1.1

3. $\vec{u}+\vec{v}+\vec{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}-3\\1\\-2\end{bmatrix}+\begin{bmatrix}2\\-3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$2\vec{u}+2\vec{v}+\vec{w} = \begin{bmatrix}2\\4\\6\end{bmatrix}+\begin{bmatrix}-6\\2\\-4\end{bmatrix}+\begin{bmatrix}2\\-3\\-1\end{bmatrix} = \begin{bmatrix}-2\\3\\1\end{bmatrix}$

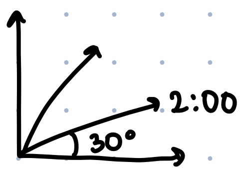
These lie in a plane because $\vec{w} = c\vec{u} + d\vec{v}$ where $c=d=-1$.

$-1+3=2$
$-2-1=-3$
$-3+2=-1$

13. a) $\vec{v} = 0$

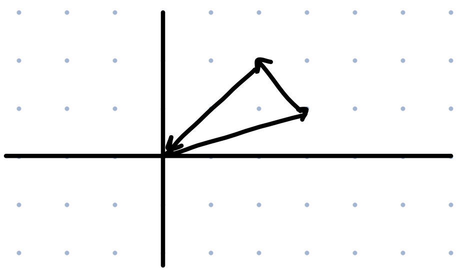
b) all other vectors cancel each other out except for 8:00 since it is missing 2:00.

c)



$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

21. $\cancel{\vec{v}} - \cancel{\vec{u}} + \cancel{\vec{w}} - \cancel{\vec{v}} + \cancel{\vec{u}} - \cancel{\vec{w}} = \vec{0}$



29. $\vec{u}=\begin{bmatrix}1\\3\end{bmatrix}$ $\vec{v}=\begin{bmatrix}2\\7\end{bmatrix}$ $\vec{w}=\begin{bmatrix}1\\5\end{bmatrix}$ $\vec{b}=\begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}1 & 2 & 1\\3 & 7 & 5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$ or $\begin{bmatrix}-2\\1\\0\end{bmatrix}$

yes, because 2 of the vectors can already fill the plane space.

there exists a free variable!

$x+2y+z=0$
$x=-2y-z$
$x=-2(1-2z)-z$
$x=-2+3z$

$3(-2y-z)+7y+5z=1$
$-6y+7y-3z+5z=1$
$y=1-2z$
now choose any z

$z=0$
$x=-2$
$y=1$

$z=1$
$x=1$
$y=-1$

1.2

5. $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\quad \vec{u}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

$\|\vec{v}\| = \sqrt{1+9} = \sqrt{10}$ $\qquad \|\vec{w}\| = \sqrt{4+1+4} = 3$

$\vec{u}_1 \cdot \vec{u}_1 = 0$ $\qquad \vec{u}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $\qquad \vec{u}_2 \cdot \vec{u}_2 = 0$ $\qquad \vec{u}_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$\frac{1}{\sqrt{10}} \cdot \vec{x}_1 + \frac{3}{\sqrt{10}} \cdot \vec{x}_2 = 0$ $\qquad \frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z = 0$

$\vec{x}_1 = -3 \quad \vec{x}_2 = 1$ $\qquad -\frac{4}{3} + \frac{+2}{3} + \frac{2}{3} = 0$

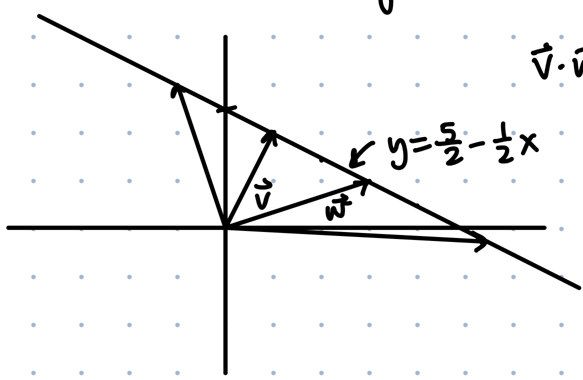
22. schwarz inequality: $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$

a) $(v_1w_1 + v_2w_2)(v_1w_1 + v_2w_2) \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$

$\cancel{v_1^2 w_1^2} + 2v_1v_2w_1w_2 + \cancel{v_2^2 w_2^2} \leq \cancel{v_1^2 w_1^2} + v_1^2 w_2^2 + v_2^2 w_1^2 + \cancel{v_2^2 w_2^2}$

b) $(v_1w_2 - v_2w_1)^2 = v_1w_2^2 + v_2^2w_1^2 - 2v_1v_2w_1w_2$ ✓

28. $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\qquad \vec{v} \cdot \vec{w} = x + 2y = 5$



$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

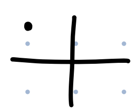
$y = \frac{5}{2} - \frac{1}{2}x$

these $\vec{w}$'s lie along a line because they all satisfy that $\vec{v} \cdot \vec{w} = 5$, which translates to a linear equation $x + 2y = 5$.

the shortest $\vec{w}$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

31. $x + y + z = 0. \qquad -2 + 1 + 1 = 0.$

$\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\quad \|\vec{v}\| = \sqrt{4+1+1} = \sqrt{6} = \|\vec{w}\|$



then $\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|} = \cos\theta = \frac{-2-2+1}{6} = -\frac{1}{2}.$ $\qquad \theta = \cos^{-1}(-\frac{1}{2}).$

$\vec{v} \cdot \vec{w} = xz + xy + yz$

$\|\vec{v}\| \cdot \|\vec{w}\| = x^2 + y^2 + z^2$

1.3

2. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$y_1 = 1$
$y_1 + y_2 = 1$
$y_1 + y_2 + y_3 = 1$

$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$

$y_1 = 1$
$y_1 + y_2 = 4$
$y_1 + y_2 + y_3 = 9$

$\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$S\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 21 \end{bmatrix}$ sum of first 5 odd #s: 21

10.
$$\Delta z = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{\text{forward diff matrix}}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$$

$z_3 = -b_3$
$-b_3 - z_2 = b_2$
$-b_3 - b_2 = z_2$

$-b_3 - b_2 - z_1 = b_1$
$-b_3 - b_2 - b_1 = z_1$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}}_{\Delta^{-1}}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

12. $C\vec{x} = \vec{b}$ find C^{-1} in $\vec{x} = C^{-1}\vec{b}$.

$x_2 = b_1$
$x_3 - x_1 = b_2$
$x_4 - x_2 = b_3$
$-x_3 = b_4$

$x_2 = b_1$
$x_4 - b_1 = b_3$ $x_4 = b_1 + b_3$
$x_3 = -b_4$
$-b_4 - b_2 = x_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

13. $C\vec{x} = \vec{b}$ where C is a centered difference. row i is $x_{i+1} - x_{i-1}$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$$\cancel{x_2} - 0 = b_1$$
$$\cancel{x_3} - x_1 = b_2$$
$$\cancel{x_4} - \cancel{x_2} = b_3$$
$$x_5 - \cancel{x_3} = b_4$$
$$0 - \cancel{x_4} = b_5$$

$$-x_1 + x_5 = \underbrace{b_2 + b_4 = 0}_{\text{must occur.}}$$

2.1

5. $$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5 \end{aligned}$$ They also satisfy the third.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

solution:

$$\begin{aligned} x + y + z &= 2 \\ y &= 1 \end{aligned} \qquad x + 1 + z = 2 \quad (\text{line } L)$$

3 solutions on L:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

6. doesn't contain or intersect that line.

13. a) A matrix w/ m rows and n columns multiplies a vector with n components to produce a vector w/ m components.

$$m\overbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}^{n}\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}n = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}m$$

b) The planes from the m equations $A\vec{x} = \vec{b}$ are in n dim space.
The combo of the columns of A are in n dim space.

31. Magic matrix

$$M_3 = \begin{bmatrix} 8 & 3 & 4 \\ 3 & 4 & 8 \\ 4 & 8 & 3 \end{bmatrix} \qquad M_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$$

$$M_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 34 \\ 34 \\ 34 \\ 34 \end{bmatrix}.$$

sum of all numbers $1 \to 9$: $\frac{n(n+1)}{2} = \frac{9 \cdot 10}{2} = 45$ sum of all #s. $\frac{45}{3} = \boxed{15}$.

$1 \to 16$: $\frac{16 \cdot 17}{2} = 8 \cdot 17.$ $\frac{8 \cdot 17}{4} = \boxed{34}$

2.2

11. a) A 3x3 system of linear equations can't have exactly 2 solns. If (x,y,z) and (X,Y,Z) are both solutions, then any linear combination of these solutions must also be a solution.

b) If 25 planes meet at 2 points, they also meet on the line that contains those 2 pts.

19.
$$x+4y-2z=1 \quad (1)$$
$$x+7y-6z=6 \quad (2)$$
$$3y+qz=t \quad (3)$$

singular: there is not a full set of pivots. either ∃ no soln or ∞ solutions.

$$\cancel{x}+4y-2z=1$$
$$-(\cancel{x}+7y-6z=6)$$
$$-3y+4z=-5$$
$$+(3y-4z=t)$$
$$0 \quad 0=-5+t$$

If $q=-4$, this system would be singular.
If $t=5$, there would be an infinite # of solutions.

21.
$$2x+y=0$$
$$x+2y+z=0$$
$$y+2z+t=0$$
$$z+2t=5$$

$$\begin{bmatrix}2&1&0&0&0\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&2&5\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix}1&2&1&0&0\\2&1&0&0&0\\0&1&2&1&0\\0&0&1&2&5\end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix}1&2&1&0&0\\0&-3&-2&0&0\\0&1&2&1&0\\0&0&1&2&5\end{bmatrix}$$

$$\xrightarrow[-\frac{1}{3}R_3 \to R_3]{R_2 \leftrightarrow R_3} \begin{bmatrix}1&2&1&0&0\\0&1&2&1&0\\0&1&2/3&0&0\\0&0&1&2&5\end{bmatrix} \xrightarrow[R_3-R_2 \to R_3]{\frac{3}{4}R_3 \to R_3} \begin{bmatrix}1&2&1&0&0\\0&1&2&1&0\\0&0&1&0&0\\0&0&1&2&5\end{bmatrix} \to \begin{bmatrix}1&2&1&0&0\\0&1&2&1&0\\0&0&1&2&5\\0&0&0&1&\frac{5}{2}\end{bmatrix}$$

all pivots are 1

$t=5/2$

$z+2(\frac{5}{2})=5 \quad z=0$

$y+\cancel{2z}+t=0 \quad y=-5/2$

$x+2y+\cancel{z}=0 \quad x=5$

$$\vec{b}=\begin{bmatrix}5\\-5/2\\0\\5/2\end{bmatrix} \text{ to } A\vec{x}=\vec{b}$$

$K\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} \to \overset{-1+4}{\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 5 \end{bmatrix}} \to \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -5/2 \end{bmatrix}$$

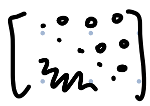
$t = -5/2$
$z = 0$
$y - \cancel{2z} + t = 0 \quad y = 5/2$
$x - 2y + \cancel{z} = 0 \quad x = 5$

$\vec{b} = \begin{bmatrix} 5 \\ 5/2 \\ 0 \\ -5/2 \end{bmatrix}$ for $K\vec{x} = \vec{b}$

31. Row j of U is a lin combo of the first j rows of A.
$U = E \cdot A$ where E is an invertible matrix describing elementary row operations.
then if $A\vec{x} = \vec{0}$, $U\vec{x} = E \cdot A\vec{x}$
$U\vec{x} = E \cdot \vec{0}$
$U\vec{x} = \vec{0} \Rightarrow$ so $U\vec{x} = \vec{0}$ too.

if $A\vec{x} = \vec{b}$, $E(A\vec{x}) = E\vec{b}$
$U\vec{x} = E\vec{b}$ and $E\vec{b} \neq \vec{b}$ (not necessarily) so no.

If A starts out lower triangular,

$A =$  then U is only diagonal since all numbers below pivots would be removed.

2.3

10. add row 3 to row 1.

$$E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{13}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1+b_3 \\ b_2 \\ b_3 \end{bmatrix}$$

b) add row 1 to row 3 and row 3 to row 1 at the same time.

$$E_{13}+E_{31} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow M\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1+b_3 \\ b_2 \\ b_1+b_3 \end{bmatrix}$$

c) add row 1 to row 3, then row 3 to row 1.

want: $M\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3+b_1+b_1 \\ b_2 \\ b_3+b_1 \end{bmatrix}$ $\qquad M = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

21. E adds row 1 to row 2.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EF = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq FE = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

27. a) no soln

$[A\ \vec{b}] = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$ $\qquad d=0 \quad c=2$
$\qquad a=3 \quad b=4$

b) infinitely many solns $d=0 \quad c=0 \quad a=3 \quad b=4.$

a and b have no impact on solvability.

30. a) $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\qquad$ b) $F = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$EM = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad FEM = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $\quad EFEM = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad EEFEM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$

$$M = B \cdot E^{-1} \cdot F^{-1} \cdot (E^2)^{-1} = B \cdot A \cdot B \cdot A^2$$

2.4

12. a) $BA = 4A$

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

b) $BA = 4B$

$BA \cdot A^{-1}$

this is only true if $B = 0$.

c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

d) $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

21.

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 0 \\ 2y \\ 2z \\ 2t \end{bmatrix} \qquad A^2\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 4z \\ 4t \end{bmatrix} \qquad A^3\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8t \end{bmatrix} \qquad A^4\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

32.

$$AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

38. a)

$$A = \begin{bmatrix} \vec{a_1}^T \\ \vec{a_2}^T \\ \vec{a_3}^T \\ \vdots \\ \vec{a_m}^T \end{bmatrix} \qquad A^TA = \begin{bmatrix} (\vec{a_1}^T)^T (\vec{a_2}^T)^T \ldots \end{bmatrix}\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \end{bmatrix} = \vec{a}_1 \cdot \vec{a}_1^T + \ldots \vec{a}_m \cdot \vec{a}_m^T$$

b)

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$\vec{a}_1^T = [2 \;\; 3]$

$\vec{a}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 20 & 26 \\ 26 & 34 \end{bmatrix}$$

$$A^TCA = A^T\begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 40 & 52 \\ 52 & 68 \end{bmatrix}$$

$A^TCA = c\vec{a}_1 \cdot \vec{a}_1^T + \ldots + c\vec{a}_m \cdot \vec{a}_m^T$

2.5 invertible: nonzero determinant

11. a) invertible A and B s.t. A+B is not invertible

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$ $A+B = \begin{bmatrix} 0 & 0 \\ 1 & 5 \end{bmatrix}$ no 2 pivots.

b) singular A and B s.t. A+B is invertible

$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

15. A matrix w/ a column of zeros would not have an inverse b/c no amount of row exchanges would establish a pivot in that column. no full set of pivots => not invertible.

29. a) true. no full set of pivots
b) false. could still exist dependent vectors
c) true. $(A^{-1})^{-1} = A$.
$(A^2)^{-1} = A^{-2} = A^{-1} \cdot A^{-1}$

43. Elim for block matrix. ?

30.

$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \sim \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix} \sim \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$ pivots = $a, a-b \neq 0$.

$C = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$ want C to be singular

C is singular for $c = \{2, 0, 7\}$

2.6 $A = LU$

1. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} = \vec{b} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x} = \vec{c}$

subtracted $l_{21} = 1$ times row 1 from row 2.
reverse step adds l_{21} times row 1 to row 2.
matrix for reverse step $= \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ l_{21} & l_{21}+1 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} 5 \\ 5l_{21}+2 \end{bmatrix}$$

$$L \cdot U\vec{x} = L \cdot \vec{c} \Rightarrow A\vec{x} = \vec{b}$$

11. $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$ A is already upper triangular

$L = I$ since A is already U. in $A = LU$. $U = A$.

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ in $A = LDU$, $U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ A is tridiagonal

$$A \overset{R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \overset{R_3 - R_2}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ echelon form $= U$

$L = E_1^{-1} \cdots E_n^{-1}$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$L_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $L_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$L_{21} \cdot L_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

check:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = A \checkmark$$

Then $U = L^T$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ — U doesn't change after division by pivots.

$A = LDL^T$

$$A = \begin{bmatrix} a & a & 0 \\ \cancel{a}\ 0 & \cancel{a+}b & b \\ 0 & \cancel{b}\ 0 & \cancel{b+}c \end{bmatrix}$$

$$L_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad L_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}$

$A = LDL^T$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$

A tridiagonal matrix has a bidirectional L and U.

20. tridiagonal = L and U factors have only 2 nonzero diags.

$$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

each row only needs one elem function to remove the nonzero below its pivot.

$$U = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

elem ops:
- add $2R_1$ to $-R_2$
- subtract R_2 from R_3
- subtract R_3 from R_4

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

check: $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} = T$ ✓

2.7

7. a) False; block matrix $\begin{bmatrix}0 & A\\ A & 0\end{bmatrix}$ is symmetric iff A is symmetric.

b) $\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}3 & 7 & 1\\ 7 & 3 & 7\\ 1 & 7 & 3\end{bmatrix} = \begin{bmatrix}13 & 17 & 9\\ 18 & 20 & 18\\ 9 & 17 & 7\end{bmatrix}$ True!

c) $\cancel{\begin{bmatrix}1 & 2 & 1\\ 2 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}}$ False. Invertible symmetric matrices have symmetric inverses.
$A \cdot A^{-1} = I$.

d) $(ABC)^T = C^T \cdot B^T \cdot A^T = CBA$ for symmetric matrices A, B, C. True.

17. a) S is not invertible.

$S = \begin{bmatrix}0 & 3\\ 0 & 0\end{bmatrix}$

b) S is invertible but cannot be factored into LU (needs row exchanges).

$S = \begin{bmatrix}0 & 3\\ 1 & 0\end{bmatrix}$

c) S has a negative D in LDL^T.

