Price Theory III Problem Set 1- Odds

Big Tombo and the Quaranteam.
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Problem 1

(a) This property follows directly from the fact that r is decreasing. Let w be some wage such that some workers are employed and others are not. Then there exists some unemployed worker of type $\theta_1 \in [\underline{\theta}, \overline{\theta}]$, and therefore $w < r(\theta_1)$. There also exists some employed worker of type $\theta_2 \in [\underline{\theta}, \overline{\theta}]$ such that $w \geq r(\theta_2)$. Since r is decreasing, it must be that $\theta_1 < \theta_2$. Since r is continuous, the Intermediate Value Theorem tells us there exists $\hat{\theta} \in (\theta_1, \theta_2] \subset [\underline{\theta}, \overline{\theta}]$ such that $w = r(\hat{\theta})$, and since r is *strictly* decreasing, $\hat{\theta}$ is unique. Now note that for any $\theta \in [\hat{\theta}, \overline{\theta}]$, since r is decreasing, $r(\theta) \leq r(\hat{\theta}) = w$, so these types will accept wage w. For any $\theta \in [\underline{\theta}, \hat{\theta})$, since r is strictly decreasing, $r(\theta) > r(\hat{\theta}) = w$, so these types will *not* accept wage w.

So at w, only types $[\hat{\theta}, \overline{\theta}]$ choose to work, and these are precisely the more capable workers.

(b) ....

Since $r(\theta) > \theta$, every worker's marginal productivity is less than their outside home production wage[1], so if the firm had full information about worker types, the[2] Pareto efficient outcome would be zero employment. If this outcome can be constructed as a competitive equilibrium in the imperfect information case, we are done.

Let $w \in \mathbb{R}$. If $w \geq r(\overline{\theta})$, then let $\hat{\theta} \equiv \inf_{[\underline{\theta},\overline{\theta}]}\{\theta : w \geq r(\theta)\}$, so that workers $[\hat{\theta}, \overline{\theta}]$ will choose employment. This cannot be an equilibrium, however, because all these workers are marginally less productive than their wage, so the firm will be making negative profits[3]. If $w < \overline{\theta}$, then the firm will demand workers in $[w, \overline{\theta}]$, but these wages will be unacceptable to the workers, since they attain a higher wage by home production. If $w \in [\overline{\theta}, r(\overline{\theta}))$, then the firm will demand zero labor (no worker is productive enough to warrant the wage), and the workers will provide zero supply of labor (all find home production preferable). Then the set of wages $[\overline{\theta}, r(\overline{\theta}))$, crossed with zero employment, form the set of competitive equilibria. These all correspond to the Pareto efficient outcome under full information, so they must be the Pareto efficient outcome under imperfect information as well.

(c) ....

If the firm had full information about types, they would offer wage θ to each type, and this arrangement would be acceptable to $\theta \in [\hat{\theta}, \overline{\theta}]$ only, since then $w = \theta > r(\theta)$. This competitive

[1] Per numeraire. I am assuming goods price $p = 1$ for simplicity.

[2] Restricting to cases where firms make non-negative profits

[3] Modulo measure zero. Technically at $w = r(\overline{\theta})$ 1 worker is employed, but they are payed measure zero and produce measure zero.

equilibrium in full-info world is then Pareto efficient by the First Welfare Theorem. Since the firm has imperfect information, they offer a single wage w, and if it is a competitive equilibrium wage, it must satisfy $w^* = \mathbb{E}[\theta \mid \theta \in \Theta^*]$, where Θ^* is the set of workers that find the wage acceptable. Thus, $\Theta^* \equiv \{\theta : r(\theta) \leq w^*\}$.

Now note that for any $\tilde{\theta} > \hat{\theta}$, we have $\mathbb{E}[\theta \mid r(\theta) \leq r(\tilde{\theta})] = \mathbb{E}[\theta \mid \theta \geq \tilde{\theta}] \geq \tilde{\theta} > r(\tilde{\theta})$, and $\mathbb{E}[\theta \mid r(\theta) \leq r(\hat{\theta})] = \mathbb{E}[\theta \mid \theta \geq \hat{\theta}] > \hat{\theta} = r(\hat{\theta})$, by the fact that r is strictly decreasing, and dominates θ below $\hat{\theta}$ and is dominated by θ above $\hat{\theta}$. Now note that $\tilde{w} = r(\tilde{\theta})$ is the wage required in order to achieve $\Theta^* = [\tilde{\theta}, \overline{\theta}]$. So to possibly achieve a competitive equilibrium wage, we must have a wage which yields more labor than the Pareto efficient amount, since we have just shown that the Pareto efficient amount and all amounts above it do not satisfy the $w^* = \mathbb{E}[\theta \mid \theta \in \Theta^*]$ condition. Therefore, any competitive equilibrium (if one exists) must be such that there is more employment than in the Pareto optimal outcome[4].

