Price Theory, III
Economics 33003

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Review Problems for pre-course week
This is NOT an assignment

Problem 1 Part 1: Consider a Cournot duopoly with linear demand, $p = a - bQ$, and constant unit costs, $C_i(q) = c_i q$. Both firms initially have unit costs equal to c, but with probability $\phi \in (0, 1)$, firm 1 obtains a cost reduction equal to δ before outputs are chosen. Thus, firm 1's unit cost is $c_{1h} = c$ with probability $1 - \phi$) and $c_{1l} = c - \delta$ with probability ϕ; firm two's unit cost is $c_2 = c$.

(a). Complete information: Assume that any cost reduction to firm 1 is publicly observable to both firms before they choose their outputs. Solve the Nash equilibrium outputs and firm 1's equilibrium profit following the two possible cost realizations. (Assume throughout that a is sufficiently high that each firm produces positive output in equilibrium.)

(b). Incomplete information: Suppose instead that any cost reduction is private information to firm 1. Solve for the Bayes-Nash equilibrium outputs, $(q^*_{1h}, q^*_{1l}, q^*_2)$, and compute firm one's profit following for each possible cost.

(c). Incomplete information with the possibility of evidence. Sticking with the private information assumption in (b), assume that if a cost reduction is achieved, firm 1 can choose to reveal evidence proving the reduction to firm 2 before outputs are chosen. To be precise, at date 0 nature chooses firm 1's cost type; at date 1, if firm 1's cost is low, it can reveal evidence proving this to firm 2; at date 2 outputs are simultaneously chosen by the firms. What is the equilibrium to this disclosure-output game (i.e., with what probability does the low-cost firm reveal evidence at stage 1; what are the corresponding outputs at stage 2)? [Hint: use your profit computations from (a) and (b).]

Part 2: Consider a differentiated Bertrand duopoly game with linear demand,

$$q_1 = \alpha - \beta p_1 + \gamma p_2$$
$$q_2 = \alpha - \beta p_2 + \gamma p_1,$$

and constant unit costs as in Part 1: $c_{1h} = c$ (with probability $(1 - \phi)$, $c_{1l} = c - \delta$ (with probability ϕ), and $c_2 = c$.

(d). In the incomplete information game with cost-reduction evidence, will the low-cost firm reveal evidence to firm 2? Note that evidence only exists regarding cost reductions; if a cost reduction didn't happen, firm 1 cannot prove that it didn't happen. (I.e., evidence is not symmetric.) Explain. You do not need to explicitly prove your statement using profit computations as you did in Part 1, but you do need to give the intuition. [Hint: drawing a set of reaction functions can be illuminating.]

Problem 2 Consider a seller who can supply one unit of a good at cost $c \in [0, 1]$ and a buyer who values the unit at $v \in [0, 1]$. The buyer does not know the seller's cost and the seller does not know the buyer's valuation. It is common knowledge that the seller's cost and the buyer's value are each drawn from a uniform distribution on $[0, 1]$.

The buyer and seller play the following game. Simultaneously, the buyer bids a price p_b and the seller bids a price p_s. If $p_b \geq p_s$, trade takes place at $p = \frac{p_b + p_s}{2}$; if $p_b < p_s$, trade does not take place and the game ends.

(a). Choose an arbitrary price $p^* \in (0,1)$ and show that the following is an equilibrium to this trading game: the buyer bids p^* if $v \geq p^*$ and 0 otherwise; the seller bids p^* if $p^* \geq c$ and 1 otherwise. Show that equilibrium trade is inefficient.

(b). Show that there exists a unique linear equilibrium in which the buyer bids $p_b(v) = \alpha + \beta v$ and the seller bids $p_s(c) = \delta + \gamma c$. Compute the equilibrium values of the α, β, δ and γ. Show that equilibrium trade is inefficient.

Problem 3 Consider a first-price auction (without a reserve price) between two buyers. Valuations are $\theta_h > \theta_l > 0$ and the probability that $\theta = \theta_h$ is $\phi \in (0,1)$.

There is no pure-strategy equilibrium to the first-price auction with discrete types. Find the symmetric Bayesian-Nash equilibrium in which type θ_l bids $b_l = \theta_l$, but type θ_h randomizes her bid according to some continuous equilibrium distribution $G(b_h)$ on $[\underline{b}, \overline{b}]$.

