How Do Persistent Earnings Affect the Response of Consumption to Transitory Shocks? Proof of Lemma (iv) including the cases of strictly increasing and strictly decreasing relative prudence

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Proof of Lemma (iv) with isoelastic utility. I first establish that, when $u(.)$ is isoelastic, the MPC function $(\partial c_t/\partial a_{t-1}) = (\partial c_t/\partial a_{t-1})(a_{t-1}, e^{p_t}, e^{\varepsilon_t})$ is homogeneous of degree zero in a_{t-1} and e^{p_t}. Again, the isoelastic utility case is a subcase of the set of utility functions such that relative prudence is strictly larger than one and increasing, because denoting $\rho > 0$ the relative risk aversion, relative prudence is $\rho + 1$, thus strictly larger than one and constant. By Euler's homogeneous function theorem, showing homogeneity of degree zero in a_{t-1} and e^{p_t} is equivalent to showing that:

$$0 = e^{p_t} \frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}} + a_{t-1} \frac{\partial^2 c_t}{\partial a_{t-1}^2}.$$

Rearranging, this also means showing that at any $t < T$:

$$\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}} = -\frac{a_{t-1}}{e^{p_t}} \frac{\partial^2 c_t}{\partial a_{t-1}^2}.$$

A quick way to prove this is to use the fact that, from the proof of Lemma (iii) above, when utility is isoelastic:

$$c_t = a_{t-1} \frac{\partial c_t}{\partial a_{t-1}} + e^{p_t} \frac{\partial c_t}{\partial e^{p_t}}.$$

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Differentiating both sides with respect to a change in a_{t-1} yields:

$$\frac{\partial c_t}{\partial a_{t-1}} = \frac{\partial c_t}{\partial a_{t-1}} + a_{t-1}\frac{\partial^2 c_t}{\partial a_{t-1}^2} + e^{p_t}\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}$$
$$0 = a_{t-1}\frac{\partial^2 c_t}{\partial a_{t-1}^2} + e^{p_t}\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}.$$

Now, this proof is unfortunately not suitable for inequalities extensions: the fact that $c_t \geq a_{t-1}\frac{\partial c_t}{\partial a_{t-1}} + e^{p_t}\frac{\partial c_t}{\partial e^{p_t}}$ does not imply by differentiation that $0 \geq a_{t-1}\frac{\partial^2 c_t}{\partial a_{t-1}^2} + e^{p_t}\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}$. Thus, I also take the long route to proving the result, in order to show how it works in this simpler setting and then explain more easily the inequality case. I prove it by backward induction. At $t = T$, $\frac{\partial^2 c_t}{\partial a_{t-1}^2} = \frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}} = 0$, so Lemma (iv) is true at this last period. I then assume that Lemma (iv) is true at $t+1$ to show it must then be true at t. I derive the expression of $\frac{\partial^2 c_t}{\partial a_{t-1}^2}$ by differentiating twice both sides of the Euler equation with respect to a_{t-1}, and rearrange (noting that a_{t-1} only affects c_{t+1} through its effect on a_t, and substituting $(\partial c_t/\partial a_{t-1})$ with $E_t\big[(\partial c_{t+1}/\partial a_{t-1})(-u''(c_{t+1})/-u''(c_t))\big]$):

$$\begin{aligned}
&\frac{\partial^2 c_t}{\partial a_{t-1}^2}(-u''(c_t)) - (\frac{\partial c_t}{\partial a_{t-1}})^2 u'''(c_t) \\
&\qquad = E_t\big[\frac{\partial^2 c_{t+1}}{\partial a_{t-1}^2}(-u''(c_{t+1}))\big] - E_t\big[(\frac{\partial c_{t+1}}{\partial a_{t-1}})^2 u'''(c_{t+1})\big] \\
&\frac{\partial^2 c_t}{\partial a_{t-1}^2}(-u''(c_t)) - (\frac{\partial c_t}{\partial a_{t-1}})^2 u'''(c_t) \\
&\qquad = E_t[(\frac{\partial^2 a_t}{\partial a_{t-1}^2}\frac{\partial c_{t+1}}{\partial a_t} + (\frac{\partial a_t}{\partial a_{t-1}})^2\frac{\partial^2 c_{t+1}}{\partial a_t^2})(-u''(c_{t+1}))] - E_t[(\frac{\partial c_{t+1}}{\partial a_{t-1}})^2 u'''(c_{t+1})] \\
&\frac{\partial^2 c_t}{\partial a_{t-1}^2}\left(1 + E_t\left[(\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) \\
&\qquad = E_t\big[(\frac{\partial a_t}{\partial a_{t-1}})^2\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big] \\
&\qquad - \frac{u'''(c_t)}{-u''(c_t)}\left(E_t\big[(\frac{\partial c_{t+1}}{\partial a_{t-1}})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\big] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2\right)
\end{aligned}$$

