Price Theory II Problem Set 4

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Odd: 1,3, 5.35, 6.1, 6.9, 6.19, 6.21, 6.23

1.

Consider a production economy $\mathcal{E}$ with private ownership in which all ownership shares are either 0 or 1, that is, $\theta^{ij} \in \{0,1\}$, for all $i \in \mathcal{I}$, all $j \in \mathcal{J}$. Define the core of $\mathcal{E}$ to be the set of all feasible allocations $(\mathbf{x}, \mathbf{y})$ such that there does not exist a nonempty subset S of $\mathcal{I}$ and an allocation $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ with $\tilde{y}^j \in Y^j$ for every $j \in \mathcal{J}$ such that,

(i) $\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} e^i + \sum_{i \in S} \sum_{j \in \mathcal{J}: \theta^{ij}=1} \tilde{y}^j$

(ii) $u^i(\tilde{x}^i) \geq u^i(x^i)$ for every $i \in S$ with at least one strict inequality.

(Think about how you would interpret this definition. How does the assumption that θ^{ij} must be either 0 or 1 impact your interpretation? What difficulties surface when trying to define the core when owenrship shares can be fractional?)

If all utility functions are strictly increasing, prove that every WEA of this production economy is in the core.

. . . .

The 0-1 distinction for the θ^{ij} means that each firm is owned entirely by one consumer, and therefore each consumer makes decisions based on their knowledge of the production possibilities for that firm, but *they need not consult with others.* If we allowed fractional shares of firms, then the core would be a bit weird to define, because we could foreseeably have allocations where firms are partially owned by "happy" consumers, and partially owned by consumers who would rather block. It is then tricky to think how blocking works: does a firm not cooperate if even one consumer wants to block? Does it require all consumers to consent to blocking? Can the firm be partially used for blocking? All these questions go away when we embed 0-1 ownership.

Recall that in a production economy, a Walrasian equilibrium $\mathbf{p}$ is such that there exist $\mathbf{x}^*$ and $\mathbf{y}^*$ such that[1]

[1] Adapting this definition for the 0-1 θs.

(I) $(x^i)^*$ solves $\max u^i(x^i)$ such that $\mathbf{p} \cdot x^i \leq \mathbf{p} \cdot e^i + \sum_{j \in \mathcal{J}:\theta^{ij}=1} \Pi(\mathbf{p})$ for all $i \in \mathcal{I}$

(II) $(y^j)^*$ solves $\max \mathbf{p} \cdot y^j$ such that $y^j \in Y^j$ for all $j \in \mathcal{J}$

(III) $\sum_{i \in \mathcal{I}} (x^i)^* = \sum_{j \in \mathcal{J}} (y^j)^* + \sum_{i \in \mathcal{I}} e^i$

In this case, $(\mathbf{x}^*, \mathbf{y}^*)$ is a WEA.

Now let's adapt Lemma 5.2 to this problem:

Lemma 5.2': Consider any $\mathbf{p} \in \mathbf{R}^n_+$, and $u^i : \mathbb{R}^n \to \mathbb{R}$ **strictly increasing**, with ownership share 1 in firms $J \subset \mathcal{J}$. If $((x^i)^*, \{(y^j)^*\}_{j \in J})$ maxes $u^i(x^i)$ such that $\mathbf{p} \cdot x^i \leq \mathbf{p} \cdot e^i + \sum_{j \in J} \mathbf{p} \cdot y^j$, then for all $x^i \in \mathbb{R}^n_+$

(a) $u^i(x^i) > u^i((x^i)^*)$ implies $\mathbf{p} \cdot x^i > \mathbf{p} \cdot (x^i)^*$

(b) $u^i(x^i) \geq u^i((x^i)^*)$ implies $\mathbf{p} \cdot x^i \geq \mathbf{p} \cdot (x^i)^*$

Proof: Given $\mathbf{p}$, let $(y^j)^* = \text{argmax}_{y^j \in Y^j} \mathbf{p} \cdot y^j$. Define $\tilde{e}^i = e^i + \sum_{j \in J} (y^j)^*$. Then the proof of Lemma 5.2 goes through[2], with $\tilde{e}^i$ taking the place of e^i.

