Price Theory II

Problem Set 2

Solutions

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[1]

1 Exercise 5.15

There are 100 units of x_1 and 100 units of x_2. Consumers 1 and 2 are each endowed with 50 units of each good. Consumer 1 says, "I love x_1, but I can take or leave x_2." Consumer 2 says, "I love x_2, but I can take or leave x_1."

(i) Draw an Edgeworth box for these traders and sketch their preferences.

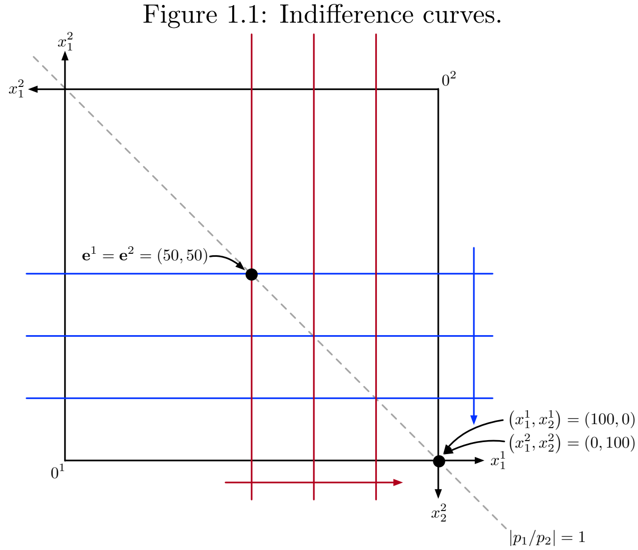
(ii) Identify the core of this economy.

(iii) Find all Walrasian equilibria for this economy.

1.1 Part (a)

The preferences imply that the indifference curves in (x_1, x_2) space is vertical (increasing to the right) for consumer 1 and horizontal for consumer 2 (increasing up). See figure below.

Figure 1.1: Indifference curves.



[1] We thank TAs from previous years, especially Takuma Habu and Eyo Herstad for solution. This version is based on their writeups.

1.2 Part (b)

The question suggests that 1's utility is increasing in x_1^1 and 2's utility is increasing in x_2^2 so that the unique Pareto optimal allocation is given by $\left(x_1^1, x_2^1\right) = (100, 0)$ and $\left(x_1^2, x_2^2\right) = (0, 100)$. Since Pareto optimality is a necessary condition for an allocation to be the core, it follows that there exists only singleton core in this set up given by the aforementioned allocation.

1.3 Part (c)

At the Pareto optimal allocation $\left(x_1^1, x_2^1\right) = (100, 0)$ and $\left(x_1^2, x_2^2\right) = (0, 100)$, both agents maximize their utility and that the allocation is feasible. The price that supports this allocation, given initial endowment of $(50, 50)$ is any $\mathbf{p}^* = (p_1^*, p_2^*) \gg \mathbf{0}$ such that $p_1^*/p_2^* = 1$. There is a unique WEA which is the Pareto optimal allocation.

Although this is probably clear from the figure, we can show this algebraically. Demand functions are given by

$$\begin{aligned} x_1^1 &= \frac{50\left(p_1 + p_2\right)}{p_1}, & x_2^1 &= 0, \\ x_1^2 &= 0, & x_2^2 &= \frac{50\left(p_1 + p_2\right)}{p_2}. \end{aligned}$$

Hence, if either of prices were zero, one of the agents will demand be unbounded. Market clearing requires that

$$\begin{aligned} \frac{50\left(p_1 + p_2\right)}{p_1} = 100, \ \frac{50\left(p_1 + p_2\right)}{p_2} &= 100 \\ \Leftrightarrow p_1 + p_2 = 2p_1 &= 2p_2 \\ \Leftrightarrow p_1 &= p_2. \end{aligned}$$

2 Exercise 5.17

Consider an exchange economy with two identical consumers. Their common utility function is $u^i(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ for $0 < \alpha < 1$. Society has 10 units of x_1 and 10 units of x_2 in all. Find endowments $\mathbf{e}^1$ and $\mathbf{e}^2$, where $\mathbf{e}^1 \neq \mathbf{e}^2$, and Walrasian equilibrium prices that will "support" as a WEA the equal division allocation giving both consumers the bundle $(5, 5)$.

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For WEA to involve equal division allocation, the marginal rate of substitution between the two consumers must be equal at $(5, 5)$. Thus, we first calculate the marginal rate of substitution:

$$MRS_{1,2}(\mathbf{x}) = \frac{MU_1(\mathbf{x})}{MU_2(\mathbf{x})} = \frac{\alpha u(\mathbf{x})/x_1}{(1-\alpha)u(\mathbf{x})/x_2} = \frac{\alpha}{1-\alpha}\frac{x_2}{x_1}.$$

The supporting equilibrium price is such that the price ratio equals the $MRS_{1,2}$ at allocation $(5, 5)$; i.e.

$$\frac{p_1}{p_2} = MRS_{1,2}(5, 5) = \frac{\alpha}{1-\alpha}.$$

Thus, any endowment vector along the line that passes through $(5, 5)$ with slope $-p_1/p_2$ will lead to equal division.

We can characterise such bundles as

$$e_2^i = -\frac{\alpha}{1-\alpha}e_1^i + \frac{5}{1-\alpha}$$

such that $\mathbf{e}^1 + \mathbf{e}^2 = (10, 10)$.[2]

[2]Recall that an equation for the line that passes through the point (x^*, y^*) with slope β is given by

$$\frac{y - y^*}{x - x^*} = \beta \Leftrightarrow y = y^* + \beta(x - x^*).$$

Here, $(x^*, y^*) = (5, 5)$ and $\beta = -\alpha/(1-\alpha)$.

Comment

An alternative method would be to solve for the demand functions and find the price vector so that that both consumers would consume the bundle $(5,5)$. Since utility functions are Cobb-Douglas, the consumer spends α will be the proportion of income that the consumer spends on good 1; i.e. for each $i \in \{1,2\}$,

$$x_1^i = \alpha \frac{p_1 e_1^i + p_2 e_2^i}{p_1}, \; x_1^i = (1-\alpha) \frac{p_1 e_1^i + p_2 e_2^i}{p_2}.$$

Setting $x_j^i = 5$ for $i, j \in \{1,2\}$ yields

$$\begin{aligned} \alpha \frac{p_1 e_1^i + p_2 e_2^i}{p_1} &= 5 = (1-\alpha) \frac{p_1 e_1^i + p_2 e_2^i}{p_2} \\ \Leftrightarrow \frac{\alpha}{1-\alpha} &= \frac{p_1}{p_2}. \end{aligned}$$

With $x_j^i = 5$, we can write

$$\begin{aligned} 5 = x_1^i &= \alpha \frac{p_1 e_1^i + p_2 e_2^i}{p_1} \\ \Leftrightarrow e_2^i &= \frac{p_1}{p_2} \left(\frac{5}{\alpha} - e_1^i \right) = \frac{\alpha}{1-\alpha} \left(\frac{5}{\alpha} - e_1^i \right) \\ &= -\frac{\alpha}{1-\alpha} e_1^i + \frac{5}{1-\alpha} \end{aligned}$$

as we found above. Finally, we just have to make sure that endowments satisfy feasibility.

3 Exercise 5.18

In two-good, two-consumer economy, utility functions are

$$u^1(x_1, x_2) = x_1 (x_2)^2,$$
$$u^2(x_1, x_2) = (x_1)^2 x_2.$$

Total endowments are $(10, 20)$.

(i) A social planner wants to allocate goods to maximize consumer 1's utility while holding consumer 2's utility at $u^2 = 8000/27$. Find the assignment of goods to consumers that solves the planner's problem and show that the solution is Pareto efficient.

(ii) Suppose, instead, that the planner just divides the endowments so that $\mathbf{e}^1 = (10, 0)$ and $\mathbf{e}^2 = (0, 20)$ and then lets the consumers transact through perfectly competitive markets. Find the Walrasian equilibrium and show that the WEAs are the same as the solution in part (a).

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3.1 Part (i)

The social planner's problem is

$$\begin{aligned} \max_{(x_1^1, x_2^1)} \quad & x_1^1 (x_2^1)^2 \\ s.t. \quad & x_1^1 + x_1^2 = 10, \\ & x_2^1 + x_2^2 = 20, \\ & (x_1^2)^2 x_2^2 = 8000/27. \end{aligned}$$

We can simplify the problem to

$$\begin{aligned} \max_{(x_1, x_2)} \quad & \ln x_1 + 2\ln x_2 \\ s.t. \quad & 2\ln(10 - x_1) + \ln(20 - x_2) = \ln(8000/27). \end{aligned}$$

The Lagrangian is

$$\mathcal{L} = \ln x_1 + 2\ln x_2 + \lambda(\ln(8000/27) - 2\ln(10 - x_1) - \ln(20 - x_2)).$$

The first-order conditions are (we can assume interior solution since utilities are Cobb-Douglas)

$$\frac{1}{x_1} + \frac{2\lambda}{10 - x_1} = 0,$$
$$\frac{2}{x_2} + \frac{\lambda}{20 - x_2} = 0.$$

From the first

$$\lambda = -\frac{1}{2}\frac{10 - x_1}{x_1}$$

Substituting into the second,

$$\frac{2}{x_2} = \frac{1}{2}\frac{10 - x_1}{x_1(20 - x_2)} \Leftrightarrow 80x_1 - 4x_1x_2 = 10x_2 - x_1x_2$$

$$\Leftrightarrow x_2 = \frac{80x_1}{10+3x_1}.$$

Substituting into the budget constraint:

$$\begin{aligned}
\ln(8000/27) &= 2\ln(10-x_1) + \ln(20-x_2) \\
&= 2\ln(10-x_1) + \ln\left(20 - \frac{80x_1}{10+3x_1}\right) \\
&= \ln\left((10-x_1)^3 \frac{20}{10+3x_1}\right) \\
\Rightarrow x_1^1 &= \frac{10}{3}, \\
x_2^1 &= \frac{40}{3}, \\
\Rightarrow x_1^2 &= \frac{20}{3}, \\
x_2^2 &= \frac{20}{3}.
\end{aligned}$$

Equilibrium is characterise by a point where the two agent's marginal utilities equal. Let us first calculate the marginal rate of substitution:

$$\begin{aligned}
MRS_{1,2}^1\left(x_1^1, x_2^1\right) &= \frac{MU_1^1}{MU_2^1} = \frac{x_2^1}{2x_1^1}, \\
MRS_{1,2}^2\left(x_1^1, x_2^1\right) &= \frac{MU_1^1}{MU_2^1} = \frac{2x_2^2}{x_1^2}.
\end{aligned}$$

At the bundle we found,

$$\begin{aligned}
MRS_{1,2}^1\left(\frac{10}{3}, \frac{40}{3}\right) &= 2, \\
MRS_{1,2}^2\left(\frac{20}{3}, \frac{20}{3}\right) &= 2.
\end{aligned}$$

Hence, the allocation we found is Pareto optimal.

3.2 Part (ii)

The agent $i \in \{1,2\}$'s problem is

$$\begin{aligned}
\max_{\left(x_1^i, x_2^i\right)} \quad & i\ln\left(x_1^i\right) + (3-i)\ln\left(x_2^i\right) \\
s.t. \quad & p_1 x_1^i + p_2 x_1^i = p_1 e_1^i + p_2 e_2^i.
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
\frac{i}{x_1^i} &= \lambda p_1, \\
\frac{3-i}{x_2^i} &= \lambda p_2, \\
\Rightarrow x_2^i &= \frac{3-i}{i}\frac{p_1}{p_2}x_1^i.
\end{aligned}$$

Substituting into the budget constraint:

$$
\begin{aligned}
x_1^i &= \frac{i}{3p_1}\left(p_1 e_1^i + p_2 e_2^i\right), \\
x_2^i &= \frac{3-i}{3p_2}\left(p_1 e_1^i + p_2 e_2^i\right).
\end{aligned}
$$

Market clearing requires that

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\begin{aligned}
10 &= \sum_{i=1}^{2} x_1^i = \frac{10}{3} + \frac{40}{3}\frac{p_2}{p_1} \\
\Rightarrow \frac{p_1}{p_2} &= 2.
\end{aligned}
$$

Then, the allocation is given by

$$
\begin{aligned}
x_1^1 &= \frac{10}{3}, \\
x_2^1 &= \frac{40}{3}, \\
x_1^2 &= \frac{20}{3}, \\
x_2^2 &= \frac{20}{3}.
\end{aligned}
$$

4 Exercise 5.23

Show that, if a firm's production set is strongly convex and the price vector is strictly positive, then there is at most one profit-maximising production plan.

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Let $\mathbf{0} \in Y^j \subseteq \mathbb{R}^n$ denote production possibility set of firm $j \in \mathcal{J}$.

