Problem Set Econ - Week 3 - DSGE models - Linear Approximation

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Exercise 1

In the Brock-Mirman model, there is only one equation, as there is no labor-leisure choice. The one Euler Equation (EE) is:

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^{\alpha} - K_{t+2}} \right\} \tag{0.1}$$

This time we want to rewrite the equation as the Γ function (using Uhlig's notation) to analytically find the values of the matrices F, G, H, L, M, N and consequently P and Q.

To do so we rewrite the above equation:

$$E_t \left\{ \beta \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} (e^{z_t}K_t^{\alpha} - K_{t+1})}{e^{z_{t+1}}K_{t+1}^{\alpha} - K_{t+2}} - 1 \right\} = 0 \tag{0.2}$$

This is equivalent to :

$$E_t \left\{ \Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t) \right\} = 0 \tag{0.3}$$

With $X_{t+1} = K_{t+2}$, $X_t = K_{t+1}$, $X_{t-1} = K_t$, $Z_{t+1} = Z_{t+1}$, $Z_t = Z_t$ and our Γ function having only 1 dimension.

We differentiate Γ wrt all parameters, then evaluate them at the steady state for z == 0.
For F:

$$\frac{d\Gamma}{dX_{t+1}} = \frac{d\Gamma}{dK_{t+2}} = E \left\{ \beta \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} (e^{z_t}K_t^{\alpha} - K_{t+1})}{(e^{z_{t+1}}K_{t+1}^{\alpha} - K_{t+2})^2} \right\} \tag{0.4}$$

$$\frac{d\Gamma}{dX} = \left\{ \beta \frac{\alpha K^{\alpha-1} (K^{\alpha} - K)}{(K^{\alpha} - K)^2} \right\} \tag{0.5}$$

$$F = \left\{ \beta \frac{\alpha K^{\alpha-1}}{K^{\alpha} - K} \right\} \tag{0.6}$$

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For G (evaluated at the steady state):

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})((\alpha-1)K^{\alpha-1}-\alpha)(K^\alpha - K) - \alpha K^{\alpha-1}(K^\alpha - 1)}{(K^\alpha - K)^2} \tag{0.7}$$

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})(K^\alpha - K)(-K^{\alpha-1}-\alpha)}{K^\alpha - 1)}(K^\alpha - K)^2 \tag{0.8}$$

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})(-K^{\alpha-1}-\alpha)}{K^\alpha - 1)}(K^\alpha - K) \tag{0.9}$$

$$G = -\frac{\beta(\alpha K_{\alpha-1})(K^{\alpha-1}+\alpha)}{K^\alpha - 1)}(K^\alpha - K) \tag{0.10}$$

For H (evaluated at the steady state):

$$\frac{d\Gamma}{dX_{t-1}} = \frac{\beta\alpha K^{\alpha-1}\alpha K^{\alpha-1}}{K^\alpha - K} \tag{0.11}$$

$$\frac{d\Gamma}{dX_{t-1}} = \frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.12}$$

$$H = \frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.13}$$

$$\tag{0.14}$$

For L (evaluated at the steady state):

$$\frac{d\Gamma}{dZ_{t+1}} = L = -\frac{\beta\alpha K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.15}$$

$$\tag{0.16}$$

For M (evaluated at the steady state):

$$\frac{d\Gamma}{dZ_t} = M = -\frac{\beta\alpha^2 K2(\alpha-1)}{K^\alpha - K} \tag{0.17}$$

$$\tag{0.18}$$

Moreover, we have

$$GP^2 + GP + H = 0 \tag{0.19}$$

$$FQN + (FP+G)Q + (LN+M) = 0 \tag{0.20}$$

Rearranging (see exercise 3 for details)

$$H = \frac{\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.21}$$

$$P = \frac{-G + / - (G^2 - 4FH)^{0.5}}{2F} \tag{0.22}$$

$$Q = -\frac{LN+M}{FN+FP+G} \tag{0.23}$$

Therefore

$$F = \left\{ -\frac{\alpha K^{\alpha-1}}{K^{\alpha} - K} \right\} \tag{0.24}$$

$$G = -\frac{(\alpha K_{\alpha-1})(K^{\alpha-1} + \alpha)}{K^{\alpha} - 1)}(K^{\alpha} - K) \tag{0.25}$$

$$H = \frac{\alpha^2 K^{2(\alpha-1)}}{K^{\alpha} - K} \tag{0.26}$$

$$L = -\frac{\alpha K_{2(\alpha-1)}}{K^{\alpha} - K} \tag{0.27}$$

$$M = -\frac{\alpha^2 K_{2(\alpha-1)}}{K^{\alpha} - K} \tag{0.28}$$

$$P = \frac{-G \pm (G^2 - 4FH)^{0.5}}{2F} \tag{0.29}$$

$$Q = -\frac{LN + M}{FN + FP + G} \tag{0.30}$$

See jupyter notebook for the rest of the exercise.

