Armington Model with CES preferences / Anderson 1979)

→ each country makes a signature good (unique provider on the int'l market)
→ comparative advantage

→ Consumers have CES pref ⇒ σ elasticity of substitut°

↳ **Consumer Demand** : $X_{ij} = \dfrac{(p_i \tau_{ij})^{1-\sigma}}{\sum_e (p_e \tau_{ej})^{1-\sigma}} X_j = \dfrac{(p_i \tau_{ij})^{1-\sigma}}{P_j^{1-\sigma}} X_j$

↳ X_j = expenditure in j

→ **Supply side** : economy endowed Q_i units ⇒ GDP Y_i

⇒ $p_i = \dfrac{Y_i}{Q_i}$

$$\Rightarrow X_{ij} = \underbrace{\frac{Y_i^{1-\sigma}}{Q_i^{1-\sigma}}}_{\text{origin term}} \underbrace{\frac{X_j}{P_j^{1-\sigma}}}_{\text{destinat° term}} \underbrace{\tau_{ij}^{1-\sigma}}_{\text{bilateral term}}$$

⇒ a Ricardian "corner" case gives us a "Gravity" equat° for multiple countries with trade costs that we can take to data.

Let's say ∃ techno → linear product funct° : $Q_i = A_i \cdot L_i$ rather than endowment

⇒ $p_i = \dfrac{w_i}{A_i}$

⇒ $Y_i = w_i L_i$ GDP, what i have to spend

Exact hat algebra (in trad "ij" equation)

↳ how to do counterfactual

$\varepsilon = \sigma - 1$

Start from market clearing condition & gravity equat°

$\lambda_{ij} \equiv \dfrac{X_{ij}}{X_j}$ expenditure share

$T_i \equiv A_i^{\varepsilon}$
↳ monotone transformat°
productivity

$$\underbrace{w_i L_i}_{\text{Nat'al income}} = \underbrace{\sum_{j=1}^N \overbrace{\lambda_{ij}}^{\text{fract spent on } i} \overbrace{w_j L_j}^{X_j}}_{\text{expenditure}} \qquad \text{Market Clearing}$$

$$\lambda_{ij} = \frac{T_i (\tau_{ij} w_i)^{-\varepsilon}}{\sum_{\ell=1}^N T_\ell (\tau_{\ell j} w_\ell)^{-\varepsilon}}$$

$\Rightarrow$ w only endogenous variable to solve for

Counterfactual : change in productivity

$$\hat{T}_i = \frac{T_i'}{T_i} \qquad \text{assume} \quad \hat{\tau} = 1 \quad \hat{L} = 1$$

$\Rightarrow$ Solve for the endogenous variables $\hat{\lambda}_{ij}$, $\hat{X}_{ij}$, $\hat{w}_i$

Define $\gamma_{ij} \equiv \dfrac{X_{ij}}{Y_i}$ sales shares.

* market clearing $\Rightarrow$ old $\quad w_i L_i = \sum_{j=1}^N \lambda_{ij} w_j L_j$

$\Rightarrow$ new $\quad w_i' L_i' = \sum_{j=1}^N \lambda_{ij}' w_j' L_j' = \sum_{j=1}^N X_{ij}'$

(1) ↳ divide $\quad \hat{w}_i \hat{L}_i = \sum_{j=1}^N \dfrac{X_{ij}'}{w_i L_i} = \sum_{j=1}^N \dfrac{X_{ij}}{w_i L_i} \hat{X}_{ij} \equiv \sum_{j=1}^N \gamma_{ij} \hat{X}_{ij}$

* Trade / Gravity Equation $\Rightarrow$ old $\quad \lambda_{ij} = \dfrac{T_i (\tau_{ij} w_i)^{-\varepsilon}}{\sum_{\ell=1}^N T_\ell (\tau_{\ell j} w_\ell)^{-\varepsilon}}$

$\Rightarrow$ New $\quad \lambda_{ij}' = \dfrac{T_i' (\tau_{ij} w_i')^{-\varepsilon}}{\sum_{\ell=1}^N T_\ell' (\tau_{\ell j} w_\ell')^{-\varepsilon}}$

↳ divide : $\quad \hat{\lambda}_{ij} \equiv \dfrac{\lambda_{ij}'}{\lambda_{ij}} = \hat{T}_i \hat{w}_i^{-\varepsilon} \hat{\tau}_{ij}^{-\varepsilon} \dfrac{\sum_{\ell=1}^N T_\ell' (\tau_{\ell j} w_\ell')^{-\varepsilon}}{\sum_{\ell=1}^N T_\ell (\tau_{\ell j} w_\ell)^{-\varepsilon}}$

$$\Rightarrow \hat{\lambda}_{ij} = \frac{\hat{T}_i \hat{w}_i^{-\varepsilon} \hat{\tau}_{ej}}{\sum_{\ell=1}^N \lambda_{\ell j} \hat{T}_\ell \hat{w}_\ell^{-\varepsilon} \hat{\tau}_{\ell j}^{-\varepsilon}} \qquad (2)$$

