# Problem Set Econ - Week 3 - DSGE models - Linear

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## Exercise 1

In the Brock-Mirman model, there is only one equation, as there is no labor-leisure choice. The one Euler Equation (EE) is:

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} \right\} \tag{0.1}$$

The strategy is to guess the policy function to be $K_{t+1} = Ae^{z_t}K_t^{\alpha}$ and replace both $K_{t+2}$ and $K_t$ in the Euler equation to only keet $K_{t+1}$:

$$\frac{1}{\frac{1}{A}K_{t+1} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - Ae^{z_{t+1}} K_{t+1}^{\alpha}} \right\} \tag{0.2}$$

$$\Leftrightarrow A^2 - (\beta\alpha + 1)A - \beta\alpha = 0 \tag{0.3}$$

$\alpha\beta, 1$ are solutions to this equation.

## Exercise 2

The given functional forms are:

$$u\left(c_t, \ell_t\right) = \ln c_t + a \ln\left(1 - \ell_t\right) \tag{0.4}$$

$$F\left(K_t, L_t, z_t\right) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{0.5}$$

We compute the FOC of these 2 equations and plug them in the baseline model. We get:

$$c_t = (1-\tau)\left[w_t \ell_t + \left(r_t - \delta\right) k_t\right] + k_t + T_t - k_{t+1} \tag{0.6}$$

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left[\left(r_{t+1} - \delta\right)(1-\tau) + 1\right] \right\} \tag{0.7}$$

$$-\frac{a}{1-l_t} = \frac{1}{c_t} w_t (1-\tau) \tag{0.8}$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \tag{0.9}$$

$$w_t = (1-\alpha)e^{z_t}K_t^{\alpha}L_t^{-\alpha} \tag{0.10}$$
$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t \tag{0.11}$$
$$z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \varepsilon_t^z \tag{0.12}$$

We cannot use the same technic as in the first exercise because we know have consumption and leisure. The random term does not cancel out if we guess a functional form, making the integral impossible to solve analytically. This is because the optimal amount of saving also depends on how much does the agent work: $l_t$ has an impact on the rate of return of the capita $k_t$ and the household's income.

## Exercise 3

The new functional forms are:

$$u\left(c_t,\ell_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\ln\left(1-\ell_t\right) \tag{0.13}$$
$$F\left(K_t,L_t,z_t\right) = e^{z_t}K_t^{\alpha}L_t^{1-\alpha} \tag{0.14}$$

The 7 characterizing equations are:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t - k_{t+1} \tag{0.15}$$
$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)\,(1-\tau)+1\right]\right\} \tag{0.16}$$
$$-\frac{a}{1-l_t} = c_t^{-\gamma}w_t(1-\tau) \tag{0.17}$$
$$r_t = \alpha e^{z_t}K_t^{\alpha-1}L_t^{1-\alpha} \tag{0.18}$$
$$w_t = (1-\alpha)e^{z_t}K_t^{\alpha}L_t^{-\alpha} \tag{0.19}$$
$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t \tag{0.20}$$
$$z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \varepsilon_t^z \tag{0.21}$$
$$\tag{0.22}$$

## Exercise 4

The new functional forms are:

$$u\left(c_t,\ell_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi}-1}{1-\xi} \tag{0.23}$$
$$F\left(K_t,L_t,z_t\right) = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}} \tag{0.24}$$

and the characterizing equations are now:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t - k_{t+1} \tag{0.25}$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} \left[ (r_{t+1} - \delta)(1-\tau) + 1 \right] \right\} \tag{0.26}$$

$$\frac{a}{(1-\ell_t)^{\xi}} = c_t^{-\gamma} w_t (1-\tau) \tag{0.27}$$

$$r_t = \alpha K_t^{\eta-1} e^{z_t} \left[ \alpha K_t^{\eta} + (1-\alpha) L_t^{\eta} \right]^{\frac{1}{\eta}-1} \tag{0.28}$$

$$w_t = (1-\alpha) L_t^{\eta-1} e^{z_t} \left[ \alpha K_t^{\eta} + (1-\alpha) L_t^{\eta} \right]^{\frac{1}{\eta}-1} \tag{0.29}$$

$$\tau \left[ w_t \ell_t + (r_t - \delta) k_t \right] = T_t \tag{0.30}$$

$$z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \varepsilon_t^z \tag{0.31}$$

$$\tag{0.32}$$

## Exercise 5

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} \tag{0.33}$$

$$F(K_t, L_t, z_t) = K_t^{\alpha} \left( e^{z_t} L_t \right)^{1-\alpha} \tag{0.34}$$

and the characterizing equations are now:

$$c_t = (1-\tau) \left[ w_t \ell_t + (r_t - \delta) k_t \right] + k_t + T_t - k_{t+1} \tag{0.35}$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} \left[ (r_{t+1} - \delta)(1-\tau) + 1 \right] \right\} \tag{0.36}$$

$$1 = c_t^{-\gamma} w_t (1-\tau) \tag{0.37}$$

$$r_t = \alpha K_t^{\alpha-1} \left( e^{z_t} \right)^{1-\alpha} \tag{0.38}$$

$$w_t = (1-\alpha) K_t^{\alpha} e^{(1-\alpha) z_t} \tag{0.39}$$

$$\tau \left[ w_t \ell_t + (r_t - \delta) k_t \right] = T_t \tag{0.40}$$

$$z_t = (1-\rho_z)\, \overline{z} + \rho_z z_{t-1} + \epsilon_t^z \tag{0.41}$$

$$\tag{0.42}$$

Solving the model at the steady state, we have:

