Price Theory III Problem Set 1- Evens

Big Tombo and the Quaranteam.
(featuring Chase Abram, Manav Chaudhary, Tom Hierons
Sidharth Sah, Jeanne Sorin, George Vojta)

April 18, 2020

Problem 2

Set-up,

- There is a continuum of used car sellers.
- Each seller has one car of quality $\theta \sim U[0,1]$. The quality is private information.
- Each seller can choose to sell their car at price p or not, the corresponding utility for the two actions is,

$$u_S(\theta, p) = \begin{cases} p & \text{if he sells} \\ r(\theta) & \text{otherwise} \end{cases}$$

  where $r'(\theta) > 0$ and $r(0) = 0$

- Buyers cannot observe the quality of car. They can choose to either buy or not buy the car at price p. The buyers ex-post utility is given by,

$$u_B(\theta, p) = \begin{cases} \theta - p & \text{if he buys} \\ 0 & \text{otherwise} \end{cases}$$

- There is more demand for used cars, than there is supply.

Part a

In a competitive equilibrium buyers and sellers take prices as given. Therefore, the set of sellers selling their cars at price p is given by,

$$\Theta(p) = \{\theta | p \geq r(\theta)\}$$

Similarly, households will buy cars if,

$$\mathbb{E}[\theta | \theta \in \Theta(p)] - p \geq 0 \implies \mathbb{E}[\theta | p \geq r(\theta)] \geq p$$

We postulate that in competitive equilibrium,

$$\mathbb{E}[\theta | p^* \geq r(\theta)] = p^*$$

Suppose $p* < \mathbb{E}[\theta|p^* \geq r(\theta)]$, then there would be all buyers would want to purchase a car, but since there is more demand than total supply of cars, this can't be a fixed point[1]. Alternatively, suppose $p* > \mathbb{E}[\theta|p^* \geq r(\theta)]$ then no buyer would want to buy a car. However, since $r(0) = 0$ and $p > 0$, some sellers would want to sell their cars, therefore this can't be a fixed point. Therefore, the only fixed point is

$$\mathbb{E}[\theta|p^* \geq r(\theta)] = p^*$$

.

Part b

If $r(\theta) = \frac{\theta}{2}$ then,

$$\mathbb{E}[\theta|2p > \theta] = \begin{cases} \int_0^{2p} \frac{u}{2p} du = p & \text{if } p \in (0, \frac{1}{2}] \\ \frac{1}{2} & \text{if } p > \frac{1}{2} \end{cases}$$

Therefore by CE result in part a, it follows that when $r(\theta) = \frac{\theta}{2}$, all $p \in (0, \frac{1}{2}]$ are equilibrium prices.

Part c

If $r(\theta) = \sqrt{\theta}$ then,

$$\mathbb{E}[\theta|\theta < p^2] = \frac{1}{p^2} \int_0^{p^2} u du = \frac{1}{2} p^2$$

Therefore in competitive equilibrium $p = \frac{1}{2}p \implies p = 0$ .i.e. no meaningful trade takes place. We also get the solution $p = 2$, but since $\sqrt{\theta} \leq p \implies p \leq 1$.

Part d

If $r(\theta) = \theta^3$ then,

$$\mathbb{E}[\theta|\theta < p^{\frac{1}{3}}] = \frac{1}{p^{\frac{1}{3}}} \int_0^{p^{\frac{1}{3}}} u du = \frac{1}{2} p^{\frac{1}{3}}$$

Therefore in competitive equilibrium $p = \frac{1}{2}p^{\frac{1}{3}} \implies p = 0$ or $\frac{1}{2\sqrt{2}}$. In one equilibrium no meaningful trade takes place, whereas in the other one trade does take place.

Part e

When $r(\theta) = \frac{\theta}{2}$,

- Under full-information $\theta > \frac{\theta}{2}, \forall \theta \in (0, 1)$ therefore full trade is efficient.
- In the range of possible CEs $p \in (0, \frac{1}{2}]$, only $p = \frac{1}{2}$ attains fulls trade, therefore it is the Pareto optimal outcome.