$S = \begin{bmatrix}-5 & 3\\ 3 & 1\end{bmatrix}$ has pivots $D = \begin{bmatrix}-2 & 0\\ 0 & 14\end{bmatrix}$

30. $A\vec{x} = \vec{y}$

$\begin{bmatrix}1 & 50\\ 40 & 1000\\ 2 & 50\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}700\\ 3\\ 3000\end{bmatrix}$ values of 1 truck and 1 plane?

$A^T\vec{y} = \begin{bmatrix}1 & 40 & 2\\ 50 & 1000 & 50\end{bmatrix}\begin{bmatrix}700\\ 3\\ 3000\end{bmatrix} = \begin{bmatrix}6820\\ 68000\end{bmatrix}$

$700 + 120 + 6000 = 6820$

$35000 + 3000 + 30000 = 68000$

37. If you take powers of a permutation matrix, why is some P^k eventually = to I?
∃ n! P's for an order n. Eventually, 2 powers of P must be the same permutation.

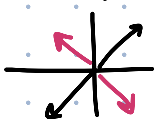
If $P^r = P^s$, $P^{r-s} = I$. and $r - s \le n!$.

3.1 spaces of vectors

4. $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \Rightarrow$ "vector" in space M of all 2x2 matrices

zero vector: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\frac{1}{2}A$: $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $-A$: $\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$

The matrices in the smallest subspace containing A are those that are equivalent to cA.



9. a) A set of vectors in $\mathbb{R}^2$ for $\vec{x}+\vec{y}$ stays in set but $\frac{1}{2}\vec{x}$ may be outside.
$\vec{x}+\vec{y}$ where the components are integers. then $\frac{1}{2}\vec{x}$ changes the value and results in it being outside the space.

b) ... for which every $c\vec{x}$ stays in plane but $\vec{x}+\vec{y}$ may be outside. NOT 2 1/4-planes!
If we remove the x axis from the xy plane, but keep the origin, then certain additions wouldn't be in the plane. $(-1,1)+(1,2)=(0,3)$

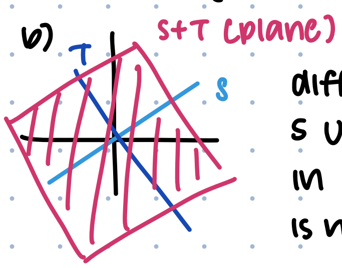
21. A and C have the same column space because the columns of C are linear combinations of cols of A.

30. S and T are 2 subspaces of a vector space V.

a) addition requirement:
$u = s_1 + t_1$
$v = s_2 + t_2$
$u+v = (s_1+s_2)+(t_1+t_2) \in S+T$

scalar mult req:
$c(s+t) \in S+T$

b)



difference btwn S+T and S∪T?
S ∪ T does not include the sums of vectors in S and T separately → the addition requirement is not met.

The span of S ∪ T is S+T, meaning that all vectors in S ∪ T are linear combinations of vectors in S+T.

3.2 The Nullspace of A

5. a) A square matrix has no free variables: false.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ a square matrix doesn't necessarily have full pivot set.

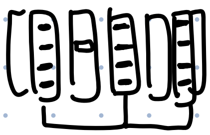
b) An invertible matrix has no free variables.
True. Must require a full set of pivots to be invertible.

c) An mxn matrix has no more than n pivot vars.

$$\begin{bmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \ \square \end{bmatrix}$$ True, since

d) 5x7 5x3
no more than the smaller of either m or n variables.

14. 4x5 matrix w/ 4 pivots:



$\vec{x} = \vec{0}$

DON'T OVERTHINK IT!
column 5 is a lin combo of previous columns, so it must not have a pivot.

special solution: one vector s b/c one free var

$$s = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$ col 5 + col 3 + col 1 = 0.

null space: contains all multiples of this vector s, so it is a line in $\mathbb{R}^5$.

32. Kirchoff's Current Law

$$A^T = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

1: $y_3 = y_1 + y_4$
2: $y_1 = y_5 + y_2$
3: $y_2 = y_3 + y_6$
4: $y_4 + y_5 + y_6 = 0$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{s}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

59. RREF form of R^TR is always R w/ extra 0 rows
R = 2x3:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \qquad R^TR = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3.3

7. $b_3 - 2b_2 + 4b_1 = 0$

$\vec{b} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$ If $\vec{b}$ is in col space, $A\vec{x} = \vec{b}$

$-2-12$

$$A|\vec{b} = \begin{bmatrix} 1 & 3 & 1 & \vdots & 4 \\ 3 & 8 & 2 & \vdots & -2 \\ 2 & 4 & 0 & \vdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & 14 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & (b_3-2b_1) - 2(b_2 - 3b_1) \end{bmatrix}$$

$b_3 + 4b_1 - 2b_2 = 0$ ✓

17. The largest possible rank of a 6×4 matrix is 4. $\exists$ a pivot in every col of U and R. The solution to $A\vec{x} = \vec{b}$ is unique (full column rank). The nullspace of A is only $\vec{0}$ (no free vars). An example is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

31. a) The only soln to $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\vec{x} = 2 \times 1$ $A \cdot \vec{x} = 3 \times 1$ A must be 3×2.

$[3 \ \ \cancel{2}][\cancel{2} \ \ 1] = [3 \ \ 1]$

$$\begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

b) The only soln to $B\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$B \cdot \vec{x} = 2 \times 1$

$[2 \ \ 3][3 \ \ 1] = [2 \ \ 1]$

$$\begin{bmatrix} -4 & 2 & 0 \\ 1 & & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

→ impossible since a 2×3 matrix must have free variables, so there would be infinite solutions

34. $A: 3 \times 4$ $\vec{s} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ is the only special soln to $A\vec{x} = \vec{0}$.

a) rank = 3.

complete soln: $\vec{x} = \begin{bmatrix} \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \ \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c) Since ~~there is a free var~~ there is full ROW RANK for A and R, there are infinite solutions.

3.4

9. suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^3$.
   a) These are linearly dependent bc they are more than # dim of $\mathbb{R}^3$.
   b) $\vec{v}_1$ and $\vec{v}_2$ will be dependent if they are linear combinations of each other.
   c) $\vec{v}_1$ and $\vec{0}$ are dependent b/c 0 times $\vec{v}_1$ is $\vec{0}$, so they are lin combos of each other.

20. Find basis for $x-2y+3z=0$ in $\mathbb{R}^3$.

$$x = 2y - 3z \qquad y = y \qquad z = z \qquad y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad z = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Find basis for intersection of the xy plane is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ (only x and y vars).
Find basis for vectors ⊥ to plane.
their dot product must be 0.

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = -2+2+0=0 \checkmark \qquad \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -3+3=0 \checkmark$$

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

alternatively, find normal vector to plane: $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$?

35. $\{1, x, x^2, x^3\}$ for cubic polynomials.

for subspace where $p(1)=0$, that means that when $x=1$, the polynomial $=0$.

$\{x-1, x^2-1, x^3-1\}$ for the basis of this subspace.

42. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ these 24 vectors span a subspace S.
find vectors $\vec{x}$ so that the dimension of S is

a) zero
↳ that means that only the zero vector is in the subspace.

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) one
↳ all values are the same.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

c) three

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

↳ all rearrangements are ⊥ to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

d) four

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

3.5 Dimensions of the Four Subspaces

13. a) If $m=n$ then the row space of A = the column space
square matrix. then the # of pivot cols = # pivot rows.
dim(row space) = rank(A) = r = dim(col space).
FALSE. though they have the same dimensions, they don't have to be equal.

$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ col space: $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ row space: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$

b) True. Negating values doesn't change the dimensions or the spaces.

c) If A and B share the same $\overset{4}{\vee}$ subspaces, then A is a multiple of B.
True. this means that they must encapsulate the same space and be dependent.

24. $A^T\vec{y}=\vec{d}$ is solvable when $\vec{d}$ is in the column space. The solution $\vec{y}$ is unique when the nullspace contains only the $\vec{0}$.

26. $a \neq 0$, b and c are given. want rank 1. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

we need $\begin{bmatrix} b \\ d \end{bmatrix}$ to be a multiple of $\begin{bmatrix} a \\ c \end{bmatrix}$

$na = b$ $\quad n = \frac{b}{a}$
$nc = d$ $\quad \boxed{d = c \cdot \frac{b}{a}}$

Then, the rowspace would be:

$\{ [1, \frac{b}{a}] \}$

$$\begin{bmatrix} 1 & \frac{b}{a} \\ c & \frac{cb}{a} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \cancel{\frac{cb}{a} - \frac{cb}{a}} \end{bmatrix}$$

The nullspace would be $\left\{ \begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix} \right\}$

$x + \frac{b}{a}y = 0$
$y = y$

$$\vec{x} = y\begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix}$$

The rowspace and nullspace are $\perp$: $\begin{bmatrix} 1 \\ b/a \end{bmatrix} \cdot \begin{bmatrix} -b/a \\ 1 \end{bmatrix} = 0$.

31. A and B have the same 4 subspaces. Both are in RREF. Then F must equal G.

$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$ Suppose that $F \neq G$. Then the rowspace of A would be $[I \;\; F]$, and for B it would be $[I \;\; G]$.
Then they wouldn't cover the same space b/c they are already in RREF. The pivots in F differ than those in G, so the vector space made w/ them would definitely be different.

4.1 orthogonality

6. $A\vec{x}=\vec{b}$ has no solutions

$$\left.\begin{aligned} x+2y+2z &= 5 \\ 2x+2y+3z &= 5 \\ 3x+4y+5z &= 9 \end{aligned}\right\} \text{by inspection: } \vec{y}=\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow 0=1.$$

$$\begin{bmatrix} 1 & 2 & 2 & 5 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & -2 & -1 & -5 \\ 0 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -5 \\ 0 & 1 & 1/2 & 5/2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 5/2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Then $\vec{y}$ is in the left nullspace of A.
$A\vec{x}=\vec{b}$ needs
$0=(\vec{y}^TA)\vec{x}=\vec{y}^T\vec{b}$ but $\vec{y}^T\vec{b}=1$ here.

19. Suppose L is a one dimensional subspace (a line) in $\mathbb{R}^3$. Its orthogonal complement $L^\perp$ is the subspace perp to L. Then $(L^\perp)^\perp$ is a subspace perp to $L^\perp$.
$(L^\perp)^\perp = L$.

23. If a subspace S is contained in a subspace V, that means $S \subseteq V$. Then $S^\perp \supseteq V^\perp$.
This is because the space perp to S is greater than space perp to V, and must include all vectors perp to V.

28. a) False. In order for two planes to be orthogonal subspaces, all vectors in each subspace must be perp to each other, not just one pair.

b) False. $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ orthogonal complement of $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \\ 4 \\ -4 \end{bmatrix}\right\}$?

The problem is that the orthogonal complement should contain EVERY vector that is $\perp$ to the subspace. Since this is $\mathbb{R}^5$, the orth. complement should contain 3 vectors.

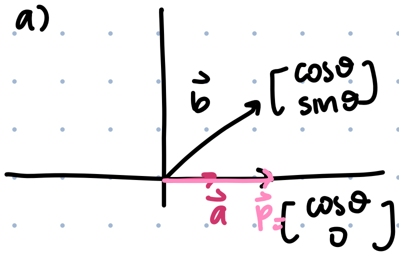
c) 2 subspaces that meet only in the $\vec{0}$ are orthogonal. False.
The vectors may not be $\perp$ to each other.

31. $N=\text{null}(A)$. $\Rightarrow$ basis for nullspace of A.
$B=\text{null}(N')$ $\Rightarrow$ basis for rowspace of A.
↳ transpose

4.2 Projections

2. Draw proj of $\vec{b}$ onto $\vec{a}$ and compute from $\vec{p}=\hat{x}\vec{a}$.

a)

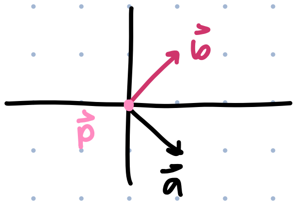


$$\vec{p}=\hat{x}\vec{a} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\cdot\vec{a}.$$

$$\vec{p}=\hat{x}\vec{a}= \frac{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}}{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}}\cdot\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{\cos\theta}{1}\cdot\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ 0\end{bmatrix}$$

$\vec{a}=\begin{bmatrix}1\\ 0\end{bmatrix}$

b) $\vec{a}=\begin{bmatrix}1\\ -1\end{bmatrix}$ $\vec{b}=\begin{bmatrix}1\\ 1\end{bmatrix}$



$$\vec{p}=\hat{x}\vec{a}=\frac{\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}}{\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}}\cdot\begin{bmatrix}1\\ -1\end{bmatrix}= \frac{0}{2}\cdot\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

15. If A is doubled, then $P = 2A(4A^TA)^{-1}2A^T = A(A^TA)^{-1}A^T$.
The col space of $2A = C(A)$. Is $\hat{x}$ the same for A and $2A$?

NO. $\hat{x}$ is the factor when $\vec{b}$ is a multiple of A. Then the value $\hat{x}$ would differ by a factor of 2 amongst projections to A vs. $2A$.

24. The nullspace of A^T is orthogonal to the col space $C(A)$. So if $A^T\vec{b}=\vec{0}$, the proj of $\vec{b}$ onto $C(A)$ should be $\vec{p}=\vec{0}$. This is because, treating $P=A^T$, the vector $\vec{b}$ is in the nullspace of A^T so if $\vec{b}$ were to be projected onto $C(A)$, it would be $\vec{0}$.

$P\vec{b} = \cancel{A(A^TA)^{-1}A^T\vec{b}} \to \vec{0} \to \vec{0}$ Then $P\vec{b}=\vec{0}$. ✓

31. In $\mathbb{R}^m$, given $\vec{b}$ and a combo $\vec{p}$ of $\vec{a}_1 \dots \vec{a}_n$. How to test if $\vec{p}$ is projection of $\vec{b}$ onto subspace spanned by the $\vec{a}$'s?

Check that error $\vec{e}=\vec{b}-\vec{p}$ is orthogonal to A (i.e. all the $\vec{a}$'s).

4.3 Least Squares Approximation

1. $b = 0, 8, 8, 20 \quad t = 0, 1, 3, 4$

$8 + 24 + 80 = 112$

$m = 4$ (4 pts)

$$A^TA = \begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \qquad = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

then $A^TA\hat{x} = A^T\vec{b}$

$$\left[\begin{array}{cc:c} 4 & 8 & 36 \\ 8 & 26 & 112 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

then $C = 1 \quad D = 4$

$e = b - p$

The line is $b = 1 + 4t$

The heights are $p_i = \{1, 5, 13, 17\}$ and the errors $e_i = \{-1, 3, -5, 3\}$

The min $E = \Sigma e_i^2 = 1 + 9 + 25 + 9 = 44$

12. $\vec{b} = (b_1, \dots b_m)$ projected onto line through $\vec{a} = (1, \dots 1)$. Solve m eqns $ax = b$ in 1 unknown

a) Solve $\vec{a}^T\vec{a}\hat{x} = \vec{a}^T\vec{b}$ to show $\hat{x}$ is mean of the $\vec{b}$'s.

$\vec{a}^T\vec{a} = m \qquad \hat{x} = \dfrac{\Sigma b_i}{m} =$ mean of b's.

$\vec{a}^T\vec{b} = \Sigma b_i$

b) $\vec{e} = \vec{b} - \vec{a}\hat{x}$ and variance $\|\vec{e}\|^2$ and STD $\|\vec{e}\|$.

$$\|\vec{e}\|^2 = (b_1 - a_1\hat{x})^2 + \dots + (b_m - a_m\hat{x})^2$$

$$\Downarrow$$

$$(b_1 - \text{mean})^2 + \dots + (b_m - \text{mean})^2$$

$\|e\| = (b_1 - \text{mean}) + \dots + (b_m - \text{mean})$

c) $\hat{b} = 3$ is closest to $\vec{b} = (1, 2, 6)$. Check that $\vec{p} = (3, 3, 3)$ is $\perp$ to $\vec{e}$ and find the 3x3 P.

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

then $\vec{p} \cdot \vec{e} = -6 - 3 + 9 = 0$ ✓

The projection matrix $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

22. Find $C + Dt$ best fitting to $b = 4, 2, -1, 0, 0$ at $t = -2, -1, 0, 1, 2$.

$$A^TA = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$4 + 1 + 0 + 1 + 4 \qquad -8 - 2$

so $C + Dt$ is $1 - 1t$.

28. Suppose cols of A are dependent. How to find matrix B s.t. $P = B(B^T B)^{-1} B^T$ gives the projection onto col space of A?

Find basis for C(A) and create matrix B whose columns are that basis.
Or, B = only pivot columns of A.

4.4 Orthonormal Bases and Gram-Schmidt

3. a) If A has 3 orthogonal columns each of length 4, what is A^TA?

$A = QR$ where R has $\begin{bmatrix} 4 & \cdot & \cdot \\ 0 & 4 & \cdot \\ 0 & 0 & 4 \end{bmatrix}$

- since the columns are orthogonal, any of the cols' dot product with another column is 0.
- each dot product of the same column would be $4 \cdot 4 = 16$.
  $(A^TA)_{ij} = \|A(:,j)\|^2 = 4^2 = 16$.
- then A^TA would have 16 on the diagonal and 0 everywhere else!

$A^TA = 16I$.

b) If A has 3 orthogonal columns of length 1, 2, 3, what is A^TA?

$A^TA = I$ with 1^2 2^2 3^2 on diagonals instead. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

15. a) Find orthonormal vectors s.t. $\vec{q}_1, \vec{q}_2$ span the colspace of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$
$\vec{q}_1, \vec{q}_2, \vec{q}_3$

$A = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{A^T\vec{b}}{A^TA} \vec{A}$ $\quad A^T\vec{b} = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 1-2-8 = -9$

$A^TA = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 1+4+4 = 9$

$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \vec{b}$ ... that won't do.

let's try making cols of A orthogonal, then finding $\vec{q}_3$.

