10
$s_{147}^{1_2}$	1,4,6,10,$\underline{11}$
$s_{147}^{2_1}$	$\underline{1}$,2,$\underline{4}$,6,11
$s_{147}^{2_2}$	$\underline{1}$,2,6,$\underline{10}$,11
$s_{147}^{2_3}$	2,$\underline{4}$,6,$\underline{10}$,11

Note: Add-on pc are underlined in each segment. In order to clarify the notation, for example, $s_{147}^{2_2}$ refers to the second note onset time in bar 147 and reports the second segment addressed to it.

procedure produces at least two pentachords (i.e. two pc segments) associated with that bar as a chord description of its pitch environment (see Table 1 for examples of pc segments). The segmentation algorithm, which produces often – but not always – overlapping segments, proceeds with the following rules:

(1) For each onset time, aim at defining an associated segment or segments of the selected cardinality.
(2) Vertical groupings are preferable.
(3) If the number of pc at the onset exceeds the selected cardinality, assign all possible pc combinations to the onset.[2]
(4) It is preferable to add missing pc from the onsets that are closest temporally.
(5) If there are equally close alternatives before and after the onset, it is preferable to search for them systematically either from one or the other direction, but not from both directions, in order to diminish evolving ‘noise’ from the resulting data. For this analysis, we preferred searching for the missing pc first *before* an onset.
(6) To be even-handed, several alternatives of equal value may have to be assigned to the same onset.

In order to illustrate the segmentation method, pentachordal segments for bar 147 of Brahms’ quartet movement are represented in Table 1, cf. note example in Figure 1. The chord in the first note onset time u_{147}^{1} consists of pc 1, 4, 6 and 10. Thus, we need one more pc to complete the

Figure 1. Bars 146–148 of the first movement of Brahms op. 51 nr. 1 which include the two-note onsets used for the illustration of the segmentation method, cf. Table 1.

cardinality. We find the nearest neighbour notes in the next onset, which includes two alternative add-on pc: 2 and 11. Therefore, we get for u^1_{147} two pc segments: $s^{1_1}_{147} = \{1, 2, 4, 6, 10\}$ and $s^{1_2}_{147} = \{1, 4, 6, 10, 11\}$. The procedure is similar for u^2_{147}, which inherits the two missing pc from the nearest neighbour chord, that of u^1_{147}, resulting in a cluster of three pc segments.

The distance at which we gather the required pc – for fulfilling the segmentation cardinality – varies depending on the number of pc at the note onset itself and the pc density of the musical texture. Our aim is to assign averaged tonal distance values – associated with pc segments – to note onsets (for details, see Section 2.3). Therefore, we want to ensure that we do not collect the required pc too far from the pivotal note onset time by using a too large segmentation cardinality. For that reason, we first evaluate in the whole score the average pc cardinality $\bar{n}_{\text{bar}}$ over score bars and select the integer value $\lfloor \bar{n}_{\text{bar}} \rfloor$ for the segmentation cardinality. It is then possible to check the average 'width' of the neighbourhoods covered by the constructed pc segments with the selected segmentation cardinality. Thereafter, we may decide to use a still smaller segmentation cardinality for the tonal analysis in order to ensure that the calculated averaged tonal distance values really function as tonal descriptions of the location in the composition to which they are indexed [3,7]. At the same time, we have to look out for the cardinality of the selected CS: the bigger the cardinality difference between the cardinality of the CS and the cardinality of segments, the lower the separating capacity of the measurements. What is even more important is to keep the cardinality of all pc segments the same: it can be shown that if the cardinality of pc segments varies, the procedure results in mutually arbitrary segment distance values [3, p. 28]. However, since all segments are of the same cardinality, the resulting values still remain commensurate. As was shown by using the so-called *sensitivity analysis*, CSA is quite a robust method as far as there is enough data available [3].

2.2. *Selection of the CS*

All pc segments are to be compared with a reference structure, the so-called CS. Since tonality is the music dimension under investigation in this paper, the CS is represented as a pc set in order to enable using, with pc segments, a tonality similarity function in the next section. In theory, any pc set could be selected for the comparison. But since we aim at somehow describing the evolution of the tonality of the analysed piece, a natural choice for CS is to take the diatonic scale related to the key signature in the tonal context (in the case of minor key, possibly the harmonic minor scale), since, intuitively speaking, it represents best the assumed main tonality. We have taken this approach in previous analyses [3,4]. However, it is possible that the tonalities of the piece are, for the most part, unrelated to the key signature. In order to attempt to extract the CS from the analysed music piece, we propose to assume that the tonality is reflected in pc segments as a whole collection. For that, we use the information included in pc segments and no other information. We thus decide to apply PCA, a simple, non-parametric method that aims at extracting relevant information from large and complicated data sets by reducing it to a lower dimensional space spanned by a few eigenvectors 'in a way that best explains the variance in the data' [11]. Thus, in our analysis, we reduce the information from a large number of pc segments.

