

Problem Set Econ - Week 3 - DSGE models

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Exercise 1

In the Brock-Mirman model, there is only one equation, as there is no labor-leisure choice. The one Euler Equation (EE) is:

$$\frac{1}{e^{z_t} K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^\alpha - K_{t+2}} \right\} \quad (0.1)$$

The strategy is to guess the policy function to be $K_{t+1} = A e^{z_t} K_t^\alpha$ and replace both K_{t+2} and K_t in the Euler equation to only keep K_{t+1} :

$$\frac{1}{\frac{1}{A} K_{t+1} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^\alpha - A e^{z_{t+1}} K_{t+1}^\alpha} \right\} \quad (0.2)$$

$$\Leftrightarrow A^2 - (\beta\alpha + 1)A - \beta\alpha = 0 \quad (0.3)$$

$\alpha\beta, 1$ are solutions to this equation.

Exercise 2

The given functional forms are:

$$u(c_t, \ell_t) = \ln c_t + a \ln(1 - \ell_t) \quad (0.4)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha} \quad (0.5)$$

We compute the FOC of these 2 equations and plug them in the baseline model. We get:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.6)$$

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (0.7)$$

$$-\frac{a}{1 - \ell_t} = \frac{1}{c_t} w_t (1 - \tau) \quad (0.8)$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \quad (0.9)$$

$$w_t = (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} \quad (0.10)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.11)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \varepsilon_t^z \quad (0.12)$$

We cannot use the same technic as in the first exercise because we know have consumption and leisure. The random term does not cancel out if we guess a functional form, making the integral impossible to solve analytically. This is because the optimal amount of saving also depends on how much does the agent work: l_t has an impact on the rate of return of the capita k_t and the household's income.

Exercise 3

The new functional forms are:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln(1 - \ell_t) \quad (0.13)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha} \quad (0.14)$$

The 7 characterizing equations are:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.15)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] \right\} \quad (0.16)$$

$$-\frac{a}{1 - \ell_t} = c_t^{-\gamma} w_t (1 - \tau) \quad (0.17)$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \quad (0.18)$$

$$w_t = (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} \quad (0.19)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.20)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \varepsilon_t^z \quad (0.21)$$

$$(0.22)$$

Exercise 4

The new functional forms are:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1-\xi} \quad (0.23)$$

$$F(K_t, L_t, z_t) = e^{z_t} [\alpha K_t^\eta + (1 - \alpha) L_t^\eta]^{\frac{1}{\eta}} \quad (0.24)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.25)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (0.26)$$

$$\frac{a}{(1 - \ell_t)^{\xi}} = c_t^{-\gamma} w_t (1 - \tau) \quad (0.27)$$

$$r_t = \alpha K_t^{\eta-1} e^{z_t} [\alpha K_t^{\eta} + (1 - \alpha) L_t^{\eta}]^{\frac{1}{\eta}-1} \quad (0.28)$$

$$w_t = (1 - \alpha) L_t^{\eta-1} e^{z_t} [\alpha K_t^{\eta} + (1 - \alpha) L_t^{\eta}]^{\frac{1}{\eta}-1} \quad (0.29)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.30)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \varepsilon_t^z \quad (0.31)$$

$$(0.32)$$

Exercise 5

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} \quad (0.33)$$

$$F(K_t, L_t, z_t) = K_t^{\alpha} (e^{z_t} L_t)^{1-\alpha} \quad (0.34)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.35)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (0.36)$$

$$1 = c_t^{-\gamma} w_t (1 - \tau) \quad (0.37)$$

$$r_t = \alpha K_t^{\alpha-1} (e^{z_t})^{1-\alpha} \quad (0.38)$$

$$w_t = (1 - \alpha) K_t^{\alpha} e^{(1-\alpha)z_t} \quad (0.39)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.40)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \varepsilon_t^z \quad (0.41)$$

$$(0.42)$$

Solving the model at the steady state, we have:

