Price Theory II Problem Set 2

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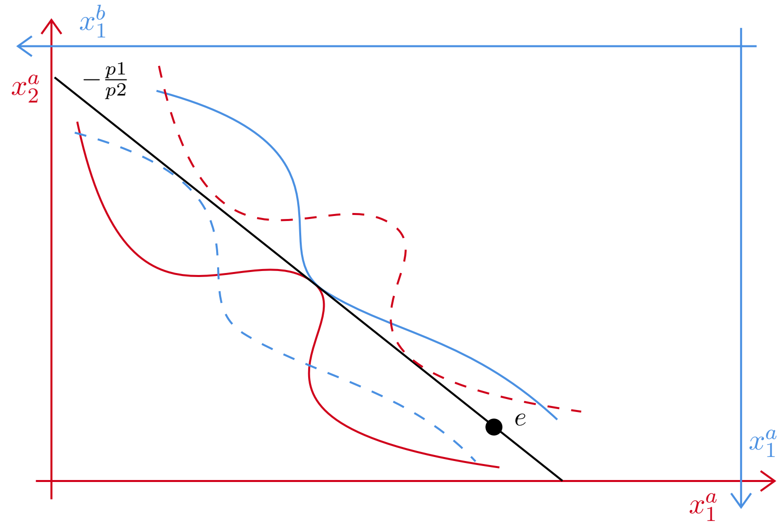
July 3, 2020

Question 1

In an Edgeworth box two-consumer, two-commodity pure exchange economy with consumption sets equal to $\mathbb{R}^2_+$ sketch a situation in which there is no competitive equilibrium because preferences are not convex.

In the example below the two consumers maximise their utility at different points. Therefore utility maximisation and market clearing do not occur simultaneously.

Figure 1: Two-consumer Edgeworth Box with Convex Preferences



Question 3

Consider a two-consumer two-good exchange economy in which $u_1(x_1; x_2) = x_1x_2; u_2(x_1; x_2) = x_1(x_2)^2; e_1 = (18; 4); and e_2 = (3; 6)$*: (a) Characterize the set of Pareto efficient allocations as completely as possible. (b) Characterize the core. (c) Find a competitive equilibrium allocation and corresponding price vector. (d) Verify that the competitive equilibrium allocation* *is in the core.*

a. The set of Pareto Efficient Allocations (PEA) consists of the set of allocations such that

- No consumer can be made better off without making another consumer worse off. With differentiable functions, this is equivalent to **equating the MRS** of the two consumers of our economy (this is valid only for interior solutions, i.e. solutions that are not on the Edgeworth box).
- Budget constraints are satisfied.

We consider the interior solutions:

$$\frac{\frac{\partial U_1}{\partial x_1}}{\frac{\partial U_1}{\partial y_1}} = \frac{\frac{\partial U_2}{\partial x_2}}{\frac{\partial U_2}{\partial y_2}}$$
$$\frac{y_1}{x_1} = \frac{y_2}{2x_2}$$

As well the corner solutions (starting from an allocation where one consumer gets everything is also PE because this consumer will be worse off from any allocation change) : (0,0) and (21,10).
S.t.

$$x_1 + x_2 \leq 18 + 3 = 21$$
$$y_1 + y_2 \leq 4 + 6 = 10$$

We can summarize these PE allocations as

$$P(\Omega) = \{(x_1, y_1), (x_2, y_2) | x_1 \in [0, 21], y_1 \in [0, 10], x_2 = 21 - x_1, y_2 = 10 - y_1, \frac{y_1}{x_1} = \frac{10 - y_1}{2(21 - x_1)}\}$$

b. The core is such that

- No agent is strictly better off by taking his initial endowment rather than any allocation on the core (Individual Rationality (IR)).
- The allocations on the core are Pareto efficient.

