Theory of Income, Spring 2020

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Problem Set 1

1 Endowment economy

Consider an OLG economy as described in the class notes. In this exercise, we focus on an endowment economy and study the effects of privatising social security in such a way that everyone remains indifferent. In particular, we want to study the evolution of savings and government debt.

An agent born in period t cares about consumption at period t and $t+1$, denoted c_t^t and c_{t+1}^t, respectively, where c_j^i refers to consumption of an agent born at date i (generation i) in period j. Agents born at date 0 are already "old" in period 1 and thus only care about consumption in period 1, c_1^0. There is a unit mass of agents in each generation.

The endowment stream of agent $t \geq 1$ is given by $\left\{e_t^t, e_{t+1}^t\right\}$ and the endowment of the initial old is e_1^0. The budget constraints of an agent of generation $t \geq 1$ are:

$$\begin{aligned} s_t + c_t^t &= e_t^t - \tau_t^t, \\ c_{t+1}^t &= s_t\left(1+r_t\right) + e_{t+1}^t - \tau_{t+1}^t, \end{aligned} \tag{1}$$

where s_t refers to savings, r_t is the interest rate between periods t and $t+1$, τ_t^t are the taxes levied on agents of generation t when young and τ_{t+1}^t are the taxes levied on the same generation when they are old. The budget constraint of the initial old is

$$c_1^0 = s_0\left(1+r_0\right) + e_1^0 - \tau_1^0, \tag{2}$$

where s_0 is given.

Let g_t denote government expenditures and B_t the stock of government assets at period t. Then, the government budget constraint is, for $t \geq 1$,

$$B_t + g_t = B_{t-1}\left(1+r_{t-1}\right) + \tau_t^t + \tau_t^{t-1}. \tag{3}$$

The market clearing conditions in the goods market is

$$g_t + c_t^t + c_t^{t-1} = e_t^t + e_t^{t-1}, \tag{4}$$

and the market clearing condition in the asset market is

$$B_t + s_t = 0.$$

We also have the initial condition $B_0 + s_0 = 0$.

Question 1. ***(Walras' Law)*** Use equations equation (1), equation (2), equation (3) and equation (4), and the initial condition $B_0 + s_0 = 0$ to deduce the market clearing condition in the asset market holds.

Now, we will analyse the social security pay-as-you-go system. The idea is that, in each period, the young agents are taxed and the earnings are redistributed to the current old. In terms of our notation, we assume:

$$\tau_t^t = \theta, \; \tau_{t+1}^t = -\theta.$$

Furthermore, consider an equilibrium with a constant interest rate $r_t = \bar{r}$, and assume

$$\begin{aligned} g_t &= 0, \\ e_t^t &= \alpha, \\ e_t^{t-1} &= 1 - \alpha, \\ s_0 &= B_0 = 0. \end{aligned}$$

Hence, we immediately see from equation (3) that $B_t = 0$ for all $t \geq 1$. Since this is an endowment economy, from the class notes, we also know that $s_t = 0$ for all $t \geq 1$.

<u>*Ricardian Equivalence and Privatization of Social Security*</u>

Consider the following environment with different taxes, for all $t \geq 1$,

$$\begin{aligned} \tilde{\tau}_1^0 &= \tau_1^0, \\ \tilde{\tau}_t^t &= \tau_t^t + \frac{\tau_{t+1}^t}{1 + \tilde{r}_t}, \\ \tilde{\tau}_{t+1}^t &= 0. \end{aligned}$$

The ˜ refers to an equilibrium with the new tax system. Guess that in the new equilibrium the interest rate does not change, that is,

$$\tilde{r}_t = r_t = \bar{r},\ \forall t.$$

Question 2. Interpret $\left\{\tilde{\tau}_t^t, \tilde{\tau}_{t+1}^t\right\}$.

Question 3. Show that $\tilde{c}_t^t = c_t^t$, $\tilde{c}_{t+1}^t = c_{t+1}^t$ for $t \geq 1$ and $\tilde{c}_1^0 = c_1^0$ is optimal.

Question 4. Show that $\tilde{B}_t = \tilde{B} = -\frac{\theta}{1+\bar{r}}$ for all $t \geq 1$. [Hint: Solve the government's budget constraint.

Question 5. Show that $\tilde{s}_t = \tilde{s} = \frac{\theta}{1+\bar{r}}$ solves the agent's problem.

