Empirical Analysis II Problem Set 3

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1 Problem 1

Consider the MA(1)

$$y_t = \epsilon_t + \alpha \epsilon_{t-1} \tag{1}$$

1.1 Rewrite this in lag-operator notation.

. . . .

$$\begin{aligned} y_t &= \epsilon_t + \alpha L \epsilon_t \\ &= (1 + \alpha L)\epsilon_t \end{aligned}$$

1.2 Suppose first that $|\alpha| < 1$. Invert the lag operator and find ξ_j in

$$\epsilon_t = \sum_{j=0}^{\infty} \xi_j y_{t-j}$$

. . . .

We interpret "inverting the lag operator" to mean the operations completed below, but recognize that until we write the sum expression, we are really just using notation, since L is not a real-valued function, but an operator on the infinite-dimensionsal observation space.

*This solution set benefited greatly from the comments and points raised by Manav Chaudhary, George Votja.

$$\begin{aligned}
\epsilon_t &= \frac{y_t}{1+\alpha L} \\
&= \frac{y_t}{1-(-\alpha L)} \\
&= (\sum_{j=0}^{\infty}(-\alpha L)^j)y_t \\
&= \sum_{j=0}^{\infty}(-\alpha)^j y_{t-j}
\end{aligned}$$

1.3 With that and $|\alpha| < 1$, show that ϵ_t is the one-step ahead prediction error, i.e. the below, and that (1) is the Wold decomposition

$$\epsilon_t = y_t - P(y_t \mid y_{t-1}, y_{t-2}, \ldots)$$

. . . .

Note

$$\begin{aligned}
\epsilon_t &= y_t - \alpha\epsilon_{t-1} \\
&= y_t - \alpha\sum_{j=0}^{\infty}(-\alpha)^j y_{t-j-1} \\
&= y_t - \sum_{j=1}^{\infty}(-\alpha)^j y_{t-j}
\end{aligned}$$

We know from above that each ϵ_t is a linear function of the current and past y_t. Therefore y_t is a linear function of past observations and contemporaneous ϵ_t. Since ϵ_t is independent of all past observations and mean zero, it is orthogonal (in the sense of the inner product in the Hilbert space of random variables in $\mathbb{R}$ with finite variance) to all past observations. So our best linear predictor of y_t is exactly

$$\begin{aligned}
\alpha\epsilon_{t-1} &= \sum_{j=1}^{\infty}(-\alpha)^j y_{t-j} \\
&= \sum_{j=0}^{\infty}(-\alpha)^{j-1} y_{t-j-1}
\end{aligned}$$

Therefore, since ϵ_t are the one-step ahead linear forecast errors, the Wold decomposition is exactly

$$
\begin{aligned}
y_t &= \mu_t + \sum_{j=0}^{\infty} c_j u_{t-j} \\
&= \epsilon_t + \alpha \epsilon_{t-1}
\end{aligned}
$$

where $\mu_t = 0$, $c_0 = 1$, $c_1 = \alpha$, $c_j = 0$ for $j > 1$, and $u_{t-j} = \epsilon_{t-j}$.

1.4 What goes wrong in this chain of argument, if $|\alpha| > 1$? A heuristic argument suffices.

....

When $|\alpha| > 1$, the inverting of the lag operator is illegal, because the series $\sum_{j=0}^{\infty}(-\alpha)^j y_{t-j}$ will not converge, so we have no series equivalence statement regarding the term $\frac{1}{1+\alpha L}$.

1.5 In case that $|\alpha| > 1$ and using lag-operator calculus, rewrite (1) in the form below, where $p(L^{-1})$ is a polynomial in the inverse of the lag operator L^{-1}: state $p(L^{-1})$.

$$
y_t = p(L^{-1})\alpha L \epsilon_t
$$

....

