Price Theory II Final
**Solutions**

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## Problem 1

**Question.** *(10 points) Suppose that there are $N = 2$ individuals and the set of social alternatives $A = \{a, b, c, d\}$. As usual, individual $i$'s strict preference relation is denoted by $P_i$. Suppose also that the social welfare function $f : \mathcal{R}^2 \to \mathcal{R}$ satisfies Arrow's independence of irrelevant alternatives (IIA) condition. Is it possible that $f$ specifies the social ranking $aPcPbPd$ when the two individual rankings are $dP_1aP_1bP_1c$ and $cP_2aP_2bP_2d$, and that $f$ specifies the social ranking $cPbPaPd$ when the two individual rankings are $cP_1dP_1aP_1b$ and $aP_2cP_2bP_2d$?*

**Solution.** No, this is not possible. Notice that $aP_1b$ and $aP_2b$ for both sets of individual rankings. In the first case social ranking specifies $aPb$. Hence, by IIA for the second set of individual rankings we should also have $aPb$, but $f$ is said to specify $bPa$. Contradiction.

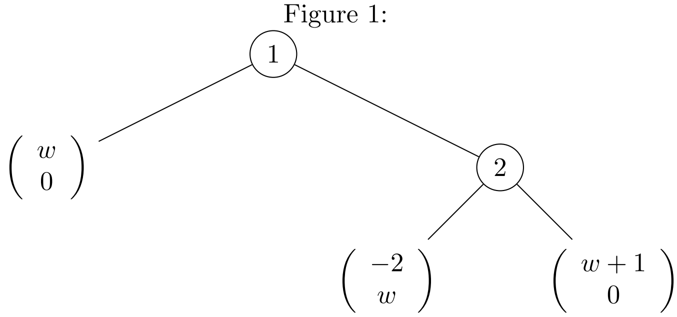
## Problem 2

**Question.** *In the game of Figure 1 below, the parameter $w \in (-\infty, \infty)$ is publicly known by the players.*

*(a) (5 points) For what values of $w \in (-\infty, \infty)$ in the game of Figure 1 does there exist a pure strategy profile that is a Nash equilibrium but that is not backward induction strategy profile?*

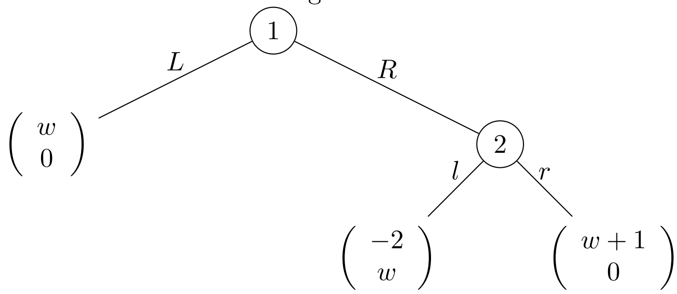
*(b) (10 points) Find a value of $w \in (-\infty, \infty)$ in the game of Figure 1 for which there exists a strictly mixed behavioral strategy profile (i.e., all players strictly mixing) that is a subgame perfect equilibrium. Provide the value of $w$ and a strictly mixed subgame perfect equilibrium behavioral strategy profile.*

Figure 1:




**Solution**. For clarity, let us specify the names of strategies as follows:

Figure 2:




**(a)** For this part of the question we consider only pure strategies of the two players.

**PSNE**:

**1 plays L**. Suppose that the first player plays $L$. Then the second player is indifferent between their action (either strategy gives a payoff of 0). Notice that playing $L$ is never optimal if the second player plays $r$. $L$ is a best-response to $l$, provided that $w \geq -2$. Hence, we have $(L, l)$ as a PSNE for $w \in [-2, \infty)$.

**1 plays R**. Suppose that the first player plays $R$. The best response of the second player to 1-st playing $R$ is:

$$BR_2(R) = \begin{cases} l, \text{ if } w > 0 \\ \{l, r\}, \text{ if } w = 0, \\ r, \text{ if } w < 0 \end{cases}$$

Hence, we have $(R, r)$ as PSNE iff $w \leq 0$.

**Backward induction equilibrium**: we start we the bottom of the tree — from the choice of player 2. At the subgame $R$, it is optimal for the second player to play $l$ iff $w \geq 0$, and optimal to play $r$ iff $w \leq 0$.

Given $w \geq 0$, if 2nd plays $l$, then the first player chooses $L$. When 2nd plays $r$, 1st chooses $R$.

Hence, we have the following backward induction pure strategy equilibria: $(L, l)$ if $w \geq 0$; $(R, r)$ if $w \leq 0$.

