Price Theory II Problem Set 7

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Question 1

A single indivisible object is being auctioned to N bidders using a first- price auction (i.e., each bidder submits a sealed bid; the high bidder wins and pays her bid; ties are resolved randomly and uniformly). Each bidder's value, v is drawn independently from a uniform distribution on $[0,1]$. All bidders are risk neutral. Note that this is a Bayesian game because each bidder's value (his "type") is private information.

a

Find a pure (bidding) strategy, $\hat{\beta} : [0,1] \to \mathbf{R}_+$ such that if every bidder employed this strategy, it would form a Bayesian-Nash equilibrium. (Because each player employs the same strategy, this is called a symmetric Bayes-Nash equilibrium.) (Hint: Assume $\hat{\beta}(v)$ takes the linear form $\hat{\beta}(v) = \alpha v$ and derive α.

Assume that $\beta_i(v) = \alpha v, \forall i$ because each player employs the same strategy: we are looking for a **symmetric Bayesian Nash equilibrium**.

Player i's chooses α in order to maximize his expected ex-interim stage utility:

$$U(v_i, v_{-i}, \alpha) = Pr(win|v_i, v_{-i}, \alpha) \cdot (v_i - \alpha v_i)$$

We see that α enters the expected ex-interim utility through two distinct objects

- $Pr(win|v_i, v_{-i}, \alpha)$ the probability for player i of winning the bid
- $(v_i - \alpha v_i)$ player i's payoff if he wins the bid

<u>Preliminary remark</u>: we convince ourselves that $\alpha \in (0,1]$: $\alpha = 0$ is not an equilibrium because player i could win the bid for sure by deviating to $\alpha = \varepsilon$, ε very small, which would give him a strictly better payoff. $\alpha \leq 1$ because the expected payoff when winning the bid would otherwise always be strictly negative.

Express the probability of winning in terms of α

$$
\begin{aligned}
Pr(win) &= Pr(\hat{B}_i \geq \cap B_j) && \forall j \neq i \\
&= Pr(\hat{B}_i \geq \cap \alpha v_j) && \text{All bidders have the same linear strategy} \\
&= Pr(\hat{B}_i \geq \cap \alpha v_j) \\
&= Pr(\hat{B}_i \geq \alpha \max v_j) \\
&= \Pi_{j\neq i} Pr(\hat{B}_i \geq \alpha v_j) \\
&= Pr(\hat{B}_i \geq \alpha v_j)^{N-1} && \text{Because iid} \\
&= Pr(\frac{\hat{B}_i}{\alpha} \geq v_j)^{N-1} \\
&= (\frac{\hat{B}_i}{\alpha})^{N-1} && \text{Because } v_j \text{ follows } U[0,1]
\end{aligned}
$$

Back to the full maximization problem in terms of α[1]:

$$U(\hat{B}_i|v_i) = (v - \hat{B}_i) \cdot (\frac{\hat{B}_i}{\alpha})^{N-1}$$

Taking the FOC wrt $\hat{B}_i$ gives :

$$
\begin{aligned}
(N-1)\hat{B}_i^{N-2}\frac{v}{\alpha^{N-1}} - \hat{B}_i^{N-1}\frac{N}{\alpha^{N-1}} &= 0 \\
(N-1)\hat{B}_i^{-1} v &= N \\
(N-1)v &= \hat{B}_i N \\
\hat{B}_i &= \frac{N-1}{N} \cdot v = \alpha \cdot v
\end{aligned}
$$

with $\alpha = \frac{N-1}{N}$.
Note that $\frac{N-1}{N} \in (0,1)$ if $N > 1$.
By symmetry we claim that $\hat{\beta}_i = \alpha \cdot v_i = \frac{N-1}{N} \cdot v_i$ is a Bayesian NE.

Let's prove that $U(v_i, v_{-i}, \hat{B}_i) \geq U(v_i, v_{-i}, B_i) \forall B_i$ in the strategy choice set.

$$
\begin{aligned}
U(\hat{B}_i|v_i) &= (v - \hat{B}_i) \cdot (\frac{\hat{B}_i}{\alpha})^{N-1} \\
&= (v - \frac{N-1}{N} v) \cdot v^{N-1} \\
&= \frac{v}{N} \cdot v^{N-1} \\
&= \frac{v^N}{N}
\end{aligned}
$$

Let's note that

- $\frac{\hat{B}_i}{\alpha} = v \leq 1$. Bidding $\hat{B}_i > \alpha$ would indeed result in winning the bid with probability 1, but with strictly negative payoff. Therefore $\hat{B}_i \leq \alpha$
- $\hat{B}_i \geq 0$ because $\alpha \geq 0$ and $v_i \in [0,1]$: we assume non-negative bids.