Definition 4.1. *(Strong convexity)* For any $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$ with $\mathbf{y}^1 \neq \mathbf{y}^2$, there exists $\bar{\mathbf{y}} \in Y^j$ such that

$$\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\,\mathbf{y}^2,\ \forall t \in (0,1)\,,$$

where the equality does not hold.

The firm's problem is

$$\max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j.$$

Suppose $\mathbf{p} \gg \mathbf{0}$. Toward a contradiction, suppose that there exists two profit maximising plans $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$ with $\mathbf{y}^1 \neq \mathbf{y}^2$. Since Y^j is strictly convex, we can find

$$\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\,\mathbf{y}^2,\ \forall t \in (0,1)\,,$$

where the equality does not hold. Since $\mathbf{p} \gg \mathbf{0}$, this implies that

$$\mathbf{p} \cdot \bar{\mathbf{y}} > \mathbf{p} \cdot \mathbf{y}^1 = \mathbf{p} \cdot \mathbf{y}^2,$$

which contradicts the initial assumption that $\mathbf{y}^1$ and $\mathbf{y}^2$ were profit maximising.

5 Exercise 5.25

Complete the proof of Theorem 5.13 by showing that $\mathbf{z}(\mathbf{p})$ in the economy with production satisfies all the properties of Theorem 5.3.

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Let's just recall what the theorems were.

Theorem 5.3 *(Aggregate excess demand and Walrasian equilibrium)* Suppose $\mathbf{z} : \mathbb{R}^n_{++} \to \mathbb{R}$ satisfies the following three conditions:

(i) (continuity) $\mathbf{z}(\cdot)$ is continuous on $\mathbb{R}^n_{++}$;

(ii) (Walras' law) $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \gg \mathbf{0}$;

(iii) if $\{\mathbf{p}^m\}$ is a sequence of prices in $\mathbb{R}^n_{++}$ and $\mathbf{p}^m \to \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k, then for some good k' with $\bar{p}_{k'} = 0$, the sequence $\{z_{k'}(\mathbf{p}^m)\}$is unbounded above.

Then, there exists $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Theorem 5.13 *(Existence of Walrasian equilibrium with Production)* Consider the economy $\mathcal{E} = \left(u^i, \mathbf{e}^i, \theta^{ij}, Y^j\right)_{i\in\mathcal{I}, j\in\mathcal{J}}$. If each u^i satisfies Assumption 5.1,[a] each Y^j satisfies Assumption 5.2,[b] and $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y} \in \sum_{j\in\mathcal{J}} Y^j$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

[a] Each u^i is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

[b] (i) $\mathbf{0} \in Y^j \subseteq \mathbb{R}^n$; (ii) Y^j is closed and bounded; and (iii) Y^j is strongly convex)i.e. for any distinct $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$, for all $t \in (0,1)$, there exists $\bar{\mathbf{y}} \in Y^j$ such that $\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\mathbf{y}^2$ and the equality does not hold).

Let us now show that

$$\mathbf{z}(\mathbf{p}) \coloneqq (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p})),$$
$$z_k(\mathbf{p}) \coloneqq \sum_{i\in\mathcal{I}} x^i_k\left(\mathbf{p}, m^i(\mathbf{p})\right) - \sum_{j\in\mathcal{J}} y^j_k(\mathbf{p}) - \sum_{i\in\mathcal{I}} e^i_k.$$

satisfies the properties of Theorem 5.3.

(i) Continuity By Theorems 5.9 and 5.12, under Assumptions 5.1 and 5.2, $\mathbf{y}^j(\mathbf{p})$ and $\mathbf{x}^i\left(\mathbf{p}, m^i(\mathbf{p})\right)$ are continuous on $\mathbb{R}^n_{++}$.[3] Since adding and subtracting are continuous operations, it follows that $\mathbf{z}(\mathbf{p})$ is continuous on $\mathbb{R}^n_{++}$.

(ii) Walras' law Since u^i is strongly increasing, at the optimum, budget constraint for each $i \in \mathcal{I}$ binds. That is,

$$\mathbf{p} \cdot \mathbf{x}^i = m^i(\mathbf{p}) = \mathbf{p} \cdot \mathbf{e}^i + \sum_{j\in\mathcal{J}} \theta^{ij} \Pi^j(\mathbf{p}), \ \forall i \in \mathcal{I}.$$

[3] Recall that

$$\mathbf{y}^j = \underset{\tilde{\mathbf{y}}^j \in Y^j}{\arg\max}\, \mathbf{p} \cdot \tilde{\mathbf{y}}^j.$$

Since we assume Y^j is compact and the objective function is continuous, $\mathbf{y}^j$ is continuous by the Theorem of Maximum.

Recall also that

$$\mathbf{x}^i\left(\mathbf{p}, m^i(\mathbf{p})\right) = \underset{\tilde{\mathbf{x}}^i \in \mathbb{R}^n_+}{\arg\max}\, u^i\left(\tilde{\mathbf{x}}^i\right) \ \ s.t.\ \mathbf{p} \cdot \tilde{\mathbf{x}}^i \leq m^i(\mathbf{p}),$$

where $m^i(\mathbf{p}) \coloneqq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j\in\mathcal{J}} \theta^{ij} \Pi^j(\mathbf{p})$ and $\Pi^j(\mathbf{p}) = \max_{\tilde{\mathbf{y}}^j \in Y^j} \mathbf{p} \cdot \tilde{\mathbf{y}}^j$. By The Theorem of Maximum, $\Pi^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$ so that $m^i(\mathbf{p})$ is continuous (since additions and multiplications are continuous operations). Then, by the Theorem of the Maximum, $\mathbf{x}^i\left(\mathbf{p}, m^i(\mathbf{p})\right)$ is continuous on $\mathbb{R}^n_{++}$ since the feasibility set is compact valued and continuous when $\mathbf{p} \gg 0$.

Summing across all i yields

$$\begin{aligned}\sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{x}^i &= \sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{e}^i + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\theta^{ij}\Pi^j(\mathbf{p}) \\ &= \sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\underbrace{\Pi^j(\mathbf{p})}_{=\mathbf{p}\cdot\mathbf{y}^j(\mathbf{p})}\underbrace{\sum_{i\in\mathcal{I}}\theta^{ij}}_{=1} \\ &= \sum_{i\in\mathcal{I}}\mathbf{p}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\mathbf{p}\cdot\mathbf{y}^j(\mathbf{p}),\end{aligned}$$

where we rearranged the order of summation and applied some definitions. Finally,

$$\begin{aligned}\mathbf{p}\cdot\sum_{i\in\mathcal{I}}\mathbf{x}^i &= \mathbf{p}\cdot\sum_{i\in\mathcal{I}}\mathbf{e}^i + \mathbf{p}\cdot\sum_{j\in\mathcal{J}}\mathbf{y}^j(\mathbf{p}) \\ \Leftrightarrow \mathbf{0} &= \mathbf{p}\cdot\left(\sum_{i\in\mathcal{I}}\mathbf{x}^i - \sum_{j\in\mathcal{J}}\mathbf{y}^j(\mathbf{p}) - \sum_{i\in\mathcal{I}}\mathbf{e}^i\right) \\ &= \mathbf{p}\cdot\mathbf{z}(\mathbf{p})\end{aligned}$$

as desired.

(iii) Unboundedness as $\mathbf{p}^r \to \bar{\mathbf{p}} \neq \mathbf{0}$ with some $\bar{p}_k = 0$ It remains to show that if $\{\mathbf{p}^r\}$ is a sequence of prices in $\mathbb{R}^n_{++}$ and $\mathbf{p}^r \to \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k, then for some good k' with $\bar{p}_{k'} = 0$, the sequence $\{z_{k'}(\mathbf{p}^r)\}$is unbounded above. By Assumption 5.2, Y^j is bounded and the endowments are bounded so that we only need to show that some consumer's demand for some good is unbounded as $\mathbf{p}^r \to \bar{\mathbf{p}}$.

Since $\mathbf{y} + \sum_{i\in\mathcal{I}}\mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y}$ and since $\bar{\mathbf{p}}$ has no negative components and $\mathbf{p} \neq \mathbf{0}$, we must have

$$\bar{\mathbf{p}}\cdot\left(\mathbf{y} + \sum_{i\in\mathcal{I}}\mathbf{e}^i\right) > 0.$$

Hence,

$$\begin{aligned}\sum_{i\in\mathcal{I}}m^i(\bar{\mathbf{p}}) &= \sum_{i\in\mathcal{I}}\left(\bar{\mathbf{p}}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\theta^{ij}\Pi^j(\bar{\mathbf{p}})\right) \\ &= \sum_{i\in\mathcal{I}}\bar{\mathbf{p}}\cdot\mathbf{e}^i + \sum_{i\in\mathcal{I}}\Pi^j(\bar{\mathbf{p}}) \\ &\geq \bar{\mathbf{p}}\cdot\left(\mathbf{y} + \sum_{i\in\mathcal{I}}\mathbf{e}^i\right) > 0, \qquad (5.1)\end{aligned}$$

where the weak inequality follows from the fact that sum of individual firm maximized profits must be at least as large as the maximized aggregate profits (Theorem 5.11), and, hence, at least as large as the aggregate profits from $\mathbf{y}$. Thus, there must exist at least one consumer whose income at prices $\bar{\mathbf{p}}$, $m^i(\bar{\mathbf{p}})$, is strictly positive. We consider such a consumer i. The rest of the proof mimics that of Theorem 5.5.

Let $\mathbf{x}^r := x^i\left(\mathbf{p}^r, m^i(\mathbf{p}^r)\right)$. We want to show that i's demand for some good k' is unbounded above. Suppose, by way of contradiction, that $\mathbf{x}^r$ is bounded. Then, there exists a converging subsequence of $\mathbf{x}^r$, which we denote as $\mathbf{x}^r$ so that $\mathbf{x}^r \to \mathbf{x}^*$. Since the budget constraint must bind at the optimal $\mathbf{x}^r$;

$$\mathbf{p}^r\cdot\mathbf{x}^r = m^i(\mathbf{p}^r),\ \forall r.$$

Since $m^i(\cdot)$ is continuous on $\mathbb{R}^n_+$ by Theorem 5.12, as $r \to \infty$,

$$\bar{\mathbf{p}} \cdot \mathbf{x}^* = m^i(\bar{\mathbf{p}}) > 0,$$

where the strict inequality follows from (5.1).

Let $\hat{\mathbf{x}} = \mathbf{x}^* + (0, \ldots, 0, 1, 0, \ldots, 0)$ denote a bundle composed of adding 1 to the kth component of $\mathbf{x}^*$. Since $\bar{p}_k = 0$ by assumption, and u^i is strongly increasing,

$$\begin{aligned} \bar{\mathbf{p}} \cdot \hat{\mathbf{x}} &= \bar{\mathbf{p}} \cdot \mathbf{x}^* > 0, \\ u^i(\hat{\mathbf{x}}) &> u^i(\mathbf{x}^*). \end{aligned}$$

By continuity of u^i, for sufficiently large $t \in (0, 1)$,

$$\begin{aligned} \bar{\mathbf{p}} \cdot t\hat{\mathbf{x}} &< \bar{\mathbf{p}} \cdot \mathbf{x}^* > 0, \\ u^i(t\hat{\mathbf{x}}) &> u^i(\mathbf{x}^*). \end{aligned}$$

Since $\mathbf{p}^r \to \bar{\mathbf{p}}$, $\mathbf{x}^r \to \mathbf{x}^*$ and u^i is continuous, for sufficiently large r,

$$\begin{aligned} \mathbf{p}^r \cdot t\hat{\mathbf{x}} &< \mathbf{p}^r \cdot \mathbf{x}^r > 0 \\ u^i(t\hat{\mathbf{x}}) &> u^i(\mathbf{x}^*), \end{aligned}$$

which contradicts that $\mathbf{x}^r$ is optimal demand at prices $\mathbf{p}^r$. Thus, $\mathbf{x}^r$ must be bounded.

Finally, since $\mathbf{x}^r$ is nonnegative, that $\{\mathbf{x}^r\}$ is unbounded means that there exists k' such that $\{\mathbf{x}^r_{k'}\}$ is unbounded above. To reconcile this with the fact that $\left\{\mathbf{p}^r \cdot m^i(\mathbf{p}^r)\right\}$ is bounded, it must be that $p^r_{k'} \to 0$.

6 Exercise 5.26

Suppose that, in a single-consumer economy, the consumer is endowed with none of the consumption good, y, and 24 hours of time, h, so that $\mathbf{e} = (24, 0)$. Suppose as well that preferences are defined over $\mathbb{R}^2_+$ and represented by $u(h, y) = hy$, and production possibilities are $Y = \left\{(-h, y) : 0 \leq h \leq b, 0 \leq y \leq \sqrt{h}\right\}$, where b is some large positive number. Let p_y and p_h be prices of the consumption good and leisure, respectively.

6.1 Part (a)

Find relative prices p_y/p_h that clear the consumption and leisure markets simultaneously.