$$c = (1 - \tau) [w + (r - \delta) k] + T \quad (0.43)$$

$$1 = \beta \{ [(r - \delta)(1 - \tau) + 1] \} \quad (0.44)$$

$$1 = c^{-\gamma} w (1 - \tau) \quad (0.45)$$

$$r = \alpha K^{\alpha-1} (e^z)^{1-\alpha} \quad (0.46)$$

$$w = (1 - \alpha) K^{\alpha} e^{(1-\alpha)z} \quad (0.47)$$

$$\tau [w + (r - \delta) k] = T \quad (0.48)$$

$$z^* = \bar{z} \quad (0.49)$$

Simplyfying, we get:

$$c = \left((1 - \tau)(1 - \alpha) \left(\frac{1 - \beta}{\beta(1 - \tau)\alpha} + \delta \right)^{\frac{1}{\alpha-1}} \right)^{\frac{1}{\gamma}} \quad (0.50)$$

$$r = \frac{1 - \beta}{\beta(1 - \tau)} + \delta \quad (0.51)$$

$$w = (1 - \alpha) \left(\frac{1 - \beta}{\beta(1 - \tau)\alpha} + \delta \right)^{\frac{1}{\alpha-1}} \quad (0.52)$$

$$k = \left(\frac{1 - \beta}{\beta(1 - \tau)\alpha} + \delta \right)^{\frac{1}{\alpha-1}} \quad (0.53)$$

$$T = \tau [w + (r - \delta) k] \quad (0.54)$$

$$z = \bar{z} \quad (0.55)$$

Exercise 6

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi} \quad (0.56)$$

$$F(K_t, L_t, z_t) = K_t^\alpha (e^{z_t} L_t)^{1-\alpha} \quad (0.57)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.58)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] \right\} \quad (0.59)$$

$$\frac{a}{(1 - \ell_t)^\xi} = c_t^{-\gamma} w_t (1 - \tau) \quad (0.60)$$

$$r_t = \alpha K_t^{\alpha-1} e^{(1-\alpha)z_t} L_t^{1-\alpha} \quad (0.61)$$

$$w_t = (1 - \alpha) K_t^\alpha e^{(1-\alpha)z_t} L_t^{1-\alpha} \quad (0.62)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.63)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \varepsilon_t^z \quad (0.64)$$

Solving for the steady state, we get the following equations:

$$c = (1 - \tau) [w \ell + (r - \delta) k] + T \quad (0.65)$$

$$1 = \beta E \left\{ [(r - \delta) (1 - \tau) + 1] \right\} \quad (0.66)$$

$$\frac{a}{(1 - \ell)^\xi} = c^{-\gamma} w (1 - \tau) \quad (0.67)$$

$$r = \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} \quad (0.68)$$

$$w = (1 - \alpha) K^\alpha e^{(1-\alpha)z} L^{1-\alpha} \quad (0.69)$$

$$\tau [w\ell + (r - \delta)k] = T \quad (0.70)$$

$$z = \bar{z} \quad (0.71)$$

and therefore:

$$c = w\ell + (r - \delta)k \quad (0.72)$$

$$r = \frac{1 - \beta}{\beta(1 - \tau)} + \delta \quad (0.73)$$

$$r = \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} \quad (0.74)$$

$$K = \frac{\alpha w}{(1 - \alpha)r} \quad (0.75)$$

$$c = \left(-\frac{w}{a} (1 - \ell)^{-\xi} (1 - \tau) \right)^{\frac{1}{\gamma}} \quad (0.76)$$

$$z = \bar{z} \quad (0.77)$$

Exercise 7

The model is now defined by:

$$\bar{c} = (1 - \tau)[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] + \bar{T}$$

$$u_c(\bar{c}, \bar{\ell}) = \beta E_t \left\{ u_c(\bar{c}, \bar{\ell}) [(\bar{r} - \delta)(1 - \tau) + 1] \right\}$$

$$-u_\ell(\bar{c}, \bar{\ell}) = u_c(\bar{c}, \bar{\ell}) \bar{w} (1 - \tau)$$

$$\bar{r} = f_K(\bar{k}, \bar{\ell}, \bar{z})$$

$$\bar{w} = f_L(\bar{k}, \bar{\ell}, \bar{z})$$

$$\tau[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] = \bar{T}$$