Price Theory II Problem Set 3

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July 3, 2020

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Consider an exchange economy with two identical consumers. Their common utility function is $u^i(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ for $0 < \alpha < 1$. Society has 10 units of x_1 and 10 units of x_2 in all. Find endowments $\mathbf{e^1}$ and $\mathbf{e^2}$, where $\mathbf{e^1} \neq \mathbf{e^2}$, and Walrasian equilibrium prices that will 'support' as a WEA the equal-division allocation giving both consumers the bundle $(5, 5)$.

By using the Lagrangian method, where

$$\mathcal{L} = x_1^\alpha x_2^{1-\alpha} - \lambda(p_1x_1 + p_2x_2 - p_1e_1 - p_2e_2)$$

we see that the FOC is

$$\frac{p_2}{p_1} = \frac{1-\alpha}{\alpha}\frac{x_1}{x_2}.$$

We want to find prices and endowments so that this equation solves to the equal-division allocation of $(5, 5)$ for both consumers. This implies $x_1^1 = x_1^2 = x_2^1 = x_2^2 = 5$. Therefore we set relative demand equal to 1. Since both consumers have the same utility function, if $\frac{x_1}{x_2} \neq 1$ whichever consumer had the higher allocation of the desired good would not be willing to sell it to the other consumer and equal division allocation would not be a solution. So, the FOC becomes

$$\frac{p_2}{p_1} = \frac{1-\alpha}{\alpha}.$$

Here we see that the relative price is dependent on the parameter α. If $\alpha < \frac{1}{2}$ then good 2 will be preferable to good 1. This is reflected in the FOC where relative price $\frac{p_2}{p_1} > 1$ means good 2 is more expensive than good 1. To end up with the equal division allocation, the market will adjust the relative price to make consuming a preferred good more costly.

Now we use the budget constraint to find the set of endowments that satisfy the FOC. $5 = \alpha e_1^1 + (1-\alpha)e_2^1$ Specifically, given any e such that $(e_1^1, e_2^1, e_1^2, e_2^2) = (e_1^1, \frac{5-\alpha e_1^1}{1-\alpha}, 10-e_1^1, \frac{5-\alpha(10-e_1^1)}{1-\alpha})$ and $0 \leq e_1^1 \leq 10, 0 \leq \frac{5-\alpha e_1^1}{1-\alpha} \leq 10$

$$p_2 = \frac{1-\alpha}{a} p_1$$

and

$$p_1 = \frac{\alpha}{1-\alpha} p_2$$

will clear the markets and create an equal division allocation.

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Throughout we use superscripts for individuals and subscripts for goods.

WLOG we let $p_1 = 1$ as we will see that all prices are strictly positive. Then $p = (1, p_2, p_3)$. It is understood that for any $\lambda > 0$, if p is a WE price vector then so is λp.

We solve each consumer's optimisation problem in turn taking p as given.

Consumer 1:

Solves

$$\max_{x_1,x_2,x_3} \min(x_1, x_2) \quad \text{s.t.} \quad x_1 + p_2 x_2 + p_3 x_3 \leq 1$$

They will choose $x_3^{1*} = 0$. They will also set $x_1^{1*} = x_2^{1*}$ and the budget constraint will bind by local nonsatiation of preferences. Putting this together we find that $x_1^{1*} = x_2^{1*} = \frac{1}{1+p_2}$.

Consumer 2:

Using the same reasoning *mutatis mutandis*, $x_1^{2*} = 0$ and $x_2^{2*} = x_3^{2*} = \frac{p_2}{p_2+p_3}$.

Consumer 3:

Similarly, $x_2^{3*} = 0$ and $x_1^{3*} = x_1^{3*} = \frac{p_3}{1+p_3}$.

Market Clearing:

We must have $x_1^{1*} + x_1^{2*} + x_1^{3*} = 1$ and $x_2^{1*} + x_2^{2*} + x_2^{3*} = 1$ and $x_3^{1*} + x_3^{2*} + x_3^{3*} = 1$. Combining these with the market demands above we find that $\frac{1}{1+p_2} + \frac{p_3}{1+p_3} = 1 \iff p_2 = p_3$ and $\frac{1}{1+p_2} + \frac{p_2}{p_2+p_3} = 1 \iff p_3 = 1$ so we have $p_2 = p_3 = 1$ which also satisfies the third market clearing condition.