When $r(\theta) = \sqrt{\theta}$,

- Under full-information $\theta < \sqrt{\theta}, \forall \theta \in (0, 1)$ therefore no trade is efficient.
- In the unique equilibrium ($p = 0$) no trade takes place, therefore it is Pareto optimal.

[1] We are also given by assumption that there are not enough cars to satisfy all potential buyers, so this cannot be a CE on those grounds.

When $r(\theta) = \theta^3$,

- Under full-information $\theta \geq \theta^3, \forall \theta \in (0, 1)$ therefore full trade is efficient.
- The equilibrium $p = 0$ is Pareto dominated by $p = \frac{1}{2\sqrt{2}}(< 1)$, however, even in this equilibrium full trade does not take place. Therefore it not Pareto optimal.

Problem 4

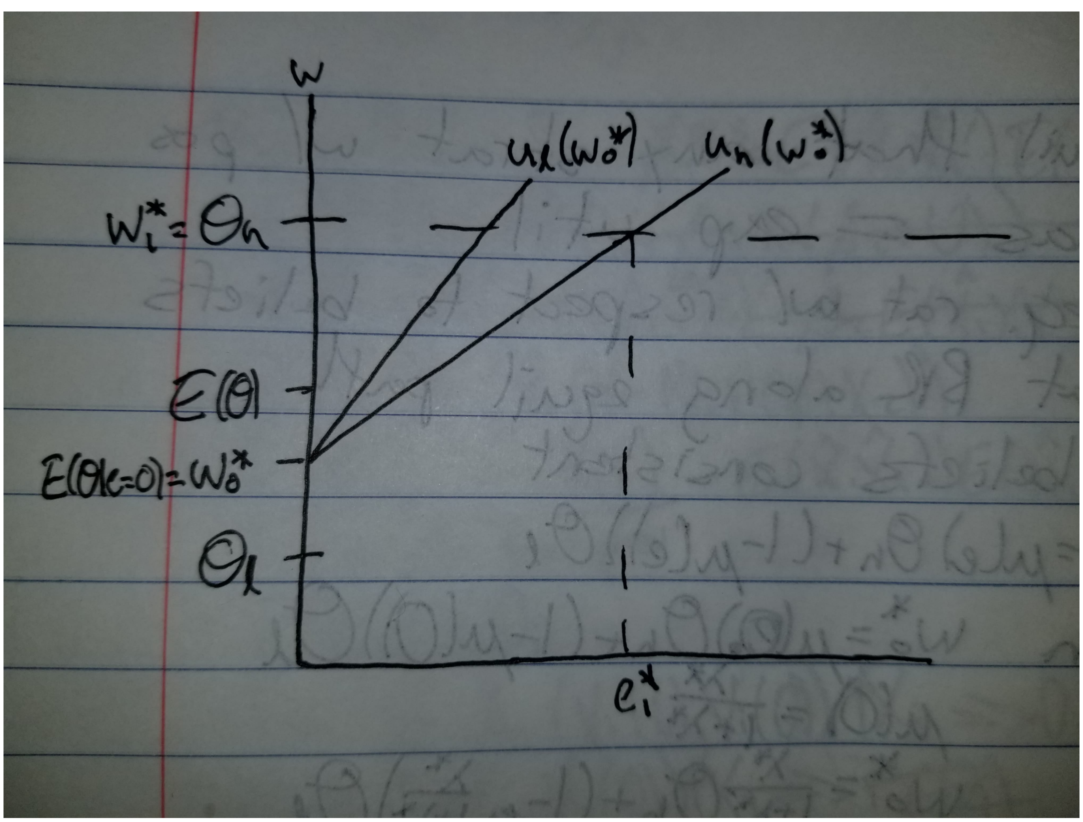
Consider the MWG labor-market signaling game from class. Assume that $c(e, \theta) = e(K - \theta)$ where K is a constant such that $K > \theta_h > \theta_l > 0$. Assume that the probability of the high type is ϕ and the probability of the low type is $(1 - \phi)$.

