Price Theory II Problem Set 4

Chase Abram, Sarah Brown,
Manav Chaudhary, Thomas Hierons,
Jeanne Sorin, George Vojta

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Even: 2, 4, 5.36, 6.3, 6.18, 6.20, 6.22

2.

There are three men and three women. Their strict preferences are as follows, where it is assumed that every individual strictly prefers to be matched to any individual on the other side of the market rather than being single.
$>_{m_1}$: w_2, w_1, w_3
$>_{m_2}$: w_1, w_3, w_2
$>_{m_3}$: w_1, w_2, w_3
$>_{w_1}$: m_1, m_3, m_2
$>_{w_2}$: m_3, m_1, m_2
$>_{w_3}$: m_1, m_3, m_2
Consider the matching
$\mu_1 = (m_1, w_1); (m_2, w_2); (m_3, w_3)$.
This matching is unstable because, for example, m_1 and w_2 form a blocking pair. If we let them divorce their partners and marry one another, and then marry their abandoned partners, we get the matching
$\mu_2 = (m_1, w_2); (m_2, w_1); (m_3, w_3)$
But this matching is blocked by m_3 and w_2 If we rematch in the same way as before, we get the matching
$\mu_3 = (m_1, w_3); (m_2, w_1); (m_3, w_2)$ which is blocked by (m_3, w_1) Rematching yet again yields, $\mu_4 = (m_1, w_3); (m_2, w_2); (m_3, w_1)$ which is blocked by (m_1, w_1) But rematching once more brings us back to μ_1!
This shows that naively trying to reach a stable match by letting blocking couples marry does not always work! However, show in this example that, by carefully choosing which blocking pairs marry (the blocking pairs are not unique), it is possible to reach stable match using this process. (It turns out that this is true in general.)

Let's first rewrite the preferences in matrix form:

$$\begin{bmatrix} & >_{w_1} & >_{w_2} & >_{w_3} \\ >_{m_1} & (2,1) & (1,2) & (3,1) \\ >_{m_2} & (1,3) & (3,3) & (2,3) \\ >_{m_3} & (1,2) & (2,1) & (3,2) \end{bmatrix}$$

where each entry is an ordered pair where the first object is how the man ranks that woman and the second object is how the woman ranks that man.

Now let's choose blocking pairs so that the pairs rating each other the highest marry first.

$$\begin{bmatrix} & >_{w_1} & >_{w_2} & >_{w_3} \\ >_{m_1} & (\mathbf{2},\mathbf{1}) & (\mathbf{1},\mathbf{2}) & (3,1) \\ >_{m_2} & (1,3) & (3,3) & (2,3) \\ >_{m_3} & (\mathbf{1},\mathbf{2}) & (\mathbf{2},\mathbf{1}) & (3,2) \end{bmatrix}$$

We see there are two possible combinations of marriages that can follow: $(m_1, w_1), (m_3, w_2)$ or $(m_1, w_2), (m_3, w_1)$ In either case, the last pairing is (m_2, w_3). Both options are stable. This can be proved by completing a man proposing DAA and woman proposing DAA to see that this result follows since any matching that DAA produces must be stable.

For u_1 let the blocking pair be (m_3, w_2). For u_2 let the blocking pair by (m_3, w_1). For u_3 let the blocking pair by (m_1, w_1). For u_4 let the blocking pair by (m_1, w_2).

Therefore by letting the blocking pairs that rate each other the highest marry first, we can produce a stable matching with these preferences.

4.

As in class we let M, W be the sets of men and women and let $n = \#M$ and $p = \#W$. Consider the two algorithms from class: deferred-acceptance-men-proposing (DAM) and deferred-acceptance-women-proposing (DAW). Denote the sets of unmatched men and women resulting from the application of each algorithm as $M^{DAM}, W^{DAM}, M^{DAW}, W^{DAW}$. Now let μ be *any arbitrary stable matching* and let M^μ, W^μ be the sets of unmatched men and women for this matching. Since μ is arbitrary, it suffices to show that $M^\mu = M^{DAM}$ and $W^\mu = W^{DAM}$.

