Price Theory II Problem Set 3

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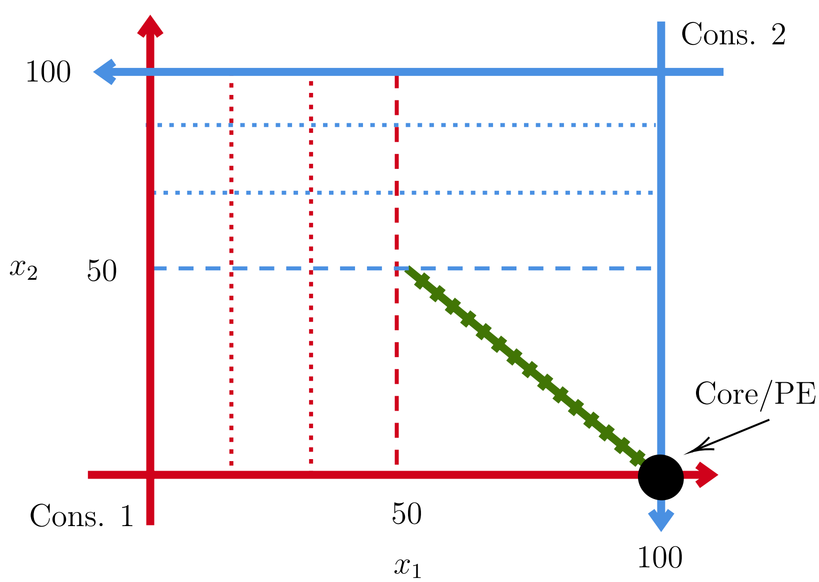
5.15

There are 100 units of x_1 and 100 units of x_2. Consumer's 1 and 2 are each endowed with 50 units of each good. Consumer 1 says "I love x_1, but I can take or leave x_2. Consumer 2 says "I love x_2, but I can take or leave x_1.

(a) *Draw an Edgeworth box for these traders and sketch their preferences.*

. . . .

I have sketched the Edgeworth box below. Consumer 1 has their origin at the bottom left, and consumer 2 has their origin at the top right. Indifference curves for 1 are vertical lines, increasing in utility as we move east, because they are indifferent between all levels for x_2, but prefer x_1. Similiarly, 2 has horizontal indifference curves, increasing as we move south.



(b) *Identify the core of this economy.*

. . . .

Recall that the core is a subset of the set of Pareto efficient allocations, since the grand coalition will block any non-Pareto efficient allocation. We can also argue that the only Pareto efficient allocation is $(100, 0)$ for 1, $(0, 100)$ for 2. Take any other allocation in the box (a, b) for 1, $(100 - a, 100 - b)$ for 2. We can increase 1's utility without taking any from 2 by giving 2's x_1 to 1. From this allocation $(100, b)$, we can increase 2's utility without taking any from 1 by given 1's x_2 to 2. So any other allocation is Pareto dominated by $(100, 0)$.

We have just shown that the grand coalition will not block. Now consider that each of the consumers has utility 50 at their endowment, but has utility 100 at $(100, 0)$, so their singleton coalitions will not block, either. So $(100, 0)$ is the core.

(c) *Find all Walrasian equilibria in this economy.*

. . . .

Normalize the price of x_1 to 1. Then the green line denotes the price vector $(1, 1)$ which is the unique Walrasian equilibrium. First, at $(1, 1)$ the consumers' demands match the Pareto efficient allocation. If we consider any other price, say $(1, p)$, then consumer 1 will demand $(50 + 50p, 0)$, which will not be feasible if $p > 1$, and consumer 2 will demand $(0, 50 + \frac{50}{p})$, which will not be feasible if $p < 1$. So we must have $p = 1$.

