Math - Problem Set 1 : Measure Theory

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July 22, 2019

Section 1

Exercise 1.3

$\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open }\}$: G1 is the set of open sets in R and is a σ algebra because:

- G1 is a non-empty set.
- G1 includes the empty set because it is an open set.
- G1 includes the complement of the empty set, R, because it is an open set in R : G1 is therefore closed under complements.
- The union of open sets is also open, so G1 is closed under finite unions.
- G1 is closed under countable unions: when n goes to infinity, the union of A_n is also in A because countable unions of open sets are also open.

$\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$ is an algebra but not a σ algebra:

- G2 is a non-empty set.
- G2 includes the empty set (for a = b) and its complement R (for $(-\inf, \infty)$.
- G2 includes intervals and their complements, as well as finite unions of intervals by construction.
- G2 does not include countable unions and is therefore not a σ algebra.

$\mathcal{G}_3 = \{A : A \text{ is a countable union of intervals of the form } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$ is a σ algebra:

- G3 is a non-empty set.
- G3 includes the empty set (for a = b) and its complement R.
- G3 includes intervals and their complements, as well as finite unions of intervals by constructions.
- G3 is closed under countable unions by construction.

Exercise 1.7

Let $A = \{\emptyset, X\}$ be the smallest σ algebra on X. Suppose $\exists S$ a σ algebra on X st $(\emptyset, X) \notin$ S. If the empty set is not in S then we have a contradiction. If X $\notin$ in S then $\emptyset^c \notin S$ then we have a contradiction. Therefore $(\emptyset, X)$ is included in S. Suppose $\exists S'$ the smallest $\sigma algebra$ on X st X $\notin (\emptyset, X)$, then S' there are some elements of S that are not included in X. S' therefore has at least 4 elements $(\emptyset, X, T, T^c)$ but S' is larger than S and we have a contradiction.

Suppose $\exists$ B, a σ algebra on X st $B \not\subset \mathcal{P}(X)$. Then $\exists$ L $\in B$ s.t. $L \notin \mathcal{P}(X)$, ie $L \notin X$ which is a contradiction. Therefore $A = \{P(X)\} = \{A : AinX\}$ is the largest σ algebra on X because in contains all other σ algebra on X.

Exercise 1.10

Let $\{S_\alpha\}$ be a family of σ-algebras on X. All S_α contain $\emptyset$ so the $\emptyset$ is also included in their interesections. Let A an element in the intersection of all S_α: if A is in S_α, then A^c is too and A^c is also included in the intersection and the family of σ algebra on X is therefore closed under complements. S^α are σ algebra so they are closed under countable unions. Their intersection is therefore also closed under countable unions (and under finite unions) and is a σ algebra.

Exercise 1.22

μ is a measure on (X,s) meaning that $\mu(\emptyset) = 0$ and $\mu(UA_i) = \sum \mu(A_i)$ for $A_i \cap A_j = \emptyset$.

μ is monotone:
Let $A \subset BtheB = A \cup DforD \subset BandD \cap A = \emptyset$. Then $\mu(B) = \mu(A \cup D) = \mu(A) + \mu(D)$ by countable additivity. and $\mu(A) + \mu(D) \geq \mu(A)$ because $\mu(D) \geq 0$.

μ is countably subbaditive: Let's take A_i and A_j in A:
$\mu(A_i \cup A_j) = \mu((A_i \cap A_j^c) \cup (A_i \subset \cap A_j) \cup (A_i \cap A_j)) = \mu(A_i \cap A_j^c) \cup \mu(A_i \text{c} \cap A_j) \cup \mu(A_i \cap A_j)$ by countable additivity.
If $A_i \cap A_j = \emptyset$ then $\mu(A_i \cap A_j) = \emptyset$ and $\mu(A_i \cup A_j) = \mu(A_i) + \mu(A_j)$.
If $A_i \cap A_j \neq \emptyset$ then $\mu(A_j \cap A_i) > \emptyset$ and $\mu(A_i \cup A_j) <= \mu(A_i) + \mu(A_j)$.

Exercise 1.23

Let $A, B \in S$ such that $\lambda(A) = \mu(A \cap B)$

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = 0$
- Let $\{A_i, A_j\}$ two elements of a family of disjoint sets in S, without loss of generality:

$$\begin{aligned}
\lambda(A_i \cup A_j) &= \mu((A_i \cup A_j) \cap B) \\
&= \mu(A_i \cup A_j \cap B) \\
&= \mu((A_i \cap B) \cup (A_j \cap B)) \\
&= \mu(A_j \cap B) + \mu(A_i \cap B) \\
&= \lambda(A_j) + \lambda(A_i)
\end{aligned}$$

So λ is also countably additive.