Also from class results we have that any man who is unmatched in the men proposing case is always unmached so that

$$M^{DAM} \subseteq M^\mu$$

And in particular:

$$M^{DAM} \subseteq M^{DAW}$$

This then implies that the number of unmatched men in the DAM case must be no larger than in any other matching so:

$$\#M^{DAM} \leq \#M^\mu$$

And in particular

$$\#M^{DAM} \leq \#M^{DAW}$$

Applying exactly the same argument to the case of women we find that:

$$W^{DAW} \subseteq W^{\mu}$$

$$W^{DAW} \subseteq W^{DAM}$$

$$\#W^{DAW} \leq \#W^{\mu}$$

$$\#W^{DAW} \leq \#W^{DAM}$$

Now notice that since matchings are one-to-one, the number of *matched* men and women in any particular matching must be the same. So we must have $\#(M \backslash M^{DAM}) = \#(W \backslash W^{DAM})$, $\#(M \backslash M^{DAW}) = \#(W \backslash W^{DAW})$ and $\#(M \backslash M^{\mu}) = \#(W \backslash W^{\mu})$. Combining these with the above inequalities for men we find that:

$$\#M - \#M^{DAM} \leq \#M - \#M^{DAW}$$

$$\#M - M^{DAW} = \#W - \#W^{DAW} \leq \#W - \#W^{DAM} = \#M - \#M^{DAM}$$

So:

$$\#M^{DAW} = \#M^{DAM}$$

Combining with the inequalities for M^{μ} we have:

$$\#M^{DAW} = \#M^{DAM} = \#M^{\mu}$$

Arguing similarly for women we have that :

$$\#W^{DAW} = \#W^{DAM} = \#W^{\mu}$$

We can now show that the unmatched sets of men and women are in fact equal across all the cases. Note that $M^{DAM} \subseteq M^{\mu}$ and $\#M^{DAM} = \#M^{\mu}$ implies that $M^{DAM} = M^{\mu}$. Also note that $W^{DAW} \subseteq W^{DAM}$ and $\#W^{DAW} = \#W^{DAM}$ implies $W^{DAW} = W^{DAM}$. So finally we have $W^{DAM} \subseteq W^{\mu}$ and $\#W^{DAM} = \#W^{\mu}$ which gives us that $W^{DAM} = W^{\mu}$. So $M^{\mu} = M^{DAM}$ and $W^{\mu} = W^{DAM}$ for arbitray μ as desired.

5.36

Part a

Remark: It wasn't clear to us if we restrict $\mathbf{x} \geq 0$. We assumed so because otherwise we could theoretically short everything and generate an asset with negative returns since $\alpha^k \in \mathbb{R}^S_+, \forall k$.

Let asset holdings be non-negative $\mathbf{x} \in \mathbb{R}^N_+$, and let A be the $(S \times N)$ matrix of pay-offs,

$$A\mathbf{x} = \begin{pmatrix} \alpha_1^1 & \cdots & \alpha_1^N \\ \vdots & \ddots & \vdots \\ \alpha_S^1 & \cdots & \alpha_S^N \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^N \alpha_1^i x_i \\ \vdots \\ \sum_{i=1}^N \alpha_S^i x_i \end{pmatrix} := \tilde{\alpha} \in \mathbb{R}_+^S$$

Part b

$$A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x}) = \begin{pmatrix} \sum_{i=1}^N x_i(\alpha_1^i - q_i) \\ \vdots \\ \sum_{i=1}^N x_i(\alpha_S^i - q_i) \end{pmatrix}$$

For each element, the first element is the real earnings in state s from holding the induced asset $\tilde{\alpha}$. While the second term is the real price paid to buy $\tilde{\alpha}$. The difference between the two gives the real net gain in state s.

Part c

As above, define the asset induced by a unit of $\mathbf{x}$ as $\tilde{\alpha} \in \mathbb{R}_+^S$. This asset costs $c_{\tilde{\alpha}} := \mathbf{q} \cdot \mathbf{x} \in \mathbb{R}$ units to construct. Let $\underline{\tilde{\alpha}} = \min_s\{\tilde{\alpha}_s\} \in \mathbb{R}$ the asset pay-off in the lowest paying state. In the question we are given the net gain is strictly positive in all states, therefore it follows that $\underline{\tilde{\alpha}} - c_{\tilde{\alpha}} > 0$.