To summarise we have the WE price vector

$$p = (1, 1, 1) \quad (\text{or any } \lambda p \text{ for } \lambda > 0$$

With corresponding WEA:

$$x_1^{1*} = x_2^{1*} = \frac{1}{2}, \quad x_3^{1*} = 0$$

$$x_2^{2*} = x_3^{2*} = \frac{1}{2}, \quad x_1^{2*} = 0$$

$$x_1^{3*} = x_3^{3*} = \frac{1}{2}, \quad x_2^{3*} = 0$$

Exercise 5.25

We left off Theorem 5.13 by needing to show that the $z(p)$ function in the economy satisfies the propositions of Theorem 5.3 in the textbook. Restating these:

- Continuity
- Walras Law $(p \cdot z(p) = 0$
- Unbounded Above

We'll start with continuity because of the three this one is the most free. Note that we define the following:

$$z(p) := \sum_i x^i(p, m^i(p)) - \sum_j y^j(p) - \sum_i e^i$$
$$m^i(p) := p \cdot e^i + \sum_{j \in J} \theta^{ij} \pi^j(p)$$

This one is is free because we previously proved by Theorem 5.12 that the demand function x^i is continuous. Now by Theorem 5.9 we also know that y^j is continuous on $\mathbb{R}^n_{++}$. Thus $z(p)$ is the sum of continuous functions making it continuous itself. Nice!

Ok, now let's prove that is satisfies Walrus Law! To do this we need to write out the whole $p \cdot z(p)$.

$$0 = p \cdot z(p) = \sum_k z_k(p) p_k = \sum_k \sum_i x_k^i(p, m^i(p) p_k - \sum_k \sum_i e_k^i p_k - \sum_k \sum_j y_k^j(p) p_k$$

Now we rearrange:

$$\sum_i \sum_k x_k^i(p, m^i(p) p_k - \sum_i \sum_k e_k^i p_k - \sum_j \sum_k y_k^j(p) p_k$$

And now we've isolated the firms profit here! $\Pi_j(p) := \sum_k y_k^j(p) p_k$:

$$\sum_i \sum_k x_k^i(p, m^i(p) p_k - \sum_i \sum_k e_k^i p_k - \sum_j \Pi_j(p)$$

And it would be really helpful now to get this in terms of individuals so lets slide in our ownership shares:

$$\sum_i \sum_k x^i_k(p, m^i(p)p_k - \sum_i \sum_k e^i_k p_k - \sum_j \sum_i \theta^{ij}\Pi_j(p)$$

More rearranging!

$$\sum_i \sum_k x^i_k(p, m^i(p)p_k - \sum_i \sum_k e^i_k p_k - \sum_i \sum_j \theta^{ij}\Pi_j(p)$$

Which we can rewrite as:

$$\sum_i [\sum_k x^i_k(p, m^i(p)p_k - \sum_k e^i_k p_k - \sum_j \theta^{ij}\Pi_j(p)]$$

But note that:

$$m^i(p) := \sum_k e^i_k p_k + \sum_j \theta^{ij}\Pi_j(p)$$

So what we get is:

$$\sum_i [\sum_k x^i_k(p, m^i(p)p_k - m^i(p)] = 0$$

Which we know the consumer will run to zero if they exhibit strongly increasing preferences, which by assumption they do so we have our second condition set. Double Nice!

Finally all we need to do is show that our $z(p)$ is unbounded above as $p_m \to p^* \neq 0, p^*_k = 0$.

To do this we start with the fact that $\sum_i m^i(p^*) > 0$ which implies that at p^* there exists an agent i, such that:

$$m^i(p^*) > 0$$

Fix this person i. Great work. Now we will show that this can't be bounded by supposing it is and getting that sweet sweet contradiction. Again we will leverage our assumption that u^i is strongly increasing:

$$p_m \cdot x^i_m = p_m \cdot e_i + \sum_j \theta^{ij}\Pi_j(p_m) = m^i(p_m)$$

Or else we would be better off by consuming that extra income. Also you can't eat what you can't afford so nice. At the limit as $p_m \to p^*$ we will approach a similar limit of $x^i_m \to x^{i*}$.