[Hint: you *may* need to solve a differential equation. If you are unfamiliar with solving *ODE*'s, try using software like *Mathematica*. Alternatively, here is an ODE formula that should work for you:

$$\alpha + f(x) = (\beta - x)f'(x) \implies f(x) = \frac{\alpha x + C}{\beta - x},$$

where C is an arbitrary constant. You can also solve this problem without needing to solve ODE.]

Problem 4 Consider a public goods game with two players, $i = 1, 2$, and private information. Each player can choose to either contribute to the public good ($s_i = 1$) or not, ($s_i = 0$), $S_i = \{0, 1\}$. If one or both players contribute, then the public good is produced and each player receives a benefit of 1. Player i's cost of contributing is $c_i \in [\underline{c}, \overline{c}]$, where $\underline{c} < 1 < \overline{c}$; c_i is independently distributed for both players according to the continuous distribution function $F(c_i)$. If player i contributes, her payoff is $1 - c_i$; if player i does not contribute, her payoff is 1 if player $j \neq i$ contributed, and 0 otherwise. Formally,

$$u_i(s_i, s_j, c_i) = \max\{s_1, s_2\} - c_i s_i.$$

A pure-strategy equilibrium in this game is a profile of contribution functions, $\{s_1^*(\cdot), s_2^*(\cdot)\}$ where $s_i^* : [\underline{c}, \overline{c}] \to \{0, 1\}$.

(a). Characterize the symmetric pure-strategy Bayesian-Nash equilibrium. Prove that it is unique and always exists.

(b). Characterize a pair of asymmetric pure-strategy equilibria in which one player never contributes. Give the condition on the distribution of costs that guarantees the existence of such equilibria.

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Answers to Review Questions

1 (a). Let's solve for the complete-information Nash equilibrium using c_1 and c_2 for the costs. Firm i solves

$$\max_{q_i} \, (a - b(q_1 + q_2) - c_i)q_i,$$

which yields the FOC

$$a - bq_j - c_i = 2bq_i,$$

and therefore firm i's reaction function is

$$R(q_j, c_i) = \frac{a - c_i}{2b} - \frac{1}{2}q_j.$$

Solving for an equilibrium point where the two reaction functions cross, we obtain (after some messy algebra)

$$q_1^* = \frac{a}{3b} - \frac{2}{3b}c_1 + \frac{1}{3b}c_2,$$
$$q_2^* = \frac{a}{3b} - \frac{2}{3b}c_2 + \frac{1}{3b}c_1.$$

For the no-cost-reduction case, $c_1 = c_2 = c$, this simplifies to

$$q_1^* = q_2^* = \frac{a - c}{3b}$$

and firm 1's profit is

$$\pi_1(q_1^*, q_2^*) = \frac{(a - c)^2}{9b}.$$

For the cost-reduction case, $c_1 = c - \delta$ and $c_2 = c$, we obtain

$$q_1^* = \frac{a - c}{3b} + \frac{2}{3b}\delta \;\text{ and }\; q_2^* = \frac{a - c}{3b} - \frac{1}{3b}\delta,$$

and firm 1's profit is

$$\pi_1 = \frac{(a - c + 2\delta)^2}{9b}.$$

(b). We now solve for the Bayesian-Nash equilibrium in which firm 2 is uncertain about firm 1's costs. Formally, we are looking for $\{q_1^*(\cdot), q_2^*\}$ where firm 1's strategy takes on two possible values as a function of costs.