Therefore, we know that the potential BR lie in $[0, \alpha]$.

Let's rule out $\hat{B}_i = 0, = \alpha$ as BNE to show that $B_i = \frac{N-1}{N} v_i$ weakly dominates any other B_i in the potential BR set $[0, \alpha]$.

- Note that $U(\hat{B}_i|v_i) = \frac{v^N}{N} \geq 0$, and is $= 0$ only if $v_i = 0$. Bidding instead $B_i = 0$ is associated to expected utility $= 0$. Therefore $\hat{B}_i = \frac{N-1}{N}$ strictly dominates $\hat{B}_i = 0, \forall v_i > 0$ (and weakly dominates for $v_i = 0$)
- On the other end of the set of potential BR, bidding $B_i = \alpha$ is weakly dominated by bidding $\hat{B}_i = \frac{N-1}{N}$ because

  $$U(B_i = \alpha | v_i) = v_i - \frac{N-1}{N} = v_i - 1 + \frac{1}{N}$$

  is equal to $U(\hat{B}_i|v_i)$ only for $v_i = 1$, and is lower than $U(\hat{B}_i|v_i)$ for all $v_i < 1$.

Thus, on the full set of potential BN strategies, $\hat{B}_i = \frac{N-1}{N} v_i$ weakly dominates all other strategies and is therefore the BNE.

b

This question was suggested by Maurizio Alouan. Suppose now that you are bidder 1 and that you are concerned that the other bidders might not be following the Bayesian-Nash equilibrium bidding strategy you derived in part (a). In fact, you believe that they might all instead be following a local tradition of simply bidding 80% of their value. Show that even if this is the case, you will do best by employing the Bayes-Nash equilibrium strategy from part (a), up to a point. Be specific about what is meant by "up to a point."

Suppose that now $\hat{B}_{j_{J \neq i}} = 0.8 \cdot v_j, \forall v_j \in [0,1]$. What is the optimal strategy for player i?

We notice that player i's best response derived in part a **does not directly depend on** α_j. In this part of the question, $\alpha_j = 0.8$, but this does not change player i's best response, unless

$$\frac{N-1}{N} > 0.8$$
$$\Leftrightarrow N > 5$$

Indeed, when $N \geq 6$, player i's BR is similar to part a until the point where she can win for sure by bidding $0.8 \leq \frac{N-1}{N} \cdot v_i$ as no player will ever bid $> 0.8 = 0.8 \cdot v_j|_{v_j = \max v_j = 1}$ under the

alternative strategy.
Therefore we have

$$
\begin{aligned}
\hat{B}_i(v_i) &= \frac{N-1}{N}v_i && \text{if } N \leq 5 \\
&= \frac{N-1}{N}v_i && \text{if } N \geq 6 \text{ and } v_i \in [0, 0.8 \cdot \frac{N}{N-1}] \\
&= 0.8 && \text{if } N \geq 6 \text{ and } v_i \in [0.8 \cdot \frac{N}{N-1}, 1]
\end{aligned}
$$

Question 3

In class we showed how to analyze games in which the players' payoffs are uncertain. But there are other sources of uncertainty that we have not considered. For example, it might be the case that one player is unsure of the set of strategies available to another player, or is unsure about the number of players in the game. Argue that both kinds of uncertainty can be captured by our model in which there is uncertainty only about payoffs.

. . . .

Below we lay out a more formal justification, but here is the gist. When there is uncertainty about the set of strategies available, we can just model as if all strategies are always available, but in certain states of the world, some strategy types are strictly dominated. When there is uncertainty about the number of players, we can just pretend that the highest number of players possible is always there, but that in some states of the world they are entirely latent, i.e. have no effect on payoffs. In both of these alterations, we are potentially adding extra scenarios that will never occur, but these extra scenarios therefore only contribute to the problem in the way we desire, and the extra fat doesn't hurt anything.

