Price Theory II Problem Set 5 - ODDS

Chase Abram, Sarah Brown,
Manav Chaudhary, Tom Hierons,
Jeanne Sorin, and George Vojta

July 3, 2020

1.

Prove that Axiom G4 implies $a_1 \succ a_n$.
<u>G.4.</u> (Monotonicity):

$$\begin{aligned} \forall \alpha, \beta \in [0,1], \{(\alpha \cdot a_1, (1-\alpha) \cdot a_n) &\succeq (\beta \cdot a_1, (1-\beta) \cdot a_n) \text{ iff } \alpha \geq \beta\} \\ &\succ (\beta \cdot a_1, (1-\beta) \cdot a_n) \text{ iff } \alpha > \beta\} \end{aligned}$$

Let's notice that these relations must hold for all $\alpha, \beta \in [0,1]$ st $\alpha \geq \beta$.
BWOC, assume that $a_n \succeq a_1$ and notice that

$$\begin{aligned} a_1 &\sim (1 \cdot a_1, 0 \cdot a_n) \\ a_n &\sim (0 \cdot a_1, 1 \cdot a_n) \end{aligned}$$

Therefore

$$\begin{aligned} &a_n \succeq a_1 \\ \Rightarrow\ &a_n \sim (0 \cdot a_1, 1 \cdot a_n) \succeq a_1 \sim (1 \cdot a_1, 0 \cdot a_n) \\ \Leftrightarrow\ &a_n \sim (\beta \cdot a_1, (1-\beta) \cdot a_n) \succeq a_1 \sim (\alpha \cdot a_1, (1-\alpha) \cdot a_n) \text{ with } \alpha = 1, \beta = 0 \end{aligned}$$

Which is a contradiction to G4, so it must be the case that $a_1 \succ a_n$.

3.

Jane has wealth equal to \$2500. With probability 1/2 she will lose \$1600 of it, while with probability 1/2 she will lose only \$900. Her VNM utility function for wealth is $u(w) = \sqrt{w}$.

(a)

What is Jane's expected utility in her present situation?

. . . .

We calculate

$$\begin{aligned}
E[u(w)] &= \frac{1}{2}\underbrace{u(2500-1600)}_{\text{Lose \$1600}} + \frac{1}{2}\underbrace{u(2500-900)}_{\text{Lose \$900}} \\
&= \frac{1}{2}\sqrt{900} + \frac{1}{2}\sqrt{1600} \\
&= \frac{1}{2}(30+40) \\
&= 35
\end{aligned}$$

(b)

Suppose that Jane can purchase an insurance policy that will cover the entire \$1600 loss in the event that it occurs (the \$900 loss cannot be insured against). What is the maximum amount of money that Jane would be willing to pay to purchase such a policy? How does this compare to the actuarially fair price of the policy?

. . . .

Jane will be willing to pay up to the amount where the insurance premium leaves her with the same expected utility as if she did not purchase the insurance. This maximum premium P solves

$$35 = \frac{1}{2}\underbrace{u(2500-P)}_{\text{\$1600 loss covered}} + \frac{1}{2}\underbrace{u(2500-900-P)}_{\text{Lose \$900}}$$

Note that in either case, Jane must pay the premium. We can solve for P.

$$\begin{aligned}
35 &= \frac{1}{2}(\sqrt{2500-P} + \sqrt{1600-P}) \\
\Rightarrow 70 &= \sqrt{2500-P} + \sqrt{1600-P} \\
\Rightarrow 4900 &= 2500 - P + 2\sqrt{(2500-P)(1600-P)} + 1600 - P \\
\Rightarrow 800 + 2P &= 2\sqrt{2500\cdot 1600 - 4100P + P^2} \\
\Rightarrow 160,000 + 800P + P^2 &= 2500\cdot 1600 - 4100P + P^2 \\
\Rightarrow P &= \frac{38,400}{49} \approx 783.67
\end{aligned}$$

The actuarially fair price is simply the expected losses, ℓ, for the insurer, because in this case profits are zero.

$$\begin{aligned}
E[\ell] &= \frac{1}{2}\cdot\underbrace{1600}_{\text{Pay for loss}} + \frac{1}{2}\cdot\underbrace{0}_{\text{Not liable}} \\
&= 800
\end{aligned}$$

Thus, the maximum amount Jane is willing to pay for the policy is less than the actuarially fair price, so she will not purchase the insurance, (assuming the insurer refuses to operate at an expected loss). The economics of this result are that fair insurance pricing here would result in the same expected *value* of wealth for Jane ($\frac{1}{2}(1700) + \frac{1}{2}(800) = \frac{1}{2}(900) + \frac{1}{2}(1600)$), but since she is risk averse, this "increased spread" lottery would actually hurt her expected *utility*.

5.

