Price Theory II Problem Set 6

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1.

A seller wishes to auction a single indivisible good by a second-price auction to one of two bidders. Thus, each of the bidders is asked to simultaneously submit a sealed bid for the object. The high bid wins and pays the second-highest bid. If both bidders submit the same bid, then the winner is determined by the toss of a fair coin and he pays the common bid. Assume that the bidders know their own and one another's value for the object. Bidder 1's value is \$20 and 2's is \$10. Finally, suppose that only bids in increments of one dollar are permitted and no bid above \$100 is permitted. If bidder i receives the object and pays p; then his (vNM) utility is $v_i - p$; where v_i is his value. If he does not receive the object and pays nothing, his utility is zero.

(a) *What are the bidders' pure strategy sets?*
For each player i with $i = \{1, 2\}$ the pure strategy set is $S_i = \{0, 1, ..., 100\}$. In other words, each player can bid any amount from 0 to 100.

(b) *What are the payoffs when bidder 1 bids 7 and bidder 2 bids 40?*
$u_1(7, 40) = 0$
$u_2(7, 40) = 3$

(c) *Show that it is a weakly dominant strategy for each bidder to bid his value.*
Here we have a payoff matrix for bidder i when they bet below, at, or above their value and the other player bets below, at or above bidder i's value and between those bets.

In other words we have $a < \underline{v_i} < b < v_i < c < \overline{v_i} < d$ for player $-i$'s bets.

$i \backslash -i$	a	$\underline{v_i}$	b	v_i	c	$\overline{v_i}$	d
$\underline{v_i}$	$v_i - a$	$\frac{1}{2}(v_i - \underline{v_i})$	0	0	0	0	0
v_i	$v_i - a$	$v_i - \underline{v_i}$	$v_i - b$	0	0	0	0
$\overline{v_i}$	$v_i - a$	$v_i - \underline{v_i}$	$v_i - b$	0	< 0	< 0	0

Player i's strategy $\hat{s}_i$ weakly dominates another of his strategies $\overline{s_i}$ if $u_i(\hat{s}_i, s_{-i}) \geq u_i(\overline{s_i}, s_{-i})$ for all $s_{-i} \in S_{-i}$ wit at least one strict inequality.

From this payoff matrix, we see that bidding your value is the weakly dominant strategy for bidder i.

3.

a.

For each $i \in \{1, 2, \ldots, N\}$ we have that $S_i = \{1, 2, \ldots, 100\}$.

b.

Let s_{-i} be any strategy profile for all individuals but i. We then have that:

$$u^i(100, s_{-i}) = \begin{cases} \frac{100}{N} & \text{if } s_{-i} = (100, \ldots, 100) \\ 0 & \text{otherwise} \end{cases}$$

Let $s \in \{1, 2, \ldots, 99\}$ be a strategy for player i (other than 100). Suppose that $s_{-i} = (100, \ldots, 100)$. Then :

$$u^i(s, s_i) = 100 > \frac{100}{N} = u^i(100, s_{-i})$$

Now suppose $s_{-i} \neq (100, \ldots, 100)$. Then

$$u^i(s, s_{-i}) \geq 0 = u^i(100, s_{-i})$$

So all of the strategies $s \in \{1, 2, \ldots, 99\}$ weakly dominate 100 for player i.

c.

As we will need something slightly more general in part d., we solve this first to avoid repeating ourselves. Suppose that some strategies have been eliminated and the highest strategy remaining is $M \geq 2$. Denote the remaining strategy set of player i as $S_i^M = \{1, 2, \ldots, M\}$ where $M \in \mathbb{N}$. The answer to this question is the case where $M = 100$ and no strategies have yet been eliminated.WLOG we assume that the index of player i is 1.

We find a threshold strategy $s_M^* \in S_1^M$ such that all strategies strictly above this strategy are weakly dominated in S_1 and all strategies at or below s_M^* are not weakly dominated. First we construct an auxilliary threshold:

$$x = \frac{(N-2)M + 2x}{3N} \iff x = \frac{M(N-2)}{3N-2}$$

Take s_M^* to be the closest integer to x. (If it is exactly between two integers then we will need to take $s_M^* = \lfloor x \rfloor$, we do not pursue this in detail).

First consider $s \in 1\{1, 2, \ldots, s_M^*\}$. We show these strategies are undominated. We have that $s \leq \frac{(N-2)M+2s}{3N}$. This means that there are strategies $(s_3, s_4, \ldots s_N)$ each between $\{1, \ldots, M\}$ such that $\frac{\sum_{i=3}^N s_i}{3N-2} = s$. When these are the strategies of players $3, \ldots, N$ and player 2 plays s, the unique best response for player 1 is to also play s as any other number will lead to him losing to player 2 whereas when he plays s he gets \$50. So any such s is undominated.