5.18

Part a

We want to solve the following social planner problem,

$$\max_{x_{11}, x_{12}, x_{21}, x_{22}} x_{11} x_{12}^2$$

subject to,

$$x_{21}^2 x_{22} = \frac{8000}{27}$$
$$x_{11} + x_{21} = 10$$
$$x_{12} + x_{22} = 20$$

Since, the solution must be in the interior, we can simply find the point on the contract curve that gives agent two utility $\frac{8000}{27}$. Along the contract curve, no agent can be made better off, without making any other agent worse off. Therefore, the allocation on the contract curve that gives agent two utility $\frac{8000}{27}$, must be the solution to the problem above.

On the contract curve

$$MRS_1 = MRS_2 \iff \frac{x_{12}}{2x_{11}} = \frac{2x_{22}}{x_{21}} \iff \frac{20 - x_{22}}{2(10 - x_{21})} = \frac{2x_{22}}{x_{21}}$$

$$\iff x_{22} = \frac{20x_{21}}{40 - 3x_{21}}$$

We can now substitute this into the utility function to find x_{11},

$$x_{21}^2 \frac{20x_{21}}{40 - 3x_{21}} = \frac{8000}{27} \implies 27x_{21}^2 + 1200x_{21} - 16000 = 0$$

Solving this we get,

$$x_{11} = \frac{10}{3}; x_{12} = \frac{40}{3}; x_{21} = \frac{20}{3}; x_{22} = \frac{20}{3}$$

By construction, this solution is Pareto optimal.

Part b

Setting $MRS_i = \frac{p_1}{p_2}$ and substituting into the budget constraint we get the following Marshillian demand curves,

$$x_{11}^* = \frac{10}{3}; x_{12}^* = \frac{20}{3}p; x_{21}^* = \frac{40}{3p}; x_{22}^* = \frac{20}{3}$$

where $p = \frac{p_1}{p_2}$.

By market clearing we know that,

$$x_{11}^* + x_{21}^* = \frac{10}{3} + \frac{40}{3p} = 10 \implies p = 2$$

Therefore,

$$x_{11} = \frac{10}{3}; x_{12} = \frac{40}{3}; x_{21} = \frac{20}{3}; x_{22} = \frac{20}{3}$$

Which is identical to part a.

5.23

Show that if a firm's production set is strongly convex and the price vector is strictly positive, then there is at most one profit-maximizing production plan.

Let's start by a gentle reminder of the definition of a **strongly convex production set**: if y^1 and $y^2 \in Y^j$, and for all $t \in (0, 1)$, there exists $\bar{y} \in Y^j$ st $\bar{y} \geq ty^1 + (1-t)y^2$, for $\bar{y} \neq ty^1 + (1-t)y^2$.
Notice that this question asks us to prove the first part of **Theorem 5.9**, so we will repeat it here for completeness: *If Y^j satisfies A.5.2. then for all $p \in \mathbf{R}_{++}^N$, a solution to the firm profit maximization problem exists and is unite, denoted $y^j(p)$. Moreover $y^j(p)$ is continuous on $\mathbf{R}_{++}^N$ and $\Pi^j(p)$ is well-defined and continuous on $\mathbf{R}_{+}^N$.*

Let's prove that the solution to the maximization problem is unique BWOC.
Assume that two plans y_1 and y_2 both solve the following maximization problem of the firm

$$\max_{y^j \in Y^j} \Pi^j(p) = p.y_j = \sum_{k=1}^{n} p_k y_k^j$$

Because both these plans y^1, y^2 belongs to Y^j, the **strongly convex** production set, then for all $t \in (0,1)$, there must exists a $\bar{y} \in Y^j$ st:

$$ty^1 + (1-t)y^2 \geq \bar{y}$$
$$ty^1 + (1-t)y^2 \neq \bar{y}$$

However, this is a contradiction because both y^1 and y^2 are already maximizing firms' profits and prices are strictly positive, so there cannot exist another affordable production plan $\bar{y}$ that does better than them in the production set, or it would be the profit maximizing plan itself.

$$p.\bar{y} > p.y^1 = p.y^2$$

Notice that this contradiction is resolved if $y^1 = y^2$, as the definition of strongly convex sets only applies to linear combinations of different plans. Therefore, if both y^1 and y^2 are profit maximizing, then it must be the case that $y^1 = y^2$.

