# ECON302 2016 Final Examination Solutions

June 16, 2016

## Problem 1

Consider a game where Player 1 must choose T or M or B, Player 2 must choose L or R, and their utility payoffs $(u_1, u_2)$ depend on their choices as follows:

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $T$ | $x, 5$ | $2, 4$ |
| | $M$ | $3, 6$ | $7, 8$ |
| | $B$ | $0, 0$ | $9, 1$ |

The payoff $x = u_1(T, L)$ has been left unspecified in the above table, so that we can consider several different assumptions about it in (a)-(c) below.

**(a)** For what range of x is the action M strictly dominated for Player 1 by some randomized strategy? (*Hint*: You can start by showing how M could be dominated if x were very high.)

**(b)** Suppose that $x = 6$. With randomization, how many Nash equilibria does this game have? Show all the equilibria, and compute the expected payoff for each player in each equilibrium.

**(c)** Now suppose that x is drawn from a Uniform distribution on the interval from 3 to 6. When they play the game, Player 1 privately knows the value of x, but Player 2 only knows that it has been drawn from a Uniform[3,6] distribution. So $x$ now may be considered Player 1's type. Find a Bayesian equilibrium in which Player 2 would randomize, using each of her possible actions with a strictly positive probability. (You do not need to show expected payoffs here.)

***Solution.***

**(a)** Looking at the game we see that $[M]$ is dominated, so there exists a behavioral strategy $\sigma_1 = q[T] + (1-q)[B] \succ_1 [M]$. Thus, we have that

$$EU_1[\sigma_1|[L]] > EU_1[[M]|[L]] \implies qx > 3$$

and

$$EU_2[\sigma_1|[R]] > EU_1[[M]|[R]] \implies 2q + 9(1-q) > 7$$

which imply $q > 3/x$ and $q < 2/7$, where solving for $x$ in $3/x = 2/7$ implies $x > 21/2$.

**(b)** First we find the PSNE: $([T],[L])$ and $([B],[R])$, which have respective payoffs of $(6,5)$ and $(9,1)$. Now we find the MSNE. After plotting Player 1's best response to mixed strategies for Player 2, we see that the only support we need to check is $\{([T],[M]),([L],[R])\}$. Thus, our game reduces to

| | | Player 2 | |
|---|---|---|---|
| | | $(q)L$ | $R$ |
| Player 1 | $(p)T$ | $x,5$ | $2,4$ |
| | $M$ | $3,6$ | $7,8$ |

from which we see that equilibria strategies would be such that $p = 2/3$ and $q = 5/8$, so that our MSNE is: $(2/3[T] + 1/3[M], 5/8[L] + 3/8[R])$.

**(c)** We draw 1's best response diagram in Figure 1. Because we are looking for an equilibrium where 2 is indifferent between $[L]$ and $[R]$ we cannot pick support

$$\{([M],[B]),([L],[R])\}$$

because 2's best response is always to pick $[R]$. Thus, we consider the support

$$\{([T],[M]),([L],[R])\}$$

for our Bayesian game.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $T$ | $x,5$ | $2,4$ |
| | $M$ | $3,6$ | $7,8$ |

Notice that Player 1 has increasing differences in his type, so we propose a cutoff of $\bar{x}$ such that if $x > \bar{x}$ then he chooses $[T]$ and if $x < \bar{x}$ he chooses $[M]$. Let $p = P(T) = P(x > \bar{x}) = (6-\bar{x})/3$. Then, Player 2 will be indifferent between $[L]$ and $[R]$ given 1's cutoff strategy $\sigma_1$, so that we can solve

$$EU_2[[L]|\sigma_1] = 5p + 6(1-p) = EU_2[[R]|\sigma_1] = 4p + 8(1-p) \implies -p + 6 = -4p + 8 \implies p = 2/3 \implies \bar{x} = 5.$$

Then, at $x = \bar{x}$, Player 1 will be indifferent between $[T]$ and $[B]$ given Player 2's mixed strategy $\sigma_2$ so that we can solve