$$\begin{aligned} p^* \cdot x^{i*} = p^* \cdot e_i + \sum_j \theta^{ij}\Pi_j(p^*) > 0 \\ \Rightarrow m^i(p^*) > 0 \\ \Rightarrow x^{i*} = x^i(p^*, m^i(p^*)) \end{aligned}$$

Now let's take an imaginary $\hat{x} = x^{i*} + \{0, 0, ..., 0, \alpha, 0, 0, ...0\}$ Where you get an $\alpha > 0$ in the k-th element . Because utility is strongly increasing we know that this bundle has to be strictly preferred to our x^{i*} bundle. However this bundle actually is affordable because remember our assumption that $p_k^* = 0$ which becomes interesting because this means:

$$p^* \cdot \hat{x} = p^* \cdot x^{i*} = p^* \cdot e_i + \sum_j \theta^{ij} \Pi_j(p^*) > 0$$

Now because the budget set itself has to be continuous we know that there exists a $t \in (0, 1)$ s.t.:

$$u_i^*(x^{i*}) < u_i^*(\hat{x})$$

And

$$tp^* \cdot < p^* \cdot e_i + \sum_j \theta^{ij} \Pi_j(p^*)$$

By the continuity of utility and the m^i expression paired with our converging prices and demand functions, if you take an m large enough in the sequence we get that:

$$u_i(x_m^i) < u_i(\hat{x})$$
$$tp^* \cdot \hat{x} < p^* \cdot e_i + \sum_j \theta^{ij} \Pi_j(p^*)$$

And this says that x_m^i is not actually solving the maximization problem and this is a contradiction. Thus we get that demand is unbounded! Now the rest is free money because by assumption the production set and endowment are bounded so if x is unbounded we know z must also be unbounded!

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Consider an exchange economy $(u^i, \mathbf{e^i})_{i \in \mathcal{I}}$ in which each u^i is continuous and quasiconcave on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}} = (\bar{\mathbf{x}}^\mathbf{1}, \bar{\mathbf{x}}^\mathbf{2}, ..., \bar{\mathbf{x}}^\mathbf{I}) \gg \mathbf{0}$ is Pareto efficient, that each u^i is continuously differentiabale in an open set containing $\bar{\mathbf{x}}^\mathbf{i}$, and that $\nabla u^i(\bar{\mathbf{x}}^\mathbf{i}) \gg \mathbf{0}$. Under these conditions, which differ somewhat than those of Theorem 5.8, follow the steps below to derive another version of the Second Welfare Theorem.

(a) *Show that for any two consumers i and j, the gradient vectors $\nabla u^i(\bar{\mathbf{x}}^\mathbf{i})$ and $\nabla u^i(\bar{\mathbf{x}}^\mathbf{j})$ must be proportional. That is, there must exist some $\alpha > 0$ (which may depend on i and j) such that $\nabla u^i(\bar{\mathbf{x}}^\mathbf{i}) = \alpha \nabla u^j(\bar{\mathbf{x}}^\mathbf{j})$. Interpret this condition in the case of the Edgeworth box economy.*

We know that pareto efficiency implies $MRS_i = MRS_j$. So choosing any goods n, m from the pareto efficient bundles of $\bar{x}^j$ and $\bar{x}^i$ we have

$$\frac{\frac{\partial u^i}{\partial \bar{x}^i_m}}{\frac{\partial u^i}{\partial \bar{x}^i_n}} = \frac{\frac{\partial u^j}{\partial \bar{x}^j_m}}{\frac{\partial u^j}{\partial \bar{x}^j_n}}$$

This is defined due to u^i being continuously differentiable and $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) \gg \mathbf{0}$. Rearranging this we can see that this is equal to a ratio of positive terms equal to some constant $\alpha_{ij} > 0$

$$\frac{\frac{\partial u^j}{\partial \bar{x}^j_m}}{\frac{\partial u^i}{\partial \bar{x}^i_m}} = a_{ij} \implies \frac{\partial u^j}{\partial \bar{x}^j_m} = a_{ij}\frac{\partial u^i}{\partial \bar{x}^i_m}$$

and

$$\frac{\frac{\partial u^j}{\partial \bar{x}^j_n}}{\frac{\partial u^i}{\partial \bar{x}^i_n}} = a_{ij} \implies \frac{\partial u^j}{\partial \bar{x}^j_n} = a_{ij}\frac{\partial u^i}{\partial \bar{x}^i_n}$$

In vectorized form over all goods we can rewrite this as $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) = \alpha_{ij}\nabla u^j(\bar{\mathbf{x}}^{\mathbf{j}})$ where $\alpha_{ij} = 1/a_{ij}$.

