Comprehension Check 3

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Preliminary setup: From the model we have

- Individuals maximizing utility by choosing their location τ and their sector σ st

$$f(\omega,\tau,\sigma) > 0 \Leftrightarrow \{\tau,\sigma\} \in \operatorname{argmax} U(\tau,\sigma,\omega) \tag{1}$$
$$U(\tau,\sigma,\omega) = T(\tau)H(\omega,\sigma)p(\sigma) - r(\tau) \tag{2}$$

- Final-good producers choose intermediate goods indexed by σ st

$$Q(\sigma) = I(\frac{P(\sigma)}{B(\sigma)})^{-\epsilon} \tag{3}$$

  Where I denotes total income as is a function of $L, q(.), p(.) f(.)$.

- Absentee Landlords choose rents $r(\tau)$ such that unoccupied locations have rental prices of 0

$$r(\tau) \times (S'(\tau) - L \int_\sigma \int_\omega f(\omega,\tau,\sigma) d\omega d\sigma)) = 0, \forall \tau \tag{4}$$

- Market clearing
  - The endogenous quantity of individuals of skill ω residing at τ and working in σ, $L \times f(\omega,\tau,\sigma)$ is st the supply of a location type $\geq$ its demand

$$S'(\tau) \geq L \int_\omega \int_\sigma f(\omega,\tau,\sigma) d\sigma d\omega, \forall \tau \tag{5}$$

  - The demand and supply of intermediate goods are equal

$$Q(\sigma) = L \int_\omega \int_\tau q(\tau,\sigma,\omega) f(\omega,\tau,\sigma) d\omega d\tau, \forall \sigma \tag{6}$$

  - Every individual is located somewhere

$$f(\omega) = \int_\sigma \int_\tau f(\omega,\tau,\sigma) d\tau d\sigma, \forall \omega \tag{7}$$

Moreover, we define

- Commuting costs

$$T(\tau) = d_1 - d_2\tau$$

- The supply of locations with innate desirability of at least τ

$$S(\tau) = \pi\tau^2$$
$$S'(\tau) = 2\pi\tau$$

- The income of workers of skill ω

$$G(\omega) = g\omega \qquad\qquad g \text{ a constant}$$

- The skill distribution

$$\omega \sim U(\underline{\omega}, \bar{\omega})$$

1. *Show that $\bar{\tau}$ and $\underline{\gamma}$ depend only on the exogenous parameters L, d_1, d_2.*

Regarding $\bar{\tau}$:
In order to get closed-form solutions, we take advantage of Appendix A.1 of the paper, that gives us that $N(.)$ gives a one-to-one mapping from τ to ω such that

$$N(\tau) = F^{-1}(\frac{L(c) - S(\tau)}{L(c)}) = \omega$$

Where $F(.)$ is the CDF of the skill distribution ω, and $S(\tau) = L\int_0^{\tau}\int_{\sigma}\int_{\omega} f(\omega, x, \sigma)d\omega d\sigma dx$. We are given that L is given and that $\omega \sim U[\underline{\omega}, \bar{\omega}]$. Therefore, $F^{-1}(x) = \underline{\omega} + (\bar{\omega} - \underline{\omega}) \times x$. With $x = \frac{L - S(\tau)}{L}$, this gives us that

$$\begin{aligned}
N(\tau) = \omega &= F^{-1}(\frac{L(c) - S(\tau)}{L(c)}) \\
&= \underline{\omega} + (\bar{\omega} - \underline{\omega}) \times \frac{L - S(\tau)}{L} \\
&= \underline{\omega} + (\bar{\omega} - \underline{\omega}) \times \frac{L - \pi\tau^2}{L} && S(\tau) \equiv \pi\tau^2 \\
\frac{\omega - \underline{\omega}}{\bar{\omega} - \underline{\omega}} &= \frac{L - \pi\tau^2}{L} \\
\tau^2 &= (1 - \frac{\omega - \underline{\omega}}{\bar{\omega} - \underline{\omega}})\frac{L}{\pi} \\
\tau &= \sqrt{(1 - \frac{\omega - \underline{\omega}}{\bar{\omega} - \underline{\omega}})\frac{L}{\pi}}
\end{aligned}$$

Given L, we have our one-to-one mapping between τ and ω, where τ is decreasing in ω So

$$\begin{aligned}
\bar{\tau} &= \tau(\underline{\omega}) = \sqrt{\frac{L}{\pi}} \\
\underline{\tau} &= \tau(\bar{\omega}) = 0
\end{aligned}$$

The city edge, characterized by $\bar{\tau}$ is therefore only a function of exogenous parameters, and in particular of L[1].