In PCA, the desired CS would be determined by calculating the first principal component (PC) for all the pc segments. PCA has been applied in different tasks in the MIR system (e.g. [12, pp. 103, 362]), and to computational music analysis, as well. Purwins et al. [13] used correspondence analysis (a similar method to PCA) in order to visualize the relation between keys and pc.

We briefly summarize the calculations to determine the first PC, i.e. the approximation of the desired CS: we convert all the pc segments s into 12-dimensional binary vectors $s_{\text{bin}} \in \mathbb{Z}_2^{12}$. For example, $s = \{0, 2, 3, 6, 7\}$ is represented as $s_{\text{bin}} = (1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0)$. Let $\bar{s}_{\text{bin}}$ be the mean 12-dimensional vector of all pc segments in their binary forms, and set $\hat{s}_{\text{bin}} = s_{\text{bin}} - \bar{s}_{\text{bin}}$.

Let then B be a $12 \times n$ matrix, in which columns are all $\hat{s}_{\text{bin}}$. The first PC is the eigenvector $\bar{v}$ (of norm 1) of BB^T corresponding to the largest eigenvalue. It is a 12-dimensional vector in $\mathbb{R}^{12}$, and its components – that are most commonly solved numerically by the computer – are most likely not integers. Thus, it does not represent any pc set and does not directly correspond to any actual musical entity. However, it can be seen as a kind of mathematical condensation of the tonalities of the piece representing the maximum variance with all the pc segments. In other words, its pc set approximation could be considered as the closest pc set structure that most stabilizes all pc segments for the analysis. For testing the procedure in another context, PCA was applied to the first 100 of Bach's 371 chorales (the segmentation cardinality being 4) and thereafter the PCs were compared with 24 KK key profiles [6] using the Pearson correlation coefficient. As a result, approximately 68 keys were able to be defined correctly, in 22 cases the procedure proposed the dominant key (I->V or i->v) and in 10 cases some other alternative key. In this study, the first PC proves to bring us back to our natural choice in defining the CS: as will be seen in Section 5, the CS turns out to give support to the theoretical tonality denoted by the key signature of the movement.

2.3. *Tonal distance*

In the next phase, we compare all pc segments s with CS using Parncutt's model that predicts the perceptual root(s) of a pc set together with the correlation distances [14].[3] In Parncutt's model, the *tonality profile of a pc set p* is introduced as

$$W(p) = (p_{\text{bin}} \cdot \sigma^0(w), \ldots, p_{\text{bin}} \cdot \sigma^{11}(w)),$$

where p_{bin} is the binary 12-dimensional vector representation of p, w is a vector of 'root-support weights' and σ^i indicates a cyclic permutation of w. In this study, we set the root-support vector w as (0.144, 0.052, 0.106, 0.095, 0.094, 0.125, 0.043, 0.129, 0.033, 0.037, 0.036, 0.106). These are not the original weights that Parncutt [14] proposes. Instead, the weights have been 'tuned' anew by applying the so-called *constraint satisfaction problem* in the Monte Carlo process as presented in [4, Chapter 2.1]. Thus, $\sigma^1(w) =$ (0.106, 0.144, 0.052, 0.106, 0.095, 0.094, 0.125, 0.043, 0.129, 0.033, 0.037, 0.036), etc. The resulting vector $W(p)$ forms a series of pc weights in relation to p for which the higher the value the more probably the associated pc represents the tonal centre of p. For example, for the C major triad, its binary vector representation is (1,0,0,0,1,0,0,1,0,0,0,0), and we obtain $W(C\text{major triad}) =$ (0.367, 0.244, 0.267, 0.183, 0.272, 0.341, 0.131, 0.306, 0.233, 0.260, 0.186, 0.210). Thus, the best candidate for the root pc of the C major triad is pc 0.

In order to calculate a *tonal distance d* between two pc sets p_1 and p_2, we set

$$d(p_1, p_2) = 1 - \text{corr}(W(p_1), W(p_2)),$$

where corr stands for the Pearson correlation coefficient between the two tonality profiles $W(p_1)$ and $W(p_2)$ [4]. While correlation values are between -1 and 1, and where 1 indicates a perfect positive linear relationship, the distance values between two pc sets are between 0 and 2. We thus decide to normalize the later values at a range of 0 to 1.