5.26

Suppose that in a single-consumer economy, the consumer is endowed with none of the consumption good y, and 24 hours of time, h, so $\mathbf{e} = (24, 0)$. Suppose as well that preferences are defined over $\mathbb{R}^2_+$ and represented by $u(h,y) = hy$, and production possibilities are $Y = \{(-h, y) \mid 0 \leq h \leq b, 0 \leq y \leq \sqrt{h}\}$, where b is some large positive number. Let p_y and p_h be prices of the consumption and leisure good, respectively.

(a) *Find relative prices $\frac{p_y}{p_h}$ that clear the consumption and leisure markets simultaneously.*

. . . .

To make our lives easier, we first rule out corners, so that we know the solutions to the optimization problems are interior. Consider if $p_h = 0$, $p_y = 0$. Then, since u is increasing in h for all $y > 0$, the demand for leisure will be infinite, and analogously the demand for y will be infinite. If only $p_h > 0$, then the consumer will always sell some amount of their labor, then have infinite demand for y. If only $p_y > 0$, then the firm will have infinite demand for labor. So we must have that both prices are positive.

We can also rule out corner solutions for labor/leisure, since the production technology roughly satisfies Inada conditions, in the sense that for any given positive prices, at least some labor will be demanded. On the other side, we may rule out the zero leisure case by noting that positive prices imply the consumer will always demand at least a little leisure, by the nature of their utility function. Technically, we also need to check that $b \geq 24$, but we are given that it is "large", so I conjecture b must be at least 100.

Normalize $p_h = 1$, so that $\frac{p_y}{p_h} = p_y \equiv p$. I let h^c denote leisure (consumption time) and h^f denote labor (firm time). Also, for notational convenience, I let π denote the firm profits, which become dividends for the consumer. The consumer solves

$$\max_{h^c \in [0,24], y \geq 0} h^c y$$
$$\text{s.t. } h^c + py = 24 + \pi$$

Then we use contrained optimization

$$\begin{aligned}
\mathcal{L}(h^c, y, \lambda) &= h^c y + \lambda[24 + \pi - h^c - py] \\
\frac{\partial \mathcal{L}}{\partial h^c} &= y - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= h^c - p\lambda \leq 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 24 + \pi - h^c - py = 0
\end{aligned}$$

We can then find consumer Marshallian demand (which maps into labor supply through the labor/leisure time market as $h^f = 24 - h^c$).

$$\begin{aligned}
\lambda &= y = \frac{h^c}{p} \\
0 &= 24 + \pi - h^c - h^c \\
\Rightarrow h^c &= \frac{1}{2}(24 + \pi) \\
y &= \frac{1}{2p}(24 + \pi)
\end{aligned}$$

The firm's profits are increasing in y, so given that they are using h as an input, they will always choose a production point on the boundary such that $y = \sqrt{h}$. Thus they solve

$$\max_{(h^f, y) \in Y} \pi = py - h^f$$

Then

$$
\begin{aligned}
\mathcal{L}(h^f, y, \mu) &= py - h^f - \mu[y - \sqrt{h^f}] \\
\frac{\partial \mathcal{L}}{\partial h^f} &= -1 + \mu \frac{1}{2}(h^f)^{-\frac{1}{2}} = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= p - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= y - \sqrt{h^f} = 0
\end{aligned}
$$

We may then find firm labor demand

$$
\begin{aligned}
\mu = 2(h^f)^{\frac{1}{2}} &= p \\
\Rightarrow h^f &= \frac{p^2}{4}
\end{aligned}
$$

We now equate supply and demand, using that everything may be written in terms of h^f, from our FONCs.

