Price Theory I
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Contents

1 Theory of demand

We focus here on the intuition and not on the technicalities. This means, for example, that we assume that the utility functions are all "nice".

In applied works, analysts often begin with demand equations rather than explicitly deriving them from utility functions. Clearly, not every set of equations relating prices to the quantities purchased by an individual are consistent with utility maximisation. Our purpose here is to show what is special about demand equations. In other words, for equations relating prices to quantities to be consistent with utility maximisation, what properties do they have to satisfy?

1.1 Utility maximisation

Let the $u : X \to \mathbb{R}$ be the consumer's utility function where $X = (X_1, X_2, \ldots, X_n) \in \mathbb{R}^N_{++}$ be the set of goods consumed by the consumer. Note that $X \in \mathbb{R}^N_{++}$ means that each of the good is being purchased by the consumer in the market. The prices of the goods are given by $P = (P_1, P_2, \ldots, P_N) \in \mathbb{R}^N_{++}$. The consumer is endowed with income M. Notice that the units have to be consistent—e.g. if M is income per week, then consumption X must also be per week.

We generally think of the consumer as having the budget constraint so that the consumer cannot spend more than his income:

$$\sum_{i=1}^{N} X_i P_i \leq M.$$

The utility maximisation problem is then given by

$$\max_{X_1, X_2, \ldots, X_N} U\left(X_1, X_2, \ldots, X_N\right) \; s.t. \; \sum_{i=1}^{N} X_i P_i \leq M. \tag{1.1}$$

We can set up the Lagrangian

$$\mathcal{L} = U\left(X_1, X_2, \ldots, X_N\right) + \lambda \left[M - \sum_{i=1}^{N} X_i P_i \right].$$

The first-order conditions are

$$\frac{\partial U}{\partial X_i} - \lambda P_i = 0 \, \forall i \tag{1.2}$$
$$\sum_{i=1}^{N} X_i P_i = M.$$

We can interpret this first N first-order conditions in at least three ways:

(i) marginal utilities are proportional to prices;

(ii) cost of consuming more of good i equals the value of consuming more good i;

(iii) utility per dollar on the margin must be equal for all goods.

We explore each interpretation below.

1.1.1 Interpretation (i): Marginal utilities are proportional to prices

The first-order condition gives us that(1.3)

$$\frac{\partial U}{\partial X_i} = \lambda P_i. \tag{1.3}$$

That is, for every good i, at an optimal, marginal utilities are proportional to prices, and that the proportion is given by λ.

Consider two states of the world. In state 1, people consume goods $X^* = (X_1^*, X_2^*, \ldots, X_N^*)$ at prices $P = (P_1, P_2, \ldots, P_N)$ and have income M. In state 2, there is some perturbation such that everything is slightly different: X^* differs by $dX = (dX_1, dX_2, \ldots, dX_N)$, P differs by $dP = (dP_1, dP_2, \ldots, dP_N)$ and M differs by dM. How does the utility differ between the two states? In other words, what is dU?

Note that

$$\begin{aligned} dU &= dU\left(X_1, X_2, \ldots, X_N\right) \\ &= \sum_{i=1}^{N} \frac{\partial U}{\partial X_i} dX_i. \end{aligned}$$

Since marginal utilities are proportion to prices by equation (1.3), we may write

$$dU = \lambda \sum_{i=1}^{N} P_i dX_i. \tag{1.4}$$

Notice that $P_i dX_i$ is observable because we can see prices and changes in consumption. That is, we can know ${}^{dU}/_{\lambda}$, which is the dollar value of how much better off the consumer is when moving from state 1 to state 2. Note that the expression above can be aggregated across consumers.

We can also use how utility changes over time. Consider time-varying consumption

$$X\left(t\right) = X_1\left(t\right), X_2\left(t\right), \ldots, X_N\left(t\right).$$

Then,

$$\begin{aligned} \frac{dU}{dt} &= \sum_{i=1}^{N} \frac{\partial U}{\partial X_i} \frac{dX_i}{dt} = \lambda \sum_{i=1}^{N} P_i \frac{dX_i}{dt} \\ \Rightarrow \frac{\frac{\partial U}{\partial t}}{\lambda} &= \sum_{i=1}^{N} P_i \frac{dX_i}{dt}. \end{aligned}$$

This tells us that we can measure the dollar value of the change in utility over time as the dollar value of the change in consumption over the same period.

1.1.2 Interpretation (ii): Cost of consuming more of good i equals the value of consuming more good i

Dividing the expression for good i 1.2 by good j gives us that

$$\frac{\frac{\partial U}{\partial X_i}}{\frac{\partial U}{\partial X_j}} = \frac{P_i}{P_j}. \tag{1.5}$$

- ▷ The right-hand side tells us that to obtain one unit of good i, we must give up ${}^{P_i}/_{P_j}$ units of good j. That is, P_i/P_j represents the cost of good i.
- ▷ The left-hand side tells us the amount the consumer is willing to give up of good j to get a unit of good i. That is, it represents the value of good i.

Putting together, the first-order conditions tells us that, at an optimal point, it must be the case that the cost of consuming more of good i equals your value of consuming more of good i; i.e. marginal cost equals marginal benefit.

The proportional relationship between prices and marginal utilities in equation (1.5) is important as it allows us to infer from something that is observable (i.e. the ratio of prices), something that is unobservable (the ratio of the marginal utilities). That is, the assumption that people are maximising allows us to infer something about preferences from observed behaviour. Notice that, here, we are using a revealed preference argument—i.e. we assume that people's preferences can be elicited from the choices they make. For example, if we observe that a consumer chooses the bundle X at prices P and $\sum_{i=1}^{N} P_i X_i = M$. Then, we know that a person prefers X to another point $\hat{X}$, where $\sum_{i=1}^{N} P_i \hat{X}_i < M$ because the bundle $\hat{X}$ was originally affordable but was not chosen.

So, if we can measure the prices people face, then we can indirectly measure their marginal utilities (up to a proportionate factor). For example, if the price of good 10 is five times the price of good 6 (i.e. $P_{10} = 5P_6$), then this means that a unit of good 10 is worth five times as much as good 6 on the margin. In fact, this is true for *everyone* in the market consuming goods 6 and 10.

1.1.3 Interpretation (iii): Utility per dollar on the margin must be equal for all goods

We can equate 1.2 for good i and j by λ to obtain

$$\lambda = \frac{\frac{\partial U}{\partial X_i}}{P_i} = \frac{\frac{\partial U}{\partial X_j}}{P_j}, \tag{1.6}$$

which says that the marginal utility per dollar must be equal to the constant λ for goods i and j (in fact, for all goods). If this were not the case, for example, if the marginal utility per dollar for good i was higher than good j, then the consumer would be better off by giving up some of good j and consuming more of good i.

Notice also that λ is the marginal utility per dollar from all goods. In other words, λ is the marginal utility that the consumer gains from receiving one dollar. In particular, we need not know how the additional dollar is spent to know the increase in marginal utility—i.e. we can talk about value independent of what the consumer actually does.

For example, suppose that dentists are considering adding fluoride to the water supply to reduce tooth cavities. It is possible that, once the fluoride is in the water, people stop brushing their teeth and get as many (or more) cavities as before. The dentists may conclude that the policy has backfired and that people did not benefit. However, an economist would say that people have in fact benefited—now they do not have to brush their teeth. To an economist, it does not matter how people take the benefit of fluoride in the water supply (in terms of fewer cavities or not brushing).

1.2 The Marshallian demand

Solving the first-order conditions for the utility maximisation problem, together with the budget constraint give us the solutions for each good i, which we denote as

$$\begin{aligned} &X_i^M (P_1, P_2, \ldots, P_N, M), \forall i \\ &\lambda^M (P_1, P_2, \ldots, P_N, M). \end{aligned}$$

The demand equation that solve the maximisation problem equation (1.1) are called the *Marshallian demand equations.*

We can consider the effect of changing P_j on good i by $\partial X_i^M / \partial P_j$ and the Marshallian demand equation allows us to see how the agent would alter the consumption of all goods in response to change in prices for a given level of income. Implicit here is the fact that re-optimisation is costless for the consumer—this might be true in the long run but may not hold in the short run (e.g. changing cars in response to changes in gas prices). Thus, we should think of the Marshallian demand as a long-run demand.