Arbitrage trading strategy

1. Short one unit of the synthetic asset $\hat{\alpha}$ which gives pay-off $\underline{\tilde{\alpha}} - \frac{\underline{\tilde{\alpha}} - c_{\tilde{\alpha}}}{2}$ in all states, at a price $c_{\tilde{\alpha}}$. By construction, $\underline{\tilde{\alpha}} - \frac{\underline{\tilde{\alpha}} - c_{\tilde{\alpha}}}{2} > c_{\tilde{\alpha}}$, since the net gain is strictly positive the demand for this asset will be positive.

2. Using the proceeds, $c_{\tilde{\alpha}}$, from shorting $\hat{\alpha}$, go long one unit of $\tilde{\alpha}$. The revenue for $\tilde{\alpha}$ in any state is greater than $\underline{\tilde{\alpha}}$, which is in turn is strictly greater than the maximum liability from shorting $\hat{\alpha}$, $\underline{\tilde{\alpha}} - \frac{\underline{\tilde{\alpha}} - c_{\tilde{\alpha}}}{2}$. Therefore, we are always guaranteed against bankruptcy in this trade.

The minimum profit made from this trade is $\frac{\underline{\tilde{\alpha}} - c_{\tilde{\alpha}}}{2} > 0$. This trading strategy can be scaled without bound to generate infinite profits.

Part d

In equilibrium we need markets to clear, therefore,

$$\sum_i \hat{x}_s^i = \sum_i e_s^i \quad \forall s$$

where $\hat{x}_s^i$ is the equilibrium allocation. From part c, we can see that strictly positive net gains would imply $\sum_i \hat{x}_s^i \to \infty, \forall s$, which clearly violates feasibility. Therefore, for markets to clear there cannot be an $\mathbf{x} \in \mathbb{R}^N$ such that the real net gain vector is strictly positive.

Part e

Remark: We think there is an error in the textbook with regards to the dimensions of y. To proceed we assume $y \in \mathbb{R}^S$.

Theorem A2.24: A Second Separation Theorem
Suppose that A and B are disjoint convex subsets of $\mathbb{R}^n$. Then, there exists a vector $\mathbf{p} \in \mathbb{R}^n$ of length one such that,

$$\mathbf{p} \cdot \mathbf{a} \geq \mathbf{p} \cdot \mathbf{b}, \quad \textit{for every } \mathbf{a} \in A \textit{ and every } \mathbf{b} \in B$$

Let $C = \{\mathbf{y} \in \mathbb{R}^S : \mathbf{y} = A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^N\}$. From part d, it follows that $\mathbf{y} \notin \mathbb{S}^S_{++}, \forall \mathbf{y} \in C \implies C \cap \mathbb{R}^S_{++} = \emptyset$. Then by the separating hyperplane theorem we have that, $\exists \hat{\mathbf{p}} \in \mathbb{R}^S$ such that,

$$\hat{\mathbf{p}} \cdot \mathbf{y} \leq \hat{\mathbf{p}} \cdot \mathbf{z}$$

for all $\mathbf{y} \in C$ and $\mathbf{z} \in \mathbb{R}^S_{++}$.

Additionally, $\hat{\mathbf{p}} \geq 0$ else the the right hand side of the equality above could be made arbitrarily negative. To see this, without loss of generality consider a price vector with $p_1 < 0$, then we could choose and arbitrarily large z_1 such that the dot product goes to negative infinity i.e. $\mathbf{z} \cdot \hat{\mathbf{p}} = p_1 \times z_1 + \sum_{i \neq 1} p_i \times z_i \to -\infty$ as $z_1 \to \infty$.

Normalize prices such that $\mathbf{1} \cdot \hat{\mathbf{p}} = 1$. i.e i.e if $\mathbf{p}$ is the un-normalized set of prices define $\hat{\mathbf{p}} = \mathbf{p} \left(\sum_i p_i\right)$.

