Assignment 1

Jeanne SORIN

1/ Discrete Time Resource Dynamics : mine deepening

Setup : p price constant
T periods
$x_1 = X > 0$ depth of mine at start $t=1$; x_t period t
w_t = vertical feet mined in period t.
q_t = amount extracted and sold in period t
r_t = reserves beginning period t ; $r_1 = R > 0$

$r_{t+1} = r_t - q_t + f(w_t, x_t)$ → $f_w > 0$, $f_{ww} < 0$, $f_x < 0$, $f_{xx} < 0$, $f(0, x_t) = 0$

$k(w_t)$ total cost of deepening the mine $k_w > 0$ $k_{ww} > 0$

$c(q_t, r_t)$ cost of extraction in t $c_q > 0$, $c_{qr} < 0$, $c_{qq} > 0$, $c_r < 0$, $c_{rr} > 0$

Miner's problem $\max_{q_t, w_t} \sum_{t=1}^{T} \delta^t (p q_t - c(q_t, r_t) - k(w_t))$ st

$$r_{t+1} = r_t + f(w_t, x_t) - q_t$$
$$x_{t+1} = x_t + w_t$$
$$r_1 = R \qquad x_1 = X$$
$$r_{T+1} \geq 0 \qquad q_t \geq 0 \qquad w_t \geq 0$$

a) State variables $= (x_t, r_t)$
Control variables $= (q_t, w_t)$

b) $$\mathcal{L} = \sum_{t=1}^{T} \delta^t (p q_t - c(q_t, r_t) - k(w_t)) - \sum_{t=1}^{T} \lambda_t (r_{t+1} - r_t - f(w_t, x_t) + q_t) + \sum_{t=1}^{T} \gamma_t (x_{t+1} - x_t - w_t)$$

↳ Kuhn-Tucker Conditions:

FOC $[q_t]$: $\delta^t (p - c_q(q_t, r_t)) - \lambda_t \leq 0$ $\quad q_t \geq 0$ $\quad$ c.s

$[w_t]$: $-\delta^t (k_w(w_t)) + \lambda_t f_w(w_t, x_t) - \gamma_t \leq 0$ $\quad w_t \geq 0$ $\quad$ cs

$[x_t]$: $\lambda_t f_x(w_t, x_t) + \gamma_{t-1} - \gamma_t = 0$

$[r_t]$ $-c_r(q_t, r_t)\, \delta^t - \lambda_{t-1} + \lambda_t = 0$

Last period's condition → $[r_{T+1}]$: $-\lambda_T \leq 0$ ⇒ $\lambda_T \geq 0$, $r_{T+1} \geq 0$ $\lambda_T r_{T+1} = 0$

$[x_{T+1}]$: $\gamma_t \geq 0$ $x_{T+1} \geq 0$ $\mu_T x_{T+1} \geq 0$

c) From the FOC for $[w_t]$:

$$\delta^t \left(k_w(w_t) \right) = \lambda_t f_w(w_t, x_t) - \gamma_t$$

this equation means that the PV of the marginal cost of digging deeper, $\delta^t k_w(w_t)$ equals the shadow value of a $f_w(w_t, x_t)$ additional unit of reserve λ_t plus the shadow value of an additional depth of mine γ_t.
↳ cost here

Note that here $\gamma_t > 0$ as the deeper the mine, the more costly it gets to dig even deeper.
Therefore, we can rewrite the above as

$$\underbrace{\delta^t \left(k_w(w_t) \right) + \gamma_t}_{\text{marginal cost}} = \underbrace{\lambda_t f_w(w_t, x_t)}_{\text{Marginal Benefit}}$$

d) From the Kuhn Tucker conditions, assuming interior solutions we have

$\gamma_t - \gamma_{t-1} = \lambda_t f_x(w_t, x_t)$ (1) $f_x < 0$ $-\delta^t c_r -$

$\lambda_t - \lambda_{t-1} = \delta^t c_r(q_t, r_t)$ (2) $c_r < 0$

⇒ $f_x < 0$, $\lambda_t \geq 0$ ⇒ $\gamma_t - \gamma_{t-1} < 0$ ⇒ $\gamma_t < \gamma_{t-1}$

⇒ $c_r < 0$ ⇒ $\lambda_t - \lambda_{t-1} < 0$ ⇒ $\lambda_{t-1} > \lambda_t$

So both λ_t and γ_t are decreasing in t. Economically this is because the closer to the last period (the larger t) the lower the (shadow) value of not mining deeper (x_t) today and of holding reserves (r_t) today for tomorrow as the number of future periods decreases.

