Price Theory II
Problem Set 1 (Odd)
Solutions

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Problem 1

Question. *Prove that if $\succeq$ is a preference relation on $\mathbb{R}^l_+$, then*

a) $\sim$ and $\succ$ are transitive

b) $x \succ y$ and $y \sim z$ imply $x \succ z$

c) for all $x, y \in \mathbb{R}^l_+$, exactly one of $x \succ y$, $x \sim y$, $y \succ x$

Solution. Note that by definition of a preference relation, $\succeq$ is complete and transitive. $\sim$ and $\succ$ are defined as follows:

- $x \sim y \Leftrightarrow x \succeq y \,\&\, y \succeq x$
- $x \succ y \Leftrightarrow x \succeq y \,\&\, y \not\succeq x$

(a) $\sim$ is transitive if $\forall x, y, z \in \mathbb{R}^l_+$, $x \sim y \,\&\, y \sim z \Rightarrow x \sim z$. Suppose that $x \sim y \,\&\, y \sim z$. From definition of $\sim$, $x \sim y \,\&\, y \sim z \Rightarrow x \succeq y \,\&\, y \succeq z$. By transitivity of $\succeq$, then, $x \succeq z$. Moreover, again by definition of $\sim$, $x \sim y \,\&\, y \sim z \Rightarrow z \succeq y \,\&\, y \succeq x$. By transitivity of $\succeq$, then $z \succeq x$. $x \succeq z \,\&\, z \succeq x \Rightarrow x \sim z$.

$\succ$ is transitive if $\forall x, y, z \in \mathbb{R}^l_+$, $x \succ y \,\&\, y \succ z \Rightarrow x \succ z$. By definition of $\succ$, $x \succ y \,\&\, y \succ z \Rightarrow x \succeq y \,\&\, y \succeq z$. By transitivity of $\succeq$, $x \succeq z$. To prove that $x \succ z$, it remains to show that $z \not\succeq x$. Suppose by contradiction that $z \succeq x$. Since $x \succeq y$, by transitivity of $\succeq$, $z \succeq y$. But since $y \succ z$, $z \not\succeq y$. Contradiction. We conclude that $z \not\succeq x$.

(b) Using the definitions of $\sim$ and $\succ$, $x \succ y \,\&\, y \sim z \Rightarrow x \succeq y \,\&\, y \succeq z$. By transitivity of $\succeq$, $x \succeq z$. To show that $x \succ z$ we need now to show that $z \not\succeq x$. Suppose by contradiction that $z \succeq x$. Then, $x \sim z$.

Notice now that $\sim$ is symmetric. That is, $\forall a, b \in \mathbb{R}^l_+$, $a \sim b \Leftrightarrow b \sim a$. Indeed, suppose that $a \sim b \Leftrightarrow b \succeq a \,\&\, a \succeq b \Leftrightarrow b \sim a$. Hence, $y \sim z \Rightarrow z \sim y$. By part (a), $\sim$ is transitive, hence $x \sim z \,\&\, z \sim y \Rightarrow x \sim y \Rightarrow y \succeq x$. Contradiction, as $x \succ y$. So we conclude that $z \not\succeq x$.

(c) By completeness of $\succeq$, for all $x, y \in \mathbb{R}^l_+$, either $x \succeq y$ or $y \succeq x$. There are three possible cases when this can be true:

(i) $x \succeq y \,\&\, y \succeq x$ In this case, by definition, $x \sim y$. Notice that $x \not\succ y$ as $y \succeq y$ and $y \not\succ x$ as $x \succeq y$.

(ii) $x \succeq y \,\&\, y \not\succeq x$ In this case, by definition $x \succ y$. Notice that $y \not\succ x$ and $x \not\sim y$ as $y \not\succeq x$.

(iii) $x \not\succeq y \,\&\, y \succeq x$ The case is symmetrical to the previous one. Only $y \succ x$ is true.

Problem 3

Question. *Let the utility function $U(\cdot)$ represent the preference relation $\succeq$ on $\mathbb{R}^l_+$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be strictly increasing. Prove that $\phi(U(\cdot))$ represents $\succeq$. Show that this is false if the word "strictly" is deleted.*

Solution. Since $U(\cdot)$ represents $\succeq$, $\forall x, y \in \mathbb{R}^l_+$, $x \succeq y \Leftrightarrow U(x) \geq U(y)$. Since $\phi(\cdot)$ is **strictly** increasing, $U(x) \geq U(y) \Leftrightarrow \phi(U(x)) \geq \phi(U(y))$. Hence, $x \succeq y \Leftrightarrow \phi(U(x)) \geq \phi(U(y))$, but then $\phi(U(\cdot))$ represents $\succeq$ by definition.

Suppose now that ϕ is not strictly increasing. Then, $\phi(U(\cdot))$ does not necessarily represents $\succeq$. Consider the following example. Suppose that $l = 1$. And define $\succeq$ as follows $x \succeq y \Leftrightarrow x \geq y$. Clearly, by construction $U(x) = x$ represents these preferences. Consider $\phi(x) = 0$. It is weakly increasing. However, $\phi(U(0)) \geq \phi(U(1))$ and $1 \succeq 0$. So $\phi(U(\cdot))$ does not represent $\succeq$.

Problem 5

Question. *Let $u(x, y) = \sqrt{xy}$. Show that u_x is strictly decreasing in x; i.e. that u exhibits strictly diminishing marginal utility for x. Exhibit a utility function representing the same preferences, but not satisfying strictly diminishing marginal utility for x.*

Solution.

$$(u_x)'_x = -\frac{1}{4}\sqrt{y}\frac{1}{\sqrt{x^3}} < 0$$

which implies that u_x is indeed decreasing in x.

By Problem 3, any strictly increasing transformation of $u(\cdot, \cdot)$ represents the same preferences. Take $g(x) = x^3$. It must be the case that function $v(\cdot, \cdot)$, such that: $v(x, y) = g(u(x, y)) = y^{\frac{3}{2}}x^{\frac{3}{2}}$ represents the same preferences as $u(x, y)$. But $v_x = y^{\frac{3}{2}}\frac{3}{2}\sqrt{x}$, $(v_x)'_x = \frac{3}{4}y^{\frac{3}{2}}\frac{1}{\sqrt{x}} > 0$, so v_x is no longer decreasing in x.

Problem 7

Question. *Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of consumption bundles in $\mathbb{R}^l_+$ converging to x and y respectively. Prove that if $x^n \sim y^n$ for all n and $\succeq$ is complete, transitive, and continuous on $\mathbb{R}^l_+$, then $x \sim y$.*

Solution. Suppose, first, by a contradiction that $x \succ y$.
Observation 1. $\succ (y) \cup \prec (x) = \mathbb{R}^l_+$.

Notice that $\forall z \in \mathbb{R}^l_+$, if $z \succeq x$, then, as $x \succ y$, $z \succ y$ (use the results from Problem 1 to see this). Hence, $\succeq (x) \subseteq \succ (y)$. By completeness, $\succeq (x) \cup \prec (x) = \mathbb{R}^l_+$, but then clearly $\succ (y) \cup \prec (x) = \mathbb{R}^l_+$ as $\succ (y)$ contains $\succeq (x)$.
Observation 2. $\forall x \in \mathbb{R}^l_+$, the sets $\succ (x)$ and $\prec (x)$ are open.

By continuity of $\succeq$, both sets $\succeq (x)$ and $\preceq (x)$ are closed, hence $\succ (x)$ and $\prec (x)$ are open as complements to closed one.
Observation 3. $\succ (y) \cap \prec (x) = \emptyset$

Indeed, suppose by contradiction that $\exists z \in \mathbb{R}^l_+$, such that $x \succ z$ and $z \succ y$. Since $x \in \succ (z)$ and $\succ (z)$ is open, there exists a neighbourhood of x, $\delta(x) \subset \succ (z)$. Since $x_n \to x$, $\exists N_x$, such that $\forall n > N_x$, $x_n \in \delta(x) \subset \succ (z)$. Similarly, $\exists N_y : \forall n > N_y$, $y_n \in \delta(y) \subset \prec (z)$. But then, for any $n > \max\{N_x, N_y\}$, $x_n \succ z$ and $z \succ y_n$. By problem 1, $\succ$ is transitive, hence $\forall n > \max\{N_x, N_y\}$ $x_n \succ y_n$. But $x_n \sim y_n$. Contradiction.
Observation 4. The sets $\succ (y)$ and $\prec (x)$ are non-empty.