Part f

Remark: Dimensions in the textbook are not correct, hence, we have assumed the question want us to show the that $(p'A - q')x \leq 0$

Let $\{\mathbf{z}_n\}_{n\in\mathbb{N}} \in \mathbb{R}^S_{++}$ be convergent sequence such that $\lim_{n\to\infty} z_n = \mathbf{0}$. Then by the separating hyperplane result in part e it follows that $\forall \mathbf{y} \in C$,

$$\begin{gathered}\hat{\mathbf{p}} \cdot \mathbf{y} \leq \hat{\mathbf{p}} \cdot \mathbf{z}_n \\ \Longleftrightarrow \\ \hat{\mathbf{p}} \cdot (A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x})) \leq \hat{\mathbf{p}} \cdot \mathbf{z}_n \\ \Longleftrightarrow \\ \hat{\mathbf{p}} \cdot (A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x})) \leq \hat{\mathbf{p}} \cdot \lim_{t\to\infty} \mathbf{z}_n = \hat{\mathbf{p}} \cdot \mathbf{0} = 0 \\ \Longleftrightarrow \\ \sum_{s=1}^{S} \sum_{i=1}^{N} p_s x_i (\alpha_s^i - q_i) = (\hat{\mathbf{p}}^T A - \mathbf{q}^T)\mathbf{x} \leq 0 \quad \forall \mathbf{x}\end{gathered}$$

Towards a contradiction, suppose the inequality is strict for some $\tilde{\mathbf{x}} \in \mathbb{R}^N$ i.e. $(\hat{\mathbf{p}}^T A - \mathbf{q}^T)\tilde{\mathbf{x}} < 0$. Clearly $-\tilde{\mathbf{x}} \in \mathbb{R}^N$, however by the previous assumption it follows that, $(\hat{\mathbf{p}}^T A - \mathbf{q}^T)(-\tilde{\mathbf{x}}) > 0$, a contradiction! Therefore we must have that,

$$(\hat{\mathbf{p}}^T A - \mathbf{q}^T)\mathbf{x} = 0 \quad \forall \mathbf{x} \implies \mathbf{q}^T = \hat{\mathbf{p}}^T A \implies q_k = \hat{\mathbf{p}} \cdot \alpha_k \quad \forall k$$

Part g

There are no arbitrage opportunities as the price of an asset is a weighted price of the underlying arrow security prices. Additionally, as can be seen from the result in part f, even if markets are incomplete the assets are implicitly priced by a set of 'complete' arrow securities.

6.1

Let $X = \{x_1, x_2\}$ and let $(R^i)_{i=1}^n$ be any collection of preference relations on X where $n \geq 2$. Since there are only two elements in the domain we define majority voting as follows. For any $x_j, x_k \in X$, $x_j R x_k$ iff $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$. Note that in this case any individual with a tie casts a "vote" for both options. We check each of Arrow's axioms in turn.

(U)
For any $(R^i)_{i=1}^n$ we want to show that majority voting gives a complete and transitive preference relation R over X.

For completeness consider any $x_j, x_k \in X$. We will have either that $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$ or $\#\{i : x_j R^i x_k\} \leq \#\{i : x_k R^i x_j\}$ so that either $x_k R x_j$ or $x_j R x_k$ and R is complete as desired.

For transitivity (which is where majority voting fails in general) let $x_j, x_k, x_l \in X$ and suppose that $x_j R x_k$ and $x_k R x_l$. This means that $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$ and $\#\{i : x_k R^i x_l\} \geq \#\{i : x_l R^i x_k\}$. Now note that since we have only two elements in the domain we will have that $x_l = x_j$ or $x_l = x_k$. In the former case, $\#\{i : x_j R^i x_l\} = \#\{i : x_l R^i x_j\}$ so that $x_j R x_l$. In the latter case we have: $\#\{i : x_j R^i x_l\} = \#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\} = \#\{i : x_l R^i x_j\}$ so that $x_j R x_l$. Conclude that R is transitive.

The other axioms are satisfied in general and do not rely on the two-element-ed-ness of the choice set:

(WP)
Let $x_j, x_k \in X$ and suppose that $x_j P x_k$ for all i. Then $\#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\}$ and $\#\{i : x_j R^i x_k\} \neq \#\{i : x_k R^i x_j\}$ so $x_j P x_k$ as desired.