2/ More discrete time resource dynamics : well drilling

Setup : world oil price P
Extractor decides to rent a drilling rig to drill his well
Drilling → one period → X units next period → λX ; $\lambda^2 X, \dots$
W = stock of wells to drill
↳ w_t = remaining nb of undrilled wells at the start of t

a_t = nb wells drilled at time t ⟺ a_t = nb rig rented
⇒ $R(a_t)$ corresponding rent $R_{a_t} > 0$

Take $\{P, \{R_t\}\}$ as given ; Rig rental market clears every period

$\frac{1}{1+r}$ discount rate $= \delta$

a) Renting a rig at $t=0$ ⇒ expected revenues :

$$\sum_{t=1}^{\infty} P(1-\lambda)^{t-1} X \delta^t = \sum_{t=1}^{\infty} PX\left[\delta(1-\lambda)\right]^t \frac{1}{1-\lambda}$$

price, decreasing productivity, initial productivity product, discounting (PV)

productivity decreases only 2nd period onwards

infinite sum

$$= \sum_{t=0}^{\infty} \frac{PX}{1-\lambda}\left[\delta(1-\lambda)\right]^t - PX$$

$$= \frac{PX}{(1-\delta(1-\lambda))(1-\lambda)} - PX = \frac{PX(1-1+\delta(1-\lambda))}{(1-\lambda)(1-\delta(1-\lambda))}$$

$$= \frac{PX\delta(1-\lambda)}{(1-\lambda)(1-\delta(1-\lambda))}$$

$$= \frac{PX\delta}{1-\delta(1-\lambda)}$$

b) BWOC let's assume that extractors prefer to drill in period t than t', then it means that the PV of their expected revenues in both periods is such that :

$$\gamma_t > \gamma_{t'}$$

$$\overbrace{\delta^t(P-R_t)}^{\gamma_t} > \overbrace{\delta^{t'}(P-R_{t'})}^{\gamma_{t'}}$$

2 cases

- $t > t'$ $\Rightarrow \delta^t (P - R_t) > \delta^{t'} (P - R_{t'}) > \delta^t (P - R_{t'})$ as $\delta < 1$

$\Leftrightarrow -R_t > -R_{t'}$

$\Leftrightarrow R_t < R_{t'}$ $\Rightarrow a_t < a_{t'}$ $\Rightarrow$ a contradiction as we started with the assumption that more extractors wanted to drill at t than t' and the rig market clears.

- $t < t'$ $\Rightarrow \delta^t (P - R_t) > \delta^{t'} (P - R_{t'})$

↳ Case 1: $\delta^t (P - R_t) > \delta^t (P - R_{t'}) > \delta^{t'} (P - R_{t'})$

$\Leftrightarrow$ same as above: $R_t < R_{t'}$ a contradiction

↳ Case 2: $\delta^t (P - R_{t'}) > \delta^t (P - R_t)$

$-R_{t'} > -R_t \Rightarrow R_{t'} < R_t$

$\Rightarrow a_{t'} < a_t$

However, if this was the case, the extractors would move from period t' to period t $\Rightarrow$ $a_{t'} \searrow$; $a_t \nearrow$

$R_{t'} \searrow$; $R_t \nearrow$ up to the point where $\delta^{t'} (P - R_{t'}) > \delta^t (P - R_t)$ $\Rightarrow \gamma_{t'} > \gamma_t$, a contradiction.