In our CSA, we are interested in calculating the tonal distance between all pc segments $s_i(p_1)$ and the selected CS (p_2). To each note onset u_b^n, we assign the mean of all the tonal distance values $d(s_b^{n_k}, \text{CS})$ for pc segments $s_b^{n_k}$ assigned to the onset u_b^n. This results in what we call the (linear interpolated) tonality CSA curve according to CS for which the horizontal axis is indexed by music onsets. The closer to 0 the value at an onset is, the closer the average tonality of the onset is to the selected CS. Thus, the CSA curve aims at representing the tonal distances from the CS and offering, as such, illustrative information about the form of the piece. Our tonality CSA approach could be comparable to a simple instance of a Riemann logic in the sense of Noll and Garbers [15] in which CSA curves would be the associated sequence of 1×1-Riemann matrices, but which values would come from a scalar product instead of the Pearson correlation coefficient.

2.4. *Relative tonal distance*

In order to determine the 'extreme' tonalities of the piece, we compare pc segments among them. For each pc segment s, we calculate its tonal distance $d(s, s_i)$ to all pc segments s_i in the piece. Similarly to the former section, we calculate the mean of average tonal distances of all pc segments s found at a given onset. We normalize these values at a range of 0 to 1. We call the resulting averaged (linear interpolation) construct the 'relative CSA curve' (i.e. for which the coordinates correspond to average distance values, between all pc segments, per onset). The closer to 0 the value at a given onset is, the closer the average tonality of the onset is to the average tonality of the piece. This curve aims at determining the most tonally extreme places of the piece. While such an approach may at least be mathematically interesting, there is no evidence that these somehow emerge perceptually from their immediate tonal context. Some empirical investigations could lead to partially address this issue, but this is beyond the scope of this paper.

If the first PC resulting from PCA (Section 2.2) is used to approximate the selected CS for the analysis (as it will turn out for the Brahms' quartet analysis – see Section 5), it is expected that both, the CSA curve according to CS and the relative CSA curve, resemble each other since the CS would then result from another kind of the averaging calculation among all pc segments.

3. The α-motif in the present study

Forte's original version of α-motif can be defined with a pitch-interval representation: $(+2, +1)$ and all its transpositions and contrapuntal transformations, i.e. $(-2, -1)$, $(+1, +2)$ and $(-1, -2)$. Huron's α-motif can be defined using a duration and pitch-interval representation: ((long,0), (short,+2), (long,+1)) with all transpositions and contrapuntal transformations. Thus, Huron's α-motif's rhythmic properties are quite broadly defined for computer-assisted analysis. For our study, we use another representation of the α-motif by taking some of Huron's further suggestions into consideration.

While considering his own definition of the α-motif, Huron suggests that 'a better rhythmic feature might arise from characterizing the precise durational proportions (e.g. the 3:1 ratio of the dotted rhythm)' [16, par. 57]. Huron also mentions that there is no reason for not including the diatonically justified pattern $\pm(2, 2)$ which occurs as a major variant of the original $\pm(2, 1)$ pattern. As far as the intervallic properties of the α-motif are concerned, Huron follows Forte's convention but mentions that 'at the time of Forte's writing, the vocabulary of the set theory was unable to accommodate diatonic intervals, and so the $(+2, +2)$ interval pattern was defined as a separate motive' [16, par. 50]. Thus in this study, we do not distinguish the otherwise similar melodic α-motifs, for example, in consecutive bars 235–236 (Figure 2).[4] However, it is more difficult to say whether the temporally immediate derivatives of α-motif in bar 124 (the chromatic (1, 1) variants) should also be taken into consideration.

Even though Huron is able to show that the rhythmic properties of α-motif are important in defining its significance among motives, the question considering the selection of the motives still remains: which of the variants are selected and which are not? Buteau and Mazzola have shown that the number of motif variants can be restricted mathematically by using a motivic similarity function and a suitable similarity threshold value [17]. However, in the present study, we consider the intuitive selection of motif variants in a quite straightforward manner. We define our α-motif using an inter-onset interval and pitch-interval representation: $((3/8, +2), (1/8, +1))$, and

Figure 2. Two motives in bars 235–236, marked in the note example, are both seen as instances of the α-motif according to the author's definition.

Figure 3. An example of the α-motif defined for the present study.

$((3/8, -2), (1/8, -1))$, $((3/8, +1), (1/8, +2))$, $((3/8, -1), (1/8, -2))$, $((3/8, +2), (1/8, +2))$, $((3/8, -2), (1/8, -2))$, where $3/8$ is a dotted quarter note duration and $1/8$ is an eighth note duration. So, the occurrences of α-motif, for example, in bars 232–236 are taken into account. Thus, in our version of α-motif for the present analysis, we emphasize the rhythmic long–short contour between the first two notes (Figure 3).