**Comparing the two equilibrium concepts**: observe that the two concepts agree on when $(R, r)$ is an equilibrium; however we have $(L, l)$ as NE for any $w \in [-2, \infty)$, but we have it as a backward induction only if $w \geq 0$.

**Final answer**: for $w \in [-2, 0)$ we have $(L, l)$ as a NE, while it is not a backward induction equilibrium.

**(b)** Recall that a behavioral strategy profile is strictly mixed, if any feasible strategy of a player at any of their information sets is played with positive probability. Denote $p$ — probability that the first player chooses $L$; $q$ — probability that the second player chooses $l$. So any behavioral strategy profile is fully described by $(p, q)$.

**Player 2 mixes**: If the second player mixes in a SPNE, then they are indifferent between the two strategies at a subgame — $R$. This is only possible if $w = 0$.

**Player 1 mixes**: The first player only mixes, when indifferent between $L$ and $R$. Hence, we must have:

$$u_1(L, q) = w = 0 = -2q + (w + 1)(1 - q) = 1 - 3q = u_1(R, q)$$

From this, $q = \dfrac{1}{3}$. Notice that we can specify $p$ to be any number in $(0, 1)$.

**Final Answer**: strictly mixed behavioral strategy profile that is a SPNE exists iff $w = 0$. In any such strategy profile $q = \dfrac{1}{3}$. An example of a strictly mixed behavioral strategy profile that is a SPNE (when $w = 0$) is: $(p, q) = \left(\dfrac{1}{2}, \dfrac{1}{3}\right)$.

# Problem 3

**Question.** *Figure 2 shows a Bayesian game BG between players 1 and 2. Player 1 has a single type and so we do not make explicit his singleton type space. Player 2's type can be any* $t_2 \in [0, 1]$ *and is uniformly distributed on* $[0, 1]$.

| | $L$ | $M$ | $R$ |
|---|---|---|---|
| $T$ | $2, 1 - 4t_2$ | $0, 1$ | $0, -1 + 4t_2$ |
| $B$ | $0, 3 - 4t_2$ | $0, 1$ | $2, -3 + 4t_2$ |

*(a) (5 points) If player 1 chooses with probability 1, then which types 2 of player 2 would have L as a best reply, which types would have M as a best reply, and which types would have R as a best reply? Conclude that there is no Bayes-Nash equilibrium of in which player 1 chooses with probability 1.*

*(b) (5 points) Use an argument similar to that used in part (i) to show that there is no Bayes-Nash equilibrium of in which player 1 chooses with probability 1. 2*

*(c) (10 points) If player 1 chooses with probability $p \in (0,1)$ then which types 2 of player 2 would have L as a best reply, which types would have M as a best reply, and which types would have R as a best reply?*

*(d) (10 points) Find a Bayes-Nash equilibrium of BG.*

**Solution**.

(a) If player 2 of type $t_2$ knows that player 1 is playing T, her payoffs for playing L, M, and R respectively are $1-4t_2$, 1, $-1+4t_2$. Comparing the payoffs, $t_2 \le 1/2$ will make M a best reply, $t_2 \ge 1/2$ makes R a best reply, and $t_2 = 0$ makes L a best reply. (We did not deduct points from indifference.)

To see why this cannot be a Bayes-Nash equilibrium, notice that $t_2 > 0$ with probability 1 ($t_2$ is distributed continuously, so the probability that $t_2 = 0$ is 0), so $1 - 4t_2 < 1$ with probability 1. It follows that probabilities of L, M, R being played respectively are 0,1/2, and 1/2, from player 1's perspective. Note that he does not know which type of player 2 he is facing against but knows the distribution of type and its type-contingent strategy. Therefore he will be better off playing B, as playing T given the best response above earns expected utility of $1/2 \times 0 + 1/2 \times 0 = 0$, whereas playing B gives $1/2 \times 0 + 1/2 \times 2 = 1$.

(b) Now $t_2 \le 1/2$ play L, and $t_2 \ge 1/2$ play M, and $t_2 = 1$ play R. By the same argument, player 1 knowing player 2's type-contingent action will find it profitable to deviate to T.

(c) Fix $t_2$. Expected payoff of playing L is

$$u_L = p(1-4t_2) + (1-p)(3-4t_2) = 3 - 2p - 4t_2,$$

whereas playing R gives

$$u_R = p(-1+4t_2) + (1-p)(-3+4t_2) = -3 + 2p + 4t_2.$$

Therefore, playing L is better than M if $3 - 2p - 4t_2 \ge 1$, or

$$\frac{1}{2}(1-p) \ge t_2, \tag{1}$$

playing R dominates M if

$$1 - \frac{1}{2}p \le t, \tag{2}$$

and playing L is better for player 2 than R if $6 - 4p - 8t_2 \ge 0$, or

$$\frac{3}{4} - \frac{1}{2}p \ge t_2. \tag{3}$$

Notice that if $u_L > 1$, then $u_L > u_R$ ((1) implies (3)), and $u_R > 1$ implies $u_R > u_L$. Therefore, if both (1) and (2) are not satisfied, or when $1 - \frac{1}{2}p \ge t_2 \ge \frac{1}{2} - \frac{1}{2}p$, then M is best for player 2. (Check if this is consistent with what we found in (a) and (b) so far!) Note that this interval always has length 1/2.

