

# Problem Set Econ - Week 3 - DSGE models

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July 21, 2019

## Exercise 1

In the Brock-Mirman model, there is only one equation, as there is no labor-leisure choice. The one Euler Equation (EE) is:

$$\frac{1}{e^{z_t} K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^\alpha - K_{t+2}} \right\} \quad (0.1)$$

This time we want to rewrite the equation as the  $\Gamma$  function (using Uhlig's notation) to analytically find the values of the matrices F, G, H, L, M, N and consequently P and Q.

To do so we rewrite the above equation:

$$E_t \left\{ \beta \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} (e^{z_t} K_t^\alpha - K_{t+1})}{e^{z_{t+1}} K_{t+1}^\alpha - K_{t+2}} - 1 \right\} = 0 \quad (0.2)$$

This is equivalent to :

$$E_t \{ \Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t) \} = 0 \quad (0.3)$$

With  $X_{t+1} = K_{t+1}$ ,  $X_t = K_t$ ,  $X_{t-1} = K_{t-1}$ ,  $Z_{t+1} = Z_{t+1}$ ,  $Z_t = Z_t$  and our  $\Gamma$  function having only 1 dimension.

We differentiate  $\Gamma$  wrt all parameters, then evaluate them at the steady state for  $z = 0$ .

For F:

$$\frac{d\Gamma}{dX_{t+1}} = \frac{d\Gamma}{dK_{t+2}} = E \left\{ \beta \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} (e^{z_t} K_t^\alpha - K_{t+1})}{(e^{z_{t+1}} K_{t+1}^\alpha - K_{t+2})^2} \right\} \quad (0.4)$$

$$\frac{d\Gamma}{dX} = \left\{ \beta \frac{\alpha K^{\alpha-1} (K^\alpha - K)}{(K^\alpha - K)^2} \right\} \quad (0.5)$$

$$F = \left\{ \beta \frac{\alpha K^{\alpha-1}}{K^\alpha \cdot K} \right\} \quad (0.6)$$

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\*In collaboration with Thomas Pellet.

For G (evaluated at the steady state):

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})((\alpha-1)K^{\alpha-1} - \alpha)(K^\alpha - K) - \alpha K^{\alpha-1}(K^\alpha - 1)}{(K^\alpha - K)^2} \quad (0.7)$$

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})(K^\alpha - K)(-K^{\alpha-1} - \alpha)}{K^\alpha - 1} (K^\alpha - K)^2 \quad (0.8)$$

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})(-K^{\alpha-1} - \alpha)}{K^\alpha - 1} (K^\alpha - K) \quad (0.9)$$

$$G = -\frac{\beta(\alpha K^{\alpha-1})(K^{\alpha-1} + \alpha)}{K^\alpha - 1} (K^\alpha - K) \quad (0.10)$$

For H (evaluated at the steady state):

$$\frac{d\Gamma}{dX_{t-1}} = \frac{\beta\alpha K^{\alpha-1}\alpha K^{\alpha-1}}{K^\alpha - K} \quad (0.11)$$

$$\frac{d\Gamma}{dX_{t-1}} = \frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \quad (0.12)$$

$$H = \frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \quad (0.13)$$

$$(0.14)$$

For L (evaluated at the steady state):

$$\frac{d\Gamma}{dZ_{t+1}} = L = -\frac{\beta\alpha K^{2(\alpha-1)}}{K^\alpha - K} \quad (0.15)$$

$$(0.16)$$

For M (evaluated at the steady state):

$$\frac{d\Gamma}{dZ_t} = M = -\frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \quad (0.17)$$

$$(0.18)$$

Moreover, we have

$$GP^2 + GP + H = 0 \quad (0.19)$$

$$FQN + (FP + G)Q + (LN + M) = 0 \quad (0.20)$$

Rearranging (see exercise 3 for details)

$$H = \frac{\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \quad (0.21)$$

$$P = \frac{-G + / - (G^2 - 4FH)^{0.5}}{2F} \quad (0.22)$$

$$Q = -\frac{LN + M}{FN + FP + G} \quad (0.23)$$

Fitting in our calibrated parameter values from the DSGE lecture:  $\beta = .98$ ,  $\alpha = .40$ ,  $z = 0$ , and having :

$$F = \left\{ -\frac{\alpha K^{\alpha-1}}{K_{\alpha} \cdot K} \right\} \quad (0.24)$$

$$G = -\frac{(\alpha K_{\alpha-1})(K^{\alpha-1} + \alpha)}{K^{\alpha} - 1}(K^{\alpha} - K) \quad (0.25)$$