Exercise 1.26

Let μ a measure on (X,S). Then μ is continuous from above: [1]

$$\begin{aligned}
\mu(\cap_{i=1}^{\infty} A_i) &= \mu\left((\cup_1^{\infty} A_i^c)^c\right) \\
&= \mu(X) - \mu\left(\cup_1^{\infty} A_i^c\right) \\
&= \mu(X) - \lim_{n\to\infty} \mu\left(A_n^c\right) \quad \text{using } i) \\
&= \mu(X) - \mu(X) + \lim_{n\to\infty} \mu\left(A_n\right) \\
&= \lim_{n\to\infty} \mu\left(A_n\right)
\end{aligned}$$

Section 2

Exercise 2.10

$$\mu^*(B) = \mu^*(B \cap E \cup E \cap E^c) \subseteq \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

By subadditivity of disjoint elements:

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

Exercise 2.14

Prove that $\mathcal{B}(\mathbb{R})$ is a subset of A, i.e. that $\sigma(\mathcal{O}) = \sigma(\mathrm{A})$ using the Cathodory extension: By definition $\mathcal{A} = \{A : A \text{ is a finite disjoint union of intervals } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$. The σ algebra generated by A contains all open and closed sets in R, and in closed under complements, so $\mathcal{B}(\mathbb{R}) \in \sigma(A)$.

Section 3

Exercise 3.1

A countable subset of the real line is a singleton because each element of the real line is surrounded by irrational numbers. Let $(x) in R$ a singleton set. x is a countable set (of size 1). Suppose that $\exists \varepsilon$ st $\mu(x) > \varepsilon$ according to μ the Lebesque measure. By construction, $(x) in (x - \frac{\varepsilon}{2}; x + \frac{\varepsilon}{2})$. By use subadditivity (because μ is a measure) and get:

$$\mu(x) \leq \mu((x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})) \leq \varepsilon$$

[1]Thanks Thomas' Pellet for debugging my $\cap$ / $\cup$ / ^{c} logical issues.

This contradicts the initial supposition, even for $\lim_O \varepsilon$. Therefore the measure of a countable subset of the real line, a singleton is 0.

Exercise 3.7

One can define a function $g : X \to R$ measurable st $g(x) = -f(x)$. For any a: $\{x \in X : f(x) < a\} = \{x \in X : -g(x) > a\}$.
$f(x) < a$ corresponds to $f(x) \subset (-\inf, a)$ so the complement of this set is also in M (because M is a σ algebra).
All sets $f^{-1}(-\inf, a]; f^{-1}(-\inf, a); f^{-1}(a, \inf), f^{-1}[a, \inf)]$ are also on M.

Exercise 3.10

See lecture notes for the proof.

Exercise 3.17

Suppose that f is bounded. $\exists c\ st, \forall x \in X \quad f(x) \leq c$ Given that f is bounded by c.

$$|s_c(x) - f(x)| = \left| \sum_{i=1}^{c \cdot 2^c} \frac{i-1}{2^c} \chi_{E_i^c} + c\chi_{E_\infty^c} - f(x) \right| \tag{0.1}$$

$$= \left| \sum_{i=1}^{c \cdot 2^c} \frac{i-1}{2^c} \chi_{E_i^c} - f(x)_{|f(x)<c} \right| \tag{0.2}$$

Note that for any x, since $f(x)$ is bounded $f(x) \in [0, c]$ so $x \in E_i^c$ for some i. Note there is an $N \geq c$ such that $\frac{1}{2^N} < \epsilon$.
Then for any $n \geq N$, $|f(x) - s_n(x)| < \epsilon$, so we have uniforme convergence.

In other words: for $x < c$ then $c\chi_{E_i^c} - f(x) = 0$ because $f(x) = c$ and one can get rid of the extra term in the equation. We reach the definition of uniform convergence as this inequality $(|s_c(x) - f(x)| < \epsilon)$ must hold for any ϵ, even arbirately small, for each x.

Lebesgue Convergence

Exercise 4.13

Because $|f| < M$ on $E \in \mathcal{M}$ and $\mu(E) < \infty$ we have using 4.7:

$$\int_E f^+ d\mu < \int_E M\chi_{E_M} d\mu = M\mu(E_M) < \infty \tag{0.3}$$

$$\int_E f^- d\mu < \int_E M\chi_{E_M} d\mu = M\mu(E_M) < \infty \tag{0.4}$$

F is therefore integrable and $f \in \mathscr{L}^1(\mu, E)$.

Exercise 4.14

Suppose $f \in \mathscr{L}^1(\mu, E)$.
Let $E_n = \{x \in X : f(x) \geq n\}$ and $E = \cup_{n=1}^{\infty} E_n$ Using the properties of the integral, we have that:

$$\int_X f d\mu \geq \int_{E_n} f d\mu \geq \int_{E_n} n\chi_{E_n} d\mu = n\mu(E_n) \geq n\mu(E) \tag{0.5}$$

$$\Rightarrow \mu(E) \leq \int_X \frac{1}{n} f d\mu \sim \mathcal{O}(\frac{1}{n}) \tag{0.6}$$

This tends to zero as $n \to \infty$

Exercise 4.15

Suppose $f, g \in \mathscr{L}^1(\mu, E)$ and $f \leq g$:

$$\left\{ \int_E s d\mu : 0 \leq s \leq f, \text{s simple} \right\} \subset \left\{ \int_E s d\mu : 0 \leq s \leq g, \text{s simple} \right\} \tag{0.7}$$

Taking the supremum and by definition of the integral, we have that

$$\int_E f d\mu \leq \int_E g d\mu \tag{0.8}$$