Price Theory II Problem Set 7 - Evens

Chase Abram, Sarah Brown, Manav Chaudhary
Thomas Hierons, Jeanne Sorin, George Votja

July 3, 2020

Question 2

Part A

We will start by writing an equation for the expected payoff for a player $i \in \mathcal{I}$:

$$\mathbb{E}(u) = (v_i - \beta_i(v_i))Pr(\beta_i > \beta_j, \forall j \neq i)$$

Now there are a few things to note. First, because values are drawn independently and privately from a uniform distribution we can write the probablity that our individual has the highest bid as:

$$\begin{aligned} Pr(\beta_i > \beta_j) = Pr((v_j \leq p) \cup (p \leq v_j \leq v_i) \\ = p + v_i - p = v_i \end{aligned}$$

Where we use the fact that because $\alpha_i = \alpha_j \ \forall i, j \in \mathcal{I}_i > \beta_j \iff v_i > v_j$. Because these α values are consistent across β_i functions as is our reserve price let's rewrite our $v_i = \beta^{-1}(\beta_i(v_i))$. Finally note that because these values are drawn from a uniform distribution we can write our probability as a function of the CDF applied $(n-1)$ times for that many "opposing" players:

$$Pr(\beta_i > \beta_j, \forall j \neq i) = F^{(n-1)}(\beta^{-1}(\beta_i(v_i)))$$

After all of this work we can rewrite our expected payoff as:

$$(v_i - \beta_i(v_i))F^{(n-1)}(\beta^{-1}(\beta_i(v_i)))$$

For simplicity we will write $\beta_i(v_i) = \beta(v_i)$ as the maximizer of the expression above. Taking the derivative with respect to $\beta_i(v_i)$ we see that:

$$\begin{aligned} \frac{\partial}{\partial \beta_i(v_i)}(v_i - \beta_i(v_i))F^{(n-1)}(\beta^{-1}(\beta_i(v_i))) = \\ 0 = v_i[(n-1)F(v_i)^{n-2}(f(v_i)\beta^{-1\prime}(\beta_i(v_i))] \\ -[\beta_i(v_i)[(n-1)F(v_i)^{n-2}(f(v_i)\beta^{-1\prime}(\beta_i(v_i)) + F^{(n-1)}(\beta^{-1}(\beta_i(v_i)))] \end{aligned}$$

Where:

$$\frac{\partial}{\partial \beta_i(v_i)} F^{(n-1)}(\beta^{-1}(\beta_i(v_i))) = [(n-1)F(v_i)^{n-2}(f(v_i)\beta^{-1\prime}(\beta_i(v_i))]$$

Rearranging terms we get that

$$v_i[(n-1)F(v_i)^{n-2}(f(v_i)\beta^{-1\prime}(\beta_i(v_i))] = [\beta_i(v_i)[(n-1)F(v_i)^{n-2}(f(v_i)\beta^{-1\prime}(\beta_i(v_i)) + F^{(n-1)}(\beta^{-1}(\beta_i(v_i)))]$$

$$\Rightarrow (n-1)F(v_i)^{n-2}(f(v_i)\beta^{-1\prime}(\beta_i(v_i))(v_i - \beta_i(v_i)) = F^{(n-1)}(v_i)$$

We know that $F(v) = v, f(v) = 1$ (because the whole interval is $= 1$) we can thus multiply both sides by $\beta'(v_i)$ to get the following expression, designating β as the solution to the F.O.C.:

$$\begin{aligned}(n-1)v_i^{n-2}(v_i - \beta(v_i)) &= v_i^{n-1}\beta'(v_i)\\ (n-1)v_i^{n-2}(v_i - \beta(v_i)) &= v_i^{n-1}\frac{\partial \beta}{\partial v_i}\\ \Rightarrow (n-1)v_i^{n-1} &= v_i^{n-1}\frac{\partial \beta}{\partial v_i} + (n-1)v_i^{n-2}\beta(v_i)\end{aligned}$$

Now for the particularly astute kids in the class we notice that this right hand side can also be written as:

$$\frac{\partial}{\partial v_i}(\beta(v_i)v_i^{n-1}) = v_i^{n-1}\frac{\partial \beta}{\partial v_i} + (n-1)v_i^{n-2}\beta(v_i)$$

So we are left with a very nice:

$$(n-1)v_i^{n-1} = \frac{\partial}{\partial v_i}(\beta(v_i)v_i^{n-1})$$

Now that we are clearly converging on an answer in which we will need to integrate both sides, we need to find some sort of place to anchor this our integral. To do this we will apply the following logic. If agent i pulls a $v_i < p$ then clearly they will bid below p if they did bid above p and win they would necessarily gain negative utility. However since they can never beat a zero utility they will just underbid. (Long winded way of saying that bidding above your value in a first price auction is strictly dominated). However if you draw a value ABOVE the price, well then it makes sense to bid somewhere in between your valuation and the reserve price, which is known by all the agents. As this gives him/her/they a probability of gaining positive utility. Thus we show that $lim_{v_i \to p}(v_i) = p$.