(d) Turning into player 1, we know that he has to mix between two strategies, and from the previous argument, for any $p$, probability of observing M from player 1's perspective is always 1/2. Let the proportion of player 2 playing L be $q \in [0, 0.5]$. Then expected payoff of playing T is $2q$, whereas expected payoff of playing B is $2(0.5 - q)$. Equating those, we get $q = 1/4$ so that player 1 is indifferent.

Now, what we need to make sure is that probabilities of observing L and R are both 1/4, which means that length of intervals (1) and (2) must be both 1/4. Taking (1), we see that $p = 1/2$ is required. Combining all, they constitute the unique (up to indifference) Bayes-Nash equilibrium of the game.

$$\begin{aligned}
P1 &: \text{Play T w.p. } \frac{1}{2} \\
P2 &: \text{Play L if } t_2 \le 1/4, \\
&\quad\ \text{Play M if } t_2 \in (1/4, 3/4), \\
&\quad\ \text{Play R if } t_2 \ge 3/4
\end{aligned}$$

## Problem 4

**Question.** *Consider the extensive form game above. Please use the Greek letters $\alpha$ and $\beta$ to specify the players' beliefs within their information sets, as indicated. Please also use $q_M$ and $q_R$ to denote the probability that player 2 chooses l, and use $q_g$ to denote the probability that player 3 chooses g. So a behavior strategy profile can be specified by $(q_M, q_R, q_l, q_g)$, a system of beliefs can be specified by $(\alpha, \beta)$, and an assessment can be specified by $(\alpha, \beta, q_M, q_R, q_l, q_g)$. Please use this notation when answering this question.*

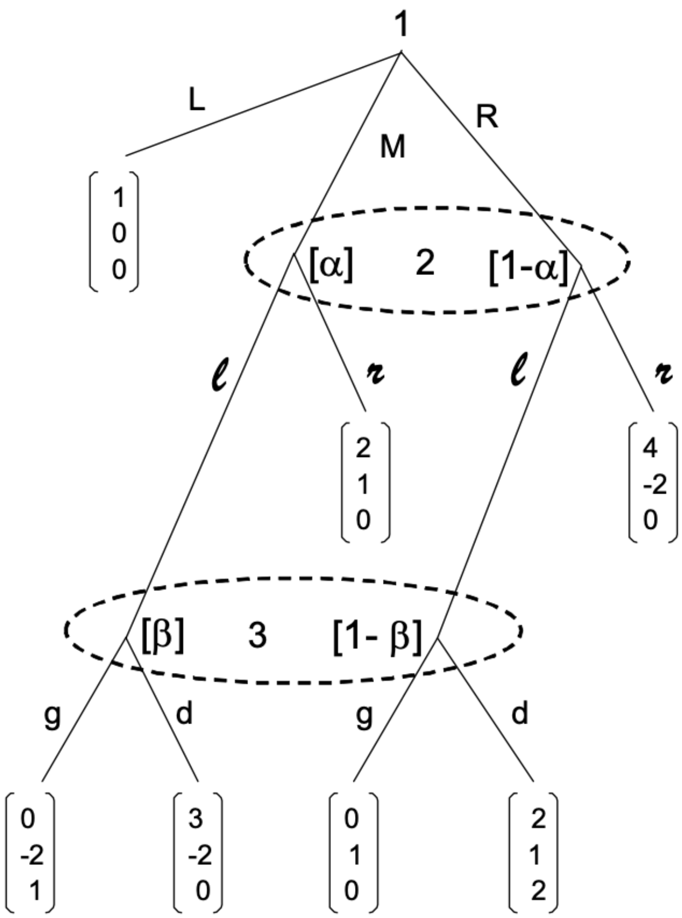
*(a) (5 points) Show that $\alpha = \beta$ in every consistent assessment.*

*(b) (5 points) Show that if a sequentially rational assessment satisfies $\alpha = \beta$, then either player 2 chooses r with probability one or player 3 chooses d with probability one, or both.*

*(c) (5 points) Show that if in a sequentially rational assessment either player 2 chooses r with probability one or player 3 chooses d with probability one, then player 1 chooses L with probability zero.*