(IIA)
Let $(R^i)_{i=1}^n$ and $(\tilde{R}^i)_{i=1}^n$ be two profiles of preferences, $R, \tilde{R}$ be the corresponding majority voting social preferences and $x_j, x_k \in X$. Suppose each individual i in the the two profiles of preferences agrees on the ranking of x_j against x_k. WLOG suppose that $x_j R x_k$. Then we have that $\#\{i : x_j \tilde{R}^i x_k\} = \#\{i : x_j R^i x_k\} \geq \#\{i : x_k R^i x_j\} = \#\{i : x_k \tilde{R}^i x_j\}$. So that $x_j \tilde{R} x_j$ and the social preferences also agree as desired.

(D)
Let $(R^i)_{i=1}^n$ be any profile of preferences and consider a particular individual i^*. We want to show that i^* is not a dictator. Let $x_j P^{i^*} x_k$. It suffices to construct a set of preferences such that it is not the case that $x_j P x_k$ in other words such that $x_k R x_j$. Let us set all $\tilde{i} \neq i^*$ to have preferences such that $x_k P^{\tilde{i}} x_j$. Note that since $n \geq 2$ there is at least one such individual. Then $\#\{i : x_k \tilde{R}^i x_j\} \geq 1 = \#\{i^*\} = \#\{i : x_j R^i x_k\}$ so $x_k R x_j$ as desired.

6.3

(a) Show that the social welfare function that coincides with individual i's preferences satisfies U, WP and IIA. Call such a social welfare function an individual i dictatorship.
Let's first define such a welfare function:

$$\begin{aligned} R = f(R^1, R^2, \ldots, R^i, \ldots, R^N) &\text{ s.t.} \\ xR^i y \Leftrightarrow xRy, \forall x, y \in X. \end{aligned}$$

Let's remind ourselves the definitions of U, WP and IIA, before showing that the SWF associated with individual i's preferences satisfies these criteria.

U (Unrestricted Domain): $f : \mathcal{R}^N \to \mathcal{R}$, which is trivially satisfied as nowhere in this question we impose restrictions of R^i, which is de facto the only individual preference that matters for social preferences. Transitivity of R is trivially satisfied as long as R^i is transitive, which it is by definition of a preference relation.

WP (Weak Pareto): $\forall x, y \in X, \forall (R^1, R^2, \ldots, R^N) \in \mathcal{R}^N$, if $xR^i y, \forall i$, then xRy.

$$xR^jy, \forall j \Rightarrow xR^iy \text{ for our dictator i}$$
$$xR^iy \Leftrightarrow xRy, \forall x, y \in X.$$

IIA (Independence of Irrelevant Alternatives): $\forall x, y \in X, \forall(R^1, R^2, \dots, R^N), (\tilde{R}^1, \dots, \tilde{R}^N) \in \mathcal{R}^N$, $[xR^jy \Leftrightarrow x\tilde{R}^jy]\forall j$ then $[xRy \Leftrightarrow x\tilde{R}y]$.

If we have both $xR^iy, x\tilde{R}^iy$, then, by dictatorship, $xRy \forall x, y \in X$.

We don't need to consider preferences of $\forall j, j \neq i$, and R^i defines by definition binary relationships, so IIA is trivially satisfied.

(b) Suppose that society ranks any two social states x and y according to individual 1's preferences unless he is indifferent in which case x and y are ranked according to 2's preferences unless he is indifferent etc. Call the resulting social welfare function a "lexicographic dictatorship". Show that a lexicographic dictatorship satisfies U, WP and IIA and that it is distinct from an individual i dictatorship.

U (Unrestricted Domain): we don't impose any restriction on the preferences of any individual, so we don't impose any restriction on the preferences of any individual that comes before the dictator, so R is complete.
There's a bit more work to show that social preferences are transitive under lexicographic dictatorship[1].
In which situation would R not be transitive, ie xPy, yPz, zPx? Because R^i are transitive for each i, it must be the case that R's preferences over x, y, z result from multiple individuals, which can result from non-strict individual preferences.
$xPy, yPz, zPx \Rightarrow xP^iy, yP^jz, zP^kx$. We can show that the only consistent ordering of $\{i, j, p\}$ is $i > j > k$ and that individuals i, j and k's preferences are

$$xP^iy, yI^iz, xI^iz$$
$$xI^jy, yP^jz, xI^jz$$
$$xI^ky, yI^kz, xP^kz$$

WP (Weak Pareto):Suppose that $\forall x, y \in X, \forall(R^1, R^2, \dots, R^N) \in \mathcal{R}^N, xR^iy, \forall i$, then either