Applying our gained knowledge with newfound hope vigor take the integral from p to v_i of the equation above to get:

$$\int_p^{v_i}(n-1)x^{n-1}\partial x = \int_p^{v_i}\frac{\partial}{\partial v_i}(\beta(x)x^{n-1})\partial x$$

$$\Rightarrow \frac{(n-1)}{n}v_i^n - \frac{(n-1)}{n}p^n = \beta(v_i)v_i^{n-1} - (\beta(p)p^{n-1})$$

And we showed above that $\beta(p) = p$ which means we can solve the problem!!! FINALLY:

$$\beta(v_i) = v_i - \frac{v_i}{n}(1 - (\frac{p}{n})^n) \Rightarrow \alpha = \frac{1}{n}$$

Which we know holds since $v_i \in (0,1), n > 1 \Rightarrow \alpha \in (0,1)$.

Part B

The revenue is merely the cash amount received by the seller should the item actually sell. This way we know that the lower bound is p and the upper bound is the bid someone employs if they draw the upper bound, or $\bar{v} = 1$ and $b(\bar{v}) = b(1)$. Thus we can write revenue as:

$$R = \int_p^1 b(x)dF^n(x)\partial x$$

Where $F^n(x)$ represents the derivative of the joint C.D.F. as a way to get the density should n number of people be playing the game. Thus we can derive it so that $\partial F^n(x) = nx^{n-1}$ Note that the values are all drawn independently so it's easy to simplify and then solve this expression:

$$R = \int_p^1 [x - \frac{x}{n}(1 - (\frac{p}{n})^n)] \cdot nx^{n-1}\partial x \Rightarrow = \frac{(n-1)}{(n+1)}(1 - p^{n+1}) + p^n(1-p)$$

Part C

To maximize the expected revenue we just need to set $\frac{\partial R}{\partial p} = 0$ or take the F.O.C. with respect to the reserve price in order to find its a maximum (we will also take the derivative again to show that it's concave.

$$\begin{aligned}\frac{\partial R}{\partial p} = (n-1)(-p^n) + np^{n-1} - (n+1)p^n = 0 \\ \Rightarrow p^n(np^{-1} + (-2n)) = 0 \\ \Rightarrow np^{-1} = 2n \\ \Rightarrow p = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial p}(n-1)(-p^n) + np^{n-1} - (n+1)p^n \\ = n(n-1)p^{n-2} - 2n^2p^{n-1}\end{aligned}$$

Evaluated at $p = \frac{1}{2}$:

$$(\frac{1}{2})^{n-1}(\frac{1}{2}n^2 - \frac{1}{2} - 2n^2) = (\frac{1}{2})^{n-1}(-\frac{1}{2}n - 1.5n^2) < 0, \forall n > 1$$

Thus we have our maximum.

Question 4

Consider the finite Bayesian game $BG = (A_i, T_i, u_i, p)_{i=1}^{N}$. Suppose that $p(t) > 0$ for all $t \in T$. Given $\hat{s}_i : T_i \to A_i$ for each player i, prove that $(\hat{s}_1, \ldots, \hat{s}_N)$ is a pure strategy Bayesian Nash equilibrium of G if and only if for every player i it is the case that, among all functions $s_i : T_i \to A_i$ $s_i : T_i \to A_i$ maximizes

$$\sum_{t \in T} p(t) u_i \left(s_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right), t_i, t_{-i}\right)$$

$$\boxed{\Longrightarrow}$$

Assume $(\hat{s}_1, \ldots, \hat{s}_N)$ is a pure strategy Bayesian Nash equilibrium of G. Then we have,

$$\sum_{t_{-i} \in T_{-i}} u_i\left(\hat{s}_i\left(t_i\right), \hat{s}_{-i}\left(t_{i-i}\right); t_i, t_{-i}\right) p_i\left(t_{-i} | t_i\right)$$

$$\geq$$

$$\sum_{t_{-i} \in T_{-i}} u_i\left(s_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_i\right) p_i\left(t_{-i} | t_i\right)$$