The idea is to recast these other forms of uncertainty into the context already covered. Consider the case where a player is unsure about the set of strategies available to another player, specifically, in a Bayesian game $G = (p_i, T_i, S_i, u_i)_{i=1}^N$, player i is unsure about the strategy sets for all other players, so that for each $j \in \{1, \ldots, N\} \backslash \{i\}$, player i has probability distribution q_{ijk} over the set of strategies $(S_k)_{k=1}^{K_{ij}}$ available to player j. Now the game $G = (p_i, T_i, (q_{ijk}, (S_k)_{k=1}^{K_{ij}})_{j=1}^N, u_i)_{i=1}^N$ is not technically a Bayesian game as defined in the notes, though we will show that it essentially is. As in the discussion before Definition 7.11 in JR, we may map our new game into a strategic form game G^* by creating a separate player for each set of strategies, and assigning payoffs such that the the probability distributions are respected. Similarly to JR, for each $i \in \{1, \ldots, N\}$, let $(t_i, S_k) \in T_i \times (S_k)_{k=1}^{K_{ij}}$ be a player in G^*, so that each player has a type t_i and a strategy profile S_k,[2], so the set of players is $\bigcup_{i \in \{1,\ldots,N\}} T_i \times (S_k)_{k=1}^{K_{ij}}$. Note that in just one step we split all players $j \in \{1, \ldots, N\}$ into their strategy types, and now will use just K to denote all the player types generated by the strategy uncertainty, which may include different numbers of strategy types across players and across players' assumptions.

[2]For simplicity, assume each type has the same strategy uncertainty, and that it is independent of type. This assumption is obviously quite specific, but it should be clear that it would be easy to map into a setting where the joint distribution of types and strategies is more interesting, without changing the underlying argument.

Let $s_{ik}(t_i) \in S_k$ denote a pure strategy chosen by player $t_i \times S_k$, and let $s^* = (s_1(t_1), \ldots, s_K(t_K)) \in S^*$ denote a joint pure strategy. Then we may define the players' payoffs as

$$v_{t_i,S_k}(s^*) = \sum_{(t_{-i},S_{-ik}) \in T_{-i} \times (S_k)_{k \neq i}} p(t_{-i} \mid t_i) q(S_{-k} \mid S_k) u_i(s_1(t_1), \ldots, s_K(t_K))$$

Then we have fully defined the strategic form game $G^* = (S^*, v_{t_i,S_k})$.

We can complete the exact flavor of exercise when there is uncertainty about the number of players in the game. I think the above argument makes clear that including the types adds no generality to the specification which cannot be added later, and it certainly make the notation worse, so here we just consider a simple game $G = (S_i, u_i)_{i=1}^N$, say where all u_i and S_i are identical, but N is uncertain for each player. Then each player has a distribution p_i over some subset of $\mathbb{N}$. Let M be the largest number of players that any any player places positive probability on. Then we may extend each p_i to be defined over $\{1, \ldots, M\}$ for each i. Now we may consider G^* as an strategic form game with M players, each with strategy set S_i, and let $s_i \in S_i$ denote pure strategy chosen by any player, and and let $s^* = (s_1, \ldots, s_M) \in S^*$ denote a joint pure strategy. We also let $u_i(s_1, \ldots, s_m)$ denote the payoff when there are m players with the associated strategies $s_1, \ldots, s_m$. We define payoffs as corresponding to the probabilities of there being each number of players

$$v_i(s^*) = \sum_{m=1}^{M} p_i(m \mid i) u_i(s_1, \ldots, s_m)$$

Note that $p(m \mid i)$ represents player i's beliefs about the probability of there being m players. Then $G^* = (S^*, v_i)$ is the associated strategic form game.

So we have shown that, in any case, we can translate the uncertainty in a game into uncertainty about payoffs, and generate a strategic form game in the same manner as we did for Bayes games.

Question 5

Part a

Definition: Game of Incomplete Information (Bayesian Game)
A game of incomplete information is a tuple $G = (p_i, T_i, S_i, u_i)_{i=1}^N$, where for each player $i = 1, \ldots, N$, the set T_i is finite, $u_i : S \times T \to \mathbb{R}$, and for each $t_i \in T_i, p_i(\cdot|t_i)$ is a probability distribution on T_{-i}. If in addition, for each player i, the strategy set S_i is finite, then G is called a finite game of incomplete information. A game of incomplete information is also called a Bayesian game.