3.1. *Testing the α-motif using the tonality CSA curve*

Important themes of a sonata movement (and the motives they are built of) appear in their tonal environment and have, in this respect, a relation to tonality. Tonal elaborations, for their part, are mostly concentrated in the development section. As Schönberg formulates it, referring to themes in the development section of the sonata form, 'themes of the first division [exposition] and their derivatives are found in a constant modulatory movement through many and even through remote regions' [18, p. 145]. He further explains the nature of the themes in the development section by mentioning that 'the elaboration normally makes use of variants of previously "exposed" themes, seldom evolving musical ideas' [19, p. 206]. In other words, when a sonata composer of the tonal era enriches music with tonal elaborations, the process may be 'balanced' by adhering mostly to the themes and their derivatives introduced earlier. Although this kind of generalization is justly open to criticism [20, pp. 30, 38, 50], we may rephrase the overall idea saying that the motives that are seen as important building units in the piece, are probably stressed in special tonal regions. As there are no comparable studies for reference concerning the mathematical relation between these two musical dimensions, we propose to simply compare the tonal regions of all occurrences of the α-motif versus those of other three-note motives. This may lead us to gain some supporting evidence of its special role in the piece.

We propose to test whether the α-motif occurs in significantly tonally distant regions. This proposed statistical analysis with the α-motif using the resulting CSA curve according to the selected CS goes as follows: calculate the mean μ of the CSA curve values for a reference and the mean $\bar{x}$ of the values associated with the occurrences of the α-motif. The latter is performed with all three variants of α-motif, i.e. Forte's, Huron's and the one proposed by the author. By comparing the sample means and the mean of all distance values, we see whether the sample distribution is greater, as is assumed, than the mean of the population. A one-sample t-test is performed to test whether each sample mean is significantly greater than the reference mean (H_0: $mu^* \leq mu$; level of significance $p = 0.05$).[5] This approach could benefit from comparing it to similar tests with different compositions, thus using a kind of an 'anticorpus', as Conklin [21] calls such a control data. In short of such data, we derive all three-note motives (with inter-onset and pitch-interval representations) from the piece and see how the α-motif compares with them.

4. Overview of the CSA approach

The main components of the analytical procedure and its implementation follow the structure illustrated in Figure 4. All algorithms are implemented in R, a language and environment for

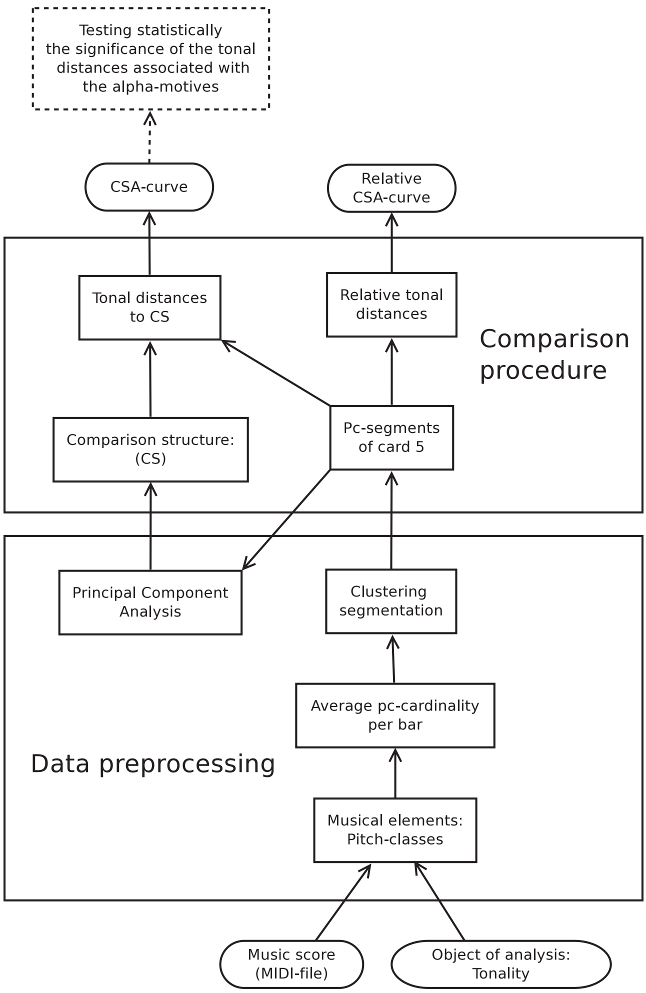

Figure 4. The overall structure of analysis performed in the present study.

statistical computing and graphics [22]. Thus, our implementation uses standard functionalities from R, such as for PCA and the t-test.