By assumption, $y \in \prec (x)$ and $x \in \succ (y)$.
But then, $\mathbb{R}^l_+$ can be partitioned into two open disjoint non-empty sets. This is impossible, as $\mathbb{R}^l_+$ is a connected space. By a symmetric argument we rule out the case that $y \succ x$. Hence, by Problem 1, $x \sim y$.

Problem 9

Question. *Consider the preferences on $\mathbb{R}^2_+$ defined by the discontinuous utility function*

$$u(x, y) = \begin{cases} 2 + xy, & \text{if } xy > 1 \\ 1 + xy, & \text{if } xy = 1 \\ xy, & \text{if } xy < 1 \end{cases}$$

Are these preferences continuous? If not, why not? If so, display a continuous utility function representing them.

Solution.

Observation. Suppose that preference relation $\succeq$ on $\mathbb{R}^l_+$ can represented by a continuous function $U : \mathbb{R}^l_+ \to \mathbb{R}$. Then, $\succeq$ is continuous.

Since $U(\cdot)$ represents the preferences, $\succeq(x) = \{y : U(y) \geq U(x)\}$ and $\preceq(x) = \{y : U(y) \leq U(x)\}$. Consider sets $\geq (U(x)) = \{a \in \mathbb{R} : a \geq U(x)\}$ and $\leq (U(x)) = \{a \in \mathbb{R} : a \leq U(x)\}$. Both of $\geq (U(x))$ and $\leq (U(x))$ are closed. As $U(\cdot)$ is continuous, a preimage of any closed set is closed. That is, the sets $\succeq(x)$ and $\preceq(x)$ are closed. So $\succeq$ is continous.

Hence, it will saffice to show that $\succeq$ can be represented by a continuous function. A candidate function for this is $v(x, y) = xy$. Let us show that $v(\cdot)$ indeed represents $\succeq$. To do that, it is enough to show that $\forall(x, y); (p, q) \in \mathbb{R}^2_+$, $v(x, y) \geq v(p, q) \Leftrightarrow u(x, y) \geq u(p, q)$.

Suppose that $v(x, y) \geq v(p, q)$, or that $xy \geq pq$.

Case 1. If $xy < 1$, then, $u(x, y) = xy \geq pq = u(p, q)$.

Case 2. If $xy = 1$, then $u(x, y) = xy + 1$. $u(p, q) = pq + \mathbb{1}\{pq = 1\}$, so $u(x, y) \geq u(p, q)$.

Case 3. If $xy > 1$, then $u(x, y) = xy + 2$. $u(p, q) = pq + 2 - \mathbb{1}\{pq = 1\} - \mathbb{1}\{pq < 1\}$. So either way $u(x, y) \geq u(p, q)$.

For now have established that $v(x, y) \geq v(p, q) \Rightarrow u(x, y) \geq u(p, q)$.

Now suppose that $u(x, y) \geq u(p, q)$. This is possible in several cases.

Case 1. $xy > 1 > pq$. Then, obviously, $xy \geq pq$.

Case 2. $1 = xy > pq$, then, again, $xy \geq pq$.

Case 3. $xy, pq > 1$ or $xy = 1 = pq$ or $xy, pq < 1$ and $xy \geq pq$.

So $u(x, y) \geq u(p, q) \Rightarrow v(x, y) \geq v(p, q)$.

Problem 11

Question. *Suppose that $\succeq$ is a preference relation on $\mathbb{R}^l_+$. Show that the continuity axiom implies that for all $x, y, z \in \mathbb{R}^l_+$, such that $x \succ y \succ z$, the line segment joining x, z must contain a point that is indifferent to y.*

To establish the result, it suffices to show that for all such x, y, z, $\exists\bar{\alpha} \in [0, 1]$, such that $\bar{\alpha}x + (1 - \bar{\alpha})z \sim y$. Denote $l = \{\alpha \in [0, 1] : \alpha x + (1 - \alpha)z \preceq y\}$, $u = \{\alpha \in [0, 1] : \alpha x + (1 - \alpha)z \succeq y\}$.

Observation 1. Both l and u are non-empty.

Indeed, we know that $0 \in l$ and $1 \in u$.

Observation 2. Both l and u are closed.

Indeed, consider a sequence $\{\alpha^n\} \in l$, such that $\alpha^n \to \alpha$ (note that, if $\alpha^n \in [0, 1]$, then $\alpha \in [0, 1]$). Since $\preceq(y)$ is closed by continuity, $\alpha^n x + (1 - \alpha^n)z \preceq y$ implies $\alpha x + (1 - \alpha)z \preceq y$. Then, $\alpha \in l$. Similarly, using $\succeq(y)$ is closed, u is closed.

Observation 3. $l \cup u = [0, 1]$

The result follows completeness of $\succeq$.

Observation 4. $l \cap u \neq \emptyset$

Indeed, since $[0, 1]$ is connected it is not possible to partition $[0, 1]$ into two disjoint closed non-empty sets.

So we have established that $\exists\bar{\alpha}$, such that $\bar{\alpha}x + (1 - \bar{\alpha})z \sim y$.

Problem 13

Question. *For each of the utility functions below, calculate the ordinary (i.e. Marshallian), Slutsky compensated, and Hicksian compensated demand functions.*

(a) $u(x, y) = x^\alpha y^{1-\alpha}, 0 < \alpha < 1$

(b) $u(x, y) = x^a + y^a, 0 < a < 1$

(c) $u(x, y) = \min(x, y)$

(d) $u(x, y) = x + y$

Solution

Definitions

• $x(p, I)$ is **Marshallian** demand iff $\forall p \gg 0, I \geq 0$ it solves **Consumer's Problem**

$$\max_{x \in \mathbb{R}^l_+} \; u(x)$$
$$\text{s.t. } p \cdot x \leq I$$

• $x^h(p, u)$ is **Hicksian** demand iff $\forall p \gg 0, \forall u \in \mathcal{U}$ (where $\mathcal{U}$ is the image of $u(\cdot)$) it solves **Cost-Minimization Problem**

$$\min_{x \in \mathbb{R}^l_+} \; p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

• x^s is **Slutsky** demand iff $x^s(p, x^0) = x(p, p \cdot x^0)$.

(a) Marshallian Demand:

Observation 1 Notice, first, that any interior point provides greater utility than any corner point. From this, we can rule out corner solutions for all $I > 0$ (as any budget set with $I > 0$ contains at least one interior point). For $I = 0$, the solution is clearly $(0, 0)$ as this is a single feasible point.

Observation 2 Moreover, notice that for all interior bundles, the utility is strictly increasing in every argument. Hence, the budget constraint should bind at the optimum whenever $I > 0$.

To find the demand, set up a Lagrangian with λ — Lagrange multiplier for budget constraint.

$$\mathcal{L}(x, y, \lambda) = x^\alpha y^{1-\alpha} + \lambda(I - p_x x - p_y y)$$

Take FOCs (which are sufficient by Observation 1):

$$\begin{aligned}
&\left[\frac{\partial \mathcal{L}}{\partial x}\right] && \alpha \left(\frac{y}{x}\right)^{1-\alpha} = \lambda p_x \\
&\left[\frac{\partial \mathcal{L}}{\partial y}\right] && 1 - \alpha \left(\frac{x}{y}\right)^{\alpha} = \lambda p_y \\
&\frac{\left[\frac{\partial \mathcal{L}}{\partial x}\right]}{\left[\frac{\partial \mathcal{L}}{\partial y}\right]} && \frac{\alpha}{1-\alpha}\frac{y}{x} = \frac{p_x}{p_y}
\end{aligned}$$

From which, $p_y y = \frac{(1-\alpha)}{\alpha} p_x x$. From earlier, budget constraint should hold with equality. Hence,

$$p_x x + p_y y = p_x x + \frac{(1-\alpha)}{\alpha} p_x x = \frac{1}{\alpha} p_x x = I$$

Or

$$p_x x = \alpha I$$
$$p_y y = (1-\alpha) I$$

This is a well-known result for Cobb-Douglas utility function: in the optimum an individual spends α of her income on x, and $1 - \alpha$ share — on y. Finally, we get the demand function:

$$\Big(x(p_x, p_y, I),\, y(p_x, p_y, I)\Big) = \left(\frac{\alpha I}{p_x},\, \frac{(1-\alpha) I}{p_y}\right)$$

Hicksian Demand

Set up the Lagrangian for the cost-minimization problem, with μ — Lagrange multiplier for the utility-value constraint:

$$\mathcal{L}(x, y, \lambda) = -p_x x - p_y y + \mu(x^\alpha y^{1-\alpha} - u)$$

We can ignore corner solutions: the only u achievable in corners is 0. So FOCs are sufficient.

