# First-arrival traveltime tomography

## First-break traveltime tomography: where are we?
## Illustration on 30 years Western-Alps database

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France

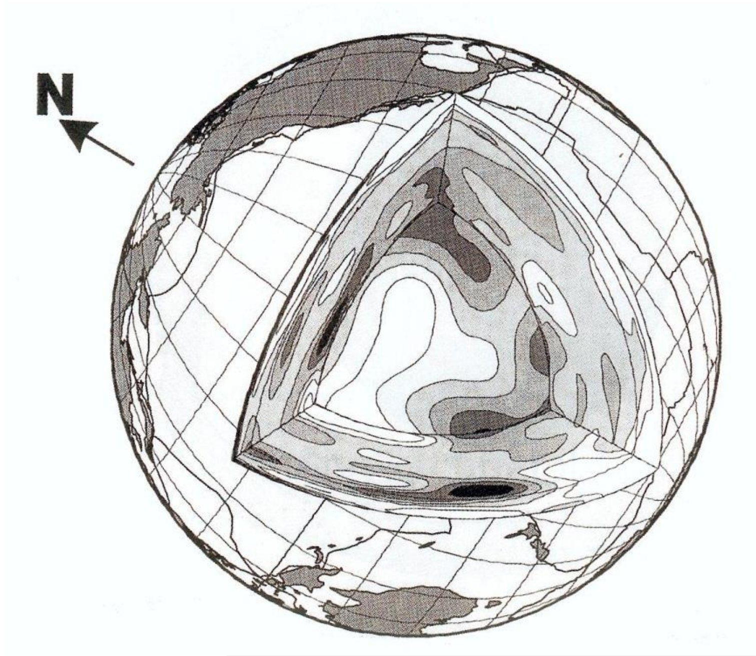
*Datase of traveltime pickings and related metadata*
*Numerous contribution from France, Switzerland and Italy*

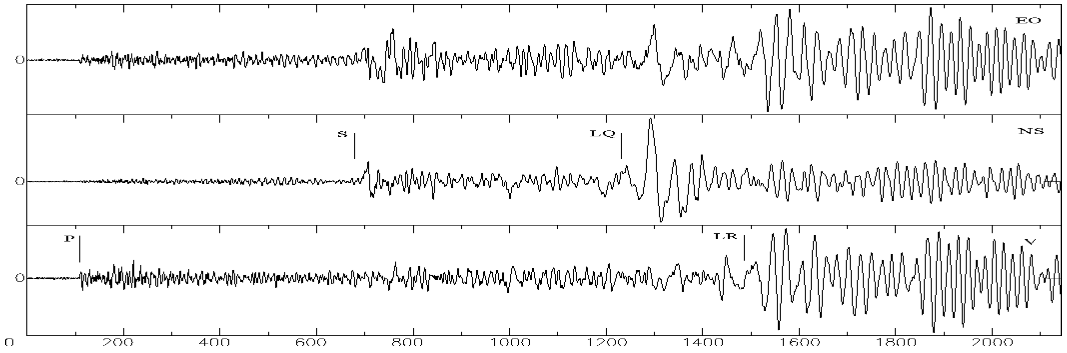
*Few names from ISTerre when using Western Alps data*
*A. Paul, Ph. Guéguen, A. Helmstetter, G. Janex, M. Langlais, B. Potin and L. Stehly*

*Still responsible of this methodological presentation*

Seiscope consortium

# Waves: phases and vibrations



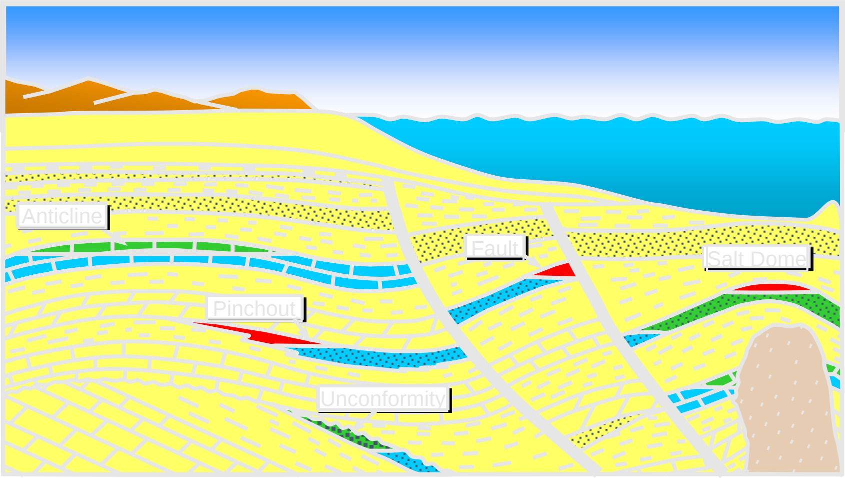

- Source time (rupture velocity)

  from 0.1 sec to 100 sec
- Wave time

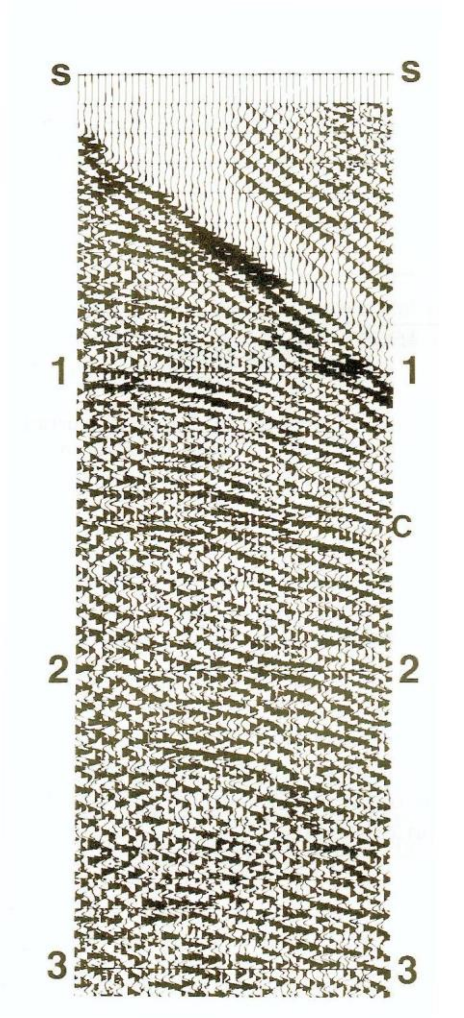
  from seconds to hours
- Window time

  from few seconds to days

days

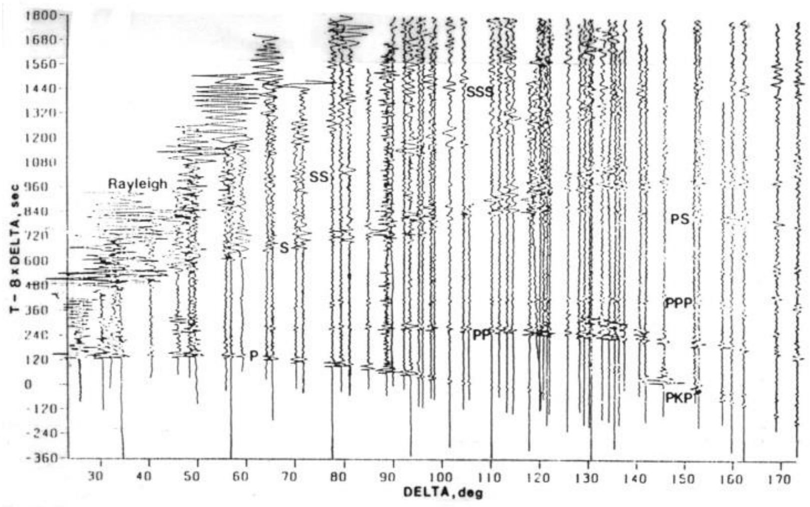


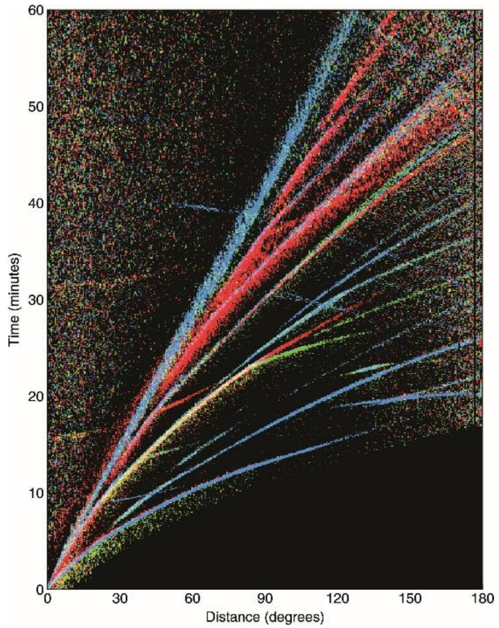

American Petroleum Institute, 1986



# Seismograms

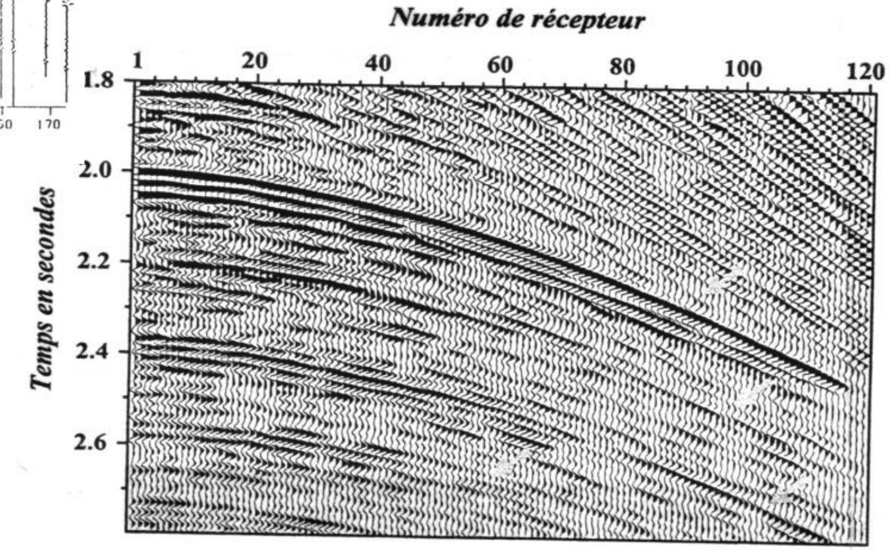
Records of a far-distance earthquake (Müller and King, 1976)



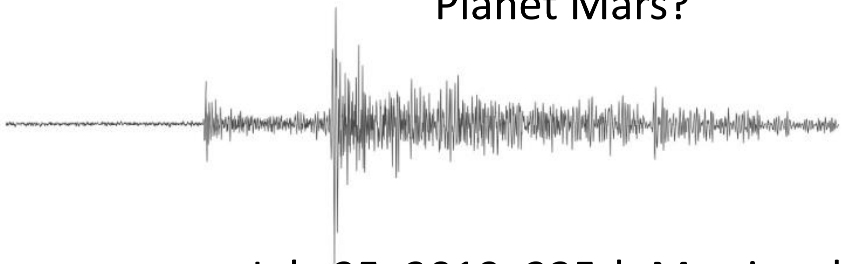




Impressive data mining over centuries: phase picking (ISC)

Traces from an oil reservoir (Thierry, 1997)

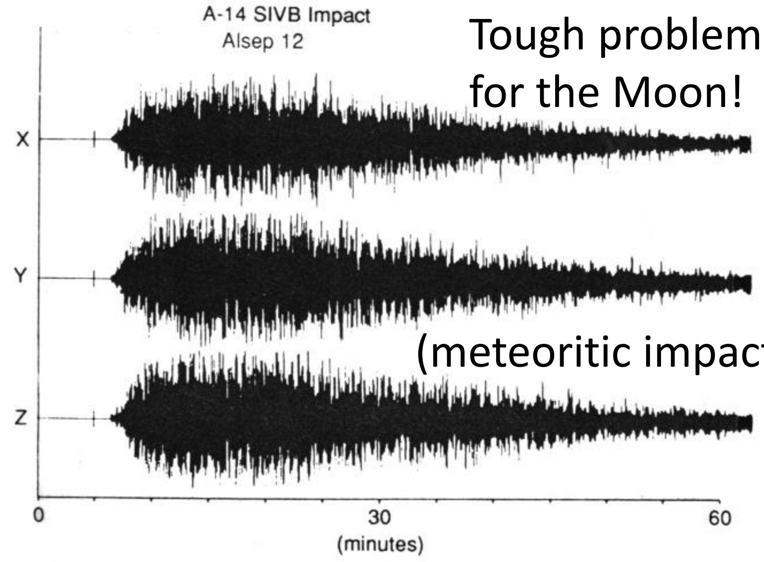



Planet Mars?



July 25, 2019, 235th Martian day

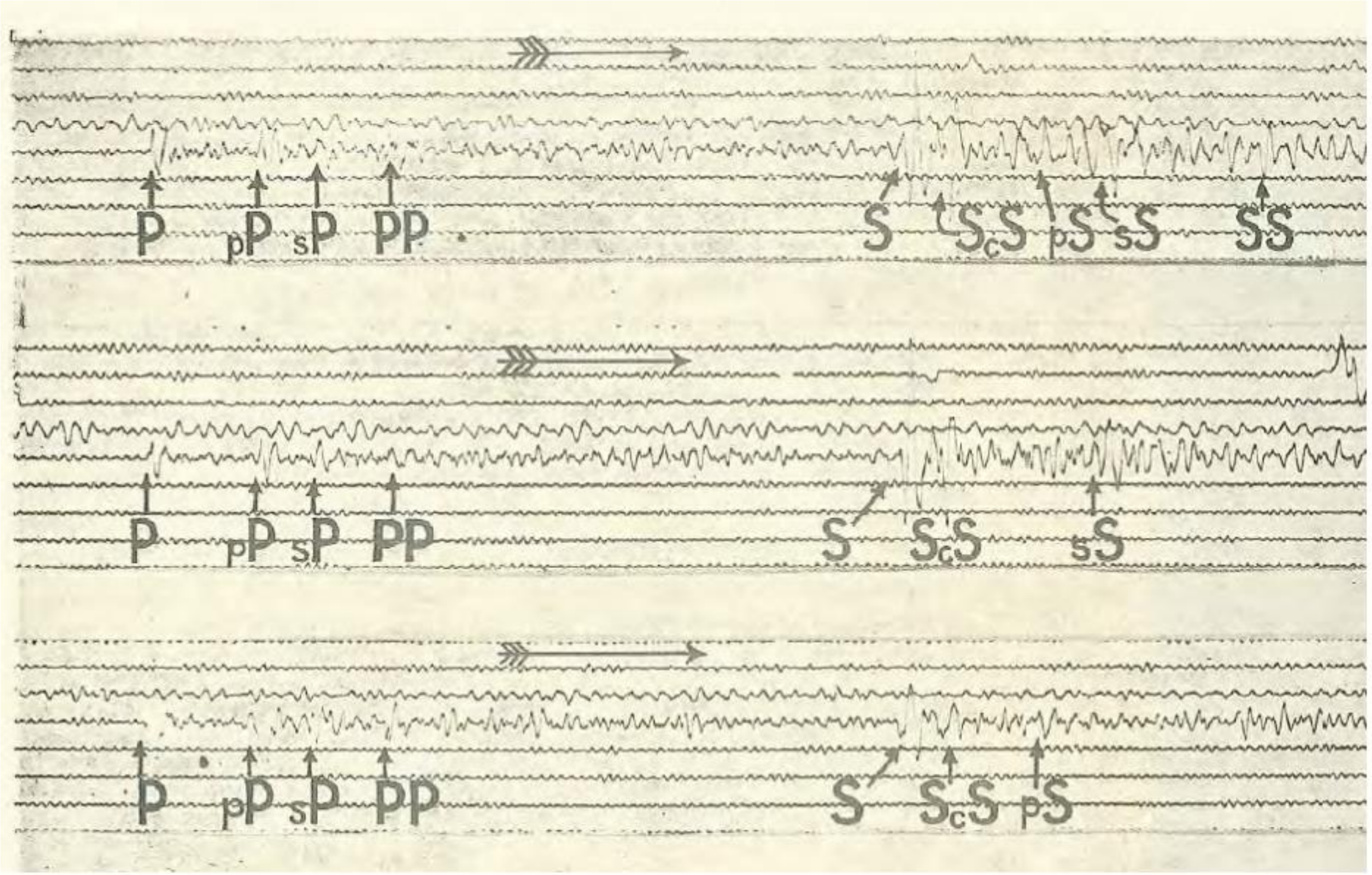
Records on the Moon (Latham et al., 1971)




Tough problem for the Moon!

(meteoritic impact)

# Onset time (date)




From Jeffreys, "The Earth"

Phase or time : a very robust observable to be extracted from seismograms

# Outline on first-arrival traveltime tomography

- Images at very different scales
- Waves and Phases: various concepts
- Few points on first-break ray-based tomography
- Illustration on 30-years Western Alps tomography
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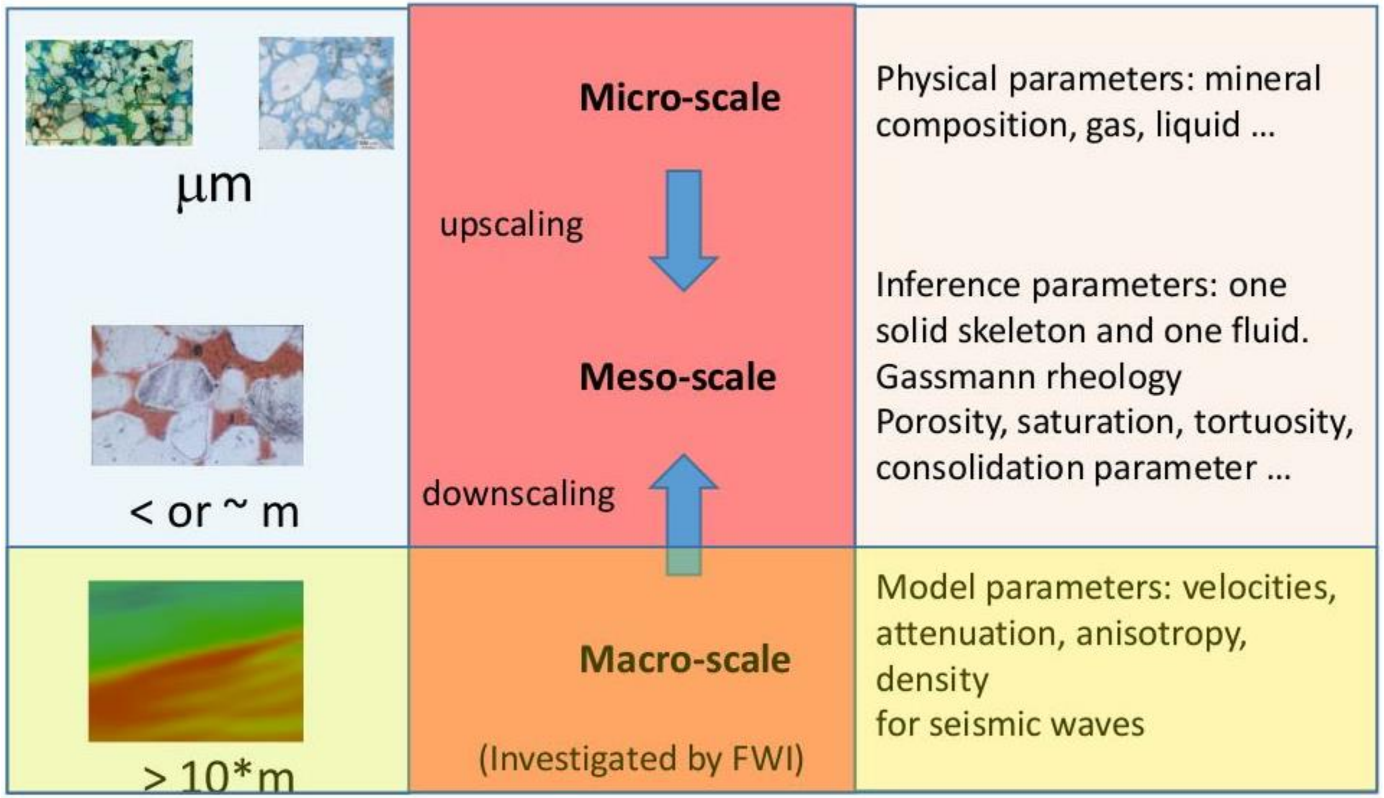
- First-break wave-equation-based tomography
- Hypocenter-velocity joint inversion
- Conclusion

# Expected resolution of seismic images

## Translucent Earth

Unaccessible target…
Intrinsic remote sensing limitation…
Seismic finite frequency & attenuation…



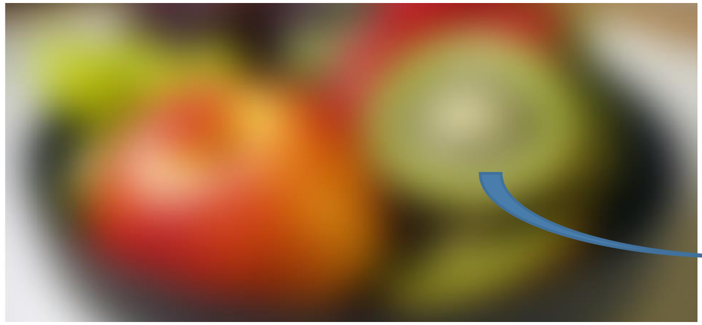

Inspired from Depuy's PhD (2011)

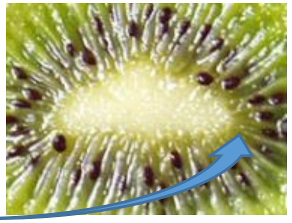
Real medium

Other information
Geology
Rock physics
Remote sensing

How far could we go with phases?

Interpretation?

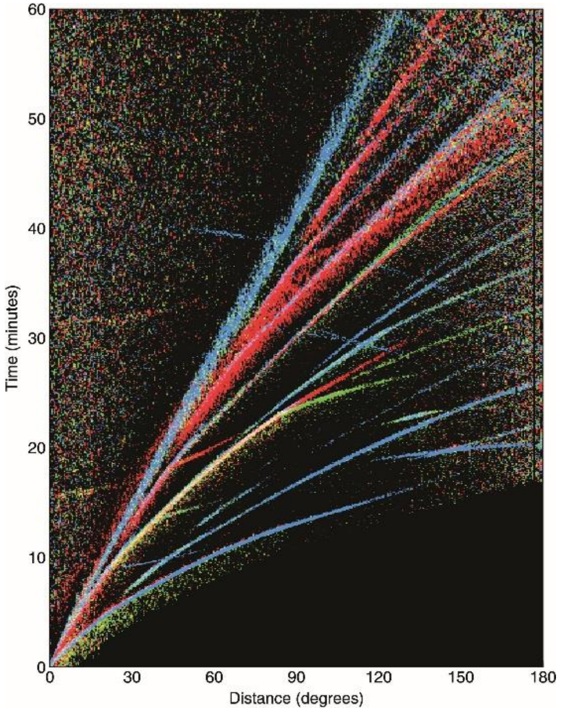




Seismic imaging

Inspired from Romanowicz's lecture

# Seismic images from time/phase picks



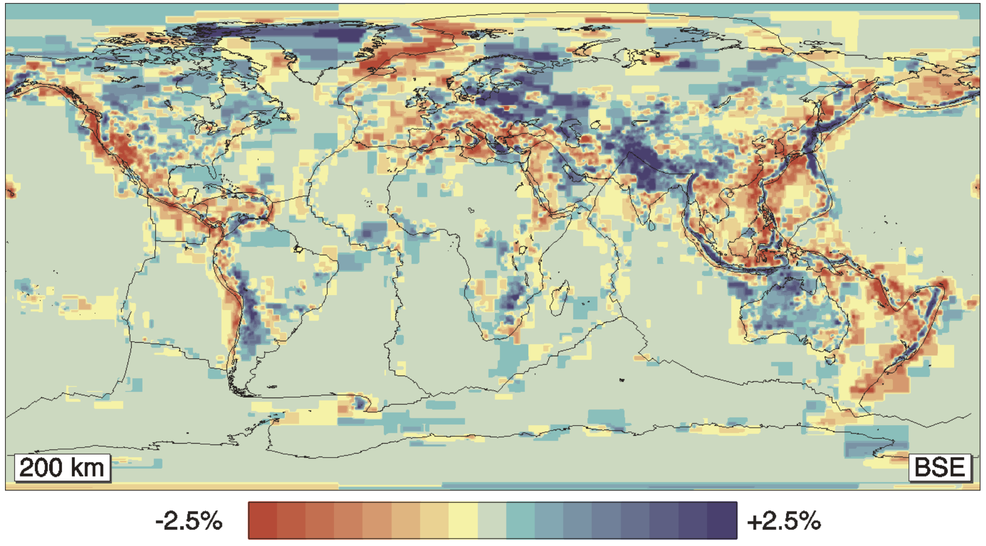

Accumulation of picks over centuries …

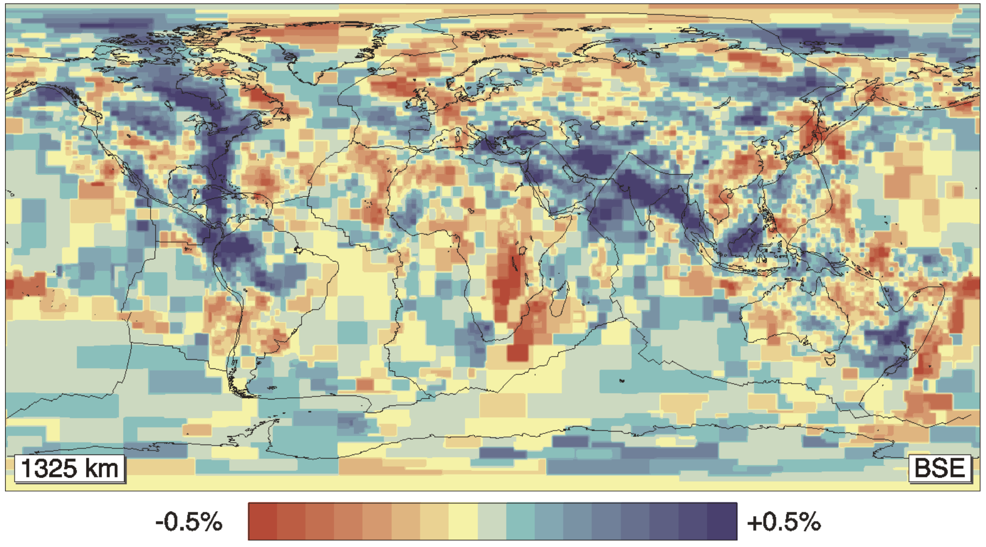
- Velocity variation at a depth of 200 km : good correlation with superficial structures.