We define below some useful concepts for the rest of the note.

1.2.1 Price elasticities

The elasticity of demand for good i with respect to the price of good j is defined as

$$\epsilon_{ij}^M := \frac{\partial X_i^M}{\partial P_j} \frac{P_j}{X_i}.$$

ϵ_{ij}^M tells us the percentage change in demand for good i for each 1% increase in the price of good j. To see this note that, we can write

$$\epsilon_{ij}^M = \frac{\frac{\partial X_i^M}{X_i}}{\frac{\partial P_j^M}{P_j}} = \frac{\%\text{ change in } X_i}{\%\text{ change in } P_j}.$$

Recall that the Marshallian demand equation gives us the quantity demanded for *a given level of income* (as well as prices). Thus, implicit above was the fact that when we change prices of good j, we hold income constant.

If $i \neq j$, then ϵ_{ij}^M is called the *cross-price elasticity* and if $i = j$, $\epsilon_i^M := \epsilon_{ii}^M$ is called the *own-price elasticity*, sometimes shortened to just *price elasticity*. Goods i and j are complements if the price increase in good i leads to a decrease in quantity demanded for quantity j. That is, $\epsilon_{ij} < 0$ if goods i and j are complements. If goods i and j are substitutes, then $\epsilon_{ij} > 0$.

1.2.2 Income elasticities

We are also often interested in the percentage change in demand for good i for each 1% increase in income M; i.e. *income elasticity of demand for good* i, denoted η_i. This is defined in a similar fashion to price elasticities:

$$\eta_i := \frac{\partial X_i^M}{\partial M}\frac{M}{X_i}.$$

We say that good i is *normal* if $\eta_i > 0$ and that it is *inferior* if $\eta_i < 0$. That is, a normal (inferior) good is one whose (Marshallian) demand increases (decreases) as M increases from a given level of M.

Good i may be both inferior and/or normal depending on the income level, and it can even "cycle". For example, one might expect that sales in Walmart to increase when income rises (i.e. demand for Walmart is normal); however, as the consumers become richer, they may stop shopping at Walmart and instead shop at "fancier" stores (i.e. demand for Walmart is now inferior). Thus, the peak demand for Walmart would happen at the income where people tend to shift their shopping to "fancier" stores.

What would you expect η_i to be? It should be 1 because it is natural to think about a 10% increase in income resulting in a 10% increase in consumption of everything. Following this intuition, we say that good i is a *necessity* if $\eta_i \in (0, 1)$ (i.e. the demand for good i changes less than proportionately to income) and that it is a *superior good* if $\eta_i > 1$ (i.e. the demand for good i changes more than proportionately to income). Note that, if we differentiate the budget constraint with respect to income, we get

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\partial X_i^M}{\partial M} P_i &= 1 \\
\Leftrightarrow \sum_{i=1}^{N} \frac{\partial X_i^M}{\partial M}\left(\frac{M}{X_i^M}\frac{X_i^M}{M}\right) P_i &= 1 \\
\Leftrightarrow \sum_{i=1}^{N} \eta_i s_i &= 1,
\end{aligned}$$

where s_i is the share of income spent on good i. This says that, on average (weighted by the shares), the income effect is 1.

1.3 The Hicksian demand

Consider the dual of the utility maximisation problem:

$$\min_{X_1, X_2, \ldots, X_N} \sum_{i=1}^{n} X_i P_i \; s.t. \, U\left(X_1, X_2, \ldots, X_N\right) = \bar{U}.$$

This problem asks the minimum amount of expenditure required to achieve a given level of utility $\bar{U}$. This contrasts with the Marshallian experiment in which we were asking for the maximum level of utility that can be achieved given a level of income.

The Lagrangian for the cost minimisation problem is

$$\mathcal{L} = \sum_{i=1}^{N} X_i P_i + \mu \left[\bar{U} - U\left(X_1 X_2, \ldots, X_N\right)\right].$$

The first-order conditions are

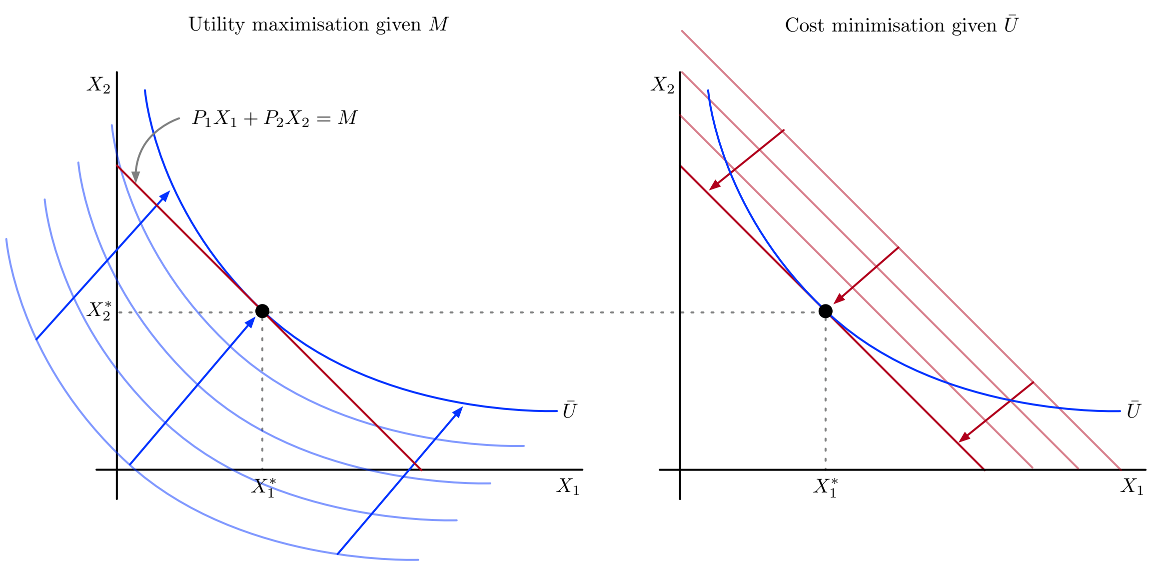
$$P_i - \mu \frac{\partial U}{\partial X_i} = 0, \forall i$$
$$\bar{U} = U\left(X_1, X_{,2}, \ldots, X_N\right).$$

Note that these are the same conditions as before—with $\mu = {}^1\!/\!_\lambda$. Recall that λ is the marginal utility that the consumer gains from receiving one dollar—similarly, μ can be interpreted as the marginal cost (in dollar terms) of an additional unit of util.

Solving the first-order conditions gives

$$X_i^H\left(P_1, P_2, \ldots, P_N, \bar{U}\right), \forall i$$
$$\mu^H\left(P_1, P_2, \ldots, P_N, \bar{U}\right).$$

If we observe that a consumer is consuming (X_1^*, X_2^*) at prices (P_1, P_2), this could have been a result of either UMP or the dual cost minimisation problem (see Figure below). This means that we are free to apply the Marshallian or the Hicksian experiment to conduct analysis independent of whether the consumer solved the UMP or cost minimisation problem.



1.3.1 Price elasticities

Price elasticities for the Hicksian demand can be defined in an analogous manner to those for the Marshallian demand:

$$\epsilon_{ij}^H := \frac{\partial X_i^H}{\partial P_j}\frac{P_j}{X_i}.$$

As we show concretely later, this is generally different from ϵ_{ij}^M. This is because the Hicksian demand equation gives us the quantity demanded for *a given level of utility* (as well as prices). Thus, implicit in ϵ_{ij}^H is the fact that when we change prices of good j, we hold utility constant, where as in the case of ϵ_{ij}^M, we were holding income constant. Moreover, it can be show that To do

$$\frac{\partial X_i^H}{\partial P_j} \leq 0.$$

Just as in the case for Marshallian demands, Hicksian demand for goods i and j can be complementary and substitutive in the same way. However, note that in a two good case, the two goods cannot be complements—if P_1 increases, then the demand for X_1 has to fall (law of demand) but since Hicksian demand function keeps utility constant, this means that the demand for X_2 has to increase; i.e. X_1 and X_2 cannot be complements in a two good case.