Now consider any arbitrary $s \in \{s_M^* + 1, \ldots, N\}$. We will show that each of these strategies is weakly dominated by playing s_M^*. Consider the case where player 2 (say) plays s_M^* and all the other players play the highest possible strategy M. In this case playing s_M^* leads player 1 to share the prize with player 2 but deviating leads him to lose and get nothing. So here s_M^* is a unique best response and strictly beats s. As player 2 raises his strategy, $\frac{1}{3}$ of the mean rises less fast than his guess so that s_M^* remains a (strict) best response. It therefore follows that if at least one player plays above s_M^* then playing s_M^* can be no worse than playing any s that is higher. Note also that whenever everyone plays below s_M^*, $\frac{1}{3}$ of the mean is also below and so player one never wins with s or s_M^* and so is indifferent between them. This completes the proof that s_M^* weakly dominates s and the choice of $s \in \{s_M^* + 1, \ldots, N\}$ was arbitrary.

d.

We recursively define an algorithm to iteratively eliminate weakly dominated strategies.

Basis
Start with $k = 0$ and set $M_0 = 100$. M_k will denote the highest uneliminated strategy for each player at step k.

Step $k \geq 1$
If $M_k \geq 2$ calculate $s_{M_k}^*$ and set M_{k+1} equal to all the undominated strategies as discussed in part c.
Else, if M_k= 1 stop.

This algorithm terminates in a finite number of steps since $s_{M_k}^*$ is strictly less than M_k whenever $M_k \geq 2$ and the number of strategies is finite. We conclude that the set of remaining strategies for each player i is $\{1\}$

5.

Question

Consider the procedure of iteratively eliminating strictly dominated strategies in an N-person game with finitely many pure strategies. Suppose that you eliminate strictly dominated strategies iteratively in any order until no further eliminations are possible. Prove that the order in which you eliminate strictly dominated strategies does not matter. That is, that the set of remaining strategies is independent of the order you choose. (Hint: If a strategy is strictly dominated at some round, is it strictly dominated in later rounds?)

Solution

First we will start by redefining a dominated strategy:

A strategy, s_i, of individual $i \in \mathcal{I}$ is strictly dominated by another strategy s_i' if $\forall s_{-i} \in S_{-i}$, $u(s_i', s_{-i}) > u(s_i, s_{-i})$.

Because this holds for all strategies the other N players utilize, we can trivially see the hint is, in fact, true: Suppose a strategy s_i^* is strictly dominated in round R, say by s_i^D. Because this holds for ALL alternative strategies $s_{-i} \in S_{-i}^R$, then even if we iteratively remove other strategies, call them S_{-i}', first, when we return to see whether s_i^* is still dominated s_i^D in round $R' > R$ it clearly is, because remaining strategies are still just a subset of the possible strategies in round R, $s_{-i} \in S_{-i} \backslash S_{-i}'$.

But what if our strictly dominating strategy was eliminated in the process?! This would be a huge problem right?? Wrong. This is if fine because that means that $\exists s_i^{DD}$ which strictly dominates s_i^D. Or $u(s_i^{DD}, s_{-i}^{R'}) > u(s_i^D, s_{-i}^{R'})$ which by transitivity gives us that $u(s_i^{DD}, s_{-i}^{R'}) > u(s_i^*, s_{-i}^{R'})$. Thus concluding our proof of the hint.

So now we turn back to the question at hand. Knowing that a strategy that is strictly dominated in one round will be strictly dominated in any other round let's set out to do a proof by contradiction. Suppose we iteratively remove strictly dominated strategies and we end up with the final set of strategies F_i. Suppose, by contradiction, that we start from the beginning again but change the order in which we iteratively remove strategies and send up with set G_i, where $\exists s_i \in G_i$ which is not in set F_i.