- Velocity variations at a depth of 1325 km : good correlation with the Geoid.

(Courtesy of W. Spakman)

Earthquake positions are fixed




Global scale

# Seismic images at all scale variations

Impressive scale variation

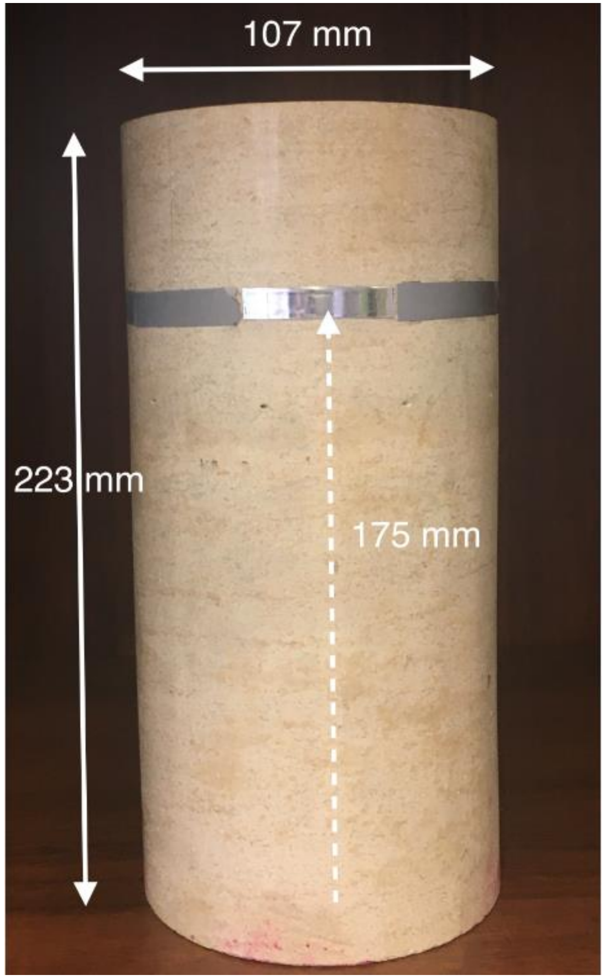
- Global scale
- Lithospheric/continental scale
- Upper crustal scale
- Near-surface scale
- Laboratory scale

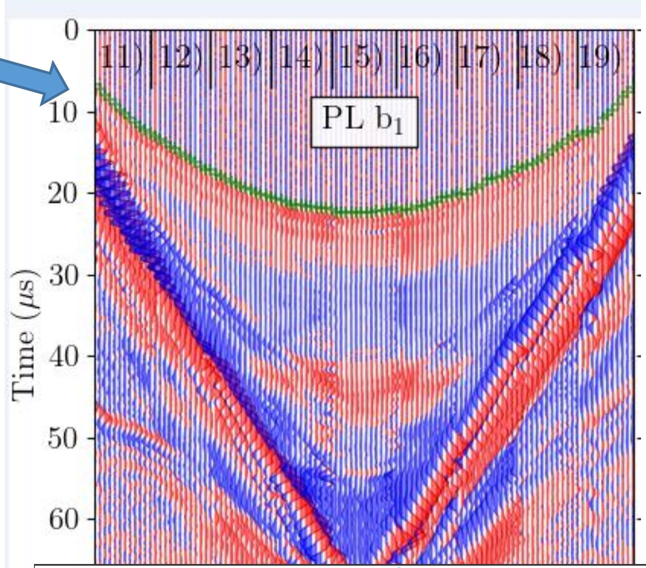
Physical upscaling of first-arrival times: crossing scale variation

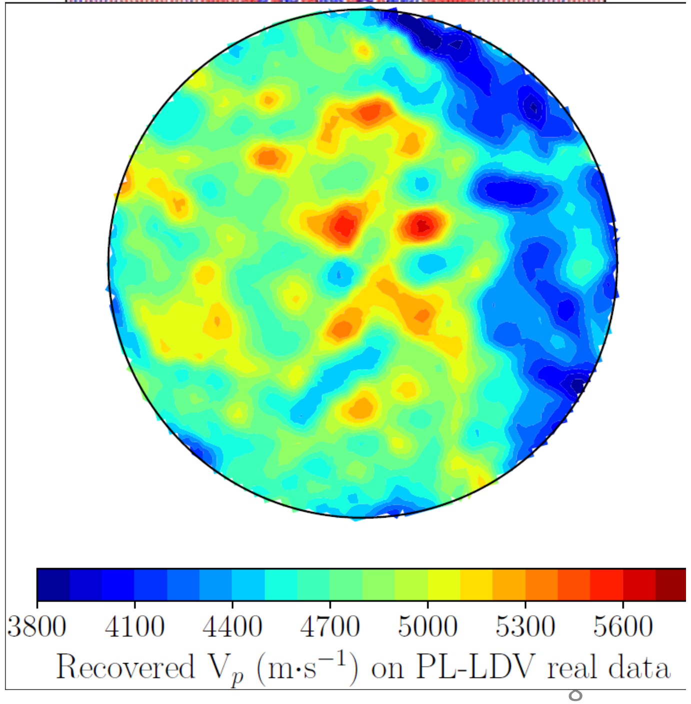
First-arrival time tomography
Results may cross scale variation …

Only first-arrival picks

Carbonate sample

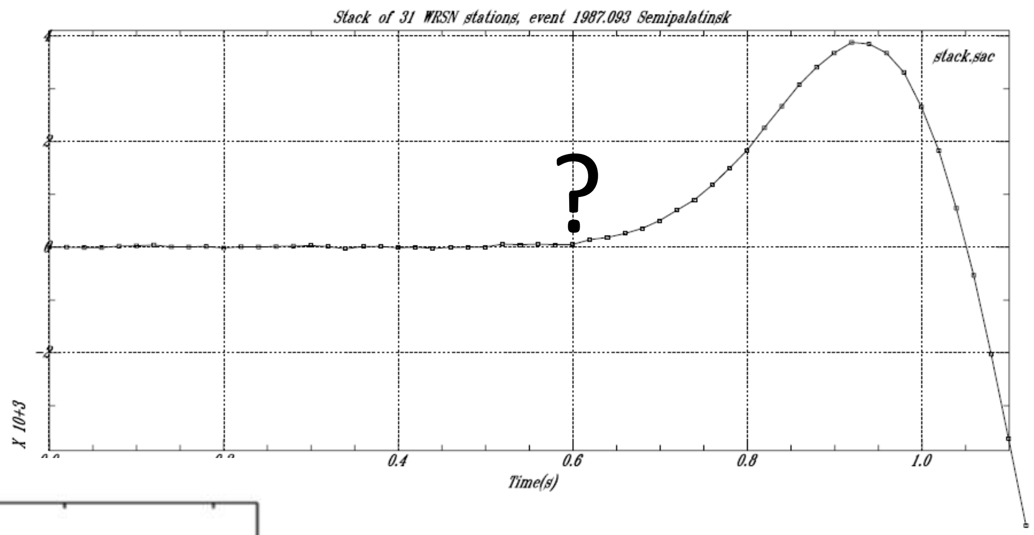



Shen et al., (in preparation)

# Picking onset time



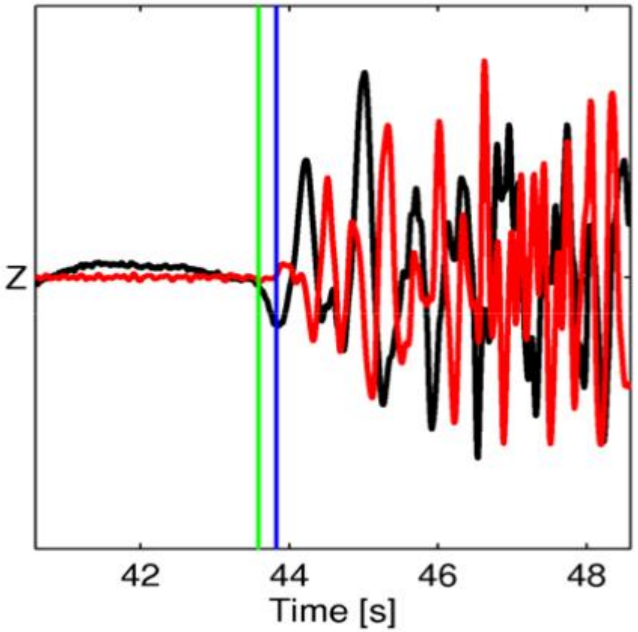

From Nolet (2010)

Onset time: picking and association (Pn, Pg …)

Agnostic to frequency content

Diffraction with wavefront healing

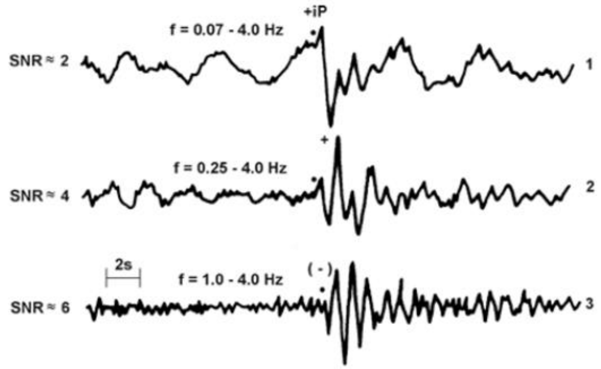
Dispersion & noise may induce phase shifts, making picking challenging (inaccurate, ambiguous association)



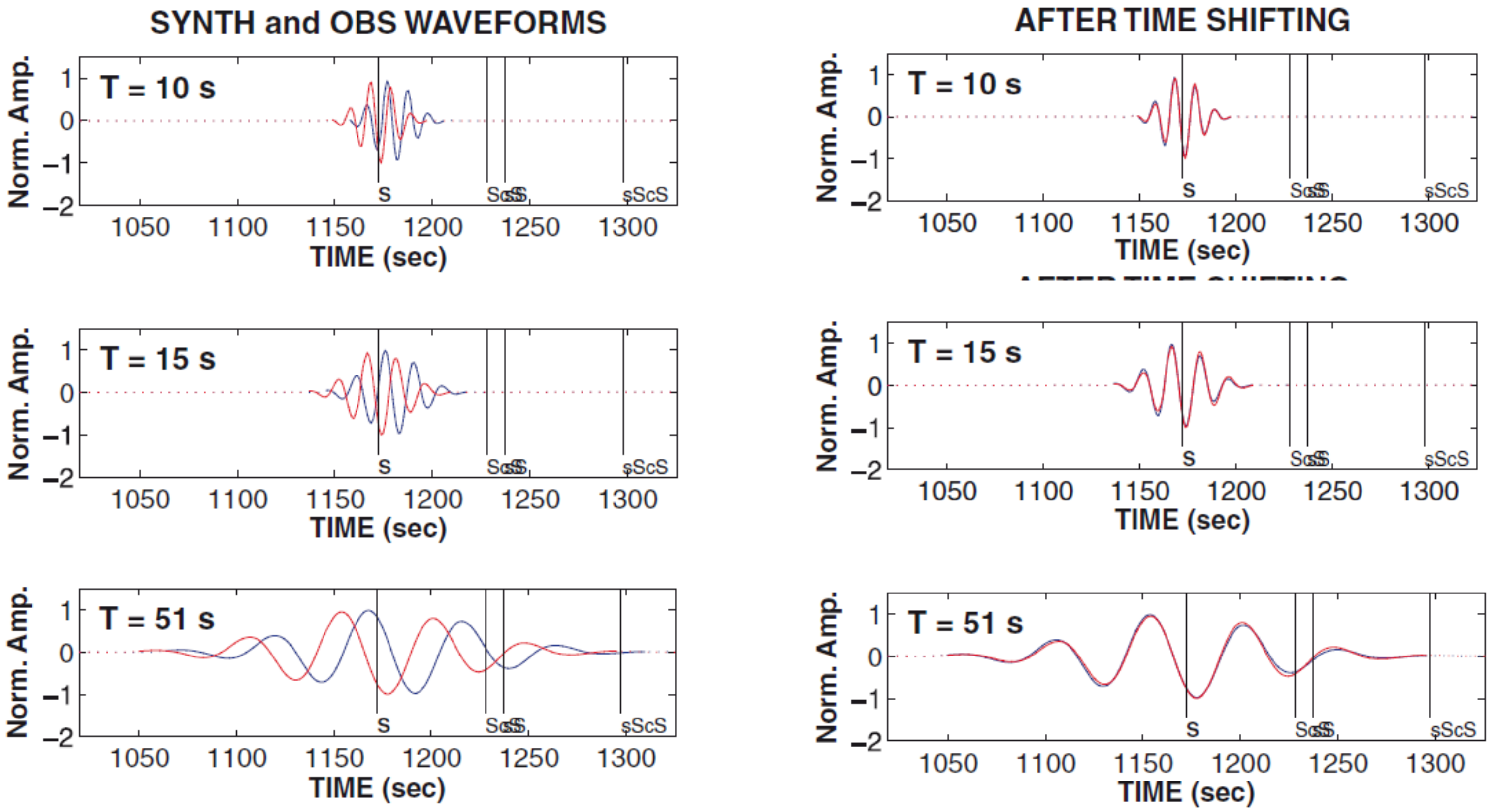

Black: unfiltered waveform (60 s – 50 Hz)
Red: Butterworht bandpass filter (2-10 Hz)

Two onset picks (green and blue) …

(From Küperkoch et al, 2010)



# Wavegroup delays: cross-correlation




Time delays by cross-correlation are frequency/period dependent (at least at long periods)

Amplitude sensitive:
well-calibrated instrument !

(Zaroli et al, GJInt, 2010)

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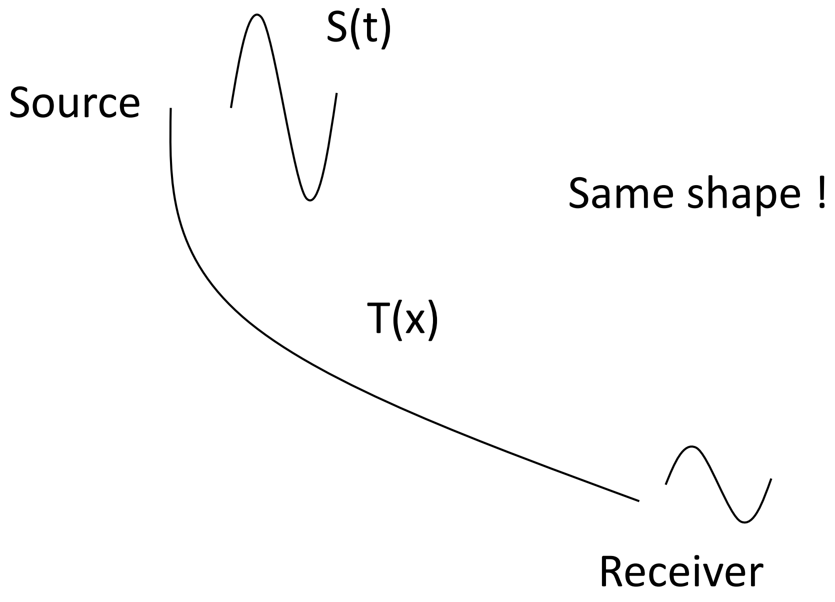
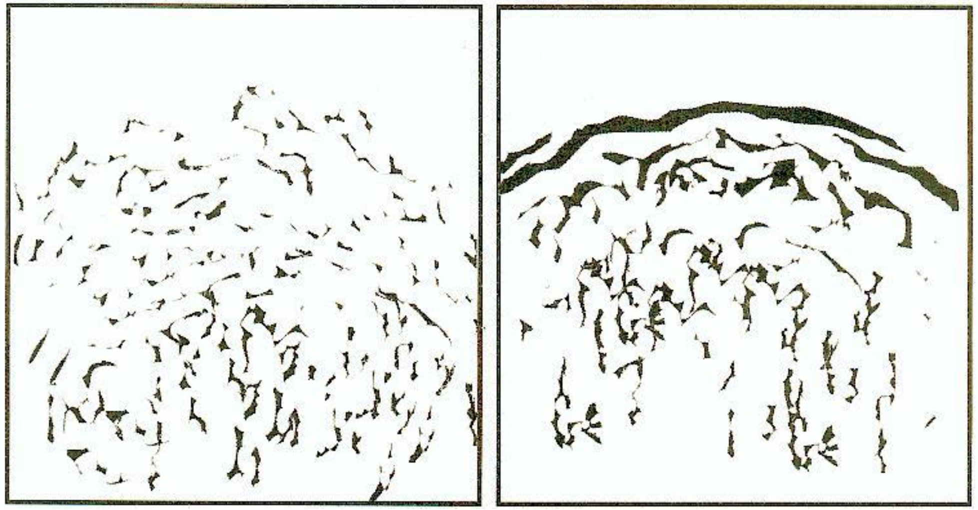
# Asymptotic solution: ray concept

$$u(x,t) = A(x)\,S\big(t - T(x)\big)$$
$$u(x,\omega) = A(x)\,S(\omega)\,e^{i\omega T(x)}$$




Travel-time $T(x)$ (phase $\omega T(x)$)
and Amplitude $A(x)$



Highly diffracting medium:
Losing wavefront coherence!

Preserved wavefront:
spatial continuity

Asymptotic approach with growing frequencies

**Diffraction still present!**

Eikonal solution – GTD (J. Keller, 1962)

**No diffraction at all!**

Ray solution ⇔ Geometrical Optics ⇔∞ frequency

⇔Singularities topology (shadow zone)

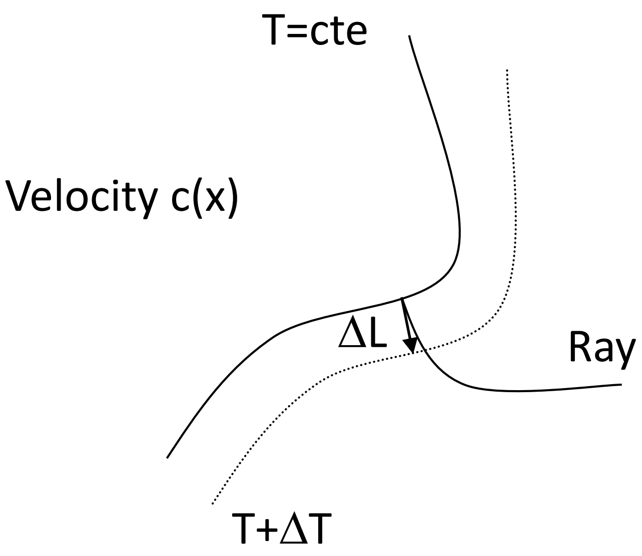
Link to the Catastrophe Theory (F. Math. René Thom)

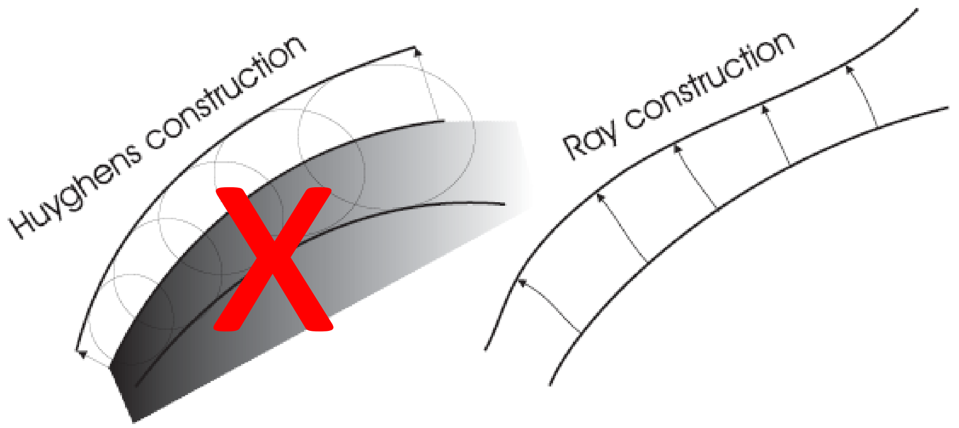
Complex phase analysis (discontinuity)…

# Ray Ansatz: $A(x)e^{i\omega T(x)}S(\omega)$ - Eikonal PDE

Two simple interpretations of wavefront evolution

Orthogonal trajectories are rays in an isotropic medium




Grad(T)=$\nabla_x T$ orthogonal to wavefront

$$c(x) = \frac{\Delta L}{\Delta T} \rightarrow \frac{\Delta T}{\Delta L} = \frac{1}{c(x)} \rightarrow \nabla_x T(x) = \frac{1}{c(x)}$$

Direction ? : abs or square

The orientation of the wavefront could not be guessed from the local information on a specific wavefront

$$(\nabla_x T(x))^2 = \frac{1}{c^2(x)}$$

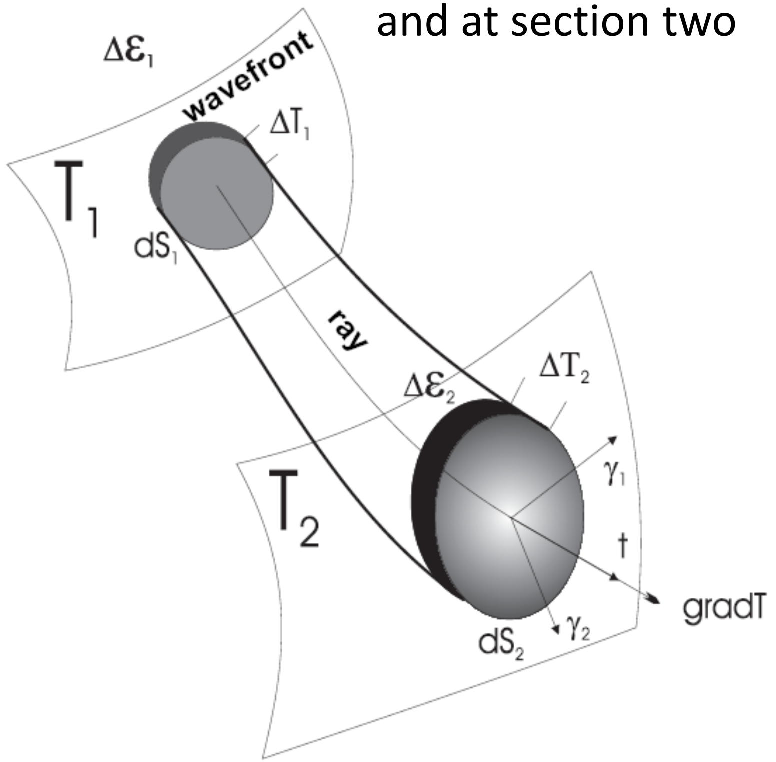
# Ray Ansatz: $A(x)e^{i\omega T(x)}S(\omega)$ - Transport PDE

Tracing neighboring rays defines a ray tube : variation of amplitude depends on energy flux conservation through sections.

$$\Delta\mathcal{E}_1 = {A_1}^2 dS_1 \Delta T_1 = {A_2}^2 dS_2 \Delta T_2 = \Delta\mathcal{E}_2$$

$$\rightarrow {A_1}^2 \nabla T_1 . \vec{n} dS_1 = {A_2}^2 \nabla T_2 . \vec{n} dS_2$$

Energy flux same at section one and at section two




$$2\nabla A(x) . \nabla T(x) + A(x)\nabla^2 T(x) = 0$$

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$$0 = \iint_{Tube}^{Ray} -{A_1}^2 \nabla T_1.\vec{n} dS_1 + {A_2}^2 \nabla T_2.\vec{n} dS_2$$

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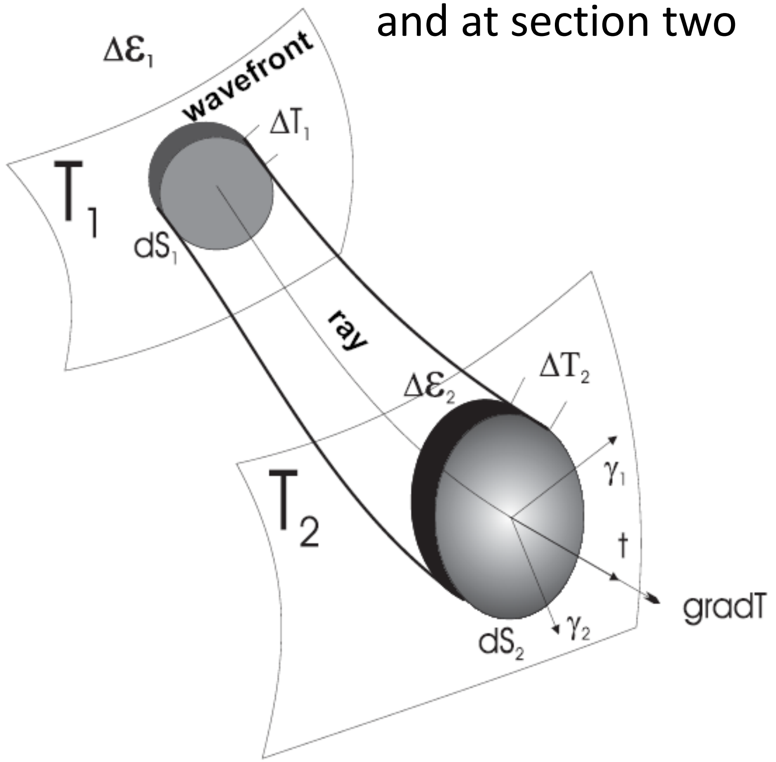