We can quickly find the equilibrium by using the reaction functions in part (a). Firm 1's optimal responses, given firm 2's choice of q_2^* are

$$q_{1h}^* = R(q_2^*, c) = \frac{a - c}{2b} - \frac{1}{2}q_2^* \quad \text{and} \quad q_{1l}^* = R(q_2^*, c - \delta) = \frac{a - c + \delta}{2b} - \frac{1}{2}q_2^*.$$

Firm 2 is unsure of firm 1's output. But note that firm 2's profit function is linear in q_1, so firm 2's best response is given by the optimal reaction to $E[q_1] = \phi q_{1l}^* + (1 - \phi)q_{1h}^*$:

$$q_2^* = R_2(E[q_1]) = \frac{a - c}{2b} - \frac{1}{2}(\phi q_{1l}^* + (1 - \phi)q_{1h}^*).$$

We have three equations and three unknowns. Solving these we obtain

$$q_{1l}^* = \frac{a-c}{3b} + \frac{\delta(3+\phi)}{6b},$$

$$q_{1h}^* = \frac{a-c}{3b} + \frac{\delta\phi}{6b},$$

$$q_2^* = \frac{a-c}{3b} - \frac{2\delta\phi}{6b}.$$

The corresponding profits of firm 1, conditional on cost realization, are

$$\pi_{1l} = \frac{\left(a-c+\frac{1}{2}\delta(3+\phi)\right)^2}{9b},$$

$$\pi_{1h} = \frac{\left(a-c+\frac{1}{2}\delta\phi\right)^2}{9b}.$$

(c). We need to be careful here and think about the equilibrium in the disclosure game. Suppose that the equilibrium is that the low-cost firm does not reveal the cost reduction. Using our computations above, firm 1 would then want to deviate and reveal the cost reduction if

$$\frac{(a-c+2\delta)^2}{9b} > \frac{\left(a-c+\frac{1}{2}\delta(3+\mu)\right)^2}{9b},$$

where μ is firm 2's belief that firm 1 is the low cost firm when the cost is not revealed. It follows that the low-cost firm would wish to reveal if $2\delta > \frac{1}{2}\delta(3+\mu)$, which is always the case if $\mu < 1$. If $\mu = 1$ and firm 2 thinks only the low-cost firm chooses not to reveal cost, then the high-cost firm would prefer to also not reveal cost; hence $\mu = 1$ is not consistent with equilibrium. We conclude that there cannot be a pure-strategy equilibrium in which the low-cost firm does not always reveal. Finally, if in equilibrium the low-cost firm always reveals evidence of low cost, then the high-cost firm will be discovered by either revealing cost or by not revealing and the deduction that a non-revealing firm must be high cost. In this separating equilibrium, the low-cost firm obtains

$$\frac{(a-c+2\delta)^2}{9b}.$$

Intuitively, the full-information equilibrium outcome emerges because Cournot reaction functions are downward sloping and a firm benefits from being thought to be low cost.

(d). The key insight here is to note that the reason the low-cost firm chose to produce cost-reduction evidence in the Cournot output game is that proving one is low cost, indicates that the firm will produce more output on average, which will lead to a reaction by the other firm to produce less output. This is because Cournot reaction functions are downward sloping. But in the differentiated Bertrand game, price-reaction functions are upward sloping. Thus, revealing that one is low cost indicates that the firm will price lower than on average, which will cause firm 2 to price lower in response. Hence, revealing low-cost evidence leads to lower profits. For this reason, if the high-cost firm could prove that she was high cost, that firm would reveal its type. And the unraveling would separate the low-cost firm too. But we have assumed in our game that evidence can only be produced showing a cost reduction, not the absence of the cost reduction. Hence, the high-cost firm 1 cannot produce evidence, and the low-cost firm 1 does not want to. The equilibrium is for no evidence to be produced and the incomplete information game analogous to (b) is played with the choice of prices.

2 (a). For concreteness, let's prove this for $p^* = \frac{3}{4}$, but note that argument will apply to any $p^* \in (0,1)$. Given $p^* = \frac{3}{4}$, the proposed equilibrium bidding functions are

$$p_b^*(v) = \begin{cases} \frac{3}{4} & \text{if } v \geq \frac{3}{4} \\ 0 & \text{if } v < \frac{3}{4}, \end{cases}$$

$$p_s^*(c) = \begin{cases} \frac{3}{4} & \text{if } c \leq \frac{3}{4} \\ 1 & \text{if } c > \frac{3}{4}. \end{cases}$$

We first how that the buyer's bidding function is an optimal response to the seller's bidding function. There are 2 cases to consider:

Case 1: $v \geq \frac{3}{4}$. By bidding according to the equilibrium strategy $\frac{3}{4}$, the buyer obtains $v - \frac{3}{4} \geq 0$. If the buyer bid $p_b > \frac{3}{4}$, then either the seller bid $p_s = 1$ and it does not matter, or the seller bid $p_s = \frac{3}{4}$ and the buyer ends up paying for than $\frac{3}{4}$. Similarly, if $p_b < \frac{3}{4}$, either the seller bid $p_s = 1$ in which case it does not matter, or the seller bid $p_s = \frac{3}{4}$, in which case no trade takes place and the buyer would have done weakly better bidding $\frac{3}{4}$.