We have

$$
\begin{aligned}
y_t &= \epsilon_t + \alpha \epsilon_{t-1} \\
&= (\frac{1}{\alpha}L^{-1} + 1)\alpha L \epsilon_t
\end{aligned}
$$

So

$$
p(L^{-1}) = \frac{1}{\alpha}L^{-1} + 1
$$

1.6 Write out the right hand side below as a convergent sum, to finally obtain ϵ_t as an infinite sum of y_ts.

$$
\alpha L \epsilon_t = \frac{1}{p(L^{-1})} y_t
$$

....

$$
\begin{aligned}
\alpha L\epsilon_t &= \frac{1}{1+\frac{1}{\alpha}L^{-1}}y_t \\
&= \frac{1}{1-(-\frac{1}{\alpha}L^{-1})}y_t \\
&= \Big(\sum_{j=0}^{\infty}(-\frac{1}{\alpha}L^{-1})^j\Big)y_t \\
&= \sum_{j=0}^{\infty}(-\frac{1}{\alpha})^j y_{t+j}
\end{aligned}
$$

We can further say

$$
\begin{aligned}
\epsilon_t &= \sum_{j=0}^{\infty}(-1)^j\frac{1}{\alpha^{j+1}}y_{t+j+1} \\
&= \sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{\alpha^j}y_{t+j}
\end{aligned}
$$

1.7 How does the result differ from your previous calculations for $|\alpha| < 1$?

. . . .

We need the $\mathcal{FUTURE}$. When we had $|\alpha| < 1$, the we could write ϵ_t as a convergent series of current and past observations, but when $|\alpha| > 1$, we need to use future observations to obtain a convergent sum.

2 Problem 2

Consider

$$y_t = 0.9y_{t-1} - 0.2y_{t-2} + \epsilon_t$$

2.1 State the characteristic polynomial and find the roots λ_1,λ_2.

....

In general we know that, for a AR(m) process

$$\rho(\lambda) = \lambda^m - \theta_1\lambda^{m-1} - \theta_2\lambda^{m-2} - \theta_3\lambda^{m-3}... - \theta_{m-1}\lambda - \theta_m$$

Which, in this case becomes

$$\rho(\lambda) = \lambda^2 - 0.9\lambda + 0.2$$

Moreover, we know that the roots of the characteristic polynomial are the eigenvalues of the stacked matrix B. Therefore we have

$$\begin{aligned}
\lambda_1 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
\lambda_2 &= \frac{-b + \sqrt{b^2 + 4ac}}{2a} \\
\Leftrightarrow \lambda_1 &= \frac{0.9 - 0.1}{2} = 0.4 \\
\lambda_2 &= \frac{0.9 + 0.1}{2} = 0.5
\end{aligned}$$

2.2 Using lag-operator calculus, find the Wold decomposition.

....

We can rewrite the AR(2) process

$$\begin{aligned}
y_t &= 0.9y_{t-1} - 0.2y_{t-2} + \varepsilon_t \\
(1 - 0.9L + 0.2L^2)y_t &= \varepsilon_t \\
y_t &= \frac{1}{1 - 0.9 + 0.2L^2}\varepsilon_t
\end{aligned}$$

From the roots of the characteristic polynomial we know that

$$(1 - \theta_1 L - \theta_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L)$$

$$\begin{aligned}
y_t &= \frac{1}{(1 - 0.5L)(1 - 0.4L)}\varepsilon_t \\
y_t &= \frac{1}{1 - 0.5L}\frac{1}{1 - 0.4L}\varepsilon_t \\
y_t &= (\frac{a}{1 - 0.5L} + \frac{b}{1 - 0.4L})\varepsilon_t
\end{aligned}$$

Solving for a, b st $\frac{a}{1-0.5L} + \frac{b}{1-0.4L} = \frac{1}{(1-0.5L)(1-0.4L)}$:

$$\frac{a}{1-0.5L} + \frac{b}{1-0.4L} = \frac{a+b-L(0.4a+0.5b)}{(1-0.5L)(1-0.4L)}$$

Matching coefficients

$$\begin{aligned} a+b &= 1 \\ 0.4a+0.5b &= 0 \\ &\Leftrightarrow a=5, b=-4 \end{aligned}$$

Therefore

$$\begin{aligned} y_t &= (\frac{5}{1-0.5L} + \frac{-4}{1-0.4L})\varepsilon_t \\ &= \sum_{j=0}^{\infty}(5(0.5)^j L^j + -4(0.4)^j L^j)\varepsilon_t \\ &= \sum_{j=0}^{\infty}(5(0.5)^j + -4(0.4)^j)\varepsilon_{t-j} \end{aligned}$$

Finally, let's check that this is actually the Wold decomposition:

- $c_0 = -4(0.5)^0 + 5(0.4)^0 = -4 + 5 = 1$
- $u_{t-j} = c_0 \varepsilon_{t-j} = \varepsilon_{t-j}$ so $V(u_t) = V(\varepsilon_t)$
- Shocks vanish over time : $lim_{j\ to\infty} - 4(0.5)^j + 5(0.4)^j = -4op(1) + 5op(1) = op(1) + op(1) = op(1)$

3 Problem 3

Consider a VAR for $x_t \in \mathbb{R}^2$, given by

$$x_t = \begin{bmatrix} 0.3 & 0.5 \\ -0.5 & 1.3 \end{bmatrix} x_{t-1} + u_t$$

Calculate the Vector Wold decomposition, i.e. provide explicit formulas for B^j. (Hint: find a useful factorization $B = VJV^{-1}$. For V, try

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

Write down J a bit more generally, and calculate J^2, J^3 to discover a pattern.)

