T, contradictory constraints arise. Since any witness of satisfiability has to assign some value to that node, we infer that it cannot be T. Thus, we may permanently assign mark F to that node. For this DAG, such an optimization does not seem to help. No test of an unmarked node detects a shared mark or a shared contradiction. Our cubic SAT solver fails for this DAG.

1.7 Exercises

Exercises 1.1

1. Use $\neg$, $\rightarrow$, $\wedge$ and $\vee$ to express the following declarative sentences in propositional logic; in each case state what your respective propositional atoms p, q, etc. mean:
   * (a) If the sun shines today, then it won't shine tomorrow.
   (b) Robert was jealous of Yvonne, or he was not in a good mood.
   (c) If the barometer falls, then either it will rain or it will snow.
   * (d) If a request occurs, then either it will eventually be acknowledged, or the requesting process won't ever be able to make progress.
   (e) Cancer will not be cured unless its cause is determined and a new drug for cancer is found.
   (f) If interest rates go up, share prices go down.
   (g) If Smith has installed central heating, then he has sold his car or he has not paid his mortgage.
   * (h) Today it will rain or shine, but not both.
   * (i) If Dick met Jane yesterday, they had a cup of coffee together, or they took a walk in the park.
   (j) No shoes, no shirt, no service.
   (k) My sister wants a black and white cat.
2. The formulas of propositional logic below implicitly assume the binding priorities of the logical connectives put forward in Convention 1.3. Make sure that you fully understand those conventions by reinserting as many brackets as possible. For example, given $p \wedge q \rightarrow r$, change it to $(p \wedge q) \rightarrow r$ since $\wedge$ binds more tightly than $\rightarrow$.
   * (a) $\neg p \wedge q \rightarrow r$
   (b) $(p \rightarrow q) \wedge \neg(r \vee p \rightarrow q)$
   * (c) $(p \rightarrow q) \rightarrow (r \rightarrow s \vee t)$
   (d) $p \vee (\neg q \rightarrow p \wedge r)$
   * (e) $p \vee q \rightarrow \neg p \wedge r$
   (f) $p \vee p \rightarrow \neg q$
   * (g) Why is the expression $p \vee q \wedge r$ problematic?

Exercises 1.2

1. Prove the validity of the following sequents:
   (a) $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$

(b) $p \wedge q \vdash q \wedge p$
* (c) $(p \wedge q) \wedge r \vdash p \wedge (q \wedge r)$
(d) $p \rightarrow (p \rightarrow q), p \vdash q$
* (e) $q \rightarrow (p \rightarrow r), \neg r, q \vdash \neg p$
* (f) $\vdash (p \wedge q) \rightarrow p$
(g) $p \vdash q \rightarrow (p \wedge q)$
* (h) $p \vdash (p \rightarrow q) \rightarrow q$
* (i) $(p \rightarrow r) \wedge (q \rightarrow r) \vdash p \wedge q \rightarrow r$
* (j) $q \rightarrow r \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)$
(k) $p \rightarrow (q \rightarrow r), p \rightarrow q \vdash p \rightarrow r$
* (l) $p \rightarrow q, r \rightarrow s \vdash p \vee r \rightarrow q \vee s$
(m) $p \vee q \vdash r \rightarrow (p \vee q) \wedge r$
* (n) $(p \vee (q \rightarrow p)) \wedge q \vdash p$
* (o) $p \rightarrow q,\ r \rightarrow s \vdash p \wedge r \rightarrow q \wedge s$
(p) $p \rightarrow q \vdash ((p \wedge q) \rightarrow p) \wedge (p \rightarrow (p \wedge q))$
(q) $\vdash q \rightarrow (p \rightarrow (p \rightarrow (q \rightarrow p)))$
* (r) $p \rightarrow q \wedge r \vdash (p \rightarrow q) \wedge (p \rightarrow r)$
(s) $(p \rightarrow q) \wedge (p \rightarrow r) \vdash p \rightarrow q \wedge r$
(t) $\vdash (p \rightarrow q) \rightarrow ((r \rightarrow s) \rightarrow (p \wedge r \rightarrow q \wedge s))$; here you might be able to 'recycle' and augment a proof from a previous exercise.
(u) $p \rightarrow q \vdash \neg q \rightarrow \neg p$
* (v) $p \vee (p \wedge q) \vdash p$
(w) $r, p \rightarrow (r \rightarrow q) \vdash p \rightarrow (q \wedge r)$
* (x) $p \rightarrow (q \vee r), q \rightarrow s, r \rightarrow s \vdash p \rightarrow s$
* (y) $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$.