I now differentiate both sides of the Euler equation twice, first with respect to a_{t-1}, second with respect to e^{p_t}:

$$\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}(-u''(c_t)) - (\frac{\partial c_t}{\partial a_{t-1}})(\frac{\partial c_t}{\partial e^{p_t}})u'''(c_t)$$
$$= E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t-1} e^{p_t}}(-u''(c_{t+1}))] - E_t[(\frac{\partial c_{t+1}}{\partial a_{t-1}})(\frac{\partial c_{t+1}}{\partial e^{p_t}})u'''(c_{t+1})]$$

$$\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}(-u''(c_t)) - (\frac{\partial c_t}{\partial a_{t-1}})(\frac{\partial c_t}{\partial e^{p_t}})u'''(c_t)$$
$$= E_t[\left(\frac{\partial^2 a_t}{\partial a_{t-1} e^{p_t}}\frac{\partial c_{t+1}}{\partial a_t} + \frac{\partial a_t}{\partial a_{t-1}}(\frac{\partial a_t}{e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t^2} + \frac{\partial e^{p_{t+1}}}{e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t e^{p_{t+1}}})\right)(-u''(c_{t+1}))]$$
$$- E_t[(\frac{\partial c_{t+1}}{\partial a_{t-1}})^2 u'''(c_{t+1})]$$

$$\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}\left(1 + E_t\left[(\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right)$$
$$= E_t[\frac{\partial a_t}{\partial a_{t-1}}(\frac{\partial a_t}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t^2} + \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t e^{p_{t+1}}})\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$-\frac{u'''(c_t)}{-u''(c_t)}\left(E_t[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{u'''(c_{t+1})}{u'''(c_t)}] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right)$$

I then proceed in two steps. First, I show that, when Lemma (iv) is true at $t+1$, then the term in red in the expression of $\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}$, which I denote A_t^{ap}, is equal to $-(a_{t-1}/e^{p_t})$ times the term in pink in the expression of $\frac{\partial^2 c_t}{\partial a_{t-1}^2}$, which I denote A_t^{aa}. Second, I show that, when Lemma (iv) is true at $t+1$, then the term in blue in the expression of $\frac{\partial^2 c_t}{\partial a_{t-1} e^{p_t}}$, which I denote B_t^{ap}, is equal to $-(a_{t-1}/e^{p_t})$ times the term in light blue in the expression of $\frac{\partial^2 c_t}{\partial a_{t-1}^2}$, which I denote B_t^{aa}. In A_t^{ap}, I substitute $\partial^2 c_{t+1}/\partial a_t e^{p_{t+1}}$ with $-(a_t/e^{p_{t+1}})(\partial^2 c_{t+1}/\partial a_t^2)$:

$$A_t^{ap} = E_t[\frac{\partial a_t}{\partial a_{t-1}}(\frac{\partial a_t}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t^2} + \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t e^{p_{t+1}}})\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$= E_t[\frac{\partial a_t}{\partial a_{t-1}}(\frac{\partial a_t}{\partial e^{p_t}} - \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{a_t}{e^{p_{t+1}}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$= E_t[\frac{\partial a_t}{\partial a_{t-1}}(e^{\varepsilon_t} - \frac{\partial c_t}{\partial e^{p_t}} - \frac{e^{p_{t+1}}}{e^{p_t}}\frac{a_t}{e^{p_{t+1}}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$= E_t[\frac{\partial a_t}{\partial a_{t-1}}(e^{\varepsilon_t} - (\frac{c_t}{e^{p_t}} - \frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}}) - \frac{e^{p_{t+1}}}{e^{p_t}}\frac{a_t}{e^{p_{t+1}}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$= E_t[\frac{\partial a_t}{\partial a_{t-1}}(\frac{a_t - (1+r)a_{t-1} + c_t}{e^{p_t}} - (\frac{c_t}{e^{p_t}} - \frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}}) - \frac{a_t}{e^{p_t}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$= -\frac{a_{t-1}}{e^{p_t}}E_t[\frac{\partial a_t}{\partial a_{t-1}}((1+r) - \frac{\partial c_t}{\partial a_{t-1}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]$$
$$= -\frac{a_{t-1}}{e^{p_t}}E_t[(\frac{\partial a_t}{\partial a_{t-1}})^2\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}] = -\frac{a_{t-1}}{e^{p_t}}A_t^{aa}.$$

Since the extension of Lemma (iii) implies that $c_t = c_t(a_{t-2}, e^{p_{t-1}}, e^{\eta_t}, e^{\varepsilon_t})$ is homogeneous of degree one in a_{t-2} and $e^{p_{t-1}}$ when utility is isoelastic, in B_t^{ap}, I substitute $(\partial c_{t+1}/\partial e^{p_t})$ with $(c_{t+1}/e^{p_t}) - (a_{t-1}/e^{p_t})(\partial c_{t+1}/\partial a_{t-1})$:

$$\begin{aligned}
B_t^{ap} &= -\frac{u'''(c_t)}{-u''(c_t)}\Bigg(E_t\Big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{u'''(c_{t+1})}{u'''(c_t)}\Big] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\Bigg)\\
&= \frac{a_{t-1}}{e^{p_t}}\frac{u'''(c_t)}{-u''(c_t)}\Bigg(E_t\Big[(\frac{\partial c_{t+1}}{\partial a_{t-1}})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\Big] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2\Bigg)\\
&- \frac{u'''(c_t)}{-u''(c_t)}\Bigg(E_t\Big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{c_{t+1}}{e^{p_t}}\frac{u'''(c_{t+1})}{u'''(c_t)}\Big] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{c_{t+1}}{e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\Bigg)\\
&= -\frac{a_{t-1}}{e^{p_t}}B_t^{aa} - \frac{u'''(c_t)}{-u''(c_t)}\frac{1}{e^{p_t}}\Bigg(E_t\Big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{c_{t+1}}{c_t}\frac{u'''(c_{t+1})}{u'''(c_t)}\Big] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{c_{t+1}}{c_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\Bigg)\\
&= -\frac{a_{t-1}}{e^{p_t}}B_t^{aa} - \frac{u'''(c_t)}{-u''(c_t)}\frac{1}{e^{p_t}}\Bigg(E_t\Big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\Big] - E_t\left[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]\Bigg)\\
&= -\frac{a_{t-1}}{e^{p_t}}B_t^{aa}.
\end{aligned}$$

As a result, it must be that:

$$\frac{\partial^2 c_t}{\partial a_{t-1}e^{p_t}} = -\frac{a_{t-1}}{e^{p_t}}\frac{\partial^2 c_t}{\partial a_{t-1}^2}.$$

Proof of Lemma (iv) with strictly increasing relative prudence. I now consider the case with a utility function displaying HARA, and a relative prudence strictly larger than one and strictly increasing. I show that, in that case, the weighted sum of the partial effects of wealth and persistent earnings on the MPC is strictly larger than zero. This means showing that at any $t < T$:

$$0 > e^{p_t}\frac{\partial^2 c_t}{\partial a_{t-1}e^{p_t}} + a_{t-1}\frac{\partial^2 c_t}{\partial a_{t-1}^2}.$$

Rearranging, this also means showing that at any $t < T$:

$$\frac{\partial^2 c_t}{\partial a_{t-1}e^{p_t}} < -\frac{a_{t-1}}{e^{p_t}}\frac{\partial^2 c_t}{\partial a_{t-1}^2}.$$

A non-strict version of Lemma (iv) is true at $t = T$ because $\frac{\partial^2 c_T}{\partial a_{T-1}^2} = \frac{\partial^2 c_T}{\partial a_{T-1}e^{p_T}} = 0$. Now I assume that a non-strict version of Lemma (iv) is true at at $t+1$ and prove that it must then be true at t.

