Price Theory II
Problem Set 2
Solutions*

Sota Ichiba Elena Istomina Christoph Schlom

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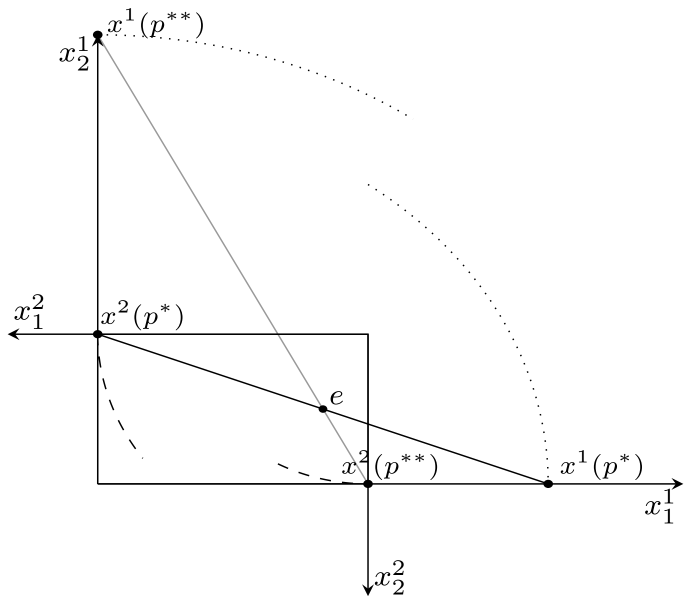
Problem 1

Question. *In an Edgeworth box two-consumer, two-commodity pure exchange economy with consumption sets equal to* $\mathbb{R}^2_+$ *sketch a situation in which there is no competitive equilibrium because preferences are not convex.*

Solution.

Figure 1: Non-Convex preferences And No Equilibrium



Note: dotted lines depict the indifference curves of the first agent, the dashed lines — the indifference curves of the second agent. e is the initial endowment. $x^i(p)$ depicts the i-th agent's demand at price p.

Suppose that the preferences of agent i are ${x_1^i}^2 + {x_2^i}^2$. Then, the only possible equilibria are the corners of Edgeworth box (since the agent always chooses either corner of her budget line). So their are two candidates of prices: such that the budget line goes through the left top corner — p^* or through right bottom corner — p^{**} [1]. But note that at p^* there is excess demand for good 1, whereas at p^{**} — excessive demand on the second good. So both of the candidates are not an equilibrium. But then there is no equilibrium for such an economy.

*We thank TAs from previous years for solution. This version is partly based on their writeups.

[1] We also use the fact that the utility functions are, in fact, strongly increasing, hence, the prices are also strictly positive

Problem 2

Question. *Prove that, in a pure exchange economy in which consumer i's preferences are represented by the utility function u^i, an allocation $\left\{\hat{\mathbf{x}}^i\right\}_{i=1}^m$ is Pareto efficient if and only if it solves the following maximization problem:*

$$\begin{aligned} \max_{\mathbf{x}^1,\ldots,\mathbf{x}^m\in\mathbb{R}^n_+} \quad & u^1\left(\mathbf{x}^1\right)+\cdots+u^m\left(\mathbf{x}^m\right) \\ s.t. \quad & u^i\left(\mathbf{x}^i\right)\geq u^i\left(\hat{\mathbf{x}}^i\right), \forall i, \\ & \sum_{i=1}^m \mathbf{x}^i \leq \sum_{i=1}^m \mathbf{e}^i. \end{aligned}$$

Solution. Consider a pure exchange economy given by

$$\mathcal{E}=\left(u^i,\mathbf{e}^i\right)_{i\in\mathcal{I}}.$$

We are not given that u^i is strongly increasing, which means that we need to change the definition of feasible allocations. For example, consider cases where no one cares about good 1, then any allocation of good 1 (including throwing some away) is Pareto efficient as long as other goods are allocated in a way that is Pareto efficient. *(Feasible allocations)*. An allocation $\left\{\mathbf{x}^i\right\}_{i=1}^m$ is feasible if

$$\sum_{i=1}^m \mathbf{x}^i \leq \sum_{i=1}^m \mathbf{e}^i.$$

(Pareto-efficient allocations). A feasible allocation $\left\{\mathbf{x}^i\right\}_{i=1}^m$ is Pareto efficient if there is no other feasible allocation $\left\{\mathbf{y}^i\right\}_{i=1}^m$ such that $u^i\left(\mathbf{y}^i\right)\geq u^i\left(\mathbf{x}^i\right)$ for all $i\in\mathcal{I}$ with at least one with strict inequality.

Proof. First note that the objective function can be written as

$$\sum_{i\in\mathcal{I}} u^i\left(\mathbf{x}^i\right).$$

(Sufficiency) Suppose $\left\{\hat{\mathbf{x}}^i\right\}_{i=1}^m$ solves the maximization problem. Then, by construction, it is a feasible allocation. To see that it is Pareto efficient, toward a contradiction, suppose that there exists another feasible allocation $\left\{\mathbf{y}^i\right\}_{i=1}^m$ for which $u^i\left(\mathbf{y}^i\right)\geq u^i\left(\hat{\mathbf{x}}^i\right)$ for all $i\in\mathcal{I}$ with at least one with strict inequality. But then

$$\sum_{i\in\mathcal{I}} u^i\left(\mathbf{y}^i\right) > \sum_{i\in\mathcal{I}} u^i\left(\hat{\mathbf{x}}^i\right),$$

which contradicts the assumption that $\left\{\hat{\mathbf{x}}^i\right\}_{i=1}^m$ solves the maximization problem.

