Price Theory II: Problem Set 4
Solutions

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5.35

(a)

Let x^i – consumption bundle of individual i, a_s^i — purchases of arrow security for state s by i, α^i — purchases of α asset by i.

In this question (and b2.2) agent's budget constraint looks like

$$\hat{p} \cdot a^i + \alpha^i p_\alpha = \hat{p} \cdot e^i$$

where p_α is the price, which we try to determine.

First, note that the asset can be replicated by purchasing α_s units of Arrow security a_s for each s, because the agents in the economy only values consumption in each state x_s.

Suppose that p_α is $\hat{p} \cdot \alpha$. We want to show that consumers are indifferent to trading in this asset. Consider any bundle $\tilde{x}^i$ that is feasible to consumer i. Then, by BC $\exists \tilde{\alpha}^i, \tilde{a}_s^i$, such that $\tilde{x}_s^i = e_s^i + \tilde{\alpha}^i \alpha_s + \tilde{a}_s^i$ and $p_\alpha \tilde{\alpha}^i + \hat{p} \cdot \tilde{a}^i = \hat{p} \cdot e^i$.

Then, we may consider another portfolio of only Arrow security $\tilde{\tilde{a}}_s^i = \tilde{\alpha}^i \alpha_s + \tilde{a}_s^i$. Notice that, clearly, $\tilde{x}_s^i = \tilde{\tilde{a}}_s^i$. Moreover, $\hat{p} \cdot \tilde{\tilde{a}}_s^i = \hat{p} \cdot \alpha \tilde{\alpha}^i + \hat{p} \cdot \tilde{a}^i = p_\alpha \tilde{\alpha}^i + \hat{p} \cdot \tilde{a}^i = 0$ whenever $p_\alpha = \hat{p} \cdot \alpha$. Hence, whenever a bundle is affordable with α present, it is also affordable in absence of it. Clearly, then, consumers are indifferent to trade α at this price.

For uniqueness, denote τ to be such that $p_\alpha = \tau \hat{p} \cdot \alpha$.

Case 1: $\tau > 1$. Suppose that $\hat{x}$ is an equilibrium allocation of goods, and $\hat{\alpha}$ — equilibrium purchases of asset α, $\hat{a}$ — equilibrium purchases of Arrow securities, associated with $\hat{p}, p_\alpha$

Consider for consumer i: $\tilde{x}^i = \hat{x}^i + (\tau - 1)\varepsilon\alpha$, for some $\varepsilon > 0$. If $\tau > 1$, then, given u^i is increasing, $\tilde{x}^i \succ_i \hat{x}^i$. We will show now that $\tilde{x}^i$ is affordable by i. Since $\hat{x}^i$ is affordable, then $\exists \hat{a}^i$, $\hat{\alpha}^i$, such that

*We thank previous TAs for this class for the solution.

$\hat{x}^i = e^i_s + \hat{a}^i_s + \hat{\alpha}^i \alpha_s$. Then, $\tilde{x}^i = e^i_s + \tilde{a}^i_s + \tilde{\alpha}^i \alpha_s$ with $\tilde{a}^i_s = \hat{\alpha}^i_s + \tau\varepsilon\alpha_s$ and $\tilde{\alpha}^i = \hat{\alpha}^i - \varepsilon$. We only have to check that BC in spot market holds for i at tilded assets purchases:

$$\hat{p} \cdot \tilde{a}^i + p_a \cdot \tilde{\alpha}^i = \varepsilon\tau\hat{p} \cdot \alpha - \varepsilon\tau\hat{p} \cdot \alpha = 0$$

But then $\hat{x}$ cannot be an equilibrium allocation. Contradiction.

Case 2: $\tau < 1$. Similarly, consider $\tilde{x}^i = \hat{x}^i + (1-\tau)\varepsilon\alpha$; $\tilde{a}^i_s = \hat{a}^i_s - \varepsilon\tau\alpha_s$, $\tilde{\alpha}^i = \hat{\alpha}^i + \varepsilon$. Same contradiction arises.

(In words, if price of $\alpha > \hat{p} \cdot \alpha$ then it's cheaper to buy Arrow securities that replicates α, so people can consume negative amount of (say, -1 unit of) the replicating portfolio and purchase positive (1 unit) amount of α, which will give them profit of $p_\alpha - \hat{p} \cdot \alpha$ without changing their consumption bundle. Same goes for the cases where price of $\alpha < \hat{p} \cdot \alpha$, in which case people short the replicating portfolio, and buy α.)

(b)

(1)

Because of the assumption of strict concavity, We have interiority of the individual consumption bundles, so it must be that for every $i \in I$

$$\frac{MU^i_j}{MU^i_k} = \frac{\pi_j v'_i(x_j)}{\pi_k v'_i(x_k)} = \frac{\pi_j}{\pi_k} = \frac{\hat{p}_j}{\hat{p}_k}$$

where the second equality forces the proposed consumption bundle $x^i_j = x^i_k$. Therefore, the proposed price induces the proposed consumption bundle. Now we need to check whether the market clears.