Some basics,

- Bob has an initial wealth $w_0 = \$100,000$
- The price of asset a is $p_a = \$6,000$, in turn the net return on asset (return - price) a is,

$$r_a = \begin{cases} \$44,000 & p = \frac{1}{2} \\ -\$3,000 & p = \frac{1}{3} \\ -\$6,000 & p = \frac{1}{6} \end{cases}$$

- Since Bob's utility is monotonically increasing in wealth and he is risk averse, the utility function satisfies the following properties for all w (i) $u'(w) > 0$, and (ii) $u''(w) < 0$.

Given the information above, we can calculate the Bob's expected utility when he purchases a share x of asset a,

$$\begin{aligned} f(x) &\equiv \mathbb{E}[u(w_0 + a - p_a)] \\ &= \frac{1}{2}u(w_0 + 44000x) + \frac{1}{3}u(w_0 - 3000x) + \frac{1}{6}u(w_0 - 6000x) \end{aligned}$$

Differentiating with respect to x, we get the marginal expected utility from infinitesimally increasing the amount of x purchased.

$$f'(x) = 1000\,[22u'(w_0 + 44000x) - u'(w_0 - 3000x) - u'(w_0 - 6000x)]$$

Similarly, we can get the second derivative,

$$f''(x) = 10^6\,[968u''(w_0 + 44000x) + 3u''(w_0 - 3000x) + 6u''(w_0 - 6000x)] < 0, \forall x > -\frac{w_0}{6000}$$

Using these relationships above, we can now answer if Bob would prefer to purchase more than $x = 0$? The answer to this question is yes, to see why consider the first derivative evaluated at 0,

$$f'(0) = 20000u'(w_0) > 0$$

Therefore, increasing one unit of x will increase Bob's marginal utility. Hence, he can't possibly be maximising utility at $x = 0$. More formally, $\lim_{x \to 0} \frac{f(x)-f(0)}{x} = f'(0) > 0 \implies \exists x > 0$ such that $f(x) > f(0)$.

In fact, the global strict concavity of $f(x)$ implies there is a unique x^* that maximizes expected utility, and given our result above we know that $x^* > 0$.

7.

Suppose that you must transport a large sum of money from your home to your office and that on any single trip there is a probability $(1-p)$ that you are robbed. Assuming that you are risk averse, show that you would prefer taking 2 trips with half of the total amount of money taken on each trip than taking all the money on a single trip. (Assume that time and effort are costless.)

Being risk averse translates into having a concave utility function, i.e. $U(\alpha a + (1-\alpha)b) \geq \alpha U(a) + (1-\alpha)U(b), \forall \alpha \in [0,1]$.
Let's call p the probability that you are not robbed.

Then we may deduce

$$\begin{aligned}
& E[1 \text{ trip}] < E[2 \text{ trips}] \\
\iff & pU(m) < p^2U(m) + 2p(1-p)U(\frac{m}{2}) \\
\iff & U(m) < pU(m) + 2(1-p)U(\frac{m}{2}) && (p > 0 \text{ else trivial}) \\
\iff & (1-p)U(m) < 2(1-p)U(\frac{m}{2}) \\
\iff & U(m) < 2U(\frac{m}{2}) && (1-p > 0 \text{ else trivial}) \\
\iff & \int_0^m U'(x)\mathrm{d}x < 2\int_0^{\frac{m}{2}} U'(x)\mathrm{d}x \\
\iff & \int_{\frac{m}{2}}^m U'(x)\mathrm{d}x < \int_0^{\frac{m}{2}} U'(x)\mathrm{d}x
\end{aligned}$$

The last line is true because U is concave, so $U'(x)$ is strictly decreasing in x.

9.

(Risk-Sharing) Two consumers, $i = 1, 2$ have strictly increasing and strictly concave VNM utility functions for wealth $u_i : [0, \infty) \to \mathbb{R}$, with $u_i' > 0$, $u_i'(0) = \infty$, and $u_i'' < 0$. But each consumer's wealth is uncertain. Specifically, consumer i's wealth, $w_i \geq 0$, is a non-negative random variable with probability density function $f_i(w_i)$. So consumer i's expected utility is $\int_0^\infty u_i(w_i)f_i(w_i)\mathrm{d}w_i$. Before finding out their wealth (i.e., before the values of w_1 and w_2 are realized), consumers 1 and 2 seek to enter into a mutually beneficial wealth-sharing agreement. Specifically, for any realized values w_1 and w_2 of their wealth, let $s_1(w_1, w_2)$ be the amount of the total wealth $w_1 + w_2$ that is given to consumer 1 and let the remainder $s_2(w_1, w_2) = w_1 + w_2 - s_1(w_1, w_2)$ be given to consumer 2. Suppose that the sharing agreement $\Big(s_1(\cdot,\cdot), s_2(\cdot,\cdot)\Big)$ that they agree to is Pareto efficient (i.e., no other sharing agreement can make one of them better off without making the other worse off).

(a)

Show that $s_1(w_1, w_2)$ depends only on their total wealth $w_1 + w_2$ and so we may write $s_1(y)$ where $y = w_1 + w_2$ is total wealth.