- xP^1y in which case 1's preferences govern the society ranking $\Leftrightarrow xRy$

[1] Alternatively, one could show BWOC that non-transitivity of social preferences would mean violating individual transitivity. More precisely, non-transitive social preferences R would imply that one good is strictly preferred by some "higher order" dictator, but that somehow later a "lower order" dictator dictates that a different good is preferred to the first good. However, this cannot happen because the "higher" dictator would prevent the "lower" dictator from ever becoming the deciding factor in that comparison.

- Or $xR^1y, x\not{P^1}y$ In which case 2's preferences govern the society ranking if $xP^2y \Leftrightarrow xRy$ Note that if $xR^2y, x\not{P^2}y$ then 3's preferences govern, etc.

Because $xR^iy, \forall i$, then either xP^jy for some $j < N \Leftrightarrow xRy$, or $xR^iy, x\not{P^i}y\forall i$, in which case the individual N decides and we know that xR^Ny so xRy is always satisfied.

IIA (Independence of Irrelevant Alternatives): if xP^1y, then xRy, no matter the other alternatives because 1 is the dictator. If $xR^1y, x\not{P}^1y$, then 2 is the dictator and we repeat the same analysis. IIA is satisfied because we sequentially compare x and y for each individual and only 1 comparison drives the social choice.

Note that except if
- $xR^iy, xP^iy, \forall x, y \in X, \forall i \in N$ (ie all individuals have exactly the same preferences with regard to the ranking of x and y)
- Individual i only has strict preferences
Then individual i will never totally decide the social choice function: for every indifference relation, we will move to individual $i + 1$ so this is distinct than individual i's dictatorship. More generally, for any $i \in N$ (but one $i \in N$ if individual i has strict preferences), individual-i's dictatorship leads to a different R than lexicographic dictatorship.

(c) Describe a social welfare function distinct from an individual i dictatorship and a lexicographic dictatorship that satisfies U, WP and IIA.
Arrow's impossibility theorem tells us that if $X \geq 3$, then there is no social welfare function that satisfies U, WP, IIA and D. As we need to satisfy U, WP and IIA we need to find a SWF that violates D.
We could define alternative individual i's dictatorship setting where $xR^iy \Leftrightarrow xRy, for i \in N$ s.t. [some rule]. Examples of such rules

- i is the oldest individual in the society
- i is the youngest individual in the society
- i is randomly chosen according to a uniform, discrete distribution on [1, N]

Showing that these rules satisfy U, WP, IIA is similar to part i.

Alternatively, we could define lexicographic dictatorship on only a subset of individuals (R is based on the lexicographic dictatorship rule only for a subset $n < N$, if members of this subgroups only have strict preferences, then $R^n \Rightarrow R$). I think this second idea is better.
Showing that this rule satisfies U, WP, IIA is similar to part ii.

6.18

Suppose that $c(\cdot)$ is a monotonic social choice function and that $c(\mathbf{R}) = x$, where $R^1, \ldots, R^N$ are each strict rankings of the social choice functions in X.

a)

Suppose that for some individual i, R^i ranks y just below x, and let $\tilde{R}^i$ be identical to R^i except that y is ranked just above x - i.e. the ranking of x and y is reversed. Prove that either $c(\tilde{R}^i, \mathbf{R}^{-i}) = x$ or $c(\tilde{R}^i, \mathbf{R}^{-i}) = y$.

. . . .

Recall that c is monotonic if $c(R^1, \dots, R^n) = x$ implies $c(\tilde{R}^1, \dots, \tilde{R}^i) = x$ whenever xR^jy implies $x\tilde{R}^jy$ for all j and all $y \neq x$.