$$
\begin{aligned}
\underbrace{h^f}_{\text{demand}} &= \underbrace{24 - h^c}_{\text{supply}} \\
&= 24 - \frac{1}{2}(24 + \pi) \\
&= 24 - \frac{1}{2}(24 + py - h^f) \\
&= 24 - \frac{1}{2}(24 + 2\sqrt{h^f}\sqrt{h^f} - h^f) \\
\Rightarrow h^f &= 8
\end{aligned}
$$

Then we may calculate the other endogenous variables

$$
\begin{aligned}
y &= \sqrt{h} = 2\sqrt{2} \\
p &= 2\sqrt{h^f} = 4\sqrt{2}
\end{aligned}
$$

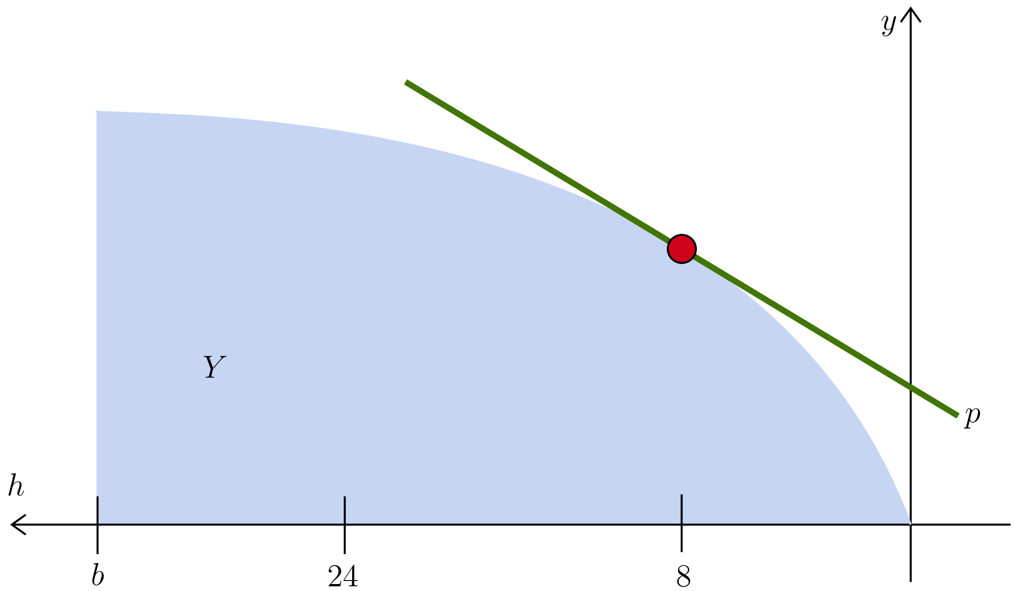
Therefore if $\frac{p_y}{p_h} = 4\sqrt{2}$, then both maximization problems are solved and markets clear.

(b) *Calculate the equilibrium consumption and production plans and sketch your results in* $\mathbb{R}^2_+$.

. . . .

We above demonstrated that $y = 2\sqrt{2}$, and we must have $y = c$ for the goods market to clear[1], so $c = 2\sqrt{2}$. Since we found labor $h^f = 8$, we must have leisure $h^c = 24 - h^f = 16$.

I'm not entirely sure why we are asked to sketch the equilibrium, since it is a point, so I assume this question is asking for the production set, along with the tangent price vector.



(c) *How many hours a day does the consumer work?*

. . . .

Based on the work done above, I have to go with 8.

5.28

Suppose the two agents in our economy have the following two utility functions,

$$u^a(x, y) = x^{\frac{1}{2}}$$

$$u^b(x, y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$$

Both these utility functions are continuous and quasi-concave, and hence satisfy the assumptions in question 5.27.

The allocation $\bar{x} = ((1, 0), (0, 1))$ is Pareto optimal. Since if we take any of good x away from agent a we will make him strictly worse off, regardless of how much we compensate him with good y—therefore we can make no changes to good x's allocation. Any changes to

[1] In the first part above, this substitution was made implicitly at the beginning, but we were careful to not impose market clearing before optimizing.

good y's allocation without change x will make agent b worse off.