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Let us check if prices must be strictly positive. Suppose $p_y = 0$, then demand for y goes to infinity (independent of whether p_h is positive or not since the agent is always able use up his endowment of h of 24 no matter the price of p_h so that $h > 0$). Hence, we can rule $p_y = 0$ out from the equilibrium. Consider $p_y > 0$ and $p_h = 0$. This means that the cost of labor for the firm is zero so that the firm's profit is maximized when $h = b > 24$ so the market would not clear. We therefore conclude that $p_y, p_h > 0$.

We can write production function as $y^s = \sqrt{h^d}$. The firm's problem is then

$$\Pi(\mathbf{p}) = \max_{h \in [0,b]} \quad p_y \sqrt{h} - p_h h.$$

We assume that b is sufficiently large to ensure that the firm will never choose b. The first-order condition is then

$$\begin{aligned} \frac{1}{2} p_y \left(h^d\right)^{-\frac{1}{2}} &= p_h \\ \Leftrightarrow \frac{p_y}{p_h} &= 2\sqrt{h^d}, \end{aligned}$$

and so

$$y^s = \frac{1}{2}\frac{p_y}{p_h}, \ h^d = \frac{1}{4}\left(\frac{p_y}{p_h}\right)^2.$$

where h^d is the labor demand and y^s is the supply of the good. The firm's profit is

$$\Pi(\mathbf{p}) = p_y \sqrt{\frac{1}{4}\left(\frac{p_y}{p_h}\right)^2} - p_h \frac{1}{4}\left(\frac{p_y}{p_h}\right)^2 = \frac{1}{4}\left(\frac{p_y}{p_h}\right) p_y.$$

The consumer's problem is

$$\max_{h,y} \quad hy \quad s.t. \quad p_y y + p_h h \leq 24 p_h + \frac{1}{4}\left(\frac{p_y}{p_h}\right) p_y.$$

Given Cobb-Douglas utility function, the first-order condition holds with equality at the optimal and are given by

$$\begin{aligned} h^s &= \lambda p_y, \\ y^d &= \lambda p_h, \\ \Rightarrow \frac{h^s}{y^d} &= \frac{p_y}{p_h}. \end{aligned}$$

Substituting above into the budget constraint (which must bind at the optimal given strictly increasing utility function) gives the demand functions:

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\begin{aligned}
p_y y^d + p_h \left(\frac{p_y}{p_h} y^d\right) &= 24 p_h + \frac{1}{4}\left(\frac{p_y}{p_h}\right) p_y \\
\Leftrightarrow y^d &= 12\frac{p_h}{p_y} + \frac{1}{8}\left(\frac{p_y}{p_h}\right) \\
&= \frac{p_y}{p_h}\left[\frac{1}{8} + 12\left(\frac{p_y}{p_h}\right)^{-2}\right], \\
h^s &= \frac{p_y}{p_h}\frac{p_y}{p_h}\left[\frac{1}{8} + 12\left(\frac{p_y}{p_h}\right)^{-2}\right] \\
&= 12 + \frac{1}{8}\left(\frac{p_y}{p_h}\right)^2,
\end{aligned}
$$

where h^s is the labor supply and y^d is the demand for the good.

Market clearing requires

$$
\begin{aligned}
y^d = \frac{p_y}{p_h}\left[\frac{1}{8} + 12\left(\frac{p_y}{p_h}\right)^{-2}\right] &= \frac{1}{2}\frac{p_y}{p_h} = y^s \\
\Leftrightarrow \frac{p_y}{p_h} &= 4\sqrt{2}.
\end{aligned}
$$

We can check that the market for h clears too:

$$
h^s + h^d = 12 + \frac{1}{8}\left(\frac{p_y}{p_h}\right)^2 + \frac{1}{4}\left(\frac{p_y}{p_h}\right)^2 = 12 + \frac{3}{8}\left(4\sqrt{2}\right)^2 = 24.
$$

Therefore, relative prices $p_y/p_h = 4\sqrt{2}$ clears the consumption and leisure markets simultaneously.

6.2 Part (b)

Calculate the equilibrium consumption and production plans and sketch your results in $\mathbb{R}^2_+$.

……

Consumers consume both the good, y, and leisure, h^s. Substituting the equilibrium relative prices into y^d and h^s we found yields

$$
\begin{aligned}
y^d &= \frac{p_y}{p_h}\left[\frac{1}{8} + 12\left(\frac{p_y}{p_h}\right)^{-2}\right] = 2\sqrt{2}, \\
h^s &= 12 + \frac{1}{8}\left(\frac{p_y}{p_h}\right)^2 = 16.
\end{aligned}
$$

$$
\frac{1}{4\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{8}
$$

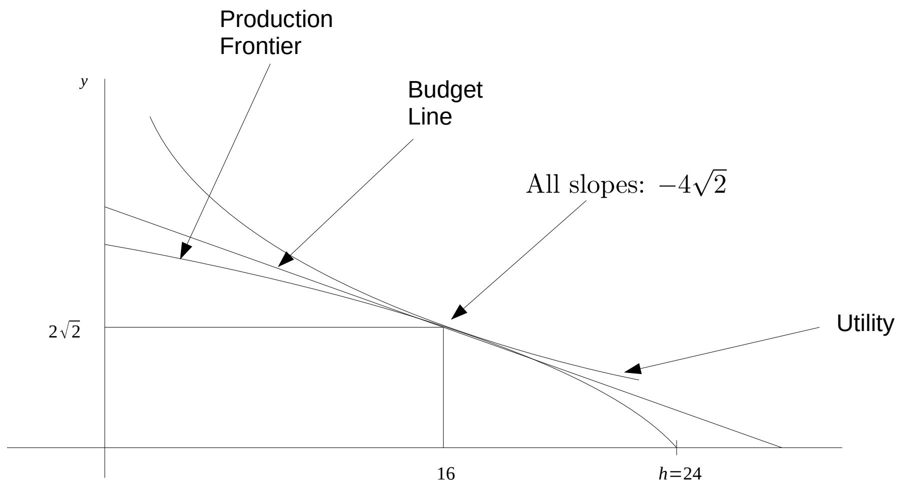
Production plans are

$$
y^s = \frac{1}{2}\frac{p_y}{p_h} = 2\sqrt{2},
$$

$$h^d = \frac{1}{4}\left(\frac{p_y}{p_h}\right)^2 = 8.$$

See the figure below.

Figure 6.1: Part (b).



6.3 Part (c)

How many hours a day does the consumer work?

......

$$h = 12 + \frac{1}{8}\left(\frac{p_y}{p_h}\right)^2 = 12 + \frac{1}{8}\left(4\sqrt{2}\right)^2$$
$$= 16.$$

Since h is leisure time, the consumer works $24 - 16 = 8$ hours a day.

7 Exercise 5.27

Consider an exchange economy $\left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ in which each u^i is continuous and quasiconcave on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}} = \left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^I\right) \gg \mathbf{0}$ is Pareto efficient, that each u^i is continuously differentiable in an open set containing $\bar{\mathbf{x}}^i$, and that $\nabla u^i\left(\bar{\mathbf{x}}^i\right) \gg \mathbf{0}$. Under these conditions, which differ somewhat from those of Theorem 5.8, follow the steps below to derive another version of the Second Welfare Theorem.

7.1 Part (a)

Show that, for any two consumers i and j, the gradient vectors $\nabla u^i\left(\bar{\mathbf{x}}^i\right)$ and $\nabla u^j\left(\bar{\mathbf{x}}^j\right)$ must be proportional. That is, there must exist some $\alpha > 0$ (which may depend on i and j) such that $\nabla u^i\left(\bar{\mathbf{x}}^i\right) = \alpha \nabla u^j\left(\bar{\mathbf{x}}^j\right)$. Interpret this condition in the case of the Edgeworth box economy.

......

Suppose, toward a contradiction that $\overline{\mathbf{x}}$ is Pareto efficient but that for some $i, j \in \mathcal{I}$, $\nabla u^i\left(\bar{\mathbf{x}}^i\right)$ and $\nabla u^j\left(\bar{\mathbf{x}}^j\right)$ are not proportional . That is, there exists goods k and ℓ such that

$$\alpha := \frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x_k}{\partial u^i\left(\overline{\mathbf{x}}^i\right)/\partial x_\ell} \neq \frac{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x_k}{\partial u^j\left(\overline{\mathbf{x}}^j\right)/\partial x_\ell} =: \beta.$$

Without loss of generality, assume $\alpha > \beta \Leftrightarrow \beta/\alpha < 1$. That is, agent i prefers good k relative to ℓ more so than agent j. Consider an alternative allocation $\hat{\mathbf{x}}$ in which we transfer some of j's good k to i; i.e. with $\mu \in (0,1)$ sufficiently close to 1 and for some small $t > 0$,

- $\hat{\mathbf{x}}^n = \overline{\mathbf{x}}^n$ for all $n \neq i, j$.
- $\hat{\mathbf{x}}^i = \overline{\mathbf{x}}^i$ except that we replace the kth and ℓth coordinates of $\overline{\mathbf{x}}^1$ with $\overline{x}^1_k + t$ and $\overline{x}^1_j - \mu t/\alpha$ respectively:
- $\hat{\mathbf{x}}^j = \overline{\mathbf{x}}^j$ except that we replace the kth and ℓth coordinates of $\overline{\mathbf{x}}^1$ with $\overline{x}^1_k - t$ and $\overline{x}^1_j + \mu t/\alpha$ respectively.

Since $\overline{\mathbf{x}}$ is Pareto efficient, in particular, the allocation is feasible. Then, by construction, $\hat{\mathbf{x}}$ is also feasible.

To study a small change from $\overline{\mathbf{x}}^i$, totally differentiating $u^i\left(\overline{\mathbf{x}}^i\right)$ gives

$$du^i\left(\overline{\mathbf{x}}^i\right) = \sum_{n=1}^{m} \frac{\partial u^i\left(\overline{\mathbf{x}}^i\right)}{\partial x_n} d\overline{x}_n$$

Note that since $\nabla u^i\left(\bar{\mathbf{x}}^i\right) \gg \mathbf{0}$, $\partial u^i\left(\overline{\mathbf{x}}\right)/\partial x_n > 0$ for all i and n. The change in utility from moving from $\overline{\mathbf{x}}^i$ to $\hat{\mathbf{x}}^i$ is given by

$$\begin{aligned} du^i &= \frac{\partial u^i\left(\overline{\mathbf{x}}\right)}{\partial x_k} t + \frac{\partial u^i\left(\overline{\mathbf{x}}\right)}{\partial x_\ell}\left(-\frac{\mu}{\alpha}t\right) = \frac{\partial u^i\left(\overline{\mathbf{x}}\right)}{\partial x_k}\left[t + \underbrace{\frac{\partial u^i\left(\overline{\mathbf{x}}\right)/\partial x_k}{\partial u^i\left(\overline{\mathbf{x}}\right)/\partial x_\ell}}_{=\alpha}\left(-\frac{\mu}{\alpha}t\right)\right] \\ &= \underbrace{\frac{\partial u^i\left(\overline{\mathbf{x}}\right)}{\partial x_k}}_{>0}\underbrace{(1-\mu)}_{>0} t > 0. \end{aligned}$$

Similarly, for j, we have

$$du^j = \frac{\partial u^j\left(\overline{\mathbf{x}}\right)}{\partial x_k}(-t) + \frac{\partial u^j\left(\overline{\mathbf{x}}\right)}{\partial x_\ell}\left(\frac{\mu}{\alpha}t\right) = \frac{\partial u^j\left(\overline{\mathbf{x}}\right)}{\partial x_k}\left[(-t) + \underbrace{\frac{\partial u^j\left(\overline{\mathbf{x}}\right)/\partial x_\ell}{\partial u^j\left(\overline{\mathbf{x}}\right)/\partial x_k}}_{=\beta}\left(\frac{\mu}{\alpha}t\right)\right]$$

$$= \underbrace{\frac{\partial u^j(\overline{\mathbf{x}})}{\partial x_k}}_{>0} \underbrace{\left(\frac{\beta}{\alpha}\mu - 1\right)}_{>0} t > 0.$$

All other agent's utilities remain unchanged so that above implies that $\hat{\mathbf{x}}$ is a feasible allocation that leaves i and j strictly better off relative to $\overline{\mathbf{x}}$—contradicting that $\overline{\mathbf{x}}$ was Pareto efficient. Therefore, it follows that there exists $\alpha_{ij} > 0$ such that

$$\nabla u^i\left(\bar{\mathbf{x}}^i\right) = \alpha_{ij} \nabla u^j\left(\bar{\mathbf{x}}^j\right), \ \forall i, j \in \mathcal{I}.$$

In an Edgeworth box economy, an interior Pareto efficient allocation (since $\overline{\mathbf{x}} \gg \mathbf{0}$) is one in which the condition each agent's marginal rate of substitution between any pairs of goods are equalized.

Here is the proof that uses the hint in the book.