The original music file is in MIDI format that is entered as a comma-separated values file into R. Depending on the selected musical dimension for the analysis, music elements from the score are reduced and/or transformed. In the present analysis, the dimension is tonality, thus the notes' pitch information is reduced to pc. The average number $\bar{n}_b$ of pc in bars is calculated and yields the proposed segmentation cardinality $n_{\text{seg}} = \lfloor \bar{n}_b \rfloor$. Note, however, that the analyst may change the segmentation cardinality, if wished. The clustering segmentation results in all pc segments of cardinality n_{seg} for the analysis. It follows the selection of the CS. In this study, the CS is determined by the use of PCA, but it could directly be provided by the analyst. The chosen CS is compared with all the pc segments by calculating their tonal distances. This yields the resulting discrete tonality CSA curve according to the CS. In addition, tonal distance measures between all pairs of pc segments are calculated yielding the relative CSA curve. Note that for visual representation purposes (automatically carried out by the implementation), we may decide to average per bar values from both CSA curves since the number of different onsets, which would result in a sharp-edged curve, is rather large. Finally, for the specific purpose of this study, we extend our CSA approach and perform statistical tests with the α-motif related to their associated tonal regions.

5. Measurements and analytical findings

We present an elementary analysis of the first movement of Brahms' op. 51 nr. 1 String Quartet by using our computational method. There are altogether 2418 different onset times in 260 bars. The average number of pc per bar is $\bar{n}_{\text{bar}} = 5.196$, thus $n_{\text{seg}} = 5$ is tentatively selected. The maximum number of different pc in a bar is 11 and happens only once, in the development, bar 124 (pc 10 is missing). Segmenting all the pc to pentachordal pc clusters produces 3315 pc segments altogether. Their mean segment width in quarter notes is 4.08, remembering that bars 1–223 consist of even six quarter notes and bars 224–260 of four quarter notes. Thus, we decided to keep $n_{\text{seg}} = 5$, since in fact, 83% of the bars, the segment widths are smaller than the bar widths. The latter is highlighted in the CSA curve (see Figure 5(a) and (b)) as a bolder line, whereas for the remaining bars where the pc cardinality exceeds 4, the curve is represented with dots.

The first PC of the 3315 pentachordal pc segments is calculated and yields $(0.54, -0.42, 0.10, 0.23, -0.23, 0.12, -0.42, 0.27, 0.21, -0.24, -0.22, 0.05)$. It is illustrated in Figure 6. The pc that have greater values are in order: 0, 7, 3, 8, 5, 2, 11, 10, 4, 9, 1 and 6, and it results in the tentatively selecting CS $= \{0, 2, 3, 5, 7, 8, 11\}$ as the tonality comparative structure when strictly using positive values from the first PC. In fact, it seems to emphasize the harmonic C minor scale, the key signature of the piece, and we decide to keep this selected CS for the analysis. We emphasize that this is a surprisingly positive result which is not self-evident in the light of much shorter and tonally simpler Bach chorales, where the right key was predicted only approximately in 70% of the cases. It should be mentioned that in our analysis, the CS is of different cardinality than the segments. This is not the most optimal situation, since we cannot find an exact match and thus, all the resulting distance values are strictly positive. At the same time, this means that the separating capacity is slightly diminished, since lower the segmentation cardinality compared with the cardinality of CS, the more probable there are different subsets that may produce the same distance value.

The resulting tonality CSA curve according to CS $= \{0, 2, 3, 5, 7, 8, 11\}$, averaged per music bar for visualization purposes, is illustrated in Figure 5(a). According to the curve, the music stays near the C minor tonality only shortly in all four main sections (Exposition, Development, Recapitulation and Coda). The curve attains the lowest values – the nearest tonal areas compared