Question 6. Show that $\tilde{B}_t$ and $\tilde{s}_t$ satisfy the asset market clearing condition.

Question 7. Interpret what happens in the asset market at time $t = 1$ and at $t \geq 2$ (who saves? why? etc.)

2 Production economy

In this problem, we consider a production economy with capital and labor. Agents live for two periods. Young agents inelastically supply one unit of labour, consume and save, while old agents are retired and consume out of their savings. We will use the same notation as in the last problem, for example, c_j^i denotes consumption of a person born at date i in period j. Aggregate consumption in period t is given by

$$C_t = c_t^t + c_t^{t-1}.$$

We use K_t and I_t to denote the stock of capital installed at the beginning of period t and aggregate investment during period t respectively. So the stock of capital evolves as

$$K_{t+1} = I_t + (1-\delta) K_t,$$

where $\delta \in (0,1)$ is the depreciation rate

The utility function of an agent of generation-$t \geq 1$ is given by $u\left(c_t^t, c_{t+1}^t\right)$, while the utility function of generation-0 (i.e. the initial old) is simply c_1^0. The budget constraints of an agent born at t in periods t and $t+1$ are:

$$\begin{aligned} c_t^t + s_t &= w_t - \tau_t^t, \\ c_{t+1}^t &= s_t (1 + r_t) + \tau_{t+1}^t, \end{aligned}$$

where w_t is the wage rate at t. The initial old's budget constraint is given by

$$c_1^0 = s_0 (1 + r_0) - \tau_1^0,$$

where s_0 is given exogenously.

The government's budget constraint is given by

$$B_t + g_t = B_{t-1} (1 + r_{t-1}) + \tau_t^t + \tau_t^{t-1}.$$

Finally, output is produced with a constant return to scale technology $F(K_t, L_t)$. Technological feasibility is given by

$$g_t + I_t + c_t = F(K_t, L_t).$$

Firms rent capital and labour from the households. Let v_t denote the rental rate of capital, then firm's problem in any period t is

$$\max_{K_t, L_t} F(K_t, L_t) - w_t L_t - v_t K_t.$$

Question 1. Show that the firm's problem implies:

$$F_L(K_t, L_t) = w_t,$$
$$F_K(K_t, L_t) = v_t.$$

Question 2. Show, by an arbitrage argument, that in any equilibrium, we must have $v_{t+1} = r_t + \delta$.

The agent's problem is

$$\max_{c_t^t, c_{t+1}^t} \quad u\left(c_t^t, c_{t+1}^t\right)$$
$$s.t. \quad c_t + \frac{c_{t+1}}{1+r_t} = w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}.$$

Question 3. Argue that the solution to the agent's problem implies that savings can be written as a function of the interest rate r_t, current income $w_t - \tau_t^t$, and total wealth $w_t - \tau_t^t - \tau_{t+1}^t/(1+r_t)$:

$$s_t = s\left(r_t, w_t - \tau_t^t, w_t - \tau_t^t - \frac{\tau_{t+1}^t}{1+r_t}\right).$$

Remark: If we formulate the problem in a different way, it is possible to write the savings policy function as $s_t = s\left(r_t, w_t - \tau_t^t, \tau_{t+1}^t\right)$, but for future purposes it is better to have it as above.

Given $B_0 + s_0 = K_1$, the quantities $\left\{K_t, c_t^t, c_t^{t-1}, I_t, B_t, g_t\right\}_{t=1}^{\infty}$ and prices $\{w_t, r_t, v_t\}_{t=1}^{\infty}$ are an equilibrium if:

1. agent's maximise utility;
2. firm's maximise profits;
3. goods mark clear;
4. government budget constraint holds;
5. agents supply their unit endowment of labour, $L_t = 1$;
6. no arbitrage holds.

Question 4. Show that in equilibrium, the Walras' Law holds; i.e.

$$B_t + s_t = K_{t+1}, \forall t \geq 1.$$

Assume that $\tau_t^t = \tau_{t+1}^t = 0$ and $g_t = 0$ for all $t \geq 1$, and $B_0 = 0$. The government's budget constraint then implies that $B_t = 0$ for all $t \geq 1$.