Fact 7.1. *For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, Cauchy-Schwarz inequality says that*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|,$$

where the inequality holds with equality if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent—i.e. if one vector is a scalar multiple of another; in other words, if $\mathbf{x}$ and $\mathbf{y}$ are proportional, and $|\mathbf{x} \cdot \mathbf{y}|$ is a dot/scalar product and $\|\mathbf{x}\|$ is the norm of the vector:

$$|\mathbf{x} \cdot \mathbf{y}| = \left|\sum_{k=1}^{n} x_k y_k\right|,$$
$$\|\mathbf{x}\| = \sqrt{\sum_{k=1}^{n} x_k^2}.$$

We proceed by a way of contradiction. In particular, we will show that if there exists two consumers, say i and j, for whom the gradient vectors $\nabla u^i\left(\bar{\mathbf{x}}^i\right)$ and $\nabla u^j\left(\bar{\mathbf{x}}^j\right)$ at the Pareto efficient allocation are not proportional, then we can improve the utility of at least one of them without affecting others utility, which would contradict the fact that the allocation is Pareto efficient.
Suppose, toward a contradiction, that $\nabla u^i\left(\bar{\mathbf{x}}^i\right)$ and $\nabla u^j\left(\bar{\mathbf{x}}^j\right)$ are not proportional. Then, by Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\cdot\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right| &= \nabla u^i\left(\bar{\mathbf{x}}^i\right)\cdot\nabla u^j\left(\bar{\mathbf{x}}^j\right)\\ &< \left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|, \qquad (7.1)\end{aligned}$$

where the equality uses the fact that we are given that the two gradient vectors $\nabla u^i\left(\bar{\mathbf{x}}^i\right)$ and $\nabla u^j\left(\bar{\mathbf{x}}^j\right)$ have strictly positive elements.
Now, consider the quantity $\mathbf{z}\in\mathbb{R}^n$ defined as

$$\mathbf{z} \coloneqq \frac{\nabla u^i\left(\bar{\mathbf{x}}^i\right)}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\nabla u^j\left(\bar{\mathbf{x}}^j\right)}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|}$$

and the allocation $\tilde{\mathbf{x}} = \left(\tilde{\mathbf{x}}^1,\ldots,\tilde{\mathbf{x}}^I\right)$, where

$$\begin{aligned}&\tilde{\mathbf{x}}^k = \bar{\mathbf{x}}^k,\ \forall k\neq i, k\neq j,\\ &\tilde{\mathbf{x}}^i = \bar{\mathbf{x}}^i + t\mathbf{z},\\ &\tilde{\mathbf{x}}^j = \bar{\mathbf{x}}^j - t\mathbf{z}\end{aligned}$$

for some $t\in\mathbb{R}$. Observe that:

- $\tilde{\mathbf{x}}$ is feasible because it is a result of transferring of goods between (solely) i and j, and $\bar{\mathbf{x}}$ is feasible (by the assumption that $\bar{\mathbf{x}}$ is Pareto efficient);
- $\tilde{\mathbf{x}} \gg \mathbf{0}$ for some $t\in\mathbb{R}$ since we are given that $\bar{\mathbf{x}}\gg\mathbf{0}$.

We now consider the change in utility for i and j for some "small" t. We want to find that at least one consumer is strictly better off from such a transfer while the other is at least weakly better off—this would contradict the assumption that $\bar{\mathbf{x}}$ is Pareto efficient.
Since u^i is continuously differentiable in an open set containing $\bar{\mathbf{x}}^i$, for sufficiently small t, the following derivative is well-defined:

$$
\begin{aligned}
\frac{\partial u^i\left(\tilde{\mathbf{x}}^i\right)}{\partial t} &= \sum_{k=1}^{n} \frac{\partial u^i\left(\tilde{\mathbf{x}}^i\right)}{\partial \tilde{x}_k^i} \frac{\partial \tilde{x}_k^i}{\partial t} = \sum_{k=1}^{n} \frac{\partial u^i\left(\tilde{\mathbf{x}}^i\right)}{\partial \tilde{x}_k^i} z_k \\
&= \nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \mathbf{z} = \nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \left(\frac{\nabla u^i\left(\bar{\mathbf{x}}^i\right)}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\nabla u^j\left(\bar{\mathbf{x}}^j\right)}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|} \right) \\
&= \frac{\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^i\left(\bar{\mathbf{x}}^i\right)}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^j\left(\bar{\mathbf{x}}^j\right)}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|} \\
&= \frac{\left\|\nabla u^i\left(\tilde{\mathbf{x}}^i\right)\right\|^2}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^j\left(\bar{\mathbf{x}}^j\right)}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|},
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^i\left(\bar{\mathbf{x}}^i\right) &= \sum_{k=1}^{n} \left(\frac{\partial u^i\left(\tilde{\mathbf{x}}^i\right)}{\partial \tilde{x}_k^i} \right)^2 = \left(\sqrt{\sum_{k=1}^{n} \left(\frac{\partial u^i\left(\tilde{\mathbf{x}}^i\right)}{\partial \tilde{x}_k^i} \right)^2} \right)^2 \\
&= \left\|\nabla u^i\left(\tilde{\mathbf{x}}^i\right)\right\|^2.
\end{aligned}
$$

From (**??**), we know that

$$
\begin{aligned}
-\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^j\left(\bar{\mathbf{x}}^j\right) &= -\left|\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^j\left(\bar{\mathbf{x}}^j\right)\right| \\
&> -\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\| \left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|.
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial u^i\left(\tilde{\mathbf{x}}^i\right)}{\partial t} &= \frac{\left\|\nabla u^i\left(\tilde{\mathbf{x}}^i\right)\right\|^2}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\nabla u^i\left(\tilde{\mathbf{x}}^i\right) \cdot \nabla u^j\left(\bar{\mathbf{x}}^j\right)}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|} \\
&> \frac{\left\|\nabla u^i\left(\tilde{\mathbf{x}}^i\right)\right\|^2}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\| \left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|} \\
&= \left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\| - \left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\| = 0.
\end{aligned}
$$

For consumer j,

$$
\begin{aligned}
\frac{\partial u^j\left(\tilde{\mathbf{x}}^j\right)}{\partial t} &= -\nabla u^j\left(\tilde{\mathbf{x}}^j\right) \cdot \mathbf{z} = -\nabla u^j\left(\tilde{\mathbf{x}}^j\right) \cdot \left(\frac{\nabla u^i\left(\bar{\mathbf{x}}^i\right)}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \frac{\nabla u^j\left(\bar{\mathbf{x}}^j\right)}{\left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|} \right) \\
&= -\left(\frac{\nabla u^j\left(\tilde{\mathbf{x}}^j\right) \cdot \nabla u^i\left(\bar{\mathbf{x}}^i\right)}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\| \right) \\
&> -\left(\frac{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\| \left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\|}{\left\|\nabla u^i\left(\bar{\mathbf{x}}^i\right)\right\|} - \left\|\nabla u^j\left(\bar{\mathbf{x}}^j\right)\right\| \right) = 0.
\end{aligned}
$$

Thus, we established that, for some neighborhood of allocation $\bar{\mathbf{x}}$, there is a transfer between individuals i and j (only) that is both feasible and strictly increases the utilities of both individuals. This contradicts that $\bar{\mathbf{x}}$ is Pareto efficient.

7.2 Part (b)

Define $\bar{\mathbf{p}} = \nabla u^1\left(\bar{\mathbf{x}}^1\right) \gg \mathbf{0}$. Show that, for every consumer i, there exists $\lambda_i > 0$ such that $\nabla u^i\left(\bar{\mathbf{x}}^i\right) = \lambda_i \bar{\mathbf{p}}$.

$$\cdots\cdots$$

From Part (a),

$$\begin{aligned} \bar{\mathbf{p}} = \nabla u^1\left(\bar{\mathbf{x}}^1\right) &= \alpha_{1j} \nabla u^j\left(\bar{\mathbf{x}}^j\right), \ \forall j \in \mathcal{I}. \\ \Rightarrow \frac{1}{\alpha_{1j}} \bar{\mathbf{p}} &= \nabla u^j\left(\bar{\mathbf{x}}^j\right), \ \forall j \in \mathcal{I}. \end{aligned}$$

Define $\lambda_j \coloneqq 1/\alpha_{1j}$ for all $j \in \mathcal{I}$. Then, we have that

$$\lambda_i \bar{\mathbf{p}} = \nabla u^i\left(\bar{\mathbf{x}}^i\right), \ \forall i \in \mathcal{I}.$$

7.3 Part (c)

Use Theorem 1.4 to argue that, for every consumer i, $\bar{\mathbf{x}}^i$ solves

$$\max_{\mathbf{x}^i} \quad u^i\left(\mathbf{x}^i\right) \quad s.t. \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}}^i \cdot \bar{\mathbf{x}}^i.$$

$$\cdots\cdots$$

> **Theorem 1.4** *(Sufficiency of consumer's first-order conditions)* Consider the following problem
>
> $$\max_{\mathbf{x}^i} \quad u(\mathbf{x}) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x} \leq y,$$
>
> where the Lagrangian is given by
>
> $$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(\mathbf{p} \cdot \mathbf{x} - y).$$
>
> Suppose that $u(\mathbf{x})$ is continuous and quasiconcave on $\mathbb{R}^n_+$, and that $(\mathbf{p}, y) \gg \mathbf{0}$. If u is differentiable at $\mathbf{x}^*$, and $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ solves
>
> $$\frac{\partial u(\mathbf{x}^*)}{\partial x_k} - \lambda^* p_k = 0, \forall k = 1, 2, \ldots, n, \tag{7.2}$$
>
> $$\mathbf{p} \cdot \mathbf{x}^* = y. \tag{7.3}$$
>
> Then, $\mathbf{x}^*$ solves the consumer's maximisation problem at prices $\mathbf{p}$ and income y.

We are given in the question that each $u^i\left(\mathbf{x}^i\right)$ is continuous and quasiconcave $\mathbb{R}^n_+$ and that each u^i is differentiable at $\bar{\mathbf{x}}^i$. Let $\bar{\mathbf{p}} \coloneqq \nabla u^1\left(\bar{\mathbf{x}}^1\right) \gg \mathbf{0}$. Together with the assumption that $\bar{\mathbf{x}} \gg \mathbf{0}$, we know that $\bar{\mathbf{p}}^i \cdot \bar{\mathbf{x}}^i > 0$. Define

$$y = \bar{\mathbf{p}}^i \cdot \bar{\mathbf{x}}^i > 0.$$

Then, $(\mathbf{p}, y) \gg \mathbf{0}$.

From part (b), we know that,

$$\lambda_i \bar{\mathbf{p}} = \nabla u^i\left(\bar{\mathbf{x}}^i\right), \forall i \in \mathcal{I},$$

where $\lambda_i = 1/\alpha_{1i} > 0$ for all $i \in \mathcal{I}$. Hence, $\left(\bar{\mathbf{x}}^i, \lambda_i\right) \gg \mathbf{0}$. We also have that $\bar{\mathbf{x}}^i$ satisfies (7.3):

$$\bar{\mathbf{p}}^i \cdot \bar{\mathbf{x}}^i = y = \bar{\mathbf{p}}^i \cdot \bar{\mathbf{x}}^i.$$

By Theorem 1.4, therefore, we know that $\bar{\mathbf{x}}^i$ is the solution to the problem given in the question for each $i \in \mathcal{I}$.

What did we just show? We have shown that, if agents start with the endowments corresponding to any (strictly positive) Pareto efficient allocation, then there exists a strictly positive price vector $\overline{\mathbf{p}}$ such that each consumer's optimal choice is to consume his endowment, given these prices and the market value of said endowment. Markets also, trivially, clear in this case (since a Pareto efficient allocation is feasible by definition) and so $\overline{\mathbf{p}}$ is a Walrasian equilibrium price vector that supports the Pareto efficient allocation as a Walrasian equilibrium allocation.

This is precisely a restatement, under different conditions and so a different version, of the Second Welfare Theorem as proved in class. As shown in class, we can always define appropriate lump-sum transfers to move from any "initial" economy, to a new economy in which individuals' endowments correspond to any arbitrary Pareto efficient allocation.