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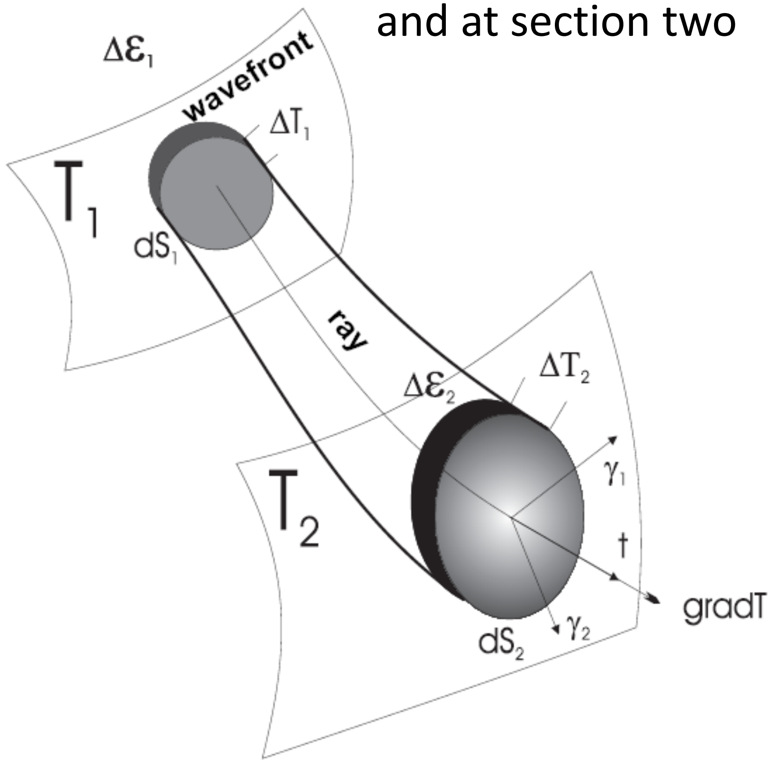
Outward normal orientation

$$0 = \iint_{Section}^{Ray} {A_1}^2 \nabla T_1 . \vec{n'} dS_1 + {A_2}^2 \nabla T_2 . \vec{n} dS_2$$

Energy flux same at section one and at section two

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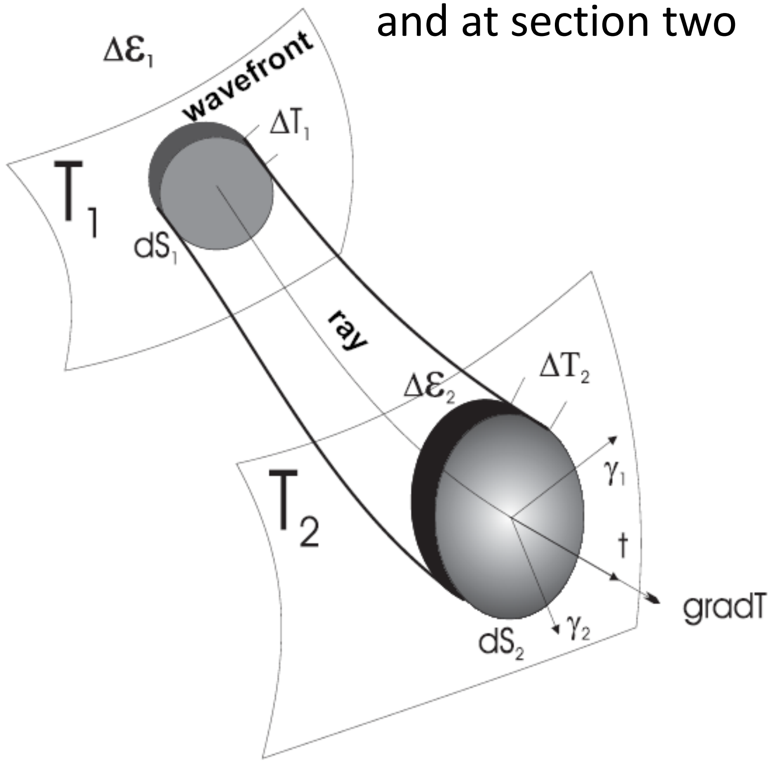
$$0 = \iint_{Section}^{Ray} {A_1}^2 \nabla T_1 . \vec{n'} dS_1 + {A_2}^2 \nabla T_2 . \vec{n} dS_2$$

*Adding the flux contribution along the tube which is zero*

$$0 = \iint_{Tube}^{Ray} \nabla T_c . \vec{n_c} dS_c$$

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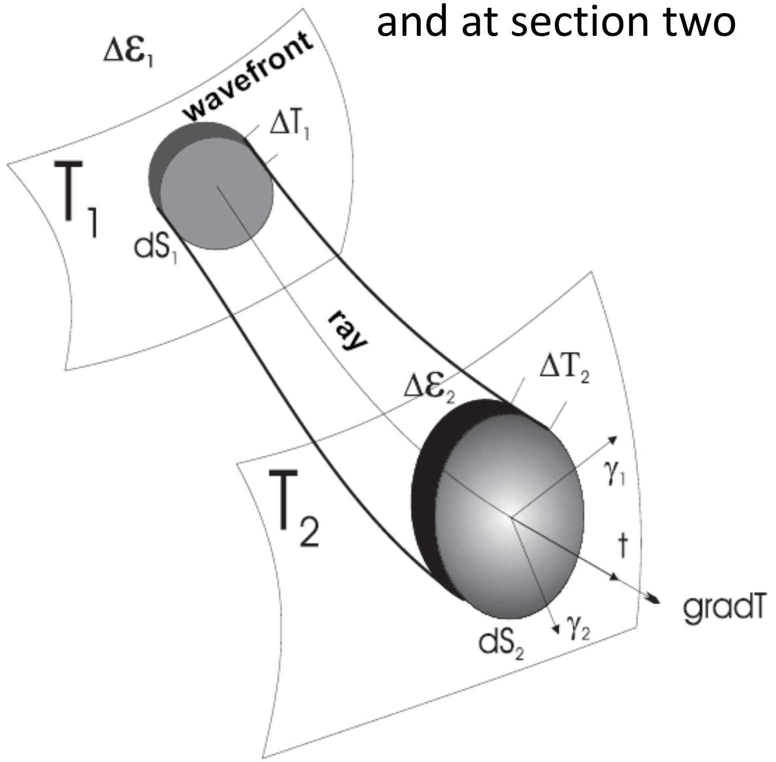


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*Application of the volume integral*

$$0 = \iiint_{Volume}^{Ray} div(A^2\, \nabla T) dV \Longrightarrow div(A^2 \nabla T) = 0$$

$$A(x)(2\nabla A(x) . \nabla T(x) + A(x) \nabla^2 T(x)) = 0$$

$$2\nabla A(x) . \nabla T(x) + A(x) \nabla^2 T(x) = 0$$

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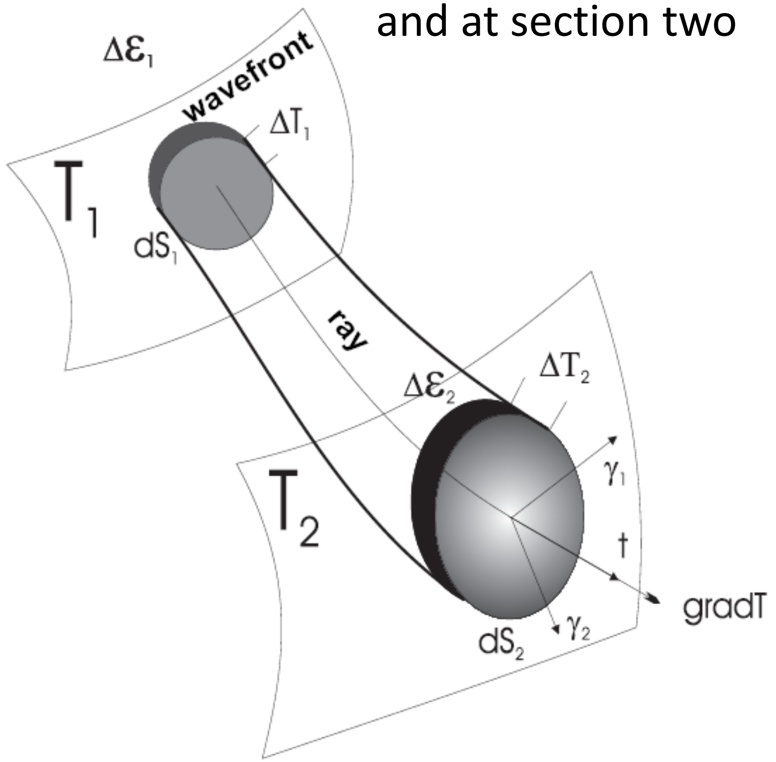


# A more mathematical approach

Scalar wave equation: $\nabla^2 p(x,t) = \frac{1}{c(x)} \ddot{p}(x,t)$ with velocity $c(x)$ and a constant density

Fourier transform: $p(x,t) = \exp[-i\omega t]\, p(x,\omega)$ with the circular frequency $\omega$

**Ray Ansatz**: an approximate high-frequency solution: $p(x,\omega) = A(x) e^{i\omega T(x)}$

The quantities A($x$) and T($x$) are presumably smooth scalar functions of coordinates

$$\nabla^2 p = \nabla . \nabla p = \{i\omega(\nabla A + i\omega A \nabla T). \nabla T + (\nabla^2 A + i\omega \nabla T. \nabla A + i\omega A \nabla^2 T)\} e^{i\omega T}$$

$$-\omega^2 A \left[ (\nabla T)^2 - \frac{1}{c^2} \right] + i\omega [2\nabla A. \nabla T + A \nabla^2 T] + \boldsymbol{\nabla^2 A} = 0; \ \forall \omega$$

$$(\nabla T)^2 - \frac{1}{c^2} = 0$$

Eikonal equation: non-linear PDE

$$2\nabla A. \nabla T + A \nabla^2 T = \nabla (A^2 \nabla T) = 0$$

Transport equation: linear PDE

# Ray-series solution

What to do with the term $\boldsymbol{\nabla^2 A}$ ?    Often considered as a corrective term

Amplitude may be considered as a series in inverse powers of frequency

$$A(x,\omega) = A_0(x) + \frac{1}{i\omega}A_1(x) + \frac{1}{(i\omega)^2}A_2(x)$$

Remark: fractional power
Needed for advanced concept?

No more the term $\boldsymbol{\nabla^2 A}$ but a series of transport equations for terms $A_l(x)$

Many other interpretations: hyper-eikonal or frequency-dependent eikonal

Very few applications with these transport equations

# Methods of (bi)characteristics

Differential geometry (Courant & Hilbert, 1966)

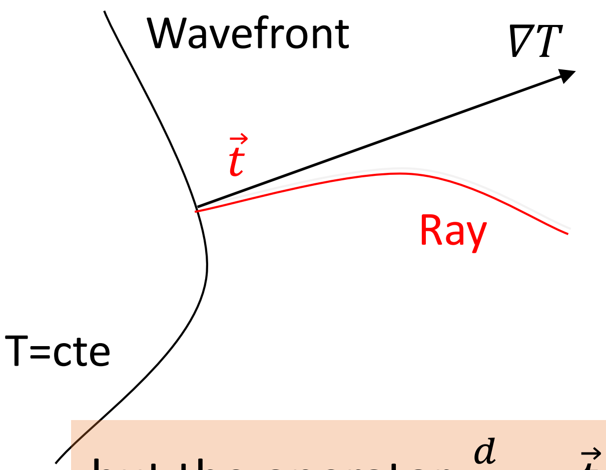
Wavefronts are geometrical characteristics

- Non-linear ordinary differential equations
- Lagrangian formulation as we integrate along rays (bicharacteristics)

In opposition to Eulerian formulation where we compute quantities at fixed positions

or mixed Eulerian-Lagrangian formulation using Eulerian wavefront tracking with Lagrangian ray evolution

# Ray equations: bicharacteristics




$$\overrightarrow{x(s)}\ with\ s\ as\ curvilinear\ abscisse\ \vec{t} = \frac{d\vec{x}}{ds} \rightarrow \|\vec{t}\| = 1$$

- Evolution of $\vec{x}$ is given by $\frac{d\vec{x}}{ds}$: $\frac{d\vec{x}}{ds} \parallel \overrightarrow{\nabla T} \rightarrow \boxed{\frac{d\vec{x}}{ds} = c(x)\overrightarrow{\nabla T}(x)}$

but the operator $\frac{d}{ds} = \vec{t}.\vec{\nabla} = c\overrightarrow{\nabla T}.\vec{\nabla}$

and, therefore, $\frac{d\overrightarrow{\nabla T}}{ds} = c\overrightarrow{\nabla T}.\vec{\nabla}(\overrightarrow{\nabla T})$

leading to $\frac{d\overrightarrow{\nabla T}}{ds} = \frac{c}{2}\vec{\nabla}(\overrightarrow{\nabla T}.\overrightarrow{\nabla T}) = \frac{c}{2}\vec{\nabla}(\overrightarrow{\nabla T})^2 = \frac{c}{2}\vec{\nabla}(\frac{1}{c^2})$

Ray equations

- Evolution of $\overrightarrow{\nabla T}$ is given by $\frac{d\overrightarrow{\nabla T}}{ds}$ $\rightarrow \boxed{\frac{d\overrightarrow{\nabla T}}{ds} = \vec{\nabla}(\frac{1}{c(x)})}$

Remark: characteristics are wavefronts

# Ray equations: position and slowness

Two quantities along the ray:

- ❑ slowness vector $\vec{p} = \overrightarrow{\nabla T}(s)$
- ❑ position vector $\vec{q} = \vec{x}(s)$

Slowness can be removed, leading to the so-called curvature equation

$$\frac{d}{ds}\left(\frac{1}{c(s)}\frac{d\vec{q}}{ds}\right) = \vec{\nabla}\left(\frac{1}{c(s)}\right)$$

Various non-linear ray equations: which ODE to choose for integration? Time or Particule

*Curvilinear stepping*

$$\frac{d\vec{q}(s)}{ds} = c(\vec{q})\vec{p}$$
$$\frac{d\vec{p}(s)}{ds} = \nabla_{\vec{q}}\frac{1}{c(\vec{q})}$$
$$\frac{dT(s)}{ds} = \frac{1}{c(\vec{q})}$$

*Time stepping*

$$\frac{d\vec{q}(t)}{dt} = c^2(\vec{q})\vec{p}$$
$$\frac{d\vec{p}(t)}{dt} = c(\vec{q})\nabla\frac{1}{c(\vec{q})}$$

*Particule stepping*

$$\frac{d\vec{q}(\xi)}{d\xi} = \vec{p}$$
$$\frac{d\vec{p}(\xi)}{d\xi} = \frac{1}{c(\vec{q})}\nabla\frac{1}{c(\vec{q})}$$
$$\frac{dT(\xi)}{d\xi} = \frac{1}{c^2(\vec{q})}$$

The simplest set

# Ray equations: position and slowness

*Curvilinear stepping*

$$\frac{d\vec{q}(s)}{ds} = c(\vec{q})\vec{p}$$
$$\frac{d\vec{p}(s)}{ds} = \nabla_{\vec{q}} \frac{1}{c(\vec{q})}$$
$$\frac{dT(s)}{ds} = \frac{1}{c(\vec{q})}$$

*Time stepping*

$$\frac{d\vec{q}(t)}{dt} = c^2(\vec{q})\vec{p}$$
$$\frac{d\vec{p}(t)}{dt} = c(\vec{q})\nabla \frac{1}{c(\vec{q})}$$

$$dt = \frac{1}{c(\vec{q})} ds = \frac{1}{c(\vec{q})^2} d\xi$$

*Particule stepping*

$$\frac{d\vec{q}(\xi)}{d\xi} = \vec{p}$$
$$\frac{d\vec{p}(\xi)}{d\xi} = \frac{1}{c(\vec{q})} \nabla \frac{1}{c(\vec{q})}$$
$$\frac{dT(\xi)}{d\xi} = \frac{1}{c^2(\vec{q})}$$

Any numerical integration tool: Runge-Kutta or Predictor-Corrector schemes. However, Eikonal quantity $p^2 = 1/c^2(\vec{q})$ may be used for quality control. No need of automatic control of schemes.

Many analytical solutions (gradient of velocity; gradient of slowness square …)

# Anisotropic media: ray equations

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*Particule stepping*

$$\frac{dq_i\,(\xi)}{d\xi} = p_i + \frac{1}{c^3(q_i,p_i)}\frac{\partial c(q_i,p_i)}{\partial p_i}$$

$$\frac{dp_i\,(\xi)}{d\xi} = \frac{1}{c(q_i,p_i)}\nabla_{q_i}\frac{1}{c(q_i,p_i)}$$

$$\frac{dT(\xi)}{d\xi} = \sum_i p_i(p_i + \frac{1}{c^3(q_i,p_i)}\frac{\partial c(q_i,p_i)}{\partial p_i})$$

Eikonal equation

$$p^2 = \frac{1}{c^2(\vec{q},\vec{p})}$$

Numerical tools can be used as well, including the perturbation theory for paraxial ray equations

*Time stepping*

$$\frac{dq_i\,(t)}{dt} = (p_i + \frac{1}{c^3(q_i,p_i)}\frac{\partial c(q_i,p_i)}{\partial p_i})\frac{1}{\sum_k p_k(p_k + \frac{1}{c^3(q_k,p_k)}\frac{\partial c(q_k,p_k)}{\partial p_k})}$$

$$\frac{dp_i\,(t)}{dt} = (\frac{1}{c(q_i,p_i)}\nabla_{q_i}\frac{1}{c(q_i,p_i)})\frac{1}{\sum_k p_i(p_k + \frac{1}{c^3(q_k,p_k)}\frac{\partial c(q_k,p_k)}{\partial p_k})}$$

# Anisotropic media: transport equations

Isotropic case: scalar functions

$$2\nabla A(x).\nabla T(x) + A(x)\nabla^2 T(x) = 0$$

$$\sum_{i=1}^{3} 2\frac{\partial A}{\partial x_i}\frac{\partial T}{\partial x_i} + A\frac{\partial^2 T}{\partial {x_i}^2} = 0$$

$$u(x,t) = A(x)\, S\big(t - T(x)\big)$$
$$\mathrm{u}(\mathrm{x},\omega) = \mathrm{A}(\mathrm{x})\, \mathrm{S}(\omega)\, e^{i\omega T(x)}$$

Anisotropic case: vectorial functions

$$\vec{\mathrm{u}}(\mathrm{x},\mathrm{t}) = \vec{\mathrm{A}}(\mathrm{x})\, \mathrm{S}\big(\mathrm{t} - \mathrm{T}(\mathrm{x})\big)$$
$$\vec{\mathrm{u}}(\mathrm{x},\omega) = \vec{\mathrm{A}}(\mathrm{x})\, \mathrm{S}(\omega)\, \mathrm{e}^{\mathrm{i}\omega\mathrm{T}(\mathrm{x})}$$

$$\forall i = 1,3$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial}{\partial x_j}\big(c_{ijkl}(x)\big)\frac{\partial A_k}{\partial x_k} + c_{ijkl}(x)\frac{\partial^2 A_k}{\partial x_j \partial x_l}\right) = 0$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left[\frac{\partial}{\partial x_j}\left(c_{ijkl}(x)A_l\frac{\partial T}{\partial x_k}\right) + c_{ijkl}(x)\frac{\partial A_k}{\partial x_l}\frac{\partial T}{\partial x_j}\right] = 0$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}(x)A_k\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_l} - \rho(x)A_i = 0$$

Slawinski, 2003, p166-167

Not easy to understand geometrically how energy flows

# Solving ray equations

How to solve these equations?

Ordinary Differential Equations

Lagrangian formulation (FAST!)

Wavefront complexity

# Simple properties of these ODEs

- Intrinsic solutions independent of the coordinate sytem used to solve it
- If a dummy variable in velocity description, use it as the variable stepping (often x coordinate)

$$\nabla_x \frac{1}{c(q_z)} = 0 \Rightarrow p_x = cte \Rightarrow q_x = {q_x}^0 + \xi p_x \text{ (1)}$$

- Eikonal equation: a good proxy for testing the accuracy of the ray tracing (not enough used)

In 3D: six or seven equations
In 2D: four or five equations

(1): rectilinear motion of a particle along this axis in mechanics

# Velocity variation $v(z)$

Ray equations are

$$\frac{dq_x}{d\tau} = p_x; \frac{dq_y}{d\tau} = p_y; \frac{dq_z}{d\tau} = p_z$$

$$\frac{dp_x}{d\tau} = 0; \frac{dp_y}{d\tau} = 0; \frac{dp_z}{d\tau} = u(z)\frac{du(z)}{dz}$$

$$u(z) = \frac{1}{v(z)}$$

The horizontal component of the slowness vector is constant: the trajectory is inside a plan which is called the plan of propagation. We may define the frame (xoz) as this plane.

$$\frac{dq_x}{dq_z} = \frac{p_x}{p_z} = \frac{p_x}{\pm\sqrt{u^2(z) - p_x^{\ 2}}}$$

where $p_x$ is a constante

$$\frac{dq_x}{d\tau} = p_x; \frac{dq_z}{d\tau} = p_z$$

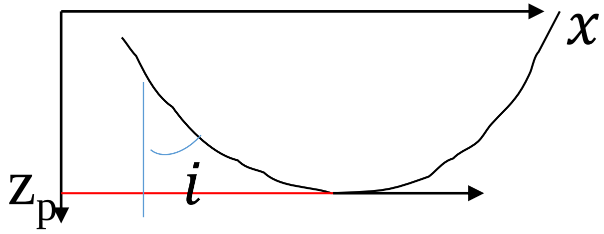
$$\frac{dp_x}{d\tau} = 0; \frac{dp_z}{d\tau} = u(z)\frac{du(z)}{dz}$$

$$q_x(z_1, p_{x1}) = q_x(z_0, p_{x0}) + \int_{z_0}^{z_1} \frac{p_x}{\sqrt{u^2(z) - p_x^{\ 2}}} dz$$

quadratic expression wrt depth

# Velocity variation $v(z)$

At a given maximum depth $z_p$, the slowness vector is horizontal following the equation $\quad p_x^2 = p^2 = u^2(z_p)$




$$q_x(z_1,p_1) = q_x(z_0,p_0) + \int_{z_0}^{z_p} \frac{p_x}{\sqrt{u^2(z) - p_x^2}} dz + \int_{z_1}^{z_p} \frac{p_x}{\sqrt{u^2(z) - p_x^2}} dz$$

$$T(z_1,p_1) = T(z_0,p_0) + \int_{z_0}^{z_p} \frac{u^2(z)}{\sqrt{u^2(z) - p_x^2}} dz + \int_{z_1}^{z_p} \frac{u^2(z)}{\sqrt{u^2(z) - p_x^2}} dz$$

Simple quadratic equations

If we consider a source at the free surface as well as the receiver, we get

In Cartesian frame

$$X(p) = 2\int_0^{z_p} \frac{p}{\sqrt{u^2(z) - p^2}} dz$$

$$T(p) = 2\int_0^{z_p} \frac{u^2(z)}{\sqrt{u^2(z) - p^2}} dz$$

$$with\ p = u\ sini$$

In Spherical frame

$$\Delta(p) = 2\int_{r_p}^{a} \frac{p}{\sqrt{r^2u^2(r) - p^2}} \frac{dr}{r}$$

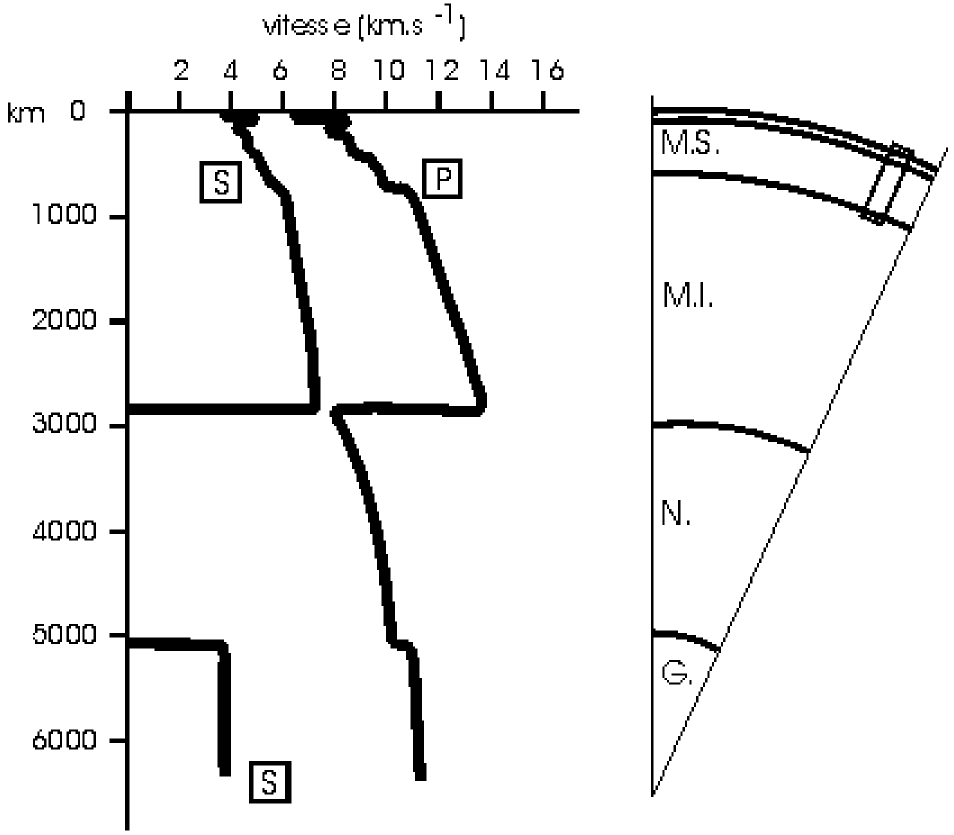
$$T(p) = 2\int_{r_p}^{a} \frac{p}{\sqrt{r^2u^2(r) - p^2}} \frac{dr}{r}$$

$$with\ p = ru\ sini$$

# Velocity structure in the Earth

Radial Structure

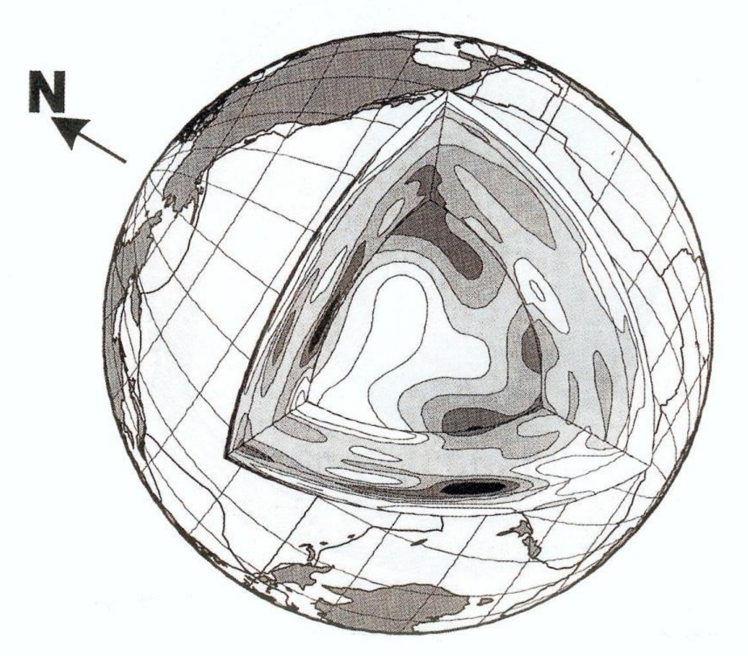
PREM – AKI135 models…




(a)

# What about velocity $v(x, y, z)$

No quadratic formulation!