The situation in the question can be modelled as incomplete information game G with:

1. N players
2. Each players type is given their marginal cost. Therefor the set of types $T = T_i = [0, 1], \forall i$, which is finite.

3. The types are drawn independently from distribution with density $f(c)$, assuming beliefs match reality, we have that $p(c_{-i}|c_i) = p(c_{-i}) = f(c_{-i}), \forall i$.

4. Each players strategy is choosing the quantity to produce, therefore the strategy set of each players is $S = S_i = [0, \infty), \forall i$. A strategy of a player is a correspondence $q_i : T \to S$.

5. The expected utility of each player is their expected profit,

$$\begin{aligned} U_i(q_i, q_{-i}|c_i) =& \mathbb{E}_f\left[q_i(\alpha - \beta Q - c_i)\right] \\ =& q_i(c_i)\left(\alpha - c_i - \beta\left(q_i(c_i) + \sum_{i \neq j} \int_0^1 q_j(c) f(c) dc\right)\right) \end{aligned}$$

Part b

Part i

Without loss of generality suppose all players except player i play a symmetric cut-off strategy i.e. $\forall j \neq i$,

$$q_j = \begin{cases} q(c_j) > 0 & \text{if } c_j < c^* \\ 0 & \text{if } c_j \geq c^* \end{cases}$$

Playes i's profit in this context is,

$$\begin{aligned} U_i(q_i, q_{-i}|c_i) =& q_i(c_i)\left(\alpha - c_i - \beta q_i(c_i) + \beta(N-1)\underbrace{\int_0^{c^*} q(c) f(c) dc}_{\equiv \bar{q}}\right) && \text{(by symmetry)} \\ =& q_i(c_i)\left(\alpha - c_i - \beta q_i(c_i) + \beta(N-1)\bar{q}\right) \end{aligned}$$

To begin with, assuming player i cannot choose to shutdown, her best response is to set,

$$\begin{aligned} \frac{\partial U_i}{\partial q_i} =& \alpha - c_i + \beta(N-1)\bar{q} - 2\beta q_i^*(c_i) = 0 \\ \implies q_i^*(c_i) =& \frac{\alpha - c_i - \beta(N-1)\bar{q}}{2\beta} \end{aligned}$$

Now allowing the firm to shutdown, given the rule derived above, it is weakly dominant to choose $q_i = 0$ if $c_i \geq \alpha - \beta(N-1)\bar{q} \equiv c^*$ (strictly if $>$). Therefore, palyer i's best response to the symmetric cut-off strategy is a symmetric cut-off strategy,

$$q_i^*(c_i) = \begin{cases} \frac{c^* - c_i}{2\beta} & \text{if } c_i < c^* \\ 0 & \text{if } c_i \geq c^* \end{cases}$$

We now need to find conditions that determine c^*, which will in turn allow us to characterize the Bayesian Nash Equilibrium. The unknown value in c^* is $\bar{q}$, which we can now make progress towards evaluating by imposing the fact that all agents are following the symmetric cut-off strategy. Namely, we have that,

$$\begin{aligned}
\int_0^{c^*} q^*(c)f(c)dc =& \bar{q} \\
\int_0^{c^*} \frac{c^*-c}{2\beta} f(c)dc =& \bar{q} \\
\frac{c^*}{2\beta}\left[F(c)\right]_0^{c^*} - \frac{1}{2\beta}\left[cF(c)\right]_0^{c^*} + \frac{1}{2\beta}\int_0^{c^*} F(c)dc =& \bar{q} \\
\frac{1}{2\beta}\int_0^{c^*} F(c)dc =& \bar{q}
\end{aligned}$$