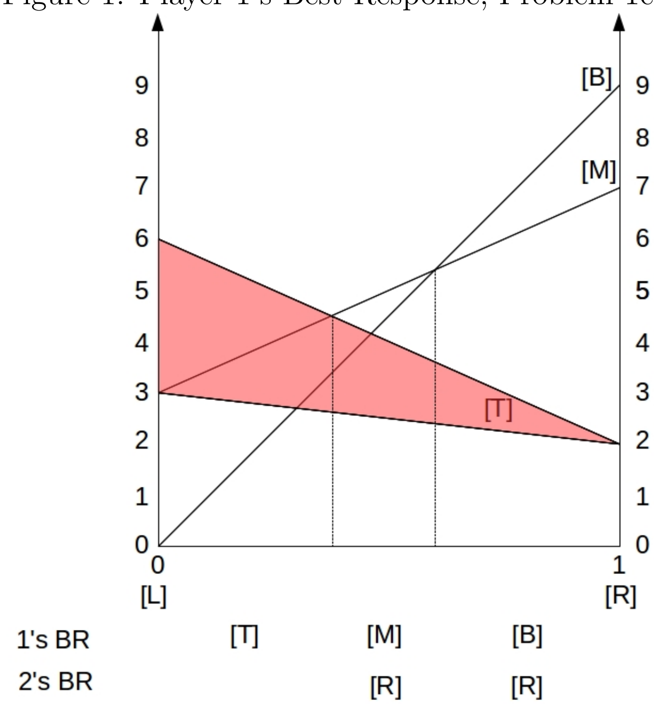
$$EU_1[[T]|\sigma_2, x = \bar{x}] = \bar{x}q + 2(1-q) =$$
$$EU_1[[M]|\sigma_2, x = \bar{x}] = 3q + 7(1-q) \implies 3q + 2 = -4q + 7 \implies q = 5/7.$$

Further, we can check that the expected utility of playing $[B]$ for Player 1 is less than playing $[T]$ or $[M]$ given these strategies. We have $EU_1[[B]|\sigma_2] = 18/7$ and $EU_1[[T]|\sigma_2, \sigma_1] = 29/7 > 18/7$ and $EU_1[[M]|\sigma_2, \sigma_1] = 29/7 > 18/7$). Thus, our Bayesian Nash equilibrium is

$$\sigma_1 = \begin{cases} [T] & \text{if } x > \bar{x} = 5 \\ [M] & \text{if } x < \bar{x} = 5 \end{cases}$$

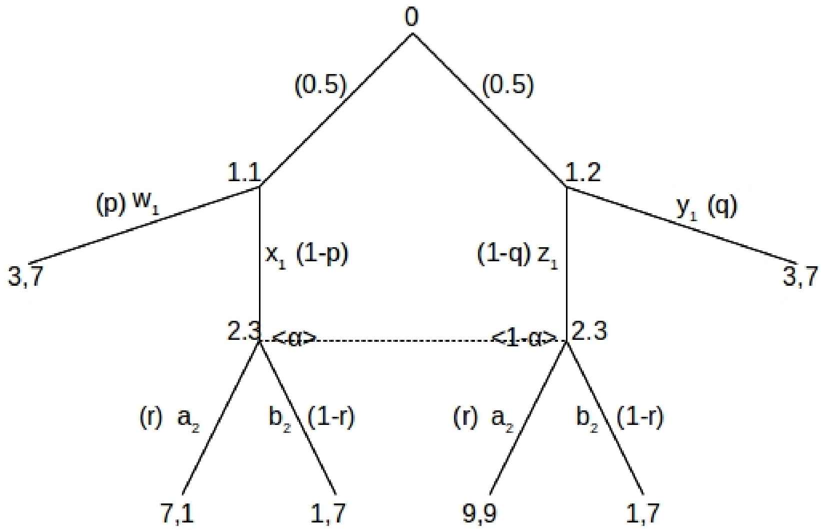
and $\sigma_2 = 5/7[L] + 2/7[R]$.