Moreover we know that extractors discount the future so conditional on making γ_t profit, they'd rather drill today than tomorrow.

c/ As noted above, the present marginal value of drilling γ_t must be equal over time. As it is discounted over time, and P is fixed, it must be that $R_t \searrow$. As $\frac{dR_t}{da_t} > 0$ $\Rightarrow a_t \searrow$ over time

d) $F_t \Rightarrow F_0 = 0$

$F_{t+1} = F_t(1-\lambda) + a_t X \qquad t > 0$

F_{t+1} increases with a_t, and we just showed that a_t decreases with time. Moreover, once $a_t = 0$ (which we proved happens for $t < \infty$), $F_{t+1} < F_t$. Thus, eventually, $F_{t+1} \to 0$.

In this model it is possible for aggregate oil flows to increase over time initially if a_t decreases relatively slower than $(1-\lambda)$; ie if the increase in F_{t+1} from the extensive margin (a_t) is bigger than the decrease in F_{t+1} from the intensive margin ($(1-\lambda) F_t$).
Note that because $a_t \searrow$ and eventually reaches 0, this initial increase in F_{t+1} (if it happens) must vanish at some point.

If $R(a_t)$ was constant in a_t, let's say $R(a_t) = R$, then

$\gamma_t = \gamma_{t'} \Rightarrow \delta^t(\theta P - R) = \delta^{t'}(\theta P - R)$ which is a contradiction because $t < t'$ WLOG.

$\hookrightarrow$ in this case we would have $\gamma_0 > \gamma_1 > \cdots \Rightarrow$ all wells are drilled at $t = 0 \Rightarrow a_0 = W$

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= a_0 X = WX \\
F_2 &= WX(1-\lambda) + 0 = WX \\
F_3 &= WX(1-\lambda)^2 \\
&\vdots \\
F_\infty &
\end{aligned}$$

monotonically decreasing after the initial jump from period 0 to period 1.

e) $\mathcal{L} = \sum_{t=0}^{T} \delta^t (PF_t - a_t R_t) - \sum_{t=0}^{T} \theta_t [F_{t+1} - F_t(1-\lambda) - a_t X]$

$\qquad - \sum_{t=0}^{T} \gamma_t [W_{t+1} - W_t + a_t]$

$\hookrightarrow [a_t] : -\delta_t R_t + \theta_t X - \gamma_t \leq 0 \qquad a_t \geq 0$, CS

$[F_t] : \delta^t P + \theta_t(1-\lambda) - \theta_{t-1} = 0$

$\theta_t \geq 0 \qquad \theta_t = \dfrac{\theta_{t-1} - \delta^t P}{1-\lambda}$

$\Rightarrow \theta_{t-1} > \theta_t$

$[W_t]: -\gamma_t + \gamma_{t+1} = 0 \qquad \gamma_T \geq 0$

TC : $\theta_T \geq 0 \qquad F_{T+1} \geq 0 \qquad \theta_T F_{T+1} = 0$

: $\gamma_T \geq 0 \qquad W_{T+1} \geq 0 \qquad \gamma_T W_{T+1} = 0$

f/ $[a_t]\ \delta^t R_t = \theta_t X - \gamma_t$

↳ at an interior solution ($\delta_t R_t = -\theta_t X + \gamma_t$), the marginal cost of drilling $\delta^t R_t$ (in PV) = the marginal benefit of adding X units to F_{t+1} (the shadow value θ_t times the number of oil units available after drilling X), minus the shadow value of foregoing one unit of well in the future (γ_t).

Notice that $\gamma_t > 0$ so ≈ the marginal value of an additional unit of W_{t+1}, which you give up by drilling today, so the negative sign.

θ_t is the shadow value of an additional well today on next period's oil flow.

