Price Theory II Problem Set 6

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2.

Part a

Consider the case where $b_{-i} > v_i$, then regardless of the reservation price player i would never want to bid in such a way as to win the auction. Since, winning would result in a strictly negative pay-off $v_i - \max\{b_{-i}, p_r\}$. Player i can guarantee not winning by bidding $b_i = v_i$, but so can any strategy $b_i < b_{-i}$. Since all these strategies give the same pay-off as $b_i = v_i$, bidding your own value is not a strictly dominant strategy (just weakly dominant like question 1).

Part b

Since $10 \leq v_i \leq 20, \forall i \in \{1, 2\}$, bidding \$100 is strictly dominated for both players.

In all cases player i makes a strictly negative expected pay-off regardless of p_r and b_{-i},

1. Case $b_{-i} = 100$: In this case the auction will be necessarily settled with a coin flip. Therefore, $\mathbb{E}[u_1|b_{-i} = 100] = \frac{1}{2} \times v_i - \frac{1}{2} \times 100 + \frac{1}{2} \times 0 = \frac{1}{2}(v_i - 100) < 0$. Therefore, player i would be strictly better off not winning.

2. Case $b_{-i} < 100$: Player i has the expected pay-off $\mathbb{E}[u_1|b_{-i} < 100] = \left(\frac{99}{100} + \frac{1}{2} \times \frac{1}{100}\right) v_i - \frac{99}{100} \times 100 + \frac{1}{2} \times \frac{1}{100} \times 100 - \frac{1}{2} \times \frac{1}{100} \times 0 = \frac{199}{200}(v_i - 100) < 0$. Like before, player i would be strictly better off by not winning.

Now consider the strategy $b_i = 0$ the pay-offs for this strategy are as follows,

1. Case: $b_{-i} = 0$. $\mathbb{E}[u_1|b_{-i} = 0] = \frac{1}{100} \times \frac{1}{2}(v_i) > 0$

2. Case: $b_{-i} > 0$. $\mathbb{E}[u_1|b_{-i} = 0] = 0$

Since playing $b_i = 100$ gives strictly negative expected payoffs whereas playing $b_i = 0$ always gives weakly positive payoffs, playing $b_i = 0$ strictly dominates playing $b_i = 100$. Therefore, we can eliminate $b_i = 100$ from the strategy space of both players.

Part c

Notice the strict dominance argument above relied on $b_i > v_i$. Therefore, applying it iteratively and excluding strategies, we reduce player 1's set of strategy set to $s_1 = \{0, 1, \ldots, 20\}$. We can't reduce this further since all of these bids are best responses to $b_2 = 20$. In turn, we can reduce player 2's set to $s_2 = \{0, 1, \ldots, 19\}$. At this point, we can no longer appeal to the argument in part b.

Given the new reduced strategy space, we have that b_2 is always less than or equal to 19. Therefore the strictly dominant strategy for player 1 is $b_1 = 20$—the expected pay-off of player 1 playing $b_1 = 20$ is strictly greater than $b_1 < 20$. With this final strict dominance argument, we are left with the following strategy space,

$$b_1 \in \{20\} \quad \text{and} \quad b_2 \in \{0, 1, \ldots, 19\}$$

4.

Consider the two person game in the figure below. Show that the order in which weakly dominated strategies are eliminated can make a difference to the set of strategies that remain.

1\2	L	R
U	$1, 1$	$2, 1$
D	$1, 2$	$0, 0$

Let's investigate two orders according to which we eliminate strictly dominated strategies.

Starting from player 1: player 1 strictly prefers U to D because the payoff associated to U is always $\geq$ the payoff associated to D, with at least one strict inequality : $\pi_1(U, L) = \pi_1(D, L); \pi_1(U, R) > \pi_1(U, R)$. The set of strategies remaining after eliminating D are

1\2	L	R
U	$1, 1$	$2, 1$

Starting now from player 2: player 2 strictly prefers L to R: $\pi_2(U, L) = \pi_2(U, R); \pi_2(D, L) > \pi_2(D, R)$. The set of strategies remaining after eliminating R are

1\2	L
U	$1, 1$
D	$1, 2$

We therefore end up with 2 different sets of strategies.