[4] We have not excluded the possible case where there is full employment, which would occur if, say, r is decreasing "a lot" over the interval $[\underline{\theta}, \hat{\theta}]$, or if there is a a relatively higher mass of workers concentrated at lower θ.

Problem 3

a)

We solve the problem using backwards induction. The seller will accept any offer higher than their valuation and reject anything lower. When the offer is equal to their valuation they are indifferent. We can then write the strategy of a type v_s seller receiving an offer of p as:

$$a_s(v_s, p) = \begin{cases} \text{accept} & \text{if } p \geq v_s \\ \text{reject} & \text{if } p < v_s \end{cases}$$

The buyer then chooses the offer p to maximise their ex-ante expected payoff given nature's distribution of v_s and the strategy above of the sellers. This gives us:

$$\begin{aligned} p^* &= \text{argmax}_{p\in\mathbb{R}}\, \mathbb{E}\Big[1_{\{p\geq v_s\}}(v_b - p)\Big] \\ &= \text{argmax}_{p\in\mathbb{R}} \int_0^1 1_{\{p\geq v_s\}}(kv_s - p)dv_s \\ &= \text{argmax}_{p\in[0,1]} \int_0^p (kv_s - p)dv_s \\ &= \text{argmax}_{p\in[0,1]} \frac{1}{2}kp^2 - p^2 \\ &= \begin{cases} 0 & \text{if } k < 2 \\ [0,1] & \text{if } k = 2 \\ 1 & \text{if } k > 2 \end{cases} \end{aligned}$$

b)

We can find the pooling equilibria as follows. Let $\bar{p}$ be given.

Consider the following conditional beliefs for the buyers after observing a price $\bar{p}$ (where $\delta_0(.)$ denotes a point mass placed on 0):

$$\mu(v_s|p) = \begin{cases} U[0,\bar{p}] & \text{if } p = \bar{p} \text{ and } \bar{p} < 1 \\ U[0,1] & \text{if } p = \bar{p} \text{ and } \bar{p} \in [1, \frac{k}{2}] \\ \delta_0(.) & \text{if } p \neq \bar{p} \end{cases}$$

Given these beliefs the payoff for the buyer of accepting will be:

$$\mathbb{E}_\mu[v_s k - p] = \begin{cases} -p & \text{if } p \neq \bar{p} \\ \bar{p}\frac{k}{2} - \bar{p} & \text{if } \bar{p} \in [0,1] \\ \frac{k}{2} - \bar{p} & \text{if } \bar{p} > 1 \end{cases}$$

If $k < 2$ then none of these are positive and the buyer always rejects the price offered. The seller cannot benefit by deviating in their offered price as in all cases they earn 0. No trade occurs in equilibrium.

If $k > 2$ then for $\bar{p} \in [0, 1]$ the expected payoffs of accepting an offer $\bar{p}$ for the buyer given their beliefs are always strictly positive so they accept whenever $p = \bar{p}$. Moreover, the sellers with valuations $v_s \in [0, \bar{p}]$ will be better off making the trade at $\bar{p}$ than failing to trade (which occurs if $p \neq \bar{p}$) so they will offer $\bar{p}$. Finally the buyer's conditional beliefs μ obey Bayes' rule along the equilibrium path since the types that offer $\bar{p}$ are uniformly distributed with valuations $v_s \in [0, \bar{p}]$. Trade occurs whenever $v_s \leq \bar{p}$

If $k > 2$ then for $\bar{p} \in [1, \frac{k}{2}]$ then again the expected payoffs of accepting an offer $\bar{p}$ for the buyer given their beliefs are always strictly positive so they accept whenever $p = \bar{p}$. However, this time all sellers will be better off making the trade at $\bar{p}$ so all sellers will make an offer at $\bar{p}$ and again the buyers' conditional beliefs are correct along the equilibrium path. Trade occurs for any $v_s \in [0, 1]$.

If $k > 2$ and $\bar{p} > \frac{k}{2}$ then the buyer will always reject as their expected payoff is negative. There is also no profitable deviation for the seller as they earn zero for any offered price.

We can rule out separating equilibria[5] as follows. Suppose that there were a separating equilibrium in which trade happens so that sellers of some type v_1 offered $p(v_1)$ and sellers of some type v_2 offered $p(v_2)$ with both of these prices accepted by the buyer. Without loss, suppose $p(v_1) > p(v_2)$. It would then be profitable for the v_2 type buyer to deviate and pretend to be a v_1 type since this earns them a strictly higher price and is accepted by the buyer.

[5]Meaning equilibria in which trade occurs at two different prices. There could be different prices off path prices by type or different prices by type for which trade does not occur.

