How Do Persistent Earnings Affect the Response of Consumption to Transitory Shocks?
Proof of Lemma (i) with time-varying demographics, discount factors, and interest rates

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In a model with time-varying demographic characteristics, discount factors and interest rates, the Euler equation becomes:

$$u'(c_t) = E_t[u'(c_{t+1})]R_{t,t+1}, \tag{??}$$

with $R_{t,t+s} \equiv \Pi_{k=1}^{s}\beta_{t+k}(1+r_{t+k-1})e^{\delta(z_{t+k}-z_{t+k-1})}$. Thus, an optimizing household equalizes its expected marginal utility over time, weighted by the strength of intertemporal substitution motives. The last period is unchanged. Under perfect foresight at t:

$$u'(c_t^{PF_t}) = u'(c_{t+1}^{PF_t})R_{t,t+1}. \tag{0.1}$$

I also differentiate both sides of (0.1) with respect to a change in a_{t-1} and rearrange:

$$\begin{aligned}
\frac{\partial c_t^{PF_t}}{\partial a_{t-1}} &= \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t-1}} \frac{-u''(c_{t+1}^{PF_t})}{-u''(c_t^{PF_t})}\\
\frac{\partial c_t^{PF_t}}{\partial a_{t-1}} &= \frac{\partial a_t^{PF_t}}{\partial a_{t-1}} \frac{\partial c_{t+1}^{PF_t}}{\partial a_t^{PF_t}} \frac{-u''(c_{t+1}^{PF_t})}{-u''(c_t^{PF_t})}\\
\frac{\partial c_t^{PF_t}}{\partial a_{t-1}} &= \big((1+r) - \frac{\partial c_t^{PF_t}}{\partial a_{t-1}}\big) \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t-1}} \frac{-u''(c_{t+1}^{PF_t})}{-u''(c_t^{PF_t})}\\
\frac{\partial c_t^{PF_t}}{\partial a_{t-1}} &= (1+r)g\left(\frac{\partial c_{t+1}^{PF_t}}{\partial a_t^{PF_t}} \frac{-u''(c_{t+1}^{PF_t})}{-u''(c_t^{PF_t})} R_{t,t+1}\right)\\
\frac{\partial c_t^{PF_t}}{\partial a_{t-1}} &= (1+r)g\left(\frac{\partial c_{t+1}^{PF_t}}{\partial a_t^{PF_t}} \frac{-u''\big((u')^{-1}(u'(c_t^{PF_t})R_{t,t+1}^{-1})\big)}{-u''(c_t^{PF_t})} R_{t,t+1}\right)
\end{aligned}$$

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Considering the general case with uncertainty, i also differentiate both sides of (**??**) with respect to a change in a_{t-1}, and do a similar rearranging:

$$\frac{\partial c_t}{\partial a_{t-1}} = (1+r)g\left(E_t\big[\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R_{t,t+1}\right) \tag{0.2}$$

$$\geq (1+r)g\left(\frac{\partial c_{t+1}^{PF_{t+1}}}{\partial a_t}\frac{E_t\big[-u''(c_{t+1})\big]}{-u''(c_t)}R_{t,t+1}\right) \tag{0.3}$$

$$> (1+r)g\left(\frac{\partial c_{t+1}^{PF_t}}{\partial a_t^{PF_t}}\frac{-u''\big((u')^{-1}(u'(c_t)R_{t,t+1}^{-1})\big)R_{t,t+1}}{-u''(c_t)}\right) \tag{0.4}$$

$$> (1+r)g\left(\frac{\partial c_{t+1}^{PF_t}}{\partial a_t^{PF_t}}\frac{-u''\big((u')^{-1}(u'(c_t^{PF_t})R_{t,t+1}^{-1})\big)R_{t,t+1}}{-u''(c_t^{PF_t})}\right) = \frac{\partial c_t^{PF_t}}{\partial a_{t-1}} \tag{0.5}$$

I move from the first to the second line using that a non-strict version of Lemma (i) holds true at $t+1$. I move from the second to the third line using that the MPC at $t+1$ is the same under perfect foresight at $t+1$ as under perfect foresight at t, because the MPC is exogenous and independent of the realization of the shocks: $\frac{\partial c_{t+1}^{PF_{t+1}}}{\partial a_t} = \frac{\partial c_{t+1}^{PF_t}}{\partial a_t^{PF_t}}$ (defining the functions $c_t^{PF_t} = c_t^{PF_t}(a_{t-1}, e^{p_t})$ and $c_{t+1}^{PF_t} = c_{t+1}^{PF_t}(a_t^{PF_t}, E_t[e^{p_{t+1}}])$). In addition, because the premium φ_t^{r1} associated with $(-u'')$ is strictly larger than the premium φ_t associated with $u'(.)$:

$$E_t\big[-u''(c_{t+1})\big] = -u''(E_t[c_{t+1}] - \varphi_t^{r1}) > -u''(E_t[c_{t+1}] - \varphi_t) = -u''\big((u')^{-1}(u'(c_t)R_{t,t+1}^{-1})\big),$$

with $E_t[c_{t+1}] - \varphi_t = (u')^{-1}(u'(c_t)R_{t,t+1}^{-1})$ from a rearranging of the Euler equation. Finally, I move from the third to the fourth line using that the expression $\frac{-u''\big((u')^{-1}(u'(c_t)R_{t,t+1}^{-1})\big)R_{t,t+1}}{-u''(c_t)}$ rewrites as $l(y,R) = \frac{g(yR^{-1})R}{g(y)}$, with $g(.) = (-u'')\circ(u')^{-1}(.)$, $y = u'(c_t)$, and $R = R_{t,t+1}$. This function $l(y,R)$ is constant or increasing in y when $g(.)$, y and R are such that $\frac{g'(yR^{-1})}{g'(y)(R^{-1})^{-1}} - \frac{g(yR^{-1})}{g(y)} \geq 0$.[1] When $l(y,R)$ is constant or increasing in y then $l(u'(c_t), R_{t,t+1}) \geq l(u'(c_t^{PF_t}), R_{t,t+1})$ because $c_t < c_t^{PF_t}$ and $u'(.)$ is strictly decreasing.

[1] Since $h(u'(c_t)) = (-u'')(c_t) > 0$ and $g'(u'(c_t)) = u'''(c_t)/(-u''(c_t) > 0$, the condition $\frac{g'(yR^{-1})}{g'(y)(R^{-1})^{-1}} - \frac{g(yR^{-1})}{g(y)} \geq 0$ is sufficient to ensure that $\frac{\partial l(y,R)}{\partial y} = \frac{g'(y)R}{g(y)}\big(\frac{g'(yR^{-1})}{g'(y)R} - \frac{g(yR^{-1})}{g(y)}\big) \geq 0$. Note that this difference is zero when $g(.) = (-u'')\circ(u')^{-1}(.)$ is separable, that is, such that $g(yR^{-1}) = g_y(y)g_R(R^{-1})$. Indeed, then $(R^{-1})g'(yR^{-1}) = \frac{\partial g(yR^{-1})}{\partial y} = \frac{\partial g_y(y)g_R(R^{-1})}{\partial y} = g_y'(y)g_R(R^{-1})$ and $\frac{g'(yR^{-1})}{g'(y)(R^{-1})^{-1}} - \frac{g(yR^{-1})}{g(y)} = \frac{g_y'(y)g_R(R^{-1})(R^{-1})^{-1}}{g_y'(y)g_R(1)(R^{-1})^{-1}} - \frac{g_y(y)g_R(R^{-1})}{g_y(y)g_R(1)} = 0$.