The property at play here is the indifference of player 1 between U and D if player 2 chooses L, and the indifference of player 2 between L and R if player 1 chooses U. As a result, one can note that R was a weakly dominated strategy for player 2 in the original game, but is not anymore once we eliminated player 1's weakly dominated strategy. Similar logic for player 1's D strategy.

6.

Two hunters are on a stag hunt. They split up in the forest and each have two strategies: hunt for a stag (S), or give up the stag hunt and instead hunt for rabbit (R). If they both hunt for a stag, they will succeed and each earn a payoff of 9. If one hunts for stag and the other gives up and hunts for rabbit, the stag hunter receives 0 and the rabbit hunter 8. If both hunt for rabbit then each receives 7. Compute all Nash equilibria for this game, called "The Stag Hunt," depicted below. Which of these equilibria do you think is "most likely" to occur? Why? Note that no such distinctions are provided by the concept of Nash equilibrium.

1\2	S	R
S	$9, 9$	$0, 8$
R	$8, 0$	$7, 7$

....

Recall the definition: **Pure Strategy Nash Equilibrium**: Given a strategic form game $G = (S_i, u_i)_{i=1}^N$, the joint strategy $\hat{s} \in S$ is a pure strategy Nash equilibrium of G if for each player i, $u_i(\hat{s}) \geq u_i(s_i, \hat{s}_{-i})$ for all $s_i \in S_i$.

We can check all pure-strategy $s \in S$, since the space is so small.

(i) (S, S): For player i, deviating to R results in less utility $(8 < 9)$. <u>So (S, S) is a Nash equilibrium</u>.

(ii) (R, S): Player 1 can improve by deviating to S $(9 > 8)$.

(iii) (S, R): Player 2 can improve by deviating to S $(9 > 8)$.

(iv) (R, R): For player i, deviating to S results in less utility $0 < 7$. <u>So (R, R) is a Nash equilibrium</u>.

Now recall the definition: **Mixed Strategy Nash Equilibria**: Given a finite strategic form game $G = (S_i, u_i)_{i=1}^N$, a joint strategy $\hat{m} \in M$ is a Nash equilibrium of G if for each player i, $u_i(\hat{m}) \geq u_i(m_i, \hat{m}_{-i})$ for all $m_i \in M_i$.

The set of mixed strategies can be parameterized by $(p, q) \in [0, 1]^2$, where p is the probability of player 1 choosing S, and q the probability of player 2 choosing S. If player 2 pursues a mixed strategy, player 1 looks for p to maximize

$$\begin{aligned} &9pq + 8(1-p)q + 7(1-p)(1-q) \\ \Rightarrow &9q - 8q - 7(1-q) \\ &= 8q - 7 \end{aligned}$$

So if $q \neq \frac{7}{8}$, a pure strategy will be optimal, in which case, as we have shown above, a pure strategy is also optimal for player 2, so we do not have a mixed-strategy equilibrium. The symmetric nature of the problem means that interchanging the ps and qs gives the problem for player 2, given player 1. Therefore the only mixed strategy equilibrium is player i has distribution $(\frac{7}{8}, \frac{1}{8})$ over (S, R). This strategy will result in expected payoff

$$(\frac{7}{8})^2 9 + \frac{1}{8}(\frac{7}{8}8 + \frac{1}{8}7) = \frac{63}{8}$$

Intuitively, we think (S, S) is "most likely" to occur, since both players are able to deduce everything about the game, and thus will realize that the other player also knows everything, and since (S, S) makes both players the best off they can be, it will be optimal to play S. Note that this line of reasoning involves each player considering optimal play from the other player, and deducing the mutual interest of arriving at the higher utility NE.

The reason the concept of NE alone does not lead us to this conclusion is that NE are simply fixed points, in some sense. So if both players knew the other player was going to play R no matter what, they would be "stuck," and thus also play R. Analogously, if one player knew the other was going to play the mixed strategy of stag-hunting $\frac{7}{8}$ of the time, they would be indifferent between which pure strategy to play, but recognize that only playing the same mixed strategy would result in a NE. The same reasoning shows why (S, S) is a fixed point, but the dominance of outcomes (not a Nash concept) is what could lead to mutual stag-hunting.