Numerical approaches
for solving differential equations.

# Solving ray equations

How to solve these equations?

Ordinary Differential Equations

Lagrangian formulation (FAST!)

Wavefront complexity

# Hamilton's ray equations

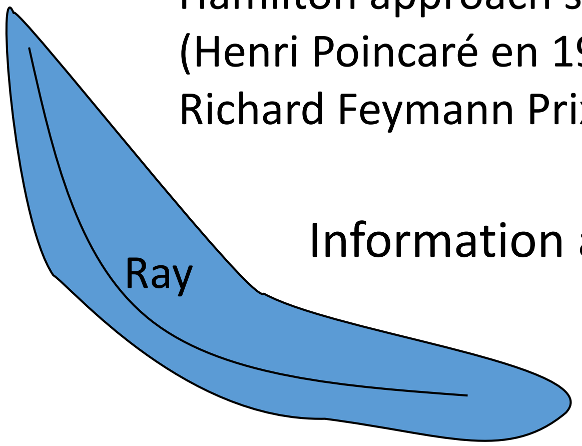
$$\frac{d\vec{q}(\xi)}{d\xi} = \vec{p}$$
$$\frac{d\vec{p}(\xi)}{d\xi} = \frac{1}{c(\vec{q})}\nabla_{\vec{q}}\frac{1}{c(\vec{q})}$$

$$\mathcal{H}(\vec{q},\vec{p}) = \frac{1}{2}(p^2 - \frac{1}{c^2(\vec{q})})$$

$$\frac{d\vec{q}(\xi)}{d\xi} = \nabla_{\vec{p}}\mathcal{H}$$
$$\frac{d\vec{p}(\xi)}{d\xi} = -\nabla_{\vec{q}}\mathcal{H}$$
$$\frac{dT}{d\xi} = \vec{p}.\nabla_{\vec{p}}\mathcal{H}$$

Mechanics : ray tracing as a particular balistic problem

sympletic structure (FUN!)




Hamilton approach suitable for perturbation
(Henri Poincaré en 1907 « Mécanique céleste »,
Richard Feymann Prix Nobel 1965)

$$\vec{q}_0 + \delta\vec{q}$$
$$\vec{p}_0 + \delta\vec{p}$$

$\delta q \; and \; \delta p \; "small"$

Information around the ray

Meaning of the neighborhood zone
Fresnel zone if finite frequency
Any zone depending on your problem Gaussian Beam summation

# Reduced Hamilton formulation

Reduction of ray equations from six to four (Cerveny, 2001, p107)

Solving the eikonal equation for obtaining $p_3$ reducing from one unknown gives

$$p_3 = -\mathcal{H}^R(x_1, x_2, x_3, p_1, p_2)$$

which is a non-linear partial differential equation of first-order known as static Hamilton-Jacobi equations.

We may select $x_3$ as the stepping variable, reducing once more from one unknown, removing two unknows and therefore two equations are cancelled.

Ray system $(x_1, x_2, p_1, p_2)$ with the evolution $x_3$ with a variable $\mathcal{H}^R$

The evolution parameter $x_3$ could not be monotonic (turning rays with an extremum in variable $x_3$).

In 3D, five equations
In 2D, three equations
including travel-time integration equation

# Hamilton’s ray equations

Turning rays leads us to consider centered ray coordinate system and trigonometric functions ! (to be avoided … as slow crunching)

This is not intrinsic to ray equations which can be written in any coordinate system (as well as the related paraxial/dynamic ray equations)

- Computational complexity for number crunching

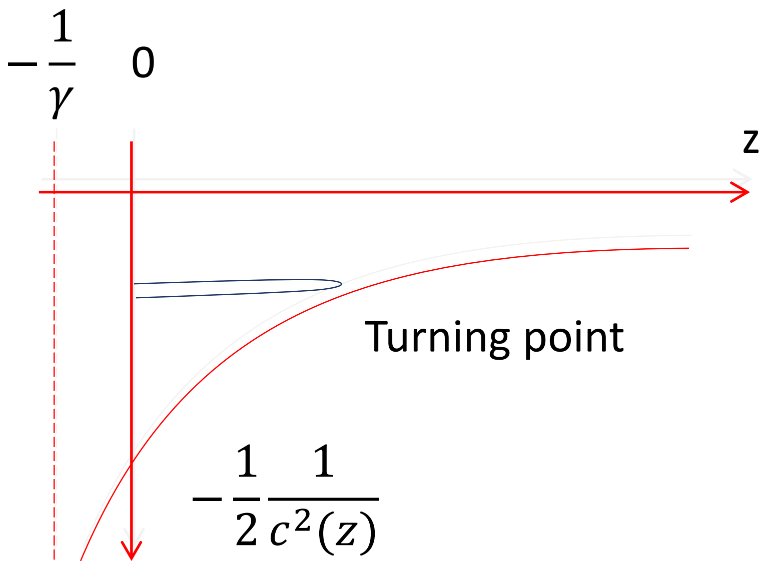
- No evaluation in the vicinity of the hyper-surface of the eikonal conservation as we live in lower-dimension space

What is best!

If you are afraid of curvilinear non-orthogonal coordinate system intrinsically linked with the ray geometry, you may forget about that.

Cartesian coordinate system leads to higher dimensions but with simpler expressions (somehow faster …)

# Mechanical point of view




Kinetic energy

Potential energy

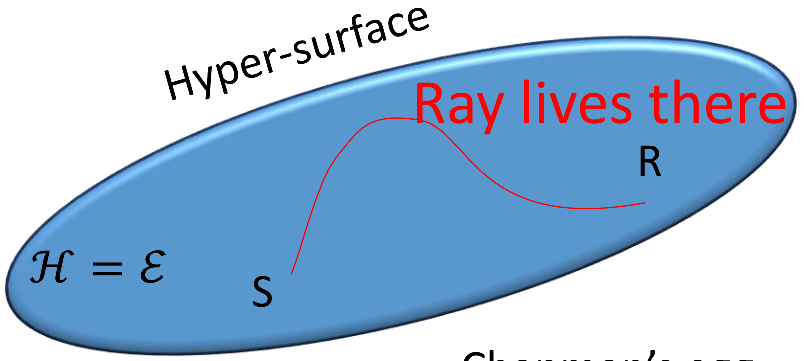
$$\mathcal{H}(\vec{q},\vec{p}) = \frac{1}{2}\frac{p^2}{m} + V(\vec{q})$$

$$\mathcal{H}(\vec{q},\vec{p}) = \frac{1}{2}p^2 - \frac{1}{2}\frac{1}{c^2(\vec{q})}$$

$$c(z) = c_0(1+\gamma z)$$

ray is a circle

I should draw a 3D curve with the x coordinate :Help !




(and not outside)

Chapman's egg

A conservative system

$$\mathcal{H}(\vec{q},\vec{p}) = 0$$

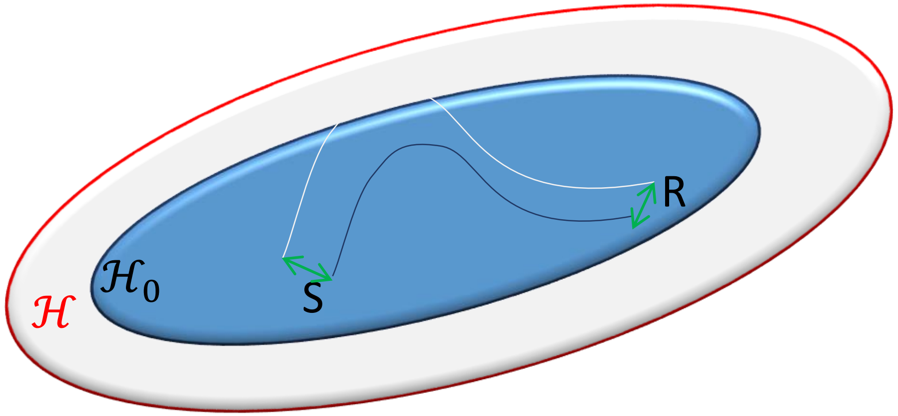
Just FUN?

A non-conservative system

$$\mathcal{H}(\vec{q},\vec{p}) = \mathcal{E}(\vec{q},\vec{p}) \Rightarrow \mathcal{H}(\vec{q},\vec{p}) - \mathcal{E}(\vec{q},\vec{p}) = 0$$

Embedding it into an isolated system !

# Mechanical point of view




If perturbation of the velocity structure, should we reset ray tracing?

$$\mathcal{H}(\vec{q},\vec{p}) = \mathcal{H}_0(\vec{q},\vec{p}) + \Delta\mathcal{H}(\vec{q},\vec{p})$$

Rays on the $\text{Egg}_0$ used for the estimation of rays on the new Egg

Shifts in the phase space of both sources and receivers: application to extended image analysis as promoted by different people: W. Symes, P. Sava among others.

Just FUN: NO!

We may feel the extra-dimensionality around a given Chapman's egg in this full Hamiltonian formulation: this is not the case when we consider the reduced Hamiltonian.

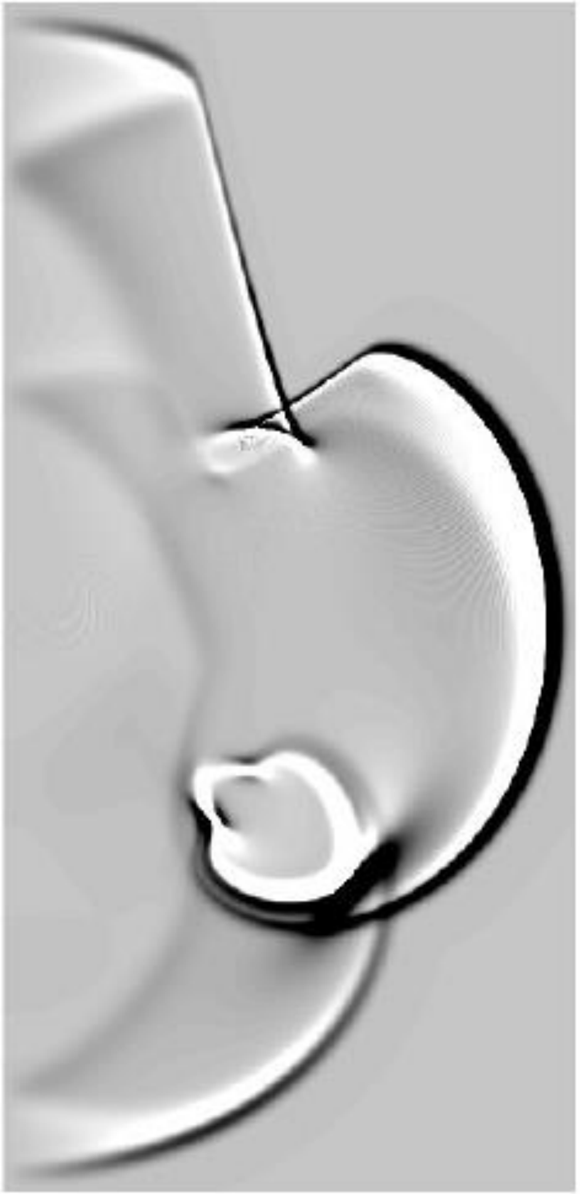
# Low complexity of ray equations

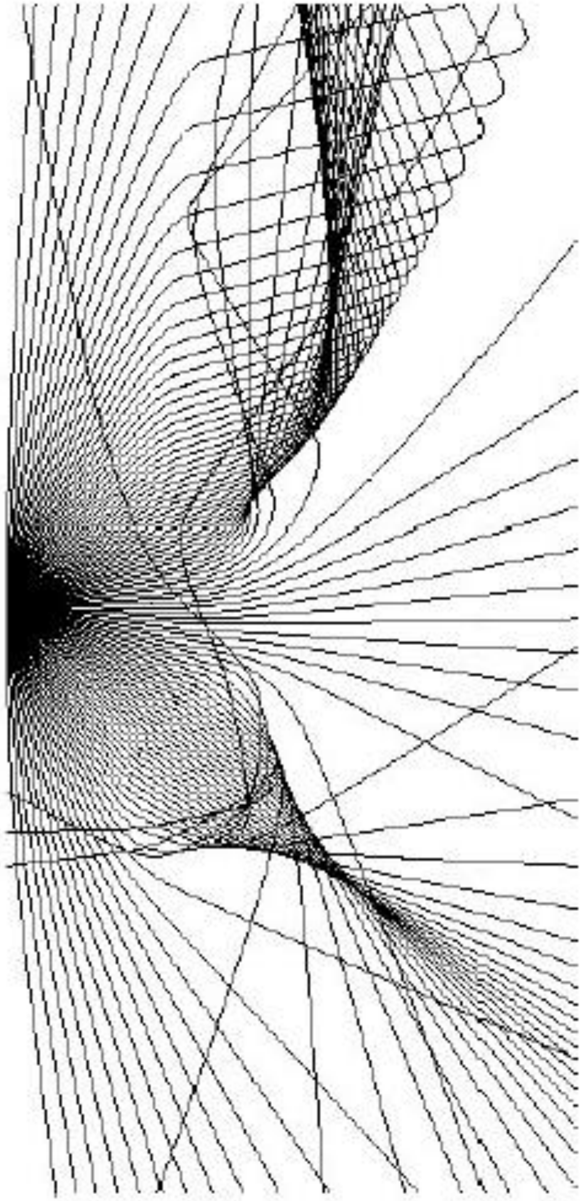
## Keeping computer complexity low!

- Ray tracing is a fast 1D integration in 2D/3D
- Ray tracing equations as ODEs may sample the model quite evenly
- They are lagrangian formulation: we follow a point while tracing rays without regarding the density of rays inside the model

# Interpolation and extrapolation challenges

Wave solution





Ray solution

# Interpolation and extrapolation challenges

How to control the ray sampling of the model?

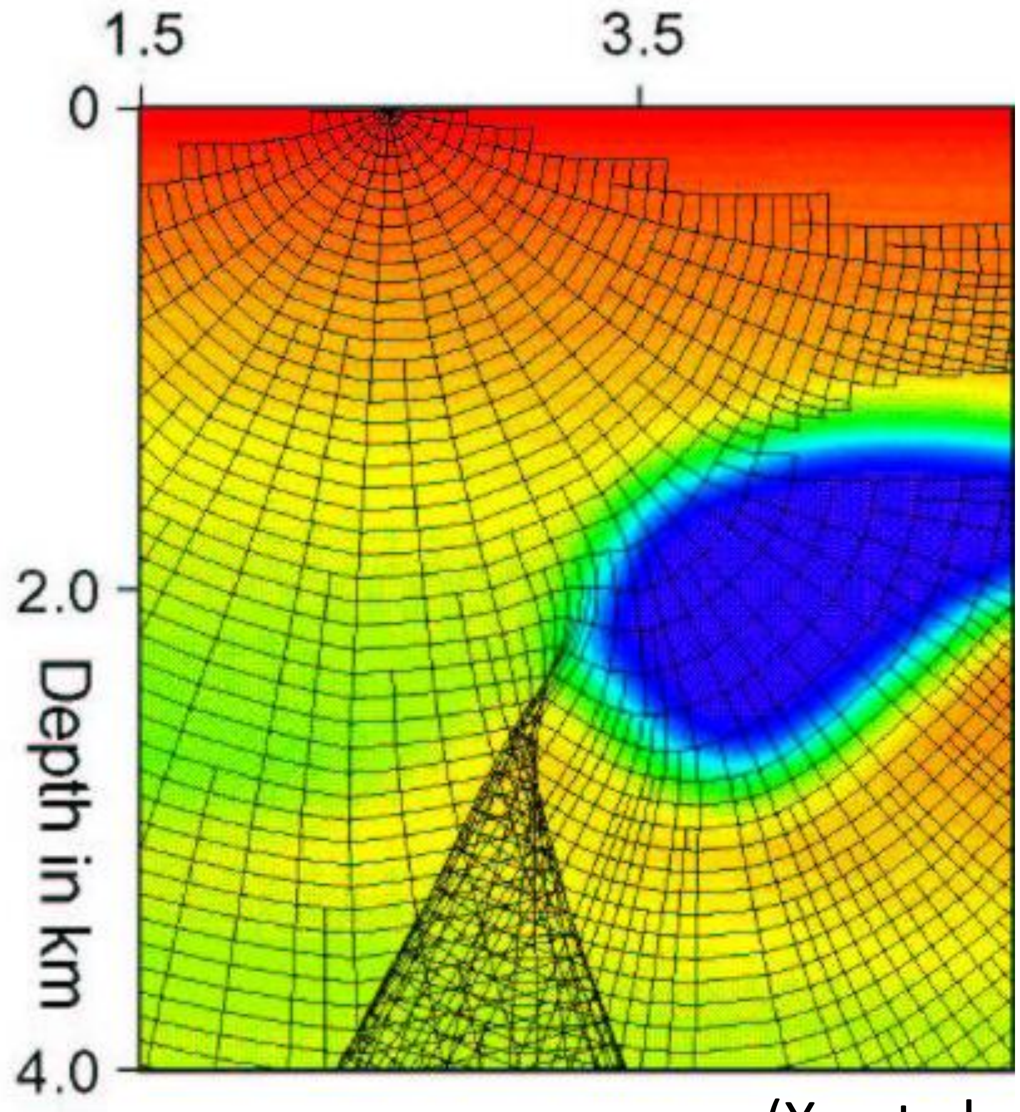
- ❑ Folding zones!

Available information

- ❑ Shadow zones!

Missing information

# Semi-lagrangian approach




(Xu et al., 1999)

Tracking the wavefront and its complexity with rays: still shadow zones … but allows the folding of rays.

Adding and suppressing rays for correct sampling of the wavefront!

HOW ?

Providing separated elemental contributions

(Jin et al, 1992; Lambaré et al, 1992)

(Vinje et al, 1993)

# ODE building

- Runge-Kutta of second order
- Write a computer program for an analytical law for the velocity: take a gradient with a component along x and a component along z

Home work : redo the same thing with a Runge-Kutta of fourth order (look after its definition) and predictor-corrector scheme.

Consider a gradient of the square of slowness and/or a vertical gradient of velocity

Consider a model defined by a grid with spline interpolation for computing spatial derivatives

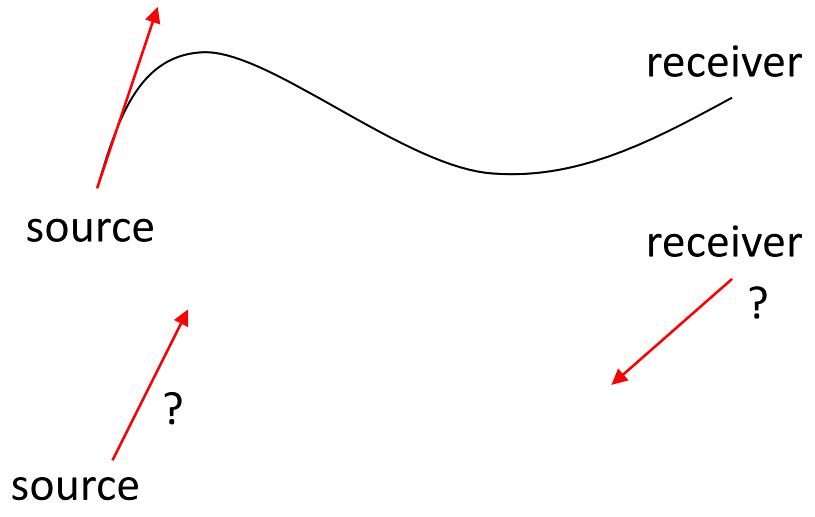
# Integration of ray equations

1D sampling of 2D/3D medium : FAST

Runge-Kutta second-order integration

Predictor-Corrector integration

A very good QC: the eikonal must be equal to zero !




Initial conditions EASY

Boundary conditions VERY DIFFICULT

Shooting $\delta p$ ?

Bending $\delta x$ ?

Continuing $\delta c$ ?

Save slowness conditions if possible !

AND FROM TIME TO TIME IT FAILS ! (inherent to geometrical optics)

But we need 2-points ray tracing because we have a source and a receiver to connect ! We even need more: branch identification (triplication for example)

# Non-linear ray tracing

## Runge-Kutta integration

$$\frac{df}{d\xi} = A(f)$$

$$f^{1/2} = f^0 + \frac{\Delta\xi}{2} A(f^0)$$

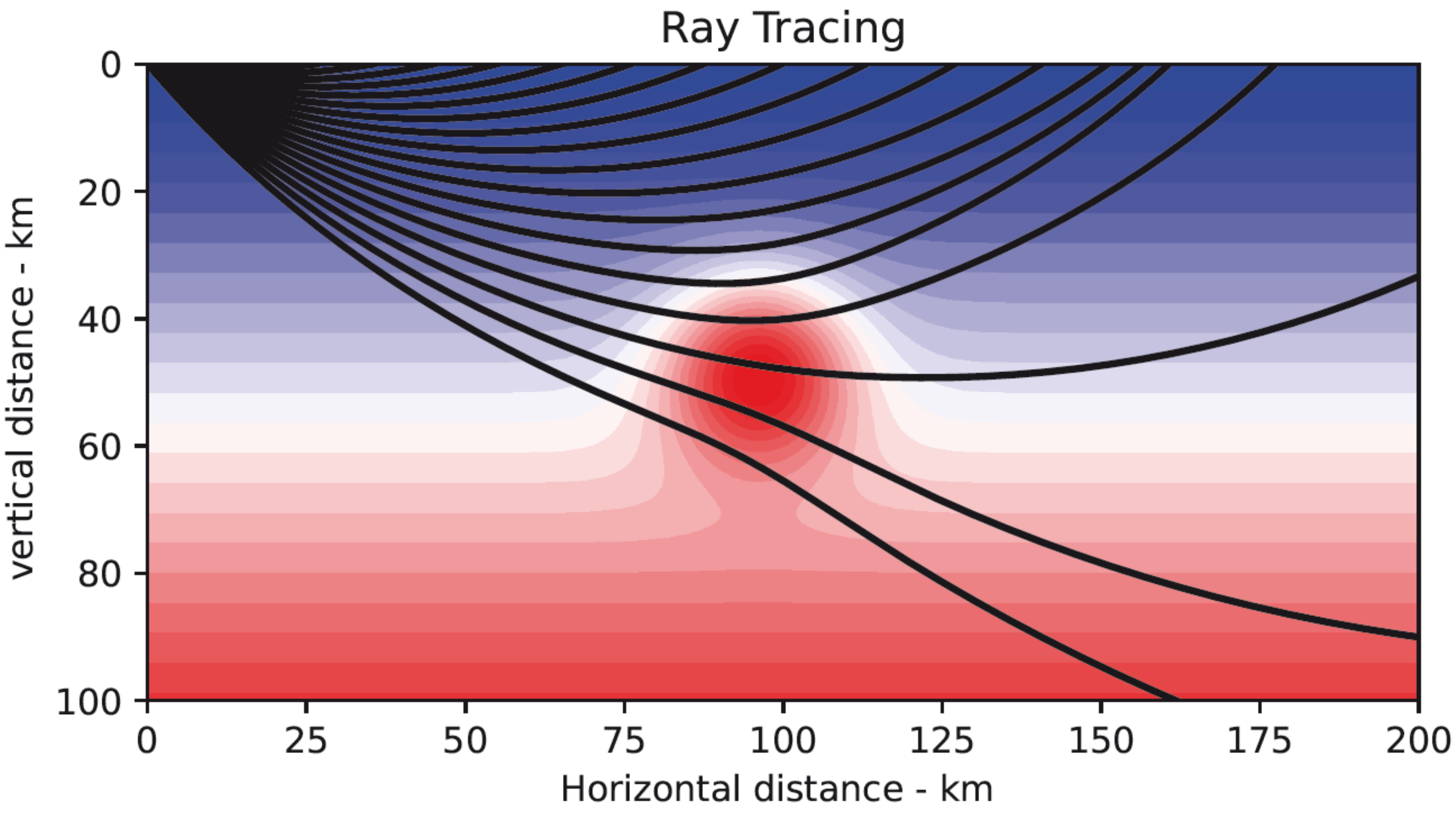
Second-order RK integration $$f^1 = f^0 + \Delta\xi \ A\left(f^{1/2}\right)$$

Fourth-order RK integration (home work!)

