**Price Theory, III** | **Professor Lars Stole**
**Economics 33003** | **Spring 2020**

# Assignment 3
## (Due Friday, May 1, prior to the start of the Review session )

**Problem 1** Consider a principal-agent model with moral hazard in which the principal is risk neutral and the agent is risk averse with $u(w) = \sqrt{100 + w}$. There are two efforts, $e_h > e_l$, with personal cost to the agent of $\psi(e_h) = 1$ and $\psi(e_l) = 0$. The agent's expected utility from wage contract $w(\cdot)$ and effort $e$ is $E[\sqrt{100 + w(x)} \mid e] - \psi(e)$.

There are two outcomes. The high-output outcome is $x_2 = 200$ and the low-outcome output is $x_1 = 100$. When the agent exerts low effort, each output is equally likely. When the agent exerts high effort, the probability of high output is $\frac{3}{4}$ (and the probability of low output is $\frac{1}{4}$). The agent's reservation utility is $\underline{U} = u(0) = 10$.

(a). Does this distribution satisfy MLRP?

(b). Solve for the optimal output-contingent wage contract, $\{w_1^*, w_2^*\}$.

**Problem 2** Consider a principal-agent model with moral hazard in which the principal is risk neutral and the agent is also risk neutral. There are two efforts, $e_h > e_l$, with personal cost to the agent of $\psi(e_h) = 10$ and $\psi(e_l) = 0$. The agent's expected utility is $E[w|e] - \psi(e)$.

There are two outcomes. The high-output payoff is $x_2 = 200$, but the low-outcome payoff represents a loss, $x_1 = -100$. When the agent exerts low effort, each output is equally likely. When the agent exerts high effort, the probability of high output is $\frac{3}{4}$ (and the probability of low output is $\frac{1}{4}$). The agent's reservation utility is $\underline{U} = 0$.

(a). Solve for an optimal output-contingent wage contract, $\{w_1^*, w_2^*\}$. Does the worker earn any surplus above his reservation value?

(b). Now suppose that the agent is still risk neutral, but legally the principal is not allowed to pay a wage that is negative. [Aside: This is sometimes called the *limited-liability* assumption. It is also equivalent to a situation in which an agent is infinitely risk averse at $w < 0$: i.e., $u(w) = w$ for $w \geq 0$ and $u(w) = -\infty$ for $w < 0$.] Solve for an optimal output-contingent wage contract, $\{w_1^*, w_2^*\}$.

What is the expected cost to the principal of the limited-liability constraint?

**Problem 3** Consider a simple moral hazard where the principal and the agent are risk neutral (i.e., $v'(\cdot) = u'(\cdot) = 1$), there is no uncertainty over output, but now output depends upon the actions of the agent and the principal. Specifically, the agent chooses $e_1 \in [0, 2]$, the principal chooses $e_2 \in [0, 2]$ and output is deterministic:

$$x = e_1 + e_2 \in \mathcal{X} = [0, 4].$$

Assume that the cost of effort for each player is $\frac{1}{2}e_i^2$, so that first-best production requires $e_i^{fb} = 1$ and $x^{fb} = 2$.

(a). Assume that the principal can offer a contract to the agent which promises $s(x)$ in payment for the outcome $x$, and the residual profit, $x - s(x)$, is kept by the principal. After the agent accepts the contract, both the principal and agent simultaneously choose their individual efforts, $e_i$. Once $x$ is revealed, payments are shared as promised with $s(x)$ going to the agent and $x - s(x)$ going to the principal.

If $s(x)$ is required to be continuously differentiable on $\mathcal{X}$, show that the first best cannot be implemented by the principal. [Hint: remember that the principal must also be given incentives. Another way to think of this problem is that there is a team of two players that need to devise a sharing rule $\{s(x), x - s(x)\}$ to give each incentives.]

(b). Suppose that instead of designing a sharing rule, $\{s(x), x - s(x)\}$, the principal can commit to giving some of the output to a third party for some values of $x$. I.e., the principal can offer $s(x)$ to the agent, and can commit to giving $z(x)$ to a third party, keeping $x - s(x) - z(x)$ for herself. Show that the first best can now be implemented as a Nash equilibrium between the principal and agent and that nothing is given to the third party *along the equilibrium path.* For this part, you may construct the implementing functions using discontinuous $s$ and $z$ functions. [Hint: think about simple contracts which split the output between principal and agent for some $x$, and give the entire output to a third party for other $x$.]

(c). Assume we are back to setting (a) without a third party, and thus the share contracts are limited to $\{s(x), x - s(x)\}$. Now, however, assume that the sets of feasible actions are discrete $e_1 \in \{0.5, 1, 1.5\}$ and $e_2 \in \{.67, 1, 1.33\}$. Can the first best $e_1^* = e_2^* = 1$ be implemented?