Now multiplying the strictly positive marginal probability of t_i we get

$$\sum_{t_{-i} \in T_{-i}} u_i\left(\hat{s}_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_{-i}\right) p_i\left(t_{i-i} | t_i\right) p\left(t_i\right)$$

$$\geq$$

$$\sum_{t_{-i} \in T_{-i}} u_i\left(s_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_{i-i}\right) p_i\left(t_{-i} | t_i\right) p\left(t_i\right)$$

Summing over all types we get

$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} u_i\left(\hat{s}_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_{-i}\right) p_i\left(t_i | t_i\right) p\left(t_i\right)$$

$$\geq$$

$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_i\right) p_i\left(t_{-i} | t_i\right) p\left(t_i\right)$$

Then rewriting as joint probability,

$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} u_i\left(\hat{s}_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_{-i}\right) p\left(t_i, t_{-i}\right)$$

$$\geq$$

$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_i\left(t_i\right), \hat{s}_{i-i}\left(t_{-i}\right); t_{i,} t_{i-i}\right) p\left(t_i, t_{-i}\right)$$

So we see that for every player i it is the case that, among all functions $s_i : T_i \to A_i$ $s_i : T_i \to A_i$ maximizes

$$\sum_{t \in T} p(t) u_i \left(s_i \left(t_i\right), \hat{s}_{-i} \left(t_{-i}\right), t_i, t_{-i}\right)$$

$$\boxed{\Longleftarrow}$$

Assume for every player i it is the case that, among all functions $s_i : T_i \to A_i$ $s_i : T_i \to A_i$ maximizes

$$\sum_{t \in T} p(t) u_i \left(s_i \left(t_i\right), \hat{s}_{-i} \left(t_{-i}\right), t_i, t_{-i}\right)$$

Now suppose for sake of contradiction that $(\hat{s}_1, \ldots, \hat{s}_N)$ is not a pure strategy Bayesian Nash equilibrium of G.

Then, there is some strategy s^* for a certain type t^* of individual i where

$$\sum_{t_{-i} \in T_{t-i}} u_i \left({s^*}_i \left({t^*}_i\right), \hat{s}_{-i} \left(t_{-i}\right); {t^*}_i, t_{-i}\right) p_i \left(t_{-i} | {t^*}_i\right)$$

$$>$$

$$\sum_{t_{-i} \in T_{-i}} u_i \left(\hat{s}_i \left({t^*}_i\right), \hat{s}_{-i} \left(t_{-i}\right); {t^*}_i, t_{-i}\right) p_i \left(t_{-i} | {t^*}_i\right)$$

Since all probabilities are greater than 0 we can multiple each side by the probability of being a certain type t^* and keep the strict inequality.

$$\sum_{t_{-i} \in T_{t-i}} u_i \left({s^*}_i \left({t^*}_i\right), \hat{s}_{-i} \left(t_{-i}\right); {t^*}_i, t_{-i}\right) p_i \left(t_{-i} | {t^*}_i\right) p(t_i^*)$$

$$>$$

$$\sum_{t_{-i} \in T_{t-i}} u_i \left(\hat{s}_i \left({t^*}_i\right), \hat{s}_{-i} \left(t_{-i}\right); {t^*}_i, t_{-i}\right) p_i \left(t_{-i} | {t^*}_i\right) p(t_i^*)$$

We can also add to each side the positive payoff of playing when individual is not of a certain type t^* but plays strategy $\hat{s}$ and keep the strict inequality.

$$\sum_{t_i \neq {t^*}_i} \sum_{t_{-i} \in T_{t-i}} u_i \left(\hat{s}_i \left(t_i\right), \hat{s}_{-i} \left(t_{-i}\right); t_i, t_{-i}\right) p_i \left(t_{-i} | t_i\right) p\left(t_i\right)$$

$$+$$

$$\sum_{t_{-i} \in T_{t-i}} u_i \left({s^*}_i \left({t^*}_i\right), \hat{s}_{-i} \left(t_{-i}\right); {t^*}_i, t_{-i}\right) p_i \left(t_{-i} | {t^*}_i\right) p(t_i^*)$$

$$>$$

$$\sum_{t_i \neq {t^*}_i} \sum_{t_{-i} \in T_{t-i}} u_i \left(\hat{s}_i \left(t_i\right), \hat{s}_{-i} \left(t_{-i}\right); t_i, t_{-i}\right) p_i \left(t_{-i} | t_i\right) p\left(t_i\right)$$

$$+$$

$$\sum_{t_{-i} \in T_{t-i}} u_i \left(\hat{s}_i \left({t^*}_i\right), \hat{s}_{-i} \left(t_{-i}\right); {t^*}_i, t_{-i}\right) p_i \left(t_{-i} | {t^*}_i\right) p(t_i^*)$$