In an edgeworth box economy, we can think of $MRS_1 = MRS_2$ as the contract curve of pareto efficient allocations.

(b) *Define* $\bar{\mathbf{p}} = \nabla u^1(\bar{\mathbf{x}}^{\mathbf{1}}) \gg \mathbf{0}$. *Show that for every consumer* i*, there exists* $\lambda_i > 0$ *such that* $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) = \lambda_i \bar{\mathbf{p}}$.

From part (a) we can make consumer $j = 1$. We have $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) = \alpha_{i1}\bar{\mathbf{p}}$. Let λ_i be the set of constants for consumer 1. Then we have $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) = \lambda_i \bar{\mathbf{p}}$

(c) *Use Theorem 1.4 to argue that for every consumer* i, $\bar{\mathbf{x}}^i$ *solves*

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^i) \quad s.t. \quad \bar{\mathbf{p}} \cdot \mathbf{x}^{\mathbf{i}} \leq \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^{\mathbf{i}}$$

Theorem 1.4 Suppose that $u(\mathbf{x})$ is continuous and quasiconcave on $\mathbb{R}^n_+$, and that $(\mathbf{p}, y) \gg \mathbf{0}$. If u is differentiable at $\mathbf{x}^*$, and $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ solves

$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial u(\mathbf{x}^*)}{\partial x_n} - \lambda^* p_n = 0$$
$$\mathbf{p} \cdot \mathbf{x}^* - y = 0$$

then $\mathbf{x}^*$ solves the consumer's maximization problem at prices $\mathbf{p}$ and income y:

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^*) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x}^* \leq y$$

Each u^i is continuous, and quasiconcave on $\mathbb{R}^n_+$. We use $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) = \lambda_i \bar{\mathbf{p}}$ and see that we've already assumed $\bar{\mathbf{p}}, \bar{\mathbf{x}}, \mathbf{e}, \lambda_i \gg 0$. And we alread have that each u^i is continuously differentiable in an open set containing $\bar{\mathbf{x}}$. This means we are already set up to meet the assumptions in Theorem 1.4.

We can also see that the Lagrangian is solved by proving $\nabla u^i(\bar{\mathbf{x}}^{\mathbf{i}}) = \lambda_i \bar{\mathbf{p}}$ which implies $\frac{\partial u(\mathbf{x}^*)}{\partial x_n} - \lambda^* p_n = 0$. Then we set $y = \mathbf{p} \cdot \mathbf{x}^*$ which is feasible since $\bar{\mathbf{x}}$ is Pareto efficient which solves the second equation. Then we can conclude that $x^* = \bar{\mathbf{x}}$, $\mathbf{p} = \bar{\mathbf{p}} = \nabla u^1(\bar{\mathbf{x}}^{\mathbf{1}}) \gg \mathbf{0}$, and $y = \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}$ will satisfy the conditions of Theorem 1.4 and solve the consumer maximization problem.

This means given an endowment equal to a pareto efficient allocation, we can find a price vector of positive elements that will support this equilibrium which leads to an autarkic economy.

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Throughout we use subscripts for goods and superscripts for individuals.

First note that since both consumers have strictly positive endowments of both goods and their utilities are strictly increasing, any zero price will lead there to be no solution to either consumer's utility maximisation problem (unbounded utility). Throughout we therefore assume that prices are strictly positive. We are therefore also free to normalise $p_1 = 1$ so that the price vector is given by $p = (1, p_2)$. It is understood that if p is a WE price vector then so is λp for any $\lambda > 0$.

a. Robinson Crusoe

Consumer 2:

Consumer 2's problem is simple. With no access to the storage firm the only feasible allocation that satisfies market clearing is $x_1^{2*} = 1$, $x_2^{2*} = 9$. There is no storage and the consumer gets utility $u^{2*} = 9$. We're not asked for the price vector so we're done (it's $((1, 9)$ from the FOC).