To this end, consider i's budget constraint

$$\hat{p} \cdot x^i = \hat{p} \cdot e^i = k_i \hat{p} \cdot \sum_j e^j$$

where $k_i > 0$, $\sum_i k_i = 1$ has interpretation of "Proportion of resources that i has" (This is without loss of generality, recall $\hat{p} \cdot e^j$ is just a scalar!). Now since we know the consumption bundle is constant across states (denote as z_i), we have

$$\hat{p} \cdot x^i = \sum_j z_i \hat{p}_j = z_i \sum_j \hat{p}_j = z_i = k_i \hat{p} \cdot \sum_j e^j$$

where the last equality utilizes $\sum_j \hat{p}_j = 1$. Summing above across i, we have for any good j

$$\sum_i x^i_j = \sum_i z_i = \sum_i k_i \hat{p} \cdot \sum_k e^k = \hat{p} \cdot \sum_k e^k = \sum_l (\hat{p}_l \sum_k e^k_l) = \sum_l (\hat{p}_l e_j) = e_j = \sum_l e^l_j,$$

where the third equality uses $\sum k_i = 1$, and the second last equality uses the assumption that $e_l = \sum_k e^k_l = \sum_k e^k_m = e_m$ for any l, m.

(2.1)

From the proof of (b) we see that forcing $x_j^i = x_k^i$ immediately results in $\hat{p}_l = \pi_l$, which makes everyone in the economy demand the same amount of good across states. Therefore the market will not clear.

(2.2)

Consider a portfolio of Arrow securities β which yields 1 regardless of state realized. Price of β is, by definition, 1.

Since people in the economy can short either α or β, in order for the market for Arrow security and α to clear, we must have

$$\frac{p_\alpha}{p_\beta} = \frac{MU_\alpha^1}{MU_\beta^1} = \cdots = \frac{MU_\alpha^I}{MU_\beta^I}.$$

Now notice that $MU_\alpha^i = \sum_j \pi_j v_i'(x_j^i)\alpha_j = E[v_i'(\tilde{x}^i)\tilde{\alpha}]$, and $MU_\beta^i = \sum_j \pi_j v_i'(x_j^i) \times 1 = E[v_i'(\tilde{x}^i)]$. Plugging those into the previous expression, we get the result.

5.36

a) i-th column of the vector is $\sum_k \alpha_i^k x_k$ which can be interpreted as amount of things the agent has in state i. So holding x is equivalent to holding one unit of Ax.

b)

$$[Ax - \mathbb{1}(q \cdot x)]_s = \sum_{k=1}^N \alpha_s^k x_k - \sum_n q_k x_k$$

The first summand is the total revenue received from portfolio x in state s, whereas the second summand is the total expenditures on the portfolio. Together, they exactly state the net gain to investor in every state.

c) If $q = 0$, then x surely can be bought infinite times. Else, $q \neq 0$. Then, there exists a vector of asset short sales y, such that $q \cdot y > 0$. Then, for every positive α, $\alpha q \cdot y > 0$. Then, the investor could always buy some share of x – say, ρ, such that $q \cdot (\rho x) \leq \alpha q \cdot y$. Then,

$$A(\rho x) - \mathbb{1} q \cdot (\rho x) \gg 0$$

. Note that given that α is arbitrary, ρ can also be set with no bounds. Provided that $Ax - \mathbb{1} q \cdot x > 0$, $\lim_{\rho \to \infty} \rho(Ax - \mathbb{1} q \cdot x) = \infty$. The agent is guaranteed from bankruptcy in every state, too, since $A(\rho x) - \mathbb{1} q \cdot (\rho x) \gg 0$ — he would always have enough money to close the short sale.

d) If such a portfolio was to exist, the consumption market would never clear, since every consumer would have unbounded income in every state.

e) Suppose q is arbitrage-free. Define:

$$C := \{y \in \mathbb{R}^N : \mathbf{y} = Ax - \mathbb{1}(q \cdot y) \text{ for some } x \in \mathbb{R}^N\}$$

(**Two sets are disjoint**). Assume that q is arbitrage-free, $Ax - \mathbb{1}q \cdot x \not> 0$. Then, $C \cap \mathbb{R}^N_{++} = \varnothing$. So C and $\mathbb{R}^N_{++}$ are disjoint.

(**Two sets are convex**). In order to use Theorem A.2.24 (Separating Hyperplane Th), we need to show that they both of them convex. $\mathbb{R}^N_{++}$: suppose $x_1 \in \mathbb{R}^N_{++}$ and $x_2 \in \mathbb{R}^N_{++}$, then $\forall \lambda \in [0,1]$: $\lambda x_1 + (1-\lambda)x_2 \geq 0 \Rightarrow \in \mathbb{R}^N_{++}$.