The notion of *Core* is therefore more restrictive than PE because the core is characterized by

$$C(\Omega) \subset P(\Omega)$$
$$P(\Omega) \cap IR(\Omega) = C(\Omega)$$
$$C(\Omega) = \{(x_1, y_1), (x_2, y_2) | x_1 y_1 \geq e_{x1} e_{y1}, x_2 y_2^2 \geq e_{x2} e_{y2}^2, \{(x_1, y_1), (x_2, y_2)\} \in P(\Omega)\}$$
$$\Leftrightarrow C(\Omega) = \{(x_1, y_1), (x_2, y_2) | x_1 y_1 \geq 72, x_2 y_2^2 \geq 108, \{(x_1, y_1), (x_2, y_2)\} \in P(\Omega)\}$$

(c) Find a competitive equilibrium allocation and corresponding price vector.
A competitive equilibrium of Ω is a price vector $p* \in \mathbf{R}^n_{++}$ and the associated allocation $x_1, x_2 \in R^N_{++}$ such that

1. (Utility Maximization) (x_i, y_i) maximizes $u_i(x_i, y_i)$ such that $p_x^* x_i + p_y^* y_i \leq p_x^* e_{x_i} + p_y^* e_{y_i}$ for $i = \{1, 2\}$
2. (Market clearing) $x_1^* + x_2^* \leq e_{x_1} + e_{x_2}$ and $y_1^* + y_2^* \leq e_{y_1} + e_{y_2}$

Each consumer maximizes his utility $\Leftrightarrow$ ratio of MRS = ratio of prices. From 3.a.

$$\begin{aligned} \frac{y_1}{x_1} &= \frac{y_2}{2x_2} = \frac{p_x}{p_y} \\ \Leftrightarrow y_1 &= \frac{x_1 p_x}{p_y} \\ y_2 &= \frac{2x_2 p_x}{p_y} \end{aligned}$$

Budget Constraints are satisfied:

$$\begin{aligned} p_x x_1 + p_y y_1 &= 18p_x + 4p_y \\ p_x x_2 + p_y y_2 &= 3p_x + 6p_y \end{aligned}$$

Solving for x_1, x_2 and plugging in the expressions for y_1, y_2 computed above gives x_1, x_2, y_1, y_2 in terms of p_x, p_y:

$$\begin{aligned} x_1 &= 9 + 2\frac{p_y}{p_x} \\ x_2 &= 1 + 2\frac{p_y}{p_x} \\ y_1 &= 9\frac{p_x}{p_y} + 2 \\ y_2 &= 3 + 2\frac{p_x}{p_y} \end{aligned}$$

Finally, recover $\frac{p_x}{p_y}$ from the market clearing conditions (what matters here is relative prices)

$$\begin{aligned} x_1 + x_2 &= 21 \\ \Rightarrow 9 + 2\frac{p_y}{p_x} + 1 + 2\frac{p_y}{p_x} &= 21 \\ \Rightarrow \frac{p_y}{p_x} &= \frac{11}{4} \end{aligned}$$

Setting $p_x = 4$ to simplify calculus (we can always normalize wrt 1 price) and plugging $p_x = 4, p_y = 11$ into the expression for the allocations we get that the competitive equilibrium is defined by

$$CE = \{(x_1, y_1), (x_2, y_2), (p_x, p_y) | (x_1, y_1) = (9 + \frac{11}{2}, 9\frac{4}{11} + 2), (x_2, y_2) = (1 + \frac{11}{2}, 2\frac{4}{11} + 4), p_x = 4, p_y = 11\}$$

(d) Verify that the competitive equilibrium allocation is in the core.
We need to show that the equilibrium allocation satisfies IR and is PE.
Individual Rationality

$$u_1(x_1^*, y_1^*) = 14.5 * 5.27 = 76.45 > u_1(e_{x_1}, e_{y_1}) = 72$$
$$u_2(x_2^*, y_2^*) = 6.5 * 4.73^2 = 145.26 > u_2(e_{x_2}, e_{y_2}) = 108$$