Now let $(\mathbf{x}^*, \mathbf{y}^*)$ be a WEA. If it is not in the core, then there exists a coalition S that blocks, meaning it satisfies the feasibility constraint (i) and the weakly preferred, with at least one strictly preferred condition (ii), from the problem setup, with $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \neq (\mathbf{x}^*, \mathbf{y}^*)$. By (i) and Lemma 5.2', we must have $\mathbf{p} \cdot \tilde{x}^i \geq \mathbf{p} \cdot (x^i)^*$ for all $i \in S$ with at least one strict. But then we may sum them and find that (ii) is not satisfied: $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is not feasible.

$$\begin{aligned} \sum_{i \in S} \mathbf{p}^* \cdot \tilde{x}^i &> \sum_{i \in S} \mathbf{p}^* \cdot (x^i)^* \\ &= \sum_{i \in S} \mathbf{p}^* \cdot e^i + \sum_{i \in S} \sum_{j \in \mathcal{J}:\theta^{ij}=1} \mathbf{p}^* \cdot \tilde{y}^j \end{aligned}$$

So $(\mathbf{x}^*, \mathbf{y}^*)$ must be in the core.

3.

The man proposing DDA matches are,

$$\mu_M := \{(m_1, w_2); (m_2, w_3); (m_3, w_1); (m_4, w_4); m_5\}$$

The woman proposing DDA matches are,

$$\mu_W := \{(m_1, w_4); (m_2, w_1); (m_3, w_2); (m_4, w_3); m_5\}$$

[2]Note that this is where the assumption of **strictly increasing** is being invoked.

5.35

(a) *We want to show* $\hat{\mathbf{p}} \cdot \alpha$ *as the price for assets clears markets and has 0 excess demand for the asset market.*

In this economy, we have only 1 good and one time period with many different S states. This means we can think of a consumer maximizing good x as maximizing an Arrow security. Then, there is another mechanism that allows consumers to buy assets that give you some return in a given state. Therefore, we can think of what the consumer demands at the end of time t is $\hat{x}^i_s + \alpha_s$. Here the consumer will have whatever arrow security in state S and assets in state S where α is return on that asset. (See MWG pg. 705). Multiplying by the price vector for arrow securities $\hat{\mathbf{p}}$ we get the market clearing condition $\sum \hat{p}_s \hat{x}^i_s + \hat{\mathbf{p}}_\mathbf{s} \alpha_s = \sum e^i_s \hat{p}_s$. Since the assets are given in their return and not units, we can multiply however many units the consumer buys of the asset by the price of the asset which is $\hat{\mathbf{p}}_\mathbf{s} \alpha_s$.

It is also important to note that by trading these assets and arrow securities, agents will have the same consumption across every state which is given in the budget constraint.

(b) *Risk Averse agents with VNM utility and uncertainty*

(i) *No asset receives a premium over and above its expected value.*

To find prices, we take FOC of the VNM utility function to see that $\frac{\pi_s v'_i(x^i_s)}{\pi_r v'_i(x^i_{r)}} = \frac{p_s}{p_r}$ where $s, r \in S$ which can be generalized to all consumers $i \in I$. We know derivatives of v are positive so these relationships are well defined. Now, consider if $\hat{\mathbf{p}} \neq (\pi_1 ... \pi_S)$ then the $\frac{v'_i(x^i_s)}{v'_i(x^i_{r)}}$ would not be equal to 1. From the previous result we know that consumption must be the same across states when summing across consumers. If this inequality were true then we would contradict this condition.

(ii) *Suppose total endowment of the good is not constant across states.*

(1) *We want to show* $\hat{\mathbf{p}} \neq (\pi_1 ... \pi_S)$ *and no consumer's consumption is constant across all states*

Similar to above, if $\hat{\mathbf{p}} = (\pi_1 ... \pi_S)$ then the market clearing condition of endowment and consumption being the same across states would hold. This is a contradiction to the supposition that endowments are not the same across states. Therefore, the $\frac{v'_i(x^i_s)}{v'_i(x^i_{r)}}$ cannot be equal to 1. So in order for the FOC to hold we must have $\hat{\mathbf{p}} \neq (\pi_1 ... \pi_S)$. In this case, the consumer will have greater marginal utilities for consuming the good in one state vs another.