Using Lemma (iv) at $t+1$, the term A_t^{ap} becomes:

$$\begin{aligned}
A_t^{ap} &= E_t\big[\frac{\partial a_t}{\partial a_{t-1}}(\frac{\partial a_t}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t^2}+\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t e^{p_{t+1}}})\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&\leq E_t\big[\frac{\partial a_t}{\partial a_{t-1}}(\frac{\partial a_t}{\partial e^{p_t}}-\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{a_t}{e^{p_{t+1}}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&\leq E_t\big[\frac{\partial a_t}{\partial a_{t-1}}(e^{\varepsilon_t}-\frac{\partial c_t}{\partial e^{p_t}}-\frac{e^{p_{t+1}}}{e^{p_t}}\frac{a_t}{e^{p_{t+1}}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&< E_t\big[\frac{\partial a_t}{\partial a_{t-1}}(e^{\varepsilon_t}-(\frac{c_t}{e^{p_t}}-\frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}})-\frac{e^{p_{t+1}}}{e^{p_t}}\frac{a_t}{e^{p_{t+1}}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&< E_t\big[\frac{\partial a_t}{\partial a_{t-1}}(\frac{a_t-(1+r)a_{t-1}+c_t}{e^{p_t}}-(\frac{c_t}{e^{p_t}}-\frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}})-\frac{a_t}{e^{p_t}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&< -\frac{a_{t-1}}{e^{p_t}}E_t\big[\frac{\partial a_t}{\partial a_{t-1}}((1+r)-\frac{\partial c_t}{\partial a_{t-1}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&< -\frac{a_{t-1}}{e^{p_t}}E_t\big[(\frac{\partial a_t}{\partial a_{t-1}})^2\frac{\partial^2 c_{t+1}}{\partial a_t^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\big] = -\frac{a_{t-1}}{e^{p_t}}A_t^{aa}.
\end{aligned}$$

The difference with the previous case is that I move the first to the second line using that, when Lemma (iv) is true at $t+1$, $\frac{\partial^2 c_{t+1}}{\partial a_t e^{p_{t+1}}} \leq -\frac{a_t}{e^{p_{t+1}}}\frac{\partial^2 c_{t+1}}{\partial a_t^2}$. I also move from the third to the fourth line using that, from Lemma (iii), when relative prudence is strictly increasing $\frac{\partial c_t}{\partial e^{p_t}} < \frac{c_t}{e^{p_t}} - \frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}}$: because utility is HARA, $\frac{\partial^2 c_{t+1}}{\partial a_t^2} < 0$, and $-\frac{\partial c_t}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_t^2}$ is strictly smaller than $-(\frac{c_t}{e^{p_t}} - \frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}})\frac{\partial^2 c_{t+1}}{\partial a_t^2}$. I now consider the term B_t^{ap}. I defined the function $G_{t+1}(x)$:

$$G_{t+1}(x) = x\times\left(\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\frac{(u'(c_{t+1}))^{-1}}{(u'(c_t))^{-1}} - E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\right),$$

with $\frac{\partial c_{t+1}}{\partial a_{t-1}}$, c_{t+1} and c_t the outcomes of the maximization problem described by (**??**)-(**??**). I compute the expected value at t of the first derivative of $G_{t+1}(.)$:

$$\begin{aligned}
E_t\big[G'_{t+1}(x)\big] &= E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\frac{(u'(c_{t+1}))^{-1}}{(u'(c_t))^{-1}}\big] - E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&= E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\frac{(u'(c_{t+1}))^{-1}}{(u'(c_t))^{-1}}\big]E_t\big[\frac{u'(c_{t+1})}{u'(c_t)}\big] - E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&= E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big] - cov_t(\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})/u'(c_{t+1})}{-u''(c_t)/u'(c_t)}, \frac{u'(c_{t+1})}{u'(c_t)})\\
&- E_t\big[\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]\\
&< 0
\end{aligned}$$