Then we arrive at a contradiction since the payoff function of an individual who plays the strategy two different strategies depending on the type will be greater than the first payoff which is assumed to be maximized. Note that multiplying conditional probability and marginal probability give joint probability.

$$\sum_{t_i \neq t^*{}_i} \sum_{t_{-i} \in T_{t-i}} u_i\left(\hat{s}_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_{-i}\right) p_i\left(t_{-i}|t_i\right) p\left(t_i\right)$$

$$+$$

$$\sum_{t_{-i} \in T_{t-i}} u_i\left(s^*{}_i\left(t^*{}_i\right), \hat{s}_{-i}\left(t_{-i}\right); t^*{}_i, t_{-i}\right) p_i\left(t_{-i}|t^*{}_i\right) p(t_i^*)$$

$$>$$

$$\sum_{t_i \in T_i, t_{-i} \in T_{-i}} u_i\left(\hat{s}_i\left(t_i\right), \hat{s}_{-i}\left(t_{-i}\right); t_i, t_{-i}\right) p\left(t_i, t_{-i}\right)$$

Question 6

In the language of the notes the players are $i \in \{1, 2\}$ each with an action set $A_i = \{0, 1\}$ and a type set $T_i = \{0, 2, 3\}$. The common prior is given by $p(t_i, t_{-i}) = \frac{1}{9}$ for each $(t_i, t_{-i}) \in \{0, 2, 3\}^2$. The VNM payoffs conditional on type are given by:

$$u_i(a_i, a_{-i}|t_i) = \begin{cases} \frac{1}{2}t_i & \text{if } a_i = 0, a_{-i} = 0 \\ t_i - 1 & \text{if } a_i = 1, a_{-i} = 0 \\ 0 & \text{if } a_i = 0, a_{-i} = 1 \\ \frac{1}{2}(t_i - 1) & \text{if } a_i = 1, a_{-i} = 1 \end{cases}$$

As in the notes, let $b_{-i}(a_{-i}|t_{-i})$ denote the (possibly random) behavioural strategy of player $-i$ conditional on their type and we look at player i's best response for each possible $t_i \in T_i$.

Case 1: $t_i = 0$
In this case playing 0 is a strictly dominant strategy for any possible behavioural strategy of the other player. To see this, note that $u_i(0, a_{-i}|0) = 0 \; \forall a_{-i}$ and following lecture notation:

$$\begin{aligned} V_i(0, b_{-i}|t_i = 0) &= 0 \\ &> \frac{1}{3}\Big(- b_{-i}(0|0) - \frac{1}{2}b_{-i}(1|0)\Big) + \frac{1}{3}\Big(- b_{-i}(0|2) - \frac{1}{2}b_{-i}(1|2)\Big) + \frac{1}{3}\Big(- b_{-i}(0|3) - \frac{1}{2}b_{-i}(1|3)\Big) \\ &= V_i(1, b_{-i}|t_i = 0) \end{aligned}$$

Since all the b_{-i} lie in $[0, 1]$.

Case 2: $t_i = 3$
We handle this case next. given that the players are both rational, they know from case 1 that each player will play 0 for sure when they are of type 0. Incorporating this, we find that 1 is the strictly dominant strategy:

$$\begin{aligned}
V_i(1, b_{-i}|t_i = 3) &= \frac{1}{3}\Big(2\Big) + \frac{1}{3}\Big(2b_{-i}(0|2) + b_{-i}(1|2)\Big) + \frac{1}{3}\Big(2b_{-i}(0|3) + b_{-i}(1|3))\Big) \\
&> \frac{1}{3}\Big(1\Big) + \frac{1}{3}\Big(\frac{3}{2}b_{-i}(0|2)\Big) + \frac{1}{3}\Big(\frac{3}{2}b_{-i}(0|3)\Big) \\
&= V_i(0, b_{-i}|t_i = 3)
\end{aligned}$$