$\|\vec{a}\| = \sqrt{1+4+4} = 3$ $\qquad \|\vec{b}\| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

$\boxed{\vec{q}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}$ $\qquad \boxed{\vec{q}_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}}$

we need a $\vec{q}_3$ s.t. $\vec{q}_3 \cdot \vec{q}_1 = 0$ and $\vec{q}_3 \cdot \vec{q}_2 = 0$.

$\vec{q}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{aligned} & x + 2y - 2z = 0 \\ -& (x - y + 4z = 0) \\ \hline & 3y - 6z = 0 \\ & y = 2z \end{aligned}$$

choose $z = 1$
$y = 2$

$x + 4 - 2 = 0$
$x = -2$
$-2 - 2 + 4 = 0$ ✓

$\vec{c} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ $\quad \|\vec{c}\| = 3$

$\boxed{\vec{q}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}$

Try it G-S algo way.

$\vec{A} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ $\boxed{\vec{q}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.}$

$$\vec{B} = \vec{b} - \frac{\vec{A}^T\vec{b}}{\vec{A}^T\vec{A}}\vec{A}. = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{\overbrace{[1\ 2\ -2]\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}}^{-9}}{\underbrace{[1\ 2\ -2]\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}_{9}}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \boxed{\vec{q}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$$

$$\vec{C} = \vec{c} - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$$

$$\begin{aligned} &x + 2y - 2z = 0 \\ + &2x + y + 2z = 0 \\ \hline &3x + 3y = 0 \\ &x = -y \end{aligned} \qquad \begin{aligned} &x = 1 \\ &y = -1 \\ &2 - 1 + 2z = 0 \\ &z = -1/2 \end{aligned} \qquad \vec{C} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \qquad \boxed{\vec{q}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}}$$

b) which of the 4 subspaces contains $\vec{q}_3$?

$\vec{q}_3$ is not in the colspace so is in the nullspace of A.

c) solve $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ by least squares.

$$R^TR\hat{x} = R^TQ^T\vec{b} \quad \underline{\text{or}} \quad R\hat{x} = Q^T\vec{b} \quad \underline{\text{or}} \quad \hat{x} = R^{-1}Q^T\vec{b}$$

$$A = QR = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\Big[$$

$$A\vec{x} = \vec{b} \qquad A^TA\hat{x} = A^T\vec{b} \qquad \hat{x} = (A^TA)^{-1}(A^T\vec{b})$$

$$A^TA = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \qquad (A^TA)^{-1} = \frac{1}{9}I$$

$$\begin{aligned} 1+4+4 &= 9 \\ 2+2-4 &= 0 \\ 2+2-4 &= 0 \\ 4+1+4 &= 9 \end{aligned} \qquad \begin{aligned} 1+4-14 &= -9 \\ 2+2+14 &= 18 \end{aligned}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -9 \\ 18 \end{bmatrix} = 9\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

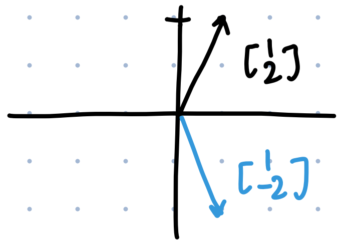
$$\hat{x} = \frac{1}{\cancel{9}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \cancel{9}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

32. If $\vec{u}$ is a unit vector, then $Q = I - 2\vec{u}\vec{u}^T$ is a reflection matrix. Find Q_1 from $\vec{u} = \begin{bmatrix}0\\1\end{bmatrix}$ and Q_2 from $\vec{u} = \begin{bmatrix}0\\ \sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix}$. Draw reflections when Q_1 and Q_2 multiply the vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

$$Q_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{u}_1^T = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

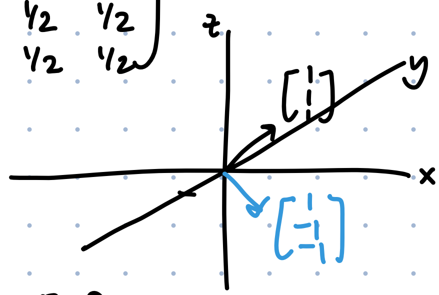
$$Q_1\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$



$$Q_2 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix}$$

$$\vec{u}_2 \cdot \vec{u}_2^T = \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 1/2 & 1/2\\0 & 1/2 & 1/2\end{bmatrix}$$

$$Q_2\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$$



37. $P = QQ^T$ is the projection onto col space of Q.
Add col $\vec{a}$ for $A = [Q \ \ \vec{a}]$. G-S replaces $\vec{a}$ w/ what vector $\vec{q}$?

Start w/ $\vec{a}$, subtract all projections of $\vec{a}$ on columns of Q ($\vec{p} = Q^T\vec{a}$) divide by length of $\vec{e} = \vec{a} - Q^T\vec{a}$ to find $\vec{q}$.

5.1 Determinants

4.

$$\det\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} = -1$$ because rows 1 and 3 of I were exchanged.

$$\det\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} = +1$$ b/c rows 1 and 4 AND rows 2 and 3 were exchanged.

15.

$$\begin{bmatrix}101&201&301\\102&202&302\\103&203&303\end{bmatrix} \sim \begin{bmatrix}101&201&301\\1&1&1\\2&2&2\end{bmatrix}$$ then rows 2 and 3 are dependent, so the matrix is singular. det = 0.

$$A=\begin{bmatrix}1&t&t^2\\t&1&t\\t^2&t&1\end{bmatrix}$$ $A = A^T$. square matrix.

$R_3 - t(R_2) \to R_3$

$t-1 \quad t^2-t \quad t^3-t^2$

$$\sim\begin{bmatrix}1&t&t^2\\t-1&1-t&t-t^2\\t^2-1&0&1-t^2\end{bmatrix}\sim\begin{bmatrix}1&t&t^2\\0&1-t^2&t-t^3\\0&t-t^3&1-t^2-t^4+t^3\end{bmatrix}\sim\begin{bmatrix}1&t&t^2\\0&1-t^2&t-t^3\\0&0&1-2t^2+t^3\end{bmatrix}$$

det=

$(1-t^2)(1-2t^2+t^3)$

$1-\cancel{t}-t^2+\cancel{t}$

$t-\cancel{t^2}-t^3+\cancel{t^2}$

$t^2-1 \quad t^3-t \quad t^4-t^3$

$1-t^2-t^4+t^3$

$0 \quad t-t^3 \quad t^2-t^4$

$1-t^2-\cancel{t^4}+t^3-t^2+\cancel{t^4}$

25. If the i,j entry of A is i·j, show that det A = 0.

example: $A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$ this setup guarantees that $A=A^T$, and the rows and cols are multiples of each other.

$A=\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$ The matrix is dependent, so det A = 0.

34. $\det A = \begin{vmatrix}R_1\\R_2\\R_3\end{vmatrix} = 6 \quad \det B = \det\begin{bmatrix}R3+R2+R1\\R2+R1\\R1\end{bmatrix} = \left(\det\begin{bmatrix}R_3\\R_2\\R_1\end{bmatrix} + \overset{0}{\det\begin{bmatrix}\cancel{R2}\\R1\\0\end{bmatrix}} + \overset{0}{\det\begin{bmatrix}\cancel{R1}\\0\\0\end{bmatrix}}\right) = \boxed{-6}$ ☺

5.2 Permutations and Cofactors

3. Show that $\det A = 0$, regardless of the 5 nonzeroes:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix} \begin{matrix} + \\ - \\ \end{matrix}$$

cofactors of row 1:

$$\left.\begin{aligned} C_{11} &= +1 \cdot 0 = 0 \\ C_{12} &= (-1) \cdot 0 = 0 \\ C_{13} &= +1 \cdot 0 = 0 \end{aligned}\right\} \text{all } 0$$

rank of A: # pivot rows = 2

6 terms in det A = $0+0+0+0+0+0=0$

15. The tridiagonal 1,1,1 matrix of order n has det E_n.

$$E_1 = |1| \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.

$$E_4 = E_3 - E_2 = [\cancel{1(0)} - 1(1) + \cancel{0} - 1(1) + 1 - 1(1) + \cancel{0} - 1(1) + \cancel{1(0)}] - [\cancel{0}]$$

$$E_3 = E_2 - E_1 = \Big[\cancel{a_{11}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} - a_{12}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + a_{13}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - a_{21}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + a_{22}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$- a_{23}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + a_{31}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - a_{32}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + a_{33}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\Big]$$
$$= \cancel{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} - |1|$$

- the 1,1 cofactor of the $n \times n$ matrix is E_{n-1}. Then the 1,2 cofactor is equivalent to $-E_{n-2}$. thus, $E_n = E_{n-1} - E_{n-2}$.

b) $E_1 = 1$, $E_2 = 0$. then for 3 till 8: $-1, -1, 0, +1, 1, 0,$

c) Find E_{100}.

1
0
−1
−1 → $E_4 = \boxed{-1 = E_{100}}$
0
1

$100 \div 6 = 16$ rem 4

24. $A = LU$ has $A_k = L_k \cdot U_k$ w/ block multiplication.

a) Suppose first 3 pivots of A are 2, 3, −1. What are the determinants of L_1, L_2, L_3 (have 1's on diagonals), U_1, U_2, U_3 and A_1, A_2, A_3?

$$A_1 = L_1 U_1 \qquad A = \begin{bmatrix} A_1 & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_1 & * \\ 0 & * \end{bmatrix}$$

after row reducing, pivots already found.

All dets of lower triangular matrices are 1.

$\det[A_3 = L_3 U_3] = (2 \cdot 3 \cdot -1) = (1 \cdot 2 \cdot 3 \cdot -1) = -6.$

$\det[A_2 = L_2 U_2] = (2 \cdot 3) = (1 \cdot 2 \cdot 3) = 6$

$\det[A_1 = L_1 U_1] = 2 = 2$

$12 \cdot 8 \quad 21 \cdot 5$

28. $D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix} = 45 + 84 + 96 - 105 - 48 - 72 =$

$\begin{matrix} - & - & - \\ & + & + & + \end{matrix}$

$129 \quad \boxed{225} - 153 = \boxed{0}$

$\begin{array}{r} 153 \\ 72 \\ \hline 225 \end{array}$

This matrix is not invertible because $2R_2 - R_1 = R_3$. Thus the rows are dependent!

5.3 Cramer's Rule, Inverses, and Volumes

2. Use Cramer's rule to solve for y.

a) $ax+by=1$
$cx+dy=0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y = \frac{\det B_y}{\det A} = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{ad-bc}$$

b) $ax+by+cz=1$
$dx+ey+fz=0$
$gx+hy+iz=0$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & 1 & c \\ d & 0 & f \\ g & 0 & i \end{vmatrix}}{D} = \frac{-1(di-gf)-d(i)-f(-g)+g(f)+i(-d)}{D}$$

$$= \frac{-di+gf-di+gf+gf-di}{D} = \frac{3gf-3di}{D}$$

10. From $AC^T = (\det A)I$ show that $\det C = (\det A)^{n-1}$.

$$\det(AC^T) = \det A \cdot \det C^T = \det \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \ \ldots \end{bmatrix}$$

$$\Downarrow$$

$$\det A \cdot \underbrace{\det C}_{=\det C^T} = (\det A)^n \Rightarrow \det C = (\det A)^{n-1} \ \checkmark$$

19. parallelograms:

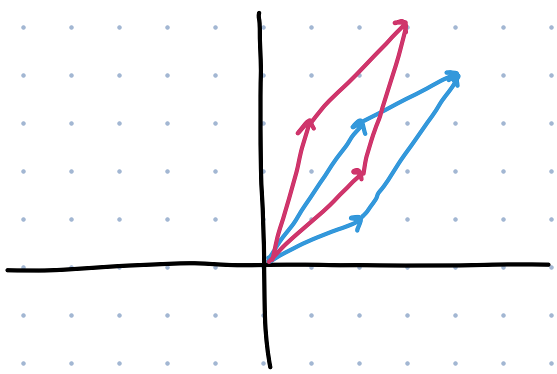
$$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6-2 = 4 \text{ area}$$

★ $(2,1)$ $(2,3)$

$\Updownarrow$ eqv area.

★ $(2,2)$ $(1,3)$

$$\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6-2=4 \text{ area}$$



39. If you know all 16 cofactors of a 4x4 invertible matrix, how would you find A?

$\frac{C^T}{\det A} = A^{-1}$. first generate C^T.
must find $\det A \rightarrow$ ~~can backwards solve w/ cofactors using "big formula"~~

$(\det A)(\det C) = (\det A)^n$.
then $\det C = (\det A)^{4-1}$.
$\sqrt[3]{(\det C)} = \det A$.

find A^{-1} and invert to find A.

6.1 Intro to Eigenvalues

16. $\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$.
then set $\lambda=0$.
Pf. $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$.

25. suppose A and B have the same evalues $\lambda_1, \ldots \lambda_n$ w/ same indp e-vectors.
Then $A=B$, bc any vector $\vec{x}$ is a combination $c_1\vec{x}_1+\ldots+c_n\vec{x}_n$.
$A\vec{x}$ is equivalent to $\lambda\vec{x} = B\vec{x}$.
$A\vec{v} = c_1\vec{x}\cdot\lambda_1 + \ldots + c_n\vec{x}\lambda_n = B\vec{v}$.

31. If we exchange $R_1 \leftrightarrow R_2$ and $c_1 \leftrightarrow c_2$, e-values don't change. Find evectors of A and B for $\lambda=11$. Rank one gives $\lambda_2=\lambda_3=0$.

$A\vec{x} - 11(I) = \vec{0}$

$$(A-11I)\vec{x} = \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$10-6+4$ $\quad -10+6+4$
$-3+15+12$ $\quad 3-15+12$
$-4-24+28$ $\quad 4+24-28$

$$\text{RREF}^{A} = \begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = 1/4$
$y = 3/4$
$z = z$

careful of signs!

$$\vec{x} = \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$B = PAP^T = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$$

$$(B-11I)\vec{x} = \begin{bmatrix} -5 & 3 & 3 \\ 2 & -10 & 1 \\ 8 & 4 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

35. $\exists$ 6 3x3 permutation matrices P. What #s can be determinants of P?
→ det's would have to be 1, since P has 3 indp columns. can also be -1. $P^TP = I$.
$(\det P)^2 = I$.
→ pivots can only be 1.
→ trace can be 0, 1, ~~2~~, or 3.

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

no ex, 1 ex, double ex

→ eigenvalues can be
$1, -1, e^{2\pi i/3}$ or $e^{-2\pi i/3}$.

$$P = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} \qquad P = \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$$

6.2 Diagonalizing a Matrix

1.

a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)(3-\lambda) = 0$

$\lambda_1 = 1, \lambda_2 = 3$

$(A - I)\vec{x} = \vec{0}$

$\begin{bmatrix} 0 & 2 & : & 0 \\ 0 & 2 & : & 0 \end{bmatrix}$ $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$2b = 0$ $b = 0$
$2b = 0$ $a = 1$

$(A - 3I)\vec{x}_2 = \vec{0}$

$\begin{bmatrix} -2 & 2 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$

$-2a + 2b = 0$
$2b = 2a$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

calculating X^{-1}: $\det X = 1$ $\frac{1}{\det X}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = X^{-1}$

$X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ so $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 3 = 0$
$\cancel{3} - 4\lambda + \lambda^2 - \cancel{3} = 0$
$\lambda(\lambda - 4) = 0$
$\lambda = \{0, 4\}$

for $\lambda = 0$,
$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$a + b = 0$ $a = -b$
$3a + 3b = 0$

for $\lambda = 4$,
$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$-3a + b = 0$ $b = 3a$
$3a - b = 0$ $3a = b$

then $X = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

$\det X = 3 + 1 = 4$ $X^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = X\Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

b) If $A = X\Lambda X^{-1}$ then $A^3 = X\Lambda^3 X^{-1}$ and $A^{-1} = X^{-1}\Lambda X$

6. Describe all matrices X that diagonalize this matrix A: (find all e-vectors).

$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$. $\det(A - \lambda I) = (4-\lambda)(2-\lambda) = 0$ $\lambda = \{2, 4\}$

$(A - 2I)\vec{x} = \vec{0}$
$\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$a = 0$ $\vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$b = 1$

$(A - 4\lambda)\vec{x} = \vec{0}$
$\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$a - 2b = 0$ $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$a = 2b$

all eigenvectors: $\left\{ c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \forall c_1, c_2 \in \mathbb{R}$

$X = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

All matrices that diagonalize A^{-1}? X^{-1}. $A^{-1} = X^{-1}\Lambda^{-1}X$

$\det X = 0-2 = -2$ $X^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

all matrices X^{-1}: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

20. suppose $A = X\Lambda X^{-1}$. Take dets to prove $\det A = \det \Lambda = \lambda_1\lambda_2 \ldots \lambda_n$.

$\det A = (\cancel{\det X})(\det \Lambda)(\det X^{-1})$

$\downarrow$

$\det X^{-1} = \frac{\cancel{1}}{\cancel{\det X}} \cdot \det[\text{mat}]$

this only works if A can be ~~inverted~~ diagonalizable. of course, since otherwise $\det \Lambda = 0$.

in order for A to be diagonalizable, A must have all unequal evalues.

11. Evalues of A are 2,2,5: matrix is certainly
   a) invertible? no, there may be dependent cols/rows true if A is 3x3.
   b) diagonalizable? no, 2=2. False, $\lambda = 2$ may only have 1 line of eigenvectors
   c) not diag? yes. False, $\lambda = 2$ may have a full set of eigenvectors.