If γ_t is constant and θ_t ↘, then the RHS ↘ over time.

g/ From our FOC we have $\gamma_t = \gamma_{t-1} \equiv \gamma$. γ is the shadow value of a marginal undrilled well. If the shadow (marginal) value is constant then the total value of undrilled wells, ie the aggregate wealth of the extractors is

γW (γ includes the discounting)

h/ WTS : $\theta_T = 0$ → as discussed in f, θ_T is the marginal value of an additional well drilled at T on F_{T+1}. However F_{T+1} cannot be consumed / sold, so it is worthless and $\theta_T = 0$

Mathematically, from the FOC & TC we have

$$F_{T+1} \geq 0 \qquad \theta_T \geq 0 \qquad \theta_T F_{T+1} = 0$$

From the FOC

$$\delta^t P + \theta_t (1-\lambda) - \theta_{t-1} = 0$$

$$\begin{aligned}\theta_{t-1} &= \delta^t P + \theta_t (1-\lambda) \\ &= \delta^t P + \left[\delta^{t+1} P + \theta_{t+1} (1-\lambda)\right](1-\lambda) \\ &\vdots \\ &= \delta^t P \left[1 + (1-\lambda)\delta + (1-\lambda)^2 \delta^2 + \cdots (1-\lambda)^{T-1} \delta^{T-1}\right]\end{aligned}$$

evaluated at t: $\theta_t = \delta^{t+1} P \sum_{t=0}^{T-t-1} (1-\lambda)^t \delta^t$

geom sum $\theta_t = \delta^{t+1} P \dfrac{1-(1-\lambda)^{T-1} \delta^{T-1}}{1-(1-\lambda)\delta}$

$$\lim_{T \to \infty} \theta_t = \underbrace{\frac{P\delta^{t+1}}{1-\delta(1-\lambda)}}_{\text{similar to (a).}} \quad \text{as} \quad (1-\lambda)^{T-1} \to 0$$

3/ Hotelling under monopoly, continuous time

a) The monopoly problem is

$$\max \; p(y_t)\, y_t \quad \text{st} \quad x_t \geq 0, \quad \dot{x}_t = -y_t$$

CV Hamiltonian $H = p(y_t)\, y_t - \mu_t\, y_t$

↳ FOC $[y_t]$: $p'(y_t)\, y_t + p(y_t) - \mu_t \leq 0 \qquad y_t \geq 0 \quad$ CS

↳ if $y_t > 0 \Rightarrow$

$$\underbrace{p(y_t)}_{\text{MB of an additional unit of } y_t} = \underbrace{\mu_t - p'(y_t)\, y_t}_{\text{MC of an additional } y_t}$$

- μ_t = foregone value of keeping it in the ground
- $\underbrace{-p'(y_t)\, y_t}_{>0}$ because the monopolist internalizes that an increase in y_t will decrease the price at which he can sell

↳ $[x_t]$: $\dot{\mu}_t = \rho \mu_t \Rightarrow$ marginal value of stock (shadow value) is increasing at ρ, following Hotelling rule.

↳ TV $\quad \lim_{t\to\infty} x_t \geq 0 \qquad \lim_{t\to\infty} \mu_t e^{-\rho t} \geq 0 \qquad \lim_{t\to\infty} e^{-\rho t} x_t = 0$

b) From constant elasticity : $\dfrac{dy_t / y_t}{dp_t / p_t} = \sigma \qquad\qquad dp_t = p_t'(y_t)$

$dy_t = 1$

$$p_t'(y_t) \cdot y_t = \frac{1}{\sigma} p_t$$

→ plugging into the FOC : $\mu_t = \dfrac{\sigma + 1}{\sigma} p_t$

$$\frac{\mu_t}{p_t} = \frac{\sigma+1}{\sigma} \quad \text{a constant} \quad \forall t$$

Rewriting $\dfrac{\mu_0 e^{\rho_1 t}}{p_0 e^{\rho_2 t}} = \dfrac{\sigma+1}{\sigma} \Rightarrow \rho_1 = \rho_2$: same growth rate

$\Rightarrow$ as in the perfect competition model : p is growing at ρ.