Pareto Efficiency: $\frac{x_1}{y_1} = \frac{2x_2}{y_2}$

$$\frac{14.5}{5.27} = \frac{6.5 * 2}{4.73} = 2.75...$$

Question 5

Theorem 1.2 *Let $\succeq$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(x)$ is a utilty function that represents it. Then $v(x)$ also represents $\succeq$ if and only if $v(x) = f(u(x))$ for every x, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.*

Proof: Suppose $v(x)$ also represents $\succeq$. A strictly increasing function means $x < y$ implies $f(x) < f(y)$. If there does not exist such an f, such that $v(x) = f(u(x))$ for all x, then there must exist some bundles x, y such that $u(x) < u(y)$, but that $v(x) > v(y)$. This is a contradiction, because both u and v represent $\succeq$, but u says $y \succ x$ and v says $x \succ y$.

Suppose $v(x) = f(u(x))$ for every x, where f is strictly increasing on the set of values taken by u. Then if $x \succeq y$, we have $u(y) \leq u(x)$, so $v(y) = f(u(y)) \leq f(u(x)) = v(x)$, and if $v(y) \leq v(x)$, then by the *strict* monotonicity of f, we know $u(y) \leq u(x)$, so $x \succeq y$. So v represents $\succeq$.

Theorem 1.3 Let $\succeq$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:

(i) *$u(x)$ is strictly increasing if and only if $\succeq$ is strictly monotonic.*

Proof: Let's recall what each of these terms means (I am grabbing from the appendix of JR, since these are not universally agreed upon). All domains are $\mathbb{R}^n_+$. A function f is increasing if $x^0 \geq x^1$ implies $f(x^0) \geq f(x^1)$. If, additionally, $x^0 >> x^1$ implies $f(x^0) > f(x^1)$, then f is strictly increasing. If, additionally, $x^0 \geq x^1$ and $x^0 \neq x^1$ implies $f(x^0) > f(x^1)$, then f is strongly increasing. The preference relation $\succeq$ is strictly monotonic if $x^0 \geq x^1$ implies $x^0 \succeq x^1$, and $x^0 >> x^1$ implies $x^0 \succ x^1$.

Suppose u is strictly increasing. Then $x^0 \geq x^1$ implies $u(x^0) \geq u(x^1)$, and since u represents, we know $x^0 \succeq x^1$. Also $x^0 >> x^1$ implies $u(x^0) > u(x^1)$, and since u represents, we know $x^0 \succ x^1$. So $\succeq$ is strictly monotonic.

Suppose $\succeq$ is strictly monotonic. Then $x^0 \geq x^1$ implies $x^0 \succeq x^1$, and since u represents, we know $u(x^0) \geq u(x^1)$. Also $x^0 >> x^1$ implies $x^0 \succ x^1$, and since u represents, we know $u(x^0) > u(x^1)$. So u is strictly increasing.

(ii) *$u(x)$ is quasiconcave if and only if $\succeq$ is convex.*

Proof: A function f is quasiconcave if (JR definition) for all $t \in (0, 1)$ $f(tx+(1-t)y) \geq \min\{f(x), f(y)\}$. The preference relation $\succeq$ is convex if $x^1 \succeq x^0$ implies $tx^1+(1-t)x^0 \succeq x^0$.

Suppose u is quasiconcave. Then if $x^1 \succeq x^0$, we know $u(x^1) \geq u(x^0)$, and we have $u(tx^1 + (1 - t)x^0) \geq \min\{u(x^1), u(x^0)\} = u(x^0)$, so $tx^1 + (1 - t)x^0 \succeq x^0$. Thus $\succeq$ is convex.

Suppose $\succeq$ is convex. Then for any x^1, x^0, without loss of generality $u(x^1) \geq u(x^0)$. So by representation $x^1 \succeq x^0$, thus $tx^1 + (1 - t)x^0 \succeq x^0$, and by representation $u(tx^1 + (1 - t)x^0) \geq u(x^0)$. So u is quasiconcave.