This would mean that in iteration 1, $\exists s_i'$ which strictly dominated s_i at some point during the iteration. By the hint above this means that for $s_i \in G_i$ it could not be dominated in any round of the elimination process. Take G_i^0 to be the starting set of strategies for agent i for the first round of sequences. Because we are talking about the same game we know that $s_i \in G_i^0 = F_i^0$ Because this strategy is eliminated in some round R for the first scenario vs the second we know that $s_i \in F_i^{R-1}, \notin F_i^R$. This means that there existed some strategy $s_i' \in F_i^{R-1}$ such that $u(s_i', s_{-i}) > u(s_i, s_{-i})$ for all strategies played by the other people in the game. However by definition of $s_i \in G_i^{R-1}, \in G_i^R$ as it is in the final set. This means we still have strategy s_i' in our final F_i but not our G_i, or else our s_i would not be there. This is only possible if there is a third strategy s_i'' that strictly dominates s_i' when played vs G_{-i}, but by transitivity of strict dominance this would still prevent $s_i \in G_i$. Thus we have a contradiction because we have mandated $s_i \notin G_i$, which it was in by assumption. Thus all iterations of deleting strictly dominated strategies yield the same set of strategies.

7.

(This question is motivated by empirical work of Pierre-Andre Chiappori.) Consider the penalty kick in soccer. There are two players, the goalie (G) and the kicker (K). The kicker has two strategies: kick to the goalie's right (R) or to the goalie's left (L). The goalie has two strategies: move left (L) or move right (R). Let α be the probability that the kick is stopped when both choose L and let β be the probability that the kick is stopped when both choose R. Assume that $0 < \alpha < \beta < 1$. Consequently, the kicker is more skilled at kicking to the goalie's left. The payoff matrix is as follows.

G\K	L	R
L	$\alpha, 1-\alpha$	$0, 1$
R	$0, 1$	$\beta, 1-\beta$

(a) *Before analyzing this game, informally answer the following questions.*

(i) *Would you expect a kicker who is more skilled at kicking to the goalie's left than to his right, score more often when he kicks to the goalie's left?*

This makes intuitive sense in the scenario that the goalie is naive. However, in a game theoretical case with rational agents and perfect information, this would not make sense. This is because any gains a kicker would make with his talent would be offset with the goalie's strategic decision to block more often on the left side.

(ii) *If a kicker's ability to score when kicking to the goalie's left rises (i.e. α decreases) how will this affect the percentage of times the kicker scores when he chooses to kick to the goalie's left? Will it affect his scoring percentage when he kicks right?*

The exact values of α and β don't really matter in this case since any change in those probabilities would be accompanied with a proportional response in the mixed strategy of the other player. In other words, an increased probability of scoring on one side will be met with an increased probability that the goalie will want to block on that side. These proportional responses between the two players result in an equilibrium where each player must be indifferent between their two strategies.

(b) *Find the unique Nash equilibrium.*

The Nash equilibrium for this game will be a mixed strategy equilibrium since no pure Nash equilibrium exists. This is because in each of the outcomes, one of the players can improve their payoff by unilaterally switching their strategy. Note that if $\alpha = 0$, a pure Nash equilibrium would exist but this is ruled out in the prompt.

To find the mixed strategy Nash equilibrium we make each player indifferent to their two strategies by randomizing the decision to use one strategy over the other. Let q be the probability of the goalie going left, $1-q$ the probability of the goalie going right, p the probability of the kicker going left and $1-p$ the probability of the kicker going right.

$$q(1-\alpha)+(1-q)=q+(1-q)(1-\beta)$$
$$q=\frac{\beta}{\beta+\alpha}$$
$$p\alpha=(1-p)\beta$$
$$p=\frac{\beta}{\beta+\alpha}$$

Therefore, the mixed strategy in which each player chooses left with probability $\frac{\beta}{\beta+\alpha}$ and right with probability $1-\frac{\beta}{\beta+\alpha}$ is the unique Nash equilibrium.

(c) *Answer again the questions in part (a). Based upon this, would it be wise to judge a kicker's relative scoring ability in kicking left versus right by comparing the fraction of times he scores when he kicks right versus the fraction of times he scores when he kicks left?*

(i) *Would you expect a kicker who is more skilled at kicking to the goalie's left than to his right, score more often when he kicks to the goalie's left?*

The intuition from part a is formalized in part b. We see that it is optimal for each player to randomize their strategy. That probability is dependent on the parameters α and β. Therefore, we would not expect the kicker to score more often due to the goalie using a mixed strategy and randomizing his choice to reflect the kicker's skill.

(ii) *If a kicker's ability to score when kicking to the goalie's left rises (i.e. α decreases) how will this affect the percentage of times the kicker scores when he chooses to kick to the goalie's left? Will it affect his scoring percentage when he kicks right?*

When looking at the expected utility of the kicker is the same for both strategies

$$\frac{\alpha+\beta-\alpha\beta}{\alpha+\beta}$$

If we decrease α the kicker's utility increases no matter what option he chooses as long as he is randomizing according to NE. So in this case, scoring percentage will increase but for both strategies.