8.

The table below presents the pay-off for the three firms for all the pure strategies. The best response strategies are in bold. There are four pure strategies Nash equilibria, namely, $\{(D_1, P_2, P_3), (P_1, D_2, P_3), (P_1, P_2, D_3), (D_1, D_2, D_3)\}$.

$1\backslash(2,3)$	$(\mathrm{P}_2, \mathrm{P}_3)$	$(\mathrm{D}_2, \mathrm{P}_3)$	$(\mathrm{P}_2, \mathrm{D}_3)$	$(\mathrm{D}_2, \mathrm{D}_3)$
P_1	$-1, -1, -1$	$\mathbf{-1, 0, -1}$	$\mathbf{-1, -1, 0}$	$-4, -3, -3$
D_1	$\mathbf{0, -1, -1}$	$-3, -3, -4$	$-3, -4, -3$	$\mathbf{-3, -3, -3}$

To find all the mixed strategy Nash equilibria, we need to consider cases where one agent randomises, two agents randomise, and all agents randomise. We begin with the latter case by assuming all the players randomise over P_i with probability π_i. Due, to the symmetry of pay-offs and strategies, we can restrict our search of mixed strategy to $\pi_1 = \pi_2 = \pi_3 = \pi$. The mixed strategy Nash equilibria make agents indifferent between their two strategies (equalize expected payoff from strategy). Therefore, in equilibria π is such that,

$$\begin{aligned} \mathbb{E}[u_i|P_i] = \mathbb{E}[u_i|D_i] \quad \forall i \in \{1, 2, 3\} \\ 3(\pi + \pi - \pi^2) - 6 = 3\pi^2 - 3 \\ 6\pi^2 - 6\pi + 1 = 0 \\ \implies \pi = \frac{1}{2} + \frac{1}{2\sqrt{3}} \text{ or } \frac{1}{2} - \frac{1}{2\sqrt{3}} \end{aligned}$$

Now lets consider the case where two players play pure strategies and one player plays a mixed strategy. Given the pay-off matrix above, it should be clear to see that the best response for the player randomising is to play the trivial mixed strategy i.e. pure strategy.

Therefore this case gives us the four pure strategy equilibrium derived above.

Finally, lets consider the case where two players randomise and one does not. WLOG suppose player 1 chooses either $\pi_1 \in \{0, 1\}$. First lets consider the case $\pi_1 = 0$ (player 1 plays D_1), then the indifference condition of player 2 and 3 imply that $\pi_2 = \pi_3 = \frac{1}{3}$. However, player 1's best response to this strategy is not to play D_i therefore it cannot be an equilibrium,

$$\mathbb{E}[u_1|P_1] = -1\left(\frac{1}{9} + \frac{2}{9} + \frac{2}{9} + 4\frac{4}{9}\right) = -\frac{7}{3} > -\frac{8}{3} = -3\left(\frac{2}{9} + \frac{2}{9} + \frac{4}{9}\right) = \mathbb{E}[u_1|D_1]$$

Now consider the case $\pi_1 = 1$ (player 1 plays P_1). In this case the indifference condition implies the following best responses $\pi_2 = \pi_3 = \frac{2}{3}$. Since, the best response of player 1 to this strategy is to play P_1, this is another mixed strategy Nash equilibrium,

$$\mathbb{E}[u_1|P_1] = -1\left(\frac{4}{9} + \frac{2}{9} + \frac{2}{9} + 4\frac{1}{9}\right) = -\frac{4}{3} > -\frac{5}{3} = -3\left(\frac{2}{9} + \frac{2}{9} + \frac{1}{9}\right) = \mathbb{E}[u_1|P_2]$$

Therefore in addition to the four pure strategy equilibria we have five mixed strategy equilibria,

$$\Bigg\{\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}}\right); \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right);$$
$$\left(1, \frac{2}{3}, \frac{2}{3}\right); \left(\frac{2}{3}, 1, \frac{2}{3}\right); \left(\frac{2}{3}, \frac{2}{3}, 1\right)\Bigg\}$$