↳ $\lambda_{j\ell}$ sales share

⇒ Combine (1) and (2) under the assumption of balanced budgets :

$\hat{Y}_i = \hat{X}_i$

national income

national expenditure

+ Notice $\hat{\tau} = \hat{L} = 1$ ⇒ system of equat° characterizing an eq^M $\hat{w}_i$ as a function of shocks

- $\hat{T}_i$
- initial eq^M shares λ_{ij}
- $\gamma_{ij} = \frac{X_{ij}}{w_i L_i}$ expenditure shares
- ε trade elasticity

$$\hat{w}_i = \sum_{j=1}^{N} \gamma_{ij} \hat{X}_{ij} = \sum_{j=1}^{N} \gamma_{ij} \hat{\lambda}_{ij} \hat{w}_j$$

$$= \sum_{j=1}^{N} \frac{\gamma_{ij} \hat{T}_i \hat{w}_i^{-\varepsilon} \hat{w}_j}{\sum_{\ell=1}^{N} \lambda_{\ell j} \hat{T}_j \hat{w}_\ell^{-\varepsilon}}$$

$$\left.\begin{array}{l}\text{Data} \Rightarrow \varepsilon, \lambda_{ij}, \gamma_{ij} \\ \text{Feed productivity shocks } \hat{T}\end{array}\right\} \Rightarrow \hat{w}.$$

The Logit Model of Discrete Choice

Individual i considers choice j

↳ $U_{ij} = V_{ij} + \varepsilon_{ij}$ utility

↳ assume errors iid T1EV : $F(\varepsilon_{ij}) = \exp(-\exp(-\varepsilon_{ij}))$

↳ choice proba

$$\Pr(U_{ij} > U_{ij'},\ \forall j' \neq j) = \frac{\exp(V_{ij})}{\sum_{j'} \exp(V_{ij'})}$$

Now, try a cost-minimizat° problem with multiplicative error term:

↳ Cost : $\ln c_{ji} = \ln c_j + \ln \tau_{ji} + \underbrace{\varepsilon_{ji}}_{\text{error term}}$

specific to pair of countries ji

↳ least cost provider proba

$$\Pr(-\ln c_{ji} > -\ln c_{j'i},\ \forall j' \neq j) = \Pr(\ln c_{ji} < \ln c_{j'i},\ \forall j' \neq j) = \frac{1/(c_j \tau_{ji})}{\sum_{j'} 1/(c_{j'} \tau_{j'i})}$$

CES preferences

Preferences $U = \left(\int_0^n q(\omega)^\rho \, d\omega\right)^{1/\rho}$ $\quad 0<\rho<1$

↳ $q(\omega)$ consumpt° of ω
↳ ρ measure of substitutability

↳ constrained optimizat° $\mathcal{L} = U^\rho - \lambda\left[\int_0^\omega p(\omega)q(\omega)\,d\omega - I\right]$ algebra trick

↳ $\dfrac{\partial \mathcal{L}}{\partial q(\omega)} = \rho\, q(\omega)^{\rho-1} - \lambda p(\omega)$

$\Rightarrow q(\omega) = \left(\dfrac{\lambda p(\omega)}{\rho}\right)^{1/\rho-1}$ Frisch demand funct°

$\Rightarrow \dfrac{q(\omega_1)}{q(\omega_2)} = \left[\dfrac{p(\omega_1)}{p(\omega_2)}\right]^{\underbrace{1/\rho-1}_{\equiv -\sigma}}$ Marshallian Demand Funct°

$\Rightarrow \sigma = \dfrac{-d\ln[q(\omega_1)/q(\omega_2)]}{d\ln[p(\omega_1)/p(\omega_2)]}$ $\equiv$ the elasticity of substitution

↳ CES demand function

Rearranging $q(\omega_1) = q(\omega_2)\left(\dfrac{p(\omega_1)}{p(\omega_2)}\right)^{-\sigma}$

$$\underbrace{\int_0^n p(\omega_1)\,q(\omega_1)\,d\omega_1}_{\text{consumer's total expenditure on all varieties}} = \int_0^n q(\omega_2)\,p(\omega_1)^{1-\sigma}\,p(\omega_2)^{\sigma}\,d(\omega_1)$$

consumer's total expenditure on all varieties
$\parallel$
consumer's income I

$\Rightarrow I = q(\omega_2)\,p(\omega_2)^{\sigma}\int_0^n p(\omega_1)^{1-\sigma}\,d\omega_1$

$\Rightarrow q(\omega_2) = \dfrac{I\,p(\omega_2)}{\int_0^n p(\omega_1)^{1-\sigma}\,d\omega_1}$ Marshallian Demand for ω_2 in terms of prices & income

Defining a price index $P \equiv \left(\int_0^n p(\omega)^{1-\sigma} \, d\omega \right)^{1/1-\sigma}$

$$\hookrightarrow q(\omega) = \frac{I}{P} \left(\frac{p(\omega)}{P} \right)^{-\sigma}$$

$$\hookrightarrow U = \frac{I}{P} = \left(\int_0^n q(\omega)^{\rho} \, d\omega \right)^{1/\rho}$$