37. suppose A_1 and A_2 are $n \times n$ invertible matrices. what matrix B shows that $A_2A_1 = B(A_1A_2)B^{-1}$? Then A_2A_1 is similar to A_1A_2: same eigenvalues.
↳ if B is the matrix of the eigenvalues

$A_2A_1 = B(A_1A_2)B^{-1}$

$A_2A_1B = BA_1A_2$

if $B = A_1^{-1}A_2^{-1}$

6.3 systems of DE

3. a) If every col of A adds to 0, that means the cols are dependent. Then $\lambda=0$ is an eigenvalue b/c the RREF of A would have a 0 as a pivot, and $\therefore$ 0 as an eigenvalue.
   The fact that $A\vec{x}=0\vec{x}$ comes from the fact that $\exists$ an eigenvector that creates the $\vec{0}$ vector from A.

   b) solve $\frac{d\vec{u}}{dt}=\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}\vec{u}$ with $\vec{u}(0)=\begin{bmatrix}4\\1\end{bmatrix}$. what is $\vec{u}(\infty)$?

$\det(A-\lambda I)=(-2-\lambda)(-3-\lambda)-6$
$\cancel{6}+2\lambda+3\lambda+\lambda^2\cancel{-6}$
$\lambda(5+\lambda)=0$
$\lambda=0 \quad \lambda=-5$

then $\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}\vec{x}_1=\begin{bmatrix}0\\0\end{bmatrix}$ $\quad$ $\begin{bmatrix}3 & 3\\ 2 & 2\end{bmatrix}\vec{x}_2=\begin{bmatrix}0\\0\end{bmatrix}$

$\vec{x}_1=\begin{bmatrix}3/2\\1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$ $\quad$ $3a=-3b$ $\quad$ $\vec{x}_2=\begin{bmatrix}1\\-1\end{bmatrix}$

$-2a+3b=0$
$2a-3b=0$
$a=\frac{3}{2}b$

1) write $\begin{bmatrix}4\\1\end{bmatrix}$ as a combo of the evectors: $\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$
   $(c_1, c_2)=(1,1)$.

2) The factors $e^{\lambda t}$ give exponential solns: $e^{0t}\vec{x}_1$ and $e^{-5t}\vec{x}_2$.

3) The combo starting from $\vec{u}(0)$ is $1e^{0t}\begin{bmatrix}3\\2\end{bmatrix}+1e^{-5t}\begin{bmatrix}1\\-1\end{bmatrix}$.

Then, λ_1 has a 0 real part, and λ_2 has a negative real part. Therefore, the $\vec{u}(t)$ is not stable, and will not approach 0. It will approach $\begin{bmatrix}3\\2\end{bmatrix}$ steady state.

11. The solution to $y''=0$ is a straight line $y=C+Dt$. convert into a matrix eqn:

$\frac{d}{dt}\begin{bmatrix}y\\y'\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y\\y'\end{bmatrix}$ has soln $\begin{bmatrix}y\\y'\end{bmatrix}=e^{At}\begin{bmatrix}y(0)\\y(0)\end{bmatrix}$

$A^2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

The series for e^{At} is $e^{At}=I+t\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+\text{zeroes}=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$.
Then $\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}y(0)\\y'(0)\end{bmatrix}=\begin{matrix}y(0)+ty'(0)\\y'(0)\end{matrix}=\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}\rightarrow\checkmark$

26. 2 reasons why the matrix exp e^{At} is never singular:
1) It's inverse is $e^{-At}=\frac{1}{e^{At}}$.

2) why are these e-values nonzero? If $AX=\lambda X$ then $e^{At}\vec{x}=e^{\lambda t}\vec{x}$. $e^{\lambda t}\neq 0$.

32. Prove that square of e^{A} is e^{2A}.
If A can be diagonalized, $e^{A}=X\cdot e^{\Lambda}\cdot X^{-1}$.
$e^{2A}=(Xe^{\Lambda}\cancel{X^{-1}})(\cancel{X}e^{\Lambda}X^{-1})=Xe^{2\Lambda}X^{-1}$.

6.4 Symmetric Matrices

6. Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ?

$$\det(S - \lambda I) = (-2-\lambda)(7-\lambda) - 36 = 0$$
$$-14 + 2\lambda - 7\lambda + \lambda^2 - 36 = \lambda^2 - 5\lambda - 50 = 0$$
$$(\lambda - 10)(\lambda + 5) = 0$$
$$\lambda = \{10, -5\}$$

$$\boxed{\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}}$$

for $\lambda = 10$: $S - 10I = \left[\begin{array}{cc:c} -12 & 6 & 0 \\ 6 & -3 & 0 \end{array}\right]$ $\quad -12x = -6y$, $6x = 3y$, $2x = y$ $\quad \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

for $\lambda = -5$: $S + 5I = \left[\begin{array}{cc:c} 3 & 6 & 0 \\ 6 & 12 & 0 \end{array}\right]$ $\quad 3x = -6y$, $6x = -12y$, $x = -2y$ $\quad \vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\boxed{Q = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}}$$

11. If $\lambda = a + ib$ is an eigenvalue of A, then $\bar{\lambda} = a - ib$ is also an eigenvalue. Why does every 3x3 matrix (real) have at least one real eigenvalue?
A real 3x3 matrix may have 2 eigenvalues that are complex conjugates. Then the third one must be real, since it'd be impossible for it <u>and</u> its complex conjugate to be a 3rd and 4th one.

23. a) A matrix w/ real eigenvalues and n real eigenvectors is symmetric.
FALSE.
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

b) A matrix w/ real eigenvalues and n orthonormal eigenvectors is symmetric.
TRUE. Then, can write $Q \Lambda Q^T$ diagonalization.

c) The inverse of an invertible matrix is symmetric.
FALSE. $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

d) The eigenvector matrix Q of a symmetric matrix is symmetric.
TRUE.
$Q \Lambda Q^T = (Q \Lambda Q^T)^T = Q^T \cdot \Lambda^T \cdot Q$ $\quad$ Then $Q = Q^T$.

33. Suppose $A^T = -A$. (real, antisymmetric matrix).

a) $x^T A x = 0$ for every real vector $\vec{x}$.
$(x^T A x)^T = 0$
$x \cdot (-A) \cdot x^T = 0$
so $x^T A x = x(-A)x^T$ and the only way this is true is if it $= 0$.

b) eigenvalues are purely imaginary. If they were real, then $\bar{z}^T A z$ is pure imaginary. real part: $x^T A x + y^T A y = 0 + 0$.

c) $\det A \geq 0$. $\det A = \lambda_1 \ldots \lambda_n \geq 0$ because pairs of complex λ's are ib and $-ib$, so it all will be positive.

6.5 Positive Definite Matrices

3. For which #s are these matrices ⊕ definite?

$S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ test 1: evalues ⊕

$a > 0$ ✓

$ac - b^2 > 0 \qquad 9 - b^2 > 0 \qquad 9 > b^2 \qquad \pm 3 > b$

$-3 < b$ and $3 > b$

$S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$ test 1:

$a > 0$ ✓

$ac - b^2 > 0 \qquad 2c - 16 > 0 \qquad 2c > 16 \qquad \boxed{c > 8}$

$S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}$ $\boxed{c > 0}$

$c^2 - b^2 > 0 \qquad \boxed{c^2 > b^2} \Rightarrow \boxed{c > b.}$

17. A diagonal entry s_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $S - s_{jj}I$ would have >0 evalues and would be ⊕ definite. But $S - s_{jj}I$ has a 0 in the (j,j) position, so this would be impossible.

24. $x^2 + xy + y^2 = 1$. Draw and find the half lengths of its axes from the evalues of the corresponding matrix S.

$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ then $S = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

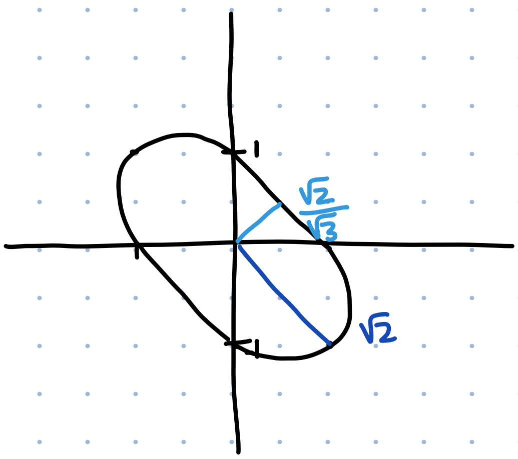
$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x + \frac{1}{2}y \\ \frac{1}{2}x + y \end{bmatrix} \qquad x^2 + \frac{1}{2}xy + \frac{1}{2}xy + y^2$ ✓

$\det(S - \lambda I) = (1-\lambda)^2 - \frac{1}{4} = 0$

$1 - \lambda = \pm\frac{1}{2} \qquad 1 \mp \frac{1}{2} = \lambda \qquad \lambda = \{\frac{3}{2}, \frac{1}{2}\}$

then half lengths:

$\frac{1}{\sqrt{\lambda_1}} = \frac{1}{\frac{\sqrt{3}}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{3}} \qquad \frac{1}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$



28. $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = Q \Lambda Q^{-1}$

a) det S: product of evalues = 10

b) eigenvalues: 2, 5

c) eigenvectors: $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

d) why is S symmetric ⊕ definite? all eigenvalues are ⊕.

7.1 Singular Value Decomposition (this was cool to learn about :))

3. Write A and B in any way as $\vec{u}_1\vec{v}_1^T + \vec{u}_2\vec{v}_2^T$

$$A_{Sweden} = A_{Finland} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 & 7 \\ 10 & 16 & 10 \\ 7 & 10 & 7 \end{bmatrix}$$ symmetric matrix ✓

~~eigenvectors:~~

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

$$B_{Benin} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}}_{\text{pivot cols}}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{rows of } R \text{ (RREF)}}$$

4. Find the trace and det of BB^T and B^TB in (3).

$$BB^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 13 \\ 13 & 19 \end{bmatrix}$$ $\det(BB^T) = 9 \cdot 19 - 13^2 = 2$

$$\Downarrow$$

$$\begin{bmatrix} 1 & \frac{13}{9} \\ 13 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{13}{9} \\ 0 & 19 - 13\left(\frac{13}{9}\right) \end{bmatrix} \sim \begin{bmatrix} 1 & 13/9 \\ 0 & 1 \end{bmatrix}$$ ~~$tr(BB^T) = 2.$~~ $tr(BB^T) = 28$

$$B^TB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ 5 & 13 & 13 \\ 5 & 13 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det(B^TB) = 2(0) - 5(0) + 5(0) = 0$ → there are no eigenvalues?

$tr(B^TB) = \cancel{2}\ 28$

$\sigma_1^2 = 28 - \frac{1}{14}$ $\sigma_2^2 = \frac{1}{14}$. ~~I don't think~~ B is compressible b/c only σ_2^2 is small, ~~yet we have~~ rank 2 matrix

B is compressible exactly because of small σ_2^2. Keep $\vec{v}_1$ and $\vec{u}_1$ of eigenvectors of B^TB and BB^T.

7. How does $AV = U\Sigma$ follow from $A = U\Sigma V^T$? $V^T \cdot V$ = eigenvectors times themselves.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = [5]$ → just a constant, the eigenvalue!

this says $A\vec{v}_i = \sigma_i\vec{u}_i$ which is the def of the right and left eigenvectors.

8. $A\vec{v}_1 = \sigma_1\vec{u}_1$ and $A\vec{v}_2 = \sigma_2\vec{u}_2$ need $\underbrace{\sigma_1 + 1 = \sigma_1^2}$ and $1 - \sigma_1 = -\sigma_2$. are these true?

$1 + \sigma_2 - \sigma_1 = 0$

$\sigma_1 = \frac{1}{2}(\sqrt{5}+1)$ $\sigma_1^2 = \lambda_1 = \frac{3+\sqrt{5}}{2}$ $\sigma_1^2 = \sigma_1^2$, so how can this be true?

$\sigma_1 + 1 = \frac{\sqrt{5}}{2} + \frac{3}{2} = \frac{3+\sqrt{5}}{2}$ ✓

$1 - \frac{1}{2}(\sqrt{5}+1) = -\frac{\sqrt{5}-1}{2}$

$\frac{1}{2} - \frac{\sqrt{5}}{2} = //$ ✓ yes!

7.2 Bases and Matrices in SVD

1. Find the eigenvalues of these matrices. Find σ_i.

$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

$A^TA = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$

$\det(A-\lambda I) = (-\lambda)^2 = 0$

$\lambda = \{0, 0\}$

$\det(A^TA-\lambda I) = (-\lambda)\cdot(16-\lambda) = 0$

$\lambda = \{0, 16\}$

$\sigma_1 = \sqrt{\lambda_1} = 4$

$\sigma_2 = \sqrt{\lambda_2} = 0$

$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

$\det(A-\lambda I) = (-\lambda)^2 - 4 = 0$

$\lambda^2 = 4$

$\pm 2 = \lambda$

$A^TA = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

$\det(A^TA-\lambda I) = (1-\lambda)(16-\lambda) = 0$

$\lambda = \{1, 16\}$ then $\sigma_1 = \sqrt{\lambda_1} = 1$

$\sigma_2 = \sqrt{\lambda_2} = 4$

4. Compute A^TA and AA^T and their eigenvalues and unit vectors for v and u.

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\det(A^TA-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = [(1-\lambda)(2-\lambda)(1-\lambda) - 1] - (1-\lambda)$

$(1-\lambda)(2-3\lambda+\lambda^2-1) - (1-\lambda)$

$(1-\lambda)(1-3\lambda+\lambda^2) - (1-\lambda)$

$= (1-\lambda)(\cancel{1}-3\lambda+\lambda^2-\cancel{1})$

$= (1-\lambda)(-3\lambda+\lambda^2) = 0$

$1 = \lambda \qquad 3\lambda = \lambda^2 \qquad \lambda = 0$

$\qquad\quad 3 = \lambda$

find eigenvectors:

$\lambda_1 = 0$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad a+b=0$, $a+2b+c=0$, $b+c=0$ $\quad a=-b$, $c=-b$ $\quad b=1$ $a=c=-1$ $\quad \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \vec{v}_1$

$\lambda_2 = 1$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad b=0$, $a+b+c=0$, $a=-c$ $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \vec{v}_2$

$\lambda_3 = 3$

$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ $\quad -2a+b=0$, $a-b+c=0$, $b-2c=0$ $\quad b=2a$ $\quad a=1$ $b=2$ $c=1$ $\quad \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \vec{v}_3$

$\det(AA^T-\lambda I) = (2-\lambda)^2 - 1 = 0$

$2-\lambda = \pm 1$

$2 \pm 1 = \lambda = \{1, 3\}$

then find eigenvectors:

$\sigma_1^2 = 3$

$\sigma_2^2 = 1$

$\lambda_1 = 1$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$ $\quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $AV = U\Sigma$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{array}{c} \begin{array}{ccc} \sigma_1 & \sigma_2 & \end{array} \\ \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

10. a) why is $tr(A^TA) = \sum_{i,j} a_{ij}^2$? The sum of all (diagonal entries of $A)^2$ is equal to the trace of A^TA because the diagonal entries of A^TA are the square of diag entries of A, and the trace sums it all together.

b) For every rank 1 matrix, why is $\sigma_1^2 =$ sum of all a_{ij}^2?
when you calculate the eigenvalues of A^TA, only one will result (σ_1^2) since it is a rank 1 matrix. Then that eigenvalue is the $tr(A^TA)$ and also the $\sum_{i,j} a_{ij}^2$.

18. If $A = QR$ with an orthogonal matrix Q, the SVD of A is almost the same as the SVD of R. Which of U, Σ, V changes b/c of Q?
The QR factorization is the Gram Schmidt factorization where Q has orthonormal columns and R has triangular rows. ~~Then the matrix Q creates different different singular values in Σ because of a changed basis~~.

If $R = U\Sigma V^T$, $QR = A = QU\Sigma V^T$. orthogonal $Q \cdot$ orthogonal $U =$ orthogonal QU.

7.3 Principal Component Analysis

1. Suppose A_0: $\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix}$

Produce $A = A_0 - \mu_i$ (i = every row). $\mu_1 = \frac{15}{5} = 3$ $\mu_2 = 0$

$A = \begin{bmatrix} 2 & 1 & 0 & -1 & -2 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix}$.

Compute the sample covariance matrix $S = AA^T/(n-1)$ and find its eigenvalues λ_1 and λ_2.

$A \cdot A^T = \begin{bmatrix} 2 & 1 & 0 & -1 & -2 \\ -1 & 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 0 \\ -1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}$ rows of A have n entries $n=5$

$S = \frac{AA^T}{(5-1)} = \frac{1}{4}\begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}$

$\det(S - \lambda I) = (\frac{10}{4} - \lambda)(1-\lambda) = 0$ then $\sigma_1^2 = \frac{10}{4}$

$\lambda = \{1, \frac{10}{4}\}$

corresponding eigenvector.

$S - \frac{10}{4}I = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix}$ $\quad a = 1$, $-\frac{3}{2}b = 0$ $\quad \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$ the line following the pts closest is the vertical line (?)

2. $A_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 3 & 2 & 1 \end{bmatrix}$ $\quad \mu_1 = \frac{1}{2}$, $\mu_2 = \frac{12}{6} = 2$ $\quad A = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 1/2 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 1/2 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 0 & 0 \\ 1/2 & 1 \\ 0 & 1 \\ 1/2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3/4 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad n=6$

$-1/2 + 1/2$

$S = \frac{AA^T}{(n-1)} = \frac{1}{5}\begin{bmatrix} 3/4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3/20 & 0 \\ 0 & 4/5 \end{bmatrix}$ $\quad \det(S - \lambda I) = (3/20 - \lambda)(4/5 - \lambda)$

$\lambda_1 = 3/20$ $\lambda_2 = 4/5$

$\sigma_1^2 = \frac{4}{5}$

$S \quad I = \begin{bmatrix} \frac{3}{20} - \frac{4}{5} & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{u}_1$

4. From eigenvectors of $S = AA^T$, find $\vec{u}_1$ and (line) $\vec{u}_1, \vec{u}_2$ (plane) closest to these 4 pts in the 3D space.