Case 2: $v < \frac{3}{4}$. By bidding according to the equilibrium strategy $p_b = 0$, the buyer obtains 0. If the buyer bid $p_b \in (0, \frac{3}{4})$, then no trade takes place and the buyer still earns 0. If the buyer bid $p_b \geq \frac{3}{4}$, then trade takes place if $p_s = \frac{3}{4}$, but in this case the buyer has a negative payoff. Hence, the buyer does weakly better by following the equilibrium strategy.

The argument for the seller's bidding function being an optimal response to the buyer's strategy is similar and left to the reader.

Efficient trade requires that trade takes place if $v \geq c$. The p^*-equilibrium generates trade only $v \geq p^* \geq c$, which is a strict subset of efficient trades. Specifically, when $v > c \geq p^*$ or $p^* > v > c$, trade is efficient but does not arise.

(b). We look for linear equilibria of the form $p_b^*(v) = \alpha + \beta v$ and $p_s^*(c) = \delta + \gamma c$. Consider first the buyer. The buyer chooses p_b to solve

$$\max_{p_b \geq 0} \left(v - \frac{p_b + E[p_s^*(c) | p_b \geq p_s^*(c)]}{2} \right) \text{Prob}\left[p_b \geq p_s^*(c)\right].$$

If the seller follows the above strategy, the seller's bid will be uniformly distributed on $[\delta, \delta + \gamma]$. Thus, this reduces to

$$\max_{p_b \geq 0} \left(v - \frac{1}{2} \left(p_b + \frac{p_b + \delta}{2} \right) \right) \left(\frac{p_b - \delta}{\gamma} \right).$$

The first-order condition gives the buyer's optimal bid as a linear function of v, as required:

$$p_b = \frac{\delta}{3} + \frac{2}{3} v.$$

Now consider the seller's program of picking p_s, and use the fact that the buyer's bid will be uniformly distributed on $[\alpha, \alpha + \beta]$:

$$\max_{p_s} \left(\frac{1}{2} \left(p_s + \left(\frac{p_s + \alpha + \beta}{2} \right) - c \right) \left(\frac{\alpha + \beta - p_s}{\beta} \right) \right).$$

The first-order condition gives the seller's optimal bid as a linear function of c,

$$p_s = \frac{1}{3}(\alpha+\beta) + \frac{2}{3}c.$$

Recall we conjectured the equilibrium functions

$$p_b^*(v) = \alpha + \beta v,$$

$$p_s^*(c) = \delta + \gamma c.$$

Using our result that $p_b = \frac{\delta}{3} + \frac{2}{3}v$ and $p_s = \frac{1}{3}(\alpha+\beta) + \frac{2}{3}c$, we conclude

$$\alpha = \frac{\delta}{3},$$

$$\beta = \frac{2}{3},$$

$$\delta = \frac{1}{3}(\alpha+\beta),$$

$$\gamma = \frac{2}{3}.$$

After some algebra, we have the solution

$$p_b^*(v) = \frac{1}{12} + \frac{2}{3}v,$$

$$p_s^*(c) = \frac{1}{4} + \frac{2}{3}c.$$

Notice that in this linear equilibrium, trade takes place only if

$$\frac{1}{12} + \frac{2}{3}v \geq \frac{1}{4} + \frac{2}{3}c,$$

or equivalently

$$v - c \geq \frac{1}{4}.$$

Thus again we have too little trade. Specifically, if $\frac{1}{4} > v - c > 0$, it is efficient to trade but trade does not occur.