(2) *The price of an asset is not so much related to its variance but rather the extent to which it is correlated with consumption.*

From previous exercise we know $\hat{\mathbf{p}} = \lambda_i \nabla u$. From part a we know that the price of an asset is $\sum_s \lambda_i \nabla u \alpha_s$. Solving for our multiplier we get $\frac{\sum_s p_s}{\lambda} = \sum_s \nabla u$ after dividing out α. We have $\frac{1}{\lambda} = E(v'(x^i))$ since in the prompt we know that the sum of prices over states is equal to 1 and by properties of the VNM utility. Therefore, the price of an asset is $\frac{\sum_s \nabla u \alpha_s}{\sum_s \nabla u}$ which is equal to $\frac{E(v'(x^i)\alpha)}{E(v'(x^i))}$

6.1

Let $X = \{x_1, x_2\}$ and let $(R^i)_{i=1}^n$ be any collection of preference relations on X where $n \geq 2$. Since there are only two elements in the domain we define majority voting as follows. For any $x_j, x_k \in X$, $x_j R x_k$ iff $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$. Note that in this case any individual with a tie casts a "vote" for both options. We check each of Arrow's axioms in turn.

(U)
For any $(R^i)_{i=1}^n$ we want to show that majority voting gives a complete and transitive preference relation R over X.

For completeness consider any $x_j, x_k \in X$. We will have either that $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$ or $\#\{i : x_j R^i x_k\} \leq \#\{i : x_k R^i x_j\}$ so that either $x_k R x_j$ or $x_j R x_k$ and R is complete as desired.

For transitivity (which is where majority voting fails in general) let $x_j, x_k, x_l \in X$ and suppose that $x_j R x_k$ and $x_k R x_l$. This means that $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$ and $\#\{i : x_k R^i x_l\} \geq \#\{i : x_l R^i x_k\}$. Now note that since we have only two elements in the domain we will have that $x_l = x_j$ or $x_l = x_k$. In the former case, $\#\{i : x_j R^i x_l\} = \#\{i : x_l R^i x_j\}$ so that $x_j R x_l$. In the latter case we have: $\#\{i : x_j R^i x_l\} = \#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\} = \#\{i : x_l R^i x_j\}$ so that $x_j R x_l$. Conclude that R is transitive.

The other axioms are satisfied in general and do not rely on the two-element-ed-ness of the choice set:

(WP)
Let $x_j, x_k \in X$ and suppose that $x_j P x_k$ for all i. Then $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$ and $\#\{i : x_j R^i x_k\} \neq \#\{i : x_k R^i x_j\}$ so $x_j P x_k$ as desired.

(IIA)
Let $(R^i)_{i=1}^n$ and $(\tilde{R}^i)_{i=1}^n$ be two profiles of preferences, $R, \tilde{R}$ be the corresponding majority voting social preferences and $x_j, x_k \in X$. Suppose each individual i in the the two profiles of preferences agrees on the ranking of x_j against x_k. WLOG suppose that $x_j R x_k$. Then we have that $\#\{i : x_j \tilde{R}^i x_k\} = \#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\} = \#\{i : x_k \tilde{R}^i x_j\}$. So that $x_j \tilde{R} x_j$ and the social preferences also agree as desired.

(D)
Let $(R^i)_{i=1}^n$ be any profile of preferences and consider a particular individual i^*. We want to show that i^* is not a dictator. Let $x_j P^{i^*} x_k$. It suffices to construct a set of preferences such that it is not the case that $x_j P x_k$ in other words such that $x_k R x_j$. Let us set all $\tilde{i} \neq i^*$ to have preferences such that $x_k P^{\tilde{i}} x_j$. Note that since $n \geq 2$ there is at least one such individual. Then $\#\{i : x_k \tilde{R}^i x_j\} \geq 1 = \#\{i^*\} = \#\{i : x_j R^i x_k\}$ so $x_k R x_j$ as desired.

6.9

Question

6.9 There are three individuals in society, $\{1, 2, 3\}$, three social states, $\{x, y, z\}$, and the domain of preferences is unrestricted. Suppose that the social preference relation, R, is given by pairwise majority voting (where voters break any indifferences by voting for x first then y then z) if this results in a transitive social order. If this does not result in a transitive social order the social order is $xPyPz$. Let f denote the social welfare function that this defines.

(a) Consider the following profiles, where P^i is individual i's strict preference relation:

Individual 1: xP^1yP^1z

Individual 2: yP^2zP^2x

Individual 3: zP^3xP^3y

What is the social order?