(a) Characterize a hybrid (mixed-strategy) equilibrium in which the high-type worker randomizes between two levels of education, $e_0 = 0$ and e_1, with probabilities λ and $(1 - \lambda)$ respectively, and the low-type plays a pure strategy always choosing $e_0 = 0$. In particular, compute w_0^*,w_1^*, and e_1^*, and give the equation that ties together the high-type's mixing probability, λ^*, with the education level chosen by the high type, e_1^*.

The requirement for BR consistent beliefs on the equilibrium path indicates that firms must be "right" about their beliefs about the likelihood of an agent being high-type at each of the two education levels that are in play. Given our setup, this indicates that they know that all workers with education level e_1^* are high-type and that the workers with 0 education are high-type with a probability that depends on ϕ and λ^*. We go ahead and define the simplest beliefs that would satisfy this requirement:

$$\mu(e) = \frac{\phi\lambda^*}{1 - \phi + \phi\lambda^*} \; \forall \, e \in [0, e_1^*) \;\; \& \;\; \mu(e) = 1 \; \forall \, e \in [e_1^*, \infty)$$

We also know, from the competitive nature of the market for workers, that firms will pay workers with education e_1^* θ_h and workers with 0 education the expected value of their productivity based on the above defined probability. At this point, it would be useful to show a graph of indifference curves for workers in the wage-education space:



We know that both sets of indifference curves are linear, with slopes equal to $(K-\theta_h)$ and $(K-\theta_h)$. The former is smaller as $\theta_h > \theta_l$. We also know that w_0^* is between θ_l and $E[\theta]$ as only a proportion of the high-type workers obtain 0 education. For equilibrium, the utility associated with the two choices of education given positive weight by high-type workers must be equal so both will lie on one indifference curve. As discussed above, the e_1^* point will be wherever this line crosses θ_h. The $e = 0$ point will be at the expected productivity at that education level. Thus, the high-type workers' indifference curve defines the relationship between λ^*, which determines the location of the intercept, and e_1^*, which occurs wherever the line with that intercept and the constant slope is equal to θ_h. To write out these various quantities in terms of each other:

$$w_0^* = \frac{\phi\lambda^*}{1-\phi+\phi\lambda^*}\theta_h + (1 - \frac{\phi\lambda^*}{1-\phi+\phi\lambda^*})\theta_l$$

$$w_0^* + e_1^*(K-\theta_h) = \theta_h$$

$$\frac{\phi\lambda^*}{1-\phi+\phi\lambda^*}\theta_h + (1 - \frac{\phi\lambda^*}{1-\phi+\phi\lambda^*})\theta_l + e_1^*(K-\theta_h) = \theta_h$$

$$\frac{\phi\lambda^*}{1-\phi+\phi\lambda^*} = 1 - \frac{e_1^*(K-\theta_h)}{\theta_h-\theta_l}$$

$$\phi\lambda^* = (1 - \frac{e_1^*(K-\theta_h)}{\theta_h-\theta_l})(1-\phi) + \phi\lambda^*(1 - \frac{e_1^*(K-\theta_h)}{\theta_h-\theta_l})$$

$$\phi\lambda^*\frac{e_1^*(K-\theta_h)}{\theta_h-\theta_l} = (1 - \frac{e_1^*(K-\theta_h)}{\theta_h-\theta_l})(1-\phi)$$

$$\lambda^* = (\frac{\theta_h-\theta_l}{e_1^*(K-\theta_h)} - 1)(\frac{1-\phi}{\phi})$$

Notice that this defines a continuum of equilibria for various $\lambda^* \in (0,1)$. Finally, we can check that this will satisfy the equilibrium conditions. Given the firms' beliefs, low-type workers will clearly not want to get more education. Hi-type workers won't want to go in between their two established levels, as they will still receive w_0^* and going beyond e_1^* also won't result in any additional benefit.

(b) Discuss the extreme cases of this hybrid equilibrium when $\lambda \to 0$ and $\lambda \to 1$.