Case 3: $t_i = 2$
Again by rationality of the players, we can incorporate the strictly dominant strategies for types 0 and 3 into the payoff calculations. Once we have done this we find that the strictly dominant strategy for a type 2 player is to play 1 for sure. This holds since:

$$\begin{aligned}
V_i(1, b_{-i}|t_i = 2) &= \frac{1}{3}\Big(1\Big) + \frac{1}{3}\Big(b_{-i}(0|2) + \frac{1}{2}b_{-i}(1|2)\Big) + \frac{1}{3}\Big(\frac{1}{2}\Big) \\
&> \frac{1}{3}\Big(1\Big) + \frac{1}{3}\Big(b_{-i}(0|2)\Big) + \frac{1}{3}\Big(0\Big) \\
&= V_i(0, b_{-i}|2)
\end{aligned}$$

Given the above inequalities are strict we find that no mixing is possible and the unique BNE is given by:

$$b_1(0|0) = b_2(0|0) = 1$$
$$b_1(1|0) = b_2(1|0) = 0$$

$$b_1(0|2) = b_2(0|2) = 0$$
$$b_1(1|2) = b_2(1|2) = 1$$

$$b_1(0|3) = b_2(0|3) = 0$$
$$b_1(1|3) = b_2(1|3) = 1$$

Book Question 7.18 (Question 8)

7.18 Reconsider the two countries from the previous exercise, but now suppose that country 1 can be one of two types, 'aggressive' or 'non-aggressive'. Country 1 knows its own type. Country 2 does not know country 1's type, but believes that country 1 is aggressive with probability $\varepsilon > 0$. The aggressive type places great importance on keeping its weapons. If it does so and country 2 spies on the aggressive type this leads to war, which the aggressive type wins and justifies because of the spying, but which is very costly for country 2. When country 1 is non-aggressive, the payoffs are as before (i.e., as in the previous exercise). The payoff matrices associated with each of the two possible types of country 1 are given below.

Country 1 is 'aggressive'
Probability ε

	Spy	Don't Spy
Keep	10, −9	5, −1
Destroy	0, 2	0, 2

Country 1 is 'non-aggressive'
Probability $1 - \varepsilon$

	Spy	Don't Spy
Keep	−1, 1	1, −1
Destroy	0, 2	0, 2

369

(a) What action must the aggressive type of country 1 take in any Bayesian-Nash equilibrium?

(b) Assuming that $\varepsilon < 1/5$, find the unique Bayes-Nash equilibrium. (Can you prove that it is unique?)

Part A

Assuming that Country 1 is aggressive we know that the country will always keep it's weapons. How do we know this? It's simple really, in this game keep strictly dominates destroy:

$$10 > 0 \Rightarrow u_1(K,S) > u_1(D,S)$$
$$5 > 0 \Rightarrow u_1(K,DS) > u_1(D,DS)$$

So it doesn't matter what the second country does, country 1 gon' keep it's weapons. This also tells us that if the second country knew country 1 was aggressive it would not spy. But that's not very interesting is it! (From an Economic perspective... From a geopolitical / war and espionage perspective it's probably a bit interesting.)

Part B

Alright this is where things get interesting. We follow the convention in class. Because we know the Nash Equilibrium in the aggressive case is (Keep, Don't Spy) we have already figured out 50 percent of the subgames.

Note that in the non-aggressive game it is unclear which action is best for country 1. If country two is spying, they (country 1) would much rather destroy their weapons. BUT country 1 would rather keep their weapons if they knew that country 2 was not spying. So here we need to find what are the probability, denoted by "p" here, that would make country 1 mix in this case:

$$\begin{aligned} u_1(K,p) = -1p + 1(1-p) = 1-2p \\ u_1(D,p) = 0 \\ \Rightarrow 1-2p >=< 0 \\ \Rightarrow p >=< \frac{1}{2} \end{aligned}$$

We can see from the eqaution above that if $p < \frac{1}{2}$, country 1 would rather keep their weapons ($q = 1$). If $p > \frac{1}{2}$ then they'd rather destroy them ($q = 0$). This leaves:

$$p = \frac{1}{2}$$

Because country 1 is indifferent to to either strategy in the non-aggressive scenario they mix at rate q (rate at which they keep).

$$u_2(S,q) = \epsilon(-9) + (1-\epsilon)[(1)q + (2)(1-q)] = -9\epsilon + (1-\epsilon)(2-q)$$
$$u_2(DS,q) = \epsilon(-1) + (1-\epsilon)[(-1)q + (2)(1-q)] = -\epsilon + (1-\epsilon)(2-3q)$$