(b) What would be the social order if individual 1's preferences in (a) were instead yP^1zP^1x? or instead zP^1yP^1x?

(c) Prove that f satisfies the Pareto property, WP.

(d) Prove that f is non-dictatorial.

(e) Conclude that f does not satisfy IIA.

(f) Show directly that f does not satisfy IIA by providing two preference profiles and their associated social preferences that are in violation of IIA.

Part A

Because we have two individuals with the preferences:

$$xP^iy, \ i = 1, 3$$

two people with the preferences:

$$yP^iz, \ i = 1, 2$$

and two people with the preferences:

$$zP^ix, \ i = 1, 2$$

we end up with the transitive social ranking:

$$xPyPzPxP...$$

And these clearly violate transitivity, as this is a Condorcet cycle of choice. Thus we have to fall back to the backup social ranking of:

$$xPyPz$$

Part B

yP^1zP^1x

If this were the case we would instead get the social ranking of:

$$yPzPx$$

which is clear to see because individuals 1 and 2 both have the same preferences, and thus their votes dominate whatever individual 3 may want... sucks for them.

zP^1yP^1x

This case is a little more involved. As we have:

$$yP^ix,\ i = 1,2$$

Thus we must rank y above x. Next we have:

$$zP^iy,\ i = 1,3$$

So z is above y, and finally:

$$zP^ix,\ i = 1,2,3$$

Thus we get the social ranking:

$$zPyPx$$

Part C

Definition of Weak Pareto (WP): For any $x, y \in X, \forall(R^1, ..., R^N) \in \mathcal{R}^N$ if $xP^iy\ \forall i$ then xPy.

With this majority voting structure, if all agents $i \in \mathcal{I}$ voted yP^ix then clearly we would have yPx. The only way this could potentially fail in this scenario is if we somehow failed to create a non-transitive social order that caused us to fallback on our back up social order mentioned by the question. For the rest of this question we will prove that it is impossible to get such a situation.

Proof: Suppose WLOG and by contradiction that we have some how have yPx (by construction as all agents voted y over x), xPz and zPy, resulting in the social ranking $yPxPzPy$.

Here we have 2 agents having voted for x over z. Otherwise it wouldn't hold that xPz. Now consider our social ranking that zPy. This is a contradiction as we already have all agents prefer yRx and we have at least two agents prefer xPz so we cannot have more than a single person have the preference structure zPy.

Part D

Suppose agent i is the dictator such that their ranking determines the social rank. We fix these rankings $\mathcal{R}^*$ - can we overthrow the dictator? Clearly suppose all other $\mathcal{I}\ \{i\}$ vote in an equal ranking $\mathcal{R}' \neq \mathcal{R}^*$. As a result we get social ranking $\mathcal{R}'$ NOT $\mathcal{R}^*$. This works for all i so there are no dictators! Hurrah.

Part E

We just showed that these preferences satisfy Weak Pareto and was non-dictatorial. We clearly also have an unrestricted domain here as all preference relations are allowed so by Arrow's Impossibility Theorem we know that the thing that must be violated is IIA. We will show below in Part F that this is indeed the case.

Part F

Below we present a clear violation of IIA. Note in R_3 of both scenarios we do not change the ranking of y vs z, but the final rankings are dramatically changed between the two of them by flipping x and z, without changing any single agent's preference between y and z. We were able to do so by exploiting the intransitivity of this voting mechanism.

Scenario 1

R_1	R_2	R_3	R
z	x	y	x
y	z	x	y
x	y	z	z

Scenario 2

R_1	R_2	R_3	R
z	x	y	z
y	z	z	y
x	y	x	x

6.19

Let $c(.)$ be a monotonic social choice function and suppose that the social choice must be x whenever all individual rankings are strict and x is at the top of individual n's ranking. So the social choice must be at least as good as x for individual n when the individual rankings are not necessarily strict and x is at least as good for individual n as any other state.

A social choice function $c(.)$ is monotonic if $c(R^1...R^N) = x$ implies $c(\tilde{R}^1...\tilde{R}^N) = x$ whenever for each individual i and every $y \in X$ distinct from x, $xR^iy \implies x\tilde{P}^iy$

Suppose not. Then, $\forall y \in X, xR^ny$ and the rest of the individuals having not necessarily strict preferences then xP^nz where z is some other choice generated by the not necessarily strict set of relations. Now we rewrite the R^i relations to be P^i relations so that $\forall i, \forall y \neq z, zR^iy \implies zP^iy$. Using monotonicity we see that $\forall y, zP^ny$. So for $y = x$ we have a contradiction.