Exercise 2

Now we redo the exercise with $k = \ln(K)$. Using the previous equations, we can replace K by e^k and take the logarithm of the gamma equation to get:

$$\ln\left(E_t \left\{ \beta \frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}}(e^{z_t+\alpha k_t} - e^{k_{t+1}})}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \right\} \right) = 0 \tag{0.31}$$

$$\ln\beta + \ln\left(\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}\right) - \ln\left(e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}\right) = 0 \tag{0.32}$$

We then have the following equations:

$$F = \frac{e^{z_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}}$$
$$G = \frac{(\alpha-1)e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} - \frac{\alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}}$$
$$H = \frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}}$$
$$L = 1 - \frac{e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}}$$
$$M = \frac{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}}$$
$$P = \frac{-G \pm (G^2 - 4FH)^{\frac{1}{2}}}{2F}$$
$$Q = -\frac{LN + M}{FN + FP + G}$$

Exercise 3

$$
\begin{aligned}
E_t\left\{F\tilde{X}_{t+1}+G\tilde{X}_t+H\tilde{X}_{t-1}+L\tilde{Z}_{t+1}+M\tilde{Z}_t\right\}&=0 && (0.33)\\
E_t\left\{F\left(P\tilde{X}_t+Q\tilde{Z}_{t+1}\right)+G\left(P\tilde{X}_{t-1}+Q\tilde{Z}_t\right)+H\tilde{X}_{t-1}+L\tilde{Z}_{t+1}+M\tilde{Z}_t\right\}&=0 && (0.34)\\
F\left[P\left(P\tilde{X}_{t-1}+Q\tilde{Z}_t\right)+QN\tilde{Z}_t\right]+G\left(P\tilde{X}_{t-1}+Q\tilde{Z}_t\right)+H\tilde{X}_{t-1}+L\tilde{Z}_{t+1}+M\tilde{Z}_t&=0 && (0.35)\\
[(FP+G)P+H]\tilde{X}_{t-1}+[(FQ+L)N+(FP+G)Q+M]\tilde{Z}_t&=0 && (0.36)
\end{aligned}
$$

Exercise 4

See Jupyter Notebook.

Exercise 5

See Jupyter Notebook.

Exercise 6

The functional forms are now

$$
\begin{aligned}
u\left(c_t,\ell_t\right)&=\frac{c_t^{1-\gamma}-1}{1-\gamma}+a\frac{\left(1-\ell_t\right)^{1-\xi}-1}{1-\xi} && (0.37)\\
F\left(K_t,L_t,z_t\right)&=K_t^{\alpha}\left(e^{z_t}L_t\right)^{1-\alpha} && (0.38)
\end{aligned}
$$

and the characterizing equations are now:

$$
\begin{aligned}
c_t&=(1-\tau)\left[w_t\ell_t+\left(r_t-\delta\right)k_t\right]+k_t+T_t-k_{t+1} && (0.39)\\
c_t^{-\gamma}&=\beta E_t\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1}-\delta\right)(1-\tau)+1\right]\right\} && (0.40)\\
\frac{a}{(1-\ell_t)^{\xi}}&=c_t^{-\gamma}w_t(1-\tau) && (0.41)\\
r_t&=\alpha K_t^{\alpha-1}e^{(1-\alpha)z_t}L_t^{1-\alpha} && (0.42)\\
w_t&=(1-\alpha)K_t^{\alpha}e^{(1-\alpha)z_t}L_t^{1-\alpha} && (0.43)\\
\tau\left[w_t\ell_t+\left(r_t-\delta\right)k_t\right]&=T_t && (0.44)\\
z_t&=(1-\rho_z)\bar{z}+\rho_z z_{t-1}+\varepsilon_t^z && (0.45)
\end{aligned}
$$

Dropping the definition equations (0.45, 0.48, 0.49, 0.50, 0.51), we are left with the following system:

$$
\beta E_t\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1}-\delta\right)(1-\tau)+1\right]-c_t^{-\gamma}\right\}=0 \qquad (0.46)
$$

$$\frac{a}{(1-\ell_t)^{\xi}} = c_t^{-\gamma} w_t (1-\tau) \tag{0.47}$$

$$\tag{0.48}$$

With $X_t = (K_{t+1}, L_t)$ and $Z_t = \{Z_t\}$, this is equivalent to $\Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t) = 0$.

Note: The gamma function should be set to 0 if one wants to do the simple linearization, and equal to 1 if one chooses the log-linearization of the model.
Here we have a system of equations equal to 0.

$$E_t \left\{ \begin{array}{l} \beta c_{t+1}^{-\gamma} \left[(r_{t+1} - \delta)(1-\tau) + 1 \right] - c_t^{-\gamma} \\ \frac{a}{(1-\ell_t)^{\xi}} - c_t^{-\gamma} w_t (1-\tau) \end{array} \right\} = 0 \tag{0.49}$$