If $u_2(S,q) > u_2(DS,q)$ then country 2 plays $p = 1$ and if $u_2(S,q) < u_2(DS,q)$ then country 2 plays $p = 0$. However, that if we simplify these expressions:

$$\begin{aligned} -9\epsilon + (1-\epsilon)(2-q) - (-\epsilon + (1-\epsilon)(2-3q)) >=< 0 \\ -8\epsilon + (1-\epsilon)(2q) >=< 0 \end{aligned}$$

Let's take the scenario where $p < \frac{1}{2}$ or country 2 picks DS more often causing country 1 to choose to keep their weapons. This sets $q = 1$. Given that $\epsilon < \frac{1}{5}$, in this case we get:

$$-8\epsilon + (1-\epsilon)2 > (-1.6) + (1.6) = 0$$

But this means that country 2 picks p = 1 now because they get a higher payoff spying. But we already imposed $p < \frac{1}{2}$ so this is necessarily a contradiction.

We get a similar dynamic when we pick $p > \frac{1}{2} \Rightarrow q = 0$:

$$-8\epsilon < 0$$

But in this case now they want NOT to spy, or $p = 0$. Contradicting the notion that $p > \frac{1}{2}$.

Having ruled out all other cases we turn our attention to the lonesome $p = \frac{1}{2}$. Here we know that q is just a mixture of some sort, the structure of which we will see in a moment, and we get that this u_2 dynamic above must be equal:

$$-8\epsilon + (1-\epsilon)(2q) = 0 \Rightarrow (1-\epsilon)(2q) = 8\epsilon$$
$$q = \frac{4\epsilon}{(1-\epsilon)}$$

Thus our Bayes-Nash Equilibrium is:

$$((K, q = \frac{4\epsilon}{1-\epsilon}), p = \frac{1}{2})$$

Book Question 7.19 (also Question 8)

A community is composed of two types of individuals, good types and bad types. A fraction $\varepsilon > 0$ are bad, while the remaining fraction, $1 - \varepsilon > 0$ are good. Bad types are wanted by the police, while good types are not. Individuals can decide what colour car to drive, red or blue. Red cars are faster than blue cars. All individuals prefer fast cars to slow cars. Each day the police decide whether to stop only red cars or to stop only blue cars, or to stop no cars at all. They cannot stop all cars. Individuals must decide what colour car to drive. Individuals do not like being stopped, and police do not like stopping good individuals. A bad individual always tries to get away if stopped by the police and is more likely to get away if driving a red car. The payoff matrices associated with this daily situation are as follows.

Bad Individual Probability ε

	Red Car	Blue Car
Stop Red	$5, -5$	$-10, 10$
Stop Blue	$-10, 15$	$10, -10$
Don't Stop	$-10, 15$	$-10, 10$

Good Individual Probability $1 - \varepsilon$

	Red Car	Blue Car
Stop Red	$-1, -1$	$0, 0$
Stop Blue	$0, 5$	$-1, -6$
Don't Stop	$0, 5$	$0, 0$

Think of this as a Bayesian game and answer the following.

(a) Suppose that $\varepsilon < 1/21$. Find two pure strategy Bayes-Nash equilibria of this game.

. . . .

We want to consider the associated strategic form game. To do so, we construct payoffs as on p. 322 of JR. So the payoff for the police, for any pure strategy profile $s = (s_p, s_b, s_g)$, where s_p is the police, s_b is bad guys, and s_g is good guys, is

$$v(s) = \sum_{j \in \{b,g\}} p(j)u(s_p, s_j)$$

For the bad and good guys, there is no randomness, so the payoffs just carry over. Then we have a $3 \times 2 \times 2$ payoff matrix, where the slices give different strategies for the bad individual, and each matrix has players police and good individual. The payoffs are ordered $(Police, Bad, Good)$:

Bad Individual Red

	Red Car	Blue Car
Stop Red	$5\epsilon - (1-\epsilon), -5, -1$	$5\epsilon + 0, -5, 0$
Stop Blue	$-10\epsilon + 0, 15, 5$	$-10\epsilon - (1-\epsilon), 15, -6$
Don't Stop	$-10\epsilon + 0, 15, 5$	$-10\epsilon + 0, 15, 0$

Bad Individual Blue

	Red Car	Blue Car
Stop Red	$-10\epsilon - (1-\epsilon), 10, -1$	$-10\epsilon + 0, 10, 0$
Stop Blue	$10\epsilon + 0, -10, 5$	$10\epsilon - (1-\epsilon), -10, -6$
Don't Stop	$-10\epsilon + 0, 10, 5$	$-10\epsilon + 0, 10, 0$

There are 12 possibilities for a pure Bayes-Nash equilibrium. Going through all the cases that are not Nash equilibria would be tedious, and merely consist of always showing that someone wants to deviate. Here are the cases where no one wants to deviate, and thus we have a Bayes-Nash equilibrium

(i) (Stop Red, Blue, Blue): If the police deviate they get payoff -10ϵ (the same) or $10\epsilon-(1-\epsilon)$, which is less than -10ϵ because $\epsilon<\frac{1}{21}$. If the baddies deviate, their payoff falls from 10 to -5. If the goodies deviate, their payoff falls from 0 to -1. So no one has an incentive to deviate.