Substituting the value of $\bar{q}$ into $c^* = \alpha - \beta(N-1)\bar{q}$ gives us the following implicit solution for c^*,

$$c^* = \alpha - \frac{(N-1)}{2}\int_0^{c^*} F(c)dc \tag{1}$$

Since $F(c)$ is a cdf it is greater than or equal to zero and weakly monotonically increasing. Therefore it follows that, $\int_0^{c^*} F(c)dc$ is greater than zero, monotonically increasing in c^*, and tends to infinity as $c^* \to \infty$. Since, α is fixed, it follows the right hand side of the implicit relationship above is decreasing in c^*, and the left hand side is increasing in c^*. Since $\alpha > 1$, c^* must be greater than zero. To see this, evaluate (1) at $c^* = 0$, this results in $0 < \alpha$, therefore for equation (1) to be satisfied c^* will need to increase. If $1 + \frac{(N-1)}{2}\int_0^1 F(c)dc > \alpha$ then there exists a unique solution to the equation above such that $c^* \in (0,1)$.[3] Whereas if $1 + \frac{(N-1)}{2}\int_0^1 F(c)dc \leq \alpha$ then the unique solution is in the set $c \in [1,\infty)$. To summarize,

$$c^* \in \begin{cases} (0,1) & \text{if } \alpha < 1 + \frac{(N-1)}{2}\int_0^1 F(c)dc \\ [1,\infty) & \text{if } \alpha \geq 1 + \frac{(N-1)}{2}\int_0^1 F(c)dc \end{cases}$$

To conclude the Bayesian Nash equilibrium of this model can be characterised as follows,

1. If $\alpha \geq 1 + \frac{(N-1)}{2}\int_0^1 F(c)dc$, then firms produce strictly positive output given by the following best response rule,

$$q_i^*(c_i) = \frac{\alpha - \frac{(N-1)}{2}\int_0^1 F(c)dc - c_i}{2\beta} > 0$$

2. If $\alpha < 1 + \frac{(N-1)}{2}\int_0^1 F(c)dc$, then firms use a threshold rule,

$$q_i^*(c_i) = \begin{cases} \frac{c^*-c_i}{2\beta} & \text{if } c_i < c^* \\ 0 & \text{if } c_i \geq c^* \end{cases}$$

where c^* solves equation (1).

[3] From the previous argument we showed that $c^* > 0$ and unique. One way of finding the solution is that we can begin increasing c^* until the equation holds with an equality, if the left hand side continues to be less than the right hand side when $c^* = 1$, then $c^* > 1$.

Part ii

As $N \to \infty$ for any given α, $\alpha < 1 + \lim_{N\to\infty}\left\{\frac{(N-1)}{2}\int_0^1 F(c)dc\right\}$. Therefore, we will be in the second type of BNE described above. Fix c^*, then by equation (1) it follows that as N increases the right hand side of equation 1 decreases, hence in turn c^* must decline to ensure equation (1) holds. If we assume that cdf the $F(c)$ is strictly increasing, then as $\lim_{N\to\infty} c^*(N) = 0$ (hence $\bar{q} \to 0$). Therefore, the amount produced by individual firms declines to zero.

Total output is given by,

$$
\begin{aligned}
N\bar{q} = (N-1)\bar{q} + \bar{q} &= \frac{\alpha - c^*(N)}{\beta} + \bar{q} \\
\implies \lim_{N\to\infty} N\bar{q} &= \frac{\alpha}{\beta} && \text{(taking limits)}
\end{aligned}
$$

Part c

Suppose agents play with some strategy $q_i(c_i), \forall i$. Then the players pay-off are given by,

$$
U_i(q_i, q_{-i}|c_i) = q_i(c_i)\left(\alpha - c_i - \beta\left(q_i(c_i) + \sum_{i\neq j}\bar{q}_j\right)\right)
$$

This gives us the following strategy from the first order conditions,

$$
q_i(c_i) = \frac{\alpha - \beta\sum_{i\neq j}\bar{q}_j - c_i}{2\beta} = \frac{\bar{c}_i - c_i}{2\beta}
$$

where $\bar{c}_i = \alpha - \beta\sum_{i\neq j}\bar{q}_j$. Similar to before, when $\bar{c}_i < c_i$ the player is strictly better off producing 0. Therefore, players only choose positive q_i when $\bar{c}_i > c_i$ in which case they produce by the rule above. Taking advantage of the symmetry of this set-up, can get an expression for $\bar{q}_j$,

$$
\begin{aligned}
\bar{q}_j &= \int_0^{t_j} q_j(c)f(c)dc \\
&= \int_0^{t_j} \frac{\bar{c}_j - c}{2\beta} f(c)dc \\
&= \frac{\bar{c}_j - t_j}{2\beta}F(t_j) + \frac{1}{2\beta}\int_0^{t_j} F(c)dc
\end{aligned}
$$

where $t_j = \min\{\bar{c}_j, 1\}$. We can use this to get,

$$
\begin{aligned}
\bar{c}_i - \bar{c}_j &= \alpha - \beta\sum_{i\neq k}\bar{q}_k - (\alpha - \beta\sum_{j\neq k}\bar{q}_k) = \beta(\bar{q}_i - \bar{q}_j) \\
&= \frac{1}{2}\left((\bar{c}_i - t_i)F(t_i) - (\bar{c}_j - t_j)F(t_j) + \int_{t_j}^{t_i} F(c)dc\right)
\end{aligned}
$$