[1]Non-closed form version (more general): Define $\bar{\tau}$ as the highest inverse desirability that is occupied in the city, and, likewise, $\underline{\gamma} = A(c)T(\bar{\tau}) = T(\bar{\tau})$ the lowest attractiveness of a location in the city that is occupied.
From the paper (without the (c) index as we only have one city here)

$$\bar{\tau} \equiv \sup_{\tau}\{\tau : f(\omega, \tau, \sigma) > 0\}$$

Besides, from equations (1) and (2) we see that $f(\omega, \tau, \sigma)$ is a function of $T(\tau) = d_1 - d_2\tau$ and $r(\tau)$ the rental rate in location τ which, according to (4), is chosen to maximize landlords' utility and such that

$$\begin{aligned}
S'(\tau) &= L\int_{\sigma}\int_{\omega} \sigma f(\omega, \sigma, \tau)d\sigma d\omega \\
\text{or } S'(\tau) &< L\int_{\sigma}\int_{\omega} \sigma f(\omega, \sigma, \tau)d\sigma d\omega \implies r(\tau) = 0
\end{aligned}$$

Regarding $\underline{\gamma}$:
Given the functional forms we are given for $T(\tau)$, and $A(c) = 1$, we have

$$\gamma = T(\tau) = d_1 - d_2\tau$$

is decreasing in τ so

$$\underline{\gamma} = d_1 - d_2\bar{\tau}$$

simply depends on the exogenous parameters d_1, d_2 directly and L through $\bar{\tau}(L)$

2. *The rent at the city edge $\bar{\tau}$ is zero. What is the rent at the center of the city $\tau = 0$?*
Let's figure out rents as a function of τ: from Appendix A.1, Lemma 7 we have

$$\begin{aligned}
r(\tau) &= -A\int_{\tau}^{\bar{\tau}} \underbrace{T'(t)}_{-d_2} G(\underbrace{N(t)}_{\omega})dt \\
&= d_2\int_{\tau}^{\bar{\tau}} G(\underline{\omega} + (\bar{\omega} - \underline{\omega}) \times \frac{L - \pi\tau^2}{L})dt \\
&= d_2\int_{\tau}^{\bar{\tau}} G(\bar{\omega} - (\bar{\omega} - \underline{\omega}) \times \frac{\pi t^2}{L})dt \\
&= d_2 \times g \times \int_{\tau}^{\bar{\tau}} \bar{\omega} - (\bar{\omega} - \underline{\omega}) \times \frac{\pi t^2}{L}dt \qquad \text{using } G(x) = g \times x \\
&= d_2 \times g \times [\bar{\omega}t - (\bar{\omega} - \underline{\omega})\frac{\pi t^3}{3L})]_{\tau}^{\bar{\tau}} \\
&= d_2 \times g \times [\bar{\omega}(\bar{\tau} - \tau) - (\bar{\omega} - \underline{\omega})\frac{\pi}{3L}(\bar{\tau}^3 - \tau^3))]
\end{aligned}$$

For $\tau = 0$

$$r(0) = d_2 \times g \times [\bar{\omega}\bar{\tau} - (\bar{\omega} - \underline{\omega})\frac{\pi}{3L}(\bar{\tau}^3)]$$

Plugging in for $\bar{\tau} = \sqrt{\frac{L}{\pi}}$ from the previous question

$$r(0) = d_2 \times g \times \sqrt{\frac{L}{\pi}}(\frac{1}{3}\underline{\omega} + \frac{2}{3}\bar{\omega})$$

At $\tau = \bar{\tau}$ it must be the case that

$$\begin{aligned}
r(\bar{\tau}) &= 0 \\
S'(\bar{\tau}) = 2\pi\bar{\tau} &= L\int_{\sigma}\int_{\omega} \sigma f(\omega, \sigma, \bar{\tau})d\sigma d\omega
\end{aligned}$$