Consumer 1 and her firm:

Consumer 1:

Letting $\pi(1, p_2)$ denote the firm's profit at the given prices the consumer solves:

$$\max_{x_1, x_2} x_1 x_2 \quad \text{s.t.} \quad x_1 + p_2 \leq 19 + p_2 + \pi(1, p_2)$$

This has no corner solutions, utility is strictly increasing and concave, the constraint binds and Lagrangean FOC's are sufficient. The problem's Lagrangean is:

$$\mathcal{L} = x_1 x_2 + \lambda[19 + p_2 + \pi(1, p_2) - x_1 - p_2 x_2]$$

The FOC's are:

$$x_2 = \lambda$$
$$x_1 = \lambda p_2$$
$$x_1 + x_2 p_2 = 19 + p_2$$

Solving we find :

$$x_1^{1*} = \frac{1}{2}\left(19 + p_2 + \pi(1, p_2)\right)$$
$$x_2^{1*} = \frac{1}{2p_2}\left(19 + p_2 + \pi(1, p_2)\right)$$

Her firm:
Turning to the firm we have that they solve:

$$\pi(1, p_2) = \max_{(y_1, y_2)} y_1 + p_2 y_2 \quad \text{s.t.} \quad (-y, y) : y \in [0, \infty)\}$$

Where we use the fact that we have already established that prices will be strictly positive so that the constraint $y_1 + y_2 \leq 0$ holds with equality and the above constraint set is WLOG.
We can distinguish three cases.

1. If $p_2 > 1$ then profits are given by $(p_2 - 1)y \to \infty$ as $y \to \infty$. So profits are unbounded and this case is incompatible with firm optimisation.

2. if $p_2 = 1$ then $\pi(1, 1) = 0$ and the firm is indifferent between all choices $(y_1^*, y_2^*) = (-y, y)$ where $y \in [0, \infty)$

3. if $p_2 < 1$ then all positive storage yields negative profits and the unique optimum is given by $(y_1^*, y_2^*) = (0, 0)$ and $\pi(1, p_2) = 0$.

Market clearing:
We must have that:

$$x_1^{1*} = y_1^* + 19$$
$$x_2^{1*} = y_2^* + 1$$

We have already ruled out $p_2 > 1$ as this was inconsistent with firm optimisation. For $p_2 > 1$ we have from the firm problem that $y_1^* = y_2^* = 0$ but then from the demand side we have the $x_2^{1*} = \frac{1}{2p_2}(19 + p_2) > \frac{1}{2}(19 + p_2) > 1$ which violates market clearing/feasibility so this case is also impossible.
We are left with the case where

$$p_2 = 1$$

. This gives us from the demand side

$$x_1^{1*} = x_2^{1*} = 10$$
$$u^{1*} = 100$$

And since the firm is indifferent between all bundles we make them set $(y_1, y_2) = (-9, 9)$ so that markets clear and 9 units of the good are carried forwards.

b. With a spot market

Exactly the same analysis as above applies for the firm and the consumer demands are given as above but adjusting in consumer 2's case for the different income in particular we have:

Consumer 1:

$$x_1^{1*} = \frac{1}{2}\left(19 + p_2 + \pi(1, p_2)\right)$$
$$x_2^{1*} = \frac{1}{2p_2}\left(19 + p_2 + \pi(1, p_2)\right)$$

Consumer 2:

$$x_1^{2*} = \frac{1}{2}\left(1 + 9p_2\right)$$
$$x_2^{2*} = \frac{1}{2p_2}\left(1 + 9p_2\right)$$

Market Clearing:

As above we will rule out $p_2 < 1$. In this $y_1^* = y_2^* = 0$ so that we then have from the demand side that $x_2^{1*} + x_2^{2*} = \frac{1}{2p_2}(20 + 10p_2) > \frac{1}{2}(20 + 10p_2) > 10$ and so this case violates feasibility. We are therefore again left with the only case that

$$p_2 = 1$$

Demand side gives us:

$$x_1^{1*} = x_2^{1*} = 10$$

With

$$u^{1*} = 100$$

$$x_1^{2*} = x_2^{2*} = 5$$

With

$$u^{2*} = 25$$

And then since the firm is indifferent between all choices that give it zero profit we choose to satisfy feasibility so that:

$$(y_1^*, y_2^*) = (-5, 5)$$

c.