(Necessity) Suppose instead that $\left\{\hat{\mathbf{x}}^i\right\}_{i=1}^m$ is a Pareto efficient allocation. Then, by definition, $\left\{\hat{\mathbf{x}}^i\right\}_{i=1}^m$ is feasible so that

$$\sum_{i=1}^m \hat{\mathbf{x}}^i \leq \sum_{i=1}^m \mathbf{e}^i.$$

By definition again, this means that there is no other feasible allocation $\left\{\mathbf{y}^i\right\}_{i=1}^m$ such that $u^i\left(\mathbf{y}^i\right)\geq u^i\left(\mathbf{x}^i\right)$ for all $i\in\mathcal{I}$ with at least one with strict inequality. That is, for any feasible allocation $\left\{\mathbf{y}^i\right\}_{i=1}^m$,

$$u^i\left(\hat{\mathbf{x}}^i\right)\geq u^i\left(\mathbf{y}^i\right),\ \forall i\in\mathcal{I}$$

with no strict equality. Summing across all agents yields

$$\sum_{i\in\mathcal{I}} u^i\left(\hat{\mathbf{x}}^i\right) \geq \sum_{i\in\mathcal{I}} u^i\left(\mathbf{y}^i\right).$$

That is, the objective function is maximized, and we already showed that $\left\{\hat{\mathbf{x}}^i\right\}_{i=1}^m$ satisfies the two constraints of the maximization problem. □

Comment
If we do not want to modify our notion of feasibility, we can instead make the following assumption. For every good k, there exists at least one agent i_k^* such that i_k^*'s utility is nondecreasing in good k. With this assumption, if the allocation $\{\tilde{x}^i\}_{i=1}^m$ does not exhaust all resources, we can define a new allocation $\{\hat{x}^i\}_{i=1}^m$ such that, for any good k with excess supply (i.e. $\sum_{i=1}^m \tilde{x}_k^i < \sum_{i=1}^m e_k^i$), we can reallocate the excess supply of this good to the individual whose utility is nondecreasing in that good. In particular, we would have

$$\hat{x}_k^{i_k^*} = \tilde{x}_k^{i_k^*} + \left(\sum_{i=1}^m e_k^i - \sum_{i=1}^m \tilde{x}_k^i \right).$$

The assumption is exactly what we need to guarantee that such an individual exists for every good. For any goods which are not in excess supply, the allocation under $\{\tilde{x}^i\}_{i=1}^m$ coincides with that under $\{\hat{x}^i\}_{i=1}^m$. By construction, allocation $\{\hat{x}^i\}_{i=1}^m$ ensures that feasibility holds with equality. The proof will goes through as before given the assumption of nondecreasing utility.

Problem 3

Question. *Consider a two-consumer two-good exchange economy in which* $u_1(x_1; x_2) = x_1 x_2$*;* $u_2(x_1; x_2) = x_1 x_2^2$*;* $e_1 = (18; 4)$*; and* $e_2 = (3; 6)$*:*

(a) *Characterize the set of Pareto efficient allocations as completely as possible.*

(b) *Characterize the core.*

(c) *Find a competitive equilibrium allocation and corresponding price vector.*

(d) *Verify that the competitive equilibrium allocation is in the core.*

Solution. Notation: x_{ji} stands for good j allocated to consumer i, $\bar{e}_j$ denotes $\sum_{i \in I} e_{ji}$.
(a) To find the set of Pareto-efficient allocations, solve the problem stated in Problem 2. Set up the Lagrangian:

$$\mathcal{L}(x_{11}, x_{21}, x_{12}, x_{22}, \lambda_1, \lambda_2, \mu) = x_{11}x_{21} + \mu(x_{12}x_{22}^2 - \bar{u}_2) + \lambda_1(\bar{e}_1 - x_{11} - x_{12}) + \lambda_2(\bar{e}_2 - x_{21} - x_{22})$$

Corner solutions. Notice that the only corners that can be a solution to this problem are the ones in which either of the consumers gets the whole endowment of the goods in the economy.

Feasibility Condition Binds. Given the above, we can see that it should always be the case that at optimal allocations all endowment gets to be consumed (as the utility of at least one consumer is strictly increasing in each good). So that the feasibility condition should bind at the optimum. Then, we can simplify the initial problem and obtain a new Lagrangian:

$$\mathcal{L}(x_{12}, x_{22}, \mu) = (\bar{e}_1 - x_{12})(\bar{e}_2 - x_{22}) + \mu(x_{12}x_{22}^2 - \bar{u}_2)$$

Interiors solutions. Take the derivatives of the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial x_{12}} = -(\bar{e}_2 - x_{22}) + \mu x_{22}^2$$
$$\frac{\partial \mathcal{L}}{\partial x_{22}} = -(\bar{e}_1 - x_{12}) + 2\mu x_{12}x_{22}$$

Notice that $\mu = 0$ only if the first consumer consumes all the goods, that is only if $\bar{u} = 0$. Otherwise, $\mu > 0$ and the constraint is binding. For interior solutions we, then, have:

$$\frac{(\bar{e}_2 - x_{22})}{(\bar{e}_1 - x_{12})} = \frac{x_{22}}{2x_{12}}$$

The set of the Pareto efficient allocations are, then, given by:

$$\mathcal{P} = \left\{ (x_{11}, x_{21}, x_{12}, x_{22}) : x_{11} = \bar{e}_1 - x_{12}, x_{11} = \bar{e}_2 - x_{12}, \frac{(\bar{e}_2 - x_{22})}{(\bar{e}_1 - x_{12})} = \frac{x_{22}}{2x_{12}} \right\} \cup \{(\bar{e}_1, \bar{e}_2, 0, 0)\} \cup \cup \{(0, 0, \bar{e}_1, \bar{e}_2)\}$$

Plug in the exact numbers for the endowmnets to finally get the solution to the problem:

$$\mathcal{P} = \left\{(x_{11}, x_{21}, x_{12}, x_{22}) : x_{11} = 21 - x_{12}, x_{11} = 10 - x_{12}, \frac{(10 - x_{22})}{(21 - x_{12})} = \frac{x_{22}}{2x_{12}}\right\} \cup \{(21, 10, 0, 0)\} \cup \cup \{(0, 0, 21, 10)\}$$

(b) An allocation x is in the core if it unblocked by any coalition. In a 2-individual economy we have only 3 coalitions: $\{1, 2\}$ and $\{1\}, \{2\}$. Notice that an allocation is unblocked by a coalition $\{1, 2\}$ iff it is Pareto-efficient (just from the definition of the latter). So an allocation is in the core if it is Pareto efficient and is not dominated by the initial allocation for each consumer (so that to take care of the two smaller coalitions).

$$\mathcal{C} = \mathcal{P} \cap \{(x_{11}, x_{12}, x_{12}, x_{22}) : x_{11} = 21 - x_{12}, x_{11} = 10 - x_{12}, x_{11}x_{21} \geq 72, x_{12}x_{22}^2 \geq 108\}$$

(c)

Making use of the First Welfare Theorem (as both preferences are lns), we may search for an equilibrium among the Pareto efficient allocations only. Moreover, from the Consumer 2 problem we have:

$$\frac{x_{22}}{2x_{12}} = \frac{p_1}{p_2}$$
$$p_1 x_{12} + p_2 x_{22} = 3p_1 + 6p_2$$