C: suppose $y_1 \in C$ and $y_2 \in C$. Then, by definition of C, $\exists x_1, x_2$: $y_1 = Ax_1 - \mathbb{1}q \cdot x_1$ and $y_2 = Ax_2 - \mathbb{1}q \cdot x_2$, then $\lambda(Ax_1 - \mathbb{1}q \cdot x_1) + (1-\lambda)(Ax_2 - \mathbb{1}q \cdot x_2) = A(\lambda x_1 + (1-\lambda)x_2) - \mathbb{1}q \cdot (\lambda x_1 + (1-\lambda)x_2)$, then $\lambda y_1 + (1-\lambda)y_2 \in C$ as we are able to find $x = \lambda x_1 + (1-\lambda)x_2$, such that $y = Ax - \mathbb{1}q \cdot x$.

(**Use Th.A.2.24**). Now we are ready to use Theorem A.2.24: it says that $\exists \hat{p} \neq 0$, such that $\hat{p} \cdot y \leq \hat{p} \cdot z$, for every $y \in C, z \in \mathbb{R}^N_{++}$.

($\hat{\mathbf{p}} \geq \mathbf{0}$). Notice that $0 \in C$ as $A\mathbf{0} - \mathbb{1}q \cdot \mathbf{0} = 0$. Suppose $\hat{p}_k < 0$ for some k. Then, we could take $z = (1, 1, ..., \varepsilon, 1, ..., 1) \in \mathbb{R}^N_{++}$, where $\varepsilon > 0$. For some ε large enough, $\hat{p} \cdot z < 0 = \hat{p} \cdot y$. Contradiction to $\hat{p} \cdot y \leq \hat{p} \cdot z$. Hence, $\hat{p} \geq 0$.

(**Normalization**) Finally, if $\hat{p} \cdot y \leq \hat{p} \cdot z$, for every $y \in C, z \in \mathbb{R}^N_{++}$, then $\frac{1}{\sum_k \hat{p}_k}\hat{p}y \leq \frac{1}{\sum_k \hat{p}_k}\hat{p}z$, but $\sum_k \frac{p_k}{\sum_k \hat{p}_k} = 1$. Hence, WLOG we can assume $\hat{p}_k$ sum to 1.

(f)

By construction of C, $\forall x$, $\exists y \in C$, where $y = Ax - \mathbb{1}q \cdot x$.

$$\hat{p} \cdot y = \hat{p} \cdot (Ax - \mathbb{1}q \cdot x) = (\hat{p}^T A - \hat{p} \cdot \mathbb{1}q^T)x$$

By the way we have earlier defined $\hat{p}$, $\hat{p} \cdot \mathbb{1} = \sum_k \hat{p}_k = 1$, hence:

$$(\hat{p}^T A - \hat{p} \cdot \mathbb{1}q^T)x = (\hat{p}^T A - q^T)x$$

From definition of $\hat{p}$:

$$\hat{p} \cdot y \leq \hat{p} \cdot z, \ \forall y \in C, z \in \mathbb{R}^N_{++}$$

Consider some $y \in$. Set a sequence $\{z^m\}$, $z^m \gg 0$, $z^m \to 0$, such that

$$\hat{p} \cdot y \leq \hat{p} \cdot z^m, \forall m$$

$$\hat{p}^T y \leq 0$$

The argument may be repeated for every $y \in C$, so we conclude that

$$\hat{p}^T y \leq 0, \forall y \in C$$

Then:

$$(\hat{p}^T A - q^T)x \leq 0, \forall x \in \mathbb{R}^N$$

Suppose that

$$\exists x \in \mathbb{R}^N : (\hat{p}^T A - q^T)x < 0$$

for some x, then for $-x$:

$$(\hat{p} \cdot A - q^T)(-x) = -(\hat{p} \cdot A - q^T)x > 0$$

which is a contradiction to the earlier derivations :

$$(\hat{p} \cdot A - q^T)x = 0, \forall x \in \mathbb{R}^N$$

Since this holds $\forall x$, this entails $\hat{p} \cdot A - q^T = 0$ and so $\forall k$:

$$q_k = \hat{p} \cdot \alpha_k$$

The result we have derived relied on the initial assumption that q is arbitrage free. Let us check this.

$$\forall x : Ax - \mathbb{1} q^T x = Ax - \mathbb{1} \hat{p} Ax = 0$$

since $\hat{p}_s$ sums to 1.

(g) In a previous exercise we obtained:

$$q_k = p \cdot \alpha_k$$

where p is the vector of A-D economy equilibrium prices.

Here we get

$$q_k = \hat{p} \cdot \alpha_k$$

We obtained q_k, so that q_k is arbitrage-free. So, by analogy, we can conclude that if the market is not complete, then we get still a vector of shadow prices for the A-D securities: which is $\hat{p}$, that could be derived from the system of equations

$$q_k = \hat{p} \cdot \alpha_k,$$

if we know arbitrage-free asset prices q.