(iii) *$u(x)$ is strictly quasiconcave if and only if $\succeq$ is strictly convex.*

Proof: A function f is strictly quasiconcave if for all $t \in (0,1)$, $x \neq y$, $f(tx + (1-t)y) > \min\{f(x), f(y)\}$. A preference relation $\succeq$ is strictly convex if $x^1 \succeq x^0$ implies $tx^1 + (1-t)x^0 \succ x^0$.

Suppose u is strictly quasiconcave. If $x^1 \succeq x^0$, then $u(x^1) \geq u(x^0)$, so $u(tx^1+(1-t)x^0) > \min\{u(x^1), u(x^0)\} = u(x^0)$. So $tx^1+(1-t)x^0 \succ x^0$, by representation. Thus $\succeq$ is strictly convex.

Suppose $\succeq$ is strictly convex. Then for any x^1, x^0, without loss of generality $u(x^1) \geq u(x^0)$. So by representation $x^1 \succeq x^0$, thus $tx^1 + (1-t)x^0 \succ x^0$, and by representation $u(tx^1 + (1-t)x^0) > u(x^0) = \min\{u(x^0), u(x^1)\}$. So u is quasiconcave.

Question 5.7

Consider an exchange economy with two goods. Suppose that its aggregate excess demand function is $z(p_1, p_2) = (-1, p_1/p_2)$ for all $(p_1, p_2) \gg (0,0)$.

1. *Show that this function satisfies conditions 1 and 2 of Theorem 5.3, but not condition 3.*
2. *Show that the conclusion of Theorem 5.3 fails here. That is, show that there is no $(p_1^*, p_2^*) \gg (0,0)$ such that $z(p_1^*, p_2^*) = (0,0)$.*

Part 1

Condition 1: *Continuity*

The first entry is a constant and hence it is continuous, the second entry is the ratio of two strictly positive real numbers which is also continuous. Therefore $z(p)$ is continuous on $\mathbb{R}^2_{++}$.

Condition 2: $p \cdot z(p) = 0, \forall p \gg 0$

$$p \cdot z(p) = -p_1 + \cancel{p_2}\frac{p_1}{\cancel{p_2}} = 0$$

Condition 3: *If $\{p_m\}$ is a sequence of price vectors in R^n_{++} converging to $p \neq 0$, and $p_k = 0$ for some good k, then for some good k' with $p'_k = 0$, the associated sequence of excess demands in the market for good k', $\{z_{k'}(p_m)\}$, is unbounded above.*

To see this condition is violated consider the sequence of prices $p_m = (\frac{1}{m}, \bar{p}_2)$, where (WLOG) $\bar{p}_2$ is a positive constant. Then p_m converges to $(0, \bar{p}_2)$ i.e. $p_{1,m} \to 0$. However, the demand for good 1 is fixed (i.e. bounded) at -1.

Part 2

This directly follows from the fact that $z_1(p) = -1, \forall p \in \mathbb{R}^2_{++}$.

Question 5.21

Consider an exchange economy with 2 consumers. Consumer 1 has utility function $u_1(x_1, x_2) = x_2$ and endowment $e_1 = (1, 1)$ and consumer 2 has utility function $u_2(x_1, x_2) = x_1 + x_2$ and endowment $e_2 = (1, 0)$

a. Which of the hypotheses of Theorem 5.4 fail in this example?

<u>**Theorem 5.4**</u>: *If each consumer's utility function satisfies Assumption 5.1 and if the aggregate endowment of each good is strictly positive ($\sum_{i \in I} e_i \gg 0$), then the aggregate excess demand satisfies conditions 1 through 3 of Theorem 5.3.*

<u>**Theorem 5.3**</u>: *Suppose $z : \mathbf{R}^N_{++} \to \mathbf{R}^N$ satisfies*
1. Continuity
2. Walras Law: $p.z(p) = 0$
3. For any sequence $\{p_m\}$ in $\mathbf{R}^N_{++}$, if $p_m \to p \neq 0$ a price vector and $\bar{p_k} = 0$ for some k, then there exists $k' : p.k' = 0$ and $z_k(p_m)$ is unbounded above.