So

$$\bar{\tau} = \frac{1}{2\ \pi}L\int_{\sigma}\int_{\omega} \sigma f(\omega, \sigma, \bar{\tau})d\sigma d\omega$$

where from (1) and (2)

$$f(\omega, \bar{\tau}, \sigma) \text{ st } \sigma \in \operatorname{argmax} U(\bar{\tau}, \sigma, \omega) = \operatorname{argmax}(d_1 - d_2\bar{\tau})H(\omega, \sigma)p(\sigma) - \underbrace{r(\bar{\tau})}_{=0}$$

Finally

$$\bar{\tau} = \frac{1}{2\ \pi}L\int_{\sigma}\int_{\omega} \sigma \underbrace{f(\omega, \sigma, \bar{\tau})}_{\text{a function of } d_1, d_2} d\sigma d\omega$$

3. *Suppose that g increases. What happens to the rent schedule? What happens to the equilibrium utility of each skill level?*

   From above, we have that $r(.)$ is increasing in g so an increase in g will lead to an increase in rents, at all τ and therefore at all ω.
   Regarding the utility:

$$\begin{aligned}
U(\tau,\sigma,\omega) &= T(\tau)H(\omega,\sigma)p(\sigma) - r(\tau) \\
&= (d_1 - d_2\tau)G(\omega) - r(\tau) && \text{using } G(\omega) = H(\omega, M(\omega))p(M(\omega)) \\
&= (d_1 - d_2\tau)g\omega - r(\tau) \\
&= (d_1 - d_2\tau)\omega g - d_2 \times g \times [\bar{\omega}(\bar{\tau} - \tau) - (\bar{\omega} - \underline{\omega})\frac{\pi}{3L}(\bar{\tau}^3 - \tau^3))] \\
&= g \times \{(d_1 - d_2\tau)\omega - d_2 \times [\bar{\omega}(\bar{\tau} - \tau) - (\bar{\omega} - \underline{\omega})\frac{\pi}{3L}(\bar{\tau}^3 - \tau^3))]\}
\end{aligned}$$

   Therefore, the utility of each worker ω is linear in g. An increase in g shifts everyone's utility up.

4. *What happens to the equilibrium utility of $\bar{\omega}$ if the value of $\underline{\omega}$ increases? (skill compression)*

   In the previous question we showed that $\bar{\tau}$ and $\underline{\gamma}$ only depend on L, d_1, d_2, which are unaffected here. Therefore, the city-edge, characterized by $\bar{\tau}$ will remain unchanged. Housing supply is fixed. Because $\omega \sim U[\underline{\omega}, \bar{\omega}]$, the increase in $\underline{\omega}$ leads to an increase in the density of higher skilled workers $\bar{\omega}$. Therefore, rents at the center of the city increase, and the utility of $\bar{\omega}$ decreases. Notice that rents at the edge of the city $r(\bar{\tau})$ are still pinned down by the boundary conditions so $r(\bar{\tau}) = 0$, unchanged.
   Formally, from 3, we have

$$U(\tau = 0, \bar{\omega}) = d_1\bar{\omega}g \underbrace{- d_2 g[\bar{\omega}\bar{\tau} - \underbrace{(\bar{\omega} - \underline{\omega})}_{\downarrow}\frac{\pi}{3L}(\bar{\tau}^3 - \tau^3))]}_{\downarrow}$$

5. *What happens to the equilibrium utility of $\underline{\omega}$ if the value of $\bar{\omega}$ increases? (skill dilation)*
   In the case of skill dilation, the equilibrium utility of $\underline{\omega}$

   Similarly, because of the uniform distribution of skills ω, an increase in $\bar{\omega}$ decreases the density of low skilled workers. However, because the city edge does not change, and the boundary condition is still valid, it must still be the case that rents at the city edge are 0. Therefore the equilibrium utility of $\underline{\omega}$ is unchanged.
   Formally, from 3, we have

$$U(\underline{\omega}, \bar{\tau}) = (d_1 - d_2)\underline{\omega}g$$

   Which doesn't depend on $\bar{\omega}$