Price Theory II Problem Set 5 - EVENS

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Question 2 (JR Question 2.20 pg 121)

2.20 Consider the following 'Independence Axiom' on a consumer's preferences, $\succsim$, over gambles: If

$$(p_1 \circ a_1, \ldots, p_n \circ a_n) \sim (q_1 \circ a_1, \ldots, q_n \circ a_n),$$

then for every $\alpha \in [0, 1]$, and every simple gamble $(r_1 \circ a_1, \ldots, r_n \circ a_n)$,

$$((\alpha p_1 + (1-\alpha)r_1) \circ a_1, \ldots, (\alpha p_n + (1-\alpha)r_n) \circ a_n)$$
$$\sim$$
$$((\alpha q_1 + (1-\alpha)r_1) \circ a_1, \ldots, (\alpha q_n + (1-\alpha)r_n) \circ a_n).$$

(Note this axiom says that when we combine each of two gambles with a third in the same way, the individual's ranking of the two new gambles is *independent* of which third gamble we used.) Show that this axiom follows from Axioms G5 and G6.

We are given the following three gambles:

$$p := (p_1 \circ a_1, ..., p_n \circ a_n)$$
$$q := (q_1 \circ a_1, ..., q_n \circ a_n)$$
$$r := (r_1 \circ r_1, ..., p_n \circ r_n)$$

We are given that $p \sim q$, and trivially $r \sim r$. Take any $\alpha \in [0, 1]$ Now let's define $\tilde{p}$ $\tilde{q}$ as the compound game that is:

$$\tilde{p} = (\alpha p, (1-\alpha)r)$$
$$\tilde{q} = (\alpha q, (1-\alpha)r)$$

By Axiom G5 we know that, because $p \sim q$ and $r \sim r$, that $\tilde{p} \sim \tilde{q}$ (because trivially $\alpha = \alpha$ and $1 - \alpha = 1 - \alpha$).

All there is to do now is notice that by Axiom G6 that we can reduce these compound games to the simple gambles induced by them. This means that:

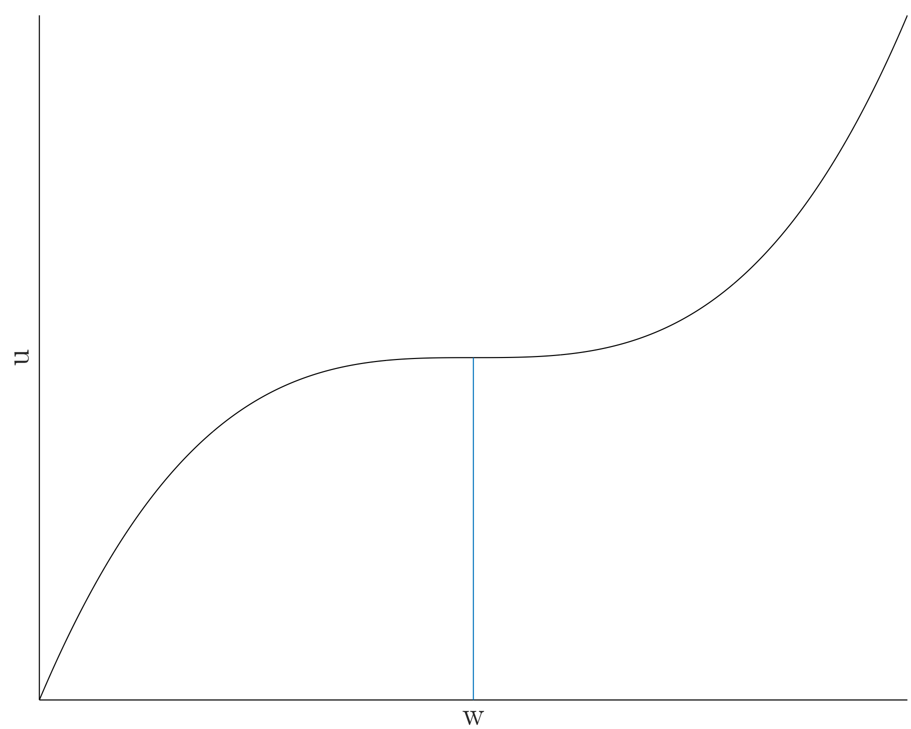
$$\tilde{p} \sim ((\alpha p_1 + (1-\alpha) r_1 \circ a_1, ..., \alpha p_n + (1-\alpha) r_n \circ a_n)$$
$$\tilde{q} \sim ((\alpha q_1 + (1-\alpha) r_1 \circ a_1, ..., \alpha q_n + (1-\alpha) r_n \circ a_n)$$

But we've already shown as well that $\tilde{p} \sim \tilde{q}$, so we will induce transitivity as well to conclude:

$$((\alpha p_1 + (1-\alpha) r_1 \circ a_1, ..., \alpha p_n + (1-\alpha) r_n \circ a_n) \sim ((\alpha q_1 + (1-\alpha) r_1 \circ a_1, ..., \alpha q_n + (1-\alpha) r_n \circ a_n)$$

4. Sarah

Graph the von Neumann-Morgenstern utility function over wealth of an indivudal who is risk averse when his wealth is low, and risk loving when his wealth is high.



A function is risk loving if the Arrow-Pratt measure of absolute risk aversion is negative and risk averse if positive.

The utility function $u(x) = (w-b)^3 + a$ is graphed here. Let b be the wealth level where the consumer switches from risk averse to risk loving. That level is denoted by the blue line. Let a be some constant so that the function is defined in the positive orthant.