Note that dimension of $\hat{p}$ is N, not S. This means that the set of price may not correspond completely to the price in 5.35. If matrix A has rank S, then those price will coincide (because some combination of assets will perfectly replicate arrow securities) — that's what we usually mean by "complete market".

6.1. Escaping Arrow's negative result: two alternatives

Claim 1. *Let the number of alternatives be restricted to two, i.e. $|X| = 2$. Then, the social welfare function induced by simple majority voting satisfies* **U**, **WP**, **IIA** *and* **D**.

Proof. Let $F : \times_{i=1}^{N} \mathscr{R}^i \to \mathscr{R}$ be a social welfare function induced by simple majority voting under $X = \{x, y\}$ and $i \in \{1, 2, \dots, N\}$. Define $n_x = \sum_{i=1}^{N} \mathbb{1}(x\ P^i\ y)$ and $n_y = \sum_{i=1}^{N} \mathbb{1}(x\ P^i\ y)$ as the number of individuals voting for x and y respectively. For any *individual* ties, let each individual i simply vote x, i.e. whenever $x\ R^i\ y$ and $y\ R^i\ x$, then i acts as if $x\ P^i\ y$.

The social welfare function F is properly defined as for *any* set of profiles in $\times_{i=1}^{N} \mathscr{R}^i$ we have either $x\ R\ y$ or $y\ R\ x$. Ties can be broken deterministically by choosing either x or y. Thus, we have **U**. Without loss of generality, if $x\ P^i\ y$ for all i, then $n_x = N > n_y = 0$ and we get $x\ P\ y$. Therefore, we get **WP** as well. Note that **IIA** is trivially satisfied. When there are only two alternatives, any two rankings R, $\tilde{R}$ that imply the same order over x and y must satisfy $R = \tilde{R}$.

Lastly, note that $x\ P^i\ y \implies x\ P\ y$ if and only if $n_x > n_y$. Thus, there is no dictator i as every outcome is *dependent* of the preference profiles of others. To see this, consider any situation where there is an individual i such that $x\ P^i\ y \implies x\ P\ y$. Then, let $\tilde{P}^j$ be such that $y\ \tilde{P}^j\ x$ for all $j \neq i$ and $x\ \tilde{P}^i\ y$. Obviously, we have $F(\tilde{P}) = y$ despite $x\ \tilde{P}^i\ y$. Thus, we must have **D**. □

6.3. Dictatorships

Claim 2 (Exercise 6.3a). *The social welfare function "individual i dictatorship" satisfies* **U**, **WP** *and* **IIA**.

Proof. Let $F : \times_{j=1}^{N} \mathscr{R}^j \to \mathscr{R}$ be a social welfare function induced by some individual i being the dictator, i.e. $F(R) = R^i$ for all $R \in \times_{j=1}^{N} \mathscr{R}^j$. Thus, F is properly defined over the whole domain of preferences, i.e. **U** holds. Also, $x\ P^j\ y, \forall j \implies x\ P^i\ y \implies x\ P\ y$. Therefore, **WP** follows almost immediately.

If *each* individual ranks some alternative x and y in the same way under R and $\tilde{R}$, then the dictator i does so too. As a result, the social order over x and y will be equivalent under R and $\tilde{R}$ as well. Thus, we have **IIA**. □

Claim 3 (Exercise 6.3b). *The social welfare function "lexicographic dictatorship" satisfies* **U**, **WP** *and* **IIA**. *Furthermore, it is distinct from "individual i" dictatorship.*

Proof. Let $F : \times_{j=1}^{N} \mathscr{R}^j \to \mathscr{R}$ be a social welfare function induced by *lexicographic dictatorship*, starting from individual 1. For any two alternatives x and y, F determines the social ranking by letting it coincide with the ranking of the *first* individual who has a *strict* preference relationship between x and y, starting from individual 1. If for some x and y, we have $x\ I^i\ y$ for all i, then we can set social ranking to be $x\ I\ y$. Therefore, F is properly defined over the whole domain of preferences over X.

Next, we want to show that the ranking constructed by this way is transitive. Suppose the result is not transitive, and we have xPy, yPz, zPx with x, y, z that are distinct. By the way constructing the ranking, there must be individual i, j, k such that xP^iy, yP^jz, zP^kx, and social ranking comes from these individuals. (Otherwise, we should get indifference as relation.) First, $i = j = k$ is impossible because this means that individual's relation is not transitive. Second, consider $i = j < k$. Then, we must have xP^iy, yP^iz, so by transitivity, xP^iz. Because $i < k$, it is impossible that k decides zPx. Third, consider $i = j > k$. In this case, k must feel xI^ky, yI^kz (otherwise, i can't decide xPy, yPz.), and by transitivity, we must have xI^kz which contradicts zP^kx. Finally, we consider i, j, k are all distance. We may assume $i > j > k$. Similarly, k must have xI^ky, yI^kz (otherwise, i, j can't make decision.), and we must know xI^kz which contradicts zP^kx. (From above, we discuss all possible cases.) By these results, we know the relation constructed by this way must be transitive. Then, we can argue that **U** is satisfied.