All trading is done at the start. p_1 is the current price of a unit of the good today, namely the spot price. p_2 is the current price of a unit of the good promised for tomorrow, namely the price of a futures contract.

d.

The firm's problem is the same throughout so we deal with it first.

Firm

If $\delta = 0$ the only feasible choice is $(0,0)$ and that is what the firm will produce. If $\delta \neq 0$ we have a situation analagous to the above. Firm solves:

$$\max_{(y_1,y_2)} y_1 + p_2 y_2 \quad \text{s.t.} \quad (y_1, y_2) \in \{(-\frac{1}{\delta}y, y) : y \in [0, \infty)\}$$

And there are three cases as above:

1. If $p_2 > \frac{1}{\delta}$ then profits are unbounded, no solution.
2. if $p_2 = \frac{1}{\delta}$ then $\pi(1,1) = 0$ and the firm is indifferent between all choices $(y_1^*, y_2^*) = (-y, y)$ where $y \in [0, \infty)$
3. if $p_2 < \frac{1}{\delta}$ then all positive storage yields negative profits and the unique optimum is given by $(y_1^*, y_2^*) = (0,0)$ and $\pi(1, p_2) = 0$.

We solve the Robinson Crusoe and Spot market cases all over again.

Robinson Crusoe

Consumer 2:

Identical to the above Robinson Crusoe case: $x_1^{2*} = 1$, $x_2^{2*} = 9$ and $u^{2*} = 9$.

Consumer 1 and her firm:

Profits are always zero in equilibrium and the consumer demands are as above:

$$x_1^{1*} = \frac{1}{2}\left(19 + p_2\right)$$

$$x_2^{1*} = \frac{1}{2p_2}\left(19 + p_2\right)$$

And market clearing requirements are the same:

$$x_1^{1*} = y_1^* + 19$$

$$x_2^{1*} = y_2^* + 1$$

We can now consider two cases: one where the firm operates and one where it does not. We will find the parameter range (in terms of δ) for which each is a possible equilibrium.

Case 1 $y_1^* = 0$: In this case firm does not operate: $y_1^* = y_2^* 0$. For this to be consistent with firm profit maximisation we must be in situation 2. or 3. of the firms problem above. So $p_2 \leq \delta$. But then from the demand side we have that $p_2 = 19$. So for this to be an equilibrium we must have that

$$\delta \in [0, \frac{1}{19}]$$

And in this case we have no storage and

$$x_1^{1*} = 19$$
$$x_2^{1*} = 1$$
$$u^{1*} = 19$$

Case 2 $y_1^* \neq 0$*:* In this case we must have $p_2 = \frac{1}{\delta}$ so that:

$$x_1^{1*} = \frac{1}{2}(19 + \frac{1}{\delta})$$
$$x_2^{1*} = \frac{\delta}{2}(19 + \frac{1}{\delta})$$

Which generates utility:

$$x_2^{1*} = \frac{\delta}{4}(19 + \frac{1}{\delta})^2$$

But then from market clearing we must have:

$$y_1^* = \frac{1}{2\delta} - \frac{19}{2}$$
$$y_2^* = \frac{19}{2\delta} - \frac{1}{2}$$

Note that this satisfies $y_2 = -\delta y_1$. But it remains to check that $y_2 < 0$ (or equivalently $y_1 > 0$). But then this holds iff $\delta > \frac{1}{19}$ so we have the following parameter restriction for this case:

$$\delta \in [\frac{1}{19}, 1)$$

Spot Market

Again we can consider two cases. One where the firm operates and one where the firm does not. We will again come up with parameter restrictions on δ where these equilibria are possible.

Case 1 $y_1^* = 0$: Then $y_1 = y_2 = 0$. The market demands are ass before and solving using market clearing as above but this time with both demands we find that $x_1^{1*} + x_1^{2*} = \frac{1}{2}(20 + 10p_2) = 20$ so we must have that:

$$p_2 = 2$$

This then gives the following demands and utilities. For consumer 1:

$$x_1^{1*} = 10.5$$
$$x_2^{1*} = 5.25$$
$$u^{1*} = 55.125$$

And for consumer 2:

$$x_1^{2*} = 9.5$$
$$x_2^{2*} = 4.75$$
$$u^{2*} = 45.125$$

This is compatible with firm profit maximisation iff we are in cases 1. or 2. above so iff:

$$\delta \in [0, \frac{1}{2}]$$

Case 2 $y_1^* \neq 0$: Then we must have $p_2 = \frac{1}{\delta}$ for compatibility with firm optimisation and the demands are:

$$x_1^{1*} = \frac{1}{2}(19 + \frac{1}{\delta})$$
$$x_2^{1*} = \frac{\delta}{2}(19 + \frac{1}{\delta})$$
$$u^{1*} = \frac{\delta}{4}(19 + \frac{1}{\delta})^2$$