Question 5. Show that

$$\begin{aligned} K_{t+1} &= s\left(F_K\left(K_{t+1}, 1\right) - \delta, F_L\left(K_t, 1\right), F_L\left(K_t, 1\right)\right), \\ K_1 &> 0 \end{aligned}$$

describes an equilibrium [Hint: use the solution to question 3].

For the rest of the problem, assume that

$$g_t = 0,\ \forall t \geq 1,$$
$$B_0 = 0,$$
$$r_t > 0,\ \forall t \geq 1,$$
$$\tau_t^t = \tau,$$
$$\tau_{t+1}^t = -\tau.$$

In words, this means that government has no expenditures, begins with stock of zero assets, interest rates are always positive and there is a social security system in which each generation pays τ when they are young and receive τ when they are old.

Question 6. Show that, if c_{t+1}^t is not an inferior good (i.e. not an inferior good), then savings $s(\cdot)$ are strictly decreasing in τ. [Hint: It may be easier to solve this question without taking derivatives. if $\tau' > \tau$ you can compute the change in wealth. Then use that in the "worst" case, the income effect is bounded by $-(\tau' - \tau)\frac{r_t}{1+r_t}$].

Question 7. Show that, if

$$\left.\frac{ds}{dr_t}\right|_{\tau=0} > 0,$$

so that the substitution effect is not strong enough, then there exists a function $s^*(\cdot)$ such that

$$K_{t+1} = s^*(K_t, \tau).$$

Question 8. Show that

$$\left.\frac{ds^*}{d\tau}\right|_{\tau=0} < 0.$$

Hint: Use the result from question 6 : that, if r_t is kept constant, then $ds/d\tau < 0$. Then, you only have to show that the indirect effect of a change in τ through its effect on the equilibrium interest rate has the correct sign.

The last question showed that, for each level of capital, next-period stock of capital will be higher in an economy with no social security system (since we showed that $K_{t+1} = s^*$ is decreasing in τ at $\tau = 0$; so increasing τ would reduce next-period capital). This, however, is not sufficient to argue that the steady-state level of capital is lower in an economy with social security. Consider an economy without social security in a steady-state $K^* > 0$. We will analyse how K^* changes if we introduce social security (for small τ).

Question 9. Assume that in a neighbourhood of K^*, we are in the "well-behaved" case:

$$0 < \frac{ds^*}{dK_t} < 1.$$

Show that an increase in τ reduces the steady-state level of capital. [Hint: a graph is sufficient].

Question 10. In addition, argue that, if close to K^*, the dynamic system has $ds^*/dK_t > 1$, then the steady-state capital is higher in the economy with social security. Is that steady-state stable? (Hint: suppose you start at that steady state, and imagine that the stock of capital is changed a little. Does the dynamic system drive you back to the original steady state?]

3 Production economy: An example

In class, we saw that, in a pure-endowment economy without population growth, if the equilibrium interest rate is positive, the introduction of social security makes the initial old better off and all the subsequent generations worse off. In this problem, we will show that this need not be the case in a production economy with capital. In particular, we will construct an example where the generation born at period 1 is strictly better off with the introduction of social security with positive interest rate.

Consider the following modification to the last problem: The production function is Cobb-Douglas, $F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$ and *consumers only care about consumption when old*, so that $u\left(c_t^t, c_{t+1}^t\right) = c_{t+1}^t$. Also assume $g_t = 0$ for all t and $B_0 = 0$.

Question 1. Argue that $s_t = w_t - \tau_t^t$ for all $t \geq 1$. Since $B_t = 0$ for all t, stock of capital is

completely owned by the old agents, so $K_{t+1} = s_t$, or

$$K_{t+1} = w_t - \tau_t^t.$$

Question 2. Show that, in equilibrium, stock of capital evolves as

$$K_{t+1} = (1-\alpha) K_t^{\alpha} - \tau_t^t$$

Question 3. Assume $\tau_t^t = \tau_{t+1}^t = 0$ for all t. Solve for the (unique) steady state with positive capital (denote it by K^*).

Suppose that, at the beginning of period 1, the economy is at the steady state without social security and the government announces that it will introduce a pay-as-you-go system with $\tau_t^t = \tau$ and $\tau_{t+1}^t = -\tau$ for all τ. Denote consumption of generation i in period j as a function of τ by $c_j^i(\tau)$. We will construct an example where agents born at period 1 are strictly better off in the new environment.