*(d) (10 points) Find a sequentially rational assessment that satisfies Bayes rule and in which all players use pure behavioral strategies, but that is not a sequential equilibrium. (Hint: Use your results from parts (a)-(c). When does Bayes' rule imply no restrictions?)*

*(e) (10 points) Find a sequential equilibrium assessment $(\alpha, \beta, q_M, q_R, q_l, q_g)$. (Hint: What do (a)-(c) imply about player 1's behavior in any sequential equilibrium?)*




**Solution**.

(a) Suppose $(\alpha, \beta, q_M, q_R, q_l, q_g)$ is a consistent assessment. Then there exists some completely mixed sequence of strategies, $(q_M^n, q_R^n, q_l^n) \to (q_M, q_R, q_l)$, such that $\alpha = \lim_{n\to\infty} \frac{q_M^n}{q_M^n + q_R^n}$ and $\beta = \lim_{n\to\infty} \frac{q_M^n q_l^n}{q_M^n q_l^n + q_R^n q_l^n}$. Cancelling $q_l^n$, we find $\alpha = \beta$.

(b) Write 3's payoff from $g$ and $d$ as functions of $\beta$:

$$g: \ \beta$$

$$d: \ 2(1-\beta)$$

So 3 will always play $d$ as long as $\beta < 2/3$.

Notice that 2's payoff from $l$ does not depend on 3's actions, so we can similarly write 2's payoffs as functions of $\alpha$:

$$\begin{aligned} l: & \ -2\alpha + (1-\alpha) \\ r: & \ \alpha - 2(1-\alpha) \end{aligned}$$

2 will always play $r$ as long as $\alpha > 1/2$.

Now, since $\alpha = \beta$, it must be that $\alpha, \beta > 1/2$ (so 2 plays $r$) or $\alpha, \beta < 2/3$ (so 3 plays $d$), or both, proving the claim.

(c) If 2 plays $r$, 1 receives $(2, 4)$ by playing $(M, R)$, respectively. If 2 plays $l$ and 3 plays $d$, then 1 receives $(3, 2)$ by playing $(M, R)$, respectively. In all cases, 1 receives more by playing $M$ or $R$ than the 1 he would receive by playing $L$.

(d) If 1 always plays $L$, Bayes rule will have no bite, since 2's and 3's information sets will never be reached. Thus, we can give them any beliefs we choose.

Now, for it to be sequentially rational for 1 to play $L$, he must expect to get $\leq 1$ by playing $M$ or $R$. To make sure this is the case, let's make sure 2 always plays $l$ and 3 always plays $g$. Beliefs ($\alpha = 0$, $\beta = 1$) will support this behavior.

So the assessment ($\alpha = 0$, $\beta = 1$, $q_M = 0$, $q_R = 0$, $q_l = 1$, $q_g = 1$) satisfies sequential rationality and Bayes rule. It is not consistent by (a), so it is not a sequential equilibrium.

Note: This answer is "unique" in that there are no assessments which satisfy sequential rationality and Bayes rule, but not consistency, in which 1 does not always play $L$.

For consistency to differ from Bayes rule, it must be that some information set is never reached. The only other way for this to happen is if 2 always plays $r$. In this case, from (b), we know that $\alpha \geq 1/2$. Since 1 is not always playing $L$, 2's information set is reached with positive probability, so we can use Bayes rule. Thus, for $\alpha \geq 1/2$, 1 must play $M$ with positive probability. But this is not optimal if 2 is always playing $r$ ($R$ is strictly better).

(e) From (a)-(c), it must be that 1 always plays $M$ or $R$. We can quickly rule out pure strategy candidates – if 1 plays $M$, 2 plays $r$, and 1 prefers $R$; if 1 plays $R$, 2 plays $l$, 3 plays $d$ and 1 prefers $M$.

So let's look for an equilibrium in which 2 or 3 is mixing. Note that they cannot both be, since by (b) at least one is playing $r$ or $d$ purely. It turns out that 2 mixing will be the right choice. For this, by (b), we require $\alpha = \beta = 1/2$, so $q_M = q_R = 1/2$.

By design, 2's indifference condition is met, and 3 strictly prefers to play $d$. Thus, we need only make sure that 1 is indifferent between $M$ and $R$. Let's write 1's payoff from $M$ and $R$ in terms of 2's mixing probability, $q_l$:

$$\begin{aligned} M: & \ 3q_l + 2(1-q_l) \\ R: & \ 2q_l + 4(1-q_l) \end{aligned}$$

Thus, 1 is indifferent if $q_l = 2/3$. Our sequential equilibrium assessment is therefore ($\alpha = 1/2$, $\beta = 1/2$, $q_M = 1/2$, $q_R = 1/2$, $q_l = 2/3$, $q_g = 0$). This is the unique sequential equilibrium.