$$\begin{aligned}
&\left[\frac{\partial \mathcal{L}}{\partial x}\right] && p_x = \mu\alpha\left(\frac{y}{x}\right)^{1-\alpha} \\
&\left[\frac{\partial \mathcal{L}}{\partial y}\right] && p_y = \mu(1-\alpha)\left(\frac{x}{y}\right)^{\alpha} \\
&\frac{\left[\frac{\partial \mathcal{L}}{\partial x}\right]}{\left[\frac{\partial \mathcal{L}}{\partial y}\right]} && \frac{\alpha}{1-\alpha}\frac{y}{x} = \frac{p_x}{p_y}
\end{aligned}$$

The constraint is clearly binding in the optimum (else, we could decrease total expenditures by slightly decreasing consumption of each good).

$$x\left(\frac{1-\alpha}{\alpha}\frac{p_x}{p_y}\right)^{1-\alpha} = u$$

$$\Big(x^h(p_x, p_y, u),\, y^h(p_x, p_y, u)\Big) = \left(u\left(\frac{\alpha p_y}{(1-\alpha)p_x}\right)^{1-\alpha},\, u\left(\frac{(1-\alpha)p_x}{\alpha p_y}\right)^{\alpha}\right)$$

Slutsky Demand: Quite straightforward,

$$\Big(x(p_x, p_y, x^0, y^0),\, y(p_x, p_y, x^0, y^0)\Big) = \left(\frac{\alpha(p_x x^0 + p_y y^0)}{p_x},\, \frac{(1-\alpha)(p_x x^0 + p_y y^0)}{p_y}\right)$$

(b) Marshallian Demand

$$\mathcal{L}(x, y, \lambda) = x^a + y^a + \lambda(I - p_x x - p_y y)$$

Notice that the utility function is strictly concave, so that FOCs are sufficient. The budget constraint is binding, as the utility function is strictly increasing in each argument.

$$\begin{aligned}
&\left[\frac{\partial \mathcal{L}}{\partial x}\right] && a x^{a-1} - \lambda p_x = 0 \\
&\left[\frac{\partial \mathcal{L}}{\partial y}\right] && a y^{a-1} - \lambda p_y = 0 \\
&\frac{\left[\frac{\partial \mathcal{L}}{\partial x}\right]}{\left[\frac{\partial \mathcal{L}}{\partial y}\right]} && \left(\frac{x}{y}\right)^{a-1} = \frac{p_x}{p_y} \\
&\frac{x}{y} = \left(\frac{p_x}{p_y}\right)^{\frac{1}{a-1}} \\
&x = y\left(\frac{p_x}{p_y}\right)^{\frac{1}{a-1}}
\end{aligned}$$

Using the budget constraint:

$$p_x x + p_y y = y\left(p_x\left(\frac{p_x}{p_y}\right)^{\frac{1}{a-1}} + p_y\right) = I$$

So we may conclude that the Marshallian demand looks as follows:

$$\Big(x(p_x, p_y, I),\, y(p_x, p_y, I)\Big) = \left(\frac{I}{p_x\left(1 + \left(\frac{p_y}{p_x}\right)^{\frac{a}{a-1}}\right)},\, \frac{I}{p_y\left(1 + \left(\frac{p_x}{p_y}\right)^{\frac{a}{a-1}}\right)}\right)$$

Hicksian Demand

$$\mathcal{L}(x, y, \lambda) = -p_x x - p_y y + \mu(x^a + y^a - u)$$

$$\begin{aligned}
\left[\frac{\partial \mathcal{L}}{\partial x}\right] \qquad & -p_x + \mu a x^{a-1} = 0 \\
\left[\frac{\partial \mathcal{L}}{\partial y}\right] \qquad & -p_y + \mu a y^{a-1} = 0 \\
\frac{\left[\frac{\partial \mathcal{L}}{\partial x}\right]}{\left[\frac{\partial \mathcal{L}}{\partial y}\right]} \qquad & \left(\frac{x}{y}\right)^{a-1} = \frac{p_x}{p_y} \\
\frac{x}{y} = \left(\frac{p_x}{p_y}\right)^{\frac{1}{a-1}} & \\
x = y\left(\frac{p_x}{p_y}\right)^{\frac{1}{a-1}} &
\end{aligned}$$

Notice that FOCs are again sufficient, as $\lim_{x\to 0} ax^{a-1} = \lim_{y\to 0} ay^{a-1} = \infty$. By same reasoning as in part (a), the constraint is binding, so we get:

$$y^a\left(1 + \left(\frac{p_x}{p_y}\right)^{\frac{a}{a-1}}\right) = u$$

$$\Big(x^h(p_x, p_y, u),\, y^h(p_x, p_y, u)\Big) = \left(u^{\frac{1}{a}}\left(1 + \left(\frac{p_y}{p_x}\right)^{\frac{a}{a-1}}\right)^{-\frac{1}{a}},\, u^{\frac{1}{a}}\left(1 + \left(\frac{p_x}{p_y}\right)^{\frac{a}{a-1}}\right)^{-\frac{1}{a}}\right)$$

Slutsky Demand

$$\Big(x^s(p_x, p_y, x^0, y^0),\, y^s(p_x, p_y, x^0, y^0)\Big) = \left(\frac{p_x x^0 + p_y y^0}{p_x\left(1 + \left(\frac{p_y}{p_x}\right)^{\frac{a}{a-1}}\right)},\, \frac{p_x x^0 + p_y y^0}{p_y\left(1 + \left(\frac{p_x}{p_y}\right)^{\frac{a}{a-1}}\right)}\right)$$

(c) Marshallian Demand

Observation. $x(p_x, p_y, I) = y(p_x, p_y, I)$

Indeed, suppose by contradiction that for some $p_x > 0, p_y > 0, I$, this does not hold, e.g. $x(p_x, p_y, I) > y(p_x, p_y, I)$. Clearly, this is impossible for $I = 0$. Suppose, then, $I > 0$. Then, $\min\{x(p_x, p_y, I), y(p_x, p_y, I)\} = y(p_x, p_y, I)$. Consider the following perturbation: $\tilde{x}(p_x, p_y, I) = x(p_x, p_y, I) - \varepsilon$ and $\tilde{y}(p_x, p_y, I) = y(p_x, p_y, I) + \dfrac{p_x}{p_y}\varepsilon$ where $\varepsilon > 0$. Notice that $\tilde{x}(p_x, p_y, I), \tilde{y}(p_x, p_y, I)$ is feasible as $x(p_x, p_y, I), y(p_x, p_y, I)$ is feasible and for ε small enough, $\min\{\tilde{x}(p_x, p_y, I), \tilde{y}(p_x, p_y, I))\} = y(p_x, p_y, I) + \dfrac{p_x}{p_y}\varepsilon > y(p_x, p_y, I) = u(x(p_x, p_y, I), y(p_x, p_y, I))$. Hence, $x(p_x, p_y, I), y(p_x, p_y, I)$ is not a solution to Consumer's Problem. Contradiction. Similarly, we can rule out the case with $y(p_x, p_y, I) > x(p_x, p_y, I)$.

Now the consumer's problem boils down to a very easy one:

$$\begin{aligned}
&\max_{x\in\mathbb{R}} \; x \\
&\text{s.t. } x(p_x + p_y) \le I
\end{aligned}$$

Clearly, then, Marhallian demand is as follows:

$$\Big(x(p_x, p_y, I),\, y(p_x, p_y, I)\Big) = \left(\frac{I}{p_x + p_y},\, \frac{I}{p_x + p_y}\right)$$

Hicksian Demand In the same fashion, clearly, $x^h(p_x, p_y, I) = y^h(p_x, p_y, I)$ (else, the same level of utility can be achieved with a cheaper bundle). The cost-minimization problem boils to:

$$\min_{x \in \mathbb{R}} \; x(p_x + p_y)$$
$$\text{s.t. } x \geq u$$

It is easy to see that:

$$\Big(x^h(p_x, p_y, u),\, y^h(p_x, p_y, u)\Big) = (u,\, u)$$

Slutsky Demand

$$\Big(x(p_x, p_y, x^0, y^0),\, y(p_x, p_y, x^0, y^0)\Big) = \left(\frac{p_x x^0 + p_y y^0}{p_x + p_y},\, \frac{p_x x^0 + p_y y^0}{p_x + p_y}\right)$$