8 Exercise 5.28

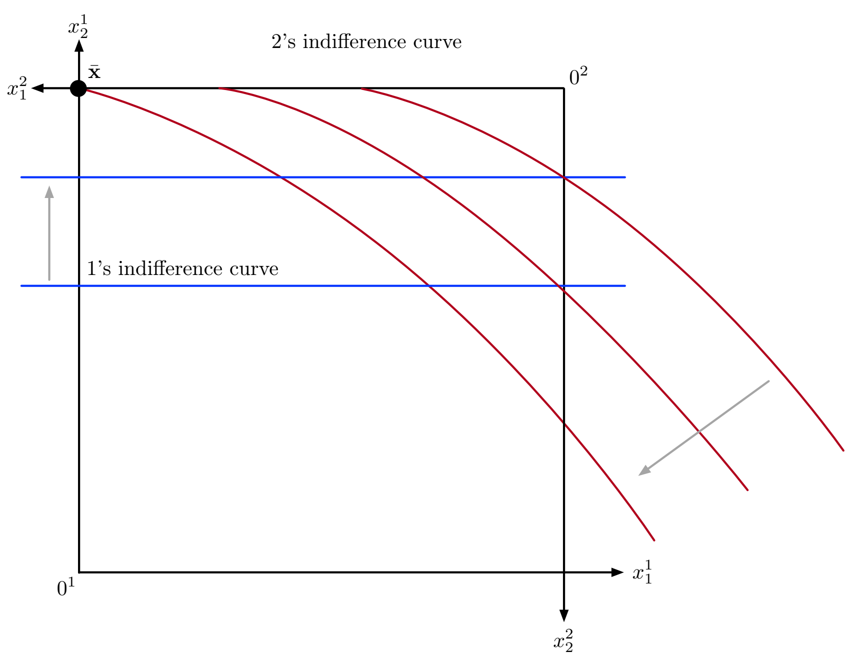
Suppose that all of the conditions in Exercise 5.27 hold, except the strict positive of $\bar{\mathbf{x}}$ and the consumers' gradient vectors. Using an Edgeworth box, provide an example showing that, in such a case, it may not be possible to support $\bar{\mathbf{x}}$ as a Walrasian equilibrium allocation. Because Theorem 5.8 does not require $\bar{\mathbf{x}}$ to be strictly positive, which hypothesis of Theorem 5.8 does your example violate?

......

Theorem 5.8 Consider an exchange economy $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i \in \mathcal{I}}$ with aggregate endowment $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, and with each utility function u^i satisfying Assumption 5.1.[a] Suppose that $\bar{\mathbf{x}}$ is a Pareto-efficient allocation for $\mathcal{E}$, and that endowments are redistributed so that the new endowment vector is $\bar{\mathbf{x}}$. Then, $\bar{\mathbf{x}}$ is a Walrasian equilibrium allocation of the resulting exchange economy $\bar{\mathcal{E}} = \left(u^i, \bar{\mathbf{x}}^i\right)_{i \in \mathcal{I}}$.

[a] Each u^i is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

Consider the case in which the endowment, $\bar{\mathbf{x}}$, lies on the boundary of the Edgeworth box so that consumer 1 has all the endowment of good 2 while consumer 2 has all the endowment of good 1. Preferences are such that consumer 2 only values good 1 while consumer 1's indifferent curve has an infinite slope (his marginal utility for good 1 is infinite) whenever he consumes zero amounts of good 1. A Walrasian equilibrium does not exist in this case because, although consumer 2 is willing to pay an "infinite" price for good 1 from consumer 1 by selling good 2, consumer 1 will not accept such a trade because he does not value good 2. Consumer 1 will always demand some of good 1 while consumer 2's demand for good 1 is one. This means that market for good 1 will never clear.



As a particular example, consider the following utility functions

$$u^1\left(x_1^1, x_2^1\right) = \sqrt{x_1^1} + x_2^1,$$
$$u^2\left(x_1^2, x_2^2\right) = x_1^2,$$

with the endowment

$$\bar{\mathbf{x}} = \left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right) = \left(\left(0,1\right), \left(1,0\right)\right).$$

Observe that $\bar{\mathbf{x}}$ is Pareto efficient since there is no "lens" formed at this endowment point. To increase u^2, we must either increase x_1^2 or x_2^2 but this would either violate feasibility condition or lower u^2. Similarly, to increase u^1, we must increase x_2^2 but this would violate feasibility.

Observe that 1's marginal utility for good 1 is given by

$$\frac{\partial u^1\left(x_1^1, x_2^1\right)}{\partial x_1} = \frac{1}{2\sqrt{x_1^1}}$$

and the marginal rate of substitution is given by

$$\frac{\partial u^1\left(x_1^1, x_2^1\right)/\partial x_1}{\partial u^1\left(x_1^1, x_2^1\right)/\partial x_2} = \frac{1}{2\sqrt{x_1^1}}.$$

Hence, the indifference curve for 1 has infinite slope when $x_1^1 = 0$ and the marginal utility for good one is infinite at this point too. This is an example of a violation of the Second Welfare Theorem (Theorem 5.8). Notice that u^2 does not satisfy Assumption 5.1 (it is neither strongly increasing nor strictly quasiconcave).

Since u^2 is strongly increasing, it follows that, in any equilibrium, prices have to be strictly positive. Moreover, the form of utility for consumer 2 means that his marginal utility is "infinite" when he consumers zero units of good 2. Hence the indifference curve we observe for consumer 2 in the figure. This means that consumer 2 is willing to pay "infinite" price for good 2 from consumer 1 by selling good 1; however, since consumer 1 does not value good 1, there is no price at which consumer 1 is willing to participate in such a trade.

9 Exercise 5.32

Consider a simple economy with two consumers, a single consumption good x, and two time periods. Consumption of the good in period t is denoted x_t for $t = 1, 2$. Intertemporal utility functions for the two consumers are

$$u_i(x_1, x_2) = x_1 x_2, \ i = 1, 2,$$

and endowments are $\mathbf{e}^1 = (19, 1)$ and $\mathbf{e}^2 = (1, 9)$. To capture the idea that the good is perfectly storable, we introduce a firm producing storage services. The firm can transform one unit of the good in period one into one unit of the good in period 2. Hence, the production set Y is the set of all vectors $(y_1, y_2) \in \mathbb{R}^2$ such that $y_1 + y_2 \leq 0$ and $y_1 \leq 0$. Consumer 1 is endowed with a 100 per cent ownership share of the firm.

9.1 Part (a)

Suppose the two consumers cannot trade with one another. That is, suppose that each consumer is in a Robinson Crusoe economy and where consumer 1 has access to his storage firm. How much does each consumer consume in each period? How well off is each consumer? How much storage takes place?

......

Let p_1 and p_2 denote the prices of consumption in periods 1 and 2 respectively. Suppose that one of the prices were zero. Since endowments are component-wise strictly larger than zero, if any one of the prices were zero, the consumer will ensure that he consumes some positive amount of the good that has a positive price, and demand infinite amount of good for which the price is zero. Since the endowments are bounded, this cannot be an equilibrium. We may therefore conclude that $p_1, p_2 > 0$. As usual, we may normalize price so that $p_1 = 1$ and $p_2 = p$.

Consumer 2 Since consumer 2 has no access to storage, there is no way for him to transfer goods from period one to period two; in other words, he lives in an exchange economy all by himself. His problem is therefore given by

$$\max_{\{x_1^2, x_2^2\} \in \mathbb{R}_+^2} x_1^2 x_2^2 \quad s.t. \quad x_1^2 = 1, x_2^2 = 9.$$

Since his utility is strictly increasing, it follows that he consumers all his endowments in each period; i.e.

$$\begin{aligned} \left(x_1^2, x_2^2\right) &= \mathbf{e}^2 = (1, 9), \\ u^1\left(\mathbf{e}^2\right) &= 9. \end{aligned}$$

Consumer 1 The firm's profit maximisation problem is

$$\Pi(p) := \max_{(\hat{y}_1, y_2) \in \mathbb{R}_+^2} p y_2 - \hat{y}_1 \quad s.t. \quad y_2 \leq \hat{y}_1,$$

where $\hat{y}_1 := -y_1$ (this seems more intuitive to me). Since $p > 0$ in any equilibrium, at the optimum, the constraint must bind, which allows us to write

$$\Pi(p) := \max_{y_2 \in \mathbb{R}_+} (p - 1) y_2.$$

Consider three cases:

- $p < 1$: if the firm chooses $y_2 > 0$, then the firm makes strictly negative profits. Thus, firm sets $y_2 = 0$ and so $\Pi(p) = 0$.

- $p = 1$: profit is zero and the firm is indifferent between any $y_2 \in \mathbb{R}_+$.

- $p > 1$: firm sets y_2 as high as possible so that $\Pi(p) \simeq \infty$.

Thus, the firm's supply is given by

$$y_2 = -y_1 \begin{cases} = 0 & \text{if } p < 1 \\ \in [0, \infty) & \text{if } p = 1 \\ \simeq \infty & \text{if } p > 1 \end{cases}.$$

In equilibrium, the market must clear. If $p > 1$, above tells us that the firm will want to produce ∞ of good 2 which requires ∞ amount of good 1. However, there is only finite amount of good 1 in the economy so that market cannot clear for good 1 if $p > 1$. Thus, in any equilibrium we must have $p \leq 1$. Note that $0 < p \leq 1$ are candidate equilibrium price although at $p < 1$, the firm does not operate.

Given the consumer's preferences (Cobb Douglas with equal weights on each good) and that $p = 1$ from above, it should be that the optimal consumption bundle is $(10, 10)$. Let us formally show this.

Consider consumer 1's budget constraint. In period 1, he may spend his endowment income on consumption good, x_1^1, or he can "save" by supplying s^1 units to the firm (equals to $-y_1^1$ from the perspective of the firm). Hence, his period-1 budget constraint is given by

$$x_1^1 + s^1 = e_1^1.$$

In period 2, he receives $p_1 s^1 / p_2$ units of consumption good from the firm, each unit of which is worth p_2, and he also has his endowment and profits from the firm. Thus, his period-2 budget constraint is given by

$$px_2^1 = pe_2^1 + s^1 + p\Pi(p)$$

We can therefore combine the budget constraints to obtain

$$\begin{aligned} x_1^1 + \left(px_2^1 - pe_2^1 - p\Pi(p)\right) &= e_1^1 \\ \Rightarrow x_1^1 + p\left(x_2^1 - e_2^1\right) &= e_1^1 + p\Pi(p) \end{aligned}$$

Then, consumer 1's problem is given by

$$\max_{\{x_1^1, x_2^1\} \in \mathbb{R}_+^2} x_1^1 x_2^1 \quad s.t. \quad x_1^1 + p\left(x_2^1 - e_2^1\right) = e_1^1 + p\Pi(p)$$

The first-order conditions imply

$$x_2^1 = \lambda, \; x_1^1 = p\lambda \Rightarrow \frac{x_1^1}{x_2^1} = p,$$

where λ is the Lagrange multiplier on the budget constraint (given Cobb-Douglas utility, we know that the optimal must be interior so that the first-order conditions must hold with equality).

The demand for goods and storage services are given by

$$\begin{aligned} x_1^1 &= \frac{1}{2}\left[e_1^1 + p\left(e_2^1 + \Pi(p)\right)\right] = \frac{1}{2}\left[19 + p\left(1 + \Pi(p)\right)\right], \\ x_2^1 &= \frac{1}{2}\left[\frac{1}{p}e_1^1 + e_2^1 + \Pi(p)\right] = \frac{1}{2}\left[\frac{1}{p}19 + 1 + \Pi(p)\right], \\ s^1 = e_1^1 - x_1^1 &= \frac{1}{2}\left[e_1^1 - p\left(e_2^1 + \Pi(p)\right)\right] = \frac{1}{2}\left[19 - p\left(1 + \Pi(p)\right)\right]. \end{aligned}$$

Consider $p = 1$. Since $\Pi(1) = 0$, then

$$
\begin{aligned}
x_1^1 &= x_2^1 = 10, \\
s^1 &= 19 - 10 = 9 \geq 0, \\
u^1(10, 10) &= 100.
\end{aligned}
$$

To show that market clears:

$$
\begin{aligned}
19 &= e_1^1 = x_1^1 + s^1 = 10 + 9, \\
1 + 9 &= e_2^1 + s^1 = x_2^1 = 10,
\end{aligned}
$$

(by Walras' law, we did not need to check for clearing in the second good.) Hence, we conclude that $p = 1$ is a Walrasian equilibrium.

Remark 9.1. We can also show that $p = 1$ in any Walrasian equilibrium. We already argued above that $p > 1$ cannot be a Walrasian equilibrium. Toward a contradiction, suppose $p < 1$ in some Walrasian equilibrium, then $\Pi(p) = 0$ still. But, in this case, the storage firm does not operate. Thus, it must be the

$$
0 = s^1 = \frac{1}{2}(19 - p) \Rightarrow p = 19.
$$

Hence, at $p < 1$ the market for storage services does not clear; i.e. a contradiction.

9.2 Part (b)

Now suppose the two consumers together with consumer 1's storage firm constitute a competitive production economy. What are the Walrasian equilibrium prices, p_1 and p_2? How much storage takes place now?

......

As before, normalize $(p_1, p_2) = (1, p)$. The firm's problem is the same as before.