1.3.2 Cost function

The minimum cost to achieve a given level of utility $\bar{U}$ with prices $P_1, P_2, \ldots, P_N$ is given by

$$C\left(P_1, P_2, \ldots, P_N, \bar{U}\right) := \left\{\min_{X_1, X_2, \ldots, X_N} \sum_{i=1}^{n} X_i P_i \, s.t. \, U\left(X_1, X_2, \ldots, X_N\right) = \bar{U}\right\}.$$

This cost function has the following properties:

(i) C is homogenous of degree 1 in prices;

(ii) C is nondecreasing in prices—i.e.
$$\frac{\partial C}{\partial P_i} \geq 0, \forall i;$$

(iii) The derivative of C with respect to P_i is the Hicksian demand for good i—i.e.
$$\frac{\partial C}{\partial P_i} = X_i^H\left(P_1, P_2, \ldots, P_N, \bar{U}\right); \tag{1.7}$$

(iv) C is concave;

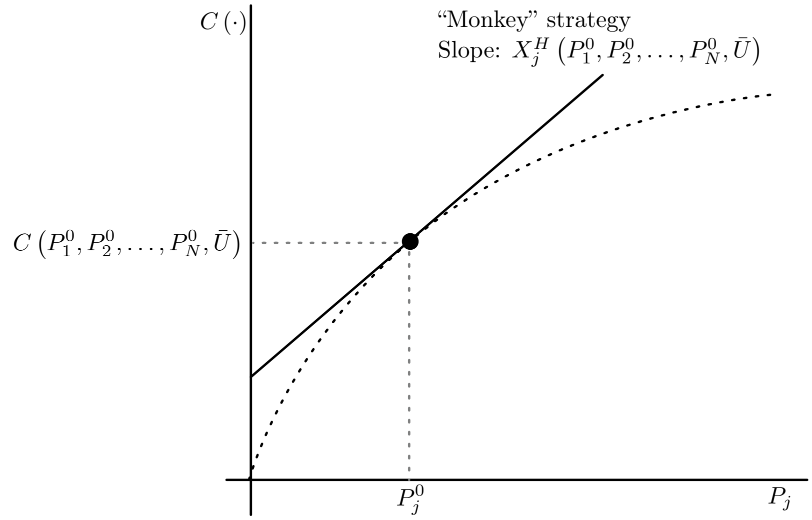
(v) C is increasing in U—i.e.
$$\frac{\partial C}{\partial U} > 0.$$

We prove properties (3) and (4) intuitively.

Consider a unidimensional case and a change in the price of good j from some initial price P_j^0 with all other prices held constant. Consider the "monkey" solution where, as the price of good j changes, the consumer does not change his consumption bundle at all. In this case, the cost of his bundle will be a straight line that is linear in P_j. For example, if the consumer was consuming ten units of good j, then, for every dollar increase in P_j, he would spend ten more dollars. Hence, the slope of the line is given by $X_j^H\left(P_1^0, P_2^0, \ldots, P_N^0, \bar{U}\right)$. That is,

$$\frac{\partial C}{\partial P_j} = X_j^H\left(P_1^0, P_2^0, \ldots, P_N^0, \bar{U}\right).$$

This is show in the figure below.



Clearly, the optimal bundle will adjust as P_j changes and the points will line below the line associated with this monkey strategy (as the new optimal bundle should not cost any more than the cost under the monkey strategy). The degree of concavity of the line associated with the optimal cost function then depends on the extent to which the optimal bundle changes with change in prices. That is, degree of concavity depends upon

$$\frac{\partial X_j^H\left(P_1^0, P_2^0, \ldots, P_N^0, \bar{U}\right)}{\partial P_j} = \frac{\partial^2 C}{\partial P_j^2}.$$

We can think of the concavity in a differently way. Consider the two price-utility vectors $\left(P_1^0, P_2^0, \ldots, P_N^0, \bar{U}\right)$ and $\left(P_1^1, P_2^1, \ldots, P_N^1, \bar{U}\right)$. Denote the optimal quantities associated with these vectors as $\left(X_1^0, X_2^0, \ldots, X_N^0\right)$ and $\left(X_1^1, X_2^1, \ldots, X_N^1\right)$, respectively. Then, it must be the case that the cost of X_i^1's must be lower at prices P_i^1's than at prices P_i^0's (since agent is cost minimising). Similarly, the cost of X_i^0's must be lower at prices P_i^0's than at prices P_i^1's. That is,

$$\sum_{i=1}^N X_i^1 P_i^0 \geq \sum_{i=1}^N X_i^0 P_i^0, \; \sum_{i=1}^N X_i^0 P_i^1 \geq \sum_{i=1}^N X_i^1 P_i^1.$$

Adding these two inequalities yields

$$\begin{aligned}
\sum_{i=1}^N X_i^1 P_i^0 + \sum_{i=1}^N X_i^0 P_i^1 &\geq \sum_{i=1}^N X_i^0 P_i^0 + \sum_{i=1}^N X_i^1 P_i^1 \\
&\Leftrightarrow 0 \geq \sum_{i=1}^N \left(X_i^0 P_i^0 + X_i^1 P_i^1 - X_i^1 P_i^0 - X_i^0 P_i^1\right) \\
&\Leftrightarrow 0 \geq \sum_{i=1}^N \left(X_i^1 - X_i^0\right)\left(P_i^1 - P_i^0\right).
\end{aligned}$$

Notice that this is a generalised version of the law of demand (property (ii)) in which all prices can be be changing at the same time, the price changes do not have to be infinitesimal, and the Hicksian demand curves need not have derivatives at the points of interest. The expression says that, on average, goods whose prices increase are consumed less (and the opposite is also true). In other words, the cross-good correlation between price changes and demand changes cannot be positive. It also says that the Hicksian demand system is concave. Notice that the law of demand comes right out of cost minimisation—which prohibits you from saying that "it got cheaper so I'm going to buy less".

We can further understand how the cost and demand functions will be related to the indifference curves. As the indifference curves become more curved, demand becomes more inelastic and the cost function will become straighter. Indeed, the cost function is closer to linear when the indifference curves have significant curvature. The opposite is also true.

To see this, suppose the following types of preferences [To do]

$$u_1(x_1,x_2) = \min\left\{\frac{x_1}{a}, \frac{x_2}{c}\right\} \Rightarrow C(p_1,p_2,u) = auP_1 + buP_2$$
$$u_2(x_1,x_2) = ax_1 + bx_2 \Rightarrow C(p_1,p_2,u) = u\min\left\{\frac{P_1}{a}, \frac{P_2}{b}\right\},$$
$$u_3(x_1,x_2) = (x_2^a + x_2^a)^{\frac{1}{a}} \Rightarrow \text{[CES cost function]}$$
$$u_4(x_1,x_2) = x_1^a x_2^b \Rightarrow \text{[in between linear and min]}.$$

1.4 Relationship between Marshallian and Hicksian demand

The Marshallian and Hicksian demand functions are equivalent if we set $M = C\left(P_1, P_2, \ldots, P_N, \bar{U}\right)$. That is,

$$X_i^M\left(P_1, P_2, \ldots, P_N, C\left(P_1, P_2, \ldots, P_N, \bar{U}\right)\right) \equiv X_i^H\left(P_1, P_2, \ldots, P_N, \bar{U}\right), \forall i.$$

This is called the *Slutsky correspondence.* Since above must hold for all P, the equality also hold in terms of (partial) derivatives: i.e.

$$\frac{\partial X_i^M}{\partial P_j} + \frac{\partial X_i^M}{\partial M}\frac{\partial C}{\partial P_j} = \frac{\partial X_i^H}{\partial P_j}.$$

Using the fact that $\partial C/\partial P_j = X_j^H = X_j^M = X_j$ at the point of evaluation, and rearranging gives the *Slutsky equation*:

$$\frac{\partial X_i^M}{\partial P_j} = \frac{\partial X_i^H}{\partial P_j} - X_j\frac{\partial X_i^M}{\partial M}.$$

This gives us a relationship between how price changes while keeping income constant (i.e. $\partial X_i^M/\partial P_j$), called the Marshallian effect, relates to price changes while keeping utility constant (i.e. $\partial X_i^H/\partial P_j$), called the Hicksian (or substitution) effect. In particular, it says that the difference between the two arises due to the income effect (i.e. $\partial X_i^M/\partial M$).