Simplify this to obtain:

$$x_{12} = 1 + 2\frac{p_2}{p_1}$$

Now use our condition for the interior Pareto-efficient allocations:

$$(10 - x_{22})x_{12} = (21 - x_{12})\frac{x_{22}}{2}$$

Plug in x_{22}:

$$\left(10 - 2\frac{p_1}{p_2}x_{12}\right) x_{12} = (21 - x_{12})\frac{p_1}{p_2}x_{12}$$
$$10 - 2\frac{p_1}{p_2}x_{12} = (21 - x_{12})\frac{p_1}{p_2}$$

Plug in x_{12}:

$$10 - 2\frac{p_1}{p_2}\left(1 + 2\frac{p_2}{p_1}\right) = \left(21 - \left(1 + 2\frac{p_2}{p_1}\right)\right)\frac{p_1}{p_2}$$

Simplify to get:

$$\frac{p_1^*}{p_2^*} = \frac{4}{11}$$

From the prices we restore the equilibrium allocation:

$$x_{12}^* = 1 + 2\frac{11}{4} = 6.5 \Rightarrow x_{11}^* = 21 - 6.5 = 14.5$$
$$x_{22}^* = 2\frac{p_1}{p_2}x_{12}^* = \frac{8}{11}\frac{13}{2} = \frac{52}{11} \Rightarrow x_{21}^* = 10 - \frac{52}{11} = \frac{58}{11}$$

Answer: the equilibrium of this economy is given by price vector $p = (4, 11)$ and allocation $\left(14.5, \frac{58}{11}, 6.5, \frac{52}{11}\right)$.

(d) Clearly, the equilibrium is Pareto-efficient (by construction). We now need to check that it is not blocked by the smaller coalitions.

$$u^1(x_{11}, x_{21}) = \frac{29}{2}\frac{58}{11} = \frac{841}{11} \approx 76.5 > 72$$
$$u^1(x_{11}, x_{22}) = \frac{13}{2}\left(\frac{52}{11}\right)^2 = \frac{17576}{121} \approx 145 > 108$$

Hence, the equilibrium is in the core.

Problem 4

Question. *Consider a pure exchange economy with three consumers and three goods. Consumers' utility functions and endowments are*

$$\begin{aligned} u^1(x_1,x_2,x_3) &= \min(x_1,x_2) \qquad \mathbf{e^1} = (1,0,0) \\ u^2(x_1,x_2,x_3) &= \min(x_2,x_3) \qquad \mathbf{e}^2 = (0,1,0) \\ u^3(x_1,x_2,x_3) &= \min(x_1,x_3) \qquad \mathbf{e}^3 = (0,0,1) \end{aligned}$$

Find a competitive equilibrium allocation and corresponding price vector for this economy.

Solution.

Since everything is symmetric, the price vector p^* must satisfy

$$p_1^* = p_2^* = p_3^*.$$

The equilibrium allocation is then

$$\begin{aligned} x_1^1 = x_2^1 = \frac{1}{2},\ x_3^1 = 0, \\ x_2^2 = x_3^2 = \frac{1}{2},\ x_1^2 = 0, \\ x_1^3 = x_3^3 = \frac{1}{2},\ x_2^3 = 0. \end{aligned}$$

More formally, note that with Leontief preferences, the optimal demands is given consuming equal amounts of the goods that the consumer values. That is,

$$x_1^{1*} = x_2^{1*},\ x_2^{2*} = x_3^{2*},\ x_1^{3*} = x_3^{3*}.$$

The budget constraints bind (if it's slack by say ϵ increase the consumption of every good the agent values by $\frac{\epsilon}{\sum p_i}$) so it must be that

$$\begin{aligned} p_1x_1^{1*} + p_2x_2^{1*} &= p_1, \\ p_2x_2^{2*} + p_3x_3^{2*} &= p_2, \\ p_1x_1^{3*} + p_3x_3^{3*} &= p_3. \end{aligned}$$

Normalizing $p_1 = 1$,[2] and imposing optimal demand

$$\begin{aligned} x_1^{1*} + p_2x_1^{1*} &= 1, \\ p_2x_2^{2*} + p_3x_2^{2*} &= p_2, \\ x_1^{3*} + p_3x_1^{3*} &= p_3. \end{aligned}$$

Market clearing conditions gives us

$$\begin{aligned} x_1^{1*} + x_1^{3*} &= 1, \\ x_2^{2*} + x_1^{1*} &= 1, \\ x_2^{2*} + x_1^{3*} &= 1. \end{aligned}$$

We have six equations and six unknowns. Solving this yields the result above (with all prices normalized to one).

Problem 5 (Theorem 1.2)

Question. *Prove Theorems 1.2 and 1.3 in (JR).*

[2] By Walras' law, the equations are not linearly independence and so we can only solve for $n-1$ prices.

Theorem 1.2 Let $\succeq$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(x)$ is a utility function that represents it. Then $v(x)$ also represents $\succeq$ if and only if $v(x) = f(u(x))$ for every x, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.

Proof. ($\Rightarrow$) By definition, since $u(\cdot)$ represents $\succeq$, $\forall x, y \in \mathbb{R}^n_+$

$$x \succeq y \Leftrightarrow u(x) \geq u(y)$$

Since f is strictly increasing on the image of $u(\cdot)$,

$$\begin{aligned} u(x) \geq u(y) \Leftrightarrow v(x) = f(u(x)) \geq f(u(y)) = v(y) \\ x \succeq y \Leftrightarrow v(x) \geq v(y) \end{aligned}$$

Hence, $v(\cdot)$ represent $\succeq$.
($\Leftarrow$) Suppose $u(x)$ and $v(x)$ represent the same preferences $\succeq$. For every $x \in \mathbb{R}^n_+$, define $f(\cdot)$ as follows: $f(u(x)) = v(x)$. $f(\cdot)$ is a well-defined function if $\nexists x, y \in \mathbb{R}^n_+$, such that $u(x) = u(y) = \bar{u}$ for some $\bar{u} \in \mathbb{R}$, but $f(u(x)) \neq f(u(y))$ — as this would entail, that $f(\bar{u})$ takes more than one value, contradicting $f(\cdot)$ is a function. Suppose by a way of contradiction that such bundles x, y do exist. By definition of $f(\cdot)$, then we should have that $v(x) \neq v(y)$. But both $v(\cdot)$ and $u(\cdot)$ represent the same preferences, hence

$$u(x) = u(y) \Leftrightarrow v(x) = v(y)$$

Contradiction. So $f(\cdot)$ is a well-defined function.