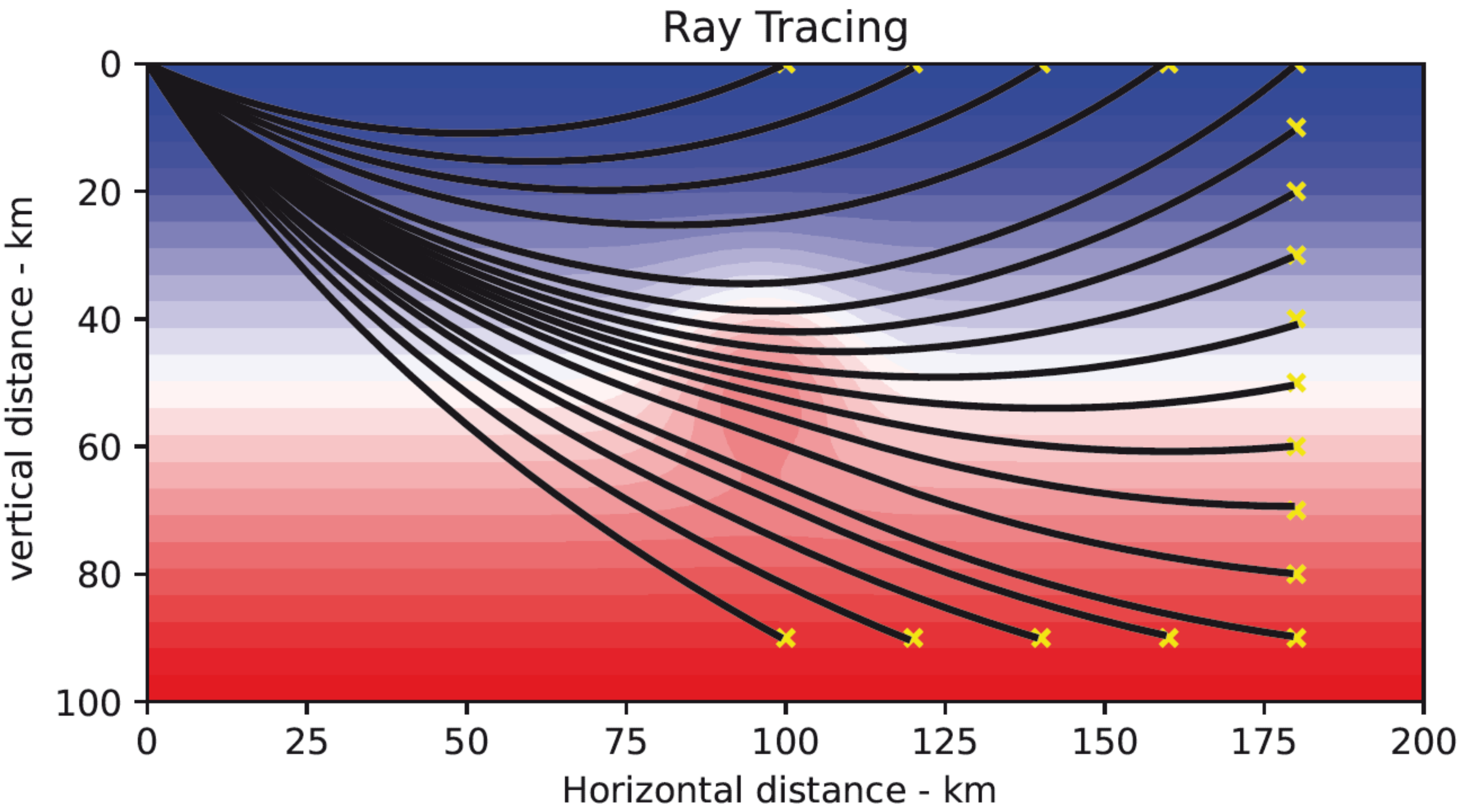
## Predictor-Corrector integration?

Show simple toy python codes for doing so.

# Ray tracing with initial conditions



# Two-points ray tracing



# Boundary conditions

How to sample the model around a given ray?

How to consider interfaces?

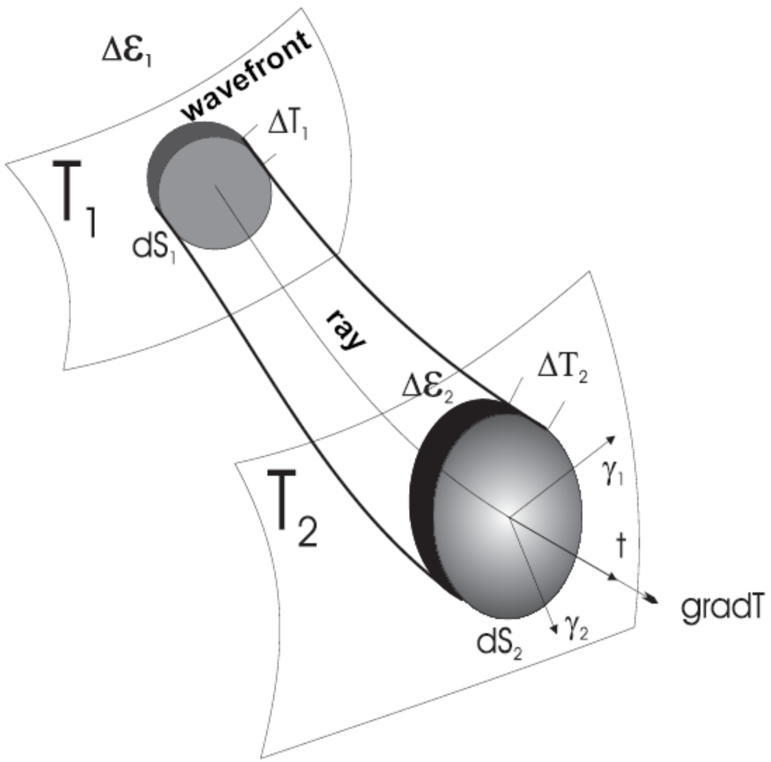
# Paraxial theory: what happens around one ray?

Paraxial ray theory

similar to

Gauss optics

# Paraxial equations: amplitude estimation




Ray tube allows amplitude estimation

No need to solve the transport equation!

One has to compute rays nearby the current ray for an estimation of the ray tube.

Keeping a ray in the neighbooring of the so-called central ray is quite difficult when using ray equations with Lagrangian formulation.

Perturbation theory is the way to go and tools from mechanics can be used …

and many other things…

# Paraxial equations: amplitude estimation

## Hamilton's ray equations

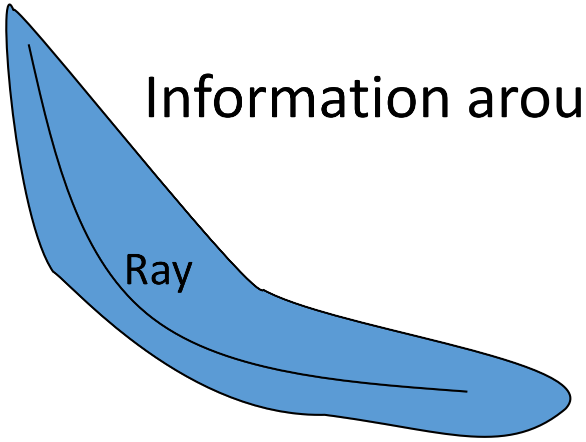
$$\frac{d\vec{q}(\xi)}{d\xi} = \vec{p}$$
$$\frac{d\vec{p}(\xi)}{d\xi} = \frac{1}{c(\vec{q})}\nabla_{\vec{q}}\frac{1}{c(\vec{q})}$$

$$\mathcal{H}(\vec{q},\vec{p}) = \frac{1}{2}\left(p^2 - \frac{1}{c^2(\vec{q})}\right)$$

$$\frac{d\vec{q}(\xi)}{d\xi} = \nabla_{\vec{p}}\mathcal{H}$$
$$\frac{d\vec{p}(\xi)}{d\xi} = -\nabla_{\vec{q}}\mathcal{H}$$
$$\frac{dT}{d\xi} = \vec{p}.\nabla_{\vec{p}}\mathcal{H}$$

$$y_0 = \begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix}$$ Central solution

Information around the ray $y_0$




$$\vec{q}_0 + \delta\vec{q}$$
$$\vec{p}_0 + \delta\vec{p}$$

$\delta\vec{q}$ *and* $\delta\vec{p}$ *"small"*

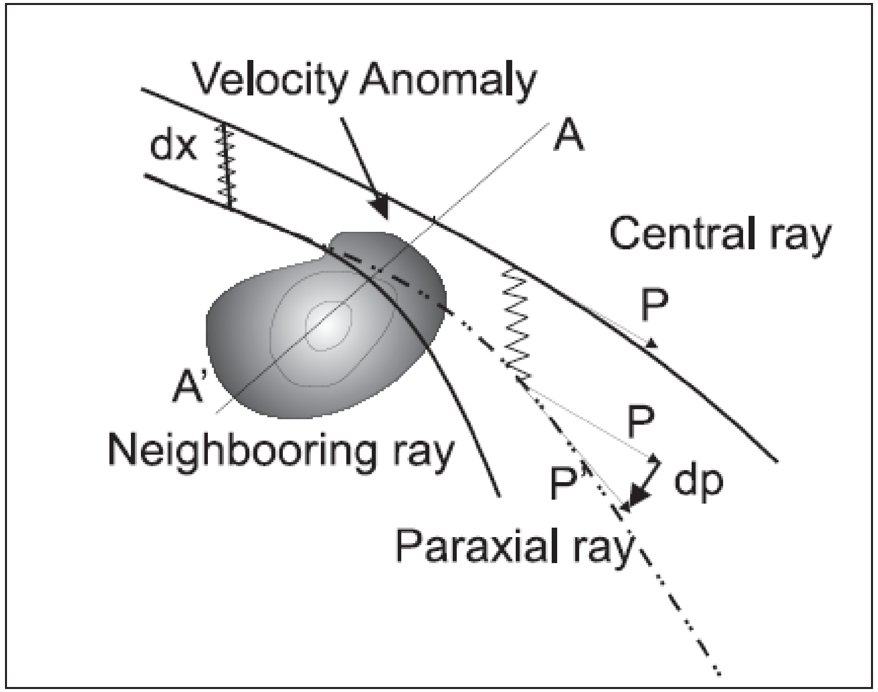
$$\delta y = \begin{pmatrix}\overrightarrow{\delta q}\\ \overrightarrow{\delta p}\end{pmatrix}$$ Perturbation solution

# Paraxial ray equations

The perturbation machinery

$$\frac{d(\overrightarrow{q_0}+\overrightarrow{\delta q})}{d\xi} = \nabla_{\overrightarrow{p_0}+\overrightarrow{\delta p}}\mathcal{H}(\overrightarrow{q_0}+\overrightarrow{\delta q},\overrightarrow{p_0}+\overrightarrow{\delta p})$$

$$\frac{d\overrightarrow{\delta q}}{d\xi} = \nabla_{\overrightarrow{p_0}}\nabla_{\overrightarrow{p_0}}\mathcal{H}(\overrightarrow{q_0},\overrightarrow{p_0})\overrightarrow{\delta p} + \nabla_{\overrightarrow{p_0}}\nabla_{\overrightarrow{q_0}}\mathcal{H}(\overrightarrow{q_0},\overrightarrow{p_0})\overrightarrow{\delta q}$$




$$\frac{d}{d\xi}\begin{bmatrix}\overrightarrow{\delta q}\\ \overrightarrow{\delta p}\end{bmatrix} = \begin{bmatrix}\nabla_{pq}\mathcal{H}^0 & \nabla_{pp}\mathcal{H}^0\\ -\nabla_{qq}\mathcal{H}^0 & -\nabla_{qp}\mathcal{H}^0\end{bmatrix}\begin{bmatrix}\overrightarrow{\delta q}\\ \overrightarrow{\delta p}\end{bmatrix}$$

$$\frac{d}{d\xi}(\overrightarrow{\delta y}) = A(\vec{y}_0)\overrightarrow{\delta y}$$

The matrix $A$ does not depend on unknown quantities $\delta y$ but only on quantities $y_0$: LINEAR PROBLEM (SIMPLE) !

Paraxial solution can be used for different purposes: *amplitude estimation, ray tube crossing (KMAH index), two-points ray tracing problem* and so on

# Paraxial ray theory

SEISCOPE

- Solutions are coordinate dependent (differential computation)
- Not restricted to the so-called ray-centered coordinate system (Cerveny, 2001)
- Cartesian formulation is much simpler to handle (Virieux & Farra, 1991)

One can think that paraxial ray is somehow a ray derivative of the central ray. It is arbitrarily near the central ray.

# 2D simple linear system

$$\frac{d}{d\xi}\begin{bmatrix}\delta q_x\\ \delta q_z\\ \delta p_x\\ \delta p_z\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0.5\,\frac{\partial^2\, 1/c^2(\vec{q}_0)}{\partial x^2} & 0.5\,\frac{\partial^2\, 1/c^2(\vec{q}_0)}{\partial x \partial z} & 0 & 0\\ 0.5\,\frac{\partial^2\, 1/c^2(\vec{q}_0)}{\partial z \partial x} & 0.5\,\frac{\partial^2\, 1/c^2(\vec{q}_0)}{\partial z^2} & 0 & 0 \end{bmatrix}\begin{bmatrix}\delta q_x\\ \delta q_z\\ \delta p_x\\ \delta p_z\end{bmatrix}$$

Linear system

More complex for anisotropic structure but still workable

Four elementary paraxial trajectories

NOT A paraxial RAY !

$\delta y1^t(0)=(1,0,0,0)$

$\delta y2^t(0)=(0,1,0,0)$

$\delta y3^t(0)=(0,0,1,0)$

$\delta y4^t(0)=(0,0,0,1)$

$$\delta y^t = (\delta q_x, \delta q_z\, , \delta p_x\, , \delta p_z\, )$$

Any paraxial ray is a linear combination of these four elementary trajectories

# Numerical integration

Second-order RK integration

$$\frac{df}{d\xi} = A(f)$$

Non-linear ray tracing

$$f^{1/2} = f^0 + \frac{\Delta\xi}{2} A(f^0)$$

$$f^1 = f^0 + \Delta\xi \; A\left(f^{1/2}\right)$$

Second-order euler integration for paraxial ray tracing is enough!

Linear paraxial ray tracing

Propagator technique

Optical Lens technique

$$\frac{d\delta f}{d\xi} = A(f)\delta f$$

$$\delta f^1 = \delta f^0 + \Delta\xi \; A(f^0)\delta f^0$$

# 2D paraxial conditions

Paraxial rays require other conservative quantities : the perturbation of the Hamiltonian should be zero (or, in other words, the eikonal perturbation is zero)

If working with the reduced hamiltonian, this is implicitly set!

$$\delta\mathcal{H}(\xi) = \delta\mathcal{H}(0) = 0$$

$$\frac{\partial\mathcal{H}}{\partial p_x}\delta p_x + \frac{\partial\mathcal{H}}{\partial p_z}\delta p_z + \frac{\partial\mathcal{H}}{\partial q_x}\delta q_x + \frac{\partial\mathcal{H}}{\partial q_z}\delta q_z = 0$$

Or in the isotropic case

$$p_x\delta p_x + p_z\delta p_z - \frac{1}{2}\frac{\partial 1/c^2(x,z)}{\partial x}\delta q_x - \frac{1}{2}\frac{\partial 1/c^2(x,z)}{\partial z}\delta q_z = 0$$

Two independent solutions

similar conditions in 3D
readily deduced for anisotropy

# Point source condition

Point source: no shift in the position when doing perturbation:

$$\delta q_x(O) = \delta q_z(O) = 0 \;\Rightarrow\; p_x(0)\delta p_x(0) + p_z(0)\delta p_z(0)=0$$

$$\delta p_x(0) = \alpha p_z(0)$$

$$\delta p_z(0) = -\alpha p_x(0)$$

This is enough to verify this condition initially

$\alpha$ *arbitrary constant (linear system)*

Point source paraxial solution $\delta y^a(\xi) = \alpha p_z(0)\, \delta y3(\xi) - \alpha p_x(0)\, \delta y4(\xi)$ elementary trajectories

From paraxial trajectories, one can combine them for paraxial rays as long as the perturbation of the Hamiltonian is zero.

For a point source, the parameter $\alpha$ could be set to an arbitrary small value: this is a derivative or plan tangent computation (Gauss optics)

# Plane source condition

Plane source: shift in the shooting direction when doing perturbation:

$$\delta p_x(O) = \delta p_z(O) = 0 \;\Rightarrow\; \frac{1}{2}\frac{\partial\, 1/c^2(x,z)}{\partial x}(0)\,\delta q_x(0) + \frac{1}{2}\frac{\partial\, 1/c^2(x,z)}{\partial z}(0)\,\delta q_z(0) = 0$$

$$\delta q_x(0) = \alpha\,\frac{1}{2}\frac{\partial\, 1/c^2(x,z)}{\partial z}(0)$$

$$\delta q_z(0) = -\alpha\,\frac{1}{2}\frac{\partial\, 1/c^2(x,z)}{\partial x}(0)$$

$\alpha$ *arbitrary constant (linear system)*

This is enough to verify this condition initially but gradient of velocity at the source could be quite arbitrariry

Cerveny's condition (both $x$ and $z$ variation)

Paraxial solution $\delta y^b(\xi) = \alpha\,\frac{1}{2}\frac{\partial 1/c^2(x,z)}{\partial z}(0)\,\delta y1(\xi) - \alpha\,\frac{1}{2}\frac{\partial 1/c^2(x,z)}{\partial x}(0)\,\delta y2(\xi)$

We combine the first two paraxial ray trajectories.

elementary trajectories

Two independent paraxial rays in 2D ($\delta y^a$ *and* $\delta y^b$): point (seismograms) and plane (beams) paraxial rays

# Chapman source condition (free surface)

Chapman condition: keep the shift positioshift in the shooting direction when doing perturbation:

$$\delta q_z(O) = 0;\ \delta p_z(O) = 0 \ \Rightarrow \frac{1}{2}\frac{\partial\, 1/c^2(x,z)}{\partial x}(0)\,\delta q_x(0) + p_x(0)\,\delta p_x(0) = 0$$

$$\delta q_x(0) = \alpha p_x(0)$$

$$\delta p_x(0) = -\alpha\frac{1}{2}\frac{\partial\, 1/c^2(x,z)}{\partial x}(0)$$

$\alpha$ *arbitrary constant (linear system)*

This is enough to verify this condition initially but gradient of velocity at the source could be quite arbitrary

Chapman's condition (only $z$ variation)

Paraxial solution $\delta y'(\xi) = \alpha p_x(0)\,\delta y1(\xi) - \alpha\frac{1}{2}\frac{\partial 1/c^2(x,z)}{\partial x}(0)\delta y3(\xi)$

We combine the first two paraxial ray trajectories.

elementary trajectories

Only perturbation along x which is important when considering source at the free surface

# Paraxial source conditions

Two independent paraxial rays in 2D ($\delta y^a$ $and$ $\delta y^b$):
point (seismograms) and plane (beams) paraxial rays

Four independent paraxial rays in 3D ($\delta y^a, \delta y^b, \delta y^c, and\ \delta y^d$):
2 point (seismograms) and 2 plane (beams) paraxial rays

*Remark: working with trajectories implies that paraxial conditions could be defined on the fly for having local conditions at different points of the model*

Remark: 3 point and 3 plane paraxial trajectories in 3D!

# KMAH index tracking through paraxial

KMAH index key element for seismograms

$$u(\vec{q},t) = A(\vec{q})e^{i\omega T(x)}e^{-i\frac{\pi}{2}sgn(\omega)KMAH}$$

# KMAH index tracking through paraxial values

❖ In 2D, the determinant $\begin{vmatrix} p_x(\xi) & \delta q_x{}^3 & \delta q_z{}^3 \\ p_z(\xi) & \delta q_x{}^4 & \delta q_z{}^4 \\ 0 & p_x(0) & p_z(0) \end{vmatrix}$ may change sign.

Increment by one the KMAH index when crossing a caustic

Point source conditions

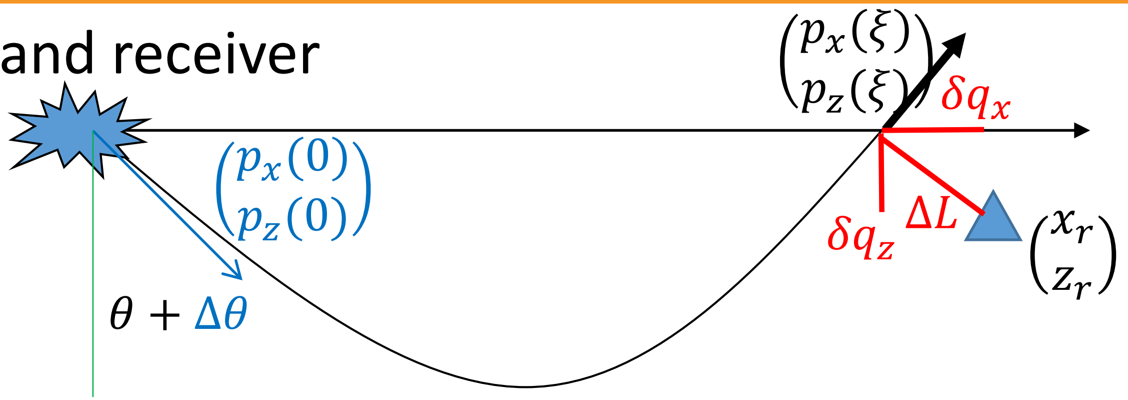
❖ In 3D, the determinant $\begin{vmatrix} p_x(\xi) & \delta q_x{}^4 & \delta q_y{}^4 & \delta q_z{}^4 \\ p_y(\xi) & \delta q_x{}^5 & \delta q_y{}^5 & \delta q_z{}^5 \\ p_z(\xi) & \delta q_x{}^6 & \delta q_y{}^6 & \delta q_z{}^6 \\ 0 & p_x(0) & p_y(0) & p_z(0) \end{vmatrix}$ may change sign.

If minor determinants do not change sign, this is a plane caustic (add 1 to KMAH). If they change sign as well, this is a point caustic (add 2 to KMAH).