What does the Slutsky equation say? Suppose P_j increases by a dollar. There will be a response to such a change holding utility constant, which is the substitution effect.[1] However, there is also an effect from the fact that the price hike has changed the consumer's income. For example, if the consumer had been consuming ten of good j, then, the consumer has, in some sense, ten fewer dollars of income after the price hike. That is, the bundle the consumer was buying before the price hike would now cost ten dollars more after the hike. The income effect tells us how responsive good i is to change in come. That is, if the consumer's income changes by ten dollars, how much does the demand for good i changes. Notice that how large the income effect depends also on how large the consumption of the good is in the budget.

We can express the Slutsky equation in terms of elasticities:

$$\frac{\partial X_i^M}{\partial P_j}\left(\frac{P_j}{X_i}\right) = \frac{\partial X_i^H}{\partial P_j}\left(\frac{P_j}{X_i}\right) - \frac{\partial X_i^M}{\partial M}\left(\frac{M}{X_i}\frac{X_jP_j}{M}\right)$$
$$\Leftrightarrow \epsilon_{ij}^M = \epsilon_{ij}^H - \eta_i s_j.$$

The expressions says that the Marshallian percentage response of a one percent increase in price of good j on good i is the Hicksian response to the price change minus the income share of the good whose price has changed times the income elasticity of the good i. Thus, the size of the income

[1] Note that the effect could be either a substitution or complementary relationship.

effect depends on both the share of good j in the budget and the responsiveness of good i to changes in income.

The law of demand holds for Marshallian demand if

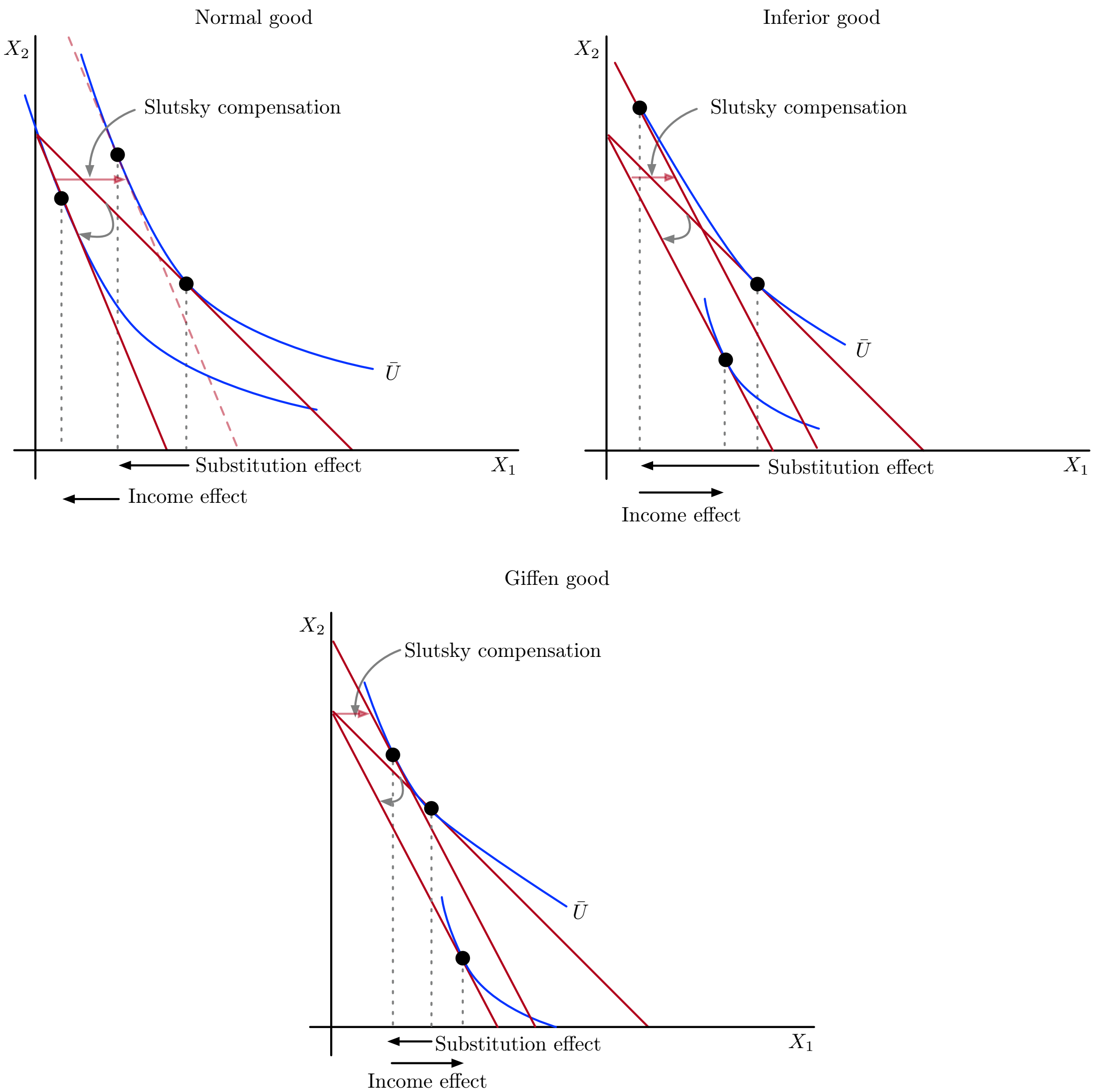
$$\frac{\partial X_i^M}{\partial P_i} \leq 0 \Leftrightarrow \epsilon_i^M \leq 0,$$

where

$$\epsilon_i^M = \epsilon_i^H - \eta_i s_i.$$

Since $\epsilon_i^H \leq 0$, in particular, the law of demand holds if $\eta_i > 0$; i.e. if good i is normal (since $s_i \in [0, 1]$). Since the law of demand always holds for the Hicksian demand, the Hicksian demand curve is always downward sloping. The Slutsky equation tells us that, if the good is normal, the Marshallian demand curve is even more responsive because the income effect reinforces the substitution effect.

However, it is possible that $\epsilon_i^M > 0$ if the income effect is sufficiently larger than the substitution effect; i.e. if the good is inferior (i.e. $\eta_i < 0$) and its share is sufficiently large. Such goods are called *Giffen* goods and are generally difficult to find in reality since most high-share goods are normal. See the figure which compares the impact of a price increase in good i on the (Marshallian demand) for X_1 for different types of good i.



1.5 Properties of demand functions

As eluded to earlier, we wish to consider the properties that makes equations relating prices to quantities be consistent with utility maximisation. Specifically, these are:

(i) *homogeneity*—that tells us the effect of a change in *all* prices (and income in the case of Marshallian demand) on the demand for a *single* good.

(ii) *adding up*—that us the the effect of a *single* price change affects the demand for *all* goods.

(iii) symmetry—that complementarity/substitutability relationships between goods are symmetric.

1.5.1 Homogeneity

Homogeneity tells us the effect of a change in *all* prices (and income in the case of Marshallian demand) on the demand for a *single* good.