Clearly, $f : \mathbb{R} \to \mathbb{R}$, as the images of both $u(\cdot)$ and $v(\cdot)$ are in $\mathbb{R}$. The only thing we now need to check is that $f(\cdot)$ is strictly increasing.

Take some $x, y \in \mathbb{R}^n_+$. It, then, suffices to show that

$$u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y))$$

Since both $v(\cdot)$ and $u(\cdot)$ represent the same preferences, we should have:

$$u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y) \Leftrightarrow f(u(x)) \geq f(u(y))$$

where the last $\Leftrightarrow$ is straight from the definition of $f(\cdot)$. This completes the proof. □

Theorem 1.3 Let $\succeq$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:

1. $u(x)$ is strictly increasing if and only if $\succeq$ is strictly monotonic.
2. $u(x)$ is quasiconcave if and only if $\succeq$ is convex.
3. $u(x)$ is strictly quasiconcave if and only if $\succeq$ is strictly convex.

Proof. (1) Assume the preference relation $\succeq$ is strictly monotone. Take some $x \geq y$. We know that $x \succeq y$, and hence $u(x) \geq u(y)$. Now take some $x \gg y$. Since $x \succ y$, it is true that $x \succeq y$ and $y \not\succeq x$, so $u(x) \geq u(y)$ and $u(y) \not\geq u(x)$, meaning that $u(x) > u(y)$. Therefore, $u(\cdot)$ is strictly increasing. In a similar way, let $u(\cdot)$ be strictly increasing. Then, for any $x \geq y$ it holds that $u(x) \geq u(y)$, so $x \succeq y$, and for any $x \gg y$ it holds that $u(x) > u(y)$, so $x \succ y$. This proves the claim.

(2) Take a pair of bundles x, y and some $t \in [0, 1]$. Let $\succeq$ be convex. Since $tx + (1-t)y \succeq \min_{\succeq}\{x, y\}$, it holds that $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, so $u(\cdot)$ is quasiconcave. Conversely, let $u(\cdot)$ be quasiconcave, so $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, implying that $tx + (1-t)y \succeq \min_{\succeq}\{x, y\}$, so $\succeq$ is convex. This proves the claim.

(3) Take a pair of bundles x, y and some $t \in (0, 1)$. Let $\succeq$ be strictly convex. By the previous claim, $u(\cdot)$ is convex, so it remains to verify the strict inequality for $t \in (0, 1)$. Since $tx + (1-t)y \succ \min_{\succeq}\{x, y\}$, it holds that $u(tx+(1-t)y) > \min\{u(x), u(y)\}$, so $u(\cdot)$ is strictly quasiconcave. Conversely, let $u(\cdot)$ be strictly quasiconcave. We know by the previous claim that $\succeq$ is convex, so it remains to verify the strict preference for $t \in (0, 1)$. Strict quasiconcavity leads to $u(tx + (1-t)y) > \min\{u(x), u(y)\}$, implying that $tx + (1-t)y \succ \min_{\succeq}\{x, y\}$, so $\succeq$ is strictly convex. This proves the claim.

□

Problem 6 (5.3)

Question. *Consider an exchange economy. Let* $\mathbf{p}$ *be a vector of prices in which the price of at least one good is nonpositive. Show that, if the consumers' utility functions are strongly increasing, then aggregate excess demand cannot be zero in every market.*

Solution. Recall that u^i is strongly increasing if $\mathbf{x}^0 \geq \mathbf{x}^1$ and $\mathbf{x}^0 \neq \mathbf{x}^1$ implies

$$u^i\left(\mathbf{x}^0\right) > u^i\left(\mathbf{x}^1\right).$$

So, if $\mathbf{x}^0 \geq \mathbf{x}^1$ and $x_k^0 > x_k^1$ for some good k, then that u^i is strongly increasing means that $u^i\left(\mathbf{x}^0\right) > u^i\left(\mathbf{x}^1\right)$.

Recall also the consumer's problem:

$$\mathbf{x}^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) \in_{\mathbf{x}\in\mathbf{X}} \quad u^i(\mathbf{x}) \quad s.t. \quad \mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\mathbf{e}^i.$$

Excess demand function for good k is defined as

$$\mathbf{z}_k(\mathbf{p}) := \sum_{i\in\mathcal{I}} x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - \sum_{i\in\mathcal{I}} \mathbf{e}_k^i,$$
$$\mathbf{z}(\mathbf{p}) := \left(\mathbf{z}_1(\mathbf{p}), \mathbf{z}_2(\mathbf{p}), \ldots, \mathbf{z}_n(\mathbf{p})\right).$$

Now suppose that price of good k, p_k, is nonpositive; i.e. $p_k \leq 0$. Consider the budget constraint in this case

$$p_k x_k + \sum_{j=1, j\neq k}^{n} p_j x_j \leq p_k e_k + \sum_{j=1, j\neq k}^{n} p_j e_j.$$

Suppose that $p_k = 0$. Then, the consumer's problem has no solution since he would set x_k to be as high as possible and since we assume $\mathbf{X} = \mathbb{R}_+^n$, which, in particular, is unbounded above. That is $x_k^i = \infty$. Since $\mathbf{e}_k^i$ are all finite, this means that $\mathbf{z}_k(\mathbf{p}) = \infty$ (since each consumers can only consumer nonnegative amounts of each good). Thus, aggregate excess demand cannot be zero in every market, in particular, it cannot be zero for good k.

Now suppose that $p_k < 0$. Again, by making x_k arbitrarily large, the agent can make the left-hand side of the budget constraint arbitrarily small. Hence, $x_k^i = \infty$ which again implies that aggregate excess demand cannot be zero in every market.

Problem 7 (5.7)

Question. *Consider an exchange economy with two goods. Suppose that its aggregate excess demand function is* $z(p_1, p2) = \left(-1, \dfrac{p_1}{p_2}\right)$ *for all* $(p_1, p_2) \gg (0, 0)$.

(a) *Show that this function satisfies conditions 1 and 2 of Theorem 5.3, but not condition 3.*

(b) *Show that the conclusion of Theorem 5.3 fails here. That is, show that there is no* $(p_1^*, p_2^*) \gg (0, 0)$ *such that* $z(p_1^*, p_2^*) = (0, 0)$.