Question 4. Show that consumption of generation 1 at period 2 is given by

$$\begin{aligned} c_2^1(\tau) &= (w_1 - \tau)(1 + r_1(\tau)) + \tau \\ &= (K^* - \tau)\left(1 - \delta + \alpha (K^* - \tau)^{\alpha - 1}\right) + \tau. \end{aligned}$$

Question 5. Show that, given $\delta > 0$,

$$\left.\frac{dc_2^1(\tau)}{d\tau}\right|_{\tau=0}$$

can be positive for sufficiently small α.

Question 6. Is it possible that *every* generation is made better off by introducing social security? [Hint: Remember that the First Welfare Theorem holds in all equilibria with positive interest rate].

Question 7. In class, we saw that the introduction of social security in an endowment economy makes everyone worse-off except the initial old, who receive the transfer without paying anything. Above, we constructed an example where generation 1 is strictly better off with the introduction of social security. Can you given intuitive explanation for this result?

4 Heterogeneity

We now modify the OLG model introduced in class so that there is heterogeneity among agents. In particular, agents are now indexed by their date of birth and their endowment:

$$\begin{aligned} e_t^{t,n} &= 1 - \alpha_n, \\ e_{t+1}^{t,n} &= \alpha_n, \\ e_s^{t,n} &= 0, \forall s \neq t, t+1, \end{aligned}$$

where $e_s^{t,n}$ denotes the endowment at time s of an individual of type n born at time t. Preferences are described by

$$u^{t,n}\left(c_1^{t,n}, c_2^{t,n}, \cdots\right) = (1-\beta)\ln\left(c_t^{t,n}\right) + \beta \ln\left(c_{t+1}^{t,n}\right), \beta \in (0,1).$$

The initial old generation receives an endowment at time $t = 1$ only:

$$\begin{aligned} e_1^{0,n} &= \alpha_n, \\ e_s^{0,n} &= 0, \forall s \neq 1. \end{aligned}$$

As usual, the initial old only care about their consumption when old, with preferences described by

$$u^{0,n}\left(c_1^{0,n}, c_2^{0,n}, \cdots\right) = c_1^{0,n}.$$

We will assume that each generation has N different types of endowments α_n so that we can write $n = 1, 2, \ldots, N$. The set of agents I in this economy consists of the pairs

$$I = \{i = (t,n) : t = 0, 1, 2, \ldots; n = 1, 2, \ldots, N\}.$$

Question 1. Write down the market clearing constraint in this economy, as it applies for any given period t.

Question 2. Show that the competitive equilibrium is such that there is no trade *across* generations. (Hint: Start by arguing that the initial old will consume its endowment. Then, explore the implication of this result for the initial young generation. Finally, use an inductive argument.)

Question 3. Find an expression for *aggregate savings*, $s\left(r; \{\alpha_n\}_{n=1}^{N}, \beta\right)$, the savings of all the young of a generation in terms of the parameters of the model (α's and β's) and the net interest rate. (Hint: Use the FOC to find an expression for $c_t^{t,n}$ and then proceed to aggregate).

Find an expression for the optimal consumption choice of an agent when young and when old if his endowment is characterised by $\alpha_n = \alpha$ and if he faces an interest rate r. Denote these optimal choices by $c_y(\alpha, r)$ and $c_0(\alpha, r)$.

Question 4. Let

$$\bar{\alpha} := \frac{1}{N}\sum_{n=1}^{N}\alpha_n.$$

Show that

$$\begin{aligned}
c_y(\bar{\alpha}, r) &= (1-\beta)\left[(1-\bar{\alpha}) + \frac{\bar{\alpha}}{1+r}\right], \\
c_o(\bar{\alpha}, r) &= \beta\left[(1+r)(1-\bar{\alpha}) + \bar{\alpha}\right], \\
\frac{c_y(\alpha, r)}{c_y(\bar{\alpha}, r)} &= \frac{1+(1-\alpha)r}{1+(1-\bar{\alpha})r}, \\
\frac{c_o(\alpha, r)}{c_o(\bar{\alpha}, r)} &= \frac{1+(1-\alpha)r}{1+(1-\bar{\alpha})r}, \\
\sum_{n=1}^{N} c_y(\alpha_n, r) &= N c_y(\bar{\alpha}, r).
\end{aligned}$$