For this question, we can look again at our graph. We see that as $\lambda^* \to 1$, w_0^* moves towards $E(\theta)$, showing the equilibrium moving towards the pooling equilibrium where no one gets any education, and everyone is paid the same wage, equal to expected productivity. As $\lambda^* \to 0$, we move towards the separating equilibrium where all types of workers are paid their productivities, low-type workers get no education, and hi-type workers get an amount of education equal that costs, to them, the exact difference between the productivities. (This is the "highest-cost separating equilibrium"; any more required education would induce the high-types to simply accept the low-type wage.) The graph also illustrates the "problem" with this equilibrium. If a firm made an offer in the triangle defined by the indifference curves and θ_h, they could capture the whole high-type work force, at a sweet discount, without attracting any low-types, and make a profit. They are only precluded from doing this in our equilibrium by their beliefs, which would not pass the Intuitive Criterion.

Problem 6

Part A

If Type and education are both observed than the arguments are pretty easy. Firm's are profit maximizing and thus pay wages to each individual:

$$w_i = \theta_i + e_i$$

Individuals get payoffs equal to the age minus the costs:

$$u_i = w_i - c(e_i, \theta_i) = \theta_i + e_i - \frac{e^2}{s\theta}$$

We can then take an F.O.C. with respect to e_i to find out how much education they will get. Note that there are no constraints on the agents so we don't need to build a Lagrangian of any sort:

$$1 - \frac{e_i}{\theta_i} = 0$$
$$\Rightarrow e_i = \theta_i$$

And this results in the high types getting education equal to their own type, and the same for the low types. As such we get a separating equilibrium when firms can observe types and education.

Part B

Part A gives us a pretty good hint that we should be looking at a separating equilibrium where each type of worker gets education equal to their type. Because $\theta_H > \theta_L$ we don't need to check make sure the high type doesn't deviate (because they get positive payoffs to education until high education and they clearly prefer to signal as high vs low to get that extra $(\theta_H - \theta_L)$ in wage.

But is it true that the low type does not want to deviate when $\theta_L > \frac{\theta_H}{3}$?

$$u_L^{dev} \overset{?}{<} u_L \Rightarrow 2\theta_H - \frac{\theta_H^2}{2\theta_L} \overset{?}{<} 2\theta_L - \frac{\theta_L}{2}$$
$$\Rightarrow 2\theta_H - \frac{3}{2}\theta_H \overset{?}{<} \frac{2}{3}\theta_H - \frac{\theta_H}{6}$$

Note that we put arrows on each side to indicate that the side is actually lower than the values indicate, because $\theta_L > \frac{\theta_H}{3}$. If it was equal, the second and third line would hold with equality. But how do we know that the left side on the first line is less than the right side? Take the derivative of both sides w.r.t. θ_L:

$$\frac{\theta_H^2}{2\theta_L^2} >? < \frac{3}{2},$$
$$(\theta_L = \frac{\theta_H}{3}),$$
$$\Rightarrow \frac{9}{2} > \frac{3}{2}$$
$$\Rightarrow \frac{\theta_H^2}{2\theta_L^2} > \frac{3}{2}$$

$$\Rightarrow 2\theta_H - \frac{3}{2}\theta_L > \frac{2}{3}\theta_H - \frac{\theta_H}{6}$$

So the low type does actually want to deviate. Note that this will tell us when $\theta_L \leq \frac{\theta_H}{3}$ the low type will NOT want to deviate[2]! So here our solution for part C is:

Quickly solving Part C:

So we conclude that our lowest cost separating-equilibrium, when $\theta_L \leq \frac{\theta_H}{3}$, is: $w_i = \begin{cases} 2\theta_L & e_i < \theta_H \\ 2\theta_H & e_i \geq \theta_H \end{cases}$

$$\mu(e) = \begin{cases} 0 & e < \theta_H \\ 1 & e \geq \theta_H \end{cases}$$

and the wage will be a rule that actually makes sure the workers follow through on their education (if we made it independent of education, then low types would just not educate).

$$w(e) = \begin{cases} \theta_L + e & e < \theta_H \\ \theta_H + e & e \geq \theta_H \end{cases}$$

What makes this special is that a), the productivity returns to education means the firm will want to use its wage-setting behavior to ensure education occurs properly, and b) θ_H is enough bigger than θ_L that low-types do not want to deviate to the high-type outcome (education at that level is too expensive).