And for consumer 2:

$$x_1^{2*} = \frac{1}{2}(1 + \frac{9}{\delta})$$
$$x_2^{2*} = \frac{\delta}{2}(1 + \frac{9}{\delta})$$
$$u^{2*} = \frac{\delta}{4}(1 + \frac{9}{\delta})^2$$

Checking market clearing as before we find that $y_2 > 0$ requires that:

$$\delta \in (\frac{1}{2}, 1)$$

And we have that the storage (net of depreciation is given by):

$$y_2^* = \frac{1}{2}(20 + \frac{10}{\delta}) - 20$$

Comparing utility across cases

Consumer 2

Is always strictly better off with the spot market.

For $\delta \in [0, \frac{1}{2})$ we have that $45.125 > 9$.

For $\delta \in [\frac{1}{2}, 1)$ we have that $\frac{\delta}{4}(1 + \frac{9}{\delta})^2 > \frac{1}{9}(1 + 9)^2 = \frac{100}{8} > 9$.

Consumer 1

Is always weakly better off with the spot market, contrary to the problem statement.

For $\delta \in [0, \frac{1}{19}]$ we have that $55.125 > 19$ so strictly better off with the spot market.

For $\delta \in (\frac{1}{19}, \frac{1}{2}]$ we have that $\frac{\delta}{4}(19 + \frac{1}{\delta})^2 \leq \frac{1}{8}(19 + 19)^2 = 45.125 < 55.125$. So strictly better off with the spot market.

For $\delta \in (\frac{1}{2}, 1)$ we have that $\frac{\delta}{4}(19 + \frac{1}{\delta})^2$ gives the utility either with or without the spot market.So indifferent.

When does storage take place?

For $\delta \in [0, \frac{1}{19}]$ no storage in either case.

For $\delta \in (\frac{1}{19}, \frac{1}{2}]$ storage in Robinson Crusoe but not Spot Market.

For $\delta \in (\frac{1}{2}, 1]$ storage in Robinson Crusoe and Spot Market.

Question 5.34

Restating the Question

We will first start by defining an Arrow Security as an asset that pays off a dollar if the states = s and if time = t, otherwise it pays off nothing. This is the only asset, or in the textbook "commodity" which can be traded apriori at time t=0.

Suppose we have a Walsrasian Equilibrium with price vector $\hat{p} >> 0$. In an economy with N goods and uncertainty about times and states of the world. At date zero the price of the arrow security itself is equal to zero.As given by the problem consumer i's budget constraint for an Arrow Security at date zero is:

$$\sum_{t,s} a_{ts}^i = 0$$

Then at each date $t \geq 1$, the consumers all know the date and state of the world, consumer's receiver their endowment in that states $e_{st}^i \in \mathbb{R}_+^N$ and markets open for the N goods with good k having a price of p_{kts}. Trading opens and the consumers can buy either goods or arrow securities such that:

$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + a^i_{ts}$$

Consumer's know the prices of goods in every state/time combinations, and decides on the trades they will make in the future at date zero, thus maximizing their utility:

$$\max_{(a^i_{ts}),(x^i_{kts})} u^i((x^i_{kts})$$

Subject to the aforementioned budget constraint:

$$\sum_{t,s} a^i_{ts} = 0$$

And the other budget constraint for goods and assets:

$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + a^i_{ts} \geq 0$$

Part A

Because the consumer is locked into a consumption path at date zero, we have implicitly assumed that the past levels of consumption are fixed at levels similar to the equivalent to all other past consumptions that did not occur. If this were not the case, you could not maximize utility over the course of one's "lifetime" at date t=0 because the consumer would be incentivized to deviate from the date t=0 plan which would not be in the spirit of the model. Similarly the "future" consumption has to be locked in, or at least some expectation does, or else depending on where you are your plans for buying different securities, etc would change based on the state, s and time, t, you actually landed on.