Question 5. Using the above results, characterise the equilibrium interest rate $\bar{r}$, i.e. write an equation for $\bar{r}$, which is the solution to

$$s\left(\bar{r}; \{\alpha_n\}_{n=1}^{N}; \beta\right) = 0.$$

Show that, if two economies have the same value of $\bar{\alpha}$, but possibly different distribution of the α's, then they will still have the same equilibrium interest rate $\bar{r}$, so that we can write $\bar{r}(\bar{\alpha}, \beta)$. Show also that if $\bar{\alpha}$ is the same for two economies, then the equilibrium average aggregate consumption for the young is the same, i.e.

$$\frac{1}{N}\sum_{i=1}^{N} c_t^{t,n} \equiv \frac{1}{N}\sum_{i=1}^{N} c_y(\alpha_n, \bar{r})$$

is the same regardless of the distribution of the α's.

Question 6. Compute the "best aggregate symmetric allocation" for this economy; i.e. the welfare-maximising allocation that only depends on whether a particular agent is young or old, so that

$$c_t^{t,n} = c_y^*, c_{t+1}^{t,n} = c_o^*, \forall t, n.$$

Question 7. Show that, if the following condition holds

$$\bar{\alpha} := \frac{1}{N}\sum_{n=1}^{N}\alpha_n < \beta,$$

then the equilibrium interest rate is negative and the allocation is not Pareto optimal. To do so, consider the following allocation:

$$\begin{aligned}
c_t^{*t,n} &= c_y^* \frac{1+(1-\alpha_n)\,\bar{r}}{1+(1-\bar{\alpha})\,\bar{r}}, \\
c_{t+1}^{*t,n} &= c_o^* \frac{1+(1-\alpha_n)\,\bar{r}}{1+(1-\bar{\alpha})\,\bar{r}},
\end{aligned}$$

for $t \geq 1$ and $n = 1, \ldots, N$ (all the current and future young) and for the initial old:

$$c_1^{*0,n} = \alpha_n + (\beta - \bar{\alpha}),$$

for $n = 1, 2, \ldots, N$, where $\bar{r}$ is the interest rate that corresponds to the equilibrium of the economy with $\bar{\alpha}$ and β.

- Show that the "*" allocation is feasible.
- Show that this allocation Pareto dominates the equilibrium allocation for the economy with $\bar{\alpha}$ and β. Make sure that you show that this allocation is preferred for each type n of the initial old and for each type n of young agents. (Hint: For the young, you have to compare the utility of the equilibrium vector

  $$(c_y(\alpha_n, \bar{r}), c_0(\alpha_n, \bar{r})) = \left(c_y(\bar{\alpha}, \bar{r})\frac{c_y(\alpha_n, \bar{r})}{c_y(\bar{\alpha}, \bar{r})}, c_o(\bar{\alpha}, \bar{r})\frac{c_o(\alpha_n, \bar{r})}{c_o(\bar{\alpha}, \bar{r})}\right)$$

  with that of the "*" allocation; i.e.

  $$\left(c_y^* \frac{1+(1-\alpha_n)\,\bar{r}}{1+(1-\bar{\alpha})\,\bar{r}}, c_o^* \frac{1+(1-\alpha_n)\,\bar{r}}{1+(1-\bar{\alpha})\,\bar{r}}\right).$$

Using your answers to parts 4, 5 and 6, conclude that the second is preferred to the first if and only if the young prefers $\left(c_y^*, c_o^*\right)$ to the bundle $\left(c_y\left(\bar{\alpha}, \bar{r}\right), c_o\left(\bar{\alpha}, \bar{r}\right)\right)$. Finally, to show the last statement, argue that $\left(c_y\left(\bar{\alpha}, \bar{r}\right), c_o\left(\bar{\alpha}, \bar{r}\right)\right)$ is a feasible symmetric allocation so that $\left(c_y^*, c_o^*\right)$ is indeed preferred by the young.)

Question 8. Consider introducing a pay-as-you-go social security system where young agents are taxed by τ when young, and receive τ when old, regardless of their type (i.e. regardless of n). Assume that the initial allocation is such that $\bar{r} < 0$. This will amount to change the endowments to $e_{t+1}^{t,n} = \alpha_n + |tau$ and $e_t^{t,n} = 1 - \alpha_n - \tau$.