$\det(AA^T - \lambda I) = (2-\lambda)(8-\lambda)(4-\lambda) = 0$

$\lambda = \{8, 4, 2\}$

$\sigma_1^2 = 8$

$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$1+1 \quad -2+2 \quad -2+2 \quad 2-$

corresponding eigenvectors:

$(AA^T - 8I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} -6a &= 0 \\ -4c &= 0 \end{aligned} \qquad \vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$(AA^T - 4I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5. Find correlation matrix DSD w/ 1's down main diagonal. D is a ⊕ diagonal matrix.

$$S = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$S \cdot D = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & & \\ & 1/4 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1 \\ 0 & 1/4 & 1 \end{bmatrix}$$

correlation matrix!

$$D \cdot S \cdot D = \begin{bmatrix} 1/2 & & \\ & 1/2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & \\ & 1/2 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1/2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & \\ & 1/2 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$$

7.4 The geometry of the SVD

1. a) Compute A^TA and its eigenvalues and unit vectors $\vec{v}_1$ and $\vec{v}_2$. Find σ_1.

$$A=\begin{bmatrix}1&2\\3&6\end{bmatrix} \qquad A^TA=\begin{bmatrix}1&3\\2&6\end{bmatrix}\begin{bmatrix}1&2\\3&6\end{bmatrix}=\begin{bmatrix}10&20\\20&40\end{bmatrix}$$

$$\det(A^TA-\lambda I)=(10-\lambda)(40-\lambda)-400=0$$

$$400-50\lambda+\lambda^2-400$$

$$\lambda(\lambda-50)=0$$

$$\lambda=\{0,50\}$$

For $\lambda=0$

$$A^TA-0I=\begin{bmatrix}10&20\\20&40\end{bmatrix} \qquad \begin{aligned}10a+20b&=0\\a&=-2b\end{aligned} \qquad \frac{1}{\sqrt5}\begin{bmatrix}-2\\1\end{bmatrix}=\vec{v}_2$$

For $\lambda=50$

$$A^TA-50I=\begin{bmatrix}-40&20\\20&-10\end{bmatrix} \qquad \begin{aligned}40a&=20b\\2a&=b\end{aligned} \qquad \frac{1}{\sqrt5}\begin{bmatrix}1\\2\end{bmatrix}=\vec{v}_1$$

$$\sigma_1^2=\lambda_1=50$$

$$\sigma_1=\sqrt{50}=5\sqrt2$$

b) Find AA^T and evalues and evectors $\vec{u}_1$ and $\vec{u}_2$.

$$AA^T=\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}1&3\\2&6\end{bmatrix}=\begin{bmatrix}5&15\\15&45\end{bmatrix}$$

$$\det(AA^T-\lambda I)=(5-\lambda)(45-\lambda)-225=0$$

$$\cancel{225}-50\lambda+\lambda^2-\cancel{225}$$

$$\lambda=\{0,50\}$$

For $\lambda=0$

$$\begin{aligned}5a+15b&=0\\a+3b&=0\end{aligned} \qquad \frac{1}{\sqrt{10}}\begin{bmatrix}3\\-1\end{bmatrix}=\vec{u}_2$$

For $\lambda=50$

$$\begin{bmatrix}-45&15\\15&-5\end{bmatrix} \qquad \begin{aligned}-45a+15b&=0\\-3a+b&=0\end{aligned} \qquad \frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}=\vec{u}_1$$

$$\sigma_1^2=50 \qquad \sigma_1=5\sqrt2$$

c) verify that $A\vec{v}_1=\sigma_1\vec{u}_1$.

$= \checkmark$

$$A\vec{v}_1=\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\cdot\frac{1}{\sqrt5}=\frac{1}{\sqrt5}\begin{bmatrix}5\\15\end{bmatrix} \qquad \sigma_1\vec{u}_1=5\sqrt2\cdot\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}=\frac{1}{\sqrt5}\begin{bmatrix}5\\15\end{bmatrix}$$

$A=U\Sigma V^T$:

$$\begin{bmatrix}1&2\\3&6\end{bmatrix}=\begin{bmatrix}1/\sqrt{10}&3/\sqrt{10}\\3/\sqrt{10}&-1/\sqrt{10}\end{bmatrix}\begin{bmatrix}5\sqrt2&\\&0\end{bmatrix}\begin{bmatrix}1/\sqrt5&-2/\sqrt5\\2/\sqrt5&1/\sqrt5\end{bmatrix}$$

2. From $\vec{u}$s and $\vec{v}$s in ①, write down the 4 orthonormal bases for the fundamental subspaces of A.

- $\vec{u}_1,\ldots\vec{u}_r$: orthonormal basis for colspace
- $\vec{u}_{r+1},\ldots\vec{u}_m$: orthonormal basis for left nullspace $N(A^T)$
- $\vec{v}_1,\ldots\vec{v}_r$: orthonormal basis for rowspace
- $\vec{v}_{r+1},\ldots\vec{v}_m$: orthonormal basis for nullspace $N(A)$.

col space: $\vec{u}_1$ row space: $\vec{v}_1$
$N(A^T)$: $\vec{u}_2$ $N(A)$: $\vec{v}_2$

then all matrices that share that are multiples of A: cA. this is b/c the subspaces are just lines.

6. AA^T has the same eigenvalues σ_1^2 and σ_2^2 as A^TA. Find unit eigenvectors $\vec{u}_1$ and $\vec{u}_2$.

$A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$ $AA^T = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$ $\det(AA^T - \lambda I) = (18-\lambda)(2-\lambda) = 0$
$\lambda = \{2, 18\}$

$\vec{u}_1$ corresponds to $\lambda_1 = 18$ $\vec{u}_2$ corresponds to $\lambda_2 = 2$

$AA^T - 18I = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $AA^T - 2I = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$

$\vec{v}_1$ corresponds to $\lambda_1 = 18$

$A^TA - 18I = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}$ $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v}_2$ corresponds to $\lambda_2 = 2$

$A^TA - 2I = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$ $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 18 & \\ & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

15. $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ $AA^T = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$ $A^TA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $A^+ = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix}$

a) $A^TA\hat{x} = A^Tb$ has many solutions because A^TA is rank 1. not full rank.
singular (not invertible).

b) verify that $x^+ = A^+b = \begin{bmatrix} 0.2b_1 + 0.1b_2 \\ 0.2b_1 + 0.1b_2 \end{bmatrix}$ solves $A^TAx^+ = A^T\vec{b}$.

$$A^TAx^+ = \begin{bmatrix} b_1 + 0.5b_2 + b_1 + 0.5b_2 \\ b_1 + 0.5b_2 + b_1 + 0.5b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 + b_2 \\ 2b_1 + b_2 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 + b_2 \\ 2b_1 + b_2 \end{bmatrix} = \checkmark$$

c) Add (1, -1) to x^+ to get another solution to $A^TA\hat{x} = A^T\vec{b}$.

$$x^{+'} = \begin{bmatrix} 0.2b_1 + 0.1b_2 + 1 \\ 0.2b_1 + 0.1b_2 - 1 \end{bmatrix}$$

$$A^TAx^{+'} = \begin{bmatrix} b_1 + 0.5b_2 + 5 + b_1 + 0.5b_2 + 5 \\ 2b_1 + b_2 + 10 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2b_1 + b_2 \\ 2b_1 + b_2 \end{bmatrix}.$$

$\|\hat{x}\|^2 = (0.2b_1 + 0.1b_2 + 1)^2 + (0.2b_1 + 0.1b_2 + 1)^2$

$\|x^+\|^2 = (0.2b_1 + 0.1b_2)^2 + (0.2b_1 + 0.1b_2)^2 \rightarrow$ shorter

8.1 The Idea of a Linear Transformation

1. A linear transformation must leave the zero vector fixed: $T(\vec{0})=\vec{0}$.
   Prove this from $T(\vec{v}+\vec{w})=T(\vec{v})+T(\vec{w})$ by choosing $\vec{w}=\vec{0}$.
   Then $T(\vec{v}+\vec{0})=T(\vec{v})+\vec{0}$ (if $T(\vec{0})=\vec{0}$).
   so $T(\vec{v})=T(\vec{v})$ ✓.
   Prove also from $T(c\vec{v})=cT(\vec{v})$ by choosing $c=0$.
   $T(0\cdot\vec{v})=\vec{0}$
   $T(\vec{0})=\vec{0}$ ✓.

8. Find the range and kernel.
   a) $T(v_1,v_2)=(v_1-v_2,0)$

kernel: inputs that lead to $\vec{0}$: if $\vec{v}_1=\vec{v}_2$. one dimensional inputs (only 1 basis vector).
line of vectors (c,c).

range: outputs: 2 dimensional vectors but 2^{nd} dimension is always 0
line of vectors $(c,0)$.

   b) $T(v_1,v_2,v_3)=(v_1,v_2)$
   kernel: $(0,0,c)$ space in $\mathbb{R}^3$.
   range: (a,b) space in $\mathbb{R}^2$.

   c) $T(v_1,v_2)=(0,0)$
   kernel: (a,b) $\mathbb{R}^2$.
   range: $(0,0)$ $\mathbb{R}^2$.

   d) $T(v_1,v_2)=(v_1,v_1)$
   kernel: $(0,c)$ in $\mathbb{R}^2$.
   range: (c,c) line.

15) Suppose $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}$. Show that I is not in the range of T. Find a nonzero matrix M s.t. $T(M)=AM=0$.
   Range = the possible outputs of T.
   If $B=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $T(B)=AB=\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a+2c&b+2d\\3a+6c&3b+6d\end{bmatrix}\stackrel{?}{=}I$

   then $a+2c=1$ and $3b+6d=1$ while $b+2d=3a+6c=0$.
   $-6d+6d\neq 1$. let's say $b=-2d$.

   $\therefore$ I is not in range.

want $\begin{bmatrix}a+2c&b+2d\\3a+6c&3b+6d\end{bmatrix}=0$ $a+2c=0$, $a=-2c$ $b+2d=0$, $b=-2d$ $\boxed{\begin{bmatrix}-2&-2\\1&1\end{bmatrix}=M}$

30) Why does every $T:\mathbb{R}^2\to\mathbb{R}^2$ take squares (and rectangles) to parallelograms?
- the linear transformation applies a dot product of vectors to the given square. then, all the resulting vectors will be in the direction of linear combinations of those vectors.
- 2 parallel lines of a square (differing by a fixed $\vec{v}$) $\Rightarrow$ 2 parallel edges (differ by $T(\vec{v})$).

8.2 Matrix of a Linear Transformation

1. Transformation S takes the 2nd derivative.
   Input basis = output basis.
   $\vec{v}_1 = 1 \quad \vec{v}_2 = x \quad \vec{v}_3 = x^2 \quad \vec{v}_4 = x^3$

$$S(\vec{v}_1) = 0 = 0w_1 + 0w_2 + 0w_3 + 0w_4 \leftarrow$$
$$S(\vec{v}_2) = 0 \quad " \quad " \leftarrow$$
$$S(\vec{v}_3) = 1 = 1w_1 + 0 \quad 0 \quad 0 \leftarrow$$
$$S(\vec{v}_4) = x = 0w_1 + 1w_2 + 0 \quad 0 \leftarrow$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. $T(\vec{v}_1) = \vec{w}_2 \quad T(\vec{v}_2) = T(\vec{v}_3) = \vec{w}_1 + \vec{w}_3$.

$$T(\vec{v}_1) = 0 \quad 1 \quad 0$$
$$T(\vec{v}_2) = 1 \quad 0 \quad 1$$
$$T(\vec{v}_3) = 1 \quad 0 \quad 1$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$T(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = 2\vec{w}_1 + \vec{w}_2 + 2\vec{w}_3.$$

12. Which are true, which are ridiculous?
   a) $T^{-1}T = I$. True.

   b) $T^{-1}(T(\vec{v}_1)) = \vec{v}_1$ — let's say $T(\vec{v}_1) = \vec{u}_1$
   $T^{-1}(\vec{u}_1) = \vec{v}_1$ — $T^{-1}(T(\vec{v}_1)) = T^{-1}(\vec{u}_1)$
   ✓ definition of an invertible transformation — $\vec{v}_1 = T^{-1}(\vec{u}_1)$.

   c) $T^{-1}T(\vec{w}_1) = \vec{w}_1$
   This is ridiculous because the $\vec{w}_i$ vectors make up the output, not the input basis. We don't know how $T(\vec{w}_1)$ behaves.

26. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for T. This means $T(\vec{v}_i) = \lambda_i \vec{v}_i$ for $i = 1, 2, 3$.
   What is the matrix for T when the input and output basis are the $\vec{v}$'s?
   The matrix A is $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

8.3 The Search for a Good Basis

1. In EX 1, what is rank$(J-3I)$?

$$J=\begin{bmatrix}2&&&\\&2&&\\&&3&1\\&&0&3\end{bmatrix}$$

rank = # pivots. = 3.
dim(nullspace) = 1. $\Big\}$ sum = 4 = n.
↳ ∃ only one indep eigenvector for $\lambda=3$.

- algebraic multiplicity is 2, b/c $\det(J-\lambda I)$ has the repeated factor $(\lambda-3)^2$.
- geometric " " is 1, b/c ∃ only 1 independent evector.

2. A_1 and A_2 are similar to J. Solve $A_1B_1=B_1J$ and $A_2B_2=B_2J$.

$J=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $A_1=\begin{bmatrix}0&4\\0&0\end{bmatrix}$

find evalues of A:
$\det(A_1-\lambda I)=(-\lambda)^2=0$
$\lambda=\{0,0\}$

find evectors for $\lambda=0$:
$A_1-0=\begin{bmatrix}0&4\\0&0\end{bmatrix}$ $\vec{x}_1=\begin{bmatrix}1\\0\end{bmatrix}$

generalized $\vec{x}_2=\begin{bmatrix}0\\0\end{bmatrix}$

$$\underset{A_1}{\begin{bmatrix}0&4\\0&0\end{bmatrix}}\underset{B_1}{\begin{bmatrix}4&0\\0&1\end{bmatrix}}=\begin{bmatrix}0&4\\0&0\end{bmatrix}=\underset{B_1}{\begin{bmatrix}4&0\\0&1\end{bmatrix}}\underset{J}{\begin{bmatrix}0&1\\0&0\end{bmatrix}}=\begin{bmatrix}0&4\\0&0\end{bmatrix}$$

$A_2=\begin{bmatrix}4&-8\\2&-4\end{bmatrix}$

$$\underset{A_2}{\begin{bmatrix}4&-8\\2&-4\end{bmatrix}}\underset{B_2}{\begin{bmatrix}0&2\\0&1\end{bmatrix}}=\begin{bmatrix}\ &\ \end{bmatrix}=\underset{B_2}{\begin{bmatrix}0&2\\0&1\end{bmatrix}}\underset{J}{\begin{bmatrix}0&1\\0&0\end{bmatrix}}=$$

$B_2=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $B_2^{-1}=\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$$\begin{bmatrix}4a-8c&4b-8d\\2a-4c&2b-4d\end{bmatrix}=\begin{bmatrix}0&a\\0&c\end{bmatrix}$$

$4a=8c$ $4b-8d=a$ $a=c=0$ $4b=8d$
$a=2c$ $2b-4d=c$ $b=2d$ $d=1$ $b=2$

$\boxed{B_2=\begin{bmatrix}0&2\\0&1\end{bmatrix}}$

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

5. If $A^3=0$ show that all $\lambda=0$, and all Jordan blocks w/ $J^3=0$ have size 1, 2, or 3.

$A\vec{x}=\lambda\vec{x}$ is the definition of λ.
$A^2\cdot A\vec{x}=A^3\vec{x}=\lambda^3\vec{x}$.
$0\vec{x}=\lambda^3\vec{x}$ and then assuming $\vec{x}$'s aren't $\vec{0}$, $\lambda=0$.

All J blocks with $J^3=0$ have size 1, 2, or 3 because if they were larger, then it could be possible that J^3 wouldn't be 0 for 4x4 matrix.
$J^3=(B^{-1}AB)(B^{-1}AB)(B^{-1}AB)=B^{-1}A^3B=0$.

$\text{rank}(A) \leq \frac{2n}{3}$. If $A^n = 0$ why is $\text{rank}(A) < n$?
The rank of A is the total number of 1's in the Jordan matrix, which is maximally n.
If the rank was $\geq n$, then A^n wouldn't be = to a 0 matrix.
A is not an invertible matrix and $\text{rank}(A) < n$.

10. Find $a_3 \cos(3x)$ in Fourier series of $f(x) = \begin{cases} 1 & \text{for } -L \leq x \leq L \\ 0 & \text{for } L \leq |x| \leq 2x \end{cases}$

$$a_3 = \frac{\int f(x)\cos(3x)\,dx}{\int \cos 3x \cos 3x\,dx} = \frac{1}{\pi}\int_{-L}^{L} (1)(\cos 3x)\,dx = \frac{1}{3\pi}[\sin 3x]_{x=-L}^{x=L} = \frac{2\sin 3L}{3\pi}$$

9.1 Complex Vectors and Matrices

5. Find $a+ib$ for numbers at $30°, 60°, 90°, 120°$ on unit circle. If $w =$ # @ $30°$, check w^2 @ $60°$. What power of $w = 1$?