Returning to Part B

When solving B one notices that actually when $\theta_L > \frac{\theta_H}{3}$, left side is actually higher than the right side, so our agent actually wants to deviate. As such the solution in b is not longer the same one as the one in c. Now we have to go looking again for the minimum education level (of the high type) in which the low type does not want to deviate.

$$u^{dev}(\theta_L, \bar{e}) = \theta_H + \bar{e} - \frac{\bar{e}^2}{2\theta_L} < 2\theta_L - \frac{\theta_L}{2} = u(\theta_L, e_L)$$
$$\Rightarrow \frac{\bar{e}^2}{2\theta_L} - \bar{e} + (2\theta_L - \theta_H - \frac{\theta_L}{2}) > 0$$

We can then use the quadratic formula where we know because of the shape of the curve we want the further (+ term) of formula:

$$\begin{aligned}\bar{e} &= \frac{1 + \sqrt{1 - 4(\frac{1}{2\theta_L})(2\theta_L - \theta_H - \frac{\theta_L}{2})}}{\frac{1}{\theta_L}} = \frac{1 + \sqrt{1 + (\frac{2}{\theta_L})(-2\theta_L + \theta_H + \frac{\theta_L}{2})}}{\frac{1}{\theta_L}} \\ &= \frac{1 + \sqrt{1 + (\frac{2}{\theta_L})(\theta_H - \frac{3\theta_L}{2})}}{\frac{1}{\theta_L}} = \frac{1 + \sqrt{1 + (\frac{2\theta_H}{\theta_L} - 3)}}{\frac{1}{\theta_L}} \\ &= \theta_L\Big[1 + \sqrt{2\Big(\frac{\theta_H}{\theta_L} - 1\Big)}\Big]\end{aligned}$$

[2] At indifference, we assume the low type will separate.

Note that because $\theta_L > \frac{\theta_H}{3} \Rightarrow \theta_L\left[1+\sqrt{2\left(\frac{\theta_H}{\theta_L}-1\right)}\right] > \theta_H$ We can show this through some math below as well with $\Rightarrow \frac{2\theta_H}{\theta_L} < 6$ Which we will call $(6-\epsilon)$. Note taht $\bar{e}$ is such that the low type is indifferent between signaling low and high, but then, by the form of the costs, it must be the case that switching to the low outcome is worse for the high types. While there is a probably a more formal and analytical way to solve this we show below a more brute force method which does the job in just the ugliest way:

$$= (1+\sqrt{1+((6-\epsilon)-3)})\theta_L = (1+(2-))\theta_L = (3-)\theta_L$$

Now lets make sure that the high type is ok with this level of education over taking the low type's:

$$\begin{aligned} u(\theta_H, \bar{e}) &> u(\theta_H, e_L)? \\ \theta_H + ((3-)\theta_H) - \frac{((3-)\theta_L)^2}{2\theta_L} &> 2\theta_L - \frac{(\theta_L)^2}{2\theta_H}? \end{aligned}$$

Plugging in for $\theta_L > \frac{\theta_H}{3}$ and $\bar{e} = (3-)\theta_L$

$$\begin{aligned} \theta_H + (\sim 1^-)\theta_H - \frac{(1^-)\theta_H}{2} &> \frac{2^-}{3}\theta_H - \frac{\theta_H^2}{(18^-)\theta_H} \\ (\frac{3}{2})\theta_H &> \frac{12-}{18}\theta_H - \frac{1}{18^-}\theta_H \\ (\frac{3}{2})\theta_H &> \frac{11-}{18}\theta_H \end{aligned}$$

So the high type doesn't want to deviate to low type at this required education. (These are approximations, thus the weird notation).

$$\mu(\theta_H|e_i) = \begin{cases} \theta_L & e_i < \bar{e} \\ \theta_H & e_i \geq \bar{e} \end{cases}$$

$$w_i = \begin{cases} \theta_L + e_i & e_i < \bar{e} \\ \theta_H + e_i & e_i \geq \bar{e} \end{cases}$$