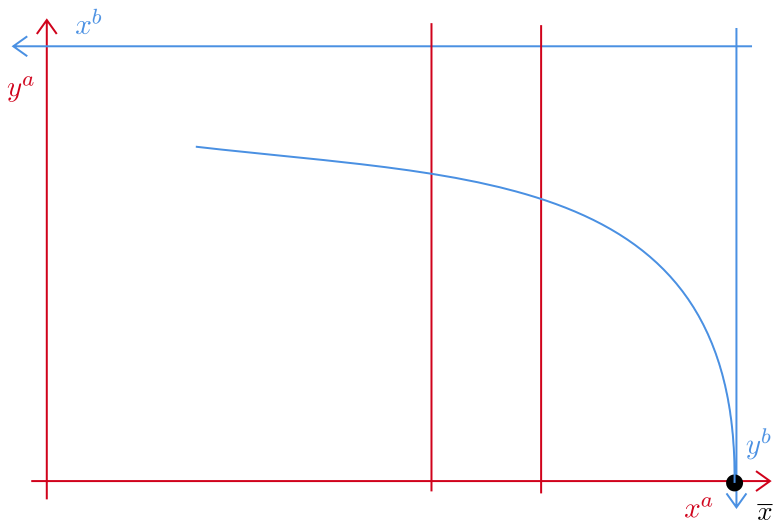
While $\bar{x} = ((1,0),(0,1))$ is Pareto optimal, it cannot be supported by a Walrasian equilibrium. Intuitively, any price less than ∞ for good x will result in agent b demanding a non-zero quantity of x (since his marginal utility goes to infinity at zero consumption). Therefore, the relative price of good x must go to infinity to support $\bar{x}$, hence there isn't a WE. Somewhat more formally, agent b has the following Marshallian demand curves,

$$x_b^* = \frac{1}{1+p}$$

where $p = \frac{p_x}{p_y}$. Notice, $p \to \infty$ for $x_b^* \to 0$.

Theorem 5.8 assumes the utilities are strongly increasing, agent a's utility violates this assumption.

Figure 1: Marginal rate of substitution at $\bar{x}$ is ∞



5.33

The contingent-commodity interpretation of our general equilibrium model permits us to consider time as well as uncertainty and more. While the trading of contracts nicely captures the idea of futures contracts and prices, one might wonder about the role that spot markets play in our theory. This exercise will guide you through thinking about this. The main result is that once the date zero contingent-commodity contracts market has cleard at Walrasian prices, there is no remaining role for spot markets. Even if spot markets were to open up for some or all goods in some or all periods and in some or all states of the world, no additional trade would take place. All agents would simply exercise the contracts they already have in hand.

a) Consider an exchange economy with I consumers, N goods, and T = 2 dates. There is no uncertainty. We will focus on one consumer whose utility

function is $u(x_1, x_2)$***, where*** $x_t \in \mathbf{R}^N_{++}$ ***is a vector of period-t consumption of the N goods. Suppose that*** $\hat{p} = (\hat{p_1}, \hat{p_2})$ ***is a Walrasian equilibrium price vector in the contingent commodity sense described in Section 5.4, where*** $\hat{p_t} \in \mathbf{R}^N_{++}$ ***is the price vector for period-t contracts on the N goods. Let*** $\hat{x} = (\hat{x_1}, \hat{x_2})$ ***be the vector of contracts that our consumer purchases prior to date 1 given the WE price vector*** $\hat{p} = (\hat{p_1}, \hat{p_2})$***.***
Suppose now that, at each date t, spot market open for trade.