. . . .

In order to calculate the vector Wold decomposition, let's **first** write x_t in lag notation

$$x_t = Bx_{t-1} + \varepsilon_t$$
$$\varepsilon_t = (1 - BL)x_t$$

Second, we want to "invert" $(1 - BL)$ and **third** rewrite x_t as geometric sum involving $B^j L^j$.
However, to do so, we need (1) to check that B is invertible and (2) find an explicit expression for B^j.
We are gonna take care of these two tasks at once by computing the Jordan Decomposition of the matrix B. The Jordan decomposition $B = VJV^{-1}$ makes it easy to take powers of B because J is an upper triangular matrix so $B^j = VJ^jV^{-1}$. Moreover, we know that the eigenvalues of B are the diagonal entries of the Jordan matrix. If the absolute value of all these entries is less than one, then we know that B is invertible and we are good!

Following the hint we have[1]

$$B = VJV^{-1}$$
$$B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -0.5 & 0.5 \end{bmatrix}$$

Solving for a, b

$$\begin{aligned} B &= \begin{bmatrix} a & 1+2b \\ a & 1+4b \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -0.5 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 2a - 0.5 - b & -a + 0.5 + b \\ 2a - 0.5 - 2b & -a + 0.5 + 2b \end{bmatrix} \\ \begin{bmatrix} 0.3 & 0.5 \\ -0.5 & 1.3 \end{bmatrix} &= \begin{bmatrix} 2a - 0.5 - b & -a + 0.5 + b \\ 2a - 0.5 - 2b & -a + 0.5 + 2b \end{bmatrix} \end{aligned}$$

[1]Technically we had no way of knowing a priori that B was not diagonalizable, but it was not hard to check that the suggested eigenvector matrix contained a generalized eigenvector, and not simple eigenvectors.

We have a system of 4 equations (2 independent equations), 2 unknowns

$$\begin{aligned} 2a - 0.5 - b &= 0.3 \\ 2a - 0.5 - 2b &= -0.5 \\ -a + 0.5 + b &= 0.5 \\ -a + 0.5 + 2b &= 1.3 \end{aligned}$$

Solving for a and b holds

$$a = b = 0.8$$

Therefore

$$\begin{aligned} B &= V \begin{bmatrix} 0.8 & 1 \\ 0 & 0.8 \end{bmatrix} V^{-1} \\ B^2 &= V \begin{bmatrix} 0.8^2 & 2 * 0.8^1 \\ 0 & 0.8^2 \end{bmatrix} V^{-1} \\ B^3 &= V \begin{bmatrix} 0.8^3 & 3 * 0.8^2 \\ 0 & 0.8^3 \end{bmatrix} V^{-1} \\ B^j &= V \begin{bmatrix} 0.8^j & j0.8^{j-1} \\ 0 & 0.8^j \end{bmatrix} V^{-1} \\ &= \begin{bmatrix} 0.8^j & j0.8^{j-1} + 2 * 0.8^j \\ 0.8^j & j0.8^{j-1} + 4 * 0.8^j \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -0.5 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.8^j - \frac{j}{2}0.8^{j-1} & \frac{j}{2}0.8^{j-1} \\ -\frac{j}{2}0.8^{j-1} & \frac{j}{2}0.8^{j-1} + 0.8^j \end{bmatrix} \end{aligned}$$

We can finally write x_t in lag notation and invert it as we know that the eigenvalues of B, the diagonal entries of J, are $0.8 < 1$ in absolute value:

$$\begin{aligned} x_t &= \frac{1}{1 - BL}\varepsilon_t \\ &= \sum_{j=0}^{\infty} B^j L^j \varepsilon_t \\ &= \sum_{j=0}^{\infty} B^j \varepsilon_{t-j} \end{aligned}$$

4 Problem 4

Consider the AR(1),

$$y_t = 0.5y_{t-1} + \epsilon_t$$

Given data $t = -n+1, \ldots, T$, let us regress y_t on n lags,

$$y_t = x_t\beta + \epsilon_t$$

where $x_t = [y_{t-1}, y_{t-2}, \ldots, y_{t-n}]$. Thus form

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix}$$

We will be interested in the asymptotic limit, as $T \to \infty$. Thus:

4.1 Calculate $E[\mathbf{X}'\mathbf{X}]$ and $E[\mathbf{X}'\mathbf{y}]$.

. . . .