# Two-points ray tracing with paraxial values

Consider the orthogonal distance between ray and receiver

$$\Delta L = (x_r - \delta q_x)p_z(\xi) - (z_r - \delta q_z)p_x(\xi)$$




Point paraxial condition

Solve iteratively $\Delta L = \frac{dq_L}{d\theta} \Delta\theta$

or $\Delta L = \left( \frac{dq_x}{d\theta} p_z(\xi) - \frac{dq_z}{d\theta} p_x(\xi) \right) \Delta\theta$

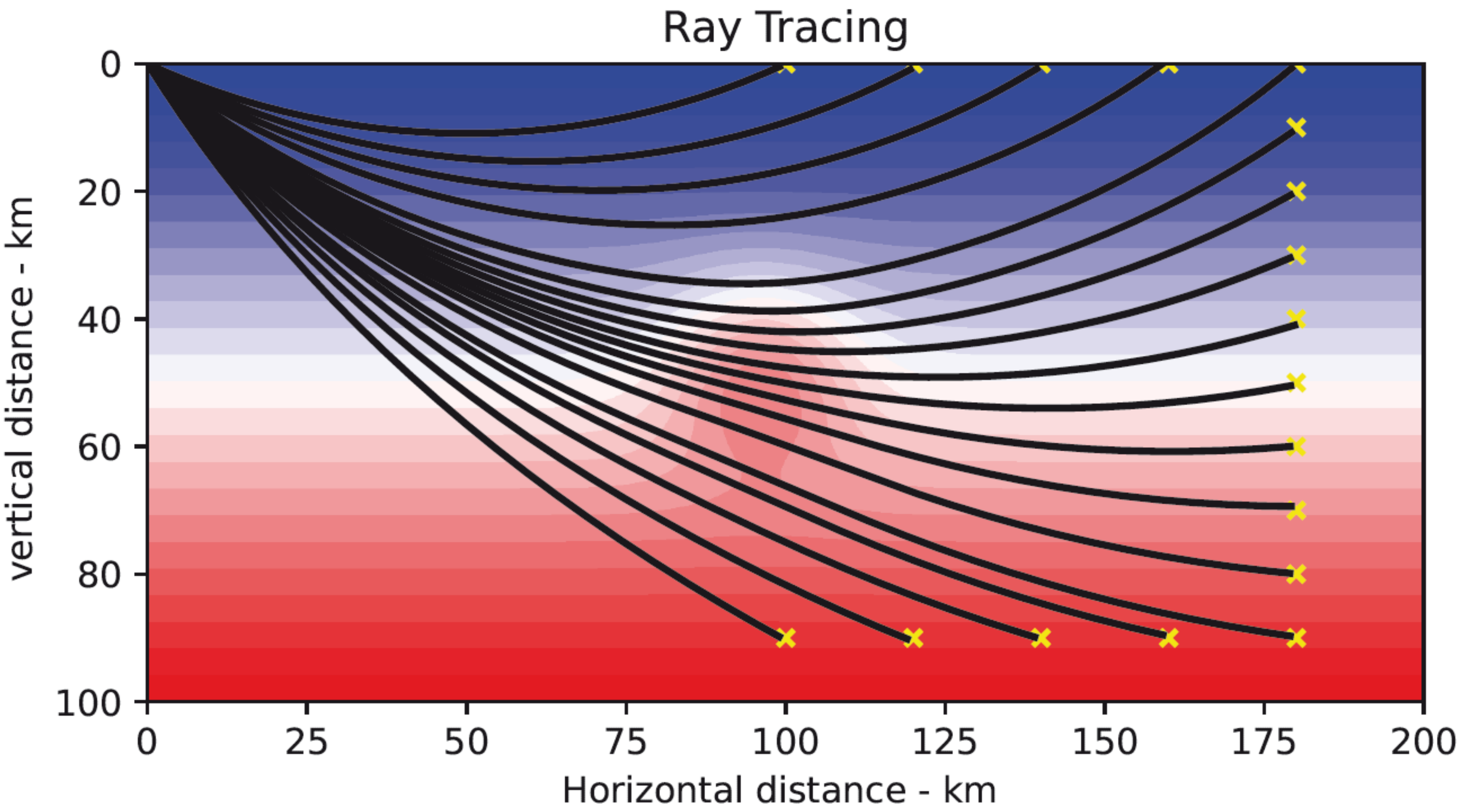
Derivative wrt shooting angle $\frac{dq_x}{d\theta} = \frac{\partial q_x}{\partial p_x} \frac{dp_x}{d\theta}(0) + \frac{\partial q_x}{\partial p_z} \frac{dp_z}{d\theta}(0)$ or $\frac{dq_x}{d\theta} = \frac{\partial q_x}{\partial p_x} p_z(0) - \frac{\partial q_x}{\partial p_z} p_x(0)$

$$\frac{dq_x}{d\theta} = \delta q_x 3 \;\; p_z(0) - \;\delta q_x 4 \;\; p_x(0) = \delta q_x$$

$$\frac{dq_z}{d\theta} = \delta q_z 3 \;\; p_z(0) - \;\delta q_z 4 \;\; p_x(0) = \delta q_z$$

$$\Delta\theta = \frac{(x_r - \delta q_x)p_z(\xi) - (z_r - \delta q_z)p_x(\xi)}{\delta q_x p_z(\xi) - \delta q_z p_x(\xi)}$$

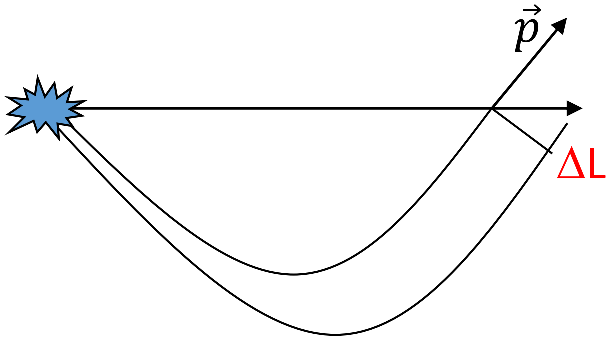
## Two-points ray tracing



# Amplitude estimation

Consider ΔL the distance between an exit point of a ray at the particule time $\xi$ and the related paraxial ray.

From point paraxial ray $\delta y^a$




$$\Delta L = \frac{\delta q_x{}^a(\xi)p_z(\xi) - \delta q_z{}^a\,(\xi)p_x(\xi)}{\sqrt{p_x(\xi)^2 + p_z(\xi)^2}}$$

From point paraxial trajectories $\delta y^a 3$ and $\delta y^a 4$

$$\frac{\Delta L}{\Delta\theta} = \frac{[\delta q_x 3(\xi)p_z(0) - \delta q_x 4(\xi)p_x(0)]p_z(\xi) - [\delta q_z 3(\xi)p_z(0) - \delta q_z 4(\xi)p_x(0)]p_x(\xi)}{\sqrt{p_x(\xi)^2 + p_z(\xi)^2}}$$

$$\frac{\Delta L}{\Delta\theta} = \frac{\begin{vmatrix} p_x(\xi) & \delta q_x 3(\xi) & \delta q_z 3(\xi) \\ p_z(\xi) & \delta q_x 4(\xi) & \delta q_z 4(\xi) \\ 0 & p_x(0) & p_z(0) \end{vmatrix}}{\sqrt{p_x(\xi)^2 + p_z(\xi)^2}}$$

Thanks to the point paraxial solutions , geometrical spreading $\Delta L/\Delta\theta$ can be computed, and the ray amplitude $A(\xi) \propto \frac{\Delta L}{\Delta\theta}$, and the KMAH index as well.

. *No need of solving the transport equation for getting the amplitude evolution*

# Beam summation

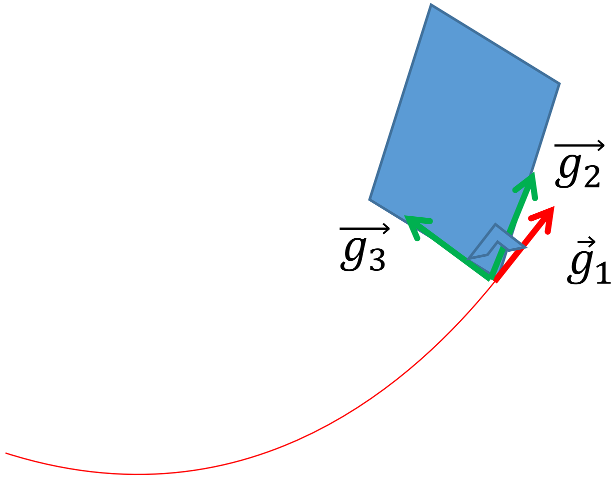
Using plane paraxial conditions with elementary solutions $\delta y1$ and $\delta y2$, we can construct any beam (as the Gaussian beam) in an ad-hoc summation …

In fact, one can combine any combination of elementary solutions $\delta y1, \delta y2$,$\delta y3, and\ \delta y4$ for focused beam summation (mixture of point and plane solutions)…

*Remark: not too much used in seismology (except GBS), while extensive use in optics, such as the non-diffracting Airy solution (Lin et al, 2015)*

# Vibration: polarization estimation

Isotropic case: shear vibrations are orthogonal to compression vibrations




It is enough to follow the evolution of the projection of elastic unitary vectors on one Cartesian coordinate: $\vec{e}_z.\vec{g}_2$ and $\vec{e}_z.\vec{g}_3$(Psencik, perso. Comm.)

One additional equation for polarization

Acoustic case: the unitary vector $\vec{g}_1 = c(\vec{q})\vec{p}$ supports the P wave vibration

Elastic case: the independent shear vibration will be along two unitary vectors $\vec{g}_2$ and $\vec{g}_3$ such that

Time stepping

$$\frac{d\vec{g}_2}{dt} = \vec{g}_2.\nabla_{\vec{q}}c \ \ \vec{g}_1$$
$$\frac{d\vec{g}_3}{dt} = \vec{g}_3.\nabla_{\vec{q}}c \ \ \vec{g}_1$$

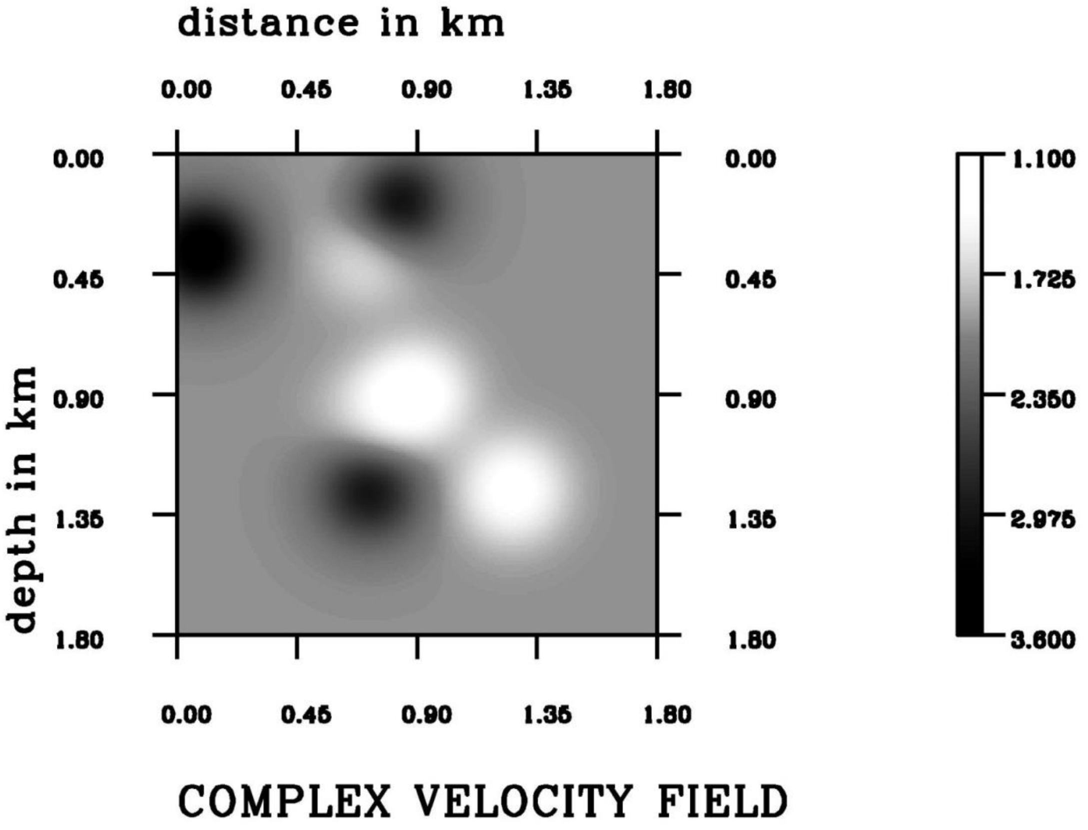
Particule stepping

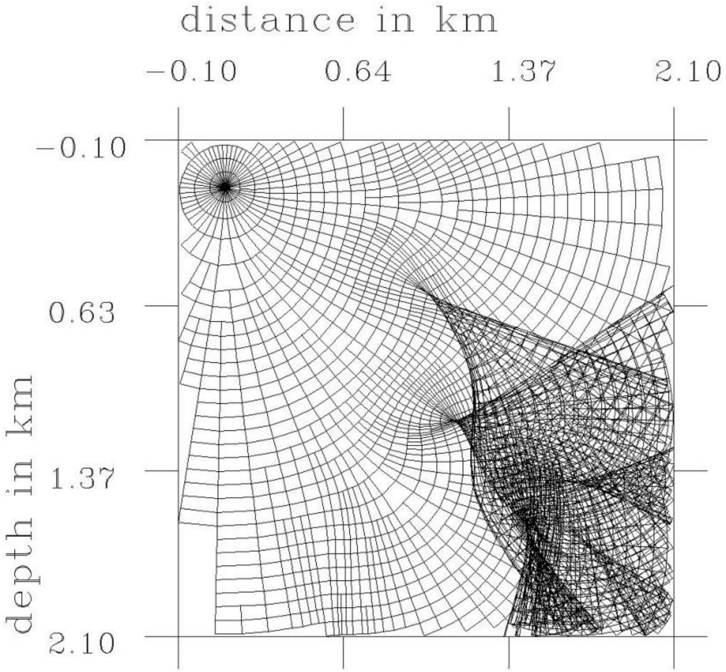
$$\frac{d\vec{g}_2}{d\xi} = \frac{\vec{g}_2.\nabla_{\vec{q}}c}{p^2}\vec{g}_1$$
$$\frac{d\vec{g}_3}{d\xi} = \frac{\vec{g}_3.\nabla_{\vec{q}}c}{p^2}\vec{g}_1$$

Chapman, 2004, p180

Example of a lagrangian-euler ray tracing

Lagrangian part: ray tracing
Eulerian part: wavefront sampling

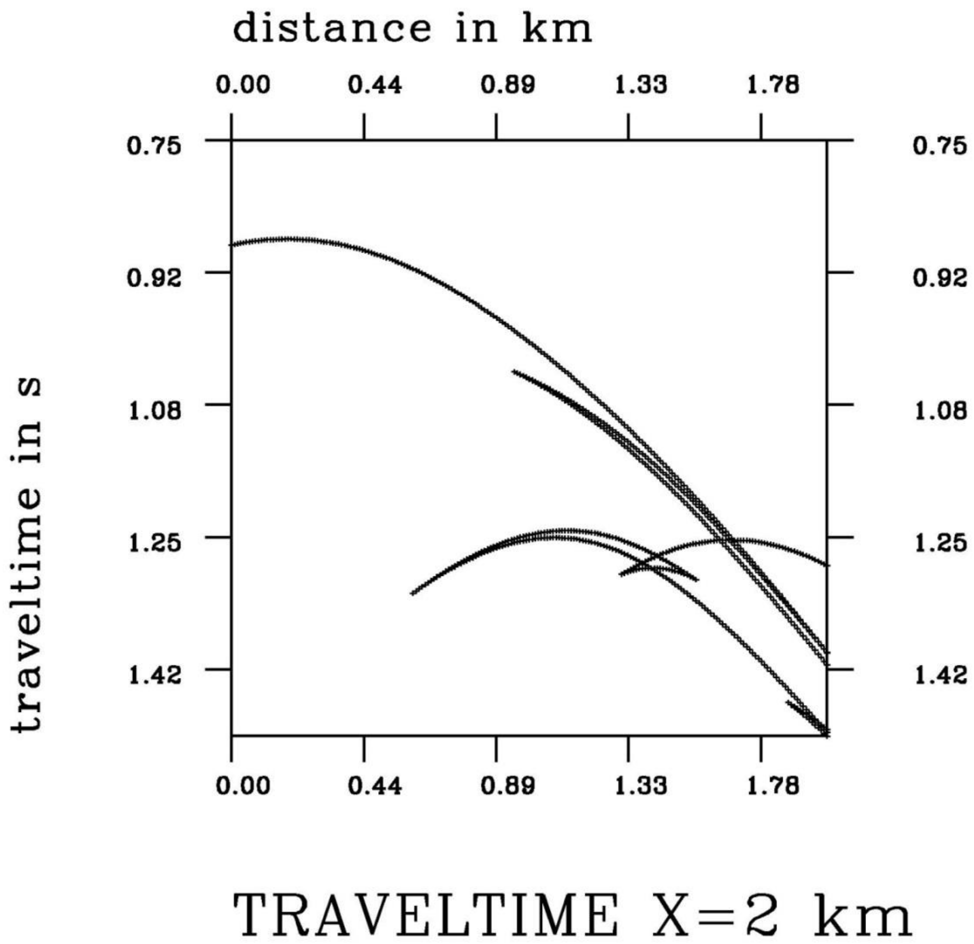


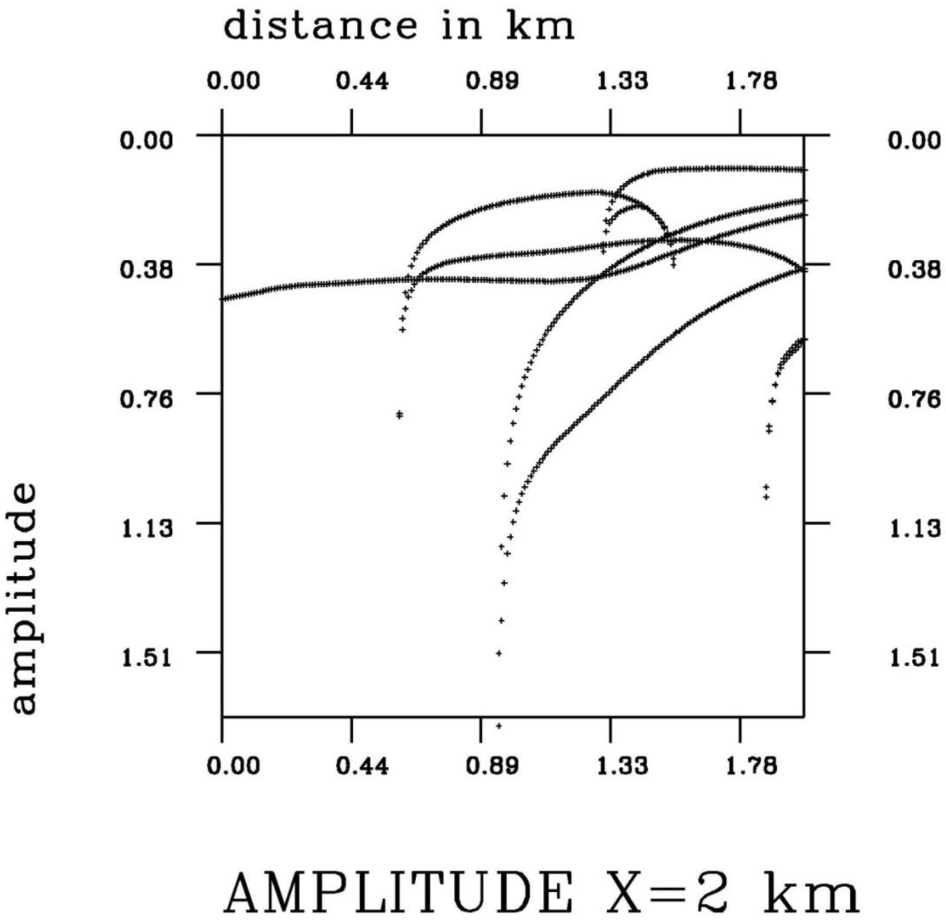




(Lambaré et al., 1996)

# Step two: seismograms (paraxial)

(Lambaré et al., 1996)



# Step three: polarization estimation (S waves)

Solving either $\vec{g}_2$ evolution or $\vec{g}_3$evolution, knowing that $\vec{g}_1$ is the direction of propagation.

The other polarization vector could be deduced from the orthonormal system $(\vec{g}_1, \vec{g}_2, \vec{g}_3)$.

Time stepping

$$\frac{d\vec{g}_2}{dt} = \vec{g}_2 . \nabla_{\vec{q}} c \;\; \vec{g}_1$$

$$\frac{d\vec{g}_3}{dt} = \vec{g}_3 . \nabla_{\vec{q}} c \;\; \vec{g}_1$$

Evolution of the polarization depends on the gradient of velocity $\nabla_{\vec{q}} c$

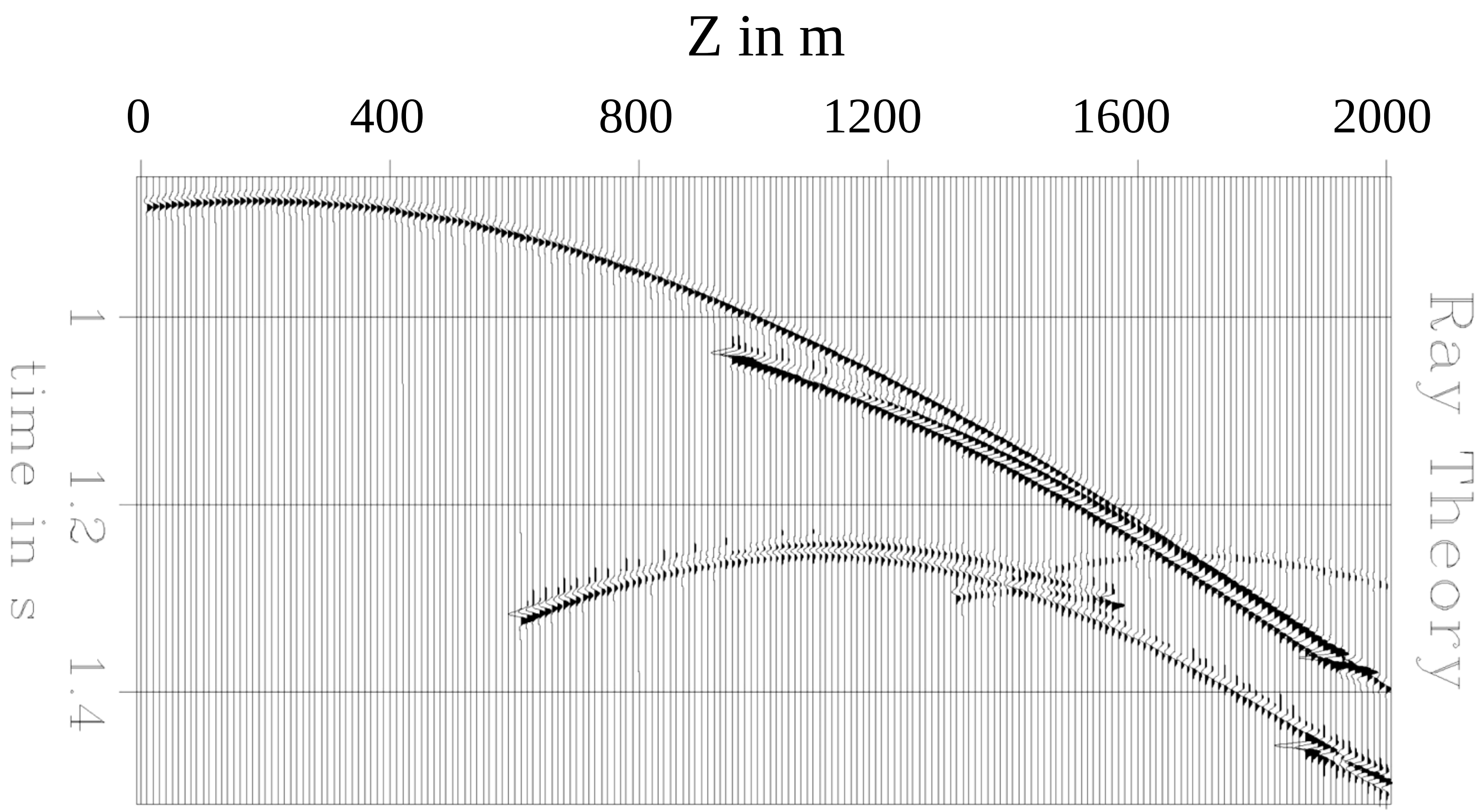
Particule stepping

$$\frac{d\vec{g}_2}{d\xi} = \frac{\vec{g}_2 . \nabla_{\vec{q}} c}{p^2} \vec{g}_1$$

$$\frac{d\vec{g}_3}{d\xi} = \frac{\vec{g}_3 . \nabla_{\vec{q}} c}{p^2} \vec{g}_1$$

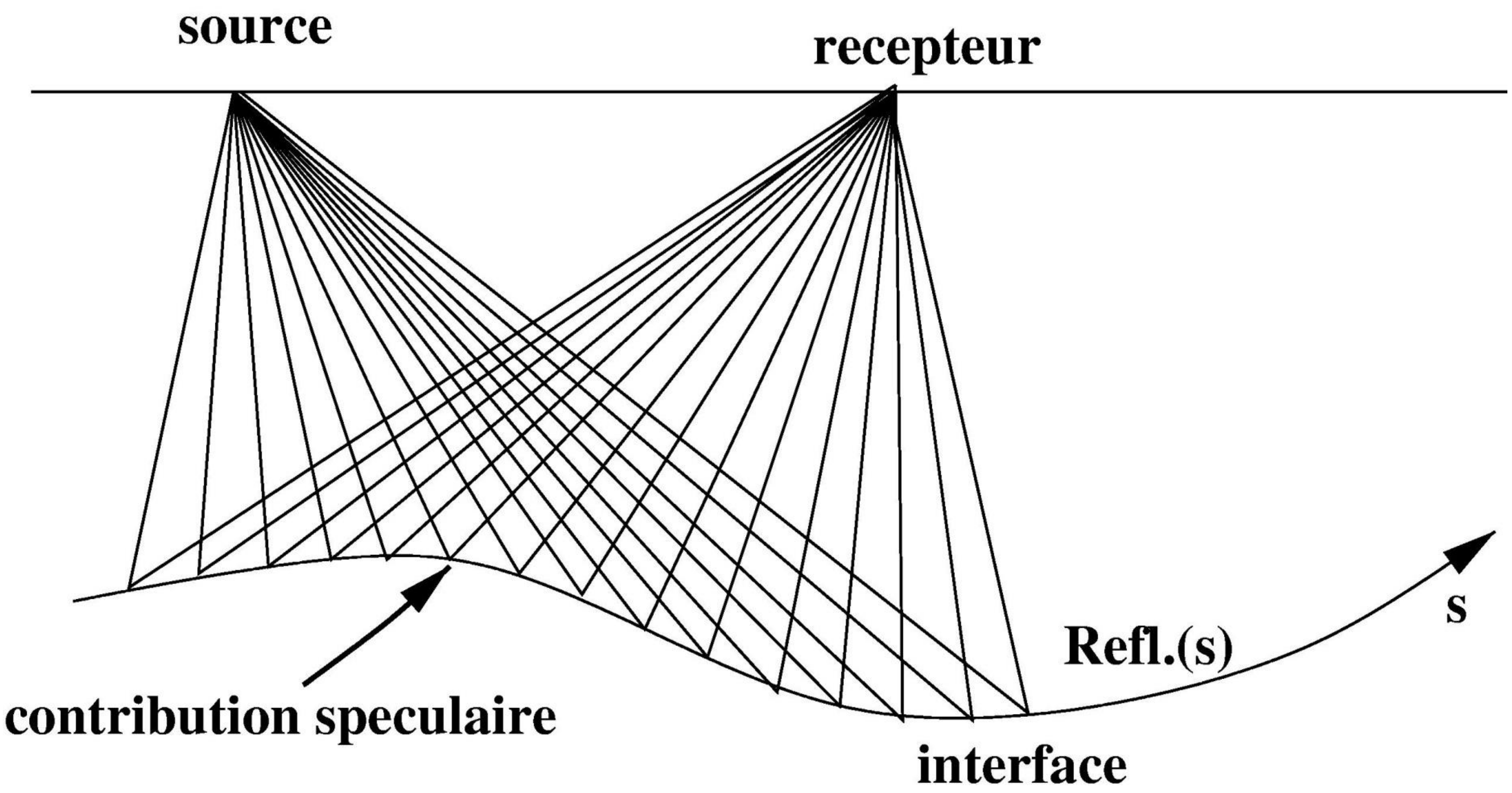
Useful for anisotropic investigation by
S receiver function theory
(Vinnik & Farra, 1992; Farra & Vinnik, 2000)