Hicksian demand What happens to the Hicksian demand function if we were to multiply all the prices by k? Since X_i^H solves the cost minimisation problem, it must be the case that X_i^H is still the cheapest bundle given prices kP. If not, then when prices was given by P, the consumer should have chosen the alternative bundle. That is, Hicksian demand is homogeneous of degree zero with respect to prices; i.e.

$$X_i^H\left(kP_1, kP_2, \dots kP_N, \bar{U}\right) = X_i^H\left(P_1, P_2, \dots P_N, \bar{U}\right).$$

Differentiating this with respect to k gives that

$$\begin{aligned} \sum_{j=1}^{N} \frac{\partial X_i^H}{\partial P_j} P_j &= 0 \\ \Leftrightarrow \sum_{j=1}^{N} \frac{\partial X_i^M}{\partial P_j} \frac{P_j}{X_i} &= 0 \\ \Leftrightarrow \sum_{j=1}^{N} \epsilon_{ij}^H &= 0. \end{aligned}$$

The condition above says that 10% increase in the prices of all goods would change the demand for good i.

Marshallian demand Suppose we multiply all the prices and the income for the consumer. by a constant $k > 0$. What happens to the Marshallian demand?

The budget constraint for the new maximisation problem is

$$\sum_{i=1}^{N} X_i \left(kP_i\right) \leq kM.$$

However, notice that the k's cancel out so that the budget constraint is in fact the same as before. Therefore, the Marshallian demand is homogeneous of degree zero with respect to prices and income; i.e.

$$X_i^M\left(kP_1, kP_2, \dots kP_N, km\right) = X_i^M\left(P_1, P_2, \dots P_N, m\right).$$

Differentiating this with respect to k gives that

$$\begin{aligned} \sum_{j=1}^{N} \frac{\partial X_i^M}{\partial P_j} P_j + \frac{\partial X_i^M}{\partial m} m &= 0 \\ \Leftrightarrow \sum_{j=1}^{N} \frac{\partial X_i^M}{\partial P_j} \frac{P_j}{X_i} + \frac{\partial X_i^M}{\partial m} \frac{m}{X_i} &= 0 \\ \Leftrightarrow \sum_{j=1}^{N} \epsilon_{ij}^M + \eta_i &= 0. \end{aligned}$$

This is the homogeneity of the Marshallian demand function expressed in terms of elasticities. It implies that the effect on good i of a 10% increase in M is equivalent to the effect due to a 10% fall in prices of all goods.

1.5.2 Adding up

In contrast to homogeneity, adding up tells us the the effect of a *single* price change affects the demand for *all* goods.

Hicksian demand Differentiating the utility constraint with respect to P_j gives

$$\sum_{i=1}^{N} \frac{\partial U}{\partial X_i^H} \frac{\partial X_i^H}{\partial P_j} = 0.$$

From the first-order condition of the expenditure minimisation problem, we know that $P_j = \mu \partial U / \partial X_j$ so that we can rewrite above as

$$\sum_{i=1}^{N} P_i \frac{\partial X_i^H}{\partial P_j} = 0.$$

This says that the change in consumption, weighted by prices, due to a change in the price of good j is zero.

We can also express this in terms of price elasticities and the share of income spent on good i.

$$\begin{aligned}
\sum_{i=1}^{N} P_i \frac{\partial X_i^H}{\partial P_j} \left(\frac{X_i}{X_i} \frac{P_j}{P_j} \frac{M}{M} \right) &= 0 \\
\Leftrightarrow \left(\frac{M}{P_j} \right) \sum_{i=1}^{N} \epsilon_{ij}^H s_i &= 0 \\
\Leftrightarrow \sum_{i=1}^{N} \epsilon_{ij}^H s_i &= 0.
\end{aligned}$$

Again, this is called the adding up as it adds the effect of a single price change (P_j) on demands for all goods $i = 1, 2, \ldots, N$. It says that, looking across all demand equations, whenever a single price changes, the demands for all the goods have to change in such a way that, weighted by their shares, the total change is zero.

Marshallian demand Differentiating the budget constraint with respect to P_j gives

$$\sum_{i=1}^{N} \frac{\partial X_i^M}{\partial P_j} P_i = -X_j^M.$$

This is like the law of demand. This says that if the price of good j increases, the utility measured in dollar terms has to fall by X_j (recall interpretation (i) of the FOC of the UMP).

We can express this in terms of price elasticities and share of income spend on goods i and j.

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\partial X_i^M}{\partial P_j} \left(\frac{P_j}{X_i^M} \frac{X_i^M}{P_j} \right) P_i \left(\frac{M}{M} \right) &= -X_j^M \\
\Leftrightarrow \sum_{i=1}^{N} \frac{\partial X_i^M}{\partial P_j} \frac{P_j}{X_i^M} \frac{P_i X_i^M}{M} \left(\frac{M}{P_j} \right) &= -X_j^M \\
\Leftrightarrow \left(\frac{M}{P_j} \right) \sum_{i=1}^{N} \epsilon_{ij} s_i &= -X_j^M \\
&= -\frac{P_j X_j^M}{M} \\
\Leftrightarrow \sum_{i=1}^{N} \epsilon_{ij} s_i &= -s_j.
\end{aligned}$$

This is called the adding up (for the Marshallian demand system) as it adds the effect of a single price change (P_j) on all demands for goods $i = 1, 2, \ldots, N$. What does this say? Suppose $s_j = 10\%$.

Then, a 10% increase in the price of good j will lead to a 1% reduction in the consumer's real income. This means that, on average, the consumer must reduce consumption of other goods by 1%.

Notice that this expression was obtained from the (binding) budget constraint—it does not require the consumer to be rational/utility maximisation. But we go to a point quite close to the law of demand—the budget constraint gives that if a good becomes more expensive, then the consumer is poorer, and that he has to consume less on average. Then, the law of demand tells us that the consumer will consume less of the good that has become more expensive.

1.5.3 Symmetry

Hicksian demand Recall that the Hicksian demand function for good i is the derivative of the cost function with respect to P_i. Recall that, by Young's Theorem (if C is continuous),

$$\frac{\partial^2 C}{\partial P_i \partial P_j} = \frac{\partial^2 C}{\partial P_j \partial P_i}.$$

Combing these together, we can write

$$\frac{\partial X_i^H}{\partial P_j} = \frac{\partial^2 C}{\partial P_i \partial P_j} = \frac{\partial^2 C}{\partial P_j \partial P_i} = \frac{\partial X_j^H}{\partial P_i}.$$

This means that if good i is a complement (substitute) to good j, then good j is a complement (substitute) to good i; i.e. that the complementarity/substitutability relationships are symmetric.

Note that this is not equivalent to saying that cross-price elasticities are symmetric. To see this, note that

$$\begin{aligned}\frac{\partial X_i^H}{\partial P_j}\left(\frac{P_j}{X_i}\right)\left(\frac{P_i}{X_j}\right) &= \frac{\partial X_j^H}{\partial P_i}\left(\frac{P_i}{X_j}\right)\left(\frac{P_j}{X_i}\right) \\ \Leftrightarrow \epsilon_{ij}^H &= \epsilon_{ji}^H \frac{P_j X_j}{X_i P_i} \\ \Leftrightarrow \epsilon_{ij}^H &= \epsilon_{ji}^H \frac{s_j}{s_i}.\end{aligned}$$

Not only does this gives us the relationship between cross-price elasticities, notice that the expression means that $\epsilon_{ij}^H > \epsilon_{ji}^H$ if $s_i > s_j$. That is, the change in demand for good i from a 1% increase in the price of j is greater than the change in demand for good j from a 1% increase in the price of i if the share of income spent on good i is larger. That is, price change in the good with a higher share is more important.