To prove that the solution in part (ii) is unique, we show that the equation above can only be satisfied for $\bar{c}_i = \bar{c}_j$ i.e. the symmetric cut-off strategy. Towards a contradiction suppose $\bar{c}_i > \bar{c}_j$. This implies $t_i \geq t_j$. Notice, that since $F(c) \leq 1, \forall c \in [t_j, t_i] \implies \int_{t_j}^{t_i} F(c)dc \leq \int_{t_j}^{t_i} 1dc = t_i - t_j$. Therefore we have,

$$\begin{aligned}
\bar{c}_i - \bar{c}_j \leq& \frac{1}{2}\left((\bar{c}_i - t_i)F(t_i) - (\bar{c}_j - t_j)F(t_j) + t_i - t_j\right) \\
(\bar{c}_i - t_i) - (\bar{c}_j - t_j) \leq& \frac{1}{2}\left((\bar{c}_i - t_i)F(t_i) - (\bar{c}_j - t_j)F(t_j) - (t_i - t_j)\right) \\
(\bar{c}_i - t_i)\left(2 - F\left(t_i\right)\right) + t_i \leq& \left(\bar{c}_j - t_j\right)\left(2 - F\left(t_j\right)\right) + t_j \\
\bar{c}_i \leq& \bar{c}_j
\end{aligned}$$

The final line contradicts our assumption $\bar{c}_i > \bar{c}_j$. Therefore, $\bar{c}_i = \bar{c}_j$, which is what we were trying to prove.[4]

Additionally to see there are no mixing strategies, note that the pay-offs are concave in the pure strategies q_i, hence mixing would reduce pay-offs (Jensen's inequality). Therefore, pure strategies strictly dominate mixed strategies.

[4] *To see why the last line follows, notice,*

$$(\bar{c}_k - t_k)\left(2 - F\left(t_k\right)\right) + t_k = \begin{cases} (\bar{c}_k - \bar{c}_k)\left(2 - F\left(\bar{c}_k\right)\right) + c_k = 0 + \bar{c}_k = \bar{c}_k & \textit{if } c_k < 1 \\ (\bar{c}_k - 1)\left(2 - F\left(1\right)\right) + 1 = \bar{c}_k - 1 + 1 = \bar{c}_k & \textit{if } c_k \geq 1 \end{cases}$$

Question 7

We are told that each player observes his own type before taking a decision. Hence, players choose their actions to maximize their interim expected utility (they don't know other players' types).

We are going to solve this problem is 4 steps. Beforehand, note that

$$Pr(t_1 = mp) = Pr(t_2 = mp) = \frac{2}{3}$$

$$Pr(t_1 = c) = Pr(t_2 = c) = \frac{1}{3}$$

Let's denote each player's probability to play each action as

$$\begin{aligned}
p_{mc,1} &= \text{probability player 1 plays } a_1 = a \text{ when } t_1 = mc \\
p_{mc,2} &= \text{probability player 2 plays } a_2 = a \text{ when } t_2 = mc \\
p_{c,1} &= \text{probability player 1 plays } a_1 = a \text{ when } t_1 = c \\
p_{c,2} &= \text{probability player 2 plays } a_2 = a \text{ when } t_2 = c
\end{aligned}$$

Therefore, the strategy of each player can be represented by the pair $(p_{mc,i}, p_{c,i}) \in [0,1]^2, \forall i = \{1,2\}$.