Figure 1: Player 1's Best Response, Problem 1c




# Problem 2

Consider the following extensive game, in which the result of the initial chance node can be observed by Player 1 but not by Player 2, who would not even get to move if 1 chose $w_1$ or $y_1$.




**(a)** Show the normal representation of this game in strategic form.

**(b)** Find a sequential equilibrium in which the probability of Player 2 getting to move is strictly positive. Explicitly show all move probabilities and belief probabilities.

**(c)** Consider the pure-strategy Nash equilibrium of this game, in which the probability of 2 getting to move is zero. Do there exist belief probabilities that would make this a sequential equilibrium? If so, find the set of all belief probabilities for the left-hand "2.3" node (following 1.1 and $x_1$) that Player 2 could have in a sequential equilibrium with this pure strategy profile.

***Solution.***

**(a)** The normal representation in strategic form is as follows:

Player 1 (rows), Player 2 (columns)

| | $a_2$ | $b_2$ |
|---|---|---|
| $w_1y_1$ | $3,7$ | $3,7$ |
| $w_1z_1$ | $6,8$ | $2,7$ |
| $x_1y_1$ | $5,4$ | $2,7$ |
| $x_1z_1$ | $8,5$ | $1,7$ |

where we notice that $([w_1y_1],[b_2])$ is a PSNE.

**(b)** We want to find a sequential equilibrium where $q,p \neq 1$. We first consider Player 2's strategy at 2.3. Since $EU_2[[a_2]|2.3] = \alpha + 9(1-\alpha)$ and $EU_2[[b_2]|2.3] = 7\alpha + (1-\alpha)7 = 7$, then 2 plays $a_2$ if and only if $\alpha < 1/4$. Thus, Player 2's strategy, satisfying sequential rationality, will be

$$\sigma_2 = \begin{cases} a_2 & \text{if } \alpha < 1/4 \\ \text{mix} & \text{if } \alpha = 1/4 \\ b_2 & \text{if } \alpha > 1/4 \end{cases}.$$

If Player 2 played $[b_2]$ for sure then Player 1 would play $w_1 y_1$ for sure, contradicting our hypothesis of this equilibrium. If Player 2 played $[a_2]$ for sure, then Player 1 would play $[x_1 z_1]$ for sure, so by Baye's consistency, we would have

$$\alpha = \frac{1/2(1-p)}{1/2(1-p) + 1/2(1-q)} = 1/2$$

but then, by sequential rationality, Player 2 would want to play $[b_2]$, contradicting our hypothesis that $[a_2]$ was played for sure. Thus, Player 2 must mix between $[a_2]$ and $[b_2]$.

If Player 1 mixed at both 1.1 and 1.2, then

$$\begin{aligned} EU_1[[w_1]|1.1, \sigma_2] = 3 = EU_1[[x_1]|1.1, \sigma_2] = 7r + 1(1-r) &\implies r = 1/3 \\ \text{and } EU_1[[y_1]|1.2, \sigma_2] = 3 = EU_1[[z_1]|1.2, \sigma_2] = 9r + 1(1-r) &\implies r = 1/4 \end{aligned}$$

which is a contradiction. Thus, we hypothesize that Player 1 mixes only at 1.1, so that $r = 1/3$, and that he plays $[z_1]$ for sure at 1.2, so that $q = 0$. Then by Baye's consistency, we have

$$\alpha = \frac{1/2(1-p)}{1/2(1-p) + 1/2(1-q)} \implies p = 2/3.$$

Thus, our sequential equilibrium is such that beliefs are $\alpha = 1/4$ with behavioral strategy

$$\begin{aligned} \sigma_1(1.1) &= 2/3[w_1] + 1/3[x_1] \\ \sigma_1(1.2) &= [z_1] \\ \sigma_2(2.3) &= 1/3[a_2] + 2/3[b_2]. \end{aligned}$$

**(c)** The PSNE we found in (a) is $([w_1 y_1], [b_2])$. From the previous part, we show that Player 2 chooses $[b_2]$ if and only if $\alpha > 1/4$.