This would make it impossible for us to compare the spot market trading models and the Arrow Debreau model because eventually the consumer would deviate from their "best" path due to the states and times falling in such another way.

Part B

We will first show that we have a vector of Arrow securities such that we satisfy both the spot market Budget Constraints as well as the Arrow Security Budget Constraint then we can satisfy:

$$\sum_{k,t,s} p_{k,t,s} x^i_{k,t,s} = \sum_{k,t,s} p_{k,t,s} e^i_{k,t,s}$$

First we take:

$$\sum_{k,t,s} p_{k,t,s} x^i_{k,t,s} = \sum_{t,s} p_{t,s} \cdot x_{t,s}$$

Then we plug in the spot market budget constraint

$$= \sum_{t,s} p_{t,s} \cdot x_{t,s} + \sum_{t,s} a_{t,s} = \sum_{t,s} p_{t,s} \cdot x_{t,s} + 0$$

Where the second equality holds by the Arrow Security Budget constraint. Now going the other way to prove that $\sum_{k,t,s} p_{k,t,s} x^i_{k,t,s} = \sum_{k,t,s} p_{k,t,s} e^i_{k,t,s} \Rightarrow$ satisfying the Arrow Security and Spot Market Budget Constraints. We probe this by first constructing a portfolio of asset securities:

$$a^i_{t,s} := \sum_{k=1}^{N} p_{k,t,s} x^i_{k,t,s} - \sum_{k=1}^{N} p_{k,t,s} e^i_{k,t,s}$$

Which by construction =0. Thus we have a market clearing Arrow Securities market where if you plug in

$$= \sum_{t,s} p_{t,s} \cdot x_{t,s} + \sum_{t,s} a_{t,s} = \sum_{t,s} p_{t,s} \cdot x_{t,s} + 0$$

You satisfy the proof.

Part C

Because in both models we use the same prices for commodities and we assume that utilities and their functions are the same across both models we know that any bundle that maximizes in one world is the same bundle that would maximize in the other. This goes both in the spot $\rightarrow$ Arrow direction and also the other way. This an allocation that is both feasible and utility maximizing which only happens if and only if this is the case in both worlds. The only thing we need to make sure that holds is that markets clear in both scenarios. For this we need in the commodity world:

$$\sum_{i \in I} x^i_{k,t,s} = \sum_{i \in I} e^i_{k,t,s}$$

Which is the same constraint in the Arrow (as securities sum to zero) and spot commodity scenarios. But if I have an arrow security profile does it go the other way and satisfy the AD world BC:

$$\begin{aligned} a^i_{t,s,} &:= \sum_{i}^{N} p_{k,t,s} x^i_{k,t,s} - \sum_{i}^{N} p_{k,t,s} e^i_{k,t,s} \\ \Rightarrow \sum_{i \in I} a^i_{t,s,} = \sum_{i \in I} [\sum_{i}^{N} p_{k,t,s} x^i_{k,t,s} - \sum_{i}^{N} p_{k,t,s} e^i_{k,t,s}] &= \sum_{i}^{N} p_{k,t,s} [\sum_{i \in I} x^i_{k,t,s} - e^i_{k,t,s}] \\ &[\sum_{i \in I} x^i_{k,t,s} - e^i_{k,t,s}] = 0 \end{aligned}$$

So we can actually show $\sum_{i \in I} a^i_{t,s,} = 0$ Which completes the proof.

Part D

An Arrow security for a t, s pair pays out 1 dollar in period t, s but not yet at the beginning of the problem. As such, the spot market budget constraints are constructed using relative prices, designated by $p_{t,s}$, which we can think of as normalized by price at t/s=0 and set = 1. This does not imply that there is no discounting: these relative prices could certainly have embedded discounting over time.

Part E

This modification implies that the security is now valued at the same price as good 1, which we call $p^1_{t,s}$. Good 1 now serves as the numeraire good. We can modify the two constraints as shown.

$$\begin{aligned} p_{t,s} \cdot x^i_{t,s} = p_{t,s} \cdot e^i_{t,s} + p^1_{t,s} a^i_{t,s} \\ \sum_{ts} p^1_{t,s} a^i_{t,s} = 0 \end{aligned}$$

Note that the budget constraints are equivalent as before.