(i) Because all existing contracts are enforced, argue that our consumer's available endowment in period t is $\hat{x_t}$
What Arrow-Debreu, fully enforceable contracts traded prior to date 1 do, is to define your endowment (and as we will see your consumption bundles as you have no incentive to trade then) at each date t. Therefore, is $\hat{x_t}$ is the consumer's allocation under the existing contracts, then it is also his endowments at date t.

(ii) Show that, if our consumer wishes to trade in some period-t spot market, and if all goods have period-t spot markets, and the period-t spot prices are $\hat{p}$***, then our consumer's period t budget-constraint is***

$$\hat{p_t}.x_t \leq \hat{p_t}.\hat{x_t}$$

As argued above, if contracts are fully enforceable, then deciding on $\hat{x_t}$ before period 1 is equivalent to choosing your endowment for period t to be $\hat{x_t}$. Moreover, we are told that period-t spot prices are the same as the prices the consumer based his $\hat{x_t}$ choice on: $\hat{p_t}$. The LHS represents the set of bundles that are affordable for the consumer at time t given the vector price $\hat{p_t}$ and his endowment $\hat{x_t}$.

(iii) Conclude that our consumer can ultimately choose any (x_1, x_2) ***such that***

$$\hat{p_1}.x_1 \leq \hat{p_1}.\hat{x_1}$$
$$\hat{p_2}.x_2 \leq \hat{p_2}.\hat{x_2}$$

Trivially, the budget constraint defined in the previous question for a general t applies to both $t = \{1, 2\}$

(iv) Prove that the consumer can do no better than to choose $x_1 = \hat{x_1}$ ***in period*** $t = 1$ ***and*** $x_2 = \hat{x_2}$ ***in period*** $t = 2$ ***by showing that any bundle that is feasible through trading in spot markets is feasible in the contingent-commodity contract market. You should assume that, in period 1, the consumer is forward-looking, knows the spot prices he will face in period 2, and that he wishes to behave so as to maximise his lifetime utility*** $u(x_1, x_2)$***. Further, assume that, if he consumes*** $\bar{x_1}$ ***in period*** $t = 1$***, his utility of consuming any bundle*** x_2 ***in period*** $t = 1$ ***is*** $u(\bar{x_1}, x_2)$***.***
Because the consumer can do no better if there are fewer spot markets open, parts (i)–(iv) show that, if there is a period-t spot market for good k and the period-t spot price of good k is $\hat{p_{kt}}$***, then our consumer has no incentive to***

trade. Since this is true for all consumers, this shows that spot markets clear at prices at which there is no trade.

We are told that the consume chooses $\hat{x_1}, \hat{x_2}$ before period 1 in order to maximize his lifetime utility. Given $\hat{p_1}, \hat{p_2}$ he therefore solves the following maximization problem

$$\max_{(x_1,x_2)\in\mathbb{R}^N_+} u(x_1,x_2) \text{ s.t. } \hat{p_1}\cdot x_1 + \hat{p_2}\cdot x_2 = \hat{p_1}\cdot\hat{x_1} + \hat{p_2}\cdot\hat{x_2} \tag{1}$$

Note that $\hat{p_1}\cdot\hat{x_1} + \hat{p_2}\cdot\hat{x_2}$ does not say anything about the optimal bundle choice itself, but simply reflects the total available income of consumers prior to trading.

We want to show that solving this problem is equivalent to solving the recursive problem $\max_{x_1} u(x_1,\hat{x_2})$ **first, and then** $\max_{x_2} u(\hat{x_1},x_2)$**. Note that this is an application of the envelope condition.**

Before period 1 the consumer aggrees to receive $\hat{x_1}$ in period 1 and $\hat{x_2}$ in period 2.
His maximization problem on the spot market in period 1 is

$$max_{x_1} u(x_1,\hat{x_2}) \text{ s.t } \hat{p_1}.x_1 \leq \hat{p_1}.\hat{x_1}$$