Once again, $x\ P^j\ y, \forall j \implies x\ P^1\ y \implies x\ P\ y$. Therefore, we have also **WP**. Also, if *each* individual ranks some alternative x and y in the same way under R and $\tilde{R}$, then the social relation between x and y is determined by the same sequence of rankings. This means we have **IIA** as well.

To distinguish *individual* i dictatorship from *lexicographic* dictatorship, we consider the simplest scenario. Let $N = \{1, 2\}$ and $X = \{x, y\}$ with $x\ I^1\ y$ and $x\ P^2\ y$. Under individual i dictatorship, we get $x\ I\ y$ if $i = 1$ and $x\ P\ y$ if $i = 2$. However, lexicographic dictatorship implies $x\ P\ y$. Therefore with $i = 1$, the two dictatorships are different.

c. Arrow's theorem implies that if we have to abide by **U**, **WP** and **IIA**, then we are stuck to some form of dictatorship. A social welfare function that is distinct from individual i dictatorship and lexicographic dictatorship, whilst still satisfying the "desired" properties, is *lexicographic dictatorship of order* q. The definition is given below.

Definition (Lexicographic dictatorship of order q). *A social welfare function* $F : \times_{j=1}^{N} \mathscr{R}^j \to \mathscr{R}$ *is a lexicographic dictatorship of order* q *if there exists a set of* q *distinct voters* $\{i_1, i_2, \ldots, i_q\}$ *and* $\forall R \in \times_{j=1}^{N} \mathscr{R}^j$, $\forall x, y \in X$ *and* $\forall k \in \{1, 2, \ldots, q\}$, *we have:*

- **I**. $x\ P^{i_k}\ y$ *and* $\forall t < k$, $x\ I^{i_t}\ y \implies x\ P\ y$.
- **II**. $\forall k < q,\ x\ I^{i_k}\ y \implies \left[x\ R\ y \iff x\ R^{i_q}\ y\right]$.

Note that the lexicographic dictatorship as defined in the exercise is equivalent to a lexicographic dictatorship of order n.

6.9. Adjusted *Condorcet* cycles

Let our setup be characterized as follows. There are three individuals and social states in society, where $i \in \{1, 2, 3\}$ and $s \in \{x, y, z\}$. Furthermore, let the domain be *unrestricted*, i.e. we have **U**. Whenever voting leads to a *transitive* social order, we construct the social order through a process of pairwise majority voting. Any individual ties are broken by the order x, y and z. If the social order is not transitive, then the social order is given by $x\ P\ y\ P\ z$. Let f denote the social welfare function that this defines. Furthermore, the individual profiles are strict and given by:

R^1	R^2	R^3
x	y	z
y	z	x
z	x	y

a. This example without adjusting for aggregate intransitivity is canonical. In the literature, it is often coined as the *Condorcet paradox*. Even though individual preferences are transitive, pairwise majority voting leads to an *intransitive* social order. To see, consider the possible pairwise voting outcomes when implementing simple majority. The pairwise vote (x, y) leads to x, (y, z) to y and (x, z) to z. Then the social order satisfies $x\ P\ y$ and $y\ P\ z$, but *not* $x\ P\ z$. Because of intransitivity, the social order becomes $x\ P\ y\ P\ z$ by construction.

b. When individual 1's preferences are $y\ P^1\ z\ P^1\ x$, then her preferences coincide with that of individual 2. Therefore, they will always form a majority. As a result, the social order is $y\ P\ z\ P\ x$. If instead, we have $z\ P^1\ y\ P^1\ x$. Then, a pairwise voting between (x, y) leads to y, (y, z) to z and (x, z) to z. Therefore, we get a transitive social order which is given by $z\ P\ y\ P\ x$.

c. Consider the pair (x, y) and assume that $y\ P^i\ x$ for all individuals i. When the resulting social order is transitive, then we are done. As a result, we should consider the "social intransitive" case. Suppose the final outcome is not transitive. For example, we can take yPx, xPz, zPy as intransitive outcome. First, by xPz, we know there are at least 2 individuals with xR^iz. (By the rule of voting, we know i votes x between (x, z) for xP^iz or xI^iz. These two cases give us xR^iz) Second, by zPy, we know there are at least 2 individuals with zP^iy. (By voting rule, i votes z between (z, y) only for zP^iy) Because there are only 3 individuals, combine all results above, we know there is at least 1 individual with xR^iz and zP^iy. By transitivity, xP^iy which contradicts that all individuals strict prefer y to x. Thus, it is impossible to get intransitive outcome here. (If you want, you can discuss all possible pair (p, q) with $p\ P^i\ q$ for all individuals i. The proof should be really similar because you can always find at least 2 individuals with weak preference between one pair and at least 2 individuals with strictly preference between the other pair. Then, you can get the contradiction.)