X is not a vector, but a vector of stacked-vectors, i.e. a matrix:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} y_0 & y_{-1} & y_{-2} & \cdots & y_{1-n} \\ \vdots \\ y_{T-1} & y_{T-2} & y_{T-3} & \cdots & y_{T-n} \end{bmatrix}$$

Therefore

$$\mathbf{E}(\mathbf{X}'\mathbf{X}) = E\begin{bmatrix} y_0 & y_1 & y_2 & \dots & y_{T-1} \\ \vdots & & & & \\ y_{1-n} & y_{1-n+1} & y_{1-n+2} & \dots & y_0 \end{bmatrix}\begin{bmatrix} y_0 & y_{-1} & y_{-2} & \dots & y_{1-n} \\ \vdots & & & & \\ y_{T-1} & y_{T-2} & y_{T-3} & \dots & y_{T-n} \end{bmatrix}$$

$$= \begin{bmatrix} y_0^2 + y_1^2 + \dots + y_{T-1}^2 & & & \\ y_0 y_{-1} + y_1 y_0 + \dots + y_T y_{T-1} & y_0^2 + y_1^2 + \dots + y_{T-1}^2 & \dots & \\ y_0 y_{-2} + y_1 y_0 + \dots & y_0 y_{-1} + \dots y_T y_{T-1} & y_0^2 + y_1^2 + \dots + y_{T-1}^2 & \dots \\ \vdots & & & \\ y_{T-1} y_{N-1} + \dots & & & \end{bmatrix}$$

$$= \begin{bmatrix} T\mathbf{E}(y_t^2) & T\mathbf{E}(y_t y_{t-1}) & T\mathbf{E}(y_t y_{t-2}) & \dots & T\mathbf{E}(y_t y_{t-N}) \\ T\mathbf{E}(y_t y_{t-1}) & T\mathbf{E}(y_t^2) & T\mathbf{E}(y_t y_{t-1}) & \dots & \\ & T\mathbf{E}(y_t y_{t-2}) & T\mathbf{E}(y_t y_{t-1}) & \dots & \\ \vdots & & & & \\ T\mathbf{E}(y_t y_{t-N}) & \dots & & & \end{bmatrix}$$

$$= \frac{T\sigma^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \rho^2 & \dots \\ \rho^2 & \rho & 1 & \rho \dots & \\ \rho^{N-1} & \dots & & & 1 \end{bmatrix}$$

Moreover, following the same procedure we find that

$$\mathbf{E}(\mathbf{X}'\mathbf{y}) = \begin{bmatrix} y_0 & y_1 & y_2 & \dots & y_{T-1} \\ \vdots & & & & \\ y_{1-n} & y_{1-n+1} & y_{1-n+2} & \dots & y_0 \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix}$$

$$= \begin{bmatrix} T\frac{\rho\sigma^2}{1-\rho^2} \\ T\frac{\rho^2\sigma^2}{1-\rho^2} \\ \vdots \\ T\frac{\rho^N\sigma^2}{1-\rho^2} \end{bmatrix}$$

$$= T\frac{\sigma^2}{1-\rho^2}\begin{bmatrix} \rho \\ \rho^2 \\ \vdots \\ \rho^N \end{bmatrix}$$

4.2 Calculate the inverse of $E[\mathbf{X}'\mathbf{X}]$. (Hint: try this numerically first for, say, $n = 4$,$n = 5$,$n = 6$. Do you see a pattern? Prove that the pattern is correct.)

....

Playing a bit in our favorite language we notice a pattern, which we formalize using the

Gaussian Elimination Strategy into the following general formula[2]:

$$\mathbf{E}(\mathbf{X}'\mathbf{X})^{-1} = (\frac{T\sigma^2}{1-\rho^2})^{-1}\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \dots & \dots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \dots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

Where $\frac{1}{1-\rho^2}$ is multiplied by 1 on the first and last diagonal entries, $1+\rho^2$ on the rest of the diagonal entries and $-\rho$ on the upper and lower off-diagonals.