Step 1. Compute player 1's expected interim utility

$\underline{t_1 = mc}$

$$\begin{aligned}
E[U_{1,interim}(a_1 = a|t_1 = mp)] &= Pr(t_2 = mp) \cdot U(a_1 = a|t_1 = mp, t_2 = mp) + Pr(t_2 = c) \cdot U(a_1 = a|t_1 = m \\
&= \frac{2}{3} \cdot U(a_1 = a|t_1 = mp, t_2 = mp) + \frac{1}{3} \cdot U(a_1 = a|t_1 = mp, t_2 = c) \\
&= \frac{2}{3} \cdot (1 \cdot p_{mc,2} + (-1) \cdot (1 - p_{mc,2})) + \frac{1}{3} \cdot (1 \cdot p_{c,2} + (-1) \cdot (1 - p_{c,2})) \\
&= \frac{2}{3} \cdot (p_{mc,2} - (1 - p_{mc,2})) + \frac{1}{3} \cdot (p_{c,2} - (1 - p_{c,2}))
\end{aligned}$$

$$\begin{aligned}
E[U_{1,interim}(a_1 = b|t_1 = mp)] &= Pr(t_2 = mp) \cdot U(a_1 = b|t_1 = mp, t_2 = mp) + Pr(t_2 = c) \cdot U(a_1 = b|t_1 = mp \\
&= \frac{2}{3} \cdot (-p_{mc,2} + (1 - p_{mc,2})) + \frac{1}{3} \cdot (-p_{c,2} + (1 - p_{c,2}))
\end{aligned}$$

$\underline{t_1 = c}$

$$\begin{aligned}
E[U_{1,interim}(a_1 = a|t_1 = c)] &= Pr(t_2 = mp) \cdot U(a_1 = b|t_1 = c, t_2 = mp) + Pr(t_2 = c) \cdot U(a_1 = b|t_1 = c, t_2 = \\
&= \frac{2}{3} \cdot p_{mc,2} + \frac{1}{3} \cdot p_{c,2}
\end{aligned}$$

$$\begin{aligned}
E[U_{1,interim}(a_1 = b|t_1 = c)] &= Pr(t_2 = mp) \cdot U(a_1 = b|t_1 = c, t_2 = mp) + Pr(t_2 = c) \cdot U(a_1 = b|t_1 = c, t_2 = \\
&= \frac{2}{3} \cdot (1 - p_{mc,2}) + \frac{1}{3} \cdot ((1 - p_{c,2})
\end{aligned}$$

Step 2. Compute player 2's expected interim utility

Following the same logic we have

$$\begin{aligned}
E[U_{2,interim}(a_2 = a|t_2 = mp)] &= \frac{2}{3} \cdot (-p_{mc,1} + (1 - p_{mc,1})) + \frac{1}{3} \cdot (-p_{c,1} + (1 - p_{c,1})) \\
E[U_{2,interim}(a_2 = b|t_2 = mp)] &= \frac{2}{3} \cdot (p_{mc,2} - (1 - p_{mc,2})) + \frac{1}{3} \cdot (p_{c,2} - (1 - p_{c,2})) \\
E[U_{2,interim}(a_2 = a|t_2 = c)] &= \frac{2}{3} \cdot p_{mc,1} + \frac{1}{3} \cdot p_{c,1} \\
E[U_{2,interim}(a_2 = b|t_2 = c)] &= \frac{2}{3} \cdot (1 - p_{mc,1}) + \frac{1}{3} \cdot ((1 - p_{c,1})
\end{aligned}$$

Step 3. Determine each player's BR

We need to consider 3 cases for each player : when is $p_{mc,1} = 1$ (plays $a_1 = a$ with probability 1 when his type is mc), $p_{mc,1} = 0$, $p_{mc,1} \in [0, 1]$ (player 1 mixes between $a_1 = a, b$) ; when is $p_{mc,2} = 1, = 0, \in [0, 1]$.
While we will be able to appeal to the symmetry argument, we can't avoid computing how (WLOG) player 1 will set $p_{mc,1}, p_{c,1}$ as functions of $p_{mc,2}, p_{c,2}$.