$$H = \frac{\alpha^2 K^{2(\alpha-1)}}{K^{\alpha} - K} \quad (0.26)$$

$$L = -\frac{\alpha K_{2(\alpha-1)}}{K^{\alpha} - K} \quad (0.27)$$

$$M = -\frac{\alpha^2 K_{2(\alpha-1)}}{K^{\alpha} - K} \quad (0.28)$$

$$P = \frac{-G \pm (G^2 - 4FH)^{0.5}}{2F} \quad (0.29)$$

$$Q = -\frac{LN + M}{FN + FP + G} \quad (0.30)$$

becomes

$$F = \left\{ -\frac{.40K^{-0.60}}{K_{.40} \cdot K} \right\} \quad (0.31)$$

$$G = -\frac{(\alpha K_{-0.60})(K^{-0.60} + 0.40)}{K^{.40} - 1}(K^{.40} - K) \quad (0.32)$$

$$H = \frac{0.40^2 K^{-1.2}}{K(0.40) - K} \quad (0.33)$$

$$L = -\frac{\alpha K_{-1.2}}{K^{.40} - K} \quad (0.34)$$

$$M = -\frac{\alpha^2 K_{-1.2}}{K^{.40} - K} \quad (0.35)$$

$$\text{SolvePQ} \quad (0.36)$$

See Notebook for more the values of P & Q

## Exercise 2

Now we redo the exercise with  $k = \ln(K)$ . Using the previous equations, we can replace  $K$  by  $e^k$  and take the logarithm of the gamma equation to get:

$$\ln \left( E_t \left\{ \beta \frac{\alpha e^{z_{t+1} + (\alpha-1)k_{t+1}} (e^{z_t + \alpha k_t} - e^{k_{t+1}})}{e^{z_{t+1} + \alpha k_{t+1}} - e^{k_{t+2}}} \right\} \right) = 0 \quad (0.37)$$

$$\ln \beta + \ln \left( \alpha e^{z_{t+1} + (\alpha-1)k_{t+1} + z_t + \alpha k_t} - \alpha e^{z_{t+1} + \alpha k_{t+1}} \right) - \ln \left( e^{z_{t+1} + \alpha k_{t+1}} - e^{k_{t+2}} \right) = 0 \quad (0.38)$$

We then have the following equations:

$$\begin{aligned}
F &= \frac{e^{z_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
G &= \frac{(\alpha - 1)e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} - \frac{\alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
H &= \frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} \\
L &= 1 - \frac{e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
M &= \frac{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} \\
P &= \frac{-G \pm (G^2 - 4FH)^{\frac{1}{2}}}{2F} \\
Q &= -\frac{LN + M}{FN + FP + G}
\end{aligned}$$

### Exercise 3

$$E_t \{ F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t \} = 0 \quad (0.39)$$

$$E_t \{ F(P\tilde{X}_t + Q\tilde{Z}_{t+1}) + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t \} = 0 \quad (0.40)$$

$$F[P(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + QN\tilde{Z}_t] + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t = 0 \quad (0.41)$$

$$[(FP + G)P + H]\tilde{X}_{t-1} + [(FQ + L)N + (FP + G)Q + M]\tilde{Z}_t = 0 \quad (0.42)$$

### Exercise 4

See Jupyter Notebook.

### Exercise 5

See Jupyter Notebook.

### Exercise 6

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi} \quad (0.43)$$

$$F(K_t, L_t, z_t) = K_t^\alpha (e^{z_t} L_t)^{1-\alpha} \quad (0.44)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.45)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] \right\} \quad (0.46)$$

$$\frac{a}{(1 - \ell_t)^\xi} = c_t^{-\gamma} w_t (1 - \tau) \quad (0.47)$$

$$r_t = \alpha K_t^{\alpha-1} e^{(1-\alpha)z_t} L_t^{1-\alpha} \quad (0.48)$$

$$w_t = (1 - \alpha) K_t^\alpha e^{(1-\alpha)z_t} L_t^{1-\alpha} \quad (0.49)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.50)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \varepsilon_t^z \quad (0.51)$$

Dropping the definition equations (0.45, 0.48, 0.49, 0.50, 0.51), we are left with the following system:

$$\beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] - c_t^{-\gamma} \right\} = 0 \quad (0.52)$$

$$\frac{a}{(1 - \ell_t)^\xi} = c_t^{-\gamma} w_t (1 - \tau) \quad (0.53)$$

$$(0.54)$$

With  $X_t = (K_{t+1}, L_t)$  and  $Z_t = \{Z_t\}$ , this is equivalent to  $\Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t) = 0$ .

Note: The gamma function should be set to 0 if one wants to do the simple linearization, and equal to 1 if one chooses the log-linearization of the model.

Here we have a system of equations equal to 0.

$$E_t \left\{ \frac{\beta c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] - c_t^{-\gamma}}{\frac{a}{(1 - \ell_t)^\xi} - c_t^{-\gamma} w_t (1 - \tau)} \right\} = 0 \quad (0.55)$$