$\Rightarrow \{y_t\}$ is also the same because the exhaustion constraint is the same.

c/ Linear demand $\Rightarrow y_t = a - b\, p_t$

$$1 = a - b \frac{dp}{dy} = -b\, p_t' \Rightarrow p_t' = -\frac{1}{b}$$

Plugging into the FOC : $p'(y_t)\, y_t + p(y_t) = \mu_t$

$$-\frac{1}{b}\left(a - b\, p_t\right) + p_t = \mu_t$$

$$-\frac{a}{b} + 2p_t = \mu_t$$

$$2\, p_t = \mu_t + \frac{a}{b} \quad ; \; a, b > 0$$

$\Rightarrow p_t$ is growing at a slower rate than μ_t as the above must hold $\forall\, t$. The initial price is higher than perfect competition and initial extraction is higher.

d/ In b because of constant demand elasticity, the monopoly problem is a "static" shift from the perfect competition problem, so we still have that p_t grows at ρ. The exhaustion constraint imposes the than both the y_t and p_t paths are the same as in the perfect competition case.
In c however, the problem becomes dynamic for the monopolist because the demand elasticity is not constant anymore. The optimal $\{y_t, p_t\}_{t=0}^{\infty}$ pairs are therefore different from the competitive equilibrium. More elastic demand for low y_t implies that market power is bigger early and decreases then. Initial prices are higher and increase more slowly. Initial extraction is lower but decreases more slowly.

4/ Uranium product' : continuous time

↳ product' of uranium, an exhaustible resource

Set up : $R_0 > 0$ initial stock of uranium

y_t rate of uranium extract' in each period

$u(y_t)$ utility from uranium consumpt'

↳ $u'(\cdot) > 0$

$u''(\cdot) < 0$

$u'(\cdot)$ finite ⇒ choke price

$c(y_t)$ extract' cost

↳ $c(0) = 0$ $c'(0) < u'(0)$

↳ $c'(0) = 0$ $c''(\cdot) > 0$

R_t stock remaining at t

∞ horizon

ρ social discount rate

a) $\max \int_0^\infty e^{-\rho t}\left[u(y_t) - c(y_t)\right]$ st

$R_t \geq 0 \quad \forall t$

R_0 given

$\dot{R}_t = -y_t$

$\lim_{t\to\infty} R_t \geq 0$

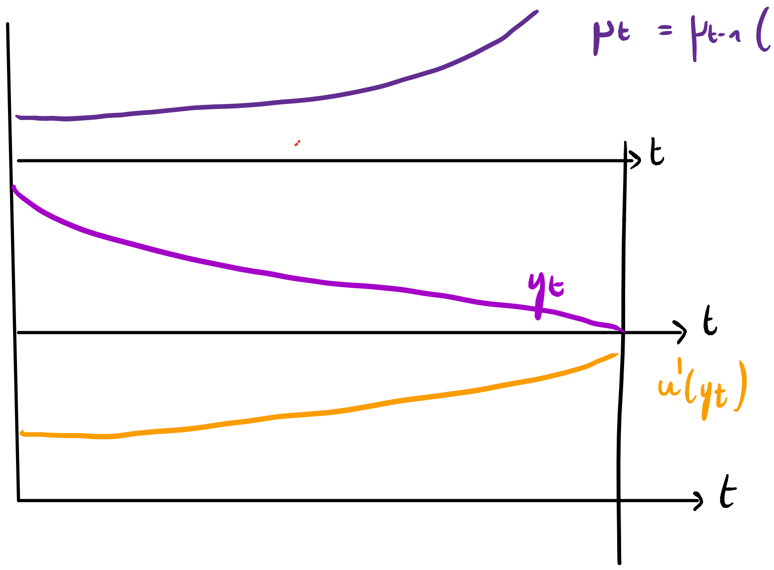
$$H = u(y_t) - c(y_t) - \underbrace{\mu_t}_{\lambda_t e^{\rho t}} y_t$$

b/ FOC $[y_t]$ $u'(y_t) - c'(y_t) \leq \mu_t$ $=$ if $y_t > 0$ $y_t \geq 0$, cs

$[R_t]$: $\dot{\mu}_t - \rho\mu_t = 0 \Rightarrow \dot{\mu}_t = \rho\mu_t$

$\lim_{t\to\infty} \mu_t e^{-\rho t} \geq 0$ $\lim_{t\to\infty} y_t \geq 0$ $\lim_{t\to\infty} y_t \mu_t e^{-\rho t} = 0$