d. Without loss of generality, suppose that individual 1 is a dictator. Then, fix his ranking and construct the rankings for individuals 2 and 3 to be the reverse of individual 1. Obviously, the former two constitute a majority and their preferred outcome will get implemented. However, this contradicts individual 1 being a dictator.

e. By Arrow's impossibility theorem, we get **U**, **WP** and **IIA** $\iff \neg$**D** whenever $|X| \geq 3$ and $N \geq 2$. Since we have **D**, then we must have either $\neg$**U**, $\neg$**WP** or $\neg$**IIA**. We showed that **U** and **WP** hold, therefore we cannot have **IIA**.

f. Consider the following two *strict* profiles R and $\tilde{R}$ and their associated social rankings according to f:

R^1	R^2	R^3	$f(R)$
y	z	x	x
x	y	z	y
z	x	y	z

$\tilde{R}^1$	$\tilde{R}^2$	$\tilde{R}^3$	$f(\tilde{R})$
y	y	z	y
z	z	x	z
x	x	y	x

Notice that R induces an intransitive social order and we therefore get $x \ P \ y \ P \ z$ by construction. The relative ranking between x and y is not changed between R and $\tilde{R}$, but we do have $x \ P \ y$ and $y \ \tilde{P} \ x$. This is a straight violation of **IIA**.

6.18. Monotonic social choice functions

Let $\mathscr{R} = \times_{j=1}^{N} \mathscr{R}^j$ be the domain of preference profiles. Suppose that the social choice function is given by $c : \mathscr{R} \to X$ and satisfies monotonicity. Let $R = (R^1, \ldots, R^N) \in \times_{j=1}^{N} \mathscr{R}^j$ be *strict* rankings and suppose that $c(R) = x$.

Claim 4 (Exercise 6.18a)**.** *Suppose that for some individual i, R^i ranks y just below x and let $\tilde{R}^i$ be identical to R^i except that y is ranked just above x, i.e. the ranking of x and y is reversed. Then, we must have $c(\tilde{R}^i, R^{-i}) \in \{x, y\}$.*

Proof. By way of contradiction, suppose that $c(\tilde{R}^i, R^{-i}) = z \notin \{x, y\}$ but $c(R) = x$. By construction, we have $\tilde{R}^{-i} = R^{-i}$. Then, we also know that for every individual j:

$$z \ \tilde{R}^j \ c \implies z \ R^j \ c \implies z \ P^j \ c, \forall c \neq z$$

as only the ranking of x and y changed for individual i and we are dealing with *strict* rankings. Since we hypothesized $c(\tilde{R}^i, R^{-i}) = z \notin \{x, y\}$, then by monotonicity we should also get:

$$c(R^i, R^{-i}) = z \notin \{x, y\}$$

but this contradicts $c(R^i, R^{-i}) = c(R) = x$. Therefore, we must have $c(\tilde{R}^i, R^{-i}) \in \{x, y\}$. □

Claim 5 (Exercise 6.18b)**.** *Let $\tilde{R} = (\tilde{R}^1, \ldots, \tilde{R}^N)$ be strict rankings such that for every individual i, the ranking of x versus any other social state is the same under $\tilde{R}^i$ as it is under R^i, then $c(\tilde{R}) = x$.*

Proof. The result follows straight from the definition. For every individual i, we have:

$$x \ R^i \ y \implies x \ \tilde{R}^i \ y \implies x \ \tilde{P}^i \ y$$

then by monotonicity, we get:

$$c(R) = x \implies c(\tilde{R}) = x$$

which is exactly what we needed to show. □

6.19

Let $c(\cdot)$ be a monotonic social choice function and suppose that the social choice must be x whenever all individual rankings are strict and x is at the top of individual n's ranking. Show the social choice must be at least as good as x for individual n when the individual rankings are not necessarily strict and x is at least as good for individual n as any other social state.

Proof:

Let R^i be preferences not necessarily strict and $c^* := c(R)$. We want to show that $c^* R^n x$, when $x R^n y$, $\forall y$.

By contradiction, suppose this is not the case, this is when $x R^n y$, $\forall y$ and R^i, we have $x P^n c^*$ (Note that this implies that $c^* \neq x$). Define a new preference profile $\tilde{R}$ as follows:

$$\forall i, y, z : x P^i y \Rightarrow x \tilde{P}^i y$$
$$\forall i, y \neq c^*, z \neq c^* : x I^i y \Rightarrow y \tilde{P}^i z \text{ if } y \text{ is first, alphabetically}$$
$$\forall i, y \neq c^* : c^* I^i y \Rightarrow c^* \tilde{P}^i y$$

$$\forall i, y : c^* R^i y \Rightarrow c^* \tilde{P}^i y$$
$$c(R) = c^* \Rightarrow c(\tilde{P}) = c^* \text{ by monotonicity assumption}$$

Notice that $\tilde{R}^i$ is now strict ordering for each i. And $x P^n c^* \Rightarrow x \tilde{P}^i c^*$, so that the text of the problem says $c(\tilde{R}) = x$. Contradiction.