Problem 5

Consider a labor-market signaling model from MWG. Recall: there are two types of workers : $\theta_H > \theta_L > 0$ and the probability of the high type is $\phi \in (0,1)$. Type-θ worker's payout from wage w is given by $u(w,e,\theta) = w - c(e,\theta)$, where $c(e,\theta) = \frac{e}{\theta}$ is the cost of obtaining (unproductive) education level $e > 0$. The worker's reservation utility is zero ($r(\theta) = 0$) and there are at least risk-neutral two firms.

(a) What is the range of education levels that can arise in a pooling equilibrium?

In a pooling equilibrium, w does not depend on the worker's type, and both workers choose the same level of education $e_h = e_l = e^*$.

Before looking at the education choices that can be sustained in a pooling equilibrium, we need to specify the firms' belief system. We assume that

$$\mu(e) = \begin{cases} 0 & \text{if } e < e_p^* \\ \phi & \text{if } e \geq e_p^* \end{cases}$$

Importantly, notice that a different belief system would imply different conditions for the pooling equilibrium [6].

In order to determine the range of the education levels that can arise in a pooling competitive equilibrium we must have that (1) both workers are willing to work at wage $w = E(\theta) = \phi\theta_H + (1-\phi)\theta_L$ rather than staying home, and rather than signaling themselves as their real type by adopting a different education level.

$$u(w,e,\theta_L) = w - \frac{e_p^*}{\theta_L} = E(\theta) - \frac{e_p^*}{\theta_L} > r(\theta_L) = 0 \qquad \theta_L \text{ prefers pooling rather than home} \tag{1}$$

$$u(w,e,\theta_L) = w - \frac{e_p^*}{\theta_L} = E(\theta) - \frac{e_p^*}{\theta_L} > w_L = \theta_L \qquad \theta_L \text{ prefers pooling rather than signaling as a low type} \tag{2}$$

$$u(w,e,\theta_H) = w - \frac{e_p^*}{\theta_H} = E(\theta) - \frac{e_p^*}{\theta_H} > r(\theta_H) = 0 \qquad \theta_H \text{ prefers pooling rather than staying home} \tag{3}$$

Note that

- Because $\theta_H > \theta_L$ only the first inequality is binding, (3) is redundant to (1).
- Because $w_L > 0$, (1) is redundant to (2)

[6] In particular, if we had

$$\mu(e) = \begin{cases} 0 & \text{if } e < e_p^* \\ \phi & \text{if } e = e_p^* \\ 1 & = \text{ if } e > e_p j \end{cases}$$

then we would need to refine the conditions on θ_H such that θ_H has no incentives to signal itself as a high type by choosing $e > e_p^*$

- Because of the belief structure imposed above, we don't need to worry about θ_H wanting to signal himself as a high type, as the firm will never believe him.

Therefore, only (2) is binding. Rearranging holds

$$\begin{aligned}
E(\theta) - \frac{e_p^*}{\theta_L} &> \theta_L \\
E[\theta] \cdot \theta_L - e_p^* &> \theta_L^2 \\
\theta_L \cdot (E[\theta] - \theta_L) &> e_p^* \\
\theta_L \cdot (\phi\theta_H + (1-\phi)\theta_L - \theta_L) &> e_p^* \\
\theta_L \cdot \phi(\theta_H - \theta_L) &> e_p^*
\end{aligned}$$

Because WLOG education can't be negative, the range of education levels that can arise in a pooling equilibrium, for the specified pooling system is $e_p^* \in [0, \theta_L \cdot \phi(\theta_H - \theta_L)]$.

(b) In the set of all separating equilibria, what is the range of education levels that are chosen by the low type, θ_L, and what is the range of education levels chosen by the high type, θ_H?

Once again, let's start by specifying a belief system[7]:

$$\mu(e) = \begin{cases} 1 & \text{if } e \geq \tilde{e} \\ 0 & \text{if } e < \tilde{e} \end{cases}$$

Let's first notice that in a separating equilibrium, because education is costly, and the labor market is competitive, the low type will always receive $w(\theta_L) = \theta_L$ whatever education level he provides, θ_L always provides $e(\theta_L) = 0$ in a separating equilibrium.
This in mind, let's investigate the conditions under which θ_L prefers to signal himself as a low type than as a high type:

$$\begin{aligned}
w(\theta_H) - \frac{\tilde{e}}{\theta_L} &\leq w(\theta_L) - \frac{0}{\theta_L} \\
\theta_H - \frac{\tilde{e}}{\theta_L} &\leq \theta_L && \text{because competitive labor market} \\
\tilde{e} &\geq \theta_H \cdot \theta_L - \theta_L^2 \\
\tilde{e} &\geq \theta_L(\theta_H - \theta_L)
\end{aligned}$$

Where $\tilde{e}$ is the least cost separating equilibrium.