Given the symmetry between the two consumers, consumer 2's demand function is given by

$$
\begin{aligned}
x_1^2 &= \frac{1}{2}(1 + 9p), \\
x_2^2 &= \frac{1}{2}\left(\frac{1}{p} + 9\right), \\
x_1^2 - e_1^2 &= \frac{1}{2}(1 - 9p).
\end{aligned}
$$

Consider $p = 1$, then

$$
\begin{aligned}
x_1^2 &= x_2^2 = 5, \\
x_1^1 &= x_2^1 = 10.
\end{aligned}
$$

and aggregate storage is

$$
\left(x_1^2 - e_1^2\right) + \left(x_1^1 - e_1^1\right) = 9 - 4 = 5.
$$

To show that the market clears,

$$
19 + 1 = e_1^1 + e_1^2
$$

$$\begin{aligned} &= x_1^1 + x_1^2 + \left[\left(x_1^2 - e_1^2 \right) + \left(x_1^1 - e_1^1 \right) \right] \\ &= 10 + 5 + 5 = 20, \end{aligned}$$

and market for good 2 clears by Walras' law. In fact, since the storage technology is one way (you can only save for the future, not borrow against the future because of the restriction $y_1 \leq 0$), we know that consumer 1 sells 4 units of goods in period 1 to consumer 2, and store 5 units of goods, which he consumes in period 2. In this case,

$$\begin{aligned} u_1 (10, 10) &= 100, \\ u_2 (5, 5) &= 25. \end{aligned}$$

We are asked to find *the* Walrasian equilibrium prices. Hence, we must show that $p = 1$ is the unique Walrasian equilibrium. Since firm's demand for the good is unbounded if $p > 1$ as before, in any Walrasian equilibrium, we must have $p < 1$. So consider $p < 1$, then the firm does not operate and so $\Pi (p) = 0$. Market clearing requires

$$\begin{aligned} 20 = e_1^1 + e_1^2 = x_1^1 + x_1^2 &= \frac{1}{2} (19 + p) + \frac{1}{2} (1 + 9p) \\ \Rightarrow p &= 2, \end{aligned}$$

which contradicts our assumption that $p < 1$. Hence, $p < 1$ in any Walrasian equilibrium.

9.3 Part (c)

Interpret p_1 as a spot price and p_2 as a futures price.

......

In period 1, consumers trade the consumption good at price p_1. Hence, p_1 can be interpreted as the spot-market price. The storage service can be interpreted as the agent purchasing a futures contract in period 1 that promises a delivery of one unit of good in period 2. Recall the budget constraint:

$$x_1^1 + p \left(x_2^1 - e_2^1 \right) = e_1^1 + p\Pi (p),$$

where $x_2^1 - e_2^1$ is the promised delivery of goods in the future. Hence, we can see that $p = p_2$ is the price of such a contract; in other words, it is the price of good in period 1 in good 1 terms.

9.4 Part (d)

Repeat the exercise under the assumption that storage is costly; i.e. that Y is the set of vectors $(y_1, y_2) \in \mathbb{R}^2$ such that $\delta y_1 + y_2 \leq 0$ and $y_1 \leq 0$, where $\delta \in [0, 1)$. Show that the existence of spot and futures markets now make both consumers strictly better off.

......

9.4.1 Part (a')

Consumer 2 As before consumer 2 has no access to the storage technology so

$$\begin{aligned} \left(x_1^2, x_2^2 \right) &= \mathbf{e}^2 = (1, 9), \\ u^1 \left(\mathbf{e}^2 \right) &= 9. \end{aligned}$$

Consumer 1 The firm's profit maximisation problem is

$$\Pi(\mathbf{p}) := \max_{(y_1^1, y_2^1) \in \mathbb{R}^2} y_1 + py_2 \quad s.t. \quad \delta y_1 + y_2 \leq 0, y_1 \leq 0.$$

Notice that 1 unit of input y_2 turns into, at most, δ units of output y_2, while 1 unit of input costs 1 and bring in revenue of at most δp. Thus, if $1 > \delta p$, firm will set $y_1 = y_2 = 0$ since purchasing any strictly positive amount will lead to negative profits. So, we now have

- $p < \frac{1}{\delta}$: if the firm chooses $y_1 < 0$, then the firm makes strictly negative profits given $y_1 + y_2 \leq 0$. Thus, firm chooses to set $y_1 = 0$, which implies that $y_2 = 0$ and $\Pi(p) = 0$.
- $p = 1$: the firm sets $y_2 = -\delta y_1$ and the profit is zero for any feasible values of y_1 and $\Pi(p) = 0$.
- $p > 1$: firm sets y_2 as high as possible and y_1 as low as possible to satisfy the constraints so that $\Pi(p) \simeq \infty$.

Thus, the firm's supply is given by

$$y_2 = -\delta y_1 \begin{cases} = 0 & \text{if } p < 1/\delta \\ \in [0, \infty) & \text{if } p = 1/\delta \\ \simeq \infty & \text{if } p > 1/\delta \end{cases}$$

In equilibrium, the market must clear. Thus, we can immediately rule out the case $p > 1/\delta$ given that endowments are bounded. Note that $0 < p \leq 1$ are candidate equilibrium price although at $p < 1$, the firm does not operate.

Using the demand functions that we found before, if $p = 1/\delta$, then $\Pi(p) = 0$, and

$$\begin{aligned} x_1^1 &= \frac{1}{2}\left(19 + \frac{1}{\delta}\right), \\ x_2^1 &= \frac{1}{2}(19\delta + 1), \\ s^1 &= \frac{1}{2}\left(19 - \frac{1}{\delta}\right). \end{aligned}$$

For s^1 to be positive, we need

$$s^1 \geq 0 \Leftrightarrow \delta \geq \frac{1}{19}.$$

To verify that the market clears:

$$\begin{aligned} 19 = e_1^1 &= x_1^1 + s^1 \\ &= \frac{1}{2}\left(19 + \frac{1}{\delta}\right) + \frac{1}{2}\left(19 - \frac{1}{\delta}\right) = 19, \end{aligned}$$

and the market for second good clears by Walras' law. Thus, we realize that $p = 1/\delta$ is a Walrasian equilibrium if $\delta \geq 1/19$ (if $\delta = 1/19$, then $s^1 = 0$ so that consumer 1 does not use the storage service). In this case,

$$\begin{aligned} x_1^1 &= \frac{1}{2}\left(19 + \frac{1}{\delta}\right) \in (10, 19], \\ x_2^1 &= \frac{1}{2}(19\delta + 1) \in [1, 10), \\ u_1\left(x_1^1, x_2^1\right) &= \frac{1}{4}\left(19 + \frac{1}{\delta}\right)(19\delta + 1) \in [19, 100). \end{aligned}$$

If $\delta < 1/19$, then there is no equilibrium with $p = 1/\delta$.

Now, suppose that $p < 1/\delta$, then firm does not operate and $\Pi(p)$, then market clearing requires

$$\begin{aligned} 19 = e_1^1 = x_1^1 &= \frac{1}{2}(19+p) \\ \Rightarrow p = 19 &< \frac{1}{\delta}. \end{aligned}$$

Hence, there is an equilibrium with $p = 19$ if $\delta < 1/19$. In this case,

$$\begin{aligned} x_1^1 &= \frac{1}{2}(19+19) = 19, \\ x_2^1 &= \frac{1}{2}\left(\frac{1}{19}19+1\right) = 1, \\ u_1\left(x_1^1, x_2^2\right) &= 19. \end{aligned}$$

9.4.2 Part (b')

Suppose $p = 1/\delta$, consumer 2's demand function is given by

$$\begin{aligned} x_1^2 &= \frac{1}{2}(1+9p) = \frac{1}{2}\left(1+\frac{9}{\delta}\right) \\ x_2^2 &= \frac{1}{2}\left(\frac{1}{p}+9\right) = \frac{1}{2}(\delta+9), \\ x_1^2 - e_1^2 &= \frac{1}{2}(1-9p) = \frac{1}{2}\left(1-\frac{9}{\delta}\right). \end{aligned}$$

We see that good market 1 clears:

$$\begin{aligned} 20 = e_1^1 + e_1^2 &= x_1^1 + x_1^2 + \left[\left(x_1^1 - e_1^1\right) + \left(x_1^2 - e_1^2\right)\right] \\ &= \frac{1}{2}\left(19+\frac{1}{\delta}\right) + \frac{1}{2}\left(1+\frac{9}{\delta}\right) + \left[\frac{1}{2}\left(19-\frac{1}{\delta}\right) + \frac{1}{2}\left(1-\frac{9}{\delta}\right)\right] \\ &= 20, \end{aligned}$$

and by Walras' law, so does good market 2. Note that we need the storage service use to be of positive amount:

$$\begin{aligned} \left(x_1^1 - e_1^1\right) + \left(x_1^2 - e_1^2\right) &\geq 0 \\ \Leftrightarrow \frac{1}{2}\left(19-\frac{1}{\delta}\right) + \frac{1}{2}\left(1-\frac{9}{\delta}\right) &\geq 0 \\ \Rightarrow \delta &\geq \frac{1}{2}. \end{aligned}$$

There is therefore a competitive equilibrium with $p = 1/\delta$ if $\delta \geq 1/2$. In this case,

$$\begin{aligned} x_1^1 &= \frac{1}{2}\left(19+\frac{1}{\delta}\right) \in (10, 10.5], \\ x_2^1 &= \frac{1}{2}(19\delta+1) \in [5.25, 10), \\ u_1\left(x_1^1, x_2^1\right) &= \frac{1}{4}\left(19+\frac{1}{\delta}\right)(19\delta+1) \in [55.125, 100) \\ x_1^2 &= \frac{1}{2}\left(1+\frac{9}{\delta}\right) \in (5, 9.5], \end{aligned}$$

$$x_2^2 = \frac{1}{2}(\delta+9) \in [4.75, 5)$$
$$u_2\left(x_1^1, x_1^2\right) = \frac{1}{4}\left(1+\frac{9}{\delta}\right)(\delta+9) \in (25, 45.125]$$

Suppose $p < 1/\delta$, the firm does not operate and $\Pi(p) = 0$. So, market clearing implies

$$20 = e_1^1 + e_1^2 = x_1^1 + x_1^2 = \frac{1}{2}(19+p) + \frac{1}{2}(1+9p)$$
$$\Rightarrow p = 2 < \frac{1}{\delta}.$$

In this case,

$$x_1^1 = \frac{1}{2}(19+2) = 10.5,$$
$$x_2^1 = \frac{1}{2}\left(\frac{1}{2}19+1\right) = 5.25,$$
$$u_1\left(x_1^1, x_2^2\right) = 55.125,$$
$$x_1^2 = \frac{1}{2}(1+9\times 2) = 9.5,$$
$$x_2^2 = \frac{1}{2}\left(\frac{1}{2}+9\right) = 4.75,$$
$$u_2\left(x_1^2, x_2^2\right) = 45.125.$$

Thus, there is an equilibrium with $p = 2$ if $\delta < 0.5$.

We summarise the result in the table below.

Table 1: Summary.

	Without spot market		With spot market		
	Consumer 1	Consumer 2	Consumer 1	Consumer 2	Comparison
$\delta \in [0.5, 1)$	$[55.125, 100)$	9	$[55.125, 100)$	$(25, 45.125]$	2 is strictly better, 1 is indifferent
$\delta \in [1/19, 0.5)$	$[19, 55.125)$	9	55.125	45.125	Both are strictly better off
$\delta \in [0, 1/19)$	19	9	55.125	45.125	Both are strictly better off
$\delta = 1$	100	9	100	25	2 is strictly better, 1 is indifferent

10 Exercise 5.33

The contingent-commodity interpretation of our general equilibrium model permits us to consider time (as in the previous exercise) as well as uncertainty and more (e.g. location). While the trading of contracts nicely captures the idea of futures contracts and prices, one might wonder about the role that spot markets play in our theory. This exercise will guide you through thinking about this.

The main result is that, once the date zero contingent-commodity contracts market has cleared at Walrasian prices, there is no remaining role for spot markets. Even if spot markets were to open up for some or all goods in some or all periods, and in some or all states of the world, no additional trade would take place. All agents would simply exercise the contracts they already have in hand.

10.1 Part (a)

Consider an exchange economy with I consumers, N goods, and $T = 2$ dates. There is no uncertainty. We will focus on one consumer whose utility function is $u(\mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_t \in \mathbb{R}^N_+$ is a vector of period-t consumption of the good N goods.

Suppose that $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$ is a Walrasian equilibrium price vector in the contingent commodity sense described in Section 5.4, where $\hat{\mathbf{p}}_t \in \mathbb{R}^N_{++}$ is the price vector for period-t contracts on the N goods. Let $\hat{\mathbf{x}} = (\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ be the vector of contracts that our consumer purchases prior to date 1 given the Walrasian equilibrium price vector $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$.

Suppose now that, at each date t, spot market open for trade.

10.1.1 Part (i)

Because all existing contracts are enforced, argue that our consumer's available endowment in period t is $\hat{\mathbf{x}}_t$.

......

In period 0, consumer's agree to a contract that defines their consumption of N goods in each period $t = 1, 2$. Thus, in each period t, consumers have exactly $\hat{\mathbf{x}}_t$ goods to consume.