6.20

Let x and y be distinct social states. Suppose that the social choice is at least as good as x for individual i whenever x is at least as good as every other social state for i. Suppose also that the social choice is at least as good as y for individual j whenever y is at least as good as every other social state for j. Prove that $i = j$.

Note that in particular this exercise implies that there cannot be two distinct dictators for distinct social states.

Proof: Let $c^* := c(R)$ denote the social choice for social preferences R and $x \neq y$.

We are given that

(i) $c^* R^i x$ whenever $x R^i m$ $\forall m \in X$

(ii) $c^* R^j y$ whenever $y R^i m \; \forall m \in X$.

By contradiction suppose $i \neq j$. Then, because of unrestricted domain, we can consider a profile, R, in which x is strictly preferred to all other social states by i and y is strictly preferred to all other social states by j, $\{xP^i z, \forall z \neq x, z \in X\}$, $\{yP^j s, \forall s \neq y, s \in X\}$, where P denotes strict preferences. Then, since $\{xP^i z, \forall z \neq x, z \in X\}$, by (i) this implies $c^* = x$ because no other social states in X can be at least as good as x for i. Similarly, from $\{yP^j s, \forall s \neq y, s \in X\}$, $c^* = y$ because no other social states in X can be at least as good as y for j. This contradicts the fact that $x \neq y$.

Therefore it has to be that $i = j$.

6.21

Call a social choice function strongly monotonic if $c(R) = x$ implies $c(\tilde{R}) = x$ whenever for every individual i and every $y \in X$, $xR^i y \Rightarrow x\tilde{R}^i y$. Suppose there are two individuals, 1 and 2, and three social states, x, y and z. Define the social choice function $c(\cdot)$ to choose individual 1's top-ranked social state unless it is not unique, in which case the social choice of individual 2's top-ranked social state among those that are top-ranked for individual 1, unless this too is not unique, in which case, among those that are top-ranked for both individuals, choose x if it is among them, otherwise y.

(a) Prove that $c(\cdot)$ is strategy-proof.

Let us first take the case of individual 1. It is straightforward to see that, given individual's 2 reported preferences, individual 1 never has incentives to lie since his preferred choice is always the one picked by the social choice function.

Take now the case of individual 2. If individual's 1 top preference is unique, then individual 2 does not has incentives to lie because the social outcome is independent of her reported preferences.

Denote $T_i(R_i)$ — the set of alternatives that are top-ranked by individual i. And suppose that 1's top choice is not unique. Notice that in this case $\forall a \in \{x, y, z\}$, $c(R_1, R_2) = a$, only if $aR_2 b$, for every $b \in T_1(R_1)$. But by construction of the choice function, $c(R_1, \tilde{R}_2) \in T_1(R_1)$, which means that $c(R_1, R_2) R_2 c(R_1, \tilde{R}_2), \forall \tilde{R}_2$. This establishes SP for individual 2.

(b) Show by example that $c(\cdot)$ is not strongly monotonic. (Hence, strategy-proofness does not imply strong monotonicity, even though it implies monotonicity.)

Suppose the following preferences,

R^1	R^2	$c(\cdot)$
x	z	x
z	x	
y	y	

and

$\tilde{R}^1$	$\tilde{R}^2$	$c(\cdot)$
x,z	z	z
y	x	
	y	

In this case we have that $c(R) = x$ and $xR^1a \Rightarrow x\tilde{R}^1a$, $\forall a \in X$ and R^2 and $\tilde{R}^2$ are the same. However, $c(\tilde{R}) = z$, which contradicts strong monotonicity.

6.23

Show that when there are just two alternatives and an odd number of individuals, the majority rule social choice function (i.e. that which chooses the outcome that is top ranked choice for the majority of individuals) is Pareto efficient, strategy proof and non-dictatorial.

Let $X = \{x, y\}$.

Pareto efficiency. If xP^iy for every i, then $c(R) = x$ is the majority rule social choice (since there is unanimity). Therefore the majority rule social function is Pareto efficient.

Strategy proof.
For studying strategy-proofness it is useful to define a pivotal individual: we say an individual i is *pivotal* whenever the preferences of the other individuals, R^{-i}, lead to a tie.

Consider the case where individual i is not pivotal, and the social choice is $c\left(R^i, R^{-i}\right) = y$. Since i is not pivotal, the report of her own preferences does not affect the outcome and $c\left(R^i, R^{-i}\right) = c\left(\tilde{R}^i, R^{-i}\right) = y$ for any choice $\tilde{R}^i$. Since yR^iy, then the individual does not has incentives to lie about her own preferences, and strategy-proof is satisfied in this case.

Now consider the case where individual i is pivotal. Without loss of generality, let us assume xR^iy. Then $c\left(R^i, R^{-i}\right) = x$ and $c\left(\tilde{R}^i, R^{-i}\right) = y$, but xR^iy and therefore the individual does not has incentives to lie about her preferences either and strategy-proof is satisfied in this case too.