We show that the solution to this problem is $\hat{x_1}$. BWOC, assume that the solution to this problem is x_1^*. x_1^* is feasible:

$$\hat{p_1}.x_1^* \leq \hat{p_1}.\hat{x_1}$$
$$\hat{p_1}.x_1^* + \hat{p_2}.\hat{x_2} \leq \hat{p_1}.\hat{x_1} + \hat{p_2}.\hat{x_2}$$

But if $(x_1^*,\hat{x_2})$ is feasible and is chosen over $(\hat{x_1},\hat{x_2})$, it must be that $(x_1^*,\hat{x_2})$ is the solution to the lifetime maximization problem (1). This is a contradiction, so $x_1^* = \hat{x_1}$.
Similarly for period 2. Denote x_2^* the solution to the problem

$$max_{x_2} u(\hat{x_1},x_2) \text{ s.t } \hat{p_2}.x_2 \leq \hat{p_2}.\hat{x_2}$$

This implies

$$\hat{p_2}.x_2^* \leq \hat{p_2}.\hat{x_2}$$
$$\hat{p_2}.x_2^* + \hat{p_1}.\hat{x_1} \leq \hat{p_2}.\hat{x_2} + \hat{p_1}.\hat{x_1}$$
$$\Rightarrow u(\hat{x_1},x_2^*) \geq u(\hat{x_1},\hat{x_2})$$

Which is a contradiction as $(\hat{x_1},\hat{x_2})$ is the solution to problem (1).
In a nutshell, maximizing the consumer's lifetime utility is equivalent to maximizing the consumer's utility each period in an Arrow Debreu world (how surprising, this is what we do in most macro models...).

b) Repeat the exercise with uncertainty instead of time. Assume that N goods and two states of the world, $s = 1, 2$***. What is the interpretation of the assumption (analogous to that made in part (iv) of (a)) that if the consumer would have consumed bundle*** $\hat{x}_1$ ***had state*** $s = 1$ ***occurred, his utility of consuming any***

bundle x_2 ***in state*** $s = 2$ ***is*** $u(\bar{x}_1, x_2)$***?***

As we have already seen in the TA sessions, maximizing lifetime utility over multiple time periods is equivalent to maximizing ex-ante expected utility over multiple states of the world (in an Arrow-Debreu world), as well as maximizing utility over multiple goods.
The results is therefore exactly the same as above : in both states of the world, one can show that the consumer cannot do better than consuming the endowment he agreed to receive in each state of the world beforehand. The argument once again relies on the envelope condition and proving BWOC that if x_i^* maximizes the spot market problem, then we must have $x_i^* = \bar{x}_i$.
Because both states are independent, and contracts are fully enforceable, the consumer cannot decide ex-post to "borrow from the other state of the world" after he knows in which states he ends in.

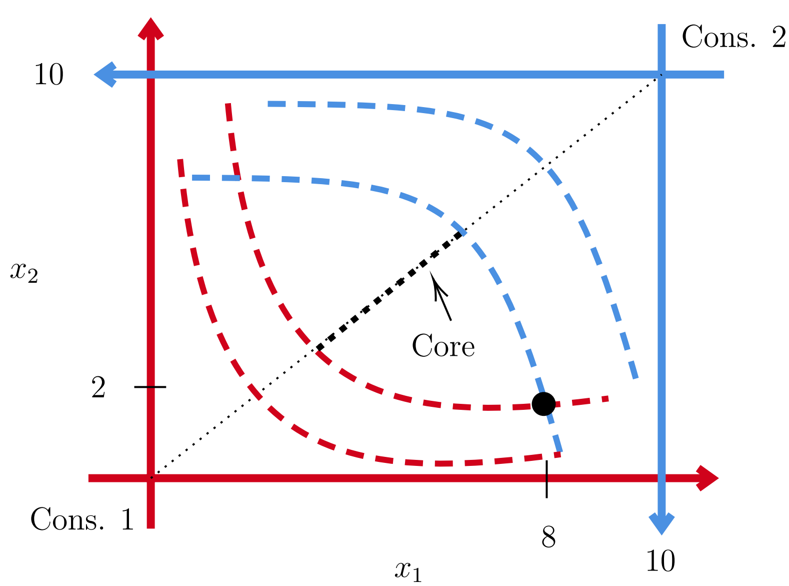
5.39

(Cornwall) In an economy with two types of consumers, each type has the respective utility function and endowments:

$$u^{1q}(x_1, x_2) = x_1 x_2, \quad \mathbf{e}^1 = (8, 2)$$
$$u^{2q}(x_1, x_2) = x_1 x_2, \quad \mathbf{e}^1 = (2, 8)$$

(a) *Draw an Edgeworth box for this economy when there is one consumer of each type.*

The dashed lines are indifference curves, the dotted line is the Pareto efficient set, and the subset of the dotted line which is dashed thickly is the core, for the given endowment.



(b) *Characterize as precisely as possible the set of possible allocations that are in the core of this two-consumer economy.*

The core is feasible and unblocked by every possible coalition. In the case where there are only two individuals, this is equivalent to the set of Paret efficient allocations which are individually rational, i.e. at least as good as the initial endowment.

We can show that the set of Pareto efficient bundles is the set[2]

$$P(\Omega) = \{(x, x) \mid x \in [0, 10]\}$$

Consider any other bundle (x, y). Then consumer 1 has utilty xy, and consumer 2 has utility $(10 - x)(10 - y) = 100 - 10y - 10x + xy$. We can then see that both consumers would agree that increasing xy, without changing the value of the sum $x + y$, would be advantageous [3]. If we consider this as the constrained optimization problem

$$\begin{aligned} &\max xy \\ \text{s.t. } &x + y = k \end{aligned}$$

the solution is $x = y = \frac{1}{2}k$. Therefore, any allocation not along $x = y$ could be improved upon by some allocation on the line $x = y$. By analogous logic, once at a bundle (x, x), to increase utility at least one good must increase, but since any allocation off $x = y$ is not Pareto efficient, the only possible candidate for a a Pareto dominant allocation must be at $(x + \epsilon, x + \epsilon)$ for some $\epsilon > 0$. But then consumer 2 is worse off, so the Pareto efficient set is exactly as above.

To find the core we need to restrict the set of Pareto efficient allocation to the set where both consumers are at least as well off as at their endowment. Note that both consumers receive utility 16 at their endowment, so the boundaries of the core (along the Pareto set) must be $16 = x^2 \Rightarrow x = 4$ and $10 - x \Rightarrow x = 6$. So the core is

$$C(\Omega) = \{(x, x) \mid x \in [4, 6]\}$$

(c) *Show that the allocation giving $x^{11} = (4, 4)$ and $x^{21} = (6, 6)$ is in the core.*

From our above argument, $(4, 4) \in C(\Omega)$, $(6, 6) \in C(\Omega)$. We could simply note that both of these allocations give one consumer the same utilty they have at their endowment (and the other consumer gets more, and they are along the Pareto curve, so there is no way to improve both consumers' utility from either of these points.

[2] I am parameterizing the box with consumer 1's goods vector.

[3] This is what their coalition is requiring.

(d) *Now replicate this economy once, so there are two consumers of each type, for a total of four consumers in the economy. Show that the double copy of the previous allocation, giving* $x^{11} = x^{12} = (4,4)$ *and* $x^{21} = x^{22} = (6,6)$*, is not in the core of the replicated economy.*

Consider the coalition $\{x^{11}, x^{12}, x^{21}\}$. Their total endowment is $(18, 12)$, so if they "went it alone", they could allocate[4] $x^{11} = x^{12} = (4.4, 4)$, and $x^{21} = (9.2, 4)$, and all would strictly improve upon the suggested allocation. Therefore this coalition will block the suggested allocation, and it cannot reside in the core.

[4]This is not a Pareto optimal allocation, but it is sufficient for establishing the suggested allocation is not in the core.