<u>**Assumption 5.1**</u>: *Each u_i is continuous, strongly increasing and strictly quasi-concave on $\mathbf{R}^N_+$*

Let's look at the hypotheses of Theorem 5.4.
First, <u>continuity</u> (from A.1) is trivially satisfied.
Second, <u>strong increasing functions</u> (from A.1): $\frac{\partial u_2}{\partial x_1} > 0, \frac{\partial u_2}{\partial x_2} > 0, \frac{\partial u_1}{\partial x_2} > 0, \frac{\partial u_1}{\partial x_1} = 0$ so u_1 is not strongly increasing in x_1.
Third, <u>strict quasi-concavity</u> (from A.1). As a reminder, a function is strictly quasi-concave iff $f(tx_1 + (1-t)x_2) > \min(f(x_1), f(x_2))$ for $t \in [0, 1], x_1 \neq x_2$.
From this definition we see that neither u_1 nor u_2 are strictly quasi-concave:

$$
\begin{aligned}
u_1(x_1^A, x_2^A) &= x_2^A \\
u_1(x_1^B, x_2^B) &= x_2^B \\
&= tx_2^A + (1-t)x_2^B \text{ even if } x^A \neq x^B \text{ through different } x_1 \text{ only}
\end{aligned}
$$

And

$$
\begin{aligned}
u_2(x_1^A, x_2^A) &= x_1^A + x_2^A \\
u_2(x_1^B, x_2^B) &= x_1^B + x_2^B \\
u_2(tx^A + (1-t)x^B) &= tx_1^A + (1-t)x_1^B + tx_2^A + (1-t)x_2^B \\
&= u_2(x^A) = u_2(x^B) \text{ if } x_1^A + x_2^A = x_1^B + x_2^B
\end{aligned}
$$

Fourth, <u>strictly positive aggregate endowment</u> is satisfied as $e_1 = 2$, $e_2 = 1$.

b. Show that there does not exist a Walrasian Equilibrium in this economy.
Let's try to show a Walrasian Equilibrium and see where we run into problems.

From the utility functions:
- Agent 1's utility only depends on x_2 so the agent will spend all his income on good 1. From his budget constraint[1]:

$$0p_1 + x_2p_2 = e_1p_1 + e_2p_2$$
$$x_2 = p_1 + p_2$$

Normalizing one of the prices we have $(p_1, p_2) = (p, 1)$ such that, on agent 1's side $x_2 = 1+p$.

- Agent 2's utility is strongly increasing in both goods, so we know by Theorem 5.3 that if one of the prices is 0, then demand for this good will be unbounded above. However, as we have finite endowment, we must consider strictly positive prices only. Moreover, x_1 and x_2 are perfect substitutes for agent 2, so he will spend all his income on the cheaper of those goods[2].
Agent's 2 total income is $e_1 * p + e_2 * 1 = p$.
Agent 2's demand for each good is

$(1, 0)$ if $p \in (0, 1)$: good 1 is cheaper and not demanded by agent 1 so agent 2 will consume all of good 1
$(x_1, x_2)|x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1$ if $p = 1$
$(0, p)$if $p > 1$ as both agents will bargain for good 2.

Why is this not a Walrasian Equilibrium for any of the price vectors $\mathbf{p}$? Because markets don't clear:
- $p > 1$: demand for x_1 is 0, but the economy is endowed with 2 units of x_1.
- $p = 1$: the economy is only endowed with 1 unit of x_2, but any positive demand from agent 2 will lead total demand for $x_2 > 1$.
- $p \in (0, 1)$: aggregate demand for good 1 is 1 but total supply is 2.

Unlike in "standard" situations, notice that the initial endowments are **not** Pareto optimal because agent 1 does not value x_1 at all: agent 2 could be made better off by receiving agent 1's initial endowments of x_1 without making agent 1's worse off.

[1]Note that because his utility is strongly increasing in x_1 his BC will always be binding.
[2]Strongly increasing utility function (in both goods this time) ensures that agent 2's budget constraint is binding.