Marshallian demand From the Slutsky equation, we have that

$$\frac{\partial X_i^M}{\partial P_j} = \frac{\partial X_i^H}{\partial P_j} - X_j \frac{\partial X_i^M}{\partial M} \Leftrightarrow \frac{\partial X_i^H}{\partial P_j} = \frac{\partial X_i^M}{\partial P_j} + X_j \frac{\partial X_i^M}{\partial M}$$

Since the Hicksian demand function is symmetric, then

$$\frac{\partial X_i^H}{\partial P_j} = \frac{\partial X_i^M}{\partial P_j} + X_j \frac{\partial X_i^M}{\partial M} = \frac{\partial X_j^M}{\partial P_i} + X_i \frac{\partial X_j^M}{\partial M} = \frac{\partial X_j^H}{\partial P_i}.$$

Hence, symmetry holds for the Marshallian demand if

$$\begin{aligned}X_j \frac{\partial X_i^M}{\partial M} &= X_i \frac{\partial X_j^M}{\partial M} \\ \Leftrightarrow \frac{\partial X_i^M}{\partial M}\frac{M}{X_i} &= \frac{\partial X_j^M}{\partial M}\frac{M}{X_j} \\ \Leftrightarrow \eta_i &= \eta_j.\end{aligned}$$

That is, symmetry holds for the Marshallian demand if the two goods have equal income elasticities. That is,

$$\eta_i = \eta_j \Rightarrow \frac{\partial X_i^M}{\partial P_j} = \frac{\partial X_j^M}{\partial P_i}.$$

Again, this does not cross-price elasticities are symmetric when $\eta_i = \eta_j$. By analogous argument to that for the Hicksian demand, we have that

$$\epsilon_{ij}^M = \epsilon_{ji}^M \frac{s_j}{s_i}.$$

That is, price change in the good with a higher share is more important.

1.5.4 Relationship between homogeneity, adding up and symmetry

Consider the two good Marshallian case. The demand system is parameterised by:

- ▷ shares of expenditures: s_1, s_2;
- ▷ income elasticities: η_1, η_2;
- ▷ price elasticities: $\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{21}$.

If we know s_1, then we can calculate $s_2 = 1 - s_1$. Similarly, if we know η_1, then we can calculate η_2 using the fact that $\sum_{i=1}^N s_i \eta_i = 1$. In addition, if we know ϵ_{11}, by homogeneity ($\sum_{i=1}^N \epsilon_{ij}^M + \eta_i = 0$), we can calculate ϵ_{12}. Then, by adding up ($\sum_{i=1}^N \epsilon_{ij} s_i = -s_j$), we can calculate ϵ_{21} and, in turn, by homogeneity, calculate ϵ_{22}. Therefore, we would need 3 parameters (e.g. s_1,η_1 and ϵ_{11}) to estimate the Marshallian demand system. This is also true for estimating the Hicksian demand system. Notice that, in the two good case, we did not need to rely on symmetry.

Now consider the three good Marshallian case. The demand system is parametrised by:

- ▷ shares of expenditures: s_1, s_2, s_3;
- ▷ income elasticities: η_1, η_2, η_3;
- ▷ price elasticities:

$$\boldsymbol{\epsilon} := \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{23} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}.$$

If we know s_1 and s_2, then we can calculate s_3. Similarly, if we know η_1 and η_2, then we can calculate η_3. What about ϵ? If we know two of the items in a row, then we can obtain the third by homogeneity. Similarly, if we know two of the entries in a column, then we cna calculate the thid by adding up. Thus, with homogeneity and adding up, we would need at least four elasticities (e.g. if we are given the entries in bold)

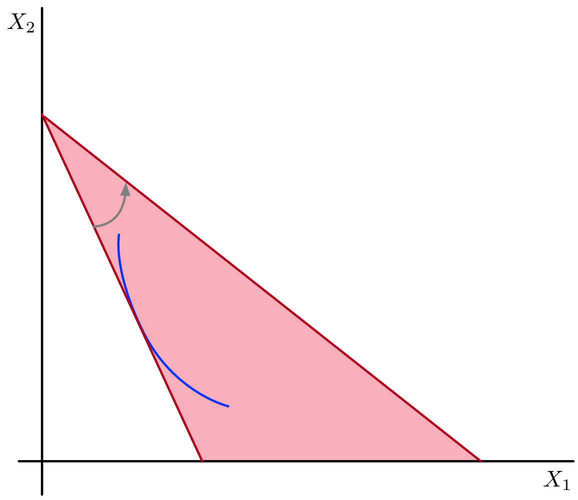
$$\boldsymbol{\epsilon} :=_{\text{adding up}} \Bigg\downarrow \overset{\xrightarrow{\text{homogeneity}}}{\begin{pmatrix} \boldsymbol{\epsilon_{11}} & \boldsymbol{\epsilon_{12}} & \epsilon_{23} \\ \boldsymbol{\epsilon_{21}} & \boldsymbol{\epsilon_{22}} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}}.$$

However, notice that we can use symmetry to reduce the number of parameters required. Symmetry means that the matrix $\boldsymbol{\epsilon}$ is symmetric. Hence, we only need the upper or lower diagonals. Then, by homogeneity and/or adding up, we can calculate the elements in the diagonal if we were given one of the diagonals. Hence, the number of parameters required is reduced to the number of elements in one diagonal; i.e. in this case 3 (e.g. $\epsilon_{21}, \epsilon_{31}, \epsilon_{32}$).

Empirically, the income elasticities are relatively easy to estimate. Since, in a market, individuals with differing income levels face the same price. Shares of income are also relatively easy to estimate. However, price elasticities ϵ_{ij} are difficult to estimate. In order to estimate ϵ_{ij}, one requires changes in a single price holding income or utility constant, which is unusual to observe in reality.

1.5.5 Opportunities when price change

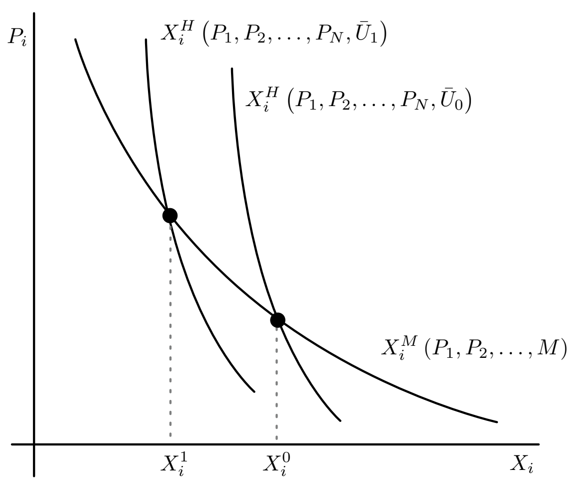
In a two good case, suppose that price of good 1 falls. Then, the new opportunities will mostly arise for bundles where a lot of the cheaper good, i.e. good 1, is consumed. That is, after P_1 falls, the "centre of gravity" moves towards good 1—if we pick a bundle within the budget set at random, it will contain more of good 1 on average.



1.6 Demand curves

Recall that demand curves shows how quantities react to prices—it is a curve drawn in (x, p) space. First, consider a Marshallian demand curve and take a point X_i^0. We know that, if X_i is a normal good, then the Hicksian demand curve is steeper because the income effect will reinforce the substitution effect.

Now, consider the point $X_i^1 < X_i^0$ on the Marshallian demand curve. Will the consumer be better or worse off relative to X_i^0. Worse off, unambiguously, as there will be a second Hicksian demand curve corresponding to a lower level of utility.



Although the figure shows the Hicksian demand curve to be much steeper than the Marshallian demand curve, in reality, the difference in slope is likely to be small. Recall the Slutsky's equation, which says that, if the income elasticity is 1 and the share of the good is 1%, then Marshallian and Hicksian elasticities will differ only by 0.01.

1.7 Aggregating demand

We have so far analysed demand at the individual level. We can of course use it to think about aggregate demand.

Consider a situation in which different make make different choices along a given budget constraint. Then we can think of a the average choice. Now, suppose that prices change in such a way that we pivot around the average point. In the case of the figure, P_1 falls, while P_2 increases. The consumers who were consuming more of good 2 may not like the new prices. More generally, even though incomes are constant on average, they are being reattributed from the group consuming more of the good whose price increased to the other; i.e. there is a redistribution of income from heavy X_2 consumers to the heavy X_1 consumers. Suppose further that indifference curves for individuals are perfect complements so that individuals do not substitute. In aggregate, we will nevertheless get substitution because, effectively, we make the people who like X_1 richer, and the people who like X_2 poorer.

We can also construct a story where people have enough X_1 in the first world, so that when it becomes cheaper, they spend all their additional income on X_2.

In general, there is a theorem that says that, because of these types of income effects, one can get just about any kind of aggregate demand function.