. . . .

Since we are dealing with expectations, all of the statements we will want to make need only be true on sets of positive measure (probability), so that these statements could technically be false on sets of measure (probability) zero. This point means that we will need to make statements about sets with positive probability under the joint density for wealth draws, not simply "points," which could have probability zero and therefore be irrelevant.

Suppose $s_1(w_1, w_2)$ does not depend only on total wealth with positive probability. Then there exist sets A and B, of positive probability, such that for all $(w_1, w_2) \in A$ and all $(w_1', w_2') \in B$, if $w_1 + w_2 = w_1' + w_2'$, then $s_1(w_1, w_2) \neq s_1(w_1', w_2')$.

Recall the expected utility of consumer i is

$$E[u_i] = \int_0^\infty \int_0^\infty u_i(s_i(w_1, w_2)) f_1(w_1) f_2(w_2) \mathrm{d}w_1 \mathrm{d}w_2$$

Now we may use the strict concavity of the u_i to note that, in expectation (which is the objective in this context), both consumers would prefer if, for all $(w_1, w_2) \in A$ and all $(w_1', w_2') \in B$, we had $s_1(w_1, w_2) = s_1(w_1', w_2') = \frac{f_1(w_1)f_2(w_2)s_1(w_1,w_2)+f_1(w_1')f_2(w_2')s_1(w_1',w_2')}{f_1(w_1)f_2(w_2)+f_1(w_1')f_2(w_2')}$, since this sharing agreement "removes spread" from the wealth distribution for both (risk-averse) consumers, as opposed to the differential sharing agreements[1]. More rigorously, Jensen's Inequality for a *strictly* concave function u implies $E[u(X)] < u(E[X])$, and with our proposed new agreement we are moving from the left side to the right side of the inequality. In other words, for the same $w_1 + w_2$ values, both consumers are better off if their share is constant, as compared to the two possibly different values for s_i. Therefore positive probability of differential shares for the same $w_1 + w_2$ value cannot be Pareto efficient, and we must have that $s_1(w_1, w_2) = s_1(w_1 + w_2) = s_1(y)$ almost surely.

(b)

Show that $\frac{u_1'(s_1(y))}{u_2'(s_2(y))}$ is constant as a function of y.

. . . .

We again want to argue that, otherwise, the sharing plan would not be Pareto efficient. Let

$$F(y) = \frac{u_1'(s_1(y))}{u_2'(s_2(y))}$$

Suppose the above fraction is, with positive probability, not constant. Then there exist sets A and B, of positive probability, such that for all $(w_1, w_2) \in A$ and all $(w_1', w_2') \in B$,

[1] Note that this statement, on its own, is not saying that this agreement is the proper one, but merely that it is better than the original agreement which was not solely dependent on total wealth

we have $w_1 + w_2 = y < y' = w_1' + w_2'$,[2] and (without loss of generality) $F(y) < F(y')$. By the knowledge that $F(y) < F(y')$, there exist $\delta_1, \delta_2 > 0$ such that we may increase $s_2(y)$ by δ_1 for all outcomes in A, and decrease $s_2(y')$ by δ_2 for all outcomes in B, and increase expected utility for *both* consumers. The reasoning is that for all draws in A, the marginal δ increase will increase expected utility for consumer 2, and decrease it for consumer 1, *but* $F(y) < F(y')$ implies that, comparatively, we may pick δ_2 so that the δ_2 decrease impacts expected utility *less* than the δ_1 bump. By making these analogous arguments in reverse for consumer 1, whose marginal utilities are represented reciprocally to consumer 2, we can see that (provided we are careful enough in our choosing of δs), this new sharing agreement also makes consumer 1 better off[3] (in terms of expected utility, obviously)! Therefore F not being constant cannot be Pareto efficient, because we were able to improve upon this plan, so we must have $F(y)$ is constant almost surely, as desired.

(c)

What is the solution if consumer 1 is risk neutral?

. . . .

If consumer 1 is risk-neutral, then u_1' is some constant $\bar{u_1'}$. From above, we then must have

$$\frac{\bar{u_1'}}{u_2'(s_2(y))}$$

is constant. Assuming we maintain the strict concavity of u_2, the above term can only be constant for all y if $s_2(y)$ is constant for y (again, by "all" we mean with probability 1). Therefore, if consumer 1 is risk-neutral, the solution will be, for *any* given joint wealth draw, to give consumer 2 the same amount of wealth (in level), and give the remainder to consumer 1. Note that for this solution to make sense, we technically also need to assume that the level $s_2(y)$ will be feasible regardless of the draw for w_1, w_2. Therefore a solution is to simply always give all the wealth to consumer 1, where we then allow the convention that the above fraction is zero because the denominator goes to ∞.

[2]The sup of the possible total wealth draws in A is less than the inf of the possible total wealth draws in B.

[3]All that we need is to make consumer 1 as well off as before we started tinkering, but this stronger result can be reached, and it seems to make the argument intuitively cleaner.