(ii) (Stop Blue, Red, Red): If the police deviate to Don't Stop, they get the same payoff, and if they deivate to Stop Red, they get payoff $5\epsilon-(1-\epsilon)$, which is less than -10ϵ because $\epsilon<\frac{1}{21}$. If the bads deviate, they move from 15 to -10. If the goods deviate, they move from 5 to -6. Again, no incentives to deviate.

(iii) (Don't Stop, Red, Red): By the exact argument above, the police will not deviate. If the bads deviate, they move from 15 to 10, and if the goods deviate, they move from 5 to 0. No incentive to deviate.

(b) Suppose that $\varepsilon>1/16$. Show that 'Don't Stop' is strictly dominated for the police by a mixed strategy.

. . . .

The increased probability of catching bad people has made some of the strategies involving stopping have higher payoffs. We can search for a mixed strategy such that, regardless of what the good and bad guys do, the police still have a higher expected payoff than just purely playing Don't Stop. Let the probability of Stop Red be r, Stop Blue be b, and Don't Stop $1-r-b$, so that we are showing there exists a mixed strategy which strictly dominate $r=b=0$. Now we must consider the cases for the choices of bad and good guys.

(a) Bad Red, Good Red: Note the payoff for Stop Red: $5\epsilon-(1-\epsilon)=6\epsilon-1>-10\epsilon$ because $\epsilon>\frac{1}{16}$. Then, since the payoff for both Stop Blue and Don't Stop is -10ϵ, any strategy with $r>0$ will strictly improve upon pure Don't Stop.

(b) Bad Red, Good Blue: The police payoff is

$$
\begin{aligned}
r(5\epsilon)+b(-10\epsilon-(1-\epsilon))+(1-r-b)(-10\epsilon) &= r(15\epsilon)+b(-(1-\epsilon))-10\epsilon \\
&\Rightarrow r(15\epsilon)-b(1-\epsilon)>0 \quad \text{(To Improve on Don't Stop)} \\
&\Rightarrow r>b\frac{1-\epsilon}{15\epsilon}
\end{aligned}
$$

(c) Bad Blue, Good Red: The police payoff is

$$
\begin{aligned}
r(-10\epsilon-(1-\epsilon))+b(10\epsilon)+(1-r-b)(-10\epsilon) &= r(-(1-\epsilon)+b(20\epsilon)-10\epsilon \\
&\Rightarrow r(-(1-\epsilon)+b(20\epsilon)>0 \qquad \text{(Improve on Don't Stop)} \\
&\Rightarrow b>r\frac{1-\epsilon}{20\epsilon}
\end{aligned}
$$

(d) Bad Blue, Good Blue: In this case, $10\epsilon+(1-\epsilon)=11\epsilon-1>-10\epsilon$, since $\epsilon>\frac{1}{16}$, so any strategy with $b>0$ will dominate pure Don't Stop.

Now we just need to find a strategy that satisfies all the above conditions. The middle ones imply the outer ones, and there is no unique mix that works, as long as the inequalities above aren't violated. We can consider making the inequalities as stringent as possible by playing with ϵ, and we note that both $\frac{1-\epsilon}{15\epsilon}$ and $\frac{1-\epsilon}{20\epsilon}$ are decreasing over $(0, 1]$. We can also note that giving positive weight to Don't Stop does not help us achieve any of the above conditions, so we will give it zero weight and set $b = 1 - r$. Therefore our bounds imply

$$
\begin{aligned}
r > 1 - r &\Rightarrow r > \frac{1}{2} \\
r < \frac{4}{3}(1 - r) &\Rightarrow r < \frac{4}{7} \\
\Rightarrow \frac{1}{2} < r < \frac{4}{7}
\end{aligned}
$$

So we have shown that the mixed strategy of playing Stop Red with probability $r \in (\frac{1}{2}, \frac{4}{7})$, and Stop Blue with probability $1 - r$ dominates the pure strategy of playing Don't Stop.