$$c = (1-\tau) \left[ w + (r - \delta) k \right] + T \tag{0.43}$$

$$1 = \beta \left\{ \left[ (r - \delta)(1-\tau) + 1 \right] \right\} \tag{0.44}$$

$$1 = c^{-\gamma} w (1-\tau) \tag{0.45}$$

$$r = \alpha K^{\alpha-1} \left( e^{z} \right)^{1-\alpha} \tag{0.46}$$

$$w = (1-\alpha) K^{\alpha} e^{(1-\alpha)} z \tag{0.47}$$

$$\tau \left[ w + (r - \delta) k \right] = T \tag{0.48}$$

$$z^* = \overline{z} \tag{0.49}$$

Simplyfying, we get:

$$
\begin{aligned}
c &= \left((1-\tau)(1-\alpha)\left(\frac{1-\beta}{\beta(1-\tau)\alpha}+\delta\right)^{\frac{1}{\alpha-1}}\right)^{\frac{1}{\gamma}} && (0.50)\\
r &= \frac{1-\beta}{\beta(1-\tau)}+\delta && (0.51)\\
w &= (1-\alpha)\left(\frac{1-\beta}{\beta(1-\tau)\alpha}+\delta\right)^{\frac{1}{\alpha-1}} && (0.52)\\
k &= \left(\frac{1-\beta}{\beta(1-\tau)\alpha}+\delta\right)^{\frac{1}{\alpha-1}} && (0.53)\\
T &= \tau\left[w+(r-\delta)\,k\right] && (0.54)\\
z &= \bar{z} && (0.55)
\end{aligned}
$$

## Exercise 6

The functional forms are now

$$
\begin{aligned}
u\left(c_t,\ell_t\right) &= \frac{c_t^{1-\gamma}-1}{1-\gamma}+a\frac{\left(1-\ell_t\right)^{1-\xi}-1}{1-\xi} && (0.56)\\
F\left(K_t,L_t,z_t\right) &= K_t^{\alpha}\left(e^{z_t}L_t\right)^{1-\alpha} && (0.57)
\end{aligned}
$$

and the characterizing equations are now:

$$
\begin{aligned}
c_t &= (1-\tau)\left[w_t\ell_t+\left(r_t-\delta\right)k_t\right]+k_t+T_t-k_{t+1} && (0.58)\\
c_t^{-\gamma} &= \beta E_t\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1}-\delta\right)(1-\tau)+1\right]\right\} && (0.59)\\
\frac{a}{(1-\ell_t)^{\xi}} &= c_t^{-\gamma}w_t(1-\tau) && (0.60)\\
r_t &= \alpha K_t^{\alpha-1}e^{(1-\alpha)z_t}L_t^{1-\alpha} && (0.61)\\
w_t &= (1-\alpha)K_t^{\alpha}e^{(1-\alpha)z_t}L_t^{1-\alpha} && (0.62)\\
\tau\left[w_t\ell_t+\left(r_t-\delta\right)k_t\right] &= T_t && (0.63)\\
z_t &= (1-\rho_z)\bar{z}+\rho_z z_{t-1}+\varepsilon_t^z && (0.64)
\end{aligned}
$$

Solving for the steady state, we get the following equations:

$$
\begin{aligned}
c &= (1-\tau)\left[w\ell+(r-\delta)\,k\right]+T && (0.65)\\
1 &= \beta E\left\{\left[(r-\delta)(1-\tau)+1\right]\right\} && (0.66)\\
\frac{a}{(1-\ell)^{\xi}} &= c^{-\gamma}w(1-\tau) && (0.67)
\end{aligned}
$$

$$
\begin{aligned}
r &= \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} && (0.68)\\
w &= (1-\alpha) K^{\alpha} e^{(1-\alpha)z} L^{1-\alpha} && (0.69)\\
\tau \left[ w\ell + (r-\delta)\, k \right] &= T && (0.70)\\
z &= \overline{z} && (0.71)
\end{aligned}
$$

and therefore:

$$
\begin{aligned}
c &= w\ell + (r-\delta)k && (0.72)\\
r &= \frac{1-\beta}{\beta(1-\tau)} + \delta && (0.73)\\
r &= \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} && (0.74)\\
K &= \frac{\alpha w}{(1-\alpha) r} && (0.75)\\
c &= \left( -\frac{w}{a} (1-\ell)^{-\xi} (1-\tau) \right)^{\frac{1}{\gamma}} && (0.76)\\
z &= \overline{z} && (0.77)
\end{aligned}
$$

## Exercise 7

The model is now defined by:

$$
\begin{aligned}
\bar{c} &= (1-\tau)[\overline{w}\overline{\ell} + (\bar{r}-\delta)\overline{k}] + \overline{T}\\
u_c(\bar{c}, \overline{\ell}) &= \beta E_t \left\{ u_c(\bar{c}, \overline{\ell}) [(\bar{r}-\delta)(1-\tau) + 1] \right\}\\
-u_{\ell}(\bar{c}, \overline{\ell}) &= u_c(\bar{c}, \overline{\ell}) \overline{w} (1-\tau)\\
\bar{r} &= f_K(\overline{k}, \overline{\ell}, \bar{z})\\
\overline{w} &= f_L(\overline{k}, \overline{\ell}, \bar{z})\\
\tau[\overline{w}\overline{\ell} + (\bar{r}-\delta)\overline{k}] &= \overline{T}
\end{aligned}
$$