# Problem 3

Consider a Bayesian game where Player 1 must choose T or B, Player 2 must choose L or R, each Player i privately learns a type $t_i$ that is drawn independently from a Uniform distribution on the interval from 0 to 1, and their payoffs $(u_1, u_2)$ depend on their types and choices as follows:

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | $L$ | $R$ |
| Player 1 | $T$ | $t_1, -t_1$ | $0, 0$ |
|  | $B$ | $0, 0$ | $t_2, -t_2$ |

That is, the stakes at $(T, L)$ and $(B, R)$ are drawn independently from the Uniform[0,1] distribution, but only Player 1 knows the stakes at $(T, L)$, and only Player 2 knows the stakes at $(B, R)$.

**(a)** A student suggested that "Player 2 should choose [R] if $t_2 \leq 2/3$, but 2 should choose [L] if $t_2 > 2/3$." Show a strategy for Player 1 that would be his best response to this strategy for Player 2.

**(b)** Show a strategy for Player 2 that would be her best response to the strategy for Player 1 that you found in part (a).

**(c)** Characterize a Bayesian equilibrium of this game in terms of two parameters (you may call them $\theta_1$ and $\theta_2$) that satisfy a system of two equations. You should sow these two equations in a form which could be solved numerically, but DO NOT attempt to solve them here. Be sure to fully characterize the player's equilibrium strategies in terms of these two parameters.

***Solution.***

**(a)** For Player 1, the payoff from choosing $[T]$ is $1/3t_1 + 2/3 \cdot 0 = 1/3t_1$ and the payoff from choosing $[B]$ is $0 + E[t_2|t_2 \leq 2/3] \cdot 2/3 = (1/3) \cdot (2/3) = 2/9$ so that if $1/3t_1 > 2/9 \implies t > 2/3$ then Player 1 chooses $[T]$. Thus, Player 1's best response to Player 2's strategy would be

$$B(t_1|\sigma_2) = \begin{cases} [T] & \text{if } t_1 > 2/3 \\ [B] & \text{if } t_1 < 2/3 \end{cases}.$$

**(b)** Player 2's payoff to playing $[L]$ given Player 1's strategy above is $EU_2[[L]|\sigma_1] = -E[t_1|t_1 > 2/3]P(t_1 > 2/3) + 0 = -5/18$. His payoff to playing $[R]$ is $EU_2[[R]|\sigma_1] = 0 - t_2 \cdot 2/3 = -2/3t_2$ which implies that Player 2's best response to Player 1's strategy above is

$$B(t_2|\sigma_1) = \begin{cases} [L] & \text{if } t_2 > 5/12 \\ [R] & \text{if } t_2 < 5/12 \end{cases}$$

so that the strategy given to Player 2 in part (a) cannot hold in equilibrium.

**(c)** Because both players' payoffs satisfy increasing differences we hypothesize that they play a cutoff strategy whereby Player 1 chooses $[T]$ if $t_1 > \theta_1$ and $[B]$ if $t_1 < \theta_1$, and Player 2 chooses $[L]$ if $t_2 > \theta_2$ and $[R]$ if $t_2 < \theta_2$. Moreover, at the cutoff they will be indifferent to their choices. Thus, the two equations come from setting expected payoffs equal for both Players at the cutoff:

$$\begin{aligned} & EU_1[[B]|\sigma_2, t_1 = \theta_1] = E[t_2|t_2 \leq \theta_2]P(t_2 \leq \theta_2) + 0 = 1/2\theta_2^2 = \\ & EU_1[[T]|\sigma_2, t_1 = \theta_1] = \theta_1 P(t_2 \geq \theta_2) + 0 = \theta_1(1-\theta_2) \\ \text{and } & EU_2[[L]|\sigma_1, t_2 = \theta_2] = -E[t_1|t_1 > \theta_1]P(t_1 > \theta_1) + 0 = -(1+\theta_1)(1-\theta_1)/2 = \\ & EU_2[[R]|\sigma_1, t_2 = \theta_2] = 0 - \theta_2 P(t_1 < \theta_1) = -\theta_1\theta_2. \end{aligned}$$