Solution.
(a) Condition 1: $z(\cdot)$ in continuous on $\mathbb{R}_{++}^2$.
For a vector function to be continuous, each of its component must be continuous. $z_1(p) = -1$ is trivially continuous. $z_2(p) = \dfrac{p_1}{p_2}$ is continuous on $\mathbb{R}_{++}^2$.
Condition 2: $p\cdot z(p) = 0$, $\forall p \gg 0$.
$p\cdot z(p) = -p_1 + p_2\dfrac{p_1}{p_2} = 0$ for any positive p_1, p_2. So this condition is met.
Condition 3: If $\{p_m\}$ is a sequence of price vectors in $\mathbb{R}_{++}^n$ converging to $\bar{p} \neq 0$ and $\bar{p}_k = 0$ for some good k, then for some good k' with $\bar{p}_{k'} = 0$ the associated sequence of excess demands in the market for good k', $\{z_{k'}(p_m)\}$ is unbounded above.
Consider a sequence of prices $\{p^m\}$, where $(p_1^m, p_2^m) = \left(\frac{1}{m}, 1\right)$. Notice that $\lim_{m\to\infty}(p_1^m, p_2^m) = (0, 1) \neq 0$. Then, if $z(p)$ satisfies the condition 3, it must be the case that the sequence $\{z(p^m)_k\}$ is unbounded from above for some good k. But $z(p^m) = \left(-1, \dfrac{1}{m}\right)$, so $z_k(p^m)$ is bounded from above by 2. Which is a contradiction.
(b) Notice that $\forall p \gg 0$, $z_1(p) = -1$. Then, clearly, $\nexists p^* \gg 0$, such that $z(p^*) = 0$, as at least $z_1(p^*) = -1$.

Problem 8 (5.14)

Question. *Suppose that each consumer i has strictly positive endowment vector, $\mathbf{e}^i$, and a Cobb-Douglas utility function on $\mathbb{R}^n_+$ of the form $u^i(\mathbf{x}) = x_1^{\alpha_1^i} x_2^{\alpha_2^i} \cdots x_n^{\alpha_n^i}$, where $\alpha_k^i > 0$ for all consumers i, and goods k, and $\sum_{k=1}^n \alpha_k^i = 1$ for all i.*

1. *Show that no consumer's utility function is strongly increasing on $\mathbb{R}^n_+$, so that one cannot apply Theorem 5.5 to conclude that this economy possesses a Walrasian economy.*

2. *Show that conditions 1, 2 and 3 of Theorem 5.3 are satisfied so that one can nevertheless use Theorem 5.3 directly to conclude that a Walrasian equilibrium exists here.*

Utility u^i is continuous, strongly increasing, and strictly quasiconcave in $\mathbb{R}^n_+$. If each consumer's utility function satisfies Assumption 5.1, and $\sum_{i=1}^I \mathbf{e}^i \gg \mathbf{0}$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$, such that $\mathbf{z}(\mathbf{p}^*) = 0$. Suppose $\mathbf{z} : \mathbb{R}^n_{++} \to \mathbb{R}$ satisfies the following three conditions:

1. *(continuity) $\mathbf{z}(\cdot)$ is continuous on $\mathbb{R}^n_{++}$;*

2. *(Walras' law) $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \gg \mathbf{0}$;*

3. *if $\{\mathbf{p}^m\}$ is a sequence of prices in $\mathbb{R}^n_{++}$ and $\mathbf{p}^m \to \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k, then for some good k' with $\bar{p}_{k'} = 0$, the sequence $\{z_{k'}(\mathbf{p}^m)\}$is unbounded above.*[3]

Then, there exists $\mathbf{p}^* \gg 0$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Solution.

Part (a)

Strongly increasing requires that for any $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, $\mathbf{x}^0 \geq \mathbf{x}^1$ and $\mathbf{x}^0 \neq \mathbf{x}^1$ implies that $u^i(\mathbf{x}^0) > u^i(\mathbf{x}^1)$. Consider $\mathbf{x}^0 = (0, 1, 1, \ldots, 1)$ and $\mathbf{x}^1 = (0, 0, \ldots 0)$. Clearly, $\mathbf{x}^0 \geq \mathbf{x}^1$ and $\mathbf{x}^0 \neq \mathbf{x}^1$ but

$$u^i(\mathbf{x}^0) = u^i(\mathbf{x}^1) = 0.$$

Thus, u^i does not satisfy strong increasingness.

Part (b)

The consumer solves

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} \quad x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad s.t. \quad \sum_{k=1}^n p_k x_k \leq \sum_{k=1}^n p_k e_k.$$

We cannot have corner solutions for the same reason as in Q3. Thus, the solution satisfies the first-order condition:

$$\begin{aligned} \frac{\alpha_k}{x_k} u(\mathbf{x}) &= \lambda p_k, \forall k \\ \Rightarrow \frac{\alpha_k}{\alpha_j} \frac{x_j}{x_k} &= \frac{p_k}{p_j} \\ \Leftrightarrow x_k &= \frac{\alpha_k}{\alpha_j} \frac{p_j}{p_k} x_j. \end{aligned}$$

Without loss of generality, suppose that $p_1 = 1$, then

$$x_k = \frac{\alpha_k}{\alpha_1} \frac{1}{p_k} x_1, \forall k.$$

Substituting into the budget constraint (which holds with equality at the optimal):

$$\sum_{k=1}^n p_k e_k = \sum_{k=1}^n p_k \left(\frac{\alpha_k}{\alpha_1} \frac{1}{p_k} x_1 \right)$$

[3] The last condition says that if prices of some, but not all goods, is arbitrarily close to zero, then the (excess) demand for at least one of those goods is arbitrarily high.

$$
\begin{aligned}
&= \frac{x_1}{\alpha_1}\sum_{k=1}^{n}\alpha_k \\
x_1 &= \alpha_1\left(\sum_{k=1}^{n}p_k e_k\right). \\
\Rightarrow x_k &= \frac{\alpha_k}{p_k}\underbrace{\left(\sum_{j=1}^{n}p_j e_j\right)}_{:=w} = \frac{\alpha_k w}{p_k}.
\end{aligned}
$$

The aggregate excess demand functions are then defined as

$$
\begin{aligned}
z_k(\mathbf{p}) &:= \sum_{i\in\mathcal{I}}x_k^i - \sum_{i\in\mathcal{I}}e_k^i \\
&= \frac{1}{p_k}\sum_{i\in\mathcal{I}}\alpha_k^i w^i - \sum_{i\in\mathcal{I}}e_k^i, \\
\mathbf{z}(\mathbf{p}) &:= (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p})).
\end{aligned}
$$

Observe that:

- Since $\mathbf{p}$ is bounded away from zero, the function z_k is continuous on $\mathbb{R}^n_{++}$;
- Walras' law holds:

$$
\begin{aligned}
\mathbf{p}\cdot\mathbf{z}(\mathbf{p}) &= \sum_{k=1}^{n}p_k z_k(\mathbf{p}) \\
&= \sum_{k=1}^{n}p_k\left(\frac{1}{p_k}\sum_{i\in\mathcal{I}}\alpha_k^i w^i - \sum_{i\in\mathcal{I}}e_k^i\right) \\
&= \sum_{k=1}^{n}\sum_{i\in\mathcal{I}}\alpha_k^i w^i - \sum_{k=1}^{n}\sum_{i\in\mathcal{I}}p_k e_k^i \\
&= \sum_{i\in\mathcal{I}}w^i\underbrace{\sum_{k=1}^{n}\alpha_k^i}_{=1} - \sum_{i\in\mathcal{I}}\underbrace{\sum_{k=1}^{n}p_k e_k^i}_{=w^i} = 0;
\end{aligned}
$$

- Let $\{\mathbf{p}^m\}$ be sequence of prices in $\mathbb{R}^n_{++}$ and $\mathbf{p}^m \to \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k, then

$$
\lim_{p_k^m\to 0} z_k(\mathbf{p}^m) = \infty
$$

  so that condition (iii) is satisfied.

It follows from the theorem that there exists $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Problem 9 (5.21)

Question. *Consider an exchange economy with the two consumers. Consumer 1 has utility function $u_1(x_1, x_2) = x_2$ and endowment $e_1 = (1, 1)$ and consumer 2 has utility function $u_2(x_1, x_2) = x_1 + x_2$ and endowment $e_2 = (1, 0)$.*

(a) Which of the hypotheses of Theorem 5.4 fail in this example?

(b) Show that there does not exist a Walrasian equilibrium in this exchange economy.

Solution.
(a) Conditions of Theorem 5.4: each individual i has a utility function that satisfies Assumption 5.1 and $\sum_{i\in\mathcal{I}}e^i \gg 0$.
Assumption 5.1: utility u^i is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

Consumer 1: $u^1(x_1, x_2) = x_2$. This function is clearly continuous in $\mathbb{R}^n_+$, but it is not strongly increasing. Indeed, consider two bundles $(1,1)$ and $(0,1)$.Then,

$$(1,1) \geq (0,1)\,\&\,(1,1) \neq (0,1), \text{ but } u^1(1,1) = u^1(0,1) = 1$$

u^1 is not strictly quasiconcave. Indeed, consider, for example, the same two bundles.

$$u^1(\lambda(1,1) + (1-\lambda)(0,1)) = 1 = \min\{u^1(1,1), u^1(0,1)\}$$

.

Consumer 2: $u^1(x_1, x_2) = x_1 + x_2$. This function is continuous in $\mathbb{R}^n_+$. It is strongly increasing. Indeed, consider any two bundles (x_1'', x_2'') and (x_1', x_2'), such that $(x_1'', x_2'') \geq (x_1', x_2')$ and $(x_1'', x_2'') \neq (x_1', x_2')$ Then,

$$u^2(x_1'', x_2'') = x_1'' + x_2'' > x_1' + x_2' = u^2(x_1', x_2') \text{ as either } x_1'' > x_1' \text{ or } x_2'' > x_2',$$

This utility function is not strictly quasiconcave, though. Consider, for example two bundles $(1,0)$ and $(0,1)$, as before.

$$u^2(\lambda(1,0) + (1-\lambda)(0,1)) = 1 = \min\{u^2(1,0), u^2(0,1)\}$$

which contradicts the definition of a strictly quasiconcave function.

Finally, the condition $\sum_{i \in \mathcal{I}} e^i \gg 0$ is clearly met.

(b) The demands for the two consumers are:

$$x^1(p) = \left(0, 1 + \frac{p_1}{p_2}\right)$$

$$x^2(p) = \begin{cases} \left(0, \dfrac{p_1}{p_2}\right), \text{ if } p_1 > p_2 \\ (1,0)\,, \text{ if } p_2 > p_1 \\ \{(1 \geq x_1 \geq 0, 1 \geq x_2 \geq 0) : x_1 + x_2 = 1\} \text{ if } p_1 = p_2 \end{cases}$$

Then,

$$z(p) = \begin{cases} \left(-2, 1 + 2\dfrac{p_1}{p_2}\right) \text{ if } p_1 > p_2 \\ \left(-1, \dfrac{p_1}{p_2}\right) \text{ if } p_2 > p_1 \\ (x_1 - 2, 2 - x_1) \text{ if } p_1 = p_2 \end{cases}$$

Hence, it cannot be the case that $p_1 > p_2$ (since $z_1 = -2$). $p_2 > p_1$ is not possible either, as $z_1 = -1$. Finally, if $p_1 = p_2$, then $z(p) = 0 \Leftrightarrow x_1 = 2$ but $x_1 \leq 1$. Contradiction. We run out of possible values of p, hence, there is no equilibrium for this economy.

Problem 10 (5.22)

Question. *This exercise will guide you through a proof of a version of Theorem 5.4 when the consumer's utility function is quasiconcave instead of strictly quasiconcave and strictly increasing instead of strongly increasing.*

1. *If the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous, quasiconcave and strictly increasing, show that, for every $\varepsilon \in (0,1)$, the approximating utility function $v_\varepsilon : \mathbb{R}^n_+ \to \mathbb{R}$ defined by*

$$v_\varepsilon(\mathbf{x}) = u\left((x_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (x_i)^\varepsilon, \ldots, (x_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (x_i)^\varepsilon\right)$$

is continuous, strictly quasiconcave and strongly increasing. Note that the approximation to $u(\cdot)$ becomes better and better as $\varepsilon \to 1$ because $v_\varepsilon(\mathbf{x}) \to u(\mathbf{x})$ as $\varepsilon \to 1$.

2. *Show that if, in an exchange economy with positive endowment of each good, each consumer's utility function is continuous, quasiconcave and strictly increasing on $\mathbb{R}^n_+$, there are approximating utility functions as in part (a) that define an exchange economy with the same endowments and possessing a Walrasian equilibrium. If, in addition, each consumer's endowment gives him a positive amount of each good, show that any limit of such Walrasian equilibria, as the approximation becomes better and better (e.g., as $\varepsilon \to 1$ in the approximations in part (a)), is a Walrasian equilibrium of the original exchange economy.*

3. *Show that such a limit of Walrasian equilibria as described in part (b) exists. You will then have prove the following result:*

   > *If each consumer in an exchange economy is endowed with a positive amount of each good and has a continuous, quasiconcave and strictly increasing utility function, a Walrasian equilibrium exists.*

4. *Which hypotheses of the Walrasian equilibrium existence result proved in part (b) fail to hold in the exchange economy in Exercise 5.21?*

Solution.