Non dictatorial.
For any i and any social state $a \in X$, the social choice can be a even if bR^ia, $\forall b \in X$ as long as a is the top-ranked alternative for at least $\frac{n+1}{2}$ individuals. Thus, the majority rule social choice function is not dictatorial.

Question 1

The restriction $\theta_{ij} \in \{0,1\}$ is needed because otherwise concept of a blocking coalition will require that everyone with positive ownership of a firm need to dislike the current allocation to form a coalition (so that we don't have to "cut the firm").

Now turning into the question, if a Walrasian equilibrium allocation is blocked by some group of people with allocation $\tilde{x}, \tilde{y}$, by Lemma 5.2 of JR, we have to have that for all $j \in I$

$$p \cdot \tilde{x}^j > p \cdot (e^j + \sum_k \theta^{jk} \tilde{y}^k)$$

with at least one strict inequality. Adding them up,

$$p \cdot \sum_j \tilde{x}^j > p \cdot (\sum_j e^j + \sum_j \sum_k \theta^{jk} \tilde{y}^k).$$

Since $p >> 0$ we have at least one component l such that $\sum_j \tilde{x}_l^j > (\sum_j e_l^j + \sum_j \sum_k \theta^{jk} \tilde{y}_l^k)$. It either means that it is infeasible, or $\tilde{y}$ maximizes firms' profit (as opposed to y), both of which are contradiction.

Question 2

Here we just need to another pair, at some point of the process, who would block the matching.

Starting with μ_1, by eyeballing we see that other than (m_1, w_2), we have (m_3, w_2) as a potential blocking pair. So let's consider picking (m_3, w_2). Then we have $\mu_{2'} = (m_1, w_1)\ (m_2, w_3)\ (m_3, w_2)$, which is stable.

Question 3

We simply move through the algorithm, with every round the single men propose to their most preferred woman.

1. m_1, m_4, m_5 propose to w_1, who accepts m_1. m_2, m_3 propose to w_4, who accepts m_2. So after round 2, m_1, w_1 are together and m_2, w_4 are together.

2. m_3 proposes to w_3, who accepts. m_4 proposes to w_4, who accepts and releases m_2. m_5 proposes to w_2, who accepts. We now have $(m_1, w_1), (m_2, m_2), (m_3, w_3), (m_4, w_4), (m_5, w_2)$. only unmatched is m_2.

3. m_2 proposes to w_3, who accepts and releases m_3.

4. m_3 proposes to w_1, who accepts and releases m_1.

5. m_1 proposes to w_2, who accepts and releases m_5.

6. m_5 proposes to w_4, who rejects him.
7. m_5 prefers to be unmatched over being matched to w_3, so the algorithm ends with

$$(m_1, w_2), (m_2, w_3), (m_3, w_1), (m_4, w_4), (m_5, m_5)$$

Repeat the exercise with the women proposing

1. Round one gives $(w_1, m_2), (w_2, m_3), (w_3, w_3), (w_4, m_1)$, as w_3 is rejected by m_5.
2. w_3 proposes to m_4, who accepts her. All women are matched so final matching is

$$(w_1, m_2), (w_2, m_3), (w_3, m_4), (w_4, m_1), (m_5, m_5)$$

Question 4

Suppose for contradiction there are matching functions μ, μ' such that both are stable, and some $m_1 \in M$ such that $\mu(m_1) = m_1, \mu'(m_1) = w_1$.

As μ stable, it must be that $\mu(w_1) = m_2$ strictly prefers his new match to w_1, i.e. $w_2 = \mu'(m_2) \succ_{m_2} w_1$, otherwise μ' would be blocked by (m_2, w_1).

However as m_2 prefers w_2 to w_1, for μ to be stable, w_2 must be matched with someone better than m_2 under μ, so $m_3 = \mu(w_2) \succ_{w_2} m_2$.

But then again, for μ to be stable, it must be that m_3 prefers his new match to w_2, i.e. $w_3 = \mu'(m_3) \succ_{m_3} w_2$, otherwise μ' would be blocked by (m_3, w_2).

Continuing this, it becomes clear that all the men must prefer μ', all women must prefer μ.

However note that since both are stable, all people matched under μ or μ' prefer their match to being alone. This means that all men matched under μ must be matched in μ', and all women matched in μ' must be matched in μ. As matches are one to one, the first means that $K_\mu = \#$ matches in $\mu \leq K_{\mu'}$. The second implies $K_{\mu'} \leq K_\mu$, so we must have the same number of matches in every stable matching, $K_\mu = K_{\mu'}$.

But since we started with a man who was unmatched in μ getting matched in μ', it must be that some man is unmatched in μ' but matched in μ. But this contradicts every man being better off in μ', so we have a contradiction. As we started out with an arbitrary unmatched man and matching functions, it must be that all unmatched men are unmatched in all stable matches. The equivalent argument hold for the women, so the set of unmatched individuals is the same across stable matches.