10.1.2 Part (ii)

Show that, if our consumer wishes to trade in some period-t spot market, and if all good have period-t spot markets, and the period-t spot prices are $\hat{\mathbf{p}}_t$, then our consumer's period-t budget constraint is,

$$\hat{\mathbf{p}}_t \cdot \mathbf{x}_t \leq \hat{\mathbf{p}}_t \cdot \hat{\mathbf{x}}_t.$$

......

As argued in Part (a)(i), consumer's endowment is $\hat{\mathbf{x}}_t$, which, given period-t spot prices of $\hat{\mathbf{p}}_t$, is worth $\hat{\mathbf{p}}_t \cdot \hat{\mathbf{x}}_t$. Letting $\mathbf{x}_t$ denote a consumption bundle the consumer can purchase on the spot market, the cost of acquiring such a bundle is $\hat{\mathbf{p}}_t \cdot \mathbf{x}_t$. Of course, the consumer cannot spend more than his income, so

$$\hat{\mathbf{p}}_t \cdot \mathbf{x}_t \leq \hat{\mathbf{p}}_t \cdot \hat{\mathbf{x}}_t.$$

10.1.3 Part (iii)

Conclude that our consumer can ultimately choose any $(\mathbf{x}_1, \mathbf{x}_2)$ such that

$$\hat{\mathbf{p}}_1 \cdot \mathbf{x}_1 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1,$$

$$\hat{\mathbf{p}}_2 \cdot \mathbf{x}_2 \leq \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2.$$

......

The budget constraint in Part (a)(ii) is applicable for each $t = 1, 2$. Consumer can choose any bundle that cost less than his income on the spot market.

10.1.4 Part (iv)

Prove that the consumer can do no better than to choose $\mathbf{x}_1 = \hat{\mathbf{x}}_1$ in period $t = 1$ and $\mathbf{x}_2 = \hat{\mathbf{x}}_2$ in period $t = 2$ by showing that any bundle that is feasible through trading in spot markets is feasible in the contingent-commodity contract market. You should assume that, in period 1, the consumer is forward-looking, knows the spot prices he will face in period 2, and that he wishes to behave so as to maximize his lifetime utility $u(\mathbf{x}_1, \mathbf{x}_2)$. Further, assume that, if he consumes $\bar{\mathbf{x}}_1$ in period $t = 1$, his utility of consuming any bundle $\mathbf{x}_2$ in period $t = 2$ is $u(\bar{\mathbf{x}}_1, \mathbf{x}_2)$.

Because the consumer can do no better if there are fewer spot markets open, parts (i)–(iv) show that, if there is a period-t spot market for good k and the period-t spot price of good k is $\hat{p}_{kt}$, then our consumer has no incentive to trade. Since this is true for all consumers, this shows that spot markets clear at prices at which there is no trade.

......

In period $t = 0$, before any spot markets open, the consumer optimizes his life-time utility by choosing $\hat{\mathbf{x}}$, which is a contract that specifies his consumption in each period. Given the Walrasian equilibrium price vector in the contingent-commodity contract market $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$, $\hat{\mathbf{x}} = (\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ is the solution to the following problem:

$$\max_{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^N_+} u(\mathbf{x}_1, \mathbf{x}_2) \quad s.t. \quad \hat{\mathbf{p}}_1 \cdot \mathbf{x}_1 + \hat{\mathbf{p}}_2 \cdot \mathbf{x}_2 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1 + \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2. \tag{10.1}$$

These are decisions that are made ex ante which is why there could be a role for spot markets.

In period $t = 1$, the consumer's problem for the period-1 spot market is given by

$$\max_{\mathbf{x}_1 \in \mathbb{R}^N_+} u(\mathbf{x}_1, \hat{\mathbf{x}}_2) \quad s.t. \quad \hat{\mathbf{p}}_1 \cdot \mathbf{x}_1 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1. \tag{10.2}$$

Since the contract is enforceable, the consumer still would have consumption $\hat{\mathbf{x}}_2$ in period 2. However, he could potentially use his consumption today $\hat{\mathbf{x}}_1$ (which can be thought of as his endowment) in the market sport and trade. Denote the solution to this problem by $\bar{\mathbf{x}}_1$. We want to show that $\bar{\mathbf{x}}_1 = \hat{\mathbf{x}}_1$. Suppose, by way of contradiction, that $\bar{\mathbf{x}}_1 \neq \hat{\mathbf{x}}_1$. Then, this implies that

$$u(\bar{\mathbf{x}}_1, \hat{\mathbf{x}}_2) > u(\bar{\mathbf{x}}_1, \hat{\mathbf{x}}_2).$$

But, since $\bar{\mathbf{x}}_1$ solves (10.2),

$$\hat{\mathbf{p}}_1 \cdot \bar{\mathbf{x}}_1 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1$$
$$\Rightarrow \hat{\mathbf{p}}_1 \cdot \bar{\mathbf{x}}_1 + \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1 + \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2.$$

Together, they imply that $(\bar{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ was a feasible choice under problem (10.1) that gives strict higher utility than the bundle $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$—contradicting the assumption that $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ is the solution to (10.1). Thus, $\bar{\mathbf{x}}_1 = \hat{\mathbf{x}}_1$.

Now consider period $t = 2$. Given the previous result, we know that the consumer consumes $\hat{\mathbf{x}}_1$ in period 1. The consumer's problem for the period-2 spot market is:[4]

$$\max_{\mathbf{x}_2 \in \mathbb{R}^N_+} u\left(\hat{\mathbf{x}}_1, \mathbf{x}_2\right) \quad s.t. \quad \hat{\mathbf{p}}_2 \cdot \mathbf{x}_2 \leq \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2. \tag{10.3}$$

Denote the solution to this problem by $\bar{\mathbf{x}}_2$, where we wish to wish that $\bar{\mathbf{x}}_2 = \hat{\mathbf{x}}_2$. Suppose instead that $\bar{\mathbf{x}}_2 \neq \hat{\mathbf{x}}_2$, then

$$u\left(\hat{\mathbf{x}}_1, \bar{\mathbf{x}}_2\right) > u\left(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2\right),$$
$$\hat{\mathbf{p}}_2 \cdot \bar{\mathbf{x}}_2 \leq \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2.$$

The latter implies that

$$\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1 + \hat{\mathbf{p}}_2 \cdot \bar{\mathbf{x}}_2 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1 + \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2.$$

Thus, we obtain the same contradiction before so we conclude that $\bar{\mathbf{x}}_2 = \hat{\mathbf{x}}_2$.

The idea is that any spot market outcome is feasible ex ante so that the ex ante budget constraint in (10.1) contains a larger set of allocations than those implied by the spot-market budget constraints. Therefore, the ex ante contracts are the first-best outcome; no consumer can do better than this.

10.2 Part (b)

Repeat the exercise with uncertainty instead of time. Assume that N goods and two states of the world, $s = 1, 2$. What is the interpretation of the assumption (analogous to that made in part (iv) of (a)) that if the consumer would have consumed bundle $\hat{\mathbf{x}}_1$ had state $s = 1$ occurred, his utility of consuming any bundle $\mathbf{x}_2$ in state $s = 2$ is $u\left(\bar{\mathbf{x}}_1, \mathbf{x}_2\right)$?

. . . . . .

This is simply a change of notation. However, we do have to reinterpret the meaning of our utility function: $u\left(x_1, x_2\right)$ is now the ex ante expected utility of consumption. Furthermore, $u\left(\overline{x}_1, x_2\right)$ is the consumer's state-contingent utility conditional on fixing utility as if $\overline{x}_1$ were consumed in state $s = 1$. The assumption of perfectly enforceable Arrow-Debreu contracts (i.e. commitment) and perfect foresight, in which spot prices are correctly anticipated, are crucial for the result.

[4] Notice that consumption in period 1 from the perspective in consumer 1 is sunk. We usually assume that sunk "costs" should not affect decisions—here, we assume the opposite.

11 Exercise 5.34: Arrow securities

Exercise 5.33 shows that when there are opportunities to trade a priori in any commodity contingent on any date, state etc., there is no remaining role for spot markets. Here, we show that, if not all commodities can be traded contingent on every date and state, then spot markets do have a role. We will in fact suppose that there is only one "commodity" that can be traded a priori, an *Arrow security*. An Arrow security for date t and state s entitles the bearer to one dollar at date t and in state s and nothing otherwise.

We wish to guide you towards showing that if $\hat{\mathbf{p}} \gg \mathbf{0}$ is a Walrasian equilibrium price in the contingent-commodity sense of Section 5.4 when there are N goods as well as time and uncertainty, and $\hat{\mathbf{x}} \geq \mathbf{0}$ is the corresponding Walrasian allocation, then the same prices and allocation arise when only Arrow securities can be traded a priori and all other goods must be traded on spot markets. This shows that, as long as there is a contingent-commodity market for a unit of account (money), the full contingent-commodity Walrasian equilibrium can be implemented with the aid of spot markets. We will specialize our attention to exchange economies.

Consider then the following market structure and timing. At date zero, there is a market for trade in Arrow securities contingent on any date and any state. The price of each Arrow security is one dollar, and each date t and state s security entitles the bearer to one dollar at date t in state s, and nothing otherwise. Let a^i_{ts} denote consumer i's quantity of date t and state s Arrow securities. No consumer is endowed with any Arrow securities. Hence, consumer i's budget constraint for Arrow securities at date zero is

$$\sum_{t,s} a^i_{ts} = 0.$$

At each date $t \geq 1$, the date-t event s_t is realized and all consumers are informed of the date-t state of the world $s = (s_1, \ldots, s_t)$. Each consumer i receives his endowment $\mathbf{e}^i_{st} \in \mathbb{R}^N_+$ of the N goods. Spot markets open for each of the N goods. If the spot price of good k is p_{kts}, then consumer i's date-t and state-s budget constraint is

$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + a^i_{ts}.$$

Each consumer i is assumed to know all current and future spot prices for every good in every state. Consequently, at date zero, consumer i can decide on the trades he will actually make in each spot market for each good at every future date and in every state. At date zero, consumer i therefore solves

$$\max_{(a^i_{ts}),(x^i_{kts})} u^i\left(\left(x^i_{kts}\right)\right)$$

subject to the Arrow-security budget constraint,

$$\sum_{t,s} a^i_{ts} = 0, \tag{11.1}$$

and subject to the spot-market budget constraint,

$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + a^i_{ts} \geq 0 \tag{11.2}$$

for each date t and state s. (Note the inequality in the date-t and state-s constraints. This ensures that there is no bankruptcy.)

11.1 Part (a)

Argue that the above formulation implicitly assumes that, at any date t, current and future utility in any state is given by $u^i(\cdot)$ where past consumption is fixed at actual levels and consumption in states that did not occur are fixed at the levels that would have been chosen had they occurred.

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In period zero, the consumer decides on the trade/consumption for each good at every future date in every state. This means that, if he find himself in a particular period $\bar{t}$ and state $\bar{s} \in (s_1, \dots, s_t)$, he should not regret his decision in period zero, and continue to follow his contingent plan. In other words, if he were to solve the maximisation problem in period $\bar{t}$ and state $\bar{s}$ for all goods in all future periods/states, he must arrive at the same solution as in his period-0 problem so that he is *time consistent*. Moreover, this must be true for all $\bar{t}$ and $\bar{s}$. To ensure this holds, it must be the case that past consumption is fixed at actual levels and consumption in states that did not occur are fixed at the levels that would have been chosen had they occurred.

We actually made the same assumption in the previous exercise when we were considering scope for spot trading in period 2.

11.2 Part (b)

The consumer's budget constraint in the contingent-commodity model of Section 5.4 specialized to exchange economies is

$$\sum_{k,t,s} p_{kts} x_{kts}^i = \sum_{k,t,s} p_{kts} e_{kts}^i. \tag{11.3}$$

Show that $\left(x_{kts}^i\right)$ satisfies this budget constraint if and only if there is a vector of Arrow securities $\left(a_{st}^i\right)$ such that $\left(x_{kts}^i\right)$ and $\left(a_{st}^i\right)$ together satisfy the Arrow security budget constraint and each of the spot-market budget constraints.