# Step four: ray seismograms




(Lambaré et al., 1996)

# Two-points ray tracing with boundary conditions



# Interfaces in short!

***Discontinuous model properties incompatible with ray ansatz:***
***two independent ray tracing on both sides of the interface to be connected***

- ODE: resume solutions at the interface and restart the integration (Snell-Descartes law and paraxial Snell Descartes law)

(Cerveny et al, 1974; Farra et al, 1989)

- PDE: semi-lagrangian procedure back to ODE

(Rawlison & Sambdrige, 2003)

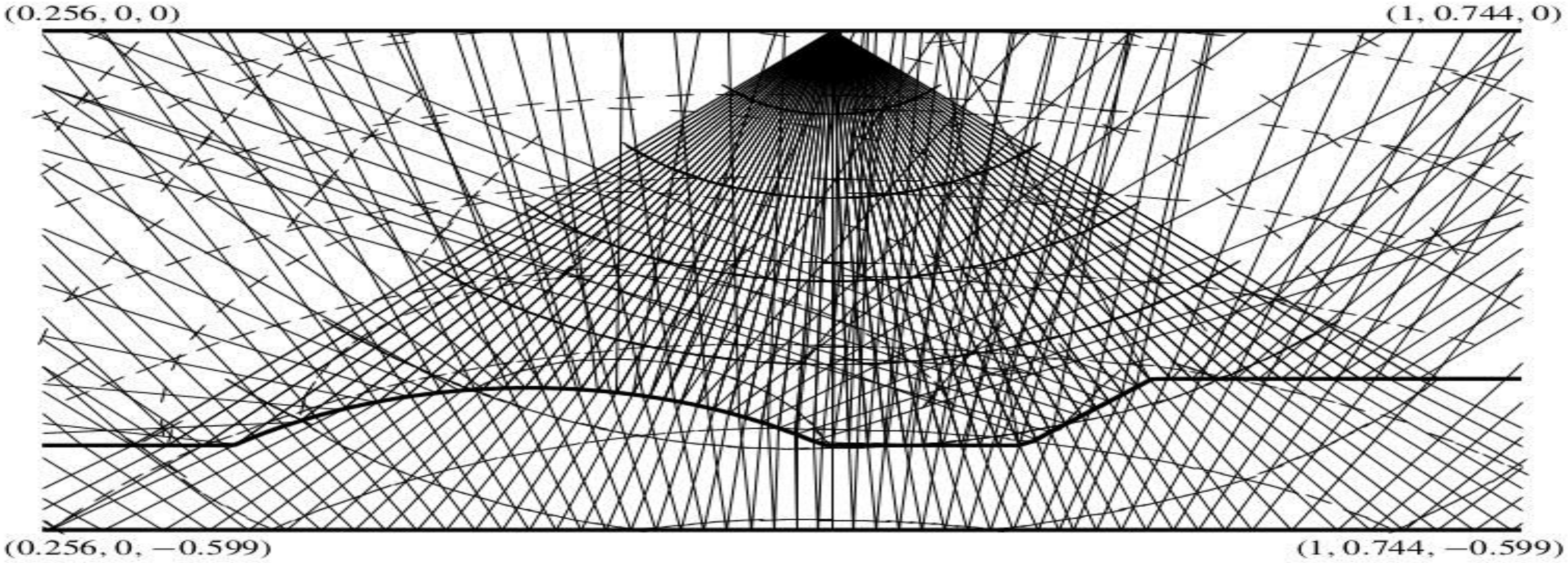
- PDE: rely on discontinuous methods

(Cheng & Shu, 2007)

# Keeping complexity low?

More and more demanding on computer resources

3D French's model




Solutions: moving from ODEs to PDEs

(Osher et al, 2002) for adequate spatial sampling of the wavefront. Grids control the complexity!

# Ray tracing: partial lessons to take away

***Rays: a quite useful tool for interpretation and understanding***

- Geometrical optics: ODE versus PDE
  - Choose PDE when possible !

- ODE: tracing one (paraxial) ray is fast
  - Please always trace paraxial rays as incremental cost

- Keep complexity low (seismic waves are finite frequency waves)
  - Do not drown yourself into the no-scale « optical » infinite-frequency singularities

- Rays help the identification of phases: key interpretation
  - PDE does not allow easy interpretation! (maybe work in progress …)

# Toy computer codes for ray tracing

In the directory « practices »,

Sub-directory: ray_tracing_analytic

comparison between analytic solutions and runge-kutta solutions

Sub-directory: ray-tracing_grid

ray tracing over a velocity grid using bspline interpolation

Sub-directory: two_points_ray_tracing

ray tracing including paraxial ray tracing with an illustration when hitting receiver

Simple codes based on python3 for practical understanding of the different equations of this presentation

# Ray ODE vs Eikonal PDE

- ❑ Scalar wave equation PDE

  - ❖ Linear partial differential equation
  - ❖ Eulerian formulation

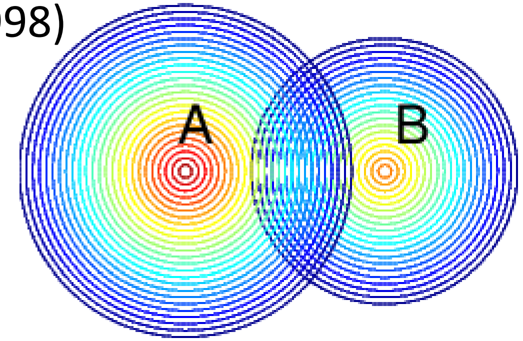
- ❑ Ray ODE: Methods of characteristics (Courant & Hilbert, 1966)

  - ❖Non-linear ordinary differential equations
  - ❖Lagrangian formulation as we integrate along rays

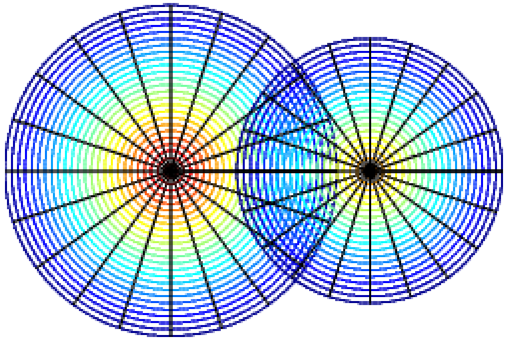
- ❑ Eikonal PDE: challenging equation (complete solution – the Graal)!

  - ❖ Non-linear partial differential equations
  - ❖ Eulerian formulation as we compute quantities at fixed positions
  - ❖ Fastest solution through fast marching or fast sweeping methods

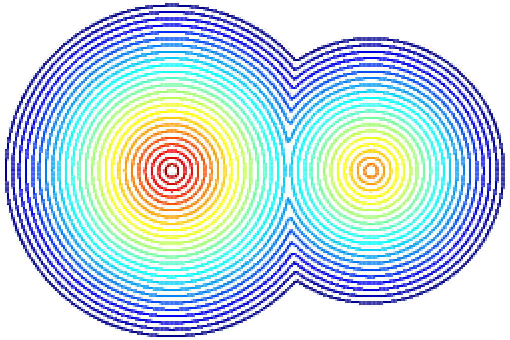
(after Runborg, 1998)




Wave solution



Ray solution



Eikonal solution

# Asymptotic solution: beyond ray solution!

***Geometrical theory of diffraction (Keller, 1962)***

- Ray solution is one asymptotic solution among many other expansions.

  Ray Ansatz is limited to integer power of frequency

  Zero-out any diffraction effect (fractional frequency power)

- Airy, Bessel, Mathieu expansions (complicated formulations)!

  alternative expansions

  with or without diffraction

- <u>Fastest solution known as viscous solution (Crandall & Lions, 1983)</u>

# Asymptotic solution: beyond ray solution!

Viscous solution: efficient tools exist for computing it!

- Fast sweeping method O(N) for travel-times and for amplitudes
  - Amplitude equations have to be designed

- Finite element methods put into the scene
  - Stencils are moving to higher orders and h-adaptivity

- Discontinuous Galerkin methods
  - This is the road to take for interface investigation in the frame of PDE.

# Ray solution versus first-arrival solution

### 3.8.1 Ray Theory Travel Times and First-Arrival Travel Times

(Cerveny, 2001)

In this section, we shall define the ray-theory travel times and first-arrival travel times and explain the main differences between them.

One extracted property of first-arrival solution mentioned by Cerveny (2001)

**c.** The first-arrival travel time is a *unique* function of position. It is defined at any point of the model. There are no shadow zones. Moreover, the first-arrival travel time is a continuous function of coordinates. The first spatial derivatives of the first-arrival travel time, however, may be discontinuous. They may be discontinuous even at points where the velocities are continuous. (Example of such discontinuity include the intersection of the wavefronts of direct and head waves.)

# Viscous solution!

Understanding this asymptotic solution!

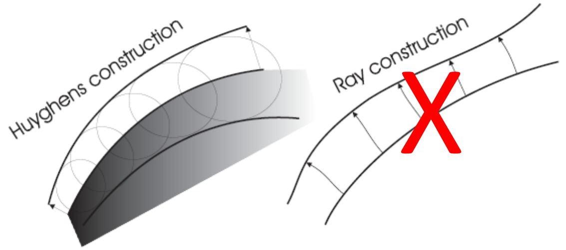
Defining its validity domain!

Crandall & Lions (1983)

# Fermat/Huygens principles to Eikonal equation

First-arrival traveltimes follow Fermat principle of minimum time along any trajectory connecting the starting point and the end point. This principle is highly connected to the Huygens principle related to the wavefront construction

The related variational problem can be written




$$\delta \int u(s)ds = 0$$

$$slowness\ u = {}^{1}/_{c}$$

where $u(s)$ is the slowness at a given point and the curvilinear coordinate at this point is given by the quantity $s$.

The Eikonal equation is thought of the related PDE of this variational problem, leading to the Hamilton-Jacobi(-Bellman) equation.

(Kalaba, 1961 (isotropic case); Brandstatter, 1974 (anisotropic case)).

# Fermat principle to Eikonal equation

In a 2D medium defined by coordinates $(x, z)$, the path of the ray (perpendicular to the wavefront) is such that

$$\mathrm{T}(x,z) = \min_{l} \int u\big(x(l), z(l)\big) dl \quad (1)$$

We consider an infinitesimal path $\Delta l$ from $(x - sin\theta\ \Delta l, \mathrm{z} - cos\theta\ \Delta l)$ where the angle $\theta$ is the tangent angle to the current trajectory.

We get

$$\mathrm{T}(x,z) = \min_{\theta}[T(x - sin\theta\ \Delta l\ , \mathrm{z} - cos\theta\ \Delta l) + u(x - sin\theta\ \Delta l\ , \mathrm{z} - cos\theta\ \Delta l)\Delta l + \sigma(\Delta l)]$$

Expanding in Taylor series, we get

$$T(x,z) = \min_{\theta}\left[T(x, \mathrm{z}) - sin\theta\Delta l \frac{\partial T}{\partial x} - cos\theta \frac{\partial T}{\partial z} + u(x, z)\Delta l + \sigma(\Delta l)\right]$$

Or

$$\min_{\theta}[-sin\theta\ \Delta l\ T_x - cos\theta\ \Delta l\ T_z + u(x, z)\Delta l + \sigma(\Delta l)] = 0$$

When $\Delta l$ is small, we get

$$sin\theta\ T_x + cos\theta\ T_z = u(x, z) \quad (2)$$

From Lakshminarayanan and Varadharajan (1997)

with compact notation $T_x = \frac{\partial T}{\partial x}$ and $T_z = \frac{\partial T}{\partial z}$

# Fermat principle to Eikonal equation

Minimizing with respect to $\theta$ gives

$$\sin\theta = T_x/\sqrt{T_x^2 + T_z^2} \text{ and } \cos\theta = T_z/\sqrt{T_x^2 + T_z^2}$$

Putting these expressions into equation (2) gives the eikonal equation (3)

$$T_x^2 + T_z^2 = u^2(x,z) \quad (3)$$

This can be extended to 3D geometry as well

Fermat principle

$$T(x,z) = \min_l \int u\big(x(l), z(l)\big)dl$$

Non linear Eikonal equation

$$T_x^2 + T_z^2 = u^2(x,z)$$

First-arrival time matches the zero-order ray time when it exists.
However, such time could be evaluated when there is no ray time.

No ray ansatz and frequency power expansion

From Lakshminarayanan and Varadharajan (1997)

# Viscous solution versus ray solution

Fermat principle

$$T(x,z) = \min_{s} \int u\big(x(s), z(s)\big) ds$$

Non-linear Eikonal equation

$$T_x^2 + T_z^2 = u^2(x,z)$$

also in 3D

First-arrival time matches the zero-order ray time when it exists.

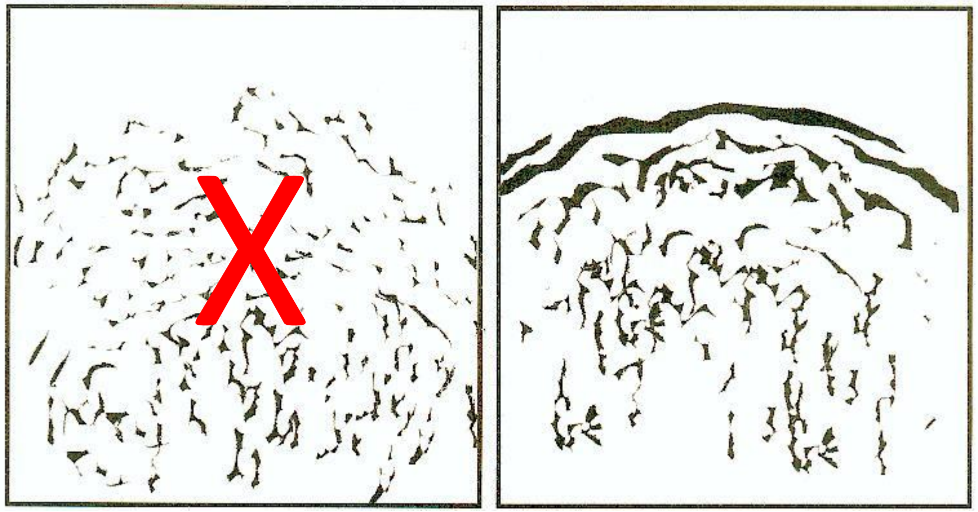
However, such time could be obtained when there is no ray time. No shadow zone!

No ray ansatz and no high-frequency asymptotic solutions. Still coherent wavefront!

Such Eikonal solution is sometimes called viscous solution. Only one-value solution!

Multi-values « viscous » solution: the Graal!

# Viscous solution with wavefront continuity




Viscous solution: first-arrival solution when wavefront continuity is preserved (maybe not differentiable!).

Diffraction is included: no shadow zone!

# Viscous solution – first-arrival solution

A non-familiar interpretation of first phases

(often associated to HF approximation)

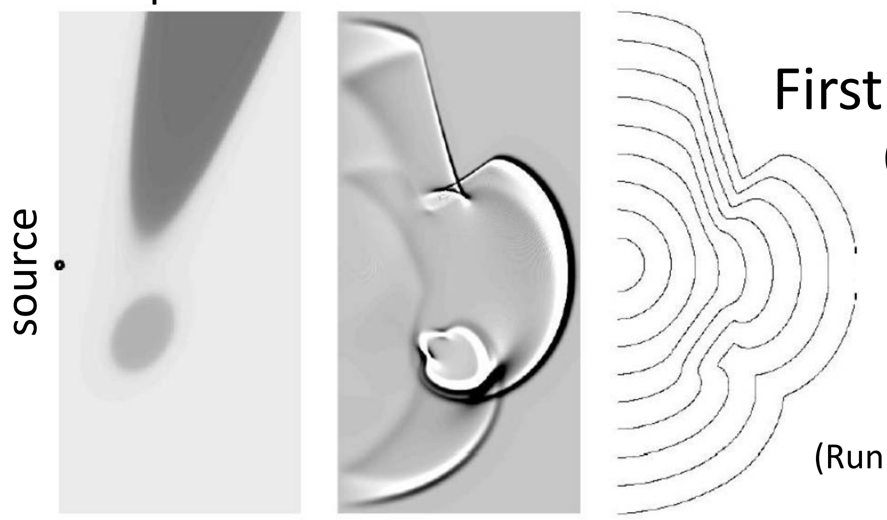
Wave disturbance (field discontinuity)

valid for any media

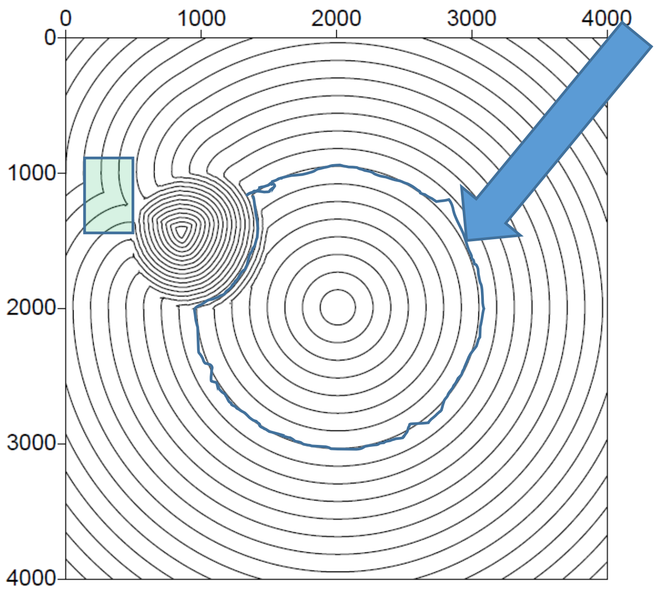
single value and always an answer

observable: continuous wavefronts

but possible discontinuous derivative




(Runborg, 2007)



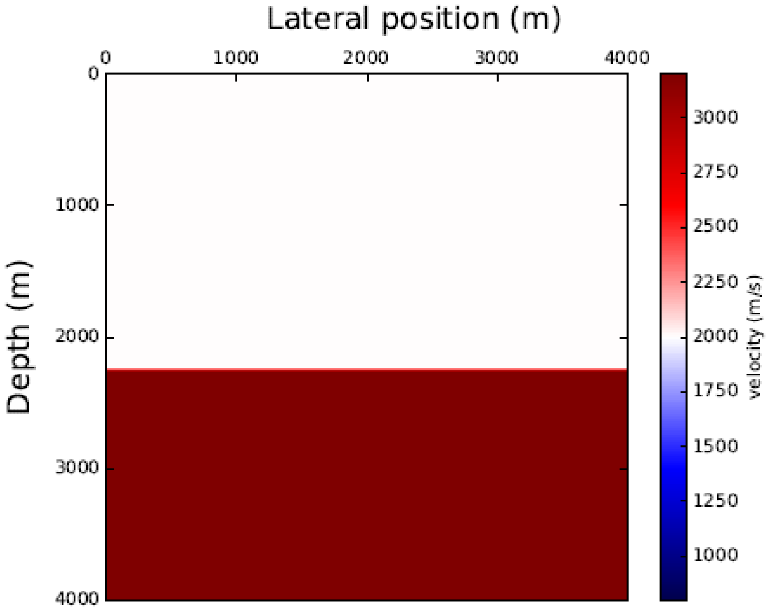

(Le Bouteiller, 2018)

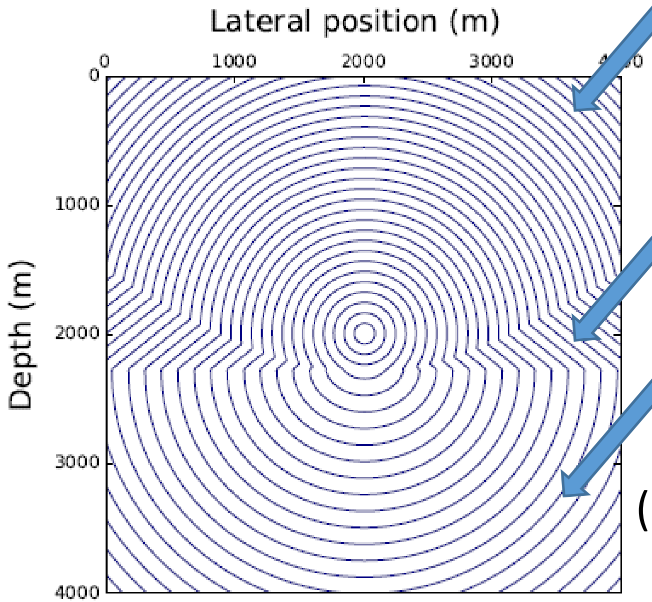
Line of same dansers

Wavefront: particles moving a synchronized way. They are in phase.

Diffraction effects included ($u \propto \omega^{1/2}$)

# Example of continuous wavefront




Direct phase

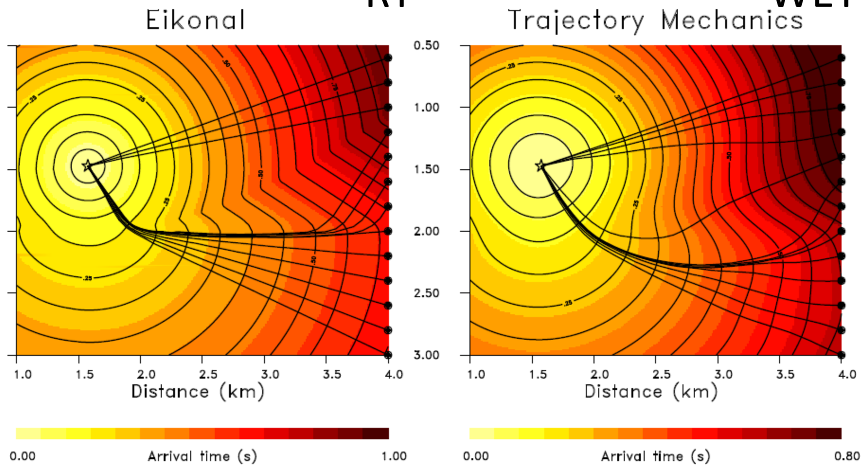
Head phase

Transmission phase

(Le Bouteiller, 2019)

RT

WET

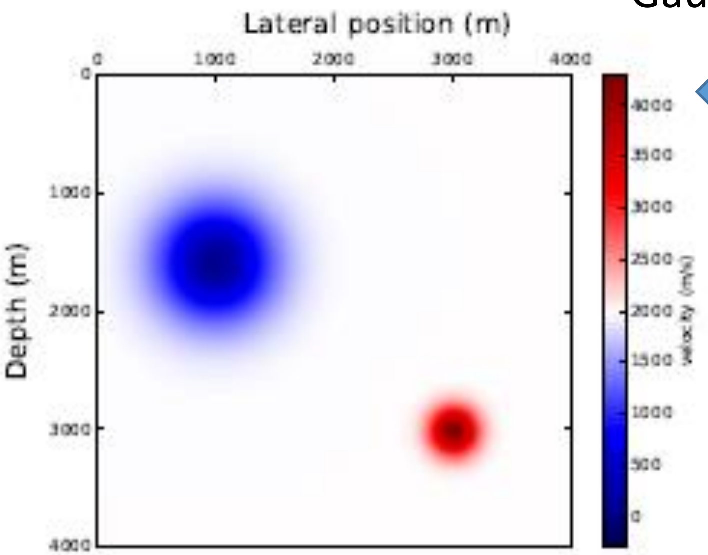


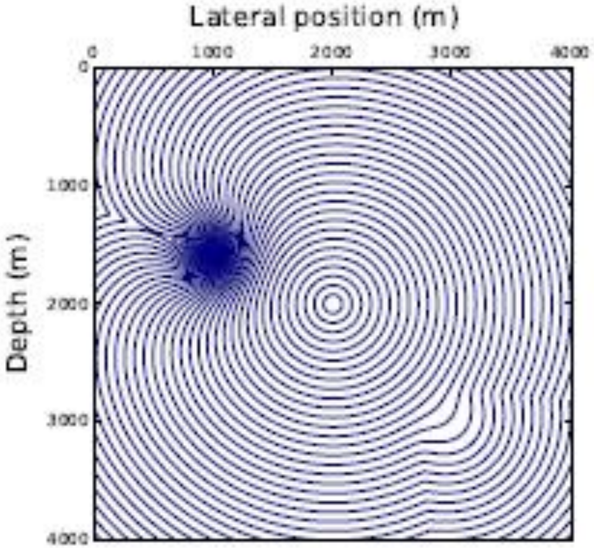

Meaning of trajectories when considering Eikonal solutions with sharp interfaces!
ray or trajectory!