**Problem 4** Consider a risk-averse individual with utility function of money $u(\cdot)$ with initial wealth $y$ who faces the risk of having an accident and losing an amount $x$ of her wealth. She has access to a perfectly competitive market of risk-neutral insurers who can offer coverage schedules $b(x)$ (i.e., in the event of loss $x$, the insurance company pays out $b(x)$) in exchange for an insurance premium, $p$. Assume that the distribution of $x$, which depends on accident-prevention effort $e$, has an atom at $x = 0$:

$$f(x|e) = \begin{cases} 1 - \phi(e) & \text{if } x = 0 \\ \phi(e)g(x) & \text{if } x > 0, \text{ where } g(\cdot) \text{ is a probability density} \end{cases}$$

Assume $\phi'(e) < 0 < \phi''(e)$. Also assume that the individual's (increasing and convex) cost of effort, separable from her utility of money, is $\psi(e)$.

(a). What is the full information insurance contract when $e$ is contractible? Specifically, what is $e$ and $b(x)$?

(b). If $e$ is not observable, prove that the optimal contract consists of a premium and a deductible; i.e., $b(x) = x - \delta$. You may assume that a solution exists and the first-order approach is valid.

**Problem 5** Consider a principal-agent setting with moral hazard and a finite number of outputs and efforts. There are $n$ possible outputs, $x_i \in \mathcal{X} = \{x_1, \ldots, x_n\}$, $x_n > x_{n-1} > \cdots > x_1$. There are $m$ possible effort levels, $e_j \in \mathcal{E} = \{e_1, \ldots, e_m\}$, $e_m > e_{m-1} > \cdots > e_1$. The probability of output $x_i$ given effort $e_j$ is $\phi_i(e) = \phi(x_i|e_j) > k > 0$ (i.e., bounded away from zero). Assume that $\phi_i(\cdot)$ satisfies MLRP: i.e., for all $e > \tilde{e}$,

$$\frac{\phi_i(e) - \phi_i(\tilde{e})}{\phi_i(e)}, \quad \text{is increasing in } i.$$

Assume that the principal is risk neutral and the agent is risk averse with preferences $U = u(w) - \psi(e)$, where $u(\cdot)$ is increasing and concave, and $\psi(\cdot)$ is increasing and convex. Assume the agent's outside option is $\underline{U} = 0$ and $\psi(e_1) = 0$. Finally, the principal's wage schedule must satisfy $w_i = w(x_i) \in [\underline{w}, \overline{w}]$, but assume that these constraints are slack at the optimum and ignore them.

(a). State the principal's program for the optimal $e$ and $(w_1, \ldots, w_n)$, and write the Lagrangian for the program using $\{\mu_j\}_j$ as multipliers for the IC constraints and $\lambda$ as the multiplier for the IR constraint.

(b). Prove that the agent's IR constraint is binding (i.e., $\lambda > 0$) in the solution to the program.

(c). Suppose that the optimal choice of effort is $e^* = e_m$, the maximum effort level. Prove that the optimal wage schedule which induces $e^m$ is monotonic, $w_1^* \le w_2^* \cdots \le w_n^*$ with strict inequality for some $i$ (i.e., $w_i^* < w_{i+1}^*$).

(d). Prove that the MLRP implies first-order stochastic dominance for the case of a discrete distribution, $(\phi_1(e), \ldots, \phi_n(e))$.

(e). Suppose that the optimal choice of effort $e^* > e_1$. Prove that there must exist some $i$ such that $w_i^* < w_{i+1}^*$ (i.e., prove that $w_1^* \ge w_2^* \ge \cdots \ge w_n^*$ cannot be optimal). [Hint: use the fact in (d) above.]

**Problem 6** Consider principal-agent problem with three exogenous states of nature, $\theta_1$, $\theta_2$, and $\theta_3$; two effort levels, $e_L$ and $e_H$; and two output levels, distributed as follows as a function of the state of nature, $\theta$, and the effort level, $e$:

| State of nature | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|
| Probability of state | $\phi_1 = 0.25$ | $\phi_2 = 0.50$ | $\phi_3 = 0.25$ |
| Output under $e_H$ | 1800 | 1800 | 100 |
| Output under $e_L$ | 1800 | 100 | 100 |

The principal is risk neutral, while the agent has utility function $u(w) = \sqrt{w}$ when receiving monetary compensation $w$, minus the cost of effort, which is normalized to 0 for $e_L$ and to $\Delta = 10$ for $e_H$. The agent's reservation expected utility is $\underline{U} = 10$.

(a). Derive the first-best, full-information contract.

(b). Derive the second-best contract when only output levels are observable.

(c). A salesman comes by (in the second-best world in which only the cash outcome is observed) and offers to sell the principal an information system. The information will be observed by both the principal and the agent after the act is implemented. (We interpret the information system as a monitor.) The system in question will tell whether or not the state is $\theta_3$; in other words it will provide the following partition of information $\{\{\theta_1, \theta_2\}, \{\theta_3\}\}$. How much will the principal be willing to pay for the monitoring system?