(d) Marshallian Demand

$$\mathcal{L}(x, y, \lambda) = x + y + \lambda(I - p_x x - p_y y)$$

Kuhn-Tucker Conditions:

$$\left[\frac{\partial \mathcal{L}}{\partial x}\right] \qquad 1 - \lambda p_x \begin{cases} \leq 0 \text{ if } x = 0 \\ = 0 \text{ if } x \in \left(0, \frac{I}{p_x}\right) \\ \geq 0 \text{ if } x = \frac{I}{p_x} \end{cases}$$

$$\left[\frac{\partial \mathcal{L}}{\partial y}\right] \qquad 1 - \lambda p_y \begin{cases} \leq 0 \text{ if } y = 0 \\ = 0 \text{ if } y \in \left(0, \frac{I}{p_y}\right) \\ \geq 0 \text{ if } y = \frac{I}{p_y} \end{cases}$$

From this, $x(p_x, p_y, I)$ and $y(p_x, p_y, I)$ can both be interior only when $p_x = p_y$. Otherwise, suppose that $x(p_x, p_y, I) = 0 \Rightarrow \lambda \geq \dfrac{1}{p_x}$. If $x(p_x, p_y, I) = 0 \Rightarrow$ (from budget constraint) $y(p_x, p_y, I) = \dfrac{I}{p_y}$. Hence, $\lambda \leq \dfrac{1}{p_y}$. This is only feasible when $\dfrac{1}{p_y} \geq \dfrac{1}{p_x} \Leftrightarrow p_x \geq p_y$. Similarly, another possible solution is $y(p_x, p_y, I) = 0$, $x(p_x, p_y, I) = \dfrac{I}{p_x}$ whenever $p_x \geq p_y$. So Marshallian Demand is:

$$\Big(x(p_x, p_y, I),\, y(p_x, p_y, I)\Big) = \begin{cases} \left(\dfrac{I}{p_x},\, 0\right) & \text{if } p_x < p_y \\ \left(0,\, \dfrac{I}{p_y}\right) & \text{if } p_y < p_x \\ \big\{(x, y) \in \mathbb{R}^2_+, \text{such that } p_x x + p_y y = I\big\} & \text{if } p_y = p_x \end{cases}$$

Notice that for $p_x = p_y$ the demand is not a singleton!

Hicksian Demand

$$\mathcal{L}(x, y, \lambda) = -p_x x - p_y y + \mu(x + y - u)$$

Kuhn-Tucker Conditions:

$$\left[\frac{\partial \mathcal{L}}{\partial x}\right] \qquad - p_x + \mu \begin{cases} \leq 0 \text{ if } x = 0 \\ = 0 \text{ if } x \in (0, u) \\ \geq 0 \text{ if } x = u \end{cases}$$

$$\left[\frac{\partial \mathcal{L}}{\partial y}\right] \qquad - p_y + \mu \begin{cases} \leq 0 \text{ if } y = 0 \\ = 0 \text{ if } y \in (0, u) \\ \geq 0 \text{ if } y = u \end{cases}$$

By same kind of derivations, we conclude:

$$\left(x^h(p_x, p_y, I),\ y^h(p_x, p_y, I)\right) = \begin{cases} (u,\ 0) \ \text{ if } p_x < p_y \\ (0,\ u) \ \text{ if } p_y < p_x \\ \left\{(x, y) \in \mathbb{R}^2_+, \text{such that } x + y = u\right\} \ \text{ if } p_y = p_x \end{cases}$$

Slutsky Demand

$$\left(x^s(p_x, p_y, x^0, y^0),\ y^s(p_x, p_y, x^0, y^0)\right) = \begin{cases} \left(\dfrac{p_x x^0 + p_y y^0}{p_x},\ 0\right) \ \text{ if } p_x < p_y \\ \left(0,\ \dfrac{p_x x^0 + p_y y^0}{p_y}\right) \ \text{ if } p_y < p_x \\ \left\{(x, y) \in \mathbb{R}^2_+, \text{such that } p_x x + p_y y = p_x x^0 + p_y y^0\right\} \ \text{ if } p_y = p_x \end{cases}$$

Price Theory II
Problem Set 1 (Even)
Solutions

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Problem 2

There are two commodities.

1. Sketch some preferences which have a bliss point and which can be represented by a continuous utility function. (A bliss point is a point which is preferred or indifferent to any other point.)
2. No matter what his income, Jones will demand fifteen units of commodity one provided he can afford it and provided the price of commodity 2 exceeds that of commodity one. Is this consistent with Axioms 1-5 and also Axioms 4' and 5'?

. . . . . .

Part (i) Suppose that an agent has a "bliss point" at $(\overline{x}, \overline{y}) \in \mathbb{R}^2_+$; i.e. agent's preference is such that

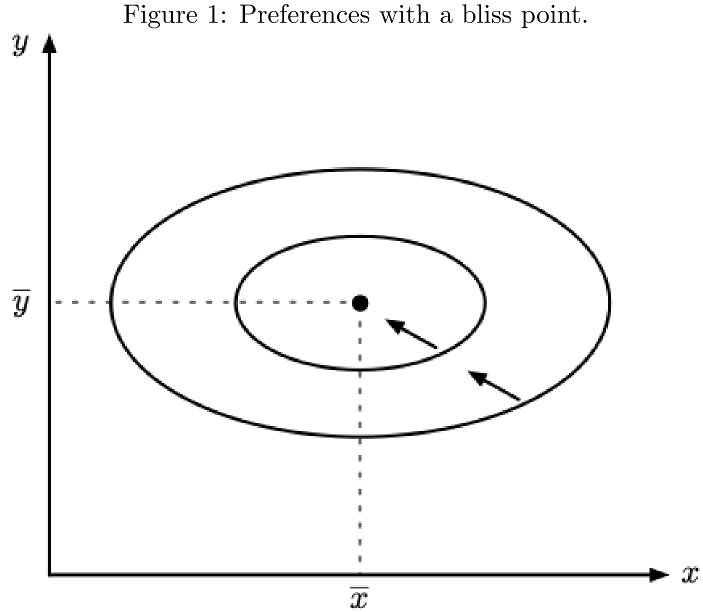
$$(x, y) \succsim (\tilde{x}, \tilde{y}) \Leftrightarrow d\left((x, y), (\overline{x}, \overline{y})\right) \leq d\left((\tilde{x}, \tilde{y}), (\overline{x}, \overline{y})\right),$$

where d is the metric on the space $\mathbb{R}^2_+$ (e.g. take it to be the Euclidean distance). This preference is represented by a continuous utility function

$$u(x, y) = -(x - \overline{x}_1)^2 - (x - \overline{y})^2.$$

The indifference curve is then a circle and the bliss point is given by the bundle (x, y). See the figure below.

Figure 1: Preferences with a bliss point.



Part (ii) Recall that Axioms are: A1. Completeness; A2. Transitivity; A3. Continuity; A4. Strict monotonicity; A5. Strict convexity; A4'. Local non-satiation; A5. Convexity.

We first show that Jones' behaviour cannot satisfy Axioms A1 to A5 simultaneously.

Claim 1. Jones' behaviour cannot satisfy Axioms A1 to A5 simultaneously.

Proof. By way of contradiction, suppose that Jones' behaviour satisfy Axioms A1 to A5.

By strict monotonicity (A4),

$$(16, a) \succsim (15, a) .$$

Let the price of commodity 1 be $p_1 = 1$ and $p_2 = n$. and suppose Jones' income is $M_n = 15 + an$. Note that the bundle $(16, a - 1/n)$ costs

$$16p_1 + \left(a - \frac{1}{n}\right) p_2 = 15 + an - 1 = M_n$$

so that this bundle is affordable for all n. Jones' preference imply that

$$(15, a) \succsim \left(16, a - \frac{1}{n}\right), \ \forall n > 1$$

(in fact, above holds with $\sim$). By continuity (A3) of $\succsim$,

$$(15, a) \succsim \lim_{n\to\infty} \left(16, a - \frac{1}{n}\right) \Rightarrow (15, a) \succsim (16, a) .$$

Using completeness (A1), and transitivity (A2), we have that

$$(16, a) \sim (15, a) .$$

But, by strict convexity (A5), for all $\lambda \in (0, 1)$,

$$(16\lambda + (1 - \lambda) 15, a) = (15 + \lambda, a) \succ (16, a) .$$

But this violates strict monotonicity (A4). Therefore, it is not possible satisfy Axioms 1–5 simultaneously. $\square$

Then Jones could be indifferent between any bundle that has $x_1 \geq 15$ for any given x_2 means that his indifference curve is linear in parts. Thus, his preference violates strict convexity (A5). However, if we were to replace A5 with A5' (convexity), Jones' preferences can satisfy Axioms A1, A2, A3, A4 and A5'. To be concrete, suppose

$$u(x_1, x_2) = \begin{cases} 15 \log x_1 + x_2 & \text{if } x_1 \leq 15 \\ 15 \log 15 + x_2 & \text{if } x_1 > 15 \end{cases} .$$

Alternatively, good one can become a "bad" once Jones consumers more than 15 units of it. In this case, his preference would violate strict monotonicity (A4). But we may replace it with A4' (LNS). To be concrete, suppose

$$u(x_1, x_2) = \begin{cases} 15 \log x_1 + x_2 & \text{if } x_1 \leq 15 \\ 15 \log 15 - (x_1 - 15)^2 & \text{if } x_1 > 15 \end{cases} .$$

Observe that both examples we found satisfies A1–3, A4' and A5'.