4.3 Calculate

$$\beta = (E[\mathbf{X}'\mathbf{X}])^{-1}E[\mathbf{X}'\mathbf{y}]$$

Are you surprised by your result, or could you have guessed the answer from the beginning?

. . . .

$$\begin{aligned}
\mathbf{E}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{E}(\mathbf{X}'\mathbf{y}) &= (\frac{T\sigma^2}{1-\rho^2})^{-1}\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \dots & \dots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \dots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & \dots & \dots & \dots & \dots & 1 \end{bmatrix}\frac{T\sigma^2}{1-\rho^2}\begin{bmatrix} \rho \\ \rho^2 \\ \vdots \\ \rho^N \end{bmatrix} \\
&= \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \dots & \dots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \dots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & \dots & \dots & \dots & \dots & 1 \end{bmatrix}\begin{bmatrix} \rho \\ \rho^2 \\ \vdots \\ \rho^N \end{bmatrix} \\
&= \frac{1}{1-\rho^2}\begin{bmatrix} \rho-\rho^3 \\ -\rho^2+\rho^2(1+\rho^2)-\rho^4 \\ \vdots \end{bmatrix}
\end{aligned}$$

[2]An easy proof that this works is to, for an arbitrary T, test $\mathbf{E}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{E}(\mathbf{X}'\mathbf{X})$, using our conjectured form. We find the identity matrix.

$$\beta = \frac{1}{1-\rho^2}\begin{bmatrix} \rho(1-\rho^2) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \rho \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

As one could have guessed, we end up with β being a vector of length N with ρ in the first entry and 0 everywhere else. Because we are looking at an AR(1) process, so y_t only depends on ϵ_t and y_{t-1}, we should indeed not find any direct impact of $y_{t-2}, y_{t-3}\ldots$ on y_t (except through y_{t-1}, which is captured by the regression coefficient on y_{t-1}).

5 Problem 5

Let y_t be a covariance stationary process. Let z be a complex number with absolute value 1, i.e. $z\bar{z} = 1$, where $\bar{z}$ denotes the complex conjugate to z.

5.1 Consider the complex-valued process

$$v_t = y_{t+1}z + y_t + y_{t-1}\bar{z}$$

(a) State the complex conjugate $\bar{v}_t$.

....

We may use that the conjugates of real numbers are themselves, and the conjugate of a sum is the sum of conjugates

$$\bar{v}_t = y_{t+1}\bar{z} + y_t + y_{t-1}z$$

(b) Calculate the product $v_t\bar{v}_t$.

....

$$\begin{aligned} v_t\bar{v}_t &= (y_{t+1}z + y_t + y_{t-1}\bar{z})(y_{t+1}\bar{z} + y_t + y_{t-1}z) \\ &= y_{t+1}^2|z|^2 + y_{t+1}y_tz + y_{t+1}y_{t-1}z^2 + y_ty_{t+1}\bar{z} + y_t^2 + y_ty_{t-1}z + y_{t-1}y_{t+1}\bar{z}^2 + y_{t-1}y_t\bar{z} + y_{t-1}^2|z|^2 \\ &= y_{t+1}^2 + y_{t+1}y_tz + y_{t+1}y_{t-1}z^2 + y_ty_{t+1}\bar{z} + y_t^2 + y_ty_{t-1}z + y_{t-1}y_{t+1}\bar{z}^2 + y_{t-1}y_t\bar{z} + y_{t-1}^2 \end{aligned}$$

(c) Calculate $E[v_t\bar{v}_t]/3$ and write it in the form

$$E[v_t\bar{v}_t]/3 = \alpha_{-2}z^2 + \alpha_{-1}z + \alpha_0 + \alpha_1\bar{z} + \alpha_2\bar{z}^2$$

What are $\alpha_{-2}, \ldots, \alpha_2$? (Hint: remember slide 21.)

....