We have that player 1 | $t_1 = mc$ will mix between $a_1 = a, b$ ($p_{mc,1} \in [0, 1]$) iff

$$\begin{gathered}
E[U_{1,interim}(a_1 = a|t_1 = mp)] = E[U_{1,interim}(a_1 = b|t_1 = mp)] \\
\frac{2}{3} \cdot (p_{mc,2} - (1 - p_{mc,2})) + \frac{1}{3} \cdot (p_{c,2} - (1 - p_{c,2})) = \frac{2}{3} \cdot (-p_{mc,2} + (1 - p_{c,2})) + \frac{1}{3} \cdot (-p_{c,2} + (1 - p_{c,2})) \\
\frac{2}{3}(2p_{mc,2} - 1) + \frac{1}{3}(2p_{c,2} - 1) = \frac{2}{3}(1 - 2p_{mc,2}) + \frac{1}{3}(1 - 2p_{c,2}) \\
8p_{mc,2} + 4p_{c,2} = 6 \\
4p_{mc,2} + 2p_{c,2} = 3
\end{gathered}$$

Therefore, and similarly

$$\begin{aligned}
p_{mc,1} &\in [0, 1] \text{ iff} & 4p_{mc,2} + 2p_{c,2} &= 3 & (2)\\
p_{mc,1} &= 1 \text{ iff} & 4p_{mc,2} + 2p_{c,2} &> 3 & (3)\\
p_{mc,1} &= 0 \text{ iff} & 4p_{mc,2} + 2p_{c,2} &< 3 & (4)\\
p_{c,1} &\in [0, 1] \text{ iff} & 4p_{mc,2} + 2p_{c,2} &= 3 & (5)\\
p_{c,1} &= 1 \text{ iff} & 4p_{mc,2} + 2p_{c,2} &> 3 & (6)\\
p_{c,1} &= 0 \text{ iff} & 4p_{mc,2} + 2p_{c,2} &< 3 & (7)\\
p_{mc,2} &\in [0, 1] \text{ iff} & 4p_{mc,1} + 2p_{c,1} &= 3 & (8)\\
p_{mc,2} &= 1 \text{ iff} & 4p_{mc,1} + 2p_{c,1} &< 3 & (9)\\
p_{mc,2} &= 0 \text{ iff} & 4p_{mc,1} + 2p_{c,1} &> 3 & (10)\\
p_{c,2} &\in [0, 1] \text{ iff} & 4p_{mc,1} + 2p_{c,1} &= 3 & (11)\\
p_{c,2} &= 1 \text{ iff} & 4p_{mc,1} + 2p_{c,1} &> 3 & (12)\\
p_{c,2} &= 0 \text{ iff} & 4p_{mc,1} + 2p_{c,1} &< 3 & (13)
\end{aligned}$$

Step 4: Consider each situation to find all BN Equilibria

We show that we have a continuum of BN satisfying equations (1), (4), (7) and (10).
BWOC, suppose equations (2) and (5) hold, i.e. $4p_{mc,2} + 2p_{c,2} > 3$. Player 1's BR is to play $p_{mc,1} = p_{c,1} = 1$. However, then, player 2's BR is to play $p_{mc,2} = 0, p_{c,2} = 1$. This implies that $4p_{mc,2} + 2p_{c,2} = 2 < 3$, a contradiction. Therefore $p_{mc,1} = p_{c,1} = 1$ is not a BN equilibrium.

Similarly, suppose equations (3) and (6) hold, i.e. $4p_{mc,2} + 2p_{c,2} < 3$. Player 1's BR is to play $p_{mc,1} = p_{c,1} = 0$. However, then, player 2's BR is to play $p_{mc,2} = 1, p_{c,2} = 0$. This implies that $4p_{mc,2} + 2p_{c,2} = 4 > 3$, a contradiction. Therefore $p_{mc,1} = p_{c,1} = 0$ is not a BN equilibrium.

Finally, for player 1, if equations (1) and (4) hold, i.e. $4p_{mc,2} + 2p_{c,2} = 3$, then player 1's BR is to play $\{p_{mc,1}, p_{c,1}\} \in [0,1]$

The same arguments follow for player 2's equations (3) & (6), (8), (9), (11) and (12).

Because the only equations that do not lead to a contradiction are equations (1), (4), (7) and (10), the BN equilibria correspond to the continuum of $p_{mc,1}, p_{c,1}, p_{mc,2}, p_{c,2} \in [0,1]^4$ such that equations (1), (4), (7) and (10) are satisfied, i.e.

$$p_{mc,1}, p_{c,1}, p_{mc,2}, p_{c,2} \in [0,1] \text{ s.t. } p_{mc,2} + 2p_{c,2} = 3, p_{mc,1} + 2p_{c,1} = 3$$

In other words, we have mixed strategy BNE for both players.