This means that stricter restrictions are required in order to ensure that law of demand holds in aggregate. To see what this is the case, suppose that the set N denotes individuals in an economy, denote $X_{n,i}^M$ as individual $n \in N$'s Marshallian demand for good i. Aggregating the Slutsky equation across the individual gives

$$\sum_{n\in N} \frac{\partial X_{n,i}^M}{\partial P_j} = \sum_{n\in N} \frac{\partial X_{n,i}^H}{\partial P_j} - \sum_{n\in N} X_{n,j} \frac{\partial X_{n,i}^M}{\partial m_n}.$$

Multiplying the income effect by $\sum_{n\in N} X_{n,j} / \sum_{n\in N} X_{n,j}$ yields

$$\sum_{n\in N} \frac{\partial X_{n,i}^M}{\partial P_j} = \sum_{n\in N} \frac{\partial X_{n,i}^H}{\partial P_j} - \sum_{n\in N} \left(X_{n,j} \frac{\sum_{n\in N} X_{n,j} \frac{\partial X_{n,i}^M}{\partial m_n}}{\sum_{n\in N} X_{n,j}} \right).$$

Notice that $\sum_{n\in N} X_{n,j}$ is simply the total consumption of good j but

$$\frac{\sum_{n\in N} X_{n,j} \frac{\partial X_{n,i}^M}{\partial m_n}}{\sum_{n\in N} X_{n,j}}$$

is not an aggregate income effect because it depends on who receives the income. Thus, if $\partial X_{n,i}^M / \partial m_n$ differs across people, the this term will change depending on who receives the income.

1.8 Price indices

Our objective here is to derive a price index.

Totally differentiating the budget constraint yields:

$$\sum_{i=1}^{N} X_i dP_i + \sum_{i=1}^{N} P_i dX_i = dM. \tag{1.8}$$

We can express this in terms of percentage changes if we divide the left hand side by $\sum_{i=1}^N P_i X_i$ and the right-hand side by M:

$$\begin{aligned} \frac{\sum_{i=1}^N X_i dP_i}{\sum_{i=1}^N P_i X_i} + \frac{\sum_{i=1}^N P_i dX_i}{\sum_{i=1}^N P_i X_i} &= \frac{dM}{M} \\ &\Rightarrow \Delta P + \Delta X = \Delta M. \end{aligned}$$

The budget constraint therefore tells us that if income increases by 10% (i.e. $\Delta M = 10\%$) and prices by 3% (i.e. $\Delta P = 3\%$), real consumption increases by $\Delta X = 7\%$.

Consider first the second term in (1.8). This can be thought of as the first-order approximation of the utility function, which measures the change in utility from changes in consumption (see (1.4)):

$$\sum_{i=1}^{N} P_i dX_i = \frac{dU}{\lambda}.$$

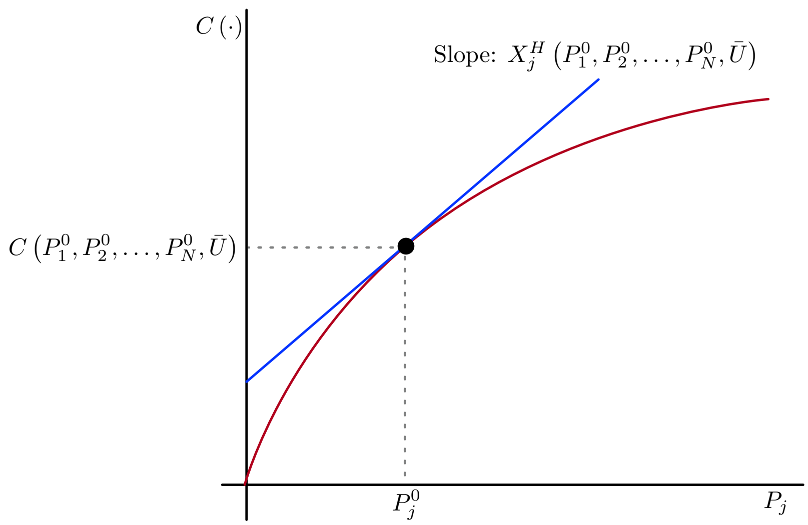
Now consider the first term in (1.8), which weighs the change in prices by the quantity consumed. Thus, it can be thought of as measuring changes in the cost of consuming the bundle that gives the same utility level:

$$\begin{aligned} dC &= dC\left(P_1, P_2, \ldots, P_N, \bar{U}\right) \\ &= \sum_{i=1}^{N} \frac{\partial C\left(P_1, P_2, \ldots, P_N, \bar{U}\right)}{\partial P_i} dP_i. \end{aligned}$$

The partial derivative, however, is the Hicksian demand (see (1.7)), so that

$$dC = \sum_{i=1}^{N} X_i dP_i,$$

This means that the first term in (1.8) can be thought of as the first-order approximation of the cost function. However, notice that, as the figure below shows, the approximation means that whenever prices increase, we will be overestimating the changes in cost and whenever there is a price decrease, we will be underestimating the changes. This occurs because when the first-order approximation ignores the consumer's ability to substitute—when price of good j increase, consumers can substitute from buying j to other goods.



Together, (1.8) can be thought of as

$$dC + \frac{dU}{\lambda} = dM \Leftrightarrow \frac{dU}{\lambda} = dM - dC,$$

where dU/λ can be thought of as the change in real income.

Income in the two periods are denoted M^1 and M^0 respectively. Since in both periods, budget constraint must be satisfied, we can write

$$\frac{M^1}{M^0} = \frac{\sum_{i=1}^{N} X_i^1 P_i^1}{\sum_{i=1}^{N} X_i^0 P_i^0}. \tag{1.9}$$

How can we measure the change in the cost of living between period 0 and 1?

1.8.1 Laspeyres index—using old bundle at new prices

Multiplying both sides of (1.9) by the cost of bundle X^0 at the new prices P^1, we can write

$$\frac{M^1}{M^0} = \frac{\sum_{i=1}^N X_i^1 P_i^1}{\sum_{i=1}^N X_i^0 P_i^0} \frac{\sum_{i=1}^N X_i^0 P_i^1}{\sum_{i=1}^N X_i^0 P_i^1} = \underbrace{\left(\frac{\sum_{i=1}^N X_i^0 P_i^1}{\sum_{i=1}^N X_i^0 P_i^0}\right)}_{\text{price index}} \underbrace{\left(\frac{\sum_{i=1}^N X_i^1 P_i^1}{\sum_{i=1}^N X_i^0 P_i^1}\right)}_{\text{quantity index}}.$$

The first term (i.e. price index) tells us how much more it would cost to purchase the period-0 bundle at period-1 prices. This is what we really mean when we say price increased since it is more expensive to buy today what we bought yesterday. Notice that we the percentage change in the price index is given by:

$$\Pi_P^L := \frac{\sum_{i=1}^N X_i^0 P_i^1}{\sum_{i=1}^N X_i^0 P_i^0} - 1 = \frac{\sum_{i=1}^N X_i^0 \left(P_i^1 - P_i^0\right)}{\sum_{i=1}^N X_i^0 P_i^0} = \frac{\sum_{i=1}^N X_i^0 dP_i}{\sum_{i=1}^N X_i^0 P_i^0}.$$

so that the term can be thought of as a first-order approximation to the cost function.

Now, consider the quantity-index term. This tells us how much more it would cost to buy the new bundle compared to the old at period-1 prices. The percentage change in this index is

$$\Pi_X^L := \frac{\sum_{i=1}^N X_i^1 P_i^1}{\sum_{i=1}^N X_i^0 P_i^1} - 1 = \frac{\sum_{i=1}^N P_i^1 \left(X_i^1 - X_i^0\right)}{\sum_{i=1}^N X_i^0 P_i^1} = \frac{\sum_{i=1}^N P_i^1 dX_i}{\sum_{i=1}^N X_i^0 P_i^1.}$$

Hence, this term can be thought of as a first-order approximation to the utility function because we are weighting changes in consumption by prices.