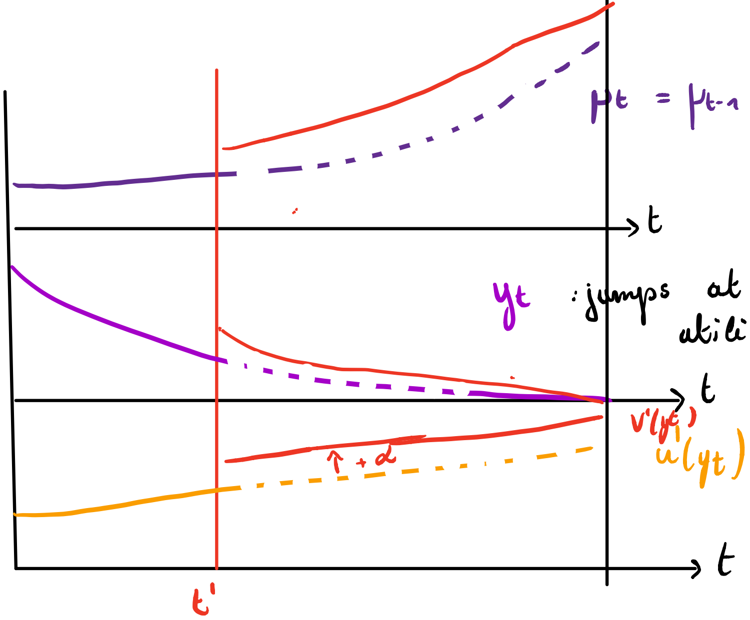
c/



$\mu_t = \mu_{t-1}(1+\rho)$

Extraction must cease at some finite T because the stock is finite at

$u'(0) > c'(0)$ ⇒ always valuable to extract something d)



$\mu_t = \mu_{t-1}(1+\rho)$ jump at t'

y_t : jumps at t' because suddenly large marginal utility

$v'(y_t)$ $\Rightarrow$ jump by definition

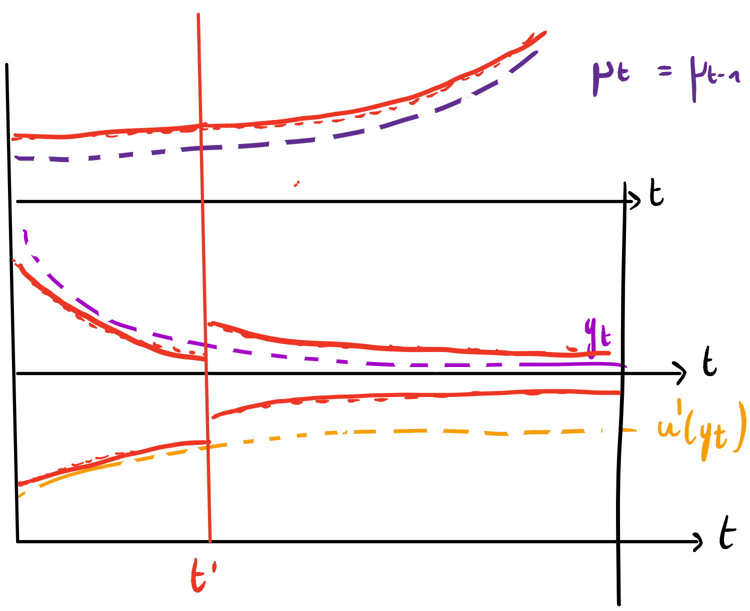
$$u'(y_t) - c'(y_t) = \mu_t = \underbrace{v'(y_t)}_{> u'(y_t)} - \underbrace{c_2'(y_t)}_{\hookrightarrow \text{ must be } > c'(y_t) \text{ original}}$$

$\Rightarrow y_t \nearrow$

Note that because $y_t^* > y_t \ \forall t \geq t'$, and the stock is finite, the $T^* < T$.