3 First, if θ_l-types bid θ_l, then it must be $\underline{b} \geq \theta_l$. Otherwise, if $\underline{b} < \theta_l$, a low type would benefit by bidding $\theta_l - \varepsilon$ which would generate a positive expected payoff which is preferred to the zero payoff of bidding θ_l. Second, it cannot be that $\underline{b} > \theta_l$ because if the high-type bidder chooses $\underline{b}$, she wins if and only if she faces a low-type bidder. (Note that a tie with another high-type bidder at $\underline{b}$ happens with probability zero because G is a continuous distribution.) The high-type bidder can do better than bidding $\underline{b}$ by reducing her bid to $\theta_l + \varepsilon$. At this bid, she will still win iff facing a low-type, but in such a case she will pay less for the good. We conclude that $\underline{b} = \theta_l$.

Third, we compute the distribution of bids by the high type. The high type to be willing to randomly bid on $[\theta_l, \bar{b}]$, it must be the case that

$$(\theta_h - b)((1-\phi) + \phi G(b))$$

is constant for all $b \in [\theta_l, \bar{b}]$. One way to proceed is to differentiate the expression and set it to zero, producing a differential equation:

$$-(1 - \phi + \phi G(b)) + (\theta_h - b)\phi g(b) = 0,$$

with our initial condition $G(\theta_l) = 0$. Together, these imply

$$G(b) = \frac{1-\phi}{\phi}\frac{b-\theta_l}{\theta_h - b}.$$

An alternative trick to avoid the ODE is to note that if

$$(\theta_h - b)((1-\phi) + \phi G(b))$$

is constant for all b, we can determine the constant by inserting $b = \theta_l$ and noting $G(\theta_l) = 0$. This yields a constant equal to $(\theta_h - \theta_l)(1-\phi)$. Hence, we have

$$(\theta_h - b)((1-\phi) + \phi G(b)) = (\theta_h - \theta_l)(1-\phi).$$

Now we can simply solve for G and get the same expression.

To determine $\bar{b}$, we solve $G(\bar{b}) = 1$:

$$1 = \frac{1-\phi}{\phi}\frac{\bar{b}-\theta_l}{\theta_h - \bar{b}},$$

which yields

$$\bar{b} = \phi\theta_h + (1-\phi)\theta_l.$$

4 Let α_j be player j's equilibrium probability of contributing: $\alpha_j = \text{Prob}[s_j^*(c_j) = 1]$. Player i will therefore choose

$$s_i(c_i) = \begin{cases} 1 & \text{if } 1 - c_i > \alpha_j \\ \in [0,1] & \text{if } 1 - c_i = \alpha_j \\ 0 & \text{if } 1 - c_i < \alpha_j. \end{cases}$$

Define $c_i^* = 1 - \alpha_j$ as the threshold cost for which player i is indifferent to contributing. Because F has no atoms, with probability 1 player i chooses a pure strategy. Notice also that there is a one-to-one mapping between c_i^* and α_i:

$$\alpha_i = F(c_i^*).$$

Using the relationship $\alpha_i = 1 - c_j^*$ fpr $i \neq j$, we thus generate the following equilibrium conditions:

$$c_j^* = 1 - F(c_i^*),$$

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(a). In a symmetric equilibrium, $c_i^* = c_j^* = c^*$. If one exists, it must satisfy

$$c^* = 1 - F(c^*).$$

Define the continuous, increasing function $\Phi(c) = c - (1 - F(c))$. Because $\Phi(\underline{c}) = \underline{c} - 1 < 0$ and $\Phi(\bar{c}) = \bar{c} > 0$, there exists a fixed point, c^*, such that $\Phi(c^*) = 0$ and hence $c^* = 1 - F(c^*)$. Because Φ is strictly increasing, there can be only one fixed point.

(b). Now consider the possibility of asymmetric equilibria in which one party does not contribute. Let's suppose that player 1 is the slacker. Thus, $\alpha_1 = 0$ (or alternatively, $c_1^* = \underline{c}$. Given this, player 2 contributes if $c_2 < 1$. Hence, $c_2^* = 1$. We now need to check that player 1 finds it optimal to never contribute. This will be true if

$$1 - c_1 \leq F(1) \text{ for all } c_1.$$

Thus, if $\underline{c} \geq 1 - F(1)$, two asymmetric equilibria exist.