Polar numbers $a+ib = re^{i\theta}$ where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$30° = \tan^{-1}\left(\frac{b}{a}\right)$ $\quad \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$ $\quad$ and $r = \sqrt{a^2+b^2}$

$1 = \sqrt{a^2+b^2} = \sqrt{\frac{3}{4}+\frac{1}{4}}$

$a+bi = \boxed{\frac{\sqrt{3}}{2} + i\frac{1}{2}}$

$60°$ $\quad a+bi = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$ $\qquad 90° = \boxed{0+i}$ $\qquad 120° = \boxed{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$w = \frac{\sqrt{3}}{2} + i\frac{1}{2} = 1e^{i\left(\frac{\pi}{6}\right)}$ $\qquad w^2 = e^{i\left(\frac{\pi}{6}\right)\cdot 2} = e^{i\left(\frac{\pi}{3}\right)} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ = value @ $60°$ ✓

What power of $w = 1$? want $1 = e^{i(2\pi)}$. then $\frac{2\pi}{\pi/6} = 12$. $\boxed{w^{12} = 1.}$

7. $M = (1+3i)(1-3i) = (a+bi)(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$

$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} = 10 + 0i = \boxed{10}.$

12. The eigenvalues of a 2x2 matrix:

$\lambda = [a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}]/2.$

a) If $a=b=d=1$, then $\lambda = [2 \pm \sqrt{\underbrace{2^2 - 4(1-c)}_{\cancel{4}-\cancel{4}+4c}}]/2 = [2 \pm 2\sqrt{c}]/2 = 1 \pm \sqrt{c}$

eigenvalues are complex when $c < 0$.

b) What are eigenvalues when $ad = bc$?

$\lambda = \frac{a+d \pm \sqrt{(a+d)^2}}{2}$ = either 0 or $a+d$.

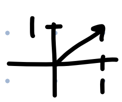
16. $z = \sin\theta + i\cos\theta$. $\quad a+ib$

$a = \sin\theta \quad b = \cos\theta \qquad r = \sqrt{a^2+b^2} = 1$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

$\tan\theta = \frac{\cos\theta}{\sin\theta} \quad \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\sin\theta}$ $\qquad \sin^2\theta = \cos^2\theta$ occurs when $\theta = 45°$.

so $z = 1e^{i\left(\frac{\pi}{4}\right)}$.



$z\cdot(\cos\theta + i\sin\theta) = \cancel{\sin\theta\cos\theta} + i\sin^2\theta + i\cos^2\theta - \cancel{\cos\theta\sin\theta} = \boxed{i}$

9.2 Hermitian and Unitary Matrices

2. A^HA and AA^H? They are both Hermitian!

$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$. $A^H = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$ $A^HA = \begin{pmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{pmatrix}$ $AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

5. Prove that A^HA is always a Hermitian matrix.

a) diagonals: are always real since it's either 2 reals or 2 imaginaries multiplying each other.
Off diagonals: always complex conjugates since all imaginary parts have been negated in A^H.
square: either A is square, so A^H and A^HA are square, or A isn't square but $m\times n \cdot n\times m$ $(A^H) = m\times m$.

b) If $Az=0$ then $A^HAz=0$. multiply $A^HAz=0$ by z^H to prove that $Az=0$.
nullspaces of A and A^H are the same! so A^HA is invertible when nullspace(A) contains only $\vec{z}=0$.

$z^H(A^HAz) = (z^H)\,0$

$(A\vec{z})^H(Az) = 0$

$\downarrow$

complex conjugate. then either $A\vec{z}=\vec{0}$ or $(Az)^H=\vec{0}$. and $(Az)^H$ is just the conjugate transpose of $A\vec{z}$, so $A\vec{z}$ must $=0$.

★ look into 15!

18. If $\vec{v}_1, \dots \vec{v}_n$ is an orthonormal basis for C^n, the matrix w/ those columns is a unitary matrix.
Any vector $= (v_1^Hz)v_1 + \dots + (v_n^Hz)v_n$ because $\vec{v}_i^H\vec{v}$ gives the length of the v_i vector, so the vector z is being constructed w/ the unitary basis for C^n!

12. How do you know the determinant of every Hermitian matrix is $\mathbb{R}$?
the eigenvalues are real, and the det $= \pi(\lambda_i)$.

9.3 Fast Fourier Transform

1. Multiply the 3 matrices of the FFT factorization and compare w/ F.

$$F_4=\begin{bmatrix}1&1&1&1\\1&i^2&i&-i\\1&1&-1&-1\\1&i^2&-i&i\end{bmatrix}\begin{bmatrix}1&&&\\&&1&\\&1&&\\&&&1\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&i&i^2&-i\\1&-1&1&-1\\1&-i&i^2&i\end{bmatrix}$$

must know that $i^2=-1$ in the 6 entries below the diagonal.

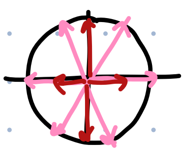
7. Put $\vec{c}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ through FFT to find $\vec{y}=F\vec{c}$. Do the same for $\vec{c}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$.

$$F_4=\begin{bmatrix}1&&1&\\&1&&i\\1&&-1&\\&1&&-i\end{bmatrix}\begin{bmatrix}1&1&&\\1&i^2&&\\&&1&1\\&&1&i^2\end{bmatrix}\begin{bmatrix}1&&&\\&&1&\\&1&&\\&&&1\end{bmatrix}$$

$\vec{y}_1=F_4\vec{c}_1=\begin{bmatrix}2\\0\\2\\0\end{bmatrix}$. $\vec{y}_2=F_4\vec{c}_2=\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}$

9. If $w=e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the 32^{nd} and 128^{th} roots of 1.

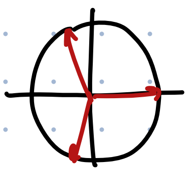
10. a) sixth roots of 1. Prove they add to 0.



they add to 0:

$1, -1, \frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, \frac{1}{2}-i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

b) 3 cube roots? also add to 0?

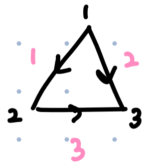


$0, \frac{2\pi}{3}, \frac{4\pi}{3}$

$1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$ still all sum to 0. ✓

10.1 Graphs & Networks

1. Find the incidence matrix A for the graph:



$$A=\begin{array}{c}\text{node}\\ \begin{array}{ccc}①&②&③\end{array}\\ \begin{bmatrix}-1&+1&0\\-1&0&+1\\0&-1&+1\end{bmatrix}\end{array}\begin{array}{l}\\ \\ 1\\2\\3\end{array}\ \text{edge}$$

edge 1 goes from node 1 to node 2.

What vectors are in its nullspace? How do you know that $(1,0,0)$ is not in its row space?

$$AX=\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_2-x_1\\x_3-x_1\\x_3-x_2\end{bmatrix}$$

In the nullspace, $A\vec{x}=\vec{0}$ meaning the difference matrix $=0$. The vector in the nullspace is $\begin{bmatrix}1\\1\\1\end{bmatrix}$ where all the voltages equal each other.

$\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\neq 0$. Since the vector isn't orthogonal to the nullspace basis, it isn't in the row space.

7. With conductances $c_1=1$ and $c_2=c_3=2$, multiply matrices to find A^TCA.

$$C=\begin{bmatrix}1&&\\&2&\\&&2\end{bmatrix}\qquad A^T=\begin{bmatrix}-1&-1&0\\1&0&-1\\0&1&1\end{bmatrix}$$

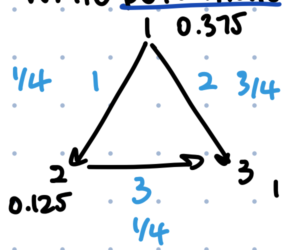
$$A^T\cdot C=\begin{bmatrix}-1&-1&0\\1&0&-1\\0&1&1\end{bmatrix}\begin{bmatrix}1&&\\&2&\\&&2\end{bmatrix}=\begin{bmatrix}-1&-2&0\\1&0&-2\\0&2&2\end{bmatrix}$$

$$A^TC\cdot A=\begin{bmatrix}-1&-2&0\\1&0&-2\\0&2&2\end{bmatrix}\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}=\begin{bmatrix}3&-1&-2\\-1&3&-2\\-2&-2&4\end{bmatrix}$$

For $f=\begin{bmatrix}1\\0\\-1\end{bmatrix}$, find a soln to $A^TCA\vec{x}=f$

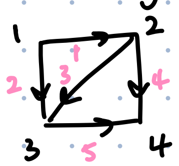
$$\left[\begin{array}{ccc|c}3&-1&-2&1\\-1&3&-2&0\\-2&-2&4&-1\end{array}\right]\sim\begin{bmatrix}1&0&-1&0.375\\0&1&-1&0.125\\0&0&0&0\end{bmatrix}\qquad \begin{aligned}x&=z+0.375\\y&=z+0.125\\z&=z\end{aligned}\qquad \vec{x}=z\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}0.375\\0.125\\0\end{bmatrix}$$

Write potentials $\vec{x}$ and currents $\vec{y}=-CA\vec{x}$ on graph. f goes into node 1 & out from node 3.



$$\vec{y}=\begin{bmatrix}-1&&\\&-2&\\&&-2\end{bmatrix}\begin{bmatrix}-1&+1&0\\-1&0&+1\\0&-1&+1\end{bmatrix}\begin{bmatrix}3/8\\1/8\\0\end{bmatrix}=\begin{bmatrix}-1&&\\&-2&\\&&-2\end{bmatrix}\begin{bmatrix}-2/8\\-3/8\\-1/8\end{bmatrix}=\begin{bmatrix}2/8\\6/8\\2/8\end{bmatrix}$$

11. Multiply matrices A^TA and guess how its entries come from the graph:



$$A=\begin{bmatrix}-1&+1&0&0\\-1&0&+1&0\\0&-1&+1&0\\0&-1&0&+1\end{bmatrix}$$

$A^TA=$ the graph Laplacian.

$$\begin{matrix} 0 & 0 & -1 & +1 \end{matrix}$$

$$A^TA = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

a) The diagonal of A^TA tells how many edges connect to each node.

b) The off diagonals -1 or 0 tell which pairs of nodes are connected. ← cool!

17. Suppose A is a 12×9 incidence matrix from a connected (but unknown) graph.

a) How many columns of A are indp?

$$\begin{array}{cc} & 9 \\ 12 & \begin{bmatrix} \quad \end{bmatrix} \end{array}$$

~~Maximally, 9 cols are indp~~

8 cols are indp — ∴ 1 dependent col always.

b) What condition on f makes it possible to solve $A^T\vec{y} = \vec{f}$?

$\vec{f}$ is in the solution space of A^T (the col space of A^T, so ∴ the row space of A). ie, $\perp$ to the nullspace of A.

c) The diag entries of A^TA give # edges into each node. What is the sum of those diagonal entries?

twice the number of edges in the graph (each edge connects 2 nodes).

$2(12 \text{ edges}) = 24$ is the sum.

10.2: Matrices in engineering

1. Show that $\det(A_0^T C_0 A_0) = c_1c_2c_3 + c_1c_3c_4 + c_1c_2c_4 + c_2c_3c_4$.
   Also find $\det(A_1^T C_1 A_1)$ in fixed-free example.

   In subscript 0: Fixed-fixed scenario.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A_0^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K_0 = A_0^T A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ ...include spring constants:

$$A_0^T C_0 A_0 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3+c_4 \end{bmatrix}$$

careful!

$$\det(A_0^T C_0 A_0) = c_1+c_2\left[(c_2+c_3)(c_3+c_4) - c_3^2\right] + c_2\left[-c_2c_3 - c_2c_4\right]$$

$$= c_1+c_2\left[c_2c_3 + c_2c_4 + \cancel{c_3^2} \cancel{-c_3^2} + c_3c_4\right] - c_2^2c_3 - c_2^2c_4$$

as expected ✓ $= c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + \cancel{c_2^2c_3} + \cancel{c_2^2c_4} + c_2c_3c_4 - \cancel{c_2^2c_3} - \cancel{c_2^2c_4}$

For fixed free example, set $c_4 = 0$.

$$A_1^T C_1 A_1 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3 \end{bmatrix} \qquad \det = c_1c_2c_3.$$

4. Both end conditions for the free-free DEQ are $\frac{du}{dx} = 0$:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \text{ w/ } \frac{du}{dx} = 0 \text{ at both ends.}$$

Integrate both sides to show $f(x)$ must balance itself, $\int f(x)\,dx = 0$, or $\nexists$ no solution.

$$\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx = \int f(x)\,dx$$

boundary cond.

$$-c(x)\frac{du}{dx}\Big]_0^1 = \int f(x)\,dx.$$

→ If $\frac{du}{dx} = 0$, then $\int f(x)\,dx$ must $= 0$.
If $f(x)$ balances itself, then the spring constant $c(x) = 1$,
and $-\frac{du}{dx} = \int f(x)\,dx$.
Otherwise, $\exists$ no soln.

7. For 5 springs and 4 masses w/ both ends fixed, what are the matrices A, C, and K? With $C = I$ solve $K\vec{u} = $ ones(4).

$$C = \begin{bmatrix} c_1 & & & & \\ & c_2 & & & \\ & & c_3 & & \\ & & & c_4 & \\ & & & & c_5 \end{bmatrix} \qquad A^TCA = \begin{bmatrix} c_1+c_2 & -c_2 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4+c_5 \end{bmatrix}$$

$C = I$.

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

diff matrix—both ends fixed.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad CA = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ -c_2 & c_2 & 0 & 0 \\ 0 & -c_3 & c_3 & 0 \\ 0 & 0 & -c_4 & c_4 \\ 0 & 0 & 0 & -c_5 \end{bmatrix}$$

$$K\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$A^TCA = K = 4\times\cancel{5\times5}\times\cancel{5\times5}\times4$

$2a - b = 1$
$-a + 2b - c = 1$
$-b + 2c - d = 1$
$-c + 2d = 1$

10. $c_1 = c_2 = c_3 = c_4 = 1$. $m_1 = 2$. $m_2 = m_3 = 1$. $A^TCA\vec{u} = (2, 1, 1)$ for fixed-fixed.
Which mass moves the most (largest $\vec{u}$)?
$C = I$. Four springs, 3 masses. do the m's play a role?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad A^TCA = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad K\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 2.25 \\ 2.5 \\ 1.75 \end{bmatrix}.$$

mass 2 moves the most.

10.3 Markov Matrices, Popl, Econ

1. Find evalues of this Markov matrix:

$A = \begin{bmatrix} 0.90 & 0.15 \\ 0.10 & 0.85 \end{bmatrix}$ $tr(A) = 1.75$
$\lambda = \{1, 0.75\}$

What is the steady state eigenvector?

$A\vec{x} = \lambda \vec{x}$ $(A - \lambda I)\vec{x} = \vec{0}$.

$(A - I)\vec{x} = \vec{0}$:

$$\begin{bmatrix} -0.10 & 0.15 \\ 0.10 & -0.15 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$-10a + 15b = 0$ $10a = 15b$
$10a - 15b = 0$ $a = \frac{3}{2}b$

2. Diagonalize the matrix to $A = X \Lambda X^{-1}$

$(A - 0.75I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} 0.15 & 0.15 & 0 \\ 0.10 & 0.10 & 0 \end{array}\right] \qquad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1.5 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & .75 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 1.5 \end{bmatrix} \cdot \frac{2}{5}$$

What is the limit of $A^k = X\Lambda^k X^{-1}$ when $\Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & 0.75^k \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$?

The limit of $A^k \Rightarrow X\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = X\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1.5 & -1.5 \\ -1 & -1 \end{bmatrix}$

8. The steady state eigenvector of a permutation matrix is $\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$. This is NOT approached when $\vec{u}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. What are $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$? What are the 4 evalues of P?

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad P - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}$$

$$\det(P - \lambda I) = (-\lambda)\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda\left[(-\lambda)(-\lambda)^2\right] - 1\left[0 - 1(-\lambda)^2\right] = \lambda^4 + \lambda^2. = 0. \quad \lambda^2(\lambda^2 + 1) = 0.$$

$\lambda = \pm 1$. $\lambda^2 = -1$. $\lambda = \pm i$.

$\lambda = \{1, -1, -i, i\}$

Starting from $\vec{u}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

$$\vec{u}_1 = P\vec{u}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\vec{u}_2 = P\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \vec{u}_0$. Just keeps permuting.

15. For which of these matrices does $I + A + A^2 + \ldots$ yield a nonnegative matrix $(I-A)^{-1}$? Then the economy can meet any demand:

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\boxed{A = \begin{bmatrix} 0 & 4 \\ .2 & 0 \end{bmatrix}}$ $A = \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$

check evalues of A. If $\lambda_{max} < 1$, $(I-A)^{-1} \geq 0$.

$\det(A - \lambda I) = (-\lambda)^2 \quad \lambda = \{0, 0\}.$ ✓

$= \lambda^2 - \left(\frac{4 \cdot 2}{10}\right) = 0. \quad \lambda = \pm\sqrt{0.8}$ ✓

$= (.5 - \lambda)(-\lambda) - 0.5 = 0$

$-\frac{1}{2}\lambda + \lambda^2 - \frac{1}{2}$

$(\lambda + \frac{1}{2})(\lambda - 1) = 0$

$\lambda = \{1, -\frac{1}{2}\}$ X

If demands are $\vec{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, what are the vectors $\vec{p} = (I-A)^{-1}\vec{y}$?

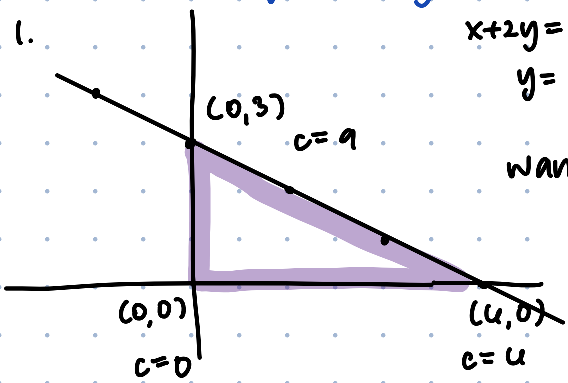
$(I-A)^{-1} = \begin{bmatrix} 1 & -4 \\ -0.2 & 1 \end{bmatrix}^{-1} = \frac{1}{0.2}\begin{bmatrix} 1 & 4 \\ 0.2 & 1 \end{bmatrix} \rightarrow \cdot\vec{y} = \frac{10}{2}\begin{bmatrix} 26 \\ 4.4 \end{bmatrix} = \begin{bmatrix} 130 \\ 32 \end{bmatrix}.$

$(I-A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \cdot\vec{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

The economy can meet any demand :: $\vec{p}$ = net production.