Problem 4

Give an example of preferences on $\mathbb{R}^\ell_+$ for which there exists a utility function, but no continuous utility function.

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Suppose that $\succsim$ is such that the agent prefers one bundle $\bar{x} \in \mathbb{R}^\ell_+$ (satiation/bliss point) over any other bundle and that he is indifferent between all else; i.e.

$$\bar{x} \succ x, \forall x \in \mathbb{R}^\ell_+ \backslash \bar{x},$$
$$x \sim y, \forall x, y \in \mathbb{R}^\ell_+ \backslash \bar{x}.$$

This preference satisfies completeness and transitivity. However, since $\succsim(x)$ for any $x \neq \bar{x}$ is not closed, it does not satisfy the continuity axiom. We can nevertheless represent this preference by (discontinuous) utility functions. For example, for any $x \in \mathbb{R}_+^\ell$, define $u : \mathbb{R}_+^\ell \to \mathbb{R}$ as

$$u(x) = \begin{cases} 1 & x = \bar{x} \\ 0 & x \neq \bar{x} \end{cases}.$$

Then, clearly,

$$x \succsim y \Leftrightarrow u(x) \geq u(y), \forall x, y \in \mathbb{R}_+^\ell.$$

We can prove that there is no continuous utility function that represents this preference by way of contradiction. Suppose that there is a continuous utility function v representing the preferences defined above. Define the sequence (of bundles) as

$$x^n := \bar{x} - \frac{1}{n}\mathbf{e},$$

where $\mathbf{e} = (1, 1, \ldots, 1) \in \mathbb{R}^\ell$ so that $x^n \to \bar{x}$ as $n \to \infty$. Since the agent is indifferent between all bundles except for $\bar{x}$, it follows that

$$v(x^n) = v\left(x^1\right), \forall n \in \mathbb{N}.$$

That v is continuous implies that:

$$v(\bar{x}) = v\left(\lim_{n\to\infty} x^n\right) = \lim_{n\to\infty} v(x^n) = v\left(x^1\right).$$

But this contradicts the agent's preference and the fact that v represents his preference; i.e.

$$\bar{x} \succ x^1 \Leftrightarrow v(\bar{x}) > v\left(x^1\right).$$

Hence, there does not exist a continuous utility function that represents this preference.

Problem 6

Show that if X is any finite set, and $\succsim$ satisfies Axioms 1 and 2 on X, then $\succsim$ can be represented by a utility function. Provide an example showing what can go wrong if Axiom 1 is not satisfied.

......

Let X be a finite set and N be the number of all possible bundles.

Axiom 1. *(Completeness). For any $x, y \in X$, either $x \succsim y$ or $y \succsim x$.*

Axiom 2. *(Transitivity). For any $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$.*

$U(\cdot)$ represents the preference relation $\succsim$ on X if

$$x \succsim y \Leftrightarrow U(x) \geq U(y), \forall x, y \in X.$$

If we can find the most-preferred bundle(s), $Z_1 \subseteq X$:

$$Z_1 = \{z \in X : z \succsim x, \forall x \in X\}.$$

Then, for any $z_1 \in Z_1$ that satisfies above, we can assign a utility value

$$U(z_1) = N, \forall z_1 \in Z_1.$$

In a similar fashion, we can find the second-most preferred bundle set, $Z_2 \subseteq X$, which satisfies

$$Z_2 = \{z \in X : z \succsim x, \forall x \in X \backslash Z_1\}.$$

Let

$$U(z_2) = N - 1, \forall z_2 \in Z_2.$$

In a similar fashion, we can define the nth most preferred bundle(s) set as

$$Z_n = \left\{ z \in X : z \succsim x, \forall x \in X \backslash \bigcup_{i=1}^{n-1} Z_i \right\} \subseteq X.$$

We can continue in this fashion to obtain utility values until $Z_{\tilde{n}} = \emptyset$. Then, clearly,

$$x \succsim y \Leftrightarrow U(x) \geq U(y), \forall x, y \in X.$$

It remains to prove that Z_1 is nonempty for any N—i.e. there exists $z \in X$ such that $z \succsim x$ for all $x \in X$ for any N. We do so by induction.

Consider the case when $N = 1$, then $X_1 = \{x_1\}$. By completeness, we have that $x_1 \succsim x$ for all $x \in X$ since $x_1 \succsim x_1$. Now suppose that X_N has a most-preferred bundle z. Consider the case X_{N+1} with $N+1$ elements, then $X_{N+1} = \{X_N\} \cup \{x_{N+1}\} = \{x_1, x_2, \ldots, x_N, x_{N+1}\}$. Clearly, $z \succsim x_n$ for all $n = 1, 2, \ldots, N$. By completeness, $z \succsim x_{N+1}$ or $x_{N+1} \succsim z$. If former, then by transitivity,

$$(z \succsim x_{N+1}) \wedge (z_N \succsim x_n, \forall n = 1, 2, \ldots, N) \Rightarrow z \succsim x_n, \forall n = 1, 2, \ldots, N+1.$$

If latter, again by transitivity,

$$(x_{N+1} \succsim z) \wedge (z_N \succsim x_n, \forall n = 1, 2, \ldots, N) \Rightarrow x_{N+1} \succsim x_n, \forall n = 1, 2, \ldots, N+1.$$

Redefine $z = x_{N+1}$ in this case. It follows by induction that z exists for all N.

In case Axiom 1 fails If Axiom 1 fails, then there exists some $\tilde{x} \in X$ such that neither $\tilde{x} \succsim y$ nor $\tilde{x} \succsim x$ for any $x \in X \backslash \{\tilde{x}\}$. However, since $U : X \to \mathbb{R}$, $\tilde{x}$ will have an associated util value. Since $\geq$ is complete in $\mathbb{R}$, then either $U(\tilde{x}) \geq U(x)$ or $U(x) \geq U(\tilde{x})$ for any $x \in X \backslash \{\tilde{x}\}$. Since U represents preference, this must imply that either $\tilde{x} \succsim x$ or $x \succsim \tilde{x}$, which is a contradiction.

In case Axiom 2 fails Suppose transitivity is not satisfied so that for any $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ does not imply $x \succsim z$. Since U represents preferences, it must be that

$$U(x) \geq U(y), U(y) \geq U(z).$$

Since $\geq$ is transitive, above implies $U(x) \geq U(z)$, which, in turn, implies $x \succsim z$, which is a contradiction.

The following shows that Completeness and Transitivity are sufficient to ensure utility representation when X is countable. We start with the following lemma. Notice that this proof is essentially the same as for the proof we provided above for finite representation.