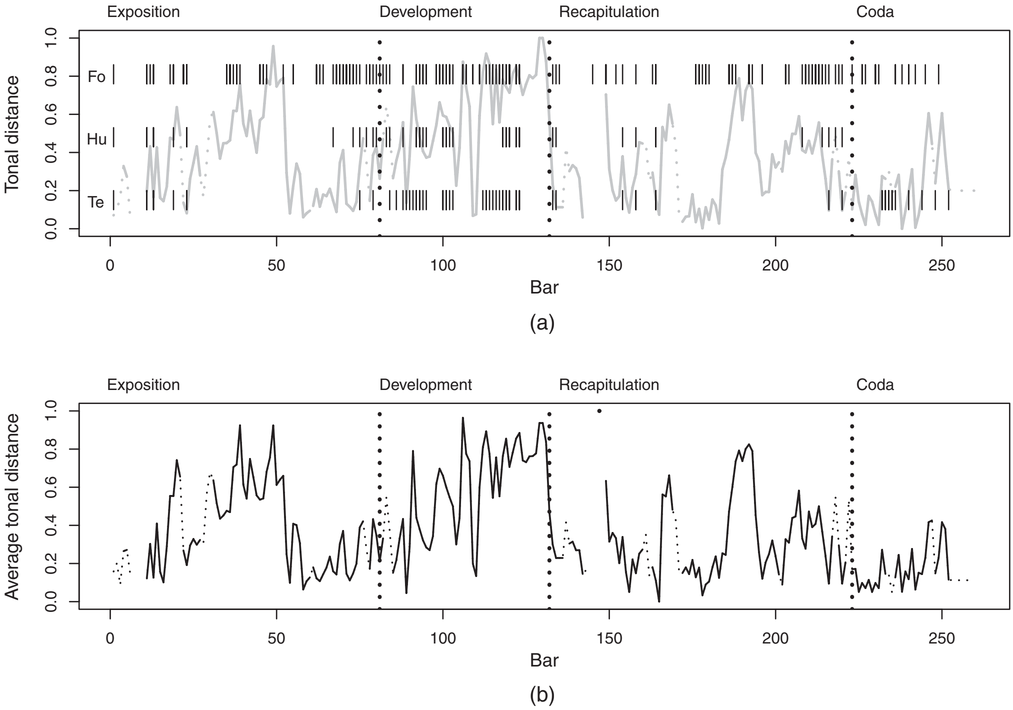

Figure 5. (a) The tonality CSA curve according to CS = {0, 2, 3, 5, 7, 8, 11}, the harmonic C minor scale, of the first movement of Brahms op. 51 nr. 1. It represents tonality distance values between all pc segments and CS, averaged over score bars for visualization purposes. The boundaries between the main sections (Exposition, Development, Recapitulation and Coda) are marked with bold dotted lines. The locations of α-motif are marked with vertical solid lines. ‘Fo’ refers to the occurrences of α-motif defined by Forte, ‘Hu’ refers to Huron’s definition of α and ‘Te’ to the occurrences of the α-motif defined by the author. (b) Relative tonality CSA curve averaged over score bars for visualization purposes. The maximum value is found in bar 147. The boundaries between the main sections are marked with bold dotted lines.

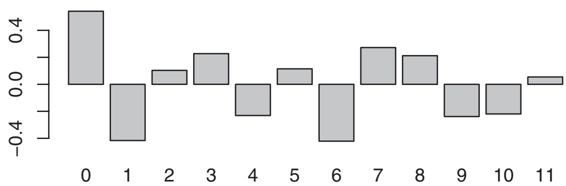

Figure 6. The first PC calculated from 3315 pentachordal pc segments. It suggests using the harmonic C minor scale as our CS.

with the C minor tonality – on bars 109–110, 150, 183, 176–178, 238 and 242, the maximum being found in the development section, others found in recapitulation or coda.[6] The highest values are reached at the end of the development section in bars 121, 112–114, 147, 106, 149, and most constantly in bars 128–131, where the music reaches the A major region (‘the main tonality of the development’ and ‘the climax of the development’ [9, p. 497, 499]), a tritone apart from Eb major, a relative key of C minor. The most salient features we observe are the ascending trends found in the exposition and development. This is the most classic example of *Durchführung* [18], but what is perhaps more striking is the sudden return from the remotest tonalities back to the C minor region. The music in the recapitulation and the coda moves closer to the C minor tonality than in the exposition and development. This may strengthen the sense of asserting the basic tonality, which is only weakly present in the previous sections. The function of such a phenomenon might be comparable to the resolution of the dissonance, but in a much larger scale [20, p. 26].

We investigate the internally 'extreme' tonalities by calculating the relative tonal CSA curve. This curve, seen in Figure 5(b), remarkably resembles that of the CSA curve in Figure 5(a). This is due to the fact that the music is intertwined around the C minor tonality, as was shown by the PCA. The most extraordinary tonalities are found in bars 118–120 (F♯ minor), 147 (B minor), 106 (V of B minor) and 124 (D major): three of them are part of the exposition. The most extreme tonality represents F♯ minor and B minor, i.e. the farthermost tonalities in the circle of fifths compared with C minor.