10.4 Linear Programming

1.



$x+2y=4$

$y=-\frac{x}{2}+3$

want to minimize $c=x+3y$

point $(0,0)$.

want to maximize c.

point $(0,3)$.

3. what are the corners of the set $x_1+2x_2-x_3=4$ w/ x_1, x_2, x_3 all ≥ 0?
show that cost x_1+2x_3 can be very negative in this feasible set.
$\exists$ 2 corners: (2 0's and 1 nonzero).
$(4,0,0)$ and $(0,2,0)$
Let $x_i \Rightarrow -\infty$, $x_2=0$ and $x_3=x_1-4$.
Then $c=x_1+2x_1-8=3x_1-8$, and can tend to $-\infty$.

4. choose a diff cost vector c so PhD gets the job. Rewrite the dual problem —
want to minimize cost s.t. $(4,0,0)$ is the answer.
$\vec{c}^T\vec{x}=x_1+5x_2+5x_3$.
$P=(4,0,0)$ $\quad c=4$
$Q=(0,4,0)$ $\quad c=20$
$R=(0,0,2)$ $\quad c=10$.
maximize the cheaters income.
min cost $c \cdot x^* =$ max income by^*

If we reduce the PhD cost to \$1 or \$2, the job goes to PhD.
$\vec{c}=(2,3,8)$. PhD takes 4hrs $(x_1+x_2+2x_3=4)$. charges \$8.
Dual problem: constraints are as follows.
$y \leq 2 \quad y \leq 3 \quad 2y \leq 8$.
$A^T\vec{y} \leq \vec{c}$. maximum has $y=2$. $\$8=4$ problems.

8. prove weak duality, that always $\vec{y}^T\vec{b} \leq \vec{c}^T\vec{x}$:
minimize c^Tx w/ $Ax \geq b$ and $x \geq 0$.
maximize y^Tb w/ $A^Ty \leq c$ and $y \geq 0$.

$y^T\vec{b} \leq y^TA\vec{x} = (A^T\vec{y})^T\vec{x} \leq c^T\vec{x}$.

10.5: Fourier Series

1. Integrate $2\cos jx \cos kx = \cos(j+k)x + \cos(j-k)x$ to show $\cos jx$ is orthogonal to $\cos kx$, provided $j \neq k$.

$$\int_0^{2\pi} 2\cos jx \cos kx \, dx = \int_0^{2\pi} [\cos(j+k)x + \cos(j-k)x] \, dx = \frac{\sin(j+k)x}{j+k} + \frac{\sin(j-k)x}{j-k} \Big]_0^{2\pi} = 0.$$

If $j = k$, then $\int_0^{2\pi} 2\cos^2 jx \, dx = \int_0^{2\pi} \frac{1+\cos(2jx)}{2} dx = \underbrace{\frac{1}{2}x\Big]_0^{2\pi}}_{\pi} + \underbrace{\int \frac{1}{2}\cos(2jx)\, dx}_{0} = \pi.$

4. Legendre polynomials: $1, x, x^2 - \frac{1}{3}$.

Choose c so $x^3 - cx$ is orthogonal to the first 3.

$$\begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can't I choose any c?

need all columns to be independent ⇒ orthogonal.

Fourier technique.

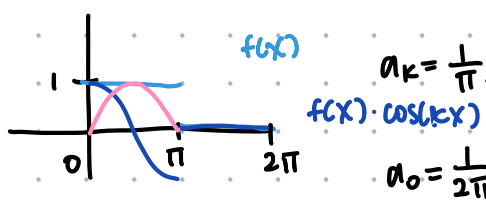
$$\left.\begin{array}{l} \int_{-1}^{1} (1)(x^3 - cx)\, dx = 0 \\ \quad \text{and} \\ \int_{-1}^{1} (x^2 - \frac{1}{3})(x^3 - cx)\, dx = 0 \end{array}\right\}$$ for all c b/c they are odd functions and already don't span the same space.

$$\int_{-1}^{1} (x)(x^3 - cx)\, dx = \int_{-1}^{1} x^4 - cx^2 \, dx = \frac{x^5}{5} - \frac{cx^3}{3}\Big]_{-1}^{1} = \frac{1}{5} - \frac{c}{3} - \left[\frac{-1}{5} + \frac{c}{3}\right]$$

$$= \frac{2}{5} - \frac{2c}{3} = 0. \quad \frac{2}{5} = \frac{2c}{3} \quad 6 = 10c \quad \boxed{\frac{3}{5} = c}.$$

9. Compute Fourier coefficients a_k and b_k for $f(x)$ from $0 \to 2\pi$.

a) $f(x) = 1$ for $0 \le x \le \pi$, $f(x) = 0$ for $\pi < x < 2\pi$.



$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx \, dx = \frac{0}{\pi} = 0.$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \frac{\pi}{2\pi} = \frac{1}{2}.$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx \, dx =$$

$$\frac{1}{\pi}\int_0^{\pi} \sin kx \, dx = \frac{-1}{\pi k}\cos(kx)\Big]_0^{\pi} = \frac{-1}{\pi k}[\cos(k\pi) - \cos(0)]$$

$$= \frac{-1}{\pi k}[(-1)^k - 1] = \frac{-(-1)^k}{\pi k} + \frac{1}{\pi k} \Rightarrow \frac{2}{\pi k} \text{ for odd ks}$$

0 for even ks.

b) $f(x) = x$. ↗ even function.

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x\cos(kx)\, dx = 0.$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x \, dx = \frac{1}{2\pi}\frac{x^2}{2}\Big]_0^{2\pi} = \frac{1}{2\pi}\frac{(2\pi)^2}{2} = \pi.$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x\sin(kx)\, dx =$$

$uv - \int v\,du$

$u = x \qquad v = -\frac{1}{k}\cos(kx)$

$du = 1 \qquad dv = \sin(kx)$

$$= -\frac{x}{k}\cos(kx) + \frac{1}{k\pi}\int_0^{2\pi} \cos(kx)\cdot dx$$

$$= -\frac{x}{k}\cos(kx)\Big]_0^{2\pi} + \frac{1}{k\pi}\left(\frac{1}{k}\sin(kx)\right]_0^{2\pi}$$

$$= \frac{-2\pi}{k} + \cancel{\frac{1}{k\pi}(0)} - \left[(0) + 0\right]$$

$$b_k = \frac{1}{\pi}\cdot\frac{-2\pi}{k} = \boxed{\frac{-2}{k}}$$

12. 1, cos x, sin x, cos 2x, sin 2x ... basis for Hilbert space.

differentiation matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}$$

$$\frac{d\cos 2x}{dx} = -\sin(2x)\cdot 2$$

10.6: Computer graphics

1. A typical pt in $\mathbb{R}^3$: $x\vec{i}+y\vec{j}+z\vec{k}$.

   $\vec{i}=(1,0,0)$

   $\vec{j}=(0,1,0)$

   $\vec{k}=(0,0,1)$

   In computer graphics, this pt is $x\vec{i}+y\vec{j}+z\vec{k}$ + origin. Its homogeneous coordinates are: $\cancel{(x,y,z,1)}$. Other coordinates for the same pt are $(\cancel{x+v_0, y+v_0, z+v_0, v_0})$.

   (cx, cy, cz, c) for $c=1$ and all $c\neq 0$.

5. Scaling matrix to produce a 1×1 square page from 8.5×11.

   x shrinks by $8.5\times$.

   y shrinks by $11\times$.

   homogeneous coords for plane scaling $\Rightarrow$ 3×3 matrix.

   then

$$S=\begin{bmatrix}\frac{1}{8.5} & & \\ & \frac{1}{11} & \\ & & 1\end{bmatrix}$$

9. What is the 3×3 projection matrix $I-nn^T$ onto the plane $\frac{2}{3}x+\frac{2}{3}y+\frac{1}{3}z=0$?

   this plane is a vector space because it contains $(0,0,0)$.

   $\vec{n}=(\frac{2}{3},\frac{2}{3},\frac{1}{3})$ is a normal vector. & it's unit!

$$\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}=1\checkmark$$

$$I-\vec{n}\cdot\vec{n}^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\underbrace{\begin{bmatrix}2/3\\2/3\\1/3\end{bmatrix}\begin{bmatrix}\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\end{bmatrix}}_{\begin{bmatrix}4/9&4/9&2/9\\4/9&4/9&2/9\\2/9&2/9&1/9\end{bmatrix}}=\begin{bmatrix}5/9&-4/9&-2/9\\-4/9&5/9&-2/9\\-2/9&-2/9&8/9\end{bmatrix}$$

15. Find the reflection of $(3,3,3)$ in the plane $\frac{2}{3}x+\frac{2}{3}y+\frac{1}{3}z=1$. Take 3 steps T_-MT_+ by using 3 4×4 matrices.

   ① Translate plane to pass through origin.

   Find a pt on the plane. $\frac{2}{3}(3)+\frac{2}{3}(-3)+\frac{1}{3}(3)=1$

   $(3,-3,3)\rightarrow(0,0,0)$

$$T=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-3&3&-3&1\end{bmatrix}$$

test: $\begin{bmatrix}3&-3&3&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-3&3&-3&1\end{bmatrix}=\begin{bmatrix}0&0&0&1\end{bmatrix}\checkmark$

$$\begin{bmatrix}3&3&3&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-3&3&-3&1\end{bmatrix}=\begin{bmatrix}0&6&0&1\end{bmatrix}$$

② Reflect translated pt in that plane.

$$M = \begin{bmatrix} I-2nn^T & \\ & 1 \end{bmatrix} \qquad \vec{n} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$2 \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = 2 \begin{bmatrix} 4/9 & 4/9 & 2/9 \\ 4/9 & 4/9 & 2/9 \\ 2/9 & 2/9 & 1/9 \end{bmatrix} = \begin{bmatrix} 8/9 & 8/9 & 4/9 \\ 8/9 & 8/9 & 4/9 \\ 4/9 & 4/9 & 2/9 \end{bmatrix}$$

$$I-2nn^T = \begin{bmatrix} 1/9 & -8/9 & -4/9 \\ -8/9 & 1/9 & -4/9 \\ -4/9 & -4/9 & 7/9 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/9 & -8/9 & -4/9 & 0 \\ -8/9 & 1/9 & -4/9 & 0 \\ -4/9 & -4/9 & 7/9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -48/9 & 6/9 & -24/9 & 1 \end{bmatrix}$$

③ Translate $(0,0,0,1) \Rightarrow (3,-3,3,1)$

$$T_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & +3 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -48/9 & 6/9 & -24/9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & +3 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -48/9-27/9 & 6/9+27/9 & -24/9-27/9 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -75/9, & 33/9, & -51/9, & 1 \end{bmatrix}$$

Find a diff pt to move to origin.

$\frac{2}{3}(0) + \frac{2}{3}(0) + \frac{1}{3}(3) = 1.$

① Translate $(0,0,3,1) \rightarrow (0,0,0,1)$

$$T_- = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 0 & -3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 & 1 \end{bmatrix}$$

② Reflect.

$$M = \begin{bmatrix} 1/9 & -8/9 & -4/9 & 0 \\ -8/9 & 1/9 & -4/9 & 0 \\ -4/9 & -4/9 & 7/9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/9 & -8/9 & -4/9 & 0 \\ -8/9 & 1/9 & -4/9 & 0 \\ -4/9 & -4/9 & 7/9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/9-24/9 & -24/9+3/9 & -12/9-12/9 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -21/9 & -21/9 & -24/9 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7/3 & -7/3 & -8/3 & 1 \end{bmatrix}$$

③ Translate $(0,0,0,1) \rightarrow (0\ 0\ 3\ 1)$

$$T_+ = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 0 & +3 & 1 \end{bmatrix} \qquad \begin{bmatrix} -7/3 & -7/3 & -8/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 0 & +3 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -7/3 & -7/3 & 1/3 & 1 \end{bmatrix}} \checkmark$$

$-8/3 + 9/3 = 1/3$

10.7 cryptography

1. If you multiply n whole numbers, when is the answer odd?
   The answer is odd iff all n numbers are odd.
   If you multiply mod 2: 0's and 1's, the answer is 1 iff all numbers are 1 mod 2.

4. $p \equiv 39$ is not prime. Find a number a that has no inverse z (mod 39).
   meaning: $az \equiv 1$ (mod 39) has no solution.
   $a = 3$ or 13! This is because no value $z \cdot (13-1)$ will be divisible by 39.
   Find a 2x2 matrix that has no inverse matrix z (mod 39). This means $Az \equiv 1$ mod 39 has no soln.

   ~~$\begin{bmatrix} 13 & 0 \\ 3 & 3 \end{bmatrix}$ because no matrix $z \cdot \begin{bmatrix} 13-1 & 0 \\ 3-1 & 3-1 \end{bmatrix}$~~ will be divisible by 39.

   Look at the determinants! z will not exist iff $(\det A)^{-1}$ does not exist mod p.
   so, we want $\det A = 3$ or 13.

   $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = A$ $\det A = 3$ and $\nexists\ (\det A)^{-1}$ mod 39.

9. THI|S WH|O LE|BOO|K IS |IN C|ODE
   [20 8 9][19 23 8][15 12 5][2 15 15][11 9 19][9 14 3][15 4 5]

   $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is encoding matrix.
   ↳ after encoding:
   37 17 9 42 31 8 32 17 5 32 30 15 39 28 19 26 17 3 24 9 5

   we can decode this by creating $L^{-1} \equiv D$ mod 26.

   $[L \mid I] \rightarrow [I \mid L^{-1}] \overset{\text{take each number mod 26}}{\Rightarrow} [I \mid D \text{ mod } 26]$

   perform all operations mod 26 to identify D.
   multiply by L^{-1}. translate back to English.

10. How to discover the matrix knowing only the plaintext and the coded message?
   - ID the block size.
   - ID the p value and convert plaintext to starting numbers.
   now you have $A\vec{x} = \vec{b}$ the $\vec{x}$ and $\vec{b}$ vectors, and you know A must be a square matrix.

11.1 Gaussian Elimination in Practice!

we can do this!

1. Find 2 pivots w/ and no row exchange to maximize the pivot.

with row exchange:

$$A=\begin{bmatrix}1 & 1000\\ .001 & 0\end{bmatrix} \xrightarrow{R_2-\frac{1}{1000}R_1\to R_2} \begin{bmatrix}\boxed{1} & 1000\\ 0 & \boxed{-1}\end{bmatrix} \xrightarrow{-R_2\to R_2} \underbrace{\begin{bmatrix}1 & 1000\\ 0 & 1\end{bmatrix}}_{U}$$

$$E_A A=\begin{bmatrix}1 & 0\\ -\frac{1}{1000} & 1\end{bmatrix}\begin{bmatrix}1 & 1000\\ .001 & 0\end{bmatrix}$$

$$E_B E_A A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix} & '' & \end{bmatrix}$$

then $L=\begin{bmatrix}1 & 0\\ \frac{1}{1000} & -1\end{bmatrix}$

check $LU=A$

$$\begin{bmatrix}1 & 0\\ \frac{1}{1000} & -1\end{bmatrix}\begin{bmatrix}1 & 1000\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1000\\ .001 & 0\end{bmatrix} \checkmark$$

why are no entries of L larger than 1?

- maximizing pivots means we only need to subtract smaller multipliers of the largest pivot to reduce A to U. all $|\ell_{ij}| = \frac{|\text{entry}|}{|\text{pivot}|} \leq 1$.

without row exchange:

$$A=\begin{bmatrix}\boxed{.001} & 0\\ 1 & \boxed{1000}\end{bmatrix} \xrightarrow{R_2-1000R_1\to R_2} \begin{bmatrix}.001 & 0\\ 0 & 1000\end{bmatrix} \xrightarrow[1000R_1\to R_1]{\frac{1}{1000}R_2\to R_2} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=U$$

we lost the 1000!

Find a 3x3 matrix A w/ all $|a_{ij}|\leq 1$ and $|\ell_{ij}|\leq 1$ but 3rd pivot = 4.

$$A=L\cdot U=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 4\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ -1 & 1 & 1\end{bmatrix}$$

8. Via partial pivoting, factor each A into $PA=LU$.

$$A=\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 2\\ 1 & 0\end{bmatrix} \xrightarrow{R_2-\frac{1}{2}R_1\to R_2} \begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}$$

$$L\cdot U=\begin{bmatrix}1 & 0\\ \frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix} = PA=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}A\end{bmatrix}$$

remember should be $-\ell_{ij}$

$$A=\begin{bmatrix}1 & 0 & 1\\ 2 & 2 & 0\\ 0 & 2 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}2 & 2 & 0\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix} \xrightarrow{R_3-\frac{1}{2}R_1\to R_3} \begin{bmatrix}2 & 2 & 0\\ 0 & 2 & 0\\ 0 & -1 & 1\end{bmatrix} \xrightarrow{R_3+\frac{1}{2}R_2\to R_3} \begin{bmatrix}2 & 2 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$L\cdot U=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \frac{1}{2} & -\frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}2 & 2 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$P \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \end{bmatrix}$$

11. a) choose $\sin\theta$ and $\cos\theta$ to triangularize A, and find R:

Givens rotation

$$Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = R.$$

$$\frac{-1}{\sqrt{1^2+3^2}} = \frac{-1}{\sqrt{10}} = \cos\theta \qquad \frac{1}{\sqrt{10}}$$

$$\frac{3}{\sqrt{1^2+3^2}} = \frac{3}{\sqrt{10}} = \sin\theta \qquad \frac{-3}{\sqrt{10}}$$

what difference is there?