Therefore, comparing the fraction of times he scores when kicking right vs. left would not make sense.

9.

We will solve for the MSNE and get PSNE as degenerate special cases. Let (π^U, π^C, π^D) and (π^L, π^M, π^R) be the probabilities that players one and two ascribe to their actions respectively. MSNE are then defined as order pairs of ordered triples: $\Big((\pi^U, \pi^C, \pi^D), (\pi^L, \pi^M, \pi^R)\Big)$ We can write the utilities for player 1 of playing each of her pure strategies conditional on player two's probabilities as:

$$u_1(U|\pi^L, \pi^M, \pi^R) = \alpha\pi^L - \pi^M + \pi^R$$

$$u_1(C|\pi^L, \pi^M, \pi^R) = \pi^L + \alpha\pi^M - \pi^R$$

$$u_1(D|\pi^L, \pi^M, \pi^R) = -\pi^L + \pi^M + \alpha\pi^R$$

Now there are three mutually exclusive and exhaustive cases: player one puts zero probability on none, one or two of her strategies. We consider each case in turn and look at how the MSNE vary with α in each case. We summarise at the end.

Case 1: none of π^U, π^C, π^D are zero

In this case player one is mixing all of her strategies with positive probability, so all of the utilities above for the pure strategies must be equal. We also have the feasibility constraint $\pi^L + \pi^M + \pi^R = 1$ giving us three linear equations in as many unknowns. Solving the system gives the unique solution:

$$(\pi^L, \pi^M, \pi^R) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

By symmetry of the problem for an MSNE we must also then have:

$$(\pi^U, \pi^C, \pi^D) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

This all holds without restriction on α.

Case 2: exactly one of π^U, π^C, π^D is zero

We solve one case, the others follow by symmetry. Say that $\pi^U = 0$ so that $\pi^C = 1 - \pi^D \in (0, 1)$. This means that player one is indifferent between C and D and prefers them (weakly) to U. From above this gives us that:

$$\pi^L + \alpha\pi^M - \pi^R = -\pi^L + \pi^M + \alpha\pi^R \geq \alpha\pi^L - \pi^M + \pi^R$$

We now claim exactly one of π^L, π^M, π^R must be zero. If none are zero, Case 1 applies (from player two's perspective) and this is does not allow for $\pi^U = 0$, $\perp$. Similarly we can get a contradiction supposing two of the probabilities are zero using the equation above. The possible cases are:

- If $\pi^L = \pi^M = 0$ then $-\pi^R \geq \alpha\pi^R$ so $\pi^R = 0$, $\perp$ (probabilities must sum to unity).
- If $\pi^L = \pi^R = 0$ then $-\pi^M \geq \pi^M$ so $\pi^M = 0$, $\perp$.
- If $\pi^M = \pi^R = 0$ then $\pi^L = -\pi^L$ so $\pi^L = 0$, $\perp$.

It now suffices to check each subcase where exactly one of π^L, π^M, π^R is zero. The joy.

Subcase i): $\pi^L = 0$ **and** $\pi^M, \pi^R > 0$

The equation above now simplifies to:

$$\alpha\pi^M - \pi^R = \pi^M + \alpha\pi^R \geq \pi^R - \pi^M$$

This entails $\alpha \neq 0$ else $\pi^R = 0$ a contradicting $\pi^R > 0$. Solving we find that@

$$\pi^R = \frac{1}{2} - \frac{1}{2\alpha} \quad \text{and} \quad \pi^M = \frac{1}{2} + \frac{1}{2\alpha}$$

Plugging into the inequality and noting that $\alpha \neq 0$ as already argued we get:

$$\alpha > 1$$

Now applying the same argument as the above to the other player we find:

$$\pi^D = \frac{1}{2} - \frac{1}{2\alpha} \quad \text{and} \quad \pi^C = \frac{1}{2} + \frac{1}{2\alpha}$$