- How would the equilibrium interest rate will change? [Hint: compute the new value of $\bar{\alpha}$]
- Is the new equilibrium allocation necessarily Pareto superior to the old equilibrium allocation? In particular, does it necessarily improve the welfare of all the young, independently of their type (n).
- What is the implication of the finding in the second bullet for the design of a pay-as-you-go social security system?

5 OLG model with multiple periods

We will now introduce agents living multiple periods in the basic model with homogeneous generations. We will allow for general preferences and endowments. In this problem, we will show that, if equilibrium interest rates are negative, then the equilibrium is not Pareto optimal.

The first time period is $t = 1$. Preferences are given by

$$u^t\left(c_1^t, c_2^t, \ldots\right) = v^t\left(c_t^t, c_{t+1}^t, \ldots, c_{t+N-1}^t\right),$$

for the generation born after $t \geq 1$, and

$$u^t\left(c_1^t, c_2^t, \ldots\right) = v^t\left(c_1^t, c_2^t, \ldots, c_{t+N-1}^t\right),$$

for the old generation born at $t = -N+2$, $-N+3$, …, -1, 0.

We denote the ages of the agents by $a = 1, 2, \ldots, N$.

Endowments are positive only while agents are alive, so that, for $t \geq 1$,

$$e_{t+a-1}^t > 0, \forall 1 \leq a \leq N,$$
$$e_{t+a-1}^t = 0, \text{otherwise},$$

and for generations born at $t = -N+2$, $-N+3$, …, -1, 0,

$$e_1^t, e_2^t, \ldots, e_{t+N-1}^t > 0,$$

and zero otherwise.

Question 1. Write down the market clearing constraint in this economy as it applies for any given period.

Question 2. Let $\{\bar{c}^t\}$ denote the equilibrium allocation of the generation born at time t. Let λ^t be the Lagrange multiplier of the budget constraint:

$$\sum_{a=1}^{N} p_{t+a-1} c_{t+a-1}^t = \sum_{a=1}^{N} p_{t+a-1} e_{t+a-1}^t, \forall t \geq 1$$

and

$$p_1 c_1^t + p_2 c_2^t + \cdots + p_{t+N-1} c_{t+N-1}^t = p_1 e_1^t + p_2 e_2^t + \cdots + p_{t+N-1} e_{t+N-1}^t$$

for $t = -N+2,\ -N+3,\ \ldots,\ -1,\ 0$.

Write down the first-order conditions that the optimal choice $\bar{c}^t$ for generation t must satisfy:

Question 3. We will define a new allocation using the equilibrium allocation $\{\bar{c}^t\}$ and two parameters τ and Y. The "*" allocation is defined as

$$c^{*t}_{t+a-1}(\tau) := \begin{cases} \bar{c}^t_{t+a-1} - \tau & \forall 1 \leq a \leq Y, \\ \bar{c}^t_{t+a-1} + \frac{Y}{N-Y}\tau & \forall Y+1 \leq a \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

when agents are young (i.e. $a \leq N$), we are reducing their consumption by an amount τ each period relative to the equilibrium consumption. When agents are old ($a > Y$), we are adding an amount $\tau Y / (N-Y)$ to their consumption.

Let $P(\tau, Y)$ denote the present value (at the time of birth) of the transfer described by parameters (τ, Y) for a generation born at time t using prices $\{p_t\}$. Show that for generations born after time $t \geq 1$,

$$P^t(\tau, Y) = \frac{1}{p_t}\left[\sum_{a=1}^{Y} p_{t+a-1}(-\tau) + \sum_{a=Y+1}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\tau\right)\right].$$

Show that for generations born at $t = -N+2,\ -N+3,\ \ldots,\ -1,\ 0$, we have two cases: one for those that at time $t=1$ are $a = 2, \ldots, Y$ years old (i.e. they were born at time t satisfying $-Y+2 \leq t \leq 0$, so they are young):

$$P^t(\tau, Y) = \frac{1}{p_1}\left[\sum_{a=2-t}^{Y} p_{t+a-1}(-\tau) + \sum_{a=Y+1}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\tau\right)\right]$$

and one for those that at time $t=1$ are $a = Y+1, \ldots, N$ years old (i.e. they were born at time t satisfying $-N+2 \leq t \leq -Y+1$, so they are old):

$$P^t(\tau, Y) = \frac{1}{p_1}\left[\sum_{a=2-t}^{N} p_{t+a-1}\left(\frac{Y}{N-Y}\tau\right)\right].$$

Argue that if the prices p_t are increasing in t for all $t \geq 1$ (so that the implied interest rates are negative) then $P^t(\tau, Y) > 0$.