Lemma 1. *(Guaranteed minimal element). Let X be a countable non-empty set and the ordering $\succsim$ over X satisfies completeness and transitivity. Then, there exists an element $x \in X$ such that $y \succsim x$ for all $y \in X$.*

Proof. We prove by induction. Let X_n denote a nonempty set with n elements. Since X_1 is nonempty, $X_1 = \{x\}$, and, by completeness, $x \succsim x$ so that we vacuously have a minimal element. Now, set up the "induction hypothesis" that $X_n = \{x_1, x_2, \ldots, x_n\}$ has a minimal element, say $\underline{x}_n$. We want to show that this implies that $X_{n+1} \{x_1, x_2, \ldots, x_{n+1}\}$ has a minimal element.
Write $X_{n+1} = X_n \cup \{x_{n+1}\}$. By the induction hypothesis, there exists $\underline{x}_n \in X_n$ such that $x \succsim \underline{x}_n$ for all $x \in X_n$. By completeness, either $x_{n+1} \succsim \underline{x}_n$ or $\underline{x}_n \succsim x_{n+1}$. If former, then $\underline{x}_n$ is the minimal element of X_{n+1} (by transitivity, same argument as above). If latter, then x_{n+1} is the minimal element of X_{n+1} (again, by transitivity). Either way, X_{n+1} has a minimal element. Then, by induction, there exists a minimal element for any countable X. □

Theorem 1. *Let X be a countable nonempty set and the order $\succsim$ over X satisfies completeness and transitivity. Then, there exists a function $u : X \to \mathbb{R}$ that represents $\succsim$.*

Proof. The previous lemma tells us that, for every countable nonempty set X, there is a minimal element. Let M_1 denote the set of such minimal elements; i.e.

$$M_1 = \{x \in X : z \succsim x, \forall x \in X\}$$

and define

$$u(x) := 1, \ \forall x \in M_1.$$

Just as we did above, for all $k \geq 1$, define

$$\begin{aligned} M_{k+1} &= \left\{ x \in X : z \succsim x, \ \forall x \in X \backslash \bigcup_{i=1}^{k} M_i \right\} \\ u(x) &:= k+1, \ \forall x \in M_{k+1}. \end{aligned}$$

(If X is finite, we stop at some $k^* \geq 1$ such that $\bigcup_{i=1}^{k^*} M_i = X$.)
It remains to show that $u : X \to \mathbb{R}$, where $u(x) := i$ for all $x \in M_i$, represents $\succsim$. This follows since, for all $k \geq 1$,

$$\begin{aligned} & x_{k+1} \succsim x_k, \ \forall x_k \in M_k, x_{k+1} \in M_{k+1} \\ \Leftrightarrow \ & u(x_{k+1}) = k+1 \geq k = u(x_k), \ \forall x_k \in M_k, x_{k+1} \in M_{k+1}. \end{aligned}$$ □

Remark 1. Although we defined the utility function iteratively from the minimal element, we can also do the same using a maximal element and assigning lower utils (which is, in fact, what we did for the finite case above).

Problem 8

Show that if $\succsim$ satisfies Axiom 2 on $\mathbb{R}_+^\ell$, then for every $x, y \in \mathbb{R}_+^\ell$, the sets $\sim(x)$ and $\sim(y)$ are either disjoint or equal. (i.e. Distinct indifference curves do not cross.)

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Take any $x \in \mathbb{R}_+^\ell$. Assume completeness, then either $x \in \sim(y)$ or $x \notin \sim(y)$.[1] If $x \in \sim(y)$, then, by definition, $x \sim y$. Then, for any $z \in \sim(x)$, since $z \sim x$ and $x \sim y$, by transitivity, $z \sim y$. Therefore, $\sim(x) \subseteq \sim(y)$. If $x \notin \sim(y)$, then, either $x \succ y$ or $y \succ x$. Then, for any $z \in \sim(x)$, either $z \succ y$ or $y \succ z$ by transitivity. Therefore, $z \notin \sim(y)$ and hence, $\sim(x) \cap \sim(y) = \emptyset$.

The argument is symmetric if we start with $y \in \mathbb{R}_+^\ell$ where either $y \in \sim(x)$ or $y \notin \sim(x)$. In this case, we would conclude that $\sim(y) \subseteq \sim(x)$ or, again, that $\sim(x) \cap \sim(y) = \emptyset$. Thus, we conclude that for any $x, y \in \mathbb{R}_+^-$,

$$\sim(x) = \sim(y) \ \text{ or } \ \sim(x) \cap \sim(y) = \emptyset.$$

[1] In fact, we don't need to assume completeness. Without completeness, it's possible that $x \in \sim(y)$ and $x \notin \sim(y)$ in which case the sets $\sim(x)$ and $\sim(y)$ are disjoint and we are done.

Problem 10

Consider two consumers' preferences over bundles in $\mathbb{R}^2_+$. Consumer 1's preferences are represented by the Cobb-Douglas utility function $u(x,y) = xy$. Consumer 2's preferences are identical except for bundles lying along the ray $x = y$. Bundles on this ray are strictly preferred by consumer 2 to distinct bundles on the Cobb-Douglas indifference curve that passes through them, and are also strictly less desirable than all bundles above that indifference curve.

1. Assuming that consumer 2's preferences are transitive, prove that they are complete and strictly monotonic.
2. Assuming that consumer 2's preferences are complete and transitive, can they be represented by a continuous utility function?

......

Part (i) Agent 2's preference is such that, for any $xy = c$ where c is a constant

$$\begin{aligned}
&(x,y) \sim (\tilde{x},\tilde{y}), \forall xy = \tilde{x}\tilde{y} = c, x \neq y, \tilde{x} \neq y, \\
&(x,y) \succ (\tilde{x},\tilde{y}), \forall xy = \tilde{x}\tilde{y} = c, x = y, \tilde{x} \neq y, \\
&(x,y) \succ (\tilde{x},\tilde{y}), \forall xy = c, x = y, \tilde{x}\tilde{y} > c.
\end{aligned}$$

Now, take any $(x,y), (\tilde{x},\tilde{y}) \in \mathbb{R}^2_+$. Then, there are six cases to consider

- $xy = \tilde{x}\tilde{y}$ and $x \neq y$, $\tilde{x} \neq \tilde{y}$, then $(x,y) \sim (\tilde{x},\tilde{y})$;
- $xy = \tilde{x}\tilde{y}$ and $x = y$, $\tilde{x} \neq \tilde{y}$, then $(x,y) \succ (\tilde{x},\tilde{y})$;
- $xy = \tilde{x}\tilde{y}$ and $x \neq y$, $\tilde{x} = \tilde{y}$, then $(\tilde{x},\tilde{y}) \succ (x,y)$;
- $xy = \tilde{x}\tilde{y}$ and $x = y$, $\tilde{x} = \tilde{y}$, then $(\tilde{x},\tilde{y}) \sim (x,y)$;
- $xy > \tilde{x}\tilde{y}$ then $(x,y) \succ (\tilde{x},\tilde{y})$ (independent of whether $x = y$ and/or $\tilde{x} = \tilde{y}$);
- $xy < \tilde{x}\tilde{y}$ then $(\tilde{x},\tilde{y}) \succ (x,y)$ (independent of whether $x = y$ and/or $\tilde{x} = \tilde{y}$).

From above, we see that

$$(x,y), (\tilde{x},\tilde{y}) \in \mathbb{R}^2_+ \Rightarrow (x,y) \succsim (\tilde{x},\tilde{y}) \text{ or } (\tilde{x},\tilde{y}) \succsim (x,y).$$

Thus, consumer 2's preferences are complete.

Strict monotonicity requires that, $(x,y), (\tilde{x},\tilde{y}) \in \mathbb{R}^2_+$,

$$(x,y) \geq (\tilde{x},\tilde{y}) \Rightarrow (x,y) \succsim (\tilde{x},\tilde{y}), \tag{1}$$

$$(x,y) \gg (\tilde{x},\tilde{y}) \Rightarrow (x,y) \succ (\tilde{x},\tilde{y}). \tag{2}$$

Consider the case when $(\tilde{x},\tilde{y}) = (0,0)$ and $(x,y) = (1,0)$. Then we have $(x,y) \geq (\tilde{x},\tilde{y})$ but since $xy = \tilde{x}\tilde{y} = 0$, $(\tilde{x},\tilde{y})$ lies on the 45 degree line while (x,y) does not, 2's preference imply that $(\tilde{x},\tilde{y}) \succ (x,y)$. Hence, (1) does not hold. Thus, 2's preference does not satisfy strict monotonicity.

However, suppose now that consumers have preferences over bundles in $\mathbb{R}^2_{++}$. Then, strict monotonicity holds. To see this, note that for any $(x,y), (\tilde{x},\tilde{y}) \in \mathbb{R}^2_{++}$,

1. If $(x,y) \geq (\tilde{x},\tilde{y})$ and $(x,y) \neq (\tilde{x},\tilde{y})$, then it must be that either $x \geq \tilde{x}$ and/or $y \geq \tilde{y}$ holds with strictly inequality. Then, since they are all strictly positive, it must be that $xy > \tilde{x}\tilde{y} \Rightarrow (x,y) \succ (\tilde{x},\tilde{y}) \Rightarrow (x,y) \succsim (\tilde{x},\tilde{y})$.
2. If $(x,y) \geq (\tilde{x},\tilde{y})$ and $(x,y) = (\tilde{x},\tilde{y})$, then $x = \tilde{x}$ and $y = \tilde{y}$ so that $xy = \tilde{x}\tilde{y} \Rightarrow (x,y) \sim (\tilde{x},\tilde{y}) \Rightarrow (x,y) \succsim (\tilde{x},\tilde{y})$.
3. If $(x,y) \gg (\tilde{x},\tilde{y})$, then $x > \tilde{x}$ and $y > \tilde{y}$ so that $xy > \tilde{x}\tilde{y} \Rightarrow (x,y) \succ (\tilde{x},\tilde{y}) \Rightarrow (x,y) \succ (\tilde{x},\tilde{y})$.