2. For the sequents below, show which ones are valid and which ones aren't:
* (a) $\neg p \rightarrow \neg q \vdash q \rightarrow p$
* (b) $\neg p \vee \neg q \vdash \neg(p \wedge q)$
* (c) $\neg p, p \vee q \vdash q$
* (d) $p \vee q, \neg q \vee r \vdash p \vee r$
* (e) $p \rightarrow (q \vee r), \neg q, \neg r \vdash \neg p$ without using the MT rule
* (f) $\neg p \wedge \neg q \vdash \neg(p \vee q)$
* (g) $p \wedge \neg p \vdash \neg(r \rightarrow q) \wedge (r \rightarrow q)$
(h) $p \rightarrow q, s \rightarrow t \vdash p \vee s \rightarrow q \wedge t$
* (i) $\neg(\neg p \vee q) \vdash p$.

3. Prove the validity of the sequents below:
(a) $\neg p \rightarrow p \vdash p$
(b) $\neg p \vdash p \rightarrow q$
(c) $p \vee q, \neg q \vdash p$
* (d) $\vdash \neg p \rightarrow (p \rightarrow (p \rightarrow q))$
(e) $\neg(p \rightarrow q) \vdash q \rightarrow p$
(f) $p \rightarrow q \vdash \neg p \vee q$
(g) $\vdash \neg p \vee q \rightarrow (p \rightarrow q)$

(h) $p \to (q \vee r),\ \neg q,\ \neg r \vdash \neg p$
(i) $(c \wedge n) \to t,\ h \wedge \neg s,\ h \wedge \neg(s \vee c) \to p \vdash (n \wedge \neg t) \to p$
(j) the two sequents implict in (1.2) on page 20
(k) $q \vdash (p \wedge q) \vee (\neg p \wedge q)$ using LEM
(l) $\neg(p \wedge q) \vdash \neg p \vee \neg q$
(m) $p \wedge q \to r \vdash (p \to r) \vee (q \to r)$
* (n) $p \wedge q \vdash \neg(\neg p \vee \neg q)$
(o) $\neg(\neg p \vee \neg q) \vdash p \wedge q$
(p) $p \to q \vdash \neg p \vee q$ possibly without using LEM?
* (q) $\vdash (p \to q) \vee (q \to r)$ using LEM
(r) $p \to q,\ \neg p \to r,\ \neg q \to \neg r \vdash q$
(s) $p \to q,\ r \to \neg t,\ q \to r \vdash p \to \neg t$
(t) $(p \to q) \to r,\ s \to \neg p,\ t,\ \neg s \wedge t \to q \vdash r$
(u) $(s \to p) \vee (t \to q) \vdash (s \to q) \vee (t \to p)$
(v) $(p \wedge q) \to r,\ r \to s,\ q \wedge \neg s \vdash \neg p$.

4. Explain why intuitionistic logicians also reject the proof rule PBC.
5. Prove the following theorems of propositional logic:
* (a) $((p \to q) \to q) \to ((q \to p) \to p)$
(b) Given a proof for the sequent of the previous item, do you now have a quick argument for $((q \to p) \to p) \to ((p \to q) \to q)$?
(c) $((p \to q) \wedge (q \to p)) \to ((p \vee q) \to (p \wedge q))$
* (d) $(p \to q) \to ((\neg p \to q) \to q)$.
6. Natural deduction is not the only possible formal framework for proofs in propositional logic. As an abbreviation, we write Γ to denote any finite sequence of formulas $\phi_1, \phi_2, \dots, \phi_n$ $(n \geq 0)$. Thus, any sequent may be written as $\Gamma \vdash \psi$ for an appropriate, possibly empty, Γ. In this exercise we propose a different notion of proof, which states rules for transforming valid sequents into valid sequents. For example, if we have already a proof for the sequent $\Gamma, \phi \vdash \psi$, then we obtain a proof of the sequent $\Gamma \vdash \phi \to \psi$ by augmenting this very proof with one application of the rule $\to$i. The new approach expresses this as an inference rule between sequents:

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to\mathrm{i}.$$

The rule 'assumption' is written as

$$\frac{}{\phi \vdash \phi} \ \text{assumption}$$

i.e. the premise is empty. Such rules are called axioms.
(a) Express all remaining proof rules of Figure 1.2 in such a form. (Hint: some of your rules may have more than one premise.)
(b) Explain why proofs of $\Gamma \vdash \psi$ in this new system have a tree-like structure with $\Gamma \vdash \psi$ as root.
(c) Prove $p \vee (p \wedge q) \vdash p$ in your new proof system.

7. Show that $\sqrt{2}$ cannot be a rational number. Proceed by proof by contradiction: assume that $\sqrt{2}$ is a fraction k/l with integers k and $l \neq 0$. On squaring both sides we get $2 = k^2/l^2$, or equivalently $2l^2 = k^2$. We may assume that any common 2 factors of k and l have been cancelled. Can you now argue that $2l^2$ has a different number of 2 factors from k^2? Why would that be a contradiction and to what?
8. There is an alternative approach to treating negation. One could simply ban the operator $\neg$ from propositional logic and think of $\phi \to \bot$ as 'being' $\neg\phi$. Naturally, such a logic cannot rely on the natural deduction rules for negation. Which of the rules $\neg$i, $\neg$e, $\neg\neg$e and $\neg\neg$i can you simulate with the remaining proof rules by letting $\neg\phi$ be $\phi \to \bot$?
9. Let us introduce a new connective $\phi \leftrightarrow \psi$ which should abbreviate $(\phi \to \psi) \wedge (\psi \to \phi)$. Design introduction and elimination rules for $\leftrightarrow$ and show that they are derived rules if $\phi \leftrightarrow \psi$ is interpreted as $(\phi \to \psi) \wedge (\psi \to \phi)$.

Exercises 1.3

In order to facilitate reading these exercises we assume below the usual conventions about binding priorities agreed upon in Convention 1.3.

1. Given the following formulas, draw their corresponding parse tree:
   (a) p
   * (b) $p \wedge q$
   (c) $p \wedge \neg q \to \neg p$
   * (d) $p \wedge (\neg q \to \neg p)$
   (e) $p \to (\neg q \vee (q \to p))$
   * (f) $\neg((\neg q \wedge (p \to r)) \wedge (r \to q))$
   (g) $\neg p \vee (p \to q)$
   (h) $(p \wedge q) \to (\neg r \vee (q \to r))$
   (i) $((s \vee (\neg p)) \to (\neg p))$
   (j) $(s \vee ((\neg p) \to (\neg p)))$
   (k) $(((s \to (r \vee l)) \vee ((\neg q) \wedge r)) \to ((\neg(p \to s)) \to r))$
   (l) $(p \to q) \wedge (\neg r \to (q \vee (\neg p \wedge r)))$.
2. For each formula below, list all its subformulas:
   * (a) $p \to (\neg p \vee (\neg\neg q \to (p \wedge q)))$
   (b) $(s \to r \vee l) \vee (\neg q \wedge r) \to (\neg(p \to s) \to r)$
   (c) $(p \to q) \wedge (\neg r \to (q \vee (\neg p \wedge r)))$.
3. Draw the parse tree of a formula ϕ of propositional logic which is
   * (a) a negation of an implication
   (b) a disjunction whose disjuncts are both conjunctions
   * (c) a conjunction of conjunctions.
4. For each formula below, draw its parse tree and list all subformulas:
   * (a) $\neg(s \to (\neg(p \to (q \vee \neg s))))$
   (b) $((p \to \neg q) \vee (p \wedge r) \to s) \vee \neg r$.