When $w < b$ the function is concave in wealth and when $w > b$ the function is convex in wealth.

The risk aversion coefficient is $R_a(w) = \frac{-2}{w-b}$. We can see that this is positive when $w < b$ and negative when $w > b$.

Question 6

Let u be the agent's VNM utility function. Since the agent is risk averse, u is strictly concave. Now since $w_3 < w_1 \leq w_2 < w_4$ we can define:

$$\lambda := \frac{w_4 - w_1}{w_4 - w_3} \in (0, 1)$$

$$\mu := \frac{w_4 - w_2}{w_4 - w_3} \in (0, 1)$$

So that

$$w_1 = \lambda w_3 + (1 - \lambda) w_4$$

$$w_2 = \mu w_3 + (1 - \mu) w_4$$

Then

$$\begin{aligned}
pu(w_1) + (1-p)u(w_2) &= pu(\lambda w_3 + (1-\lambda)w_4) + (1-p)u(\mu w_3 + (1-\mu)w_4) \\
&> p\lambda u(w_3) + p(1-\lambda)u(w_4) + (1-p)\mu u(w_3) + (1-p)(1-\mu)u(w_4) \\
&= \Big(p\lambda + (1-p)\mu\Big)u(w_3) + \Big(p(1-\lambda) + (1-p)(1-\mu)\Big)u(w_4) \\
&= pu(w_3) + (1-p)u(w_4)
\end{aligned}$$

The inequality holds by the strict concavity of u. For the final equality note that the gambles have the same expected payoff so $pw_1 + (1-p)w_2 = pw_3 + (1-p)w_4$, and plug in the definitions for λ and μ to find:

$$p\lambda + (1-p)\mu = \frac{w_4 - \big(pw_1 + (1-p)w_2\big)}{w_4 - w_3} = \frac{w_4 - pw_3 - (1-p)w_4}{w_4 - w_3} = p$$

And

$$p(1-\lambda) + (1-p)(1-\mu) = \frac{pw_1 - pw_3 + (1-p)w_2 - (1-p)w_3}{w_4 - w_3} = \frac{(1-p)w_4 - (1-p)w_3}{w_4 - w_3} = 1 - p$$

This establishes that the first gamble is strictly preferred to the second as desired.

Question 8

8. Suppose that you are risk averse and employed by a company that is competing vigorously with another. Indeed, there is only room enough for one of you in the market. Suppose further that it is equally likely that each company takes over the market and puts the other out of business. If your company goes out of business, you will be unemployed and your wealth will fall by $100,000. If your company survives (and the other doesn't) then your job is safe and your wealth will rise by $20,000. At the moment, stock prices of the two companies are identical at $1 per share. It is known that when one of the companies goes out of business, it's shares will fall to zero in value, while the surviving company's share price will rise to $2. You have decided to purchase a total of 1,000 shares. How should you divide your purchases between the two companies' stocks? (Assume that purchasing shares does not affect the share price, nor the probability that either firm goes out of business.) What does this say about the rather common practice of stock purchase packages offered by most private companies?

Suppose our agent gets the following utilities in the state of the world where u^1 is the utility from firm 1 (the firm they work at) surviving, and u^2 when their firm dies and the competitor survives. Our agent gets utility u from their total wealth, Y. Note we net the Y of the $1000 you have already decided to spend on stocks of the two companies:

$$
\begin{aligned}
u^1 &= u(Y + 20,000 + +x - (1000 - x)) = u(Y + 20,000 + (2x - 1000)) \\
u^2 &= u(Y - 100,000 - x + (1000 - x)) = u(Y - 100,000 + (1000 - 2x))
\end{aligned}
$$

Where $x \in [0, 1000]$ is the number of stocks the agent decides to purchase in company 1, their employer. Note because we net Y of the $1000 the agent just loses a dollar for every stock they buy in the dying company and gain a dollar for every stock that survives. The expected utility of our agent is:

$$U = \frac{1}{2}u^1 + \frac{1}{2}u^2$$

Because of the bound on x we can clearly see that $Y + 20,000 + (2x - 1000) > Y - 100,000 + (1000 - 2x)$:

$$
\begin{aligned}
Y + 20,000 + (2x - 1000) > Y - 100,000 + (1000 - 2x) &\iff \\
20,000 + (2x - 1000) > -100,000 + (1000 - 2x) &\iff \\
118,000 > 4x &\iff \\
x &< 29,500
\end{aligned}
$$

Which clearly checks out. We now take a derivative on our U function with respect to x:

$$\frac{\partial U}{\partial x} = u'(Y + 20,000 + (2x - 1000)) - u'(Y - 100,000 + (1000 - 2x))$$

Because a risk averse agent gets more our of a dollar the less amount of dollars they have, aka the derivative of the utility function is higher at lower values, we know that our $\frac{\partial U}{\partial x} < 0$ as we showed that clearly $Y + 20,000 + (2x - 1000) > Y - 100,000 + (1000 - 2x)$.

As such the agent will purchase exclusively stocks in the competitor and none in their own company. This makes sense from the risk aversion perspective because they will want to cushion the loss from their company going out of business.

So why do firms offer stock purchasing packages? This is because when a company is doing well an employee of the company tends to be doing better. That means that they want to hedge more against the company being in trouble. Because they'd rather own other stocks to hedge against a crises scenario the firm can actually convince them to sell their stock (perhaps at a discount) so that they can hedge.