Let $z \in X$, $z \neq x$, $z \neq y$. Suppose $c(\tilde{R}^i, \mathbf{R}^{-i}) = z$. The reshuffling of x and y for i does not change their ranking with respect to z, so zR^iz' iff $z\tilde{R}^iz'$, for all $z' \in X$ (including $z' = x$ or $z' = y$). Therefore $z\tilde{R}^jz'$ implies zR^jz' for all j and all $z' \neq z$ (when $\tilde{R}^j = R^j$ for $j \neq i$). Then, since c is monotonic, $c(\tilde{R}^i, \mathbf{R}^{-i}) = z$ implies $c(R^i, \mathbf{R}^{-i}) = z$, which contradicts that $c(\mathbf{R}) = x$. So we must have either $c(\tilde{R}^i, \mathbf{R}^{-i}) = x$ or $c(\tilde{R}^i, \mathbf{R}^{-i}) = y$.

b)

Suppose that $\tilde{R}^1, \dots, \tilde{R}^N$ are strict rankings such that for every individual i, the ranking of x versus any other social state is the same under $\tilde{R}^i$ as it is under R^i. Prove that $c(\tilde{\mathbf{R}}) = x$.

. . . .

Since the ranking of x versus $y \neq x$ is the same under $\tilde{R}^i$ and R^i for all i, we have $xR^iy \iff x\tilde{R}^iy$ for all i and $y \neq x$. Since c is monotonic, we then have $c(\mathbf{R}) = x$ implies $c(\tilde{\mathbf{R}}) = x$.

6.20

Let $c(R)$ denote the social choice for social preferences R, $x, y \in X$ be two distinct social states, and i and j be two individuals.

The question gives us the following properties of the social choice function,

1. $xR^im, \forall m \in X \implies c(R)R^ix$
2. $yR^jm, \forall m \in X \implies c(R)R^jy$

We want to prove: $i = j$.

Towards a contradiction suppose $i \neq j$. Then, because of unrestricted domains, we can construct an R in which $x \neq y$ and it satisfies the following properties,

- $xP^im, \forall m \neq x, m \in X$
- $yP^jm, \forall m \neq y, m \in X$

Then given property (1) of the choice function we have that $c(R) = x$. Similarly from property (2), it follows that $c(R) = y$. Therefore, $x = y$, which contradicts our assumption that x and y are distinct.

Therefore it must be that $i = j$.

6.22

Show that if c(.) is a monotonic social choice function and the finite set of social states is X, then for every $x \in X$ there is a profile $\mathbf{R}$, of strict rankings such that $c(\mathbf{R}) = x$. Recall that, by definition, every x in X is chosen by c(.) at some preference profile[2]

Monotonic SCF: a SCF is monotonic iff $\forall (R^1, R^2, \ldots, R^N), (\tilde{R}^1, \tilde{R}^2, \ldots, \tilde{R}^N) \in \mathbb{R}^N, \forall x \in X$, if $C(R^1, R^2, \ldots, R^N)$ and $[xR^iy \Rightarrow x\tilde{P}y, \forall y \in X, \forall i]$, then $C(\tilde{R}^1, \tilde{R}^2, \ldots, \tilde{R}^N) = x$.

Social choice function $C : \mathbb{R}^N \to X$ a mapping onto X, or equivalently as a surjective function st $\exists R$ s.t. $c(R) = x$.

As reminded above the social function is defined as $C : \mathbb{R}^N \to X$ a mapping onto X, or equivalently as a surjective function st $\exists R$.

We need to consider two cases here. First, the corresponding profile R is strict: by the surjectivity of the SCF we know that $C(R) = x$.
Second, if R is not strict then we have a little more work to do. In order to apply the definition of monotonicity we need to create a new strict ranking $\hat{P}$ from R such $xR^iy \Rightarrow x\hat{P}^iy, \forall i, x \neq y$. By construction, the definition of monotonicity gives us that $c(R) = x \Rightarrow C(\hat{P}) = x$, where $\hat{P}$ is the "profile R or strict ranking such that $c(R) = x$" we were looking for.

[2] Let's note, for our personal record, that for any SCF c, it must be surjective, meaning that for any state there must exist a preference relation R which gets mapped to that state. This question gives conditions for a stronger result: for any state there exists a *strict* preference relation which gets mapped to that state.