6.21

a

First consider individual 1. Either she has a strict top preference or she does not. If she has a strict top preference and reports truthfully she gets her top choice and cannot do any better. If she has indifference and she truthfully reports she gets one of her top preferences (depending either on player 2's choice or the tie-splitting rule) and cannot do any better by misreporting in that case either.

Now consider individual 2. If player 1 reports a strict top choice then player 2's own reported preference is irrelevant to the social choice so she cannot do any better by misreporting. If player 1 reports an indifference then player 2's choice matters. In this case, if player 2 has a strict top choice then truthfully reporting gives her it and she will do strictly worse by any misreporting. If she has equally ranked top choices and tells the truth then the tie is broken among them and she ends up with one of her top choices and so cannot do strictly better by misreporting in this case either.

Let $i \in \{1, 2\}$ and let R^i be that individual's true preferences. Let R^{-i} be any arbitrary preferences for the other individual. The above arguments have shown that $c(R^i, R^{-i})R^ic(\tilde{R}^i, R^{-i})$ for any ranking $\tilde{R}^i$ that i could report and the social choice function is strategy proof as desired.

b.

Let $\mathbf{R}$ by given by the strict ordering zP^1yP^1z for person 1 and indifference for person 2: for any $a, b \in \{x, y, z\}$ have that aR^2b. Using the choice rule we find that $c(\mathbf{R}) = z$

Now let $\tilde{\mathbf{R}}$ be total indifference for everyone. Explicitly: for any $i \in \{1, 2\}$ and any $a, b \in \{x, y, z\}$ have that $a\tilde{R}^ib$. Note that this means that the conditions for strong monotonicity are satisfied and we should have $c(\tilde{\mathbf{R}}) = z$. However, looking at the description of the choice function, total indifference means that the tie is broken such that $c(\tilde{\mathbf{R}}) = x$ and we have the desired counterexample.

6.23

Question

Show that when there are just two alternatives and an odd number of individuals, the majority rule social choice function (i.e., that which chooses the outcome that is the top ranked choice for the majority of individuals) is Pareto efficient, strategy-proof and non-dictatorial.

Solution

Pareto Efficient

Definition: Pareto Efficient
A social choice function $c(.)$ is "Pareto Efficient" if and only if $\forall x \in X$

$$[xP^iy,\ y \in X\ \{x\},\ \forall i] \Rightarrow c(R^1, ..., R^N) = x$$

.

The majority rule social choice function is clearly Pareto Efficient if $xP^iy,\ \forall i$ then $c(R) = xPy$.

Strategy Proof

Although we will show below that the social choice function is dictatorial then it cannot be strategy-proof (and visa-versa). HOWEVER because $X < 3$ this is definitely possible.

To show that this is strategy-proof we first need to define a pivotal individual: an individual, i is pivotal if and only if the preferences of the rest of society, R^i, lead to a tie.

We consider two cases, the first being when individual i is not pivotal and the other when they are. These two cases cover all possible scenarios for agent i

IF AGENT IS NOT PIVOTAL: Suppose individual i is not pivotal and the social choice is $c(R^i, R^{-i}) = x$ (WLOG). Because agent i them self is not pivotal, whatever they choose between the two does not matter, the out come will remain $c = x$. Because yR^iy agent does not have any incentive to lie, which is great.

IF AGENT IS PIVOTAL: Suppose now that the individual IS pivotal. Suppose WLOG that individual i prefers xR^iy. As a result $c(R^i, R^{-i}) = x$. Now suppose they lied and reported $y\hat{R}^ix$. Now the $c(\hat{R}^i, R^{-i}) = y$. Which is clearly worse for the agent because, as shown before xR^iy. Thus they do not have incentive to lie and thus the voting system is strategy proof.

Non-Dictatorial

This one is relatively straight forward, even though the system is strategy-proof it can also be non-dictatorial because $X < 3$. Take an i you think is the dictator. As there are only two options either xP^iy or yP^ix. WLOG suppose xP^iy, now suppose some other of the $\frac{n+1}{2}$ agents prefer yPx , then the social choice function yields yPx, thus showing that our agent is not a dictator. Here n is the odd number of agents with a vote.