Comprehension Check 2: Productivity in the Melitz (2003) model

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1. *Based on the primitives in section 2 of Melitz (2003), compute quantity per worker and revenue per worker for an active firm with productivity ϕ*
   The production function is
$$l = f + \frac{q(\phi)}{\phi}$$
   Therefore
$$\begin{aligned}\frac{q(\phi)}{l(\phi)} &= \frac{q(\phi)}{f + \frac{q(\phi)}{\phi}} \\ &= \frac{Q[\rho P \phi]^{\sigma}}{f + Q[\rho P]^{\sigma}\phi^{\sigma-1}} && \text{using (3) into (2)} \\ &= \frac{Q[\rho P]^{\sigma}}{\frac{f}{\phi^{\sigma}} + Q[\rho P]^{\sigma}\phi^{-1}}\end{aligned}$$
   Similarly for
$$\frac{r(\phi)}{l(\phi)} = \frac{R[\rho P \phi]^{\sigma-1}}{f + Q[\rho P]^{\sigma}\phi^{\sigma-1}}$$

2. *Footnote 7 in Melitz (2003) says "Higher productivity may also be thought of as producing a higher quality variety at equal cost." Answer the questions posed in the accompanying PDF.*
   The utility function is
$$U = \left[\int_{\omega\in\Omega} \varphi(\omega)^{\epsilon} q(\omega)^{\rho} \mathrm{d}\omega\right]^{1/\rho}$$
   And the price index, by definition is
$$P = \int_0^n p(\omega) q(\omega) d\omega$$
   WTS:
$$P = \left[\int_{\omega\in\Omega} \varphi(\omega)^{\epsilon\sigma} p(\omega)^{1-\sigma} \mathrm{d}\omega\right]^{\frac{1}{1-\sigma}}$$
   To do so, we find the Marshallian demand functions $q(\omega)$ as a function of $p(\omega)$ and $\varphi(\omega)$. More precisely, we solve the household's maximization problem using a Lagrangian. We notice that maximizing U is equivalent to maximizing U^{ρ}, which is quite convenient as it makes algebra (much) easier :
$$L = U^{\rho} - \lambda\left[\int_0^n p(\omega) q(\omega) d\omega - R\right]$$

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where R is the total household's expenses. Taking the FOC wrt $q(\omega_1)$ and $q(\omega_2)$ and their ratio

$$\rho q(\omega)^{\rho-1}\varphi(\omega)^{\epsilon} = \lambda p(\omega)$$
$$(\frac{q(\omega_1)}{\omega_2})^{\rho-1} = \frac{p(\omega_1)}{p(\omega_2)}(\frac{\varphi(\omega_2)}{\varphi(\omega_1)})^{\epsilon}$$

Rearranging, multiplying by $p(\omega_1)$ and integrating wrt ω_1

$$p(\omega_1)q(\omega_1) = q(\omega_2)\frac{p(\omega_2)^{\sigma}\varphi(\omega_2)^{\epsilon\sigma}}{p(\omega_1)^(\sigma-1)\varphi(\omega_1)^{-\epsilon\sigma}}$$
$$\int_{\omega_1} p(\omega_1)q(\omega_1)d\omega_1 = q(\omega_2)p(\omega_2)^{\sigma}\varphi(\omega_2)^{\epsilon\sigma}\int_{\omega_1} p(\omega_1)^{1-\sigma}\varphi(\omega_1)^{\epsilon\sigma}d\omega_1$$

Noticing that the LHS $= R$, rearranging and solving for $q(\omega_2)$

$$q(\omega_2) = \frac{\varphi(\omega_2)^{\epsilon\sigma}}{p(\omega_2)^{\sigma}}R[\int_{\omega_1} p(\omega_1)^{1-\sigma}\varphi(\omega_1)^{\epsilon\sigma}d\omega_1]^{-1}$$

Because P, the price index, is the true cost of living, it must be the case that $U = \frac{R}{P}$. Thus, plugging in for $q(\omega_2)$ into the definition of U, and using this equality, we get (using $\rho = \frac{\sigma-1}{\sigma}$)

$$\begin{aligned}
U = \frac{R}{P} &= \left[\int_{\omega\in\Omega}\varphi(\omega)^{\epsilon}[\frac{\varphi(\omega)^{\epsilon\sigma}}{p(\omega)^{\sigma}}\frac{R}{\int_{\omega_1} p(\omega_1)^{1-\sigma}\varphi(\omega_1)^{\epsilon\sigma}d\omega_1}^{\rho}\mathrm{d}\omega\right]^{1/\rho} \\
&= \frac{R}{P} = \frac{R}{\int_{\omega_1} p(\omega_1)^{1-\sigma}\varphi(\omega_1)^{\epsilon\sigma}d\omega_1}\left[\int_{\omega\in\Omega}\varphi(\omega)^{\epsilon\sigma}p(\omega)^{1-\sigma}\mathrm{d}\omega\right]^{1/\rho} \\
&= \frac{R}{[\int_{\omega\in\Omega}\varphi(\omega)^{\epsilon\sigma}p(\omega)^{1-\sigma}\mathrm{d}\omega]^{1-\frac{1}{\rho}}} \\
&= \frac{R}{[\int_{\omega\in\Omega}\varphi(\omega)^{\epsilon\sigma}p(\omega)^{1-\sigma}\mathrm{d}\omega]^{\frac{1}{1-\sigma}}} \\
\Rightarrow P &= [\int_{\omega\in\Omega}\varphi(\omega)^{\epsilon\sigma}p(\omega)^{1-\sigma}\mathrm{d}\omega]^{\frac{1}{1-\sigma}}
\end{aligned}$$