Part (a)

Continuity Since summations, multiplications and powers are continuous operators, we know that terms $x_j^\varepsilon + (1-\varepsilon)\sum_{i=1}^n x_i^\varepsilon$ are all continuous. Since u is continuous, this means that v_ε is also continuous.

Strong increasingness That u is strictly increasing means that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$,

$$\begin{aligned} \mathbf{x} \geq \mathbf{y} &\Rightarrow u(\mathbf{x}) \geq u(\mathbf{y}), \\ \mathbf{x} \gg \mathbf{y} &\Rightarrow u(\mathbf{x}) > u(\mathbf{y}). \end{aligned}$$

We want to show that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ with $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, $v_\varepsilon(\mathbf{x}) > v_\varepsilon(\mathbf{y})$.

So take any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ with $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. This means that every element in $\mathbf{x}$ is weakly larger than the elements in $\mathbf{y}$ (i.e. $x_i \geq y_i$ for all i) but there exists some elements of $\mathbf{x}$ that are strictly larger. That is,

$$\begin{aligned} & x_i \geq y_i, \forall i \in \{1 \dots n\} \\ \Rightarrow & (x_i)^\varepsilon \geq (y_i)^\varepsilon, \forall i \in \{1 \dots n\}, \varepsilon \in (0,1), \text{with at least one strict inequality} \\ \Rightarrow & \sum_{i=1}^n (x_i)^\varepsilon > \sum_{i=1}^n (y_i)^\varepsilon. \end{aligned}$$

The last two implications imply that

$$(x_j)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (x_i)^\varepsilon > (y_j)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (y_i)^\varepsilon, \forall j = 1, 2, \dots, n. \tag{1}$$

Since u is strictly increasing, this means that

$$\begin{aligned} & u\left((x_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (x_i)^\varepsilon, \dots, (x_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (x_i)^\varepsilon\right) \\ > & u\left((y_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (y_i)^\varepsilon, \dots, (y_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (y_i)^\varepsilon\right). \end{aligned}$$

That is,

$$v_\varepsilon(\mathbf{x}) > v_\varepsilon(\mathbf{y}).$$

Strict quasiconcavity That u is quasiconcave means that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$,

$$u(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{u(\mathbf{x}), u(\mathbf{y})\}, \forall t \in [0,1].$$

We want to show that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ with $\mathbf{x} \neq \mathbf{y}$,

$$v_\varepsilon(t\mathbf{x} + (1-t)\mathbf{y}) > \min\{v_\varepsilon(\mathbf{x}), v_\varepsilon(\mathbf{y})\}, \forall t \in (0,1).$$

Defining $z_i = tx_1 + (1-t)y_i$, we wish to show that

$$u\left((z_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (z_i)^\varepsilon, \dots, (z_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^n (z_i)^\varepsilon\right)$$

$$> \min \left\{ u\left(x_1 + (1-\varepsilon)\sum_{i=1}^{n} x_i, \ldots, x_n + (1-\varepsilon)\sum_{i=1}^{n} x_i \right), u\left(y_1 + (1-\varepsilon)\sum_{i=1}^{n} y_i, \ldots, y_n + (1-\varepsilon)\sum_{i=1}^{n} y_i \right) \right\}$$

Take any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ with $\mathbf{x} \neq \mathbf{y}$. Consider any element of $t\mathbf{x} + (1-t)\,\mathbf{y}$, and note that

$$(z_j)^\varepsilon = (tx_j + (1-t)\,y_j)^\varepsilon \geq t\,(x_j)^\varepsilon + (1-t)\,(y_j)^\varepsilon, \forall j \in 1, 2, \ldots, n$$

with at least one strict inequality (since taking power to $\varepsilon \in (0,1)$ is a strictly concave operation). Summing across all i's gives

$$\sum_{i=1}^{n} (z_i)^\varepsilon > t \sum_{i=1}^{n} (x_i)^\varepsilon + (1-t) \sum_{i=1}^{n} (y_i)^\varepsilon,$$

which implies that, for all $j = 1, 2, \ldots, n$,

$$\begin{aligned}
&(z_j)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (z_i)^\varepsilon \\
> &t\left((x_j)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon \right) + (1-t)\left((y_j)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon \right)
\end{aligned}$$

Hence, from the fact that u is strictly increasing,

$$\begin{aligned}
&v_\varepsilon\,(t\mathbf{x} + (1-t)\,\mathbf{y}) \\
= &u\left((z_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (z_i)^\varepsilon, \ldots, (z_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (z_i)^\varepsilon \right) \\
> &u\left(t\left((x_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon \right) + (1-t)\left((y_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon \right), \right. \\
&\quad \left. \ldots, t\left((x_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon \right) + (1-t)\left((y_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon \right) \right)
\end{aligned}$$

By the fact that u is quasiconcave,

$$\begin{aligned}
&u\left(t\left((x_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon \right) + (1-t)\left((y_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon \right), \right. \\
&\quad \left. \ldots, t\left((x_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon \right) + (1-t)\left((y_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon \right) \right) \\
\geq &\min \left\{ u\left((x_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon, \ldots, (x_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (x_i)^\varepsilon \right), \right. \\
&\quad \left. , u\left((y_1)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon, \ldots, (y_n)^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} (y_i)^\varepsilon \right) \right\} \\
= &\min \{ v_\varepsilon(\mathbf{x}), v_\varepsilon(\mathbf{y}) \}.
\end{aligned}$$

That is,

$$v_\varepsilon\,(t\mathbf{x} + (1-t)\,\mathbf{y}) > \min \{ v_\varepsilon(\mathbf{x}), v_\varepsilon(\mathbf{y}) \}.$$

Part (b) and (c)

We prove the two parts together.