$\hookrightarrow$ where " * " denotes question d.

e)



$\mu_t = \mu_{t-1}(1+\rho)$ $\rightarrow$ doesn't jump

New schedules in red

If this shock was known from $t=0$, we still have a jump in y_t at t', and an increase from $u'(y_t)$ to $v'(y_t)$ at t'.

However, now we also have an impact on the path of μ_t $\Rightarrow$ we anticipate from the beginning that the stock will be more valuable in the future $\Rightarrow$ $\mu_0 \nearrow$. This implies that
$u'(y_0) - c'(y_0) = \mu_0 \nearrow \Rightarrow y_0 \searrow \Rightarrow u'(y_0) \nearrow$.

f/ $y_t < k_t$ $\qquad \dot{k}_t = + a_t$

$$H = u(y_t) - c(y_t) - f(a_t) + \theta_t(a_t) + \varphi_t(k_t - y_t) - \mu_t y_t$$

g/ ↳ $[y_t]$ $\quad u'(y_t) - c'(y_t) \leq \mu_t + \varphi_t \qquad y_t \geq 0 \quad$ CS

$[a_t]$ $\quad -f'(a_t) + \theta_t \leq 0 \qquad a_t \geq 0 \quad$ CS

$[R_t]$ $\quad \dot{\mu}_t = \rho \mu_t$

$[k_t]$ $\quad \dot{\theta}_t = \rho \theta_t - \varphi_t$

TVC $[R_t]$ $\quad \lim_{t\to\infty} R_t \geq 0 \quad \lim e^{-\rho t}\mu_t \geq 0 \quad \lim_{t\to\infty} e^{-\rho t} R_t \mu_t = 0$

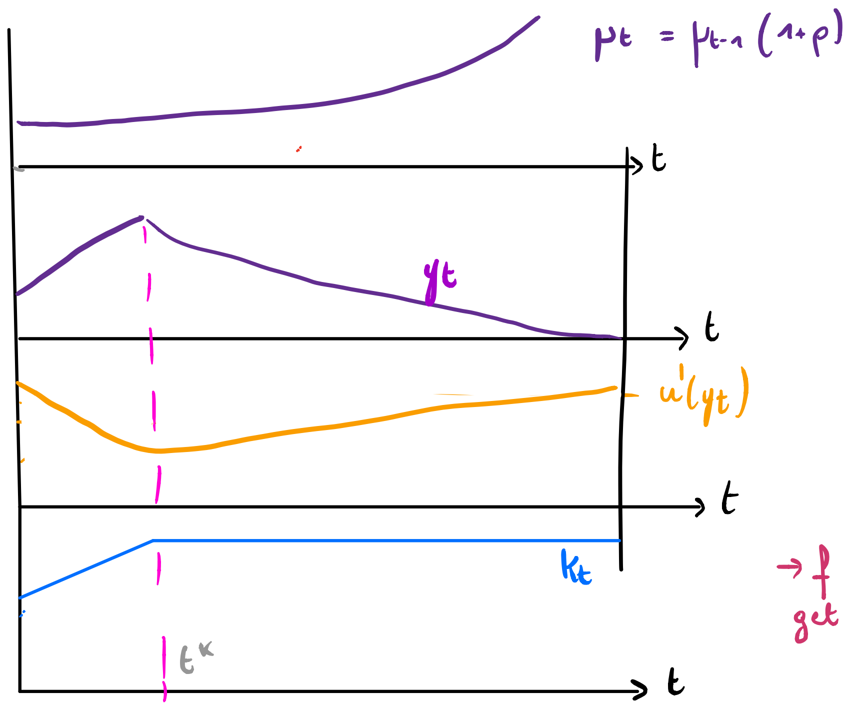
TVC $[k_t]$ $\quad \lim_{t\to\infty} k_t \geq 0 \quad \lim e^{-\rho t}\theta_t \geq 0 \quad \lim_{t\to\infty} e^{-\rho t}\theta_t k_t = 0$

h/ Remember that $\dot{k}_t = a_t \Rightarrow k_t$ never decreases. Moreover because $u'(0) - c'(0) > 0 \Rightarrow y_t > 0$ when resource is available $\Rightarrow k_t > 0$ as $y_t < k_t$.