Part (ii) For 2's preferences to be representable by a continuous utility function, it must be that $\succsim$ satisfies the continuity axiom; i.e. for any $(x, y) \in \mathbb{R}^2_+$, $\succsim (x, y)$ and $\precsim (x, y)$ are closed in $\mathbb{R}^2_+$. Thus, it suffices to show that 2's preferences violate the continuity axiom.

We do so by showing that the set $\precsim (x, y)$ is not closed and using the sequential definition of closedness. Consider the sequence $(x^n, y^n) = (c(1 + 1/n), c/(1 + 1/n))$ where c is some positive constant. Since $x^n y^n = c^2$ and $x^n \neq y^n$, for all $n \in \mathbb{N}$, $(x^n, y^n) \in \precsim (x, y)$ where $(x, y) \in \mathbb{R}^2_+$ such that $xy = c^2$ with $x \neq y$. Clearly, $\{(x^n, y^n)\} \to (c, c)$ as $n \to \infty$; however, $(c, c) \notin \precsim (x, y)$ since (c, c) lies on the 45 degree ray. That is, the set $\precsim (x, y)$ is not closed, violating the continuity axiom.

We can also prove that no continuous utility function exists directly a la Q4. Take the sequence as given above and let v be a continuous utility function representing $\succsim$. Note that

$$v(x^n, y^n) = v\left(x^1, y^1\right), \forall n \in \mathbb{N}$$

since $x^n y^n = c^2$ for all $n \in \mathbb{N}$. That v is continuous implies that

$$v(c, c) = v\left(\lim_{n\to\infty} (x^n, y^n)\right) = \lim_{n\to\infty} v(x^n, y^n) = v\left(x^1, y^1\right).$$

However, since (c, c) lies on the 45 degree ray, the given preference implies that

$$(c, c) \succ \left(x^1, y^1\right) \Leftrightarrow v(c, c) > v\left(x^1, y^1\right).$$

Hence, v cannot exist.

Problem 12

Let $u(x, y) = x^2 y + xy$ with $x, y > 0$. Determine the demand functions for the two goods using the Lagrangian first-order condition method. Is u quasi-concave? Why is this relevant for the demand functions you derived?

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The Lagrangian for this problem is $\mathcal{L} = x^2 y + xy + \lambda(I - xp_x - yp_y)$.

The FOC for this problem are:

$$\begin{aligned} 2xy + y - \lambda p_x &\leq 0 \\ x &> 0 \\ (2xy + y - \lambda p_x)\, x &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} x^2 + x - \lambda p_y &\leq 0 \\ y &> 0 \\ \left(x^2 + x - \lambda p_y\right) y &= 0 \end{aligned} \tag{4}$$

$$\begin{aligned} xp_x + yp_y - I &\leq I \\ \lambda &\geq 0 \\ (xp_x + yp_y - I)\, \lambda &= 0 \end{aligned} \tag{5}$$

Note that since x, $y > 0$ then $\lambda > 0$, otherwise equations (1) and (2) cannot hold. Then the following relation must hold:

$$\begin{aligned} \frac{2xy + y}{x^2 + x} &= \frac{p_x}{p_y} \\ \Rightarrow y &= \left(\frac{x^2 + x}{2x + 1}\right) \frac{p_x}{p_y} \end{aligned}$$

On the other hand, since $\lambda > 0$, (5) implies $xp_x + yp_y = I$. Plugging the expression found for y in the previous expression yields a quadratic expression for x. Since $x > 0$ we choose the positive root of the equation:

$$\begin{aligned} 3p_x x^2 + 2\left(p_x - I\right)x + -I &= 0 \\ \Rightarrow x &= \frac{\left(I - p_x\right) + \sqrt{\left(I - p_x\right)^2 + 3p_x I}}{3p_x}. \end{aligned}$$

Replacing this expression of x in the budget constraint and solving for y we get

$$y = \frac{2I + p_x - \sqrt{\left(I - p_x\right)^2 + 3p_x I}}{3p_y}.$$

To determine whether u is quasi-concave or not, we can check if the indifference curves of the function are convex, i.e. the absolute value of the slope of the indifference curves must be decreasing in each good. For a given utility level $\overline{u}$, an indifference curve satisfies

$$\begin{aligned} du &= u_x dx + u_y dy = 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{u_x}{u_y} \end{aligned}$$

Taking derivatives with respect to x of the absolute value of the slope of the utility curve, we have

$$\begin{aligned} \frac{\partial\left(\frac{dy}{dx}\right)}{\partial x} &= \left(\frac{u_{xx}u_y - u_x u_{yx}}{u_y^2}\right) \\ &= \left(\frac{2y\left(x^2 + x\right) - \left(2xy + y\right)\left(2x + 1\right)}{\left(x^2 + x\right)^2}\right) \\ &= -\left(\frac{2x^2 y + 2xy + y}{\left(x^2 + x\right)^2}\right) < 0. \end{aligned}$$

Therefore, the indifference curves defined by u are convex, and thus, u is quasi-concave. The quasi-concavity of u is relevant because it allows us to apply the Lagrangian method for deriving the marshallian demands previously found. In other words, the quasi-concavity of u allow us to find a solution to the maximization problem (See Theorem 1.4 in the textbook).

Problem 14

14a. Cobb-Douglas utility with normalized weights

The Marshallian demand functions are given by:

$$\begin{aligned} x^M(p_x, p_y, I) &= \frac{\alpha I}{p_x}, \\ y^M(p_x, p_y, I) &= \frac{(1-\alpha)I}{p_y}, \end{aligned}$$

whereas the Hicksian demand functions are characterized by:

$$\begin{aligned} x^H(p_x, p_y, \overline{u}) &= \left(\tfrac{\alpha}{1-\alpha}\tfrac{p_y}{p_x}\right)^{1-\alpha}\bar{u}, \\ y^H(p_x, p_y, \overline{u}) &= \left(\tfrac{1-\alpha}{\alpha}\tfrac{p_x}{p_y}\right)^{\alpha}\bar{u}. \end{aligned}$$

The *indirect* utility function is obtained by evaluating the *Marshallian* demand function in the utility function, i.e. $v(p_x, p_y, I) = u\big(x^M(p_x, p_y, I), y^M(p_x, p_y, I)\big)$. We obtain after simplifying the expression:

$$v(p_x, p_y, I) = I\left(\tfrac{\alpha}{P_x}\right)^{\alpha}\left(\tfrac{1-\alpha}{P_y}\right)^{1-\alpha}.$$

Similarly, we may obtain the *expenditure* function by evaluating the *Hicksian* demands in total costs, i.e. $e(p_x, p_y, \bar{u}) = p_x x^H(p_x, p_y, \bar{u}) + p_y y^H(p_x, p_y, \bar{u})$:

$$\begin{aligned}
e(p_x, p_y, \bar{u}) &= \bar{u}\left[p_x^{\alpha}\left(\tfrac{\alpha}{1-\alpha}p_y\right)^{1-\alpha} + p_y^{1-\alpha}\left(\tfrac{1-\alpha}{\alpha}p_x\right)^{\alpha}\right] \\
&= \bar{u}p_x^{\alpha}p_y^{1-\alpha}\left[\left(\tfrac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\tfrac{1-\alpha}{\alpha}\right)^{\alpha}\right] \\
&= \bar{u}\left(\tfrac{p_x}{\alpha}\right)^{\alpha}\left(\tfrac{p_y}{1-\alpha}\right)^{1-\alpha}.
\end{aligned}$$

By direct substitution, we see that:

$$\begin{aligned}
v(p_x, p_y, e(p_x, p_y, \bar{u})) &= \bar{u}\left(\tfrac{p_x}{\alpha}\right)^{\alpha}\left(\tfrac{p_y}{1-\alpha}\right)^{1-\alpha}\left(\tfrac{\alpha}{P_x}\right)^{\alpha}\left(\tfrac{1-\alpha}{P_y}\right)^{1-\alpha} \\
&= \bar{u}.
\end{aligned}$$

Analogously, we observe:

$$\begin{aligned}
e(p_x, p_y, v(p_x, p_y, I)) &= I\left(\tfrac{\alpha}{p_x}\right)^{\alpha}\left(\tfrac{1-\alpha}{p_y}\right)^{1-\alpha}\left(\tfrac{p_x}{\alpha}\right)^{\alpha}\left(\tfrac{p_y}{1-\alpha}\right)^{1-\alpha} \\
&= I.
\end{aligned}$$