Let $\gamma_k = E[y_ty_{t-k}]$.

$$\begin{aligned} E[v_t\bar{v}_t]/3 &= \frac{1}{3}\left[\gamma_0 + \gamma_1z + \gamma_2z^2 + \gamma_1\bar{z} + \gamma_0 + \gamma_1z + \gamma_2\bar{z}^2 + \gamma_1\bar{z} + \gamma_0\right] \\ &= \frac{\gamma_2}{3}z^2 + \frac{2\gamma_1}{3}z + \gamma_0 + \frac{2\gamma_1}{3}\bar{z} + \frac{\gamma_2}{3}\bar{z}^2 \end{aligned}$$

Therefore $\alpha_j = \frac{3-|j|}{3}\gamma_{|j|}$. So they are scaled autocovariances, with scaling decreasing for farther apart variables.

5.2 Redo the exercise and find the αs for

$$v_t = y_{t+2}z^2 + y_{t+1}z + y_t + y_{t-1}\bar{z} + y_{t-2}\bar{z}^2$$
$$E[v_t\bar{v}_t]/5 = \alpha_{-4}z^4 + \alpha_{-3}z^3 + \alpha_{-2}z^2 + \alpha_{-1}z + \alpha_0 + \alpha_1\bar{z} + \alpha_2\bar{z}^2 + \alpha_3\bar{z}^3 + \alpha_4\bar{z}^4$$

....

It takes just as much effort (perhaps less) to complete the inductive proof in 5.3. I completed that first, so the answer here directly follows from the below proof.

$$E[v_t\bar{v}_t]/5 = \gamma_0 + \sum_{j=1}^{4}\frac{5-j}{5}\gamma_j z^j + \sum_{j=1}^{4}\frac{5-j}{5}\gamma_j\bar{z}^j$$

5.3 Do you see a pattern? Provide an educated guess as to what happens for the below, when calculating the α coefficients

$$v_t = y_{t+k}z^k + \cdots + y_{t-k}\bar{z}^k$$
$$E[v_t\bar{v}_t]/(2k+1) = \alpha_{-2k}z^{2k} + \cdots + \alpha_{2k}\bar{z}^{2k}$$

....

We can just inductively prove that the coefficients are $\alpha_j = \frac{2k+1-|j|}{2k+1}\gamma_{|j|}$ for $j \in \{-2k, \ldots, 2k\}$.

(i) $k = 1$: This was shown in 5.1 above.

(ii) Assume true for $k-1$: Then

$$\begin{aligned}
E[v_t\bar{v}_t]/(2k+1) &= E[(y_{t+k}z^k + \cdots + y_{t-k}\bar{z}^k)(y_{t+k}\bar{z}^k + \cdots + y_{t-k}z^k)]/(2k+1) \\
&= E[(y_{t+k}z^k + y_{t-k}\bar{z}^k)(y_{t+k}\bar{z}^k + \cdots + y_{t-k}z^k) \\
&\quad + (y_{t+k}\bar{z}^k + y_{t-k}z^k)(y_{t+k-1}\bar{z}^{k-1} + \cdots + y_{t-k+1}z^{k-1}) \\
&\quad + (y_{t+k-1}z^{k-1} + \cdots + y_{t-k+1}\bar{z}^{k-1})(y_{t+k-1}\bar{z}^{k-1} + \cdots + y_{t-k+1}z^{k-1})]/(2k+1) \\
&= \Big[\gamma_0 + \gamma_1 z + \cdots + \gamma_k z^k + \gamma_{k+1}z^{k+1} + \cdots + \gamma_{2k}z^{2k} \\
&\quad + \gamma_{2k}\bar{z}^{2k} + \gamma_{2k-1}\bar{z}^{2k-1} + \cdots + \gamma_k\bar{z}^k + \gamma_{k-1}\bar{z}^{k-1} + \cdots + \gamma_0 \\
&\quad + \gamma_1\bar{z}^{2k-1} + \cdots + \gamma_{2k-1}\bar{z} \\
&\quad + \gamma_{2k-1}z + \cdots + \gamma_1 z^{2k-1} \\
&\quad + \gamma_{2k-1}z^{2k-1} + 2\gamma_{2k-2}z^{2k-2} + \cdots + 2\gamma_{2k-2}\bar{z}^{2k-2} + \gamma_{2k-1}\bar{z}^{2k-1}\Big]/(2k+1) \\
&= \Big[\gamma_{2k}z^{2k} + 2\gamma_{2k-1}z^{2k-1} + \cdots + 2\gamma_{2k-1}\bar{z}^{2k-1} + \gamma_{2k}\bar{z}^{2k}\Big]/(2k+1)
\end{aligned}$$

Therefore we have $\alpha_j = \frac{2k+1-|j|}{2k+1}\gamma_{|j|}$, as desired.