From our marshallian demand for $q(\omega)$ plugging in for the price index we have

$$\begin{aligned}
q(\omega) &= \frac{\varphi(\omega)^{\epsilon\sigma}}{p(\omega)^{\sigma}}\frac{R}{P^{1-\sigma}} \\
&= \varphi(\omega)^{\epsilon\sigma}[\frac{p(\omega)}{P}]^{-\sigma}\underbrace{\frac{R}{P}}_{Q} \\
&= \varphi^{\epsilon\sigma}(P\rho)^{\sigma}Q
\end{aligned}$$

Similarly

$$\begin{aligned}
r(\omega) = p(\omega)q(\omega) &= \varphi(\omega)^{\epsilon\sigma}[\frac{p(\omega)}{P}]^{1-\sigma}R \\
&= \varphi^{\epsilon\sigma}[P\rho]^{\sigma-1}R
\end{aligned}$$

Finally we let the firm's production function be $l = f + q$. Using the fact that the pricing rule is now $p(\phi) = \frac{1}{\rho} = p^1$ as the marginal cost doesn't depend on ϕ anymore, we plug in for $p(\omega)$ into the expression above

$$r(\phi) = \phi^{\epsilon\sigma}(\rho P)^{\sigma-1}R$$

[1] We can verify that this pricing rule holds by solving the firm's maximization problem for a firm in monopolistic competition choosing $p(\phi)$ to maximize it's profits, with constant markup $= \frac{1}{\rho}$

And profits

$$\begin{aligned}\pi(\phi) &= r(\phi) - l(\phi) = r(\phi) - f - q(\phi) \\ &= \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q - \phi^{\epsilon\sigma}(\rho P)^{\sigma-1} R - f \\ &= \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q - \frac{1}{\rho}\phi^{\epsilon\sigma}(\rho P)^{\sigma} Q - f \\ &= \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q(1 - \frac{1}{\rho}) - f \\ &= \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q(\frac{1}{1-\sigma}) - f \\ &= \phi^{\epsilon\sigma}[\rho P]^{\sigma-1}\frac{R}{\sigma} - f\end{aligned}$$

3. *For the quality variant of the model introduced in the previous question, compute quantity per worker and revenue per worker for an active firm with productivity ϕ*
   We first need

$$l(\phi) = f + q(\phi) = f + \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q$$

   Thus

$$\frac{q(\phi)}{l(\phi)} = \frac{\phi^{\epsilon\sigma}[\rho P]^{\sigma} Q}{f + \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q}$$

   and

$$\frac{r(\phi)}{l(\phi)} = \frac{\phi^{\epsilon\sigma}(\rho P)^{\sigma-1} R}{f + \phi^{\epsilon\sigma}[\rho P]^{\sigma} Q}$$

4. *In Melitz (2003), footnote 7 says "Given the form of product differentiation, the modeling of either type of productivity difference is isomorphic." Comment on the isomorphism's implications for quantity per worker and revenue per worker.*

   "The modeling of either type of productivity difference is isomorphic" implies that some of the equilibrium outcomes should be the same under both modeling. Comparing questions (1) and (3) we see that for $\frac{q(\phi)}{l(\phi)}$ to be equal across the two models, we would need both $\hat{\phi}^{\epsilon\sigma} = \tilde{\phi}^{\sigma}$ and $\hat{\phi}^{\epsilon\sigma} = \tilde{\phi}^{\sigma-1}$ where $\hat{\phi}$ corresponds to the quantity parameter parameter from Pr. Dingel's variant (Q3) and $\tilde{\phi}$ corresponds to the productivity parameter from Melitz (Q1). The above condition is only satisfied if under very restrictive conditions (ϕ constant and $= 1$ would work, but this is not very interesting).
   However, with regard to prices and revenues, from (1) and (3) we see that if $\hat{\phi} = \tilde{\phi}^{\frac{\sigma-1}{\epsilon\sigma}}$, then these outcomes are the same under the quantity and the quality variants.
   Therefore, it seems that there is isomorphism in this model for the price / revenue variables, but not for the quantity ones. If one is interested in the quantity equilibrium, one should be careful about their modeling choice...