We can therefore decompose the change income into two parts:

$$\frac{M^1}{M^0} = \left(1 + \Pi_P^L\right)\left(1 + \Pi_X^L\right)$$

Notice that the base period for the price index is period 1, whereas for the quantities, the base year is 1. The price index derived in this way is called the *Laspeyres index*.

1.8.2 Paasche index—using new bundle at old prices

Multiplying both sides of (1.9) by the cost of bundle X^1 at the old prices P^0, we can write

$$\frac{M^1}{M^0} = \frac{\sum_{i=1}^N X_i^1 P_i^1}{\sum_{i=1}^N X_i^0 P_i^0} \frac{\sum_{i=1}^N X_i^1 P_i^0}{\sum_{i=1}^N X_i^1 P_i^0} = \underbrace{\left(\frac{\sum_{i=1}^N X_i^1 P_i^1}{\sum_{i=1}^N X_i^1 P_i^0}\right)}_{\text{price index}} \underbrace{\left(\frac{\sum_{i=1}^N X_i^1 P_i^0}{\sum_{i=1}^N X_i^0 P_i^0}\right)}_{\text{quantity index}}.$$

The first term, which is the price index, now measures the price change as by comparing the cost of period-1 bundles at period-0 and period-1 prices. The quantity index, on the other hand, involves comparing the difference in the cost of the period-0 and period-1 bundles at period-0 prices. Denoting the percentage change in each index as Π_P^P and Π_X^p respectively, we can write

$$\frac{M^1}{M^0} = \left(1 + \Pi_P^P\right)\left(1 + \Pi_X^P\right).$$

Price index formed in this way is called the Passche index.

1.8.3 What are we measuring?

The figure below shows a Marshallian demand curve for period 0. We want to think about what each index say about how real income changes when P_1 falls.

The Laspeyres index asks how much more does the old bundle costs at the new prices. This is shown in the figure on the left. However, this is an underestimate of the gain from the fall in price because it misses out substitution

1.8.4 Measuring the change in real income

1.8.5 Objective

Now suppose there are two periods: period 0 is characterised by $X^0 = \left(X_1^0, X_2^0, \ldots, X_N^0\right)$ and $P^0 = \left(P_1^0, P_2^0, \ldots, P_N^0\right)$ while period 1 is characterised by X^1 and P^1 defined in a similar way.

The change in price is given by the change in the costs of bundle that leaves the consumer indifferent; i.e.

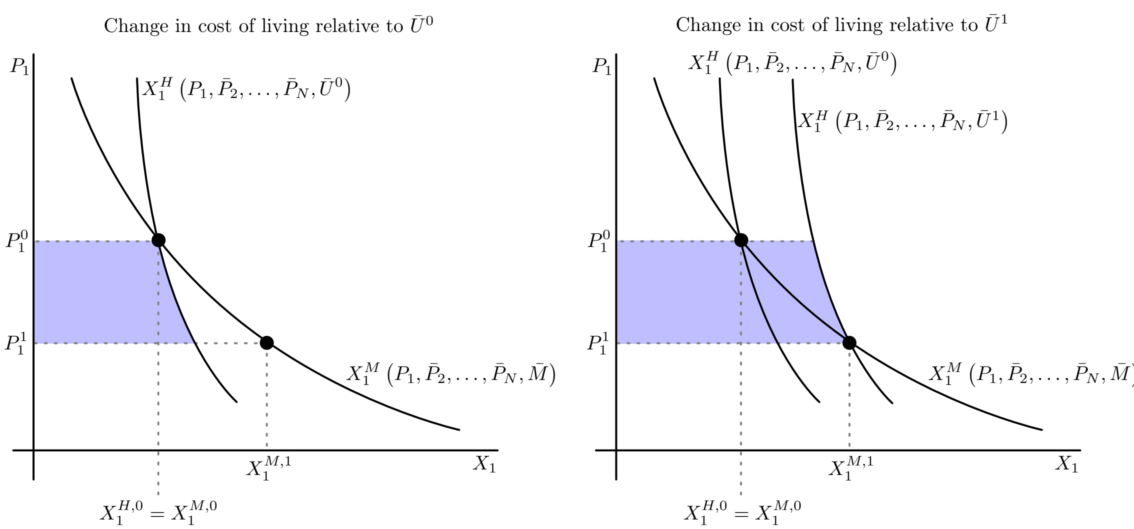
$$C\left(P_1^1, P_2^1, \ldots, P_N^1, \bar{U}\right) - C\left(P_1^0, P_2^0, \ldots, P_N^0, \bar{U}\right).$$

However, notice that we would have two Hicksian demand curves: one through the initial bundle (giving utility of $\bar{U}^0$) and the other going through the period-1 bundle (giving utility $\bar{U}^1$). Which one should we use in calculating the change in living costs? There is no right answer here. Whether we use period-0 utility or period-1 utility depends on the question we wish to answer—it depends on whether we want to know how much it costs to get period-0 utility in period 1, or how much it costs to get period-1 utility in period 0.

For simplicity, let us assume that only the price of good 1 changed between the periods and that the good 1 is normal (the latter assumption mean that the Hicksian demand curve is steeper than the Marshallian demand curve). Then, the change in the cost of the bundle is given by integrating the Hicksian demand curve along the y-axis between P_1^0 and P_1^1; i.e.

$$\begin{aligned} C\left(P_1^1, \bar{P}_2, \ldots, \bar{P}_N, \bar{U}^0\right) - C\left(P_1^0, \bar{P}_2, \ldots, \bar{P}_N, \bar{U}^0\right) &= \int_{P_1^0}^{P_1^0} \frac{\partial C}{\partial P_1} dP_1 \\ &= \int_{P_1^0}^{P_1^0} X_1^H\left(P_1, \bar{P}_2, \ldots, \bar{P}_N, \bar{U}^0\right) dP_1. \end{aligned}$$

Figure below shows the change in the cost of living depending on whether we use period-0 utility ($\bar{U}^0$) or period-1 utility ($\bar{U}^1$). Notice that the figure assumes that income did not change between the periods (hence the Marshallian demand curve does not shift). The figure shows that if were to use the Marshallian demand curve to measure the cost of living, then we would get an answer somewhere between the area under the two Hicksian curves.



2 Discrete Choice

3 A model of rental price

Suppose that people have to commute to work in the city, and people would prefer shorter commutes.

Let t be the travel time for living at distance t hours away from the city, and $R(t)$ be the rent rent for living at t. We assume that it's cheaper to live away from the city since everyone would prefer to have shorter commutes. This means that $R'(t) < 0$. In the utility function, we will not put travel time since people do not directly care about how far they have to travel—they care about how much travel will detract from leisure time, L. This is a simplifying assumption as it means that we do not have to think about whether commute time is a substitute or a complement to C or L. We therefore consider a utility function $U(C, L)$, where C denotes some aggregated consumption good. We have two constraints here:

$$\begin{aligned} L &= 24 - H - t, \\ C &= wH + A - R(t). \end{aligned}$$

The first constraint says that each agent can split 24 hours of the day into time for leisure, L, time to work, H, and time to commute t. The second says that agent's expenditure on consumption equals the wage he receives from working H hours, wH, his non-labour income, A, less the rent he pays, $R(t)$. Combining the two budget constraints, we have

$$C + R(t) = w(24 - L - t) + A.$$

The agent's maximisation problem is

$$\begin{aligned} \max_{L,C,t} \quad & U(C, L) \\ s.t. \quad & C + R(t) = w(24 - L - t) + A. \end{aligned}$$

The Lagrangian is given by

$$\mathcal{L} = U(C, L) + \lambda(w(24 - L - t) + A - C - R(t)).$$

The first-order conditions are

$$\begin{aligned} \frac{\partial U}{\partial C} &= \lambda, \\ \frac{\partial U}{\partial L} &= \lambda w, \\ -R'(t) &= w. \end{aligned}$$

The last condition says that the agent would choose to live where the savings in rent from another hour further away is equal to the wage rate. Thus, we see that we've made the restriction that travel is really the same thing as work—should you work by driving your car or should you work by working at work? ???