(c) Suppose that $\varepsilon = 1/6$. Find the unique Bayes-Nash equilibrium. (Can you prove that it is unique?)

. . . .

Our above result, that we found a mixed strategy of Stop Red and Stop Blue which strictly dominated Don't Stop, tells us that in a Bayes-Nash equilibrium, it cannot be the case that Don't Stop is played with positive probability. Then we may consider the reduced game, and also plug in $\epsilon = \frac{1}{6}$.

Bad Individual Red

	Red Car	Blue Car
Stop Red	$0, -5, -1$	$\frac{5}{6}, -5, 0$
Stop Blue	$-\frac{10}{6}, 15, 5$	$-\frac{15}{6}, 15, -6$

Bad Individual Blue

	Red Car	Blue Car
Stop Red	$-\frac{15}{6}, 10, -1$	$-\frac{10}{6}, 10, 0$
Stop Blue	$\frac{10}{6}, -10, 5$	$\frac{5}{6}, -10, -6$

It takes about to second to spot check that no completely pure strategy equilibria exist, since for each of the 8 entries above, at least one player will want to deviate. So we search for mixed strategies

We will denote p for the probability the police play Stop Red, b for a bad playing Red, and g for a good playing red. Then for any mixed strategy to be optimal it must be the case that, for the mixing player, the probabilities of the other two players make it such that both choices give the same expected payoff.

For the baddies, this means

$$\underbrace{p(-5) + (1-p)(15)}_{\text{Red}} = \underbrace{p(10) + (1-p)(-10)}_{\text{Blue}}$$
$$\Rightarrow p = \frac{5}{8}$$

For the goodies, this means

$$\underbrace{p(-1) + (1-p)(5)}_{\text{Red}} = \underbrace{p(0) + (1-p)(-6)}_{\text{Blue}}$$
$$\Rightarrow p = \frac{11}{12}$$

These two p values do not agree, which means that at least one of the drivers must optimally play a pure strategy. We are left with an exercise in case-checking all the p for equilibria, but luckily the above equations suggest a method for dividing the interval $[0, 1]$. I will make ample use of the extensive form matrices above.

(i) $p = 0$ (pure Stop Blue): In this case, the matrices above show it is clearly optimal for both drivers to choose Red. But then it would be optimal for the police to pursue a pure strategy of Stop Red, so this is not a BN eq.

(ii) $p \in (0, \frac{5}{8})$: Our above calculations show that for both the bad and good guys, it will still be optimal to play the pure strategy of Red. Then we have the same issue as above, and thus we still don't have a BN eq.

(iii) $p = \frac{5}{8}$: The good guys will still find it optimal to play the pure strategy Red, but now the bad guys are indifferent. Then we can see that for a BN eq. to exist, the bad guys must be randomizing so that the police are indifferent about which car color to stop. Since we know the good guys choose Red, and $p = \frac{5}{8}$, our condition then becomes

$$\underbrace{b(0) + (1-b)(-\frac{15}{6})}_{\text{Stop Red}} = \underbrace{b(-\frac{10}{6}) + (1-b)(\frac{10}{6})}_{\text{Stop Blue}}$$
$$\Rightarrow b = \frac{5}{7}$$

Then this mixing works, because the police are indifferent, the baddies are indifferent, and the goodies find deviating from their pure strategy suboptimal. <u>So the strategy profile where the police Stop Red with probability $\frac{5}{8}$, Stop Blue with probability $\frac{3}{8}$, and Bads drive Red with probability $\frac{5}{7}$, and Goods drive Red, is a Bayes-Nash equilibrium!</u>

(iv) $p \in (\frac{5}{8}, \frac{11}{12})$: From above, here it is optimal for Bads to play Blue, and Goods to play Red. But in this case the police will want to deviate to a pure strategy of Stop Blue. So not a BNE.

(v) $p = \frac{11}{12}$: The Bads will find Blue optimal. Then it is immediately clear from the matrix above that, given the Bads play Blue, playing Stop Blue is a strictly dominant strategy for the police, so this BNE cannot work.

(vi) $p \in (\frac{11}{12}, 1)$: Now both the drivers find it optimal to play Blue. But this means it will be optimal for the police to play Stop Blue, so this cannot be a BNE.

(vii) $p = 1$ (pure Stop Red): The exact reasoning for the previous item goes through, since it will be optimal to deviate from this pure strategy to the pure Stop Blue, given the other players' optimal reactions.

We have proven uniqueness by ruling out subcases (no Don't Stop), then exhausting all the remaining possibilities, and only finding one solution.