We know from Part (a) that we can take approximating utility functions. Consider the economy $\mathcal{E}_\varepsilon = \left(v^i_\varepsilon, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ with $\mathbf{e} \gg \mathbf{0}$. Since v^i_ε satisfies Assumption 5.1, it follows from Theorem 5.4 that there exists Walrasian equilibrium; i.e. $\mathbf{p}^*_\varepsilon \gg \mathbf{0}$ such that $\mathbf{z}\,(\mathbf{p}^*_\varepsilon) = \mathbf{0}$. Since only the price ratios matter (because demand for goods are homogenous of degree zero in prices), we can rescale $\mathbf{p}_\varepsilon^*$ such that $\sum_{i=1}^n p_{\varepsilon i}^* = 1$. Then, $\{\mathbf{p}_\varepsilon^*\}$ is a bounded sequence and therefore there exists a subsequence such that $\{\mathbf{p}_\varepsilon^*\} \to \mathbf{p}^*$ where $\mathbf{p}^* \in \mathbb{R}_+^n$ and $\mathbf{p}^* \neq \mathbf{0}$.

Now It remains to show that $\mathbf{p}^*$ is the Walrasian equilibrium of the economy $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ if each consumer has strictly positive endowment of every good. That is, we would like to show that there exists bundles $\hat{\mathbf{x}} \in \mathbb{R}_+^{nI}$ such that, if $\mathbf{e}^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$,

- $\hat{\mathbf{x}}^i$ maximizes $u^i\left(\mathbf{x}^i\right)$ such that $\mathbf{p}^* \cdot \mathbf{x}^i \leq \mathbf{p}^* \cdot \mathbf{e}^i$;
- market clears; i.e. $\sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i$.

Define the solution to the agent's maximization problem as

$$\mathbf{x}_\varepsilon^i := \mathbf{x}_\varepsilon^i\left(\mathbf{p}_\varepsilon^*, \mathbf{p}_\varepsilon^* \cdot \mathbf{e}^i\right), \forall i \in \mathcal{I}.$$

Since v_ε is continuous and the budget set is compact along the sequence $\{\mathbf{p}_\varepsilon^*\}$ (since $\mathbf{p}_\varepsilon^* \gg \mathbf{0}$ for all $\epsilon \in (0,1)$), by the Theorem of Maximum, we know that $\mathbf{x}_\varepsilon^i$ is compact-valued (i.e. closed and bounded) and continuous (on $\mathbf{p} \gg \mathbf{0}$) for all $\varepsilon \in (0,1)$. Thus, the sequence $\left\{\mathbf{x}_\varepsilon^i\right\}$ is uniformly bounded for all $\varepsilon \in (0,1)$ by the endowment (since market clearing condition holds for every ε) so that there exists a convergent subsequence—so suppose $\left\{\mathbf{x}_\varepsilon^i\right\} \to \mathbf{x}^i$ for some $\mathbf{x}^i \in \mathbb{R}_+^n$.

We now wish to show that $\mathbf{x}^i$ is WEA given the economy $\mathcal{E} = \left(u^i, \mathbf{e}^i\right)_{i\in\mathcal{I}}$ with equilibrium prices $\mathbf{p}^*$ given $\mathbf{e}^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$.

Since v_ε is strongly increasing, for any $\varepsilon \in (0,1)$, the budget constraint must bind and the income for each consumer is strictly positive since $\mathbf{p}_\varepsilon^* \gg \mathbf{0}$ and $\mathbf{e}^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$.

$$\mathbf{p}_\varepsilon^* \cdot \mathbf{x}_\varepsilon^i = \mathbf{p}_\varepsilon^* \cdot \mathbf{e}^i > 0, \forall i \in \mathcal{I}.$$

Taking the limit as $\varepsilon \to 1$,

$$\mathbf{p}^* \cdot \mathbf{x}^i = \mathbf{p}^* \cdot \mathbf{e}^i > 0, \forall i \in \mathcal{I}$$
$$\mathbf{p}^* \cdot \sum_{i\in\mathcal{I}}^n \left(\mathbf{x}^i - \mathbf{e}^i\right) = 0\cdot$$

where the strict inequality follows from our observation that $\mathbf{p}^* \in \mathbb{R}_+^n$ and $\mathbf{p}^* \neq \mathbf{0}$, and $\mathbf{e}^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$. That income for every agent is strictly positive in the limit means that agents, given u^i is strictly increasing, exhausts his strictly positive budget. From this and the fact $\mathbf{x}_\varepsilon^i$ clears the market at $\mathbf{p}_\varepsilon^*$ and $\left\{\mathbf{x}_\varepsilon^i\right\} \to \mathbf{x}^i$ implies that market clearing holds in the limit too:

$$\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i.$$

To show that each $\mathbf{x}^i$ maximizes utility u^i given the prices $\mathbf{p}^*$ and $\mathbf{e}^i$, suppose, by way of contradiction, that there exists another bundle $\mathbf{y}^i \in \mathbb{R}_+^n$ that is strictly preferred and affordable by i; i.e.

$$u^i\left(\mathbf{y}^i\right) > u^i\left(\mathbf{x}^i\right), \mathbf{p}^* \cdot \mathbf{y}^i \leq \mathbf{p}^* \cdot \mathbf{e}^i > 0.$$

Since u^i is continuous, for sufficiently large $t \in (0,1)$,

$$u^i\left(t\mathbf{y}^i\right) > u^i\left(\mathbf{x}^i\right).$$

Since $\mathbf{e}^i \gg \mathbf{0}$ and $\mathbf{p}^* \neq \mathbf{0}$, it must be that

$$\mathbf{p}^* \cdot t\mathbf{y}^i < \mathbf{p}^* \cdot \mathbf{e}^i > 0.$$

Moreover, that $\mathbf{p}_\varepsilon^* \to \mathbf{p}^*$, $\mathbf{x}_\varepsilon^i \to \mathbf{x}^i$, u^i is continuous, and v_ε converges point-wisely to u^i (given in part (a) of the problem), for ε sufficiently close to 1,

$$\mathbf{p}_\varepsilon^* \cdot t\mathbf{y}^i < \mathbf{p}_\varepsilon^* \cdot \mathbf{x}_\varepsilon^i = \mathbf{p}_\varepsilon^* \cdot \mathbf{e}^i > 0,$$
$$\Rightarrow v_\varepsilon^i\left(t\mathbf{y}^i\right) > v_\varepsilon^i\left(\mathbf{x}_\varepsilon^{i*}\right).$$

But this contradicts the fact that $\mathbf{x}_\varepsilon^{'}$ maximizes utility v_ε^i given prices $\mathbf{p}_\varepsilon^*$. Therefore, we conclude that each $\mathbf{x}^i$ maximizes utility u^i given the prices $\mathbf{p}^*$ and $\mathbf{e}^i$.

Part (d)

The hypothesis that each agent has strictly positive endowment of all goods fails in Exercise 5.21.