Finally, we need to find the maximum level of education e_1 that can be sustained by θ_H, i.e. the education level at which θ_H is better off being believed to be a low type:

$$\begin{aligned}
w(\theta_H) - \frac{e_1}{\theta_H} &\geq w(\theta_L) - \frac{0}{\theta_H} \\
\theta_H - \frac{e_1}{\theta_H} &\geq \theta_H \\
e_1 &\leq \theta_H(\theta_H - \theta_L)
\end{aligned}$$

[7] We take the freedom to define different belief system than in part (a) because it isn't specified anywhere that we should not.

Therefore the levels of education that can arise in a separating equilibrium are

$$\begin{cases} e_L & = 0 \\ e_H & \in [\theta_L(\theta_H - \theta_L); \theta_H(\theta_H - \theta_L)] \end{cases}$$

(c) Suppose now that $c = e$ (independent of θ), but the worker's marginal utility of money depends upon type. Specifically, assume the worker's payoff is

$$u(w, e, \theta) = \theta w - e$$

How does your answer in (b) change with these new preferences? Explain.

Let's assume the same belief system as in part b.
Once again, because education is costly, $e_L = 0$. For θ_L to prefer to be recognized as a low type it must be the case that

$$\begin{aligned} \theta_L \cdot w(\theta_L) - e_L &\geq \theta_L \cdot w(\theta_H) - \tilde{e} \\ \theta_L^2 &\geq \theta_L \theta_H - \tilde{e} && \text{because competitive labor market} \\ \tilde{e} &\geq \theta_L(\theta_H - \theta_L) \end{aligned}$$

Similarly, for θ_H to prefer to be recognized as a high type rather than a low type it must be that

$$\begin{aligned} \theta_H \cdot w(\theta_H) - e_1 &\geq \theta_H \cdot w(\theta_L) - 0 \\ \theta_H^2 - e_1 &\geq \theta_H \theta_L \text{ because competitive labor market} \\ e_1 &\leq \theta_H(\theta_H - \theta_L) \end{aligned}$$

Therefore the levels of education that can arise in a separating equilibrium are

$$\begin{cases} e_L & = 0 \\ e_H & \in [\theta_L(\theta_H - \theta_L); \theta_H(\theta_H - \theta_L)] \end{cases}$$

Surprise, they are the same as in part b ! Why is that?
Notice that

$$\begin{aligned} u_b(w, e, \theta) &= w - \frac{e}{\theta} \\ u_c(w, e, \theta) &= w\theta - e \\ &= u_b(w, e, \theta) \cdot \theta \end{aligned}$$

Therefore, the utility in part c is a linear transformation of the utility in part b. The education level maximizing u_b and u_c for a given (w, θ) is the same[8].

(d) Suppose again that $c = e$ (independent of θ) and the worker's marginal utility of money depends upon type, but now the worker's payoff is

$$u(w, e, \theta) = \frac{w}{\theta} - e$$

[8] The e that maximizes a function is also a maximizer of this function multiplied by any number.

How does your answer in (b) change? Explain.

Let's assume the same belief system as in part b.
Once again, because education is costly, $e_L = 0$. For θ_L to prefer to be recognized as a low type it must be the case that

$$\begin{aligned}
\frac{w(\theta_L)}{\theta_L} - e_L &\geq \frac{w(\theta_H)}{\theta_L} - \tilde{e} \\
\frac{\theta_L}{\theta_L} &\geq \frac{\theta_H}{\theta_L} - \tilde{e} && \text{because competitive labor market} \\
\tilde{e} &\geq \frac{\theta_H}{\theta_L} - 1 \\
&\geq \frac{\theta_H - \theta_L}{\theta_L}
\end{aligned}$$

Similarly, for θ_H to prefer to be recognized as a high type rather than a low type it must be that

$$\begin{aligned}
\frac{w(\theta_H)}{\theta_H} - e_1 &\geq \frac{w(\theta_L)}{\theta_H} - 0 \\
1 - e_1 &\geq \frac{\theta_L}{\theta_H} \\
e_1 &\leq \frac{\theta_H - \theta_L}{\theta_H}
\end{aligned}$$

Wait, we have a problem here: we have just shown that the education level e_h chosen by a high type such that a separating equilibrium holds is such that

$$\begin{aligned}
\tilde{e} \leq e_H &\leq e_1 \\
\frac{\theta_H - \theta_L}{\theta_L} \leq e_H &\leq \frac{\theta_H - \theta_L}{\theta_H}
\end{aligned}$$

Which is a contradiction because $\theta_H > \theta_L \rightarrow \frac{\theta_H - \theta_L}{\theta_H} < \frac{\theta_H - \theta_L}{\theta_L}$.

Therefore, no separating equilibrium can't be sustained with this utility functions. Why is that? Because now low types get more utility from money than high types. The latter have therefore less incentives to get a high wage than the former.