Question 4. Consider an equilibrium $\{\bar{c}^t, p_t\}$. Define the interest rate

$$\frac{1}{1+r_t} := \frac{p_{t+1}}{p_t}, \forall t \geq 1.$$

Show that, if the equilibrium interest rates are negative for all $t \geq 1$, then the competitive equilibrium allocation $\{\bar{c}^t\}$ is not Pareto optimal. To establish this, show that the allocation $\{c^{*t}(\tau)\}$ is feasible and, at least for small τ, $\{c^{*t}(\tau)\}$ Pareto dominates $\{\bar{c}^t\}$. In particular,

1. Show that this allocation is feasible for any τ.

2. Compute the marginal welfare gain (or loss) of changing the allocation by introducing small τ. Define the function

$$U^t(\tau) = v^t\left(c_t^{*t}(\tau), c_{t+1}^{*t}(\tau), \ldots, c_{t+N-1}^{*t}(\tau)\right)$$

   for generations $t \geq 1$ and

$$U^t(\tau) = v^t\left(c_1^{*t}(\tau), c_2^{*t}(\tau), \ldots, c_{t+N-1}^{*t}(\tau)\right)$$

   for generations $t = -N+2, -N+3, \ldots, -1, 0$. Differentiate $U^t(\tau)$ with respect to τ and evaluate this derivative at $\tau = 0$. Use your answer to Part (c) and the fact that $r_t < 0$ to argue that $dU^t/d\tau > 0$.

Question 5. Suppose that the economy has intertemporal possibilities of production, so that aggregate consumption in period t does not need to be equal to the aggregate endowment in period t or formally $Y = \emptyset$. Does your conclusion changes?, i.e. if interest rates are negative does $\{c^{*t}\}$ Pareto dominates $\{\bar{c}^t\}$? [Hint: is $\{c^{*t}\}$ feasible?]

6 Population growth and social security

Return again to the basic model where each generation lives for two periods, preferences are logarithmic and endowments are $\left(e_t^t, e_t^{t+1}\right) = (1-\alpha, \alpha)$ for all $t \geq 1$ and $e_1^0 = \alpha$. We will examine the welfare effects of changing demographic patterns in the presence of social security. To this end, let N_t denote the number of young agents at time t. Population grows at the rate n so that

$$N_{t+1} = (1+n)\, N_t, N_0 = 1,$$

where we have normalised the initial population to one.

Question 1. Write down the market clearing constraint in this economy, as it applies for any given period t.

Question 2. Briefly argue that the competitive equilibrium is such that there is no trade across and within generations.

Question 3. Find an expression for aggregate savings, $s\,(r; \alpha, \beta, N)$, the savings of all the young of a generation in terms of the parameters of the model (α, β, N) and the net interest rate, r. (Hint: Use the first-order conditions to find an expression for c_t^t and then proceed to aggregate). Characterise the equilibrium interest rate $\bar{r}$; i.e. write an equation for $\bar{r}$, which is the solution to

$$s\,(\bar{r}; \alpha, \beta, N) = 0.$$

Question 4. Compute the best symmetric allocation for this economy; i.e. the welfare-maximising allocation that only depends on whether a particular agent is young or old so that $c_t^t = c_y$ and $c_{t+1}^t = c_0$ for all t.

Question 5. Compute the level of per-capita tax collection in a pay-as-you-go system that would implement the best symmetric allocation. How does this level depend on n? Briefly explain.

Question 6. Assume that the competitive eq. with no transfers has negative real interest rates. Assume that population growth falls permanently to $n' = 0 < n$ but the tax collected from each young agent is held fixed. Draw a graph with per-capita consumption of young and old in the $x - y$ axis. Draw two lines with the corresponding sets of feasible symmetric allocations for each of the two growth rates. Locate the equilibrium with no social security in your graph, the equilibrium with the best symmetric allocation for $n > 0$, and the equilibrium with the social security system described above for the $n' = 0$ case. How is the welfare of the initial old affected when population growth decreases under the social security system with constant transfer to the old? How is the welfare of the young of the current and each future generation affected?