# Problem 4

Consider a symmetric game where each player $i \in \{1, 2\}$ chooses an action in $\{\alpha_i, \beta_i, \gamma_i\}$ and their payoffs $(u_1, u_2)$ depend on their actions and a state $\omega$ as follows:

| Player 1 \ Player 2 | $\alpha_2$ | $\beta_2$ | $\gamma_2$ |
|---|---|---|---|
| $\alpha_1$ | $3, 3$ | $8, 7$ | $2, \omega$ |
| $\beta_1$ | $7, 8$ | $4, 4$ | $2, \omega$ |
| $\gamma_1$ | $\omega, 2$ | $\omega, 2$ | $\omega, \omega$ |

Here $\omega$ is a state variable drawn form a Uniform[0,5] distribution. Each Player i observes a type $t_i = \omega + \eta_i$ where each $\eta_i$ is an independent random variable drawn from a Uniform$[-\epsilon, \epsilon]$ distribution. Here $\epsilon$ is a given parameter of the game which satisfies $0 < \epsilon < 0.1$ and is common knowledge among the players.

**(a)** Show a symmetric Bayesian equilibrium of this game in which the optimal actions would be different for different types. Be sure to fully describe the strategy that the players are symmetrically using this Bayesian equilibrium.

**(b)** Explain why, unlike some examples of global games, this game actually has another symmetric Bayesian equilibrium in which the player's behavior is independent of their types. How could you change one parameter of this example to eliminate this equilibrium and get a global game with the usual uniqueness result?

***Solution.***

**(a)** We want a symmetric equilibrium that depends on the type. From the looks of the game, we might suspect that for large types, the best choice would $\gamma$. We suppose there is a cutoff $\theta$ for which $t_i > \theta$ implies choosing $[\gamma_i]$ and for $t_i < \theta$ i plays a mixed strategy $\sigma_i = r[\alpha_i] + (1-r)[\beta_i]$ for some $r \in (0, 1)$. We can solve for r by solving the subgame

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | (r) $\alpha_2$ | $\beta_2$ |
| Player 1 | (r) $\alpha_1$ | $3, 3$ | $8, 7$ |
|  | $\beta_1$ | $7, 8$ | $4, 4$ |

which has a mixed strategy equilibrium with $r = 1/2$. Thus we have

$$\sigma_i = \begin{cases} 1/2[\alpha_i] + 1/2[\beta_i] & \text{if } t_i < \theta \\ [\gamma_i] & \text{if } t_i > \theta \end{cases}.$$

To solve for the cutoff we rewrite the game with this given strategy $t_i < \theta$ denoted by $\tilde{\sigma}_i$:

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | $\tilde{\sigma}_2$ | $\beta_2$ |
| Player 1 | $\tilde{\sigma}_1$ | $5.5, 5.5$ | $2, \omega$ |
|  | $\beta_1$ | $\omega, 2$ | $\omega, \omega$ |

.

At the cutoff, i must be indifferent between $[\tilde{\sigma}_i]$ and $[\gamma_i]$. Thus, we have

$$EU_i[[\tilde{\sigma}_i]|t_i = \theta] = 11/2 \cdot P(t_{-i} < \theta | t_i = \theta) + 2P(t_i > \theta | t_i = \theta) = 11/2 \cdot (1/2) + 2(1/2) = 15/4$$
$$EU_i[[\gamma_i]|t_i = \theta] = \theta$$

so that our equilibrium strategy is $\sigma_i$ (as defined above) using the cutoff $\theta = 15/4$.

**(b)** We could try the strategy $\sigma_i = 1/2[\alpha_i] + 1/2[\beta_i]$ and show that no individual has an incentive to deviate. But this result depends on the endpoint of the distribution of $\omega$. If the endpoint were larger (greater than 5), this may produce incentives for the players to deviate from this strategy.