Once the indirect utility function has been derived, we can use the first identity to obtain the expenditure function as follows:

$$\begin{aligned}
&\bar{u} = v(p_x, p_y, e(p_x, p_y, \bar{u})) = e(p_x, p_y, \bar{u})\left(\tfrac{\alpha}{p_x}\right)^{\alpha}\left(\tfrac{1-\alpha}{p_y}\right)^{1-\alpha} \\
&\implies e(p_x, p_y, \bar{u}) = \bar{u}\left(\tfrac{p_x}{\alpha}\right)^{\alpha}\left(\tfrac{p_y}{1-\alpha}\right)^{1-\alpha}.
\end{aligned}$$

14b. Separable concave utilities

$$v(p, m) = \left(\frac{m}{p_x\left(\left(\frac{p_x}{p_y}\right)^{\alpha/(1-\alpha)} + 1\right)}\right)^{\alpha} + \left(\frac{m}{p_y\left(\left(\frac{p_y}{p_x}\right)^{\alpha/(1-\alpha)} + 1\right)}\right)^{\alpha}$$

$$e(p, u) = \bar{u}^{\frac{1}{\alpha}}\left(\left[\frac{p_x^{\alpha}}{\left(\frac{p_x}{p_y}\right)^{\alpha/(1-\alpha)} + 1}\right]^{\frac{1}{\alpha}} + \left[\frac{p_y^{\alpha}}{\left(\frac{p_y}{p_x}\right)^{\alpha/(1-\alpha)} + 1}\right]^{\frac{1}{\alpha}}\right).$$

In the previous question, we found:

$$\begin{aligned}
x_m(p, I) &= \frac{I}{p_x\left(\left(\frac{p_x}{p_y}\right)^{\alpha/(1-\alpha)} + 1\right)} \\
y_m(p, I) &= \frac{I}{p_y\left(\left(\frac{p_y}{p_x}\right)^{\alpha/(1-\alpha)} + 1\right)}
\end{aligned}$$

We also obtained:

$$x^H(p_x, p_y, \bar{u}) = \left[\frac{\bar{u}}{\left(\frac{p_x}{p_y}\right)^{\alpha/(1-\alpha)} + 1}\right]^{\frac{1}{\alpha}}$$

$$y^H(p_x,p_y,\bar{u}) = \left[\frac{\bar{u}}{\left(\frac{p_y}{p_x}\right)^{\alpha/(1-\alpha)}+1}\right]^{\frac{1}{\alpha}}.$$

Therefore, the *indirect* utility and *expenditure* functions are given by:

$$\begin{aligned}
v(p_x,p_y,I) &= (x^M(p_x,p_y,I))^\alpha + (y^M(p_x,p_y,I))^\alpha \\
&= \left(\frac{I}{p_x\left(\left(\frac{p_x}{p_y}\right)^{\alpha/(1-\alpha)}+1\right)}\right)^\alpha + \left(\frac{I}{p_y\left(\left(\frac{p_y}{p_x}\right)^{\alpha/(1-\alpha)}+1\right)}\right)^\alpha, \\
e(p_x,p_y,\bar{u}) &= p_x x^H(p_x,p_y,\bar{u}) + p_y y^H(p_x,p_y,\bar{u}) \\
&= \bar{u}^{\frac{1}{\alpha}}\left(\left[\frac{p_x^\alpha}{\left(\frac{p_x}{p_y}\right)^{\alpha/(1-\alpha)}+1}\right]^{\frac{1}{\alpha}} + \left[\frac{p_y^\alpha}{\left(\frac{p_y}{p_x}\right)^{\alpha/(1-\alpha)}+1}\right]^{\frac{1}{\alpha}}\right).
\end{aligned}$$

After only a few simplifications, we can conclude that:

$$\begin{gathered}
v(p_x,p_y,e(p_x,p_y,\bar{u})) = \bar{u}, \\
e(p_x,p_y,v(p_x,p_y,I)) = I.
\end{gathered}$$

14c. Perfect symmetric complements

The *Marshallian* and *Hicksian* demand functions are given by:

$$\begin{gathered}
x^M(p_x,p_y,I) = y^M(p_x,p_y,I) = \tfrac{I}{p_x+p_y} \\
x^H = y^H = \bar{u}.
\end{gathered}$$

We can immediately deduce that:

$$\begin{gathered}
v(p_x,p_y,I) = \tfrac{I}{p_x+p_y}, \\
e(p_x,p_y,\bar{u}) = (p_x+p_y)\bar{u}.
\end{gathered}$$

Thus, it also follows directly that:

$$\begin{gathered}
v(p_x,p_y,e(p_x,p_y,\bar{u})) = \bar{u}, \\
e(p_x,p_y,v(p_x,p_y,I)) = I.
\end{gathered}$$

which implies that:

$$\begin{aligned}
\bar{u} &= v(p_x,p_y,e(p_x,p_y,\bar{u})) \\
&= \frac{e(p_x,p_y,\bar{u})}{p_x+p_y} \\
&\implies e(p_x,p_y,\bar{u}) = (p_x+p_y)\bar{u}.
\end{aligned}$$

14d. Perfect symmetric substitutes

The Marshallian and Hicksian demand functions are characterized by *corner* solutions. We obtained:

$$x^M(p_x,p_y,I) = \begin{cases} \frac{I}{p_x} & \text{if } p_x < p_y, \\ 0 & \text{if } p_x > p_y, \\ \in \left[0, \frac{I}{p_x}\right] & \text{if } p_x = p_y. \end{cases}$$

$$y^M(p_x,p_y,I) = \begin{cases} \frac{I}{p_y} & \text{if } p_x > p_y, \\ 0 & \text{if } p_x < p_y, \\ \frac{I-p_x x^M(p_x,p_y,I)}{p_y} & \text{if } p_x = p_y. \end{cases}$$

$$x^H(p_x, p_y, \bar{u}) = \begin{cases} \bar{u} & \text{if } p_x < p_y, \\ 0 & \text{if } p_x > p_y, \\ \in [0, \bar{u}] & \text{if } p_x = p_y. \end{cases}$$

$$y^H(p_x, p_y, \bar{u}) = \begin{cases} \bar{u} & \text{if } p_x > p_y, \\ 0 & \text{if } p_x < p_y, \\ \bar{u} - x^H(p_x, p_y, \bar{u}) & \text{if } p_x = p_y. \end{cases}$$

Therefore, we can write the indirect utility function as:

$$v(p_x, p_y, I) = \begin{cases} \frac{I}{p_x} & \text{if } p_x < p_y, \\ \frac{I}{p_y} & \text{if } p_x > p_y, \\ \frac{I}{p_x} = \frac{I}{p_y} & \text{if } p_x = p_y. \end{cases}$$

This can be simplified to:

$$v(p_x, p_y, I) = \max\left\{\frac{I}{p_x}, \frac{I}{p_y}\right\} = \frac{I}{\min\{p_x, p_y\}}.$$

The expenditure function becomes:

$$\begin{aligned} e(p_x, p_y, \bar{u}) &= \begin{cases} p_x\bar{u} & \text{if } p_x < p_y, \\ p_y\bar{u} & \text{if } p_x > p_y, \\ p_x\bar{u} = p_y\bar{u} & \text{if } p_x = p_y. \end{cases} \\ &= \bar{u}\min\{p_x, p_y\}. \end{aligned}$$

In this case, it is also easy to verify that:

$$\begin{aligned} v(p_x, p_y, e(p_x, p_y, \bar{u})) &= \frac{e(p_x, p_y, \bar{u})}{\min\{p_x, p_y\}} \\ &= \frac{\bar{u}\min\{p_x, p_y\}}{\min\{p_x, p_y\}} \\ &= \bar{u}, \\ e(p_x, p_y, v(p_x, p_y, I)) &= v(p_x, p_y, I)\min\{p_x, p_y\} \\ &= \frac{I}{\min\{p_x, p_y\}}\min\{p_x, p_y\} \\ &= I. \end{aligned}$$

Finally, we get from the first identity:

$$\begin{aligned} \bar{u} &= v(p_x, p_y, e(p_x, p_y, \bar{u})) \\ &= \frac{e(p_x, p_y, \bar{u})}{\min\{p_x, p_y\}} \\ &\implies e(p_x, p_y, \bar{u}) = \bar{u}\min\{p_x, p_y\}. \end{aligned}$$