I move from the first to the second line using that $E_t\big[\frac{u'(c_{t+1})}{u'(c_t)}\big] = 1$. I move from the second to the third line using that $E[X]E[Y] = E[XY] - cov(X,Y)$. I move from the third to the fourth line noting that, when utility is HARA, then $u'''(.)/(-u''(.)) = k(-u''(.))/u'(.)$ so decreasing relative prudence implies decreasing relative risk aversion and $(-u''(c_{t+1}))/u'(c_{t+1})$ decreases with the shocks that raise c_{t+1} (and vice-versa), while $u'(c_{t+1})$ also decreases with the shocks that raise c_{t+1} (and vice-versa). The term $\frac{\partial c_{t+1}}{\partial a_{t-1}}$ decreases with the transitory shocks that raise c_{t+1} (and vice-versa). Finally, a straightforward corollary to Theorem (ii) is that, when relative prudence is strictly above one and strictly increasing, the MPC $\frac{\partial c_{t+1}}{\partial a_{t-1}}$ decreases with the persistent shocks that raise c_{t+1} (and vice-versa). Therefore, the covariance $cov_t\big(\frac{\partial c_{t+1}}{\partial a_{t-1}}\frac{-u''(c_{t+1})/u'(c_{t+1})}{-u''(c_t)/u'(c_t)}, \frac{u'(c_{t+1})}{u'(c_t)}\big)$ must be strictly positive. The expression of $G_{t+1}(.)$ implies that :

$$B_t^{ap} = -\frac{u'''(c_t)}{-u''(c_t)}E_t[G_{t+1}\big(\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big)].$$

Recall that B_t^{aa} is such that:

$$-\frac{a_{t-1}}{e^{p_t}}B_t^{aa} = -\frac{u'''(c_t)}{-u''(c_t)}E_t[G_{t+1}\big((\frac{c_{t+1}}{e^{p_{t+1}}} - \frac{a_t}{e^{p_{t+1}}}\frac{\partial c_{t+1}}{\partial a_t})\frac{-u''(c_{t+1})}{-u''(c_t)}\big)].$$

Because $E_t\left[G'_{t+1}(x)\right] < 0$ and because $\frac{\partial c_{t+1}}{\partial e^{p_t}} < (\frac{c_{t+1}}{e^{p_{t+1}}} - \frac{a_t}{e^{p_{t+1}}}\frac{\partial c_{t+1}}{\partial a_t})$ when Lemma (iv) is true at $t+1$, then:

$$\begin{aligned} B_t^{ap} &= -\frac{u'''(c_t)}{-u''(c_t)}E_t[G_{t+1}\big(\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big)] \\ &< -\frac{u'''(c_t)}{-u''(c_t)}E_t[G_{t+1}\big((\frac{c_{t+1}}{e^{p_{t+1}}} - \frac{a_t}{e^{p_{t+1}}}\frac{\partial c_{t+1}}{\partial a_t})\frac{-u''(c_{t+1})}{-u''(c_t)}\big)] = -\frac{a_{t-1}}{e^{p_t}}B_t^{aa} \end{aligned}$$

Because $A_t^{ap} < -\frac{a_{t-1}}{e^{p_t}}A_t^{aa}$ and $B_t^{ap} < -\frac{a_{t-1}}{e^{p_t}}B_t^{aa}$, it must be that:

$$\frac{\partial^2 c_t}{\partial a_{t-1}e^{p_t}} < -\frac{a_{t-1}}{e^{p_t}}\frac{\partial^2 c_t}{\partial a_{t-1}^2}.$$

Proof of Lemma (iv) with decreasing relative prudence. The proof is the converse of the proof with strictly increasing relative prudence. In that case, when the Lemma (iv) is true at $t+1$, $\frac{\partial^2 c_{t+1}}{\partial a_t e^{p_{t+1}}} \geq -\frac{a_t}{e^{p_{t+1}}}\frac{\partial^2 c_{t+1}}{\partial a_t^2}$, and from Lemma (iii), when relative prudence is strictly decreasing $\frac{\partial c_t}{\partial e^{p_t}} > \frac{c_t}{e^{p_t}} - \frac{a_{t-1}}{e^{p_t}}\frac{\partial c_t}{\partial a_{t-1}}$.