$0.948(1) - 0.316(3) = 0$ ✓

$$Q_{21} = \begin{bmatrix} -0.316 & -0.948 \\ 0.948 & -0.316 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2.528 & -4.424 \\ 0 & -2.528 \end{bmatrix} = R.$$

b) choose $\sin\theta$ and $\cos\theta$ to make QAQ^{-1} triangular. what are the eigenvalues?

QAQ^{-1} triangular: Q are orthogonal matrices.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(5-\lambda) + 3 = 0$$
$$5 - 6\lambda + \lambda^2 + 3 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda - 4)(\lambda - 2) = 0$$

$\lambda = \{4, 2\}.$

$\lambda = 4$

$$A - 4I = \begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad -3a = b \quad \vec{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$-3a - b = 0$$
$$3a + b = 0$$

then put one of the eigenvectors into row 1 of Q:

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = Q. \qquad QAQ^{-1} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} \frac{5}{2} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

↳ follow same $\sin\theta / \cos\theta$ pattern as above.

$$\det(Q) = \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} \cdot \frac{3}{\sqrt{10}} = \frac{4}{10}$$

12. When A is multiplied by a plane rotation Q_{ij}, which entries of A are changed?

→ values under the main diagonal gain 0's.

→ all values are changed.

→ entries in ROWS i, j ($2n$ total) are changed.

When $Q_{ij}A$ is multiplied on the right by Q_{ij}^{-1}, which entries are changed now?

→ only the values under main diagonal.

→ you rotate through θ, then through $-\theta$.

→ entries in COLS i, j are changed ($2n$ total).

11.2 Norms and Condition Numbers

1. Find norms $\|A\| = \lambda_{max}$ and condition #s $c = \frac{\lambda_{max}}{\lambda_{min}}$ of these positive definite matrices:

$\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \|A\| = 2 \quad c = \frac{2}{\frac{1}{2}} = 4.$

$\det(A - \lambda I) = (0.5-\lambda)(2-\lambda)$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \|A\| = 2 \quad c = \frac{2}{2} = 1$

$\det(A - \lambda I) = (2-\lambda)(2-\lambda)$

$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \|A\| = 3 \quad c = \frac{3}{1} = 3.$

$\det(A - \lambda I) = (3-\lambda)(1-\lambda)$

8. Show that if λ is any eigenvalue of A, then $|\lambda| \le \|A\|$. Start from $A\vec{x} = \lambda\vec{x}$.

$A\vec{x} = \lambda\vec{x}$ take the norm.

$\|A\| \cdot \|\vec{x}\| = \|\lambda\vec{x}\| = |\lambda| \cdot \|\vec{x}\|$

$\frac{\|A\vec{x}\|}{\|\vec{x}\|} = |\lambda|$ for that particular vector $\vec{x}$.

when we maximize over all vectors, we get $\|A\| \ge |\lambda|$.

15. ℓ^1 norm of $\vec{x}$: $\quad \vec{x} = (x_1, \dots x_n)$

$\|x\|_1 = |x_1| + \dots + |x_n|$

ℓ^∞ norm:

$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$

compute norms $\|x\|, \|x\|_1, \|x\|_\infty$ of these 2 vectors in $\mathbb{R}^5$.

$\vec{x} = (1,1,1,1,1)$
$\|\vec{x}\| = \sqrt{5}$
$\|\vec{x}\|_1 = 5$
$\|\vec{x}\|_\infty = 1$

$\vec{x} = (.1, .7, .3, .4, .5)$
$\|\vec{x}\| = 1$
$\|\vec{x}\|_1 = 2$
$\|\vec{x}\|_\infty = .7$

17. All vector norms must satisfy triangle inequality.
Pf. $\|\vec{x} + \vec{y}\|_\infty \le \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$

$\underbrace{\max_{1 \le i \le n} |x_i + y_i|} \le \max_{1 \le i \le n} |x_i| + \max_{1 \le i \le n} |y_i|$

bounded by the right side, since $x_i + y_i \leq x_j + y_z$ where j could equal z but doesn't have to.

Pf. $\|\vec{x} + \vec{y}\|_1 \leq \|\vec{x}\|_1 + \|\vec{y}\|_1$

$|x_1+y_1| + \cdots + |x_n+y_n| \leq |x_1| + \ldots + |x_n| + |y_1| + \ldots + |y_n|$

both sides sum all components' absolute value of $\vec{x}$ and $\vec{y}$.

11.3 Iterative Methods and Preconditioners

1. Change $A\vec{x}=\vec{b}$ to $\vec{x}=(I-A)\vec{x}+\vec{b}$.
   What are S and T?

   $A = S-T$

   $A\vec{x}=\vec{b}$

   $S\vec{x} = T\vec{x}+\vec{b} \quad \Rightarrow \vec{x}=(I-A)\vec{x}+\vec{b}.$

   Then $S=I$

   $T=I-A$

   $S-T=I-(I-A)=A$ ✓

   What matrix $S^{-1}T$ controls the convergence of $\vec{x}_{k+1}=(I-A)\vec{x}_k+\vec{b}$?

   $S^{-1}=I^{-1}=I.$

   $S^{-1}T= I\cdot(I-A)=\boxed{I-A.}$

10. The Gauss-Seidel iteration @ component i uses earlier parts of $\vec{x}^{\text{new}}$:

$$x_i^{\text{new}}= x_i^{\text{old}} + \frac{1}{a_{ii}}\left(b_i-\sum_{j=1}^{i-1} a_{ij}x_j^{\text{new}}-\sum_{j=i}^{n} a_{ij}x_j^{\text{old}}\right)$$

If every $x_i^{\text{new}}=x_i^{\text{old}}$ how does this show that the soln $\vec{x}$ is correct?

$$0=\frac{1}{a_{ii}}\left(b_i-\sum_{j=1}^{i-1} a_{ij}x_j-\sum_{j=1}^{n} a_{ij}x_j\right)$$

$$0=\frac{1}{a_{ii}}\left(b_i-A\vec{x}\right)$$

$$\frac{A\vec{x}}{a_{ii}}=\frac{b_i}{a_{ii}} \quad \text{for all components } i \Rightarrow A\vec{x}=\vec{b}\ ✓.$$

How does the formula change for Jacobi's method?
Jacobi keeps only the diagonal of A as S.

$$x_i^{\text{new}}= x_i^{\text{old}} + \frac{1}{a_{ii}}\left(b_i-\sum_{j=1}^{i-1} a_{ij}x_j^{\overset{\text{old}}{\cancel{\text{new}}}}-\sum_{j=i}^{n} a_{ij}x_j^{\text{old}}\right)$$

21. Suppose A is tridiagonal and symmetric in the QR method. From $A_1=Q^{-1}AQ$ show that A_1 is symmetric.

   $A_1=Q^{-1}AQ$ $\quad$ A_1 must be symmetric b/c it is similar to A. and can be diagonalized!

   Write $A_1=RAR^{-1}$ to show that A_1 is also tridiagonal.

   $A=QR$

   $A_1= Q^{-1}(QR)Q$

   $A_1=RQ$ ✓ $\quad$ since we know Q are orthonormal, $Q^{-1}=Q^T$.

   $A_1=R(QR)R^{-1} \stackrel{?}{=} RQ$

22. For $A=\begin{bmatrix}1&&&\\&2&&\\&&3&\\&&&4\end{bmatrix}$ and vector $\vec{b}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, go through one Arnoldi step to find orthonormal vectors $\vec{q}_1$ and $\vec{q}_2$.

Arnoldi iteration

$$\vec{q}_1=\frac{\vec{b}}{\|\vec{b}\|}=\frac{1}{\sqrt{4}}\vec{b}=\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$\vec{v}=A\vec{q}_1=\begin{bmatrix}1&&&\\&2&&\\&&3&\\&&&4\end{bmatrix}\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}=\begin{bmatrix}1/2\\1\\3/2\\2\end{bmatrix}$$

for $j=1$ to $n=1$

$$h_{11}=\vec{q}_1^T\cdot\vec{v}=\begin{bmatrix}1/2&1/2&1/2&1/2\end{bmatrix}\begin{bmatrix}1/2\\1\\3/2\\2\end{bmatrix}=\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}=\frac{10}{4}=\frac{5}{2}$$

$$\vec{v}=\vec{v}-h_{11}\vec{q}_1$$

$$\begin{bmatrix}1/2\\1\\3/2\\2\end{bmatrix}-\frac{5}{2}\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}=\begin{bmatrix}2/4\\4/4\\6/4\\8/4\end{bmatrix}-\begin{bmatrix}5/4\\5/4\\5/4\\5/4\end{bmatrix}=\begin{bmatrix}-3/4\\-1/4\\1/4\\3/4\end{bmatrix}$$

$$h_{2,1}=\|\vec{v}\|=\sqrt{\frac{9}{16}+\frac{1}{16}+\frac{1}{16}+\frac{9}{16}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$$

$$\vec{q}_2=\vec{v}/h_{2,1}=\begin{bmatrix}-3/4\\-1/4\\1/4\\3/4\end{bmatrix}\cdot\frac{2}{\sqrt{5}}$$

$$\vec{q}_1=\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\qquad \vec{q}_2=\begin{bmatrix}-3/4\\-1/4\\1/4\\3/4\end{bmatrix}\cdot\frac{2}{\sqrt{5}}$$

12.1 Mean, variance, & Probability

1. Add 7 to every output.

$\mu_{new} = \mu + 7$

$\sigma_{new} = \sigma$

$E[X]_{new} = E[X] + 7.$

5. sample from 1 to 1000 w/ equal probabilities. Let x be the FIRST digit. what are the probabilities p_1 to p_9?

$p_1 =$

	10	20	30	40 ...	$\boxed{100}$	200	...	1000
1	11	:						
2	12	:						
3	:							
4	:	.						
:	.							
9	19							

# of possible answers where x = 1: $\underbrace{100}_{\text{the 100's}} + \underbrace{1}_{1} + \underbrace{1}_{1000} + \underbrace{10}_{10\text{'s}} = 112$

possible ... x = 2: $\underbrace{100}_{\text{the 200's}} + \underbrace{1}_{2} + \underbrace{10}_{20\text{'s}} = 111$

$888 + 112 = 990 + 10 = 1000$ total ✓

$p_1 = \frac{112}{1000}$ $\quad p_2 = p_3 = \ldots = p_9 = \frac{111}{1000}.$

$E[X] = p_1 x_1 + \ldots + p_n x_n$

$\frac{112}{1000} + \frac{222}{1000} + \ldots + \frac{999}{1000} = 4.996$

$\sigma^2 = E[(X-m)^2] = p_1(x_1 - m)^2 + \ldots + p_n(x_n - m)^2$

$\frac{112}{1000}(1-4.996)^2 + \ldots + \frac{111}{1000}(9-4.996)^2$

8. If all 24 samples from a pop'n produce the same age x=20, what are the μ and S^2? what if x = 20 or 21, 12 times each?

first scenario:

$\mu = 24$

$S^2 = 0$

2^{nd} scenario:

$\mu = 20.5$

$S^2 = \frac{12(20-20.5)^2 + \ldots}{23} = \frac{\frac{12}{4}}{23} = \frac{3}{23}$

11. For any function f(x) the expected value $E[f] = \Sigma p_i f(x_i)$ or $\int p(x) f(x) dx$. Suppose $E[X] = m$ and $E[(X-m)^2] = \sigma^2$. What is $E[X^2]$?

$(x-m)^2 = x^2 - 2mx + m^2$

$E[x^2 - 2mx + m^2] = \sigma^2$

$E[x^2] - E[2mx] + E[m^2] = \sigma^2$

$$E[X^2] - 2m\underbrace{E[X]}_{m} + \overset{\text{constant}}{m^2} = \sigma^2$$

$$E[X^2] = 2m^2 - m^2 + \sigma^2 = m^2 + \sigma^2$$

12.2 Covariance Matrices & Joint Probabilities

1. a) compute variance σ^2 when coin flip probs are p and $(1-p)$

$$\sigma^2 = \sum_1^n p_i (x_i - m)^2$$

$$m = 0(p) + 1(1-p) = 1-p$$

$$\sigma^2 = p(0-(1-p))^2 + (1-p)(1-(1-p))^2$$
$$= p(p-1)^2 + (1-p)p^2$$
$$p(p^2-2p+1) + p^2 - p^3$$
$$\cancel{p^3} - 2p^2 + p + p^2 - \cancel{p^3} = -p^2 + p.$$

b) sum of N indep flips (0 or 1) is count of heads after N tries. The rule (16-17-18) for variance of a sum gives

↳ var of $z = x+y$ is $\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}$

→ independent flips!
covariance matrix = diagonal
diagonal entries = $\sigma^2 = p - p^2$

→ overall variance = $[1\ 1 \ldots 1]\, V \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = N\sigma^2 = N(p-p^2)$

4. cov matrix V for M=3 indep expts w/ means m_1, m_2, m_3 and variances $\sigma_1^2, \sigma_2^2, \sigma_3^3$?

$$\begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

6. The nxn matrix P contains joint probs $p_{ij} = P(X = x_i \text{ and } Y = y_j)$.

Why does the conditional $P(Y = y_j \mid X = x_i) = \dfrac{p_{ij}}{p_{i1} + \ldots + p_{in}} = \dfrac{p_{ij}}{p_i}$?

The conditional prob = the prob of $Y = y_j$ GIVEN $X = x_i$. So denominator is probability that $X = x_i$. (aka p_i).

And the prob of both must be p_{ij}.

$\therefore \dfrac{p_{ij}}{p_i}.$

8. Explain product rule of conditional prob.

$$P_{ij} = \text{Prob}(X=x_i \text{ and } Y=y_j) = \underbrace{\text{Prob}(Y=y_j \mid X=x_i)}_{\text{when } X=x_i\text{, what are chances } Y=y_j\text{?}} \text{ times } \underbrace{\text{Prob}(X=x_i)}_{\text{what are chances } X=x_i\text{?}}.$$

together $= P(X=x_i)$ AND $P(Y=y_j)$

12.3 Multivariate Gaussian and weighted least squares

1. 2 measurements of same var x give 2 eqns: $x=b_1$ and $x=b_2$. Write 2 eqns as $A\vec{x}=\vec{b}$. Find $\hat{x}$.

$$\begin{bmatrix}1\\1\end{bmatrix}\vec{x}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$x=b_1$, $x=b_2$ is $A\vec{x}=\vec{b}$ with $A=\begin{bmatrix}1\\1\end{bmatrix}$ and $V=\begin{bmatrix}\sigma_1^2 & 0\\0 & \sigma_2^2\end{bmatrix}$.

$V^{-1/2}A\hat{x}=V^{-1/2}\vec{b}$ is $\frac{1}{\sigma_1}x=\frac{1}{\sigma_1}b_1$, $\frac{1}{\sigma_2}x=\frac{1}{\sigma_2}b_2$

$$A^TV^{-1}A\hat{x}=A^TV^{-1}\vec{b}$$

$$\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1/\sigma_1^2 & \\ & 1/\sigma_2^2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\hat{x}=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1/\sigma_1^2 & \\ & 1/\sigma_2^2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$\hat{x}=\frac{1/\sigma_1^2\, b_1+1/\sigma_2^2\, b_2}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}$$

2. a) Suppose b_2 becomes super exact $\bar{c}$ $\underline{\underline{\sigma_2^2\to 0}}$.

then $\hat{x}=\dfrac{\frac{1}{\sigma_1^2}b_1+\frac{1}{0}\cdot b_2}{\frac{1}{\sigma_1^2}+\frac{1}{0}}$ b_2 has most weight.

$\hat{x}\cong b_2$.

b) opposite case: $\sigma_2\to\infty$. no info in b_2.

$\hat{x}=\dfrac{\frac{1}{\sigma_1^2}b_1}{\frac{1}{\sigma_1^2}}=b_1$. cannot rely on b_2 at all.

5. Show inverse of 2x2 cov matrix V is

$$V^{-1}=\begin{bmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{bmatrix}^{-1}=\frac{1}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\begin{bmatrix}\sigma_2^2 & -\sigma_{12}\\ -\sigma_{12} & \sigma_1^2\end{bmatrix}$$

$\det(V)=\sigma_1^2\sigma_2^2-\sigma_{12}^2$

$\rho=\dfrac{\sigma_{12}}{\sigma_1\sigma_2}$ $\rho^2=\dfrac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}$

$$\left[\frac{1}{\frac{1}{\sigma_1^2\sigma_2^2}} \cdot \frac{1}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\right]\begin{bmatrix}\sigma_2^2 & -\sigma_{12}\\ -\sigma_{12} & \sigma_1^2\end{bmatrix}\cdot\frac{1}{\sigma_1^2\sigma_2^2}$$

$$\frac{1}{1-\rho^2}\begin{bmatrix}1/\sigma_1^2 & -\rho/\sigma_1\sigma_2\\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2\end{bmatrix} \checkmark$$

6. suppose $\hat{x}_k$ is avg of $b_1, \dots b_k$. A new measurement b_{k+1} arrives.

Kalman updates:

$$\hat{x}_{k+1} = \hat{x}_k + \frac{1}{k+1}(b_{k+1} - \hat{x}_k)$$

Verify $\hat{x}_{k+1}$ is correct avg of $b_1, \dots b_{k+1}$.

$$\hat{x}_{k+1} = \hat{x}_k + \frac{b_{k+1}}{k+1} - \frac{\hat{x}_k}{k+1}$$

$$\hat{x}_{k+1} = \left(1 - \frac{1}{k+1}\right)\hat{x}_k + \frac{b_{k+1}}{k+1}$$

gain factor.

$$\hat{x}_{k+1} = \frac{b_1 + \dots + b_k}{k} + \frac{1}{k+1}\left(b_{k+1} - \frac{b_1 + \dots + b_k}{k}\right) = \frac{b_1 + \dots + b_{k+1}}{k+1}.$$

$$\frac{1}{k} - \frac{1}{k+1}\cdot\frac{1}{k} = \frac{1}{k+1} \checkmark$$