Looking at TVC $[k_t] \Rightarrow \lim_{t\to\infty} e^{-\rho t}\theta_t = 0 \rightarrow \lim_{t\to\infty} \theta_t = 0$

This is the case because adding more capacity k becomes useless once the stock is fully depleted and $y_t = 0$ as $t \to \infty$.

i/



before t^*: $y_t <$ without constraint as too costly to build up k_t at once.

→ f is convex so potentially cannot get k^* right away

5/ Storage and uncertainty

Set up : S fixed stockpile of unobtanium
↳ divided across a large nb of firms (price takers)
$P(\varphi)$ demand
$T \Rightarrow S$ will be worthless
$t=0 \to T$: hazard $\alpha > 0$ that a substitute will be discovered, making S worthless
↳ $e^{-\alpha t}$

need to adjust for discovery risk

$\Rightarrow \dot{S}_t = -Q \quad t < T$ if no hazard

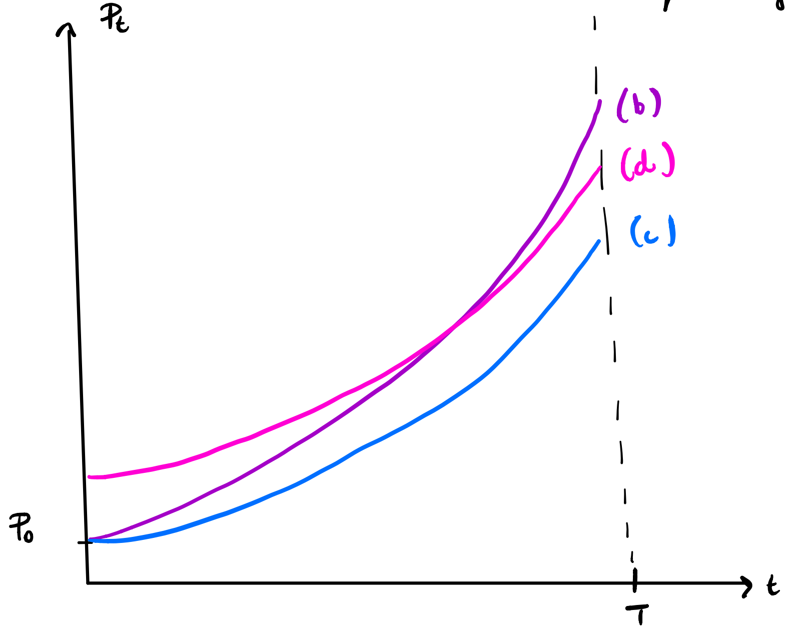
a) $$H = P_t \, \varphi_t \, e^{-\alpha t} - \mu_t \, q_t$$

↳ $[\varphi_t] \Rightarrow P_t e^{-\alpha t} \leq \mu_t \quad Q_t \geq 0 \quad CS$

$[S_t] \Rightarrow \dot{\mu}_t - r\mu_t = -\dfrac{\partial H}{\partial S_t} \Rightarrow \dot{\mu}_t = r\mu_t$

$[TC] \quad \mu_T e^{-rT} \geq 0 \quad S_T \geq 0 \quad \mu_T S_T e^{-rT} = 0$

b/ $P_t = \mu_t e^{\alpha t}$ From FOC $[\varphi_t]$
$P_t = \mu_0 e^{(\rho+\alpha)t}$ From FOC $[S_t]$ $\Rightarrow$ price grows at rate $\rho + \alpha$



c/ $$E(P_t) = \text{proba no discovery } \underbrace{\mu_0 e^{\rho t}}_{\text{price no discovery}} + \text{proba discovery} \cdot \underbrace{0}_{\text{if discovery the worthless}}$$

$$= e^{-\alpha t} \mu_0 e^{\rho t} = \mu_0 e^{(\rho - \alpha)t}$$

d/ if $\alpha = 0 \Rightarrow P_t = \mu_0 e^{\rho t}$ → same log slope as (b)
⇒ different starting point because of the exhaustion constraint