Subcase ii): $\pi^M = 0$ **and** $\pi^L, \pi^R > 0$

In this case, the requirement for mixing simplifies to:

$$\pi^L - \pi^R = \pi^L + \alpha\pi^R \geq \alpha\pi^L + \pi^R$$

Using the constraint that $\pi^R = 1 - \pi^L$ we can solve to find $\pi^L = \frac{\alpha-1}{3+\alpha}$. For this to be stricly positive we must have $\alpha > -1$ and so in particular note that $3 + \alpha > 0$. We now show a contradiction by showing that $\alpha\pi^L + \pi^R > \pi^L - \pi^R$. We have that:

$$\begin{aligned}
\alpha\pi^L + \pi^R > \pi^L - \pi^R &\iff \frac{\alpha^2 + \alpha + 2}{3 + \alpha} > \frac{\alpha - 1}{3 + \alpha} && \text{substituting in } \pi^L, \pi^R \\
&\iff \alpha^2 + \alpha + 2 > \alpha - 1 && \text{since } 3 + \alpha > 0 \\
&\iff \alpha^2 + 3 > 0 && \text{which holds, establishing a contradiction } \perp
\end{aligned}$$

Since we have a contradiction there cannot be an MSNE for this subcase.

Subcase iii): $\pi^R = 0$ and $\pi^M, \pi^L > 0$

This case also entails a contradiction in a similar manner to subcase ii) and so there is no MSNE for this subcase either.

Making the same arguments *mutatis mutandis* and using the symmetry of the problem we find the full set of MSNEs for this case to be:

$$\begin{cases} \Big\{\Big((0, \frac{1}{2\alpha}+\frac{1}{2}, \frac{1}{2}-\frac{1}{2\alpha}), (0, \frac{1}{2\alpha}+\frac{1}{2}, \frac{1}{2}-\frac{1}{2\alpha})\Big), \quad \Big((\frac{1}{2}-\frac{1}{2\alpha}, 0, \frac{1}{2\alpha}+\frac{1}{2}), (\frac{1}{2}-\frac{1}{2\alpha}, 0, \frac{1}{2\alpha}+\frac{1}{2})\Big), \\ \quad \Big((\frac{1}{2\alpha}+\frac{1}{2}, \frac{1}{2}-\frac{1}{2\alpha}, 0), (\frac{1}{2\alpha}+\frac{1}{2}, \frac{1}{2}-\frac{1}{2\alpha}, 0)\Big)\Big\} & \text{when } \alpha > 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Case 3: exactly two of π^U, π^C, π^D are zero

If $\alpha \neq 1$ then indifference cannot hold for player one between any of her pure strategies when player two is playing a pure strategy so player one will never mix. If $\alpha = 1$ then mixing could be a best response for player one but if she did mix this would be inconsistent with player two's strategy being pure by the reasoning from case 2. So the only possibilities here are PSNE.

Looking for PSNE, we directly see from the best responses that the only candidates are $\{(U, L), (C, M), (D, R)\}$ and that for these to hold we must have $\alpha \geq 1$ otherwise there are no PSNE.

Rewriting these as probabilities we have that the set of MSNE in this case is:

$$\begin{cases} \Big\{\Big((1,0,0),(1,0,0)\Big), \Big((0,1,0),(0,1,0)\Big), \Big((0,0,1),(0,0,1)\Big)\Big\} & \text{if } \alpha \geq 1 \\ \emptyset & \text{otherwise} \end{cases} \iff$$

To summarise:

Combining the results from each of the cases we find the MSNEs to be:

$$\begin{cases}
\Big\{\Big((1,0,0),(1,0,0)\Big),\Big((0,1,0),(0,1,0)\Big),\Big((0,0,1),(0,0,1)\Big), \\
\Big((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\Big),\Big((0,\frac{1}{2\alpha}+\frac{1}{2},\frac{1}{2}-\frac{1}{2\alpha}),(0,\frac{1}{2\alpha}+\frac{1}{2},\frac{1}{2}-\frac{1}{2\alpha})\Big), \\
\Big((\frac{1}{2}-\frac{1}{2\alpha},0,\frac{1}{2\alpha}+\frac{1}{2}),(\frac{1}{2}-\frac{1}{2\alpha},0,\frac{1}{2\alpha}+\frac{1}{2})\Big), \\
\Big((\frac{1}{2\alpha}+\frac{1}{2},\frac{1}{2}-\frac{1}{2\alpha},0),(\frac{1}{2\alpha}+\frac{1}{2},\frac{1}{2}-\frac{1}{2\alpha},0)\Big)\Big\} & \text{if } \alpha > 1 \\
\\
\Big\{\Big((1,0,0),(1,0,0)\Big),\Big((0,1,0),(0,1,0)\Big),\Big((0,0,1),(0,0,1)\Big), \\
\Big((1,0,0),(1,0,0)\Big),\Big((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\Big)\Big\} & \text{if } \alpha = 1 \\
\Big\{\Big((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\Big)\Big\} & \text{if } \alpha < 1
\end{cases}$$