Table 2 summarizes the total number of α-motif instances found in all four main sections of the movement and Figure 5(a) illustrates their distribution in the piece together with the CSA curve according to the harmonic C minor scale. We mark the locations of α-motif in Figure 5(a) with vertical solid lines. Independent of the definition of the α-motif, its occurrence density per bar is greatest in the development section (Forte, 1.20; Huron, 0.49 ; author, 0.69). This can be intuitively seen in Figure 5(a).

In order to test whether the α-motif occurs in tonally 'more distant regions', we calculated the mean μ=0.417 of all values of the CSA curve, and the mean $\bar{x}$ of values associated with α-motif according to its three different representations as presented in Section 3.1: 0.439 (Forte), 0.446 (Huron) and 0.463 (author). All these means are greater than μ thus hinting towards the fact that the α-motif may occur in more distant regions. The t-tests are performed with each of the three sample means $\bar{x}$. All three versions of the α-motif seem to occur significantly in more distant regions (Forte: $t = 3.20$, df $= 1459$, $p = 0.0007$; Huron: $t = 1.99$, df $= 268$, $p = 0.024$; author: $t = 3.41, df = 338, p = 0.0004$). This statistical result is independent of the segmentation cardinality.[7] We further investigated the sample means of the CSA curve values associated with the three different representations of α-motif by the use of Monte Carlo simulations[8] [23] of the sampling distribution (for sample size matching the α-motif sample under investigation) of all the CSA curve values. The results that support the results produced by the t-test are illustrated in Figure 7(a)–(c) (see the three histograms from the left).

We also confronted our results about the α-motif to similar statistical investigations on 996 different three-note motives found in the piece (with our inter-onset and pitch-interval representation, in which the four contrapuntal interval combinations are seen as the same motif). Among them, only six motives are as or more common than our α-motif (their number of occurrence being ≥ 67). Only two of them, $((1/8, 0), (1/8, \pm 1))$ and $((1/8, 0), (1/8, \pm 2))$, show statistical significance ($p < 0.05$) when compared with the CSA curve, the most common (297 occurrences) being the motif with the pitch-interval pattern (0,1) (e.g. C-C-B) produces the mean distance value of 0.641 ($t = 40.62$, df $= 1309$, $p \ll 0.0001$); see Figure 7(d) (histogram from the right) for the Monte Carlo simulation. This chromatic motif, a derivative of α (cf. [9, p. 480]) that Forte calls λ, occurs in rowing modulations (see, for instance, bars 115–131 in the score). It should be noted that these (0,1) motives include overlapping occurrences of its contrapuntal versions.

Table 2. The total number of α-motif, according to Forte's, Huron's and author's representations, in the Quartet movement.

	Exposition	Development	Recapitulation	Coda	Total
Forte	72	61	68	16	217
Huron	14	25	11	0	50
Tenkanen	10	35	9	13	67