……

Sufficiency (⇐) Suppose first that (11.1) and (11.2) hold. Summing (11.2) across t and s, we obtain

$$\begin{aligned} \sum_t \sum_s \sum_k p_{kts} x_{kts}^i &= \sum_t \sum_s \sum_k p_{kts} e_{kts}^i + \sum_t \sum_s a_{ts}^i \\ \Leftrightarrow \sum_{k,t,s} p_{kts} x_{kts}^i &= \sum_{k,t,s} p_{kts} e_{kts}^i + \sum_{t,s} a_{ts}^i. \end{aligned}$$

By (11.1), it above implies that

$$\sum_{k,t,s} p_{kts} x_{kts}^i = \sum_{k,t,s} p_{kts} e_{kts}^i.$$

Necessity (⇒) Now suppose $\left(x_{kts}^i\right)$ satisfies (11.3). Suppose, by way of contradiction, that there does not exists $\left(a_{st}^i\right)$ such that $\left(a_{st}^i\right)$ and $\left(x_{kts}^i\right)$ together does not satisfy (11.1) and (11.2). In particular, this means that, for any $\left(a_{st}^i\right)$,

$$\sum_{t,s} a_{ts}^i \gtrless 0, \tag{11.4}$$

or, there exists some (s,t) such that

$$\sum_k p_{kts} x_{kts}^i \gtrless \sum_k p_{kts} e_{kts}^i + a_{ts}^i. \tag{11.5}$$

First, suppose (11.4) is true (11.2) holds with equality for all s and t. Then, summing (11.5) t and s yields

$$\sum_{k,t,s} p_{kts} x^i_{kts} = \sum_{k,t,s} p_{kts} e^i_{kts} + \underbrace{\sum_{t,s} a^i_{ts}}_{\gtrless 0} \Rightarrow \sum_{k,t,s} p_{kts} x^i_{kts} = \sum_{k,t,s} p_{kts} e^i_{kts},$$

which is a contradiction that $\left(x^i_{kts}\right)$ satisfies (11.3).

Now suppose that there exists some (s,t) such that (11.5) is true, but (11.1) holds. Then, summing (11.5) across t and s,

$$\sum_{k,t,s} p_{kts} x^i_{kts} \gtrless \sum_{k,t,s} p_{kts} e^i_{kts} + \sum_{t,s} a^i_{ts} = \sum_{k,t,s} p_{kts} e^i_{kts},$$

which again, contradicts the assumption that $\left(x^i_{kts}\right)$ satisfies (11.3).

Finally, suppose that (11.4) is true *and* that there exists some (s,t) such that (11.5) is also true. For any such (s,t) define ε^i_{st} such that

$$\sum_{k} p_{kts} x^i_{kts} = \sum_{k} p_{kts} e^i_{kts} + a^i_{ts} + \varepsilon^i_{st}.$$

Summing across all s and t, we have

$$\sum_{k,t,s} p_{kts} x^i_{kts} = \sum_{k,t,s} p_{kts} e^i_{kts} + \sum_{t,s} a^i_{ts} + \sum_{t,s} \varepsilon^i_{st}.$$

Since we assume (11.4) holds, the only way in which (11.3) could still hold is if

$$\sum_{t,s} a^i_{ts} = \sum_{t,s} \varepsilon^i_{st}.$$

But if this is true, we can define

$$\tilde{a}^i_{ts} = a^i_{ts} - \varepsilon^i_{st}, \forall s, t$$

and $\left(\tilde{a}^i_{ts}\right)$ would satisfy both (11.1) and (11.2)—contradicting the assumption that there does not exist such $\left(\tilde{a}^i_{ts}\right)$.

11.3 Part (c)

Conclude from (b) that any Walrasian equilibrium price and allocation of the contingent-commodity model of Section 5.4 can be implemented in the spot-market model described here and that there will typically be trade in the spot markets. Show also the converse.

$\cdots\cdots$

Part (b) shows that (x_{kts}) is budget feasible in the contingent-commodity model if and only if there exists $\left(a^i_{kts}\right)$ such that the pair $\left((x_{kts}), \left(a^i_{kts}\right)\right)$ is budget feasible in the spot-market model. Since the utility functions are the same in the two models, it follows that the solution to the optimisation problems coincide with respect to (x_{kts}).

It remains to show that market clearing in one model implies market clearing in the other. The market clearing condition in the contingent-commodity model is given by

$$\sum_{i} x^i_{kts} = \sum_{i} e^i_{kts}, \ \forall k, t, s. \tag{11.6}$$

Market clearing conditions in the spot-market model are

$$\sum_{i} x^i_{kts} = \sum_{i} e^i_{kts}, \ \forall k, t, s,$$

$$\sum_i a^i_{ts} = 0,$$

since the economy is not endowed with Arrow securities.

Suppose (x_{kts}) is a Walrasian equilibrium allocation with prices (p_{kts}) so that (11.6). We may construct Arrow securities as

$$a^i_{ts} := \sum_k p_{kts} x^i_{kts} - \sum_k p_{kts} e^i_{kts}, \forall i, t, s.$$

Then, by construction, we satisfy (11.2). Note that

$$\begin{aligned}\sum_{i,t,s} a^i_{ts} &= \sum_{i,t,s} \sum_k p_{kts} x^i_{kts} - \sum_{i,t,s} \sum_k p_{kts} e^i_{kts} \\ &= \sum_k \left(p_{kts} \sum_{t,s} \underbrace{\sum_i \left(x^i_{kts} - e^i_{kts} \right)}_{=0} \right) = 0.\end{aligned}$$

Hence, we see that market clearing condition in the spot-market models are satisfied.

That market clearing condition in the spot-market model implies market clearing in the contingent-commodity model is immediately from the fact that (11.6) appears in both set of conditions.

Remark 11.1. Equilibrium with spot markets is referred to as the *Radner equilibrium* (see MWG section 19.D).

11.4 Part (d)

Explain why the price of each Arrow security is one. For example, why should the price of a security bearing entitling the bearer to a dollar today be equal to the price of a security entitling the bearer to a dollar tomorrow when it is quite possible that consumers prefer consumption today to the same consumption tomorrow? (Hint: Think about what a dollar will buy.)

. . . . . .

At each date t, state s, the dividend from the Arrow security gives one unit of account in *that* period. Therefore, its price in each date-state pair is one. In general, the unit of account in one date-state pair will not be the same as another date-state pair so the fact that that price of each Arrow security one does not violate, for example, the principle of discounting—prices (p_{tks}) take care of such features.

11.5 Part (e)

Repeat the exercise when, instead of paying the bearer in a unit of account, one date-t and state-s Arrow security pays the bearer one unit of good 1 at date t in state s and nothing otherwise. What prices must be set for Arrow securities now in order to obtain the result in part (c)? How does this affect the consumer's Arrow security and spot-market budget constraints?

. . . . . .

The price of good 1 at date t in state s is given by p_{1ts}. Since Arrow security pays the bearer one unit of good 1 in state t and s, its value must be the same as one unit of good 1 in state t and s; i.e. price must equal p_{1ts}. Since the budget constraints are defined in terms of unit of account, and $p_{1ts} a^i_{ts}$ is the value of dividend in the unit of account now, the Arrow-security and spot-market budget constraints become, respectively,

$$\sum_{t,s} p_{1ts} a^i_{ts} = 0,$$
$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + p_{1ts} a^i_{ts} \geq 0, \ \forall t, s.$$

Remark 11.2. For a full exposition, see MWG Proposition 19.D.1.

12 Exercise 5.39

(Cornwall) In an economy with two types of consumers, each type has the following respective utility functions and endowments:

$$u^{1q}(x_1, x_2) = x_1 x_2 \quad \mathbf{e}^1 = (8, 2),$$
$$u^{2q}(x_1, x_2) = x_1 x_2 \quad \mathbf{e}^2 = (2, 8).$$

12.1 Part (a)

Draw an Edgeworth box for this economy when there is one consumer of each type.

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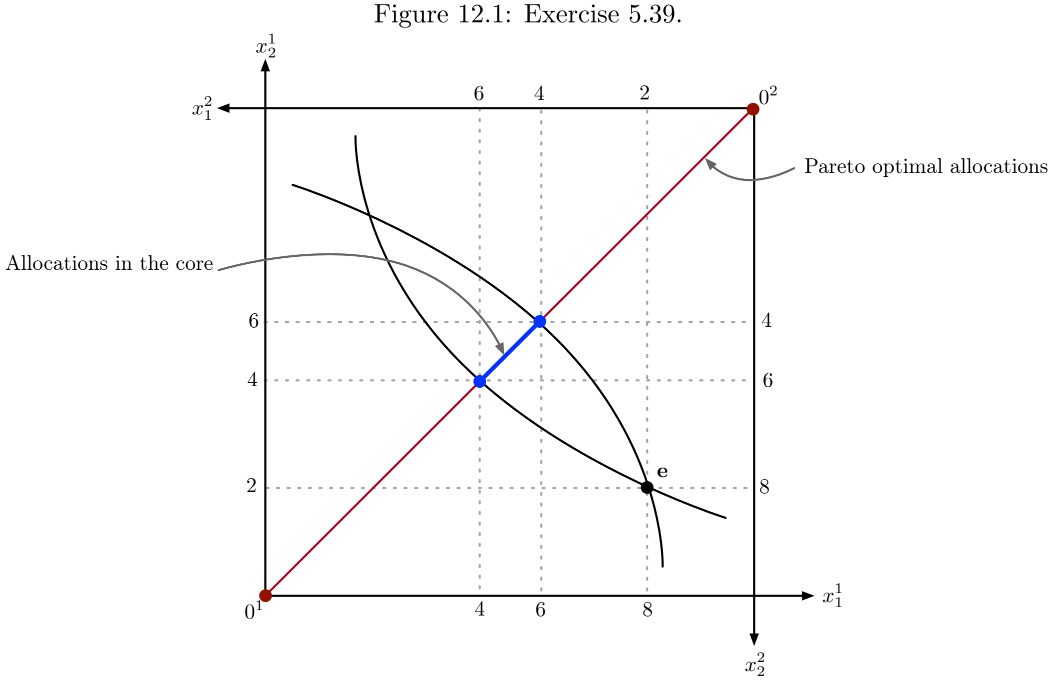
Here's the figure with everything filled.

Figure 12.1: Exercise 5.39.



12.2 Part (b)

Characterise as precisely as possible the set of allocations that are in the core this two-consumer economy.

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Since the core is a subset of Pareto optimal allocations, we first look for the set of Pareto optimal allocations. For interior allocations, Pareto optimal allocations equate the marginal rate of substitution across all agents.

$$MRS_{12}^1 = \frac{x_2^1}{x_1^1} = \frac{x_2^2}{x_1^2} = MRS_{12}^2. \tag{12.1}$$

Feasibility requires

$$x_1^1 + x_1^2 = 10,$$
$$x_2^1 + x_2^2 = 20.$$

Substituting into (12.1) yields

$$\frac{x_2^1}{x_1^1} = \frac{10 - x_2^1}{10 - x_1^1} \Leftrightarrow 10x_2^1 - x_2^1 x_1^1 = 10x_1^1 - x_2^1 x_1^1$$
$$\Leftrightarrow x_2^1 = x_1^1.$$

Similarly, we can obtain

$$x_2^2 = x_2^1.$$

Let us consider the boundary cases. Consider an allocation in which consumer i gets zero of at least one of the goods. Then, his utility is zero. Thus, Pareto optimal allocation with one consumer i receiving zero of at least one of the good is the one in which the other consumer receives everything.

The set of Pareto optimal allocations can therefore be characterised by

$$\mathbf{P} = \left(\left(x_1^1, x_2^1\right), \left(x_1^2, x_2^2\right)\right) = \left(\left(a, a\right), \left(10 - a, 10 - a\right)\right), \ \forall a \in [0, 10].$$

Since there are only two consumers in the economy, the core is a subset of $\mathbf{P}$ which yields each consumer greater utility than their endowment. Note

$$u^1\left(\mathbf{e}^1\right) = 16 = u^2\left(\mathbf{e}^2\right).$$

Hence, the core of this economy is given by

$$\mathbf{C} = \left\{\left(\left(a, a\right), \left(10 - a, 10 - a\right)\right) : a \in [0, 1], a^2 \geq 16, \left(10 - a\right)^2 \geq 16\right\}.$$

Note

$$a \geq 4 \Rightarrow a^2 \geq 16,$$
$$a \leq 6 \Rightarrow \left(10 - a\right)^2 \geq 16.$$

Hence,

$$\mathbf{C} = \left\{\left(\left(a, a\right), \left(10 - a, 10 - a\right)\right) : a \in [4, 6]\right\}.$$

12.3 Part (c)

Show that the allocation giving $\mathbf{x}^{11} = (4, 4)$ and $\mathbf{x}^{21} = (6, 6)$ is in the core.

. . . . . .

From Part (b), it is immediate the proposed allocation is in the core.

12.4 Part (d)

Now replicate this economy once so there are two consumers of each type, for a total of four consumers in the economy. Show that the double copy of the previous allocation, giving $\mathbf{x}^{11} = \mathbf{x}^{12} = (4, 4)$ and $\mathbf{x}^{21} = \mathbf{x}^{22} = (6, 6)$ is *not* in the core of the replicated economy.

......

Let us form a coalition consisting of $S = \{11, 12, 21\}$. Note that

$$\sum_{i \in S} \mathbf{e}^i = (18, 12)\,,$$
$$u^{11}(4,4) = u^{12}(4,4) = 16,$$
$$u^{21}(6,6) = 36.$$

Suppose we keep 21's bundle the same. This leaves $(12, 6)$ goods. If split this evenly between 11 and 12, each receives a utility of

$$u^{11}(6,3) = 18 > 16 = u^{11}(4,4)\,.$$

So we realize that S will block the allocation.