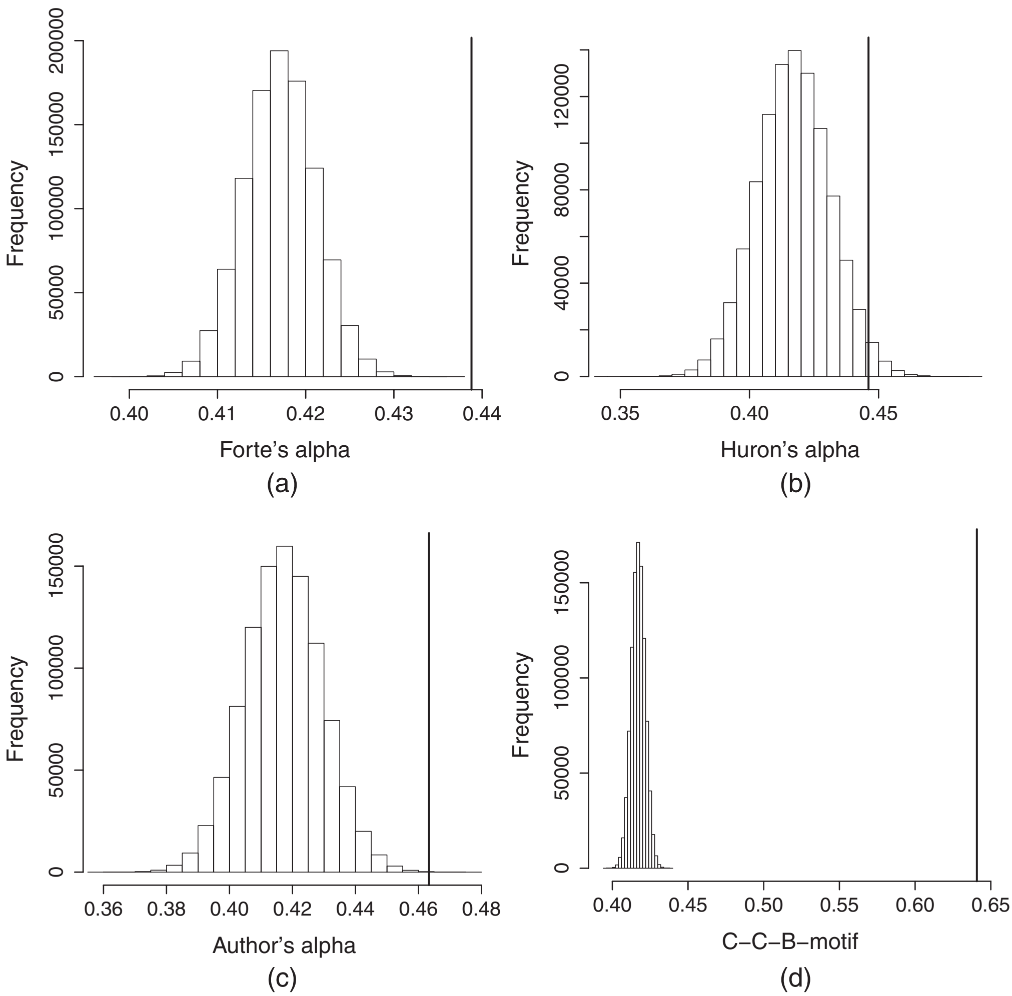

Figure 7. The means of tonal distances associated with three versions of α-motif and C-C-B-motif, compared with the simulated sampling distribution.

Finally, according to our computational approach, we agree with both Huron and Forte [9, p. 475; 16, par. 67] that the α-motif belongs to the candidates of the most salient features of the work.

6. Conclusions

In this study, we applied our CSA approach to the study of tonality. Our main focus was on tonal trends and tonal extremes in the Brahms string quartet movement which had been, for analytical purposes, segmented into pentachordal pc segment clusters. We calculated the first PC for these pc segments that revealed the music piece to be – mathematically – twined around the C minor tonality. According to the CSA, tonal elaborations seem to be organized almost systematically in the exposition and especially in the development section, in which the tonality obtains the remotest point, just before the recapitulation section. In this case, our CSA approach supports the expected tonal evolution in the sonata movement [18,20]. However, we hope that further investigations with our CSA approach may interestingly lead to compositions that do not follow tonal elaborations that are common in classical sonata movements.

When all pairs of pc segments were compared, we found the occurrences of the most extreme tonalities that proved to be F♯ minor or B minor. It is, however, improbable that these mathematically calculated occurrences can somehow be observed perceptually. We could instead measure the modulation strengths between the consecutive onsets or the bars applying the same distance function, since modulations might be more easily recognizable.

We also studied the occurrences of α-motif and found that it occurred especially in the development section independent of its three representations (Forte's, Huron's and the author's) presented in the current study. It was also discovered that the α-motif occurs in significantly more remote tonal regions, according to the reference tonal structure, the harmonic C minor scale. In comparison with many alternative motives found in the piece, the α-motif belongs to the most prevalent and to those that are more common in more remote tonal regions. These findings give support both to Forte's and Huron's concept that the α-motif plays an important role in the first movement of Brahms' String Quartet op. 51 nr. 1.

The validity of the segmentation cardinality is connected both to the cardinality of the CS and especially to the properties of the musical texture itself. The complicated issue, how the difference between the segmentation cardinality and the cardinality of CS effects results, would need a thoroughgoing study in the future.

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Notes

1. Later, Aarden and Temperley proposed improvements to the original Krumhansl-Kessler (KK) key profiles by deriving their weights straight from the music itself [24,25].
2. If the vertical harmony consists of six pc and the segmentation cardinality n_{seg} has been defined as 5, we get 6 different pc sets for that particular note onset time. The number of combinations is the binomial coefficient $\binom{n}{k} = n!/k!(n-k)!$, where n is the number of objects from which you can choose and k is a number to be chosen.
3. The version of the model used in this study is found in [4].
4. In this study, the bar numbering of the movement follows the score in [26].
5. It is assumed that with our sample sizes the mean is normally distributed.
6. In this study, we define these sections as encompassing the following bars: Exposition 1–81, Development 82–132, Recapitulation 133–223 and Coda 224–260.
7. Similar hypothesis tests were performed for segmentation cardinalities from 3 to 7, which led to small p-values, except for Forte's α-motif and segmentation cardinalities 6 and 7.
8. With 1,000,000 random samples.

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