(Vasco & Nihei, 2019)

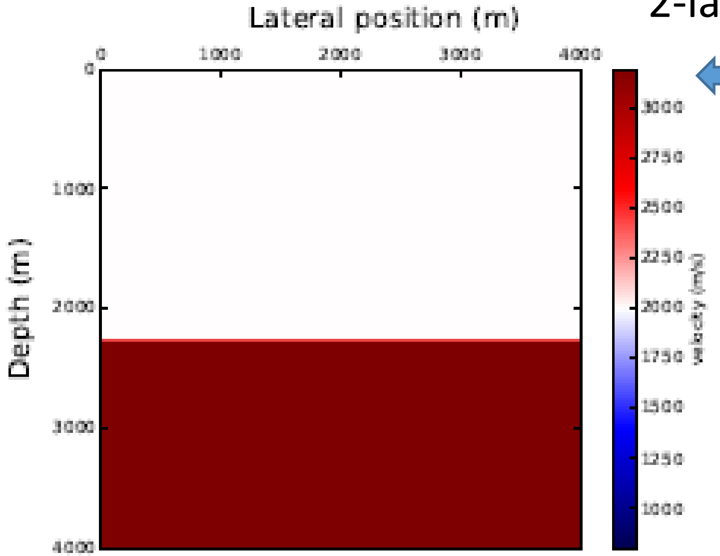
# Viscous solution: examples (# ray solution)

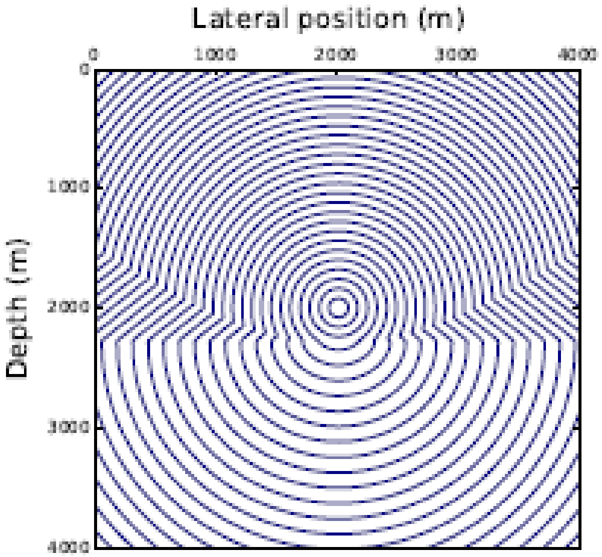
Gaussian model




2-layers model



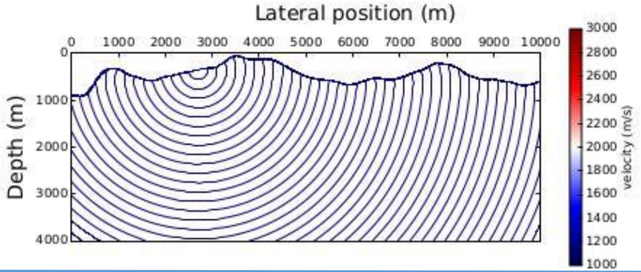

Viscous solution depends on the mesh discretization

(Le Bouteiller, 2018)

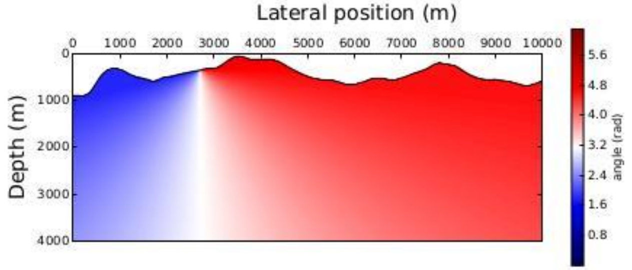
# Transport equation and related PDEs

Isotropic case

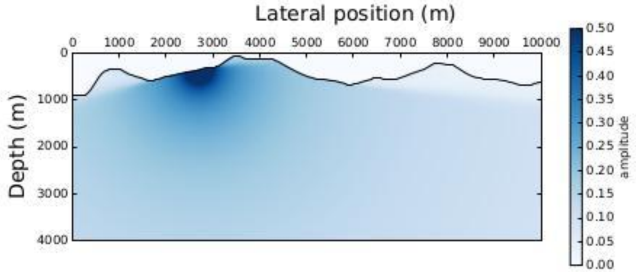
Non-linear PDE: Eikonal equation $(\nabla T)^2 - \dfrac{1}{c^2} = 0$




Linear PDE: take-off angle equation $\vec{\nabla}\varphi . \overrightarrow{\nabla T} = 0$

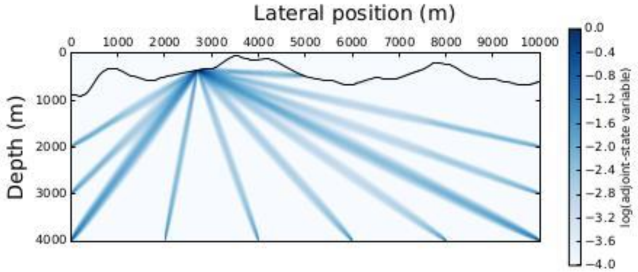



Linear PDE: amplitude equation $\vec{\nabla} . \left(A^2 \overrightarrow{\nabla T}\right) = 0$




« transport without dissipation »

Linear PDE: adjoint equation $\vec{\nabla} . \left(\lambda \overrightarrow{\nabla T}\right) = \mathcal{F}$




Le Bouteiller (2018)

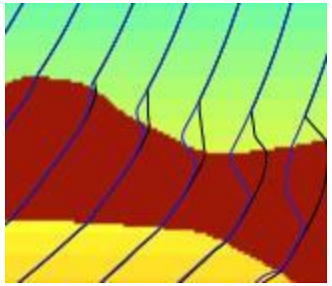
# Useful family of equations

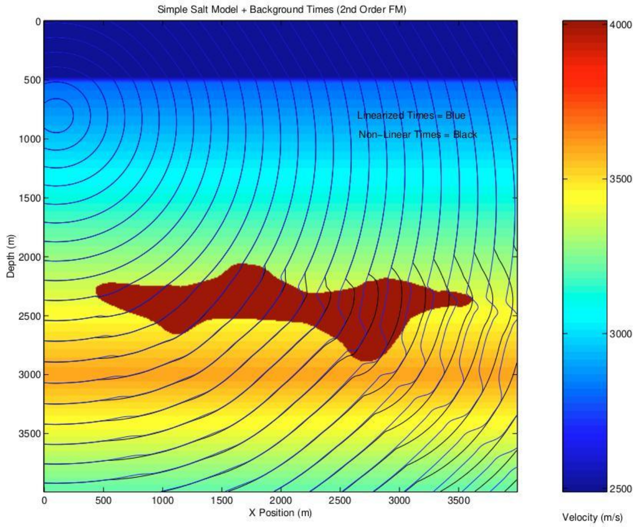
Linear Eikonal equations: time delay, angle, arclength estimation

Franklin & Harris (2001)

$$\overrightarrow{\nabla T}.\overrightarrow{\nabla \tau} = \mathcal{R}$$

Linear perturbation






Linear Transport equations: amplitude

$$\vec{\nabla}.\left(u\overrightarrow{\nabla T}\right) = 0$$

singularities!

Belayouni (2013)

Factorization for removing these singularities at the source (and at receivers …)

$$\vec{\nabla}.\left(\bar{u}u_0T_0\overrightarrow{\nabla\tau}\right) + u_0\overrightarrow{\nabla T_0}.\overrightarrow{\nabla(\bar{u}\tau)} = 0$$

$$u = \bar{u}u_0; T = \tau T_0$$

with *known (analytical) solution* $\vec{\nabla}.\left(u_0\overrightarrow{\nabla T_0}\right) = 0$

# Computer codes

## Computer codes available

Podvin & Lecomte (1996)

- Solving Eikonal equation for isotropic models: many codes

Cartesian and Spherical coordinates

- Solving Eikonal equation for anisotropic models: few codes
- Solving Eikonal equation and Sensitivity kernel: very few codes

Hamilton Fast Marching (HFM) and Adaptive Grid Discretizations (AGD) from Dr. Jean-Marie Mirebeau

HFM is written in C++17: Github repository (type Mirebeau and HFM on your browser)
Follow content of the file « Readme.md »
AGD is written in python & CUDA: Github repository (same location as the HFM software)
Recommendation of the installation from conda environment. See content of the file « Readme.md »

# HFM illustration

Interface with python langage in the directory HFM_python

Eikonal solver for an homogeneous model
Eikonal solver for Marmousi and Nankai models

The use of the FileHFM interface allows an illustration of the multiplex input/output of the C++ code HFM: it can be translated for any langage (C, Fortran …) at the expense of written files

Attractive features in 2D geometry, 3D geometry, and on curved surface (Riemann metrics)

- General anisotropy solver
- Efficient TTI Eikonal solver,
- Including topography through masks (or deformed Cartesian grid),
- Computation of sensitivity kernels
- Computation of adjoint field
- Computation of rays

DO NOT WRITE YOUR OWN CODE! (Pykonal, pyekfmm, scikit-fmm from github … among many other codes)

# Take-away message

- ❑ Ray solution (multi-valued)
  When available, fruitfull for interpretation

- ❑ Viscous solution (single-valued)
  Efficient computer codes, even for anisotropy (TTI)

- ❑ Viscous solution (multi-valued)?
  Still open problem for efficient numerical strategy
  Such solution is single-valued in the phase space!!!

# Forward problem: Temporary End

## End of the first part

New searching direction:

Neural Eikonal Solver: physics-informed Neural Network
PINNeikonal from github … (U. Waheed)
peikonal from github … (J. Calder)

Cool feature: open road for efficient tomography strategies …
(work in progress … see relation with the second part of this presentation)

# Outline on first-arrival traveltime tomography

- Images at very different scales
- Waves and Phases: various concepts
- Few points on first-break ray-based tomography
- Illustration on 30-years Western Alps tomography
- First-break eikonal-based tomography
- First-break wave-equation-based tomography
- Hypocenter-velocity joint inversion
- Conclusion

Delayed ray-based tomography based on Fréchet derivative building

# Delayed traveltime tomography

$$t(source, receiver) = \int_s^r s(x,y,z)dl$$

Finding the slowness $s(x,y,z)$ from $t(s,r)$ is a difficult problem: only integral techniques for one variable $s(z)$ (Abel) !

Consider small perturbations $\delta s(x,y,z)$ of the slowness field $s_0(x,y,z)$

$$t(s,r) = \int_s^r s(x,y,z)dl = \int_s^r s_0(x,y,z)dl + \int_s^r \delta s(x,y,z)dl$$

$$t(s,r) \approx \int_{s_0}^{r_0} s_0(x,y,z)dl + \int_{s_0}^{r_0} \delta s(x,y,z)dl$$

« frozen » ray approximation (ray connecting source/receiver for the known slowness $s_0$)

$$t(s,r) - t_0(s,r) \approx \int_{s_0}^{r_0} \delta s(x,y,z)dl$$

*No Fermat argumentation!*

$$\delta t(s,r) \approx \int_{s_0}^{r_0} \delta s(x,y,z)dl$$

LINEARIZED PROBLEM $\delta t(d) = J(d,m)\, \delta s(m)$ from the model domain to the data domain

related to linear Eikonal equation

# Discretization of the slowness perturbation

The velocity perturbation field (or the slowness field) $\delta s(x,y,z)$ can be described into a meshed cube regularly spaced in the three directions.

For each node, we specify a value $\delta s_{i,j,k}$. The interpolation will be performed with functions as step functions. For each grid point (i,j,k), shape functions $h_{i,j,k} = 1$ for i,j,k, and zero for other indices.

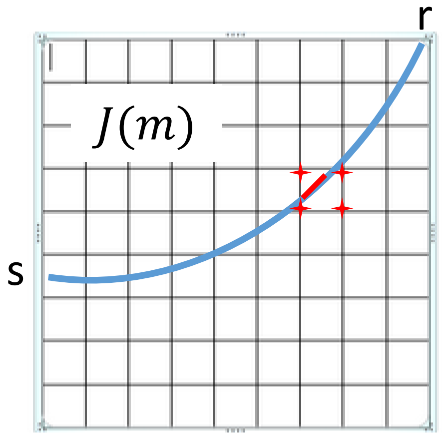
$$\delta s(x,y,z) = \sum_{cube} \delta s_{i,j,k}\, h_{i,j,k}$$

Nodal approach

Other shape functions are possible with two-end members (nodal versus modal):
fourier functions (cos,sin), chebychev, spline, wavelet … and so on

Sampling the model space is the mandatory stabilization strategy
(smoothing or damping ones)
Model discretization provide an implicit limit to the wavenumber range to be filled in

# Discrete linearized inversion problem




Discretization of the model fats the ray

$$\delta t(s,\mathrm{r}) = \int_{ray_0} \sum_{cube} \delta s_{i,j,k}\, h_{i,j,k}\, dl = \sum_{cube} \delta s_{i,j,k} \boxed{\int_{ray_0} h_{i,j,k} dl}$$

Weighted ray segment

$$\delta t(s,r) = \sum_{i,j,k} \delta s_{i,j,k} \Delta l_{i,j,k} = \sum_{i,j,k} \frac{\partial t}{\partial s_{i,j,k}} \delta s_{i,j,k}$$

$$\delta t(s,r) = \sum_{i,j,k} J_{i,j,k}\ \delta s_{i,j,k}$$

$$\delta t(n) = J(n,m)\ \delta s(m)$$

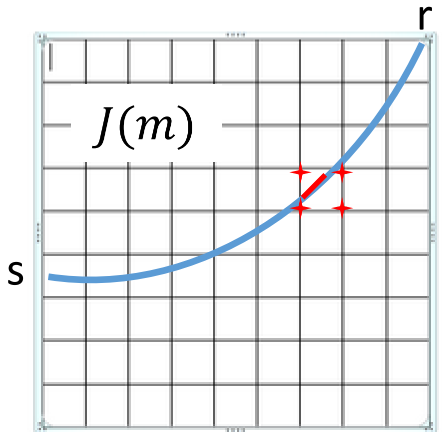
to be solved in least-squares sense

$$\begin{pmatrix} \delta t_1 \\ \delta t_2 \\ \vdots \\ \delta t_{n-1} \\ \delta t_n \end{pmatrix} = \begin{pmatrix} \frac{\partial t_1}{\partial s_1} & \cdots & \frac{\partial t_1}{\partial s_m} \\ \frac{\partial t_2}{\partial s_1} & \cdots & \frac{\partial t_2}{\partial s_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial t_{n-1}}{\partial s_1} & \cdots & \frac{\partial t_{n-1}}{\partial s_m} \\ \frac{\partial t_n}{\partial s_1} & \cdots & \frac{\partial t_n}{\partial s_m} \end{pmatrix} \begin{pmatrix} \delta s_1 \\ \delta s_2 \\ \vdots \\ \delta s_{m-1} \\ \delta s_m \end{pmatrix}$$

Sensitivity matrix J is a sparse matrix

also named Fréchet derivative or Jacobian matrix …

# Discrete linearized inversion problem




Discretization of the model fats the ray

$$\delta t(s,r) = \sum_{i,j,k} \delta s_{i,j,k} \Delta l_{i,j,k}$$

$$\delta t(s,r) = \sum_{i,j,k} J_{i,j,k} \, \delta s_{i,j,k}$$

(billion, million)

$$\delta t(n) = J(n,m)\, \delta s(m)$$

- n dimension of the data space
- m dimension of the model space

$$\begin{pmatrix} \delta t_1 \\ \delta t_2 \\ \vdots \\ \delta t_{n-1} \\ \delta t_n \end{pmatrix} = \begin{pmatrix} \frac{\partial t_1}{\partial s_1} & \cdots & \frac{\partial t_1}{\partial s_m} \\ \frac{\partial t_2}{\partial s_1} & \cdots & \frac{\partial t_2}{\partial s_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial t_{n-1}}{\partial s_1} & \cdots & \frac{\partial t_{n-1}}{\partial s_m} \\ \frac{\partial t_n}{\partial s_1} & \cdots & \frac{\partial t_n}{\partial s_m} \end{pmatrix} \begin{pmatrix} \delta s_1 \\ \delta s_2 \\ \vdots \\ \delta s_{m-1} \\ \delta s_m \end{pmatrix}$$

# Least-squares solution

The rectangular system can be recast into a square system (sometimes called normal equations).

- Solving this square linear system gives the so-called least-squares solution.

Least-squares solution

$$J^t J \delta s = J^t \delta t$$

$$\delta s = (J^t J)^{-1} J^t \delta t$$

Remark

- Another interesting solution with minimum norm

Least-norm solution

$$J J^t \delta u = \delta t$$

$$\delta s = J^t (J J^t)^{-1} \delta t$$

The system is both under-determined and over-determined depending on the considered zone (and the number of rays going through).

# Damped least-squares solution

$$\delta t = J\delta s$$
$$d = Gm$$
$$b = Ax$$

Damping parameter $\varepsilon$

$A$ is a rectangular matrix (either over- or under-determined)

$$\min_x ||Ax - b||^2 + \varepsilon\ ||x||^2$$

LSQR solves it using only products $Ax$ or $A^T b$ by considering the system

$$(A^T A + \varepsilon\ I)x = A^T b$$

widely used subroutine in travel-time tomography

LSMR solves it using only products $Ax$ or $A^T b$ by considering the system

$$(A^T A + \varepsilon\ I)x = A^T b$$

http://www.numerical.rl.ac.uk/spral/doc/latest/Fortran/

# Occam’s razor

- Do not use more complicated maths than the data deserves

- Approximate the least constrained quantity

Given: data (observed and modeled)
Assumed: wavefront propagation
Unknown: Earth structure

- Occam’s Razor: parcimonious principle

*When you have many explanations for predicting exactly the same quantities and that there is no way to distinguish them, select the simplest one… until you end up with a contradiction.*

Ockham (~1295-~1349)

Constable et al (1987)

Constrained damped least-squares solution

$$\min_x (||Ax - b||^2 + \lambda ||Dx||^2 + \varepsilon\, ||x||^2)$$

$D$ operator is a smoothing operator, such as a Laplacian operator which limit variations of the spatial second derivative of the slowness model.

Two hyper-parameters $\lambda$ and $\varepsilon$ to be selected?

Penalty approach is often selected

$$\begin{bmatrix} \mathcal{g}_k \\ \varepsilon I \end{bmatrix} [\delta s_k] = \begin{bmatrix} \dfrac{\partial t}{\partial s_k} \\ \lambda D \\ \varepsilon I \end{bmatrix} [\delta s_k] = \begin{bmatrix} \delta t_k \\ 0 \\ 0 \end{bmatrix}$$

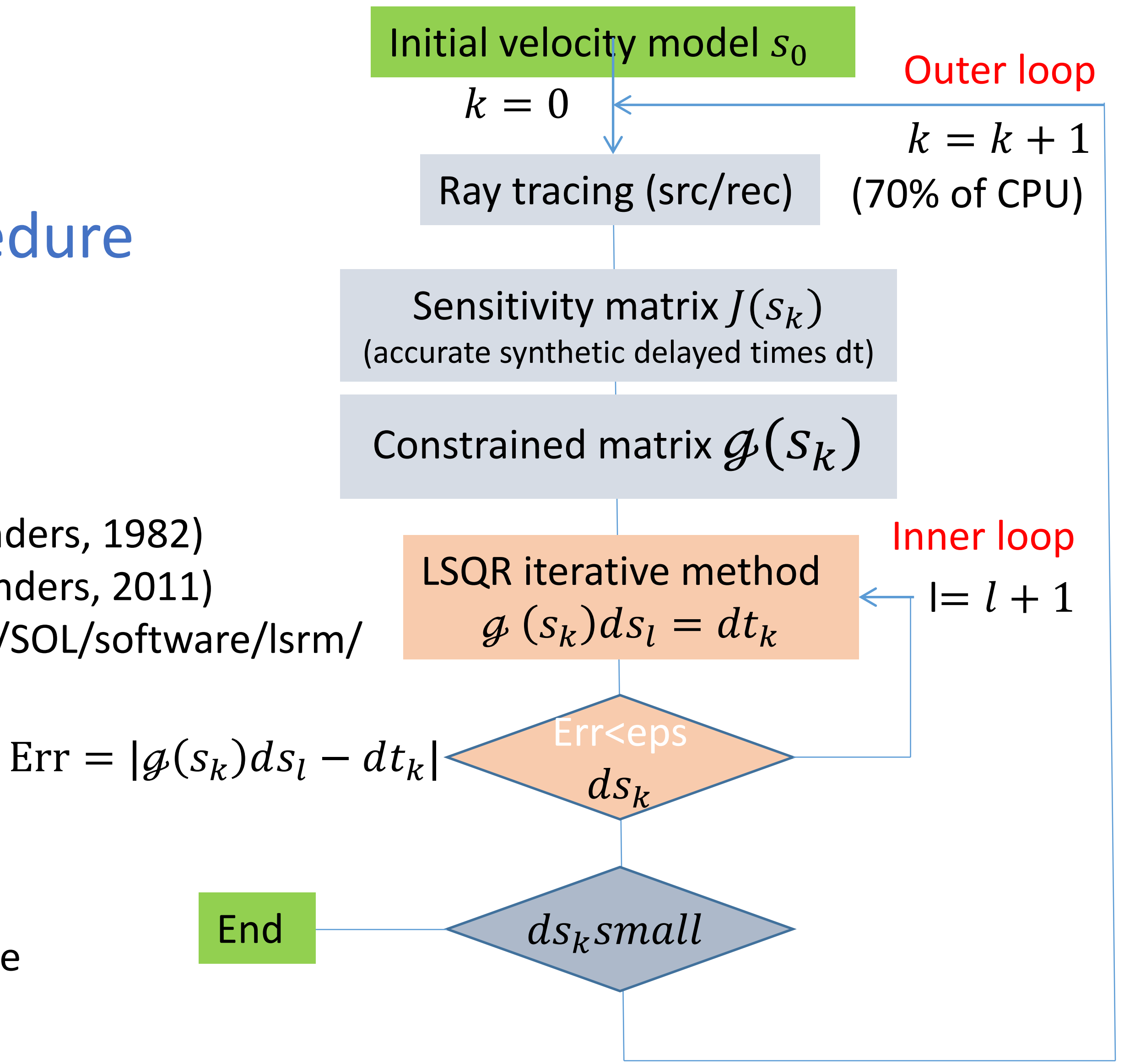
Smoothing could vary with coordinates

$$\lambda_x D_x + \lambda_y D_y + \lambda_z D_z$$

with seven-points finite-difference stencil along each direction for the laplacian

# Discrete Fréchet-ray algorithm

## Two-loops procedure




LSQR subroutine (Paige and Sanders, 1982)
LSRM subroutine (Fong and Sanders, 2011)
http://web.stanford.edu/group/SOL/software/lsrm/

LSQR or LSRM includes the damping operator

The slowness field is denoted by s and it is often the one we reconstruct

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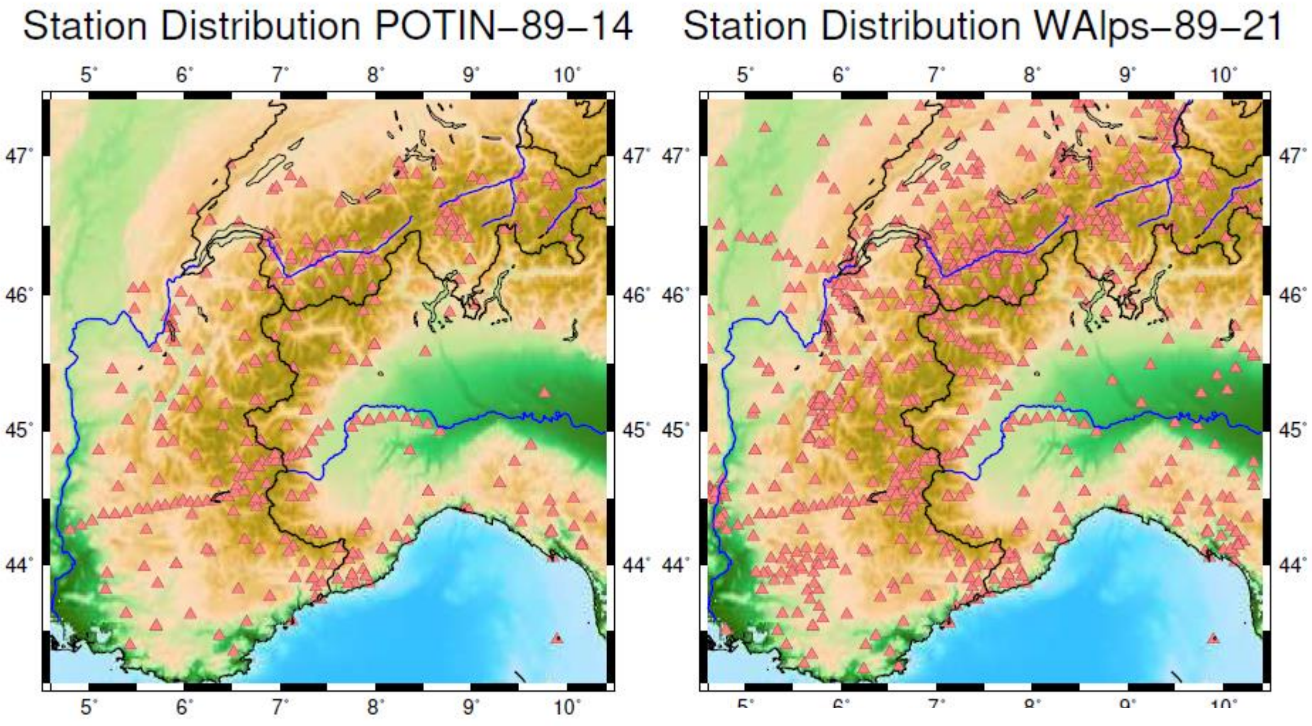
# SISMALP: 30 years Western Alps recording




Increasing number of permanent stations

Starting with manual picks performed by B. Potin: a human-eye checking

Downloading picks from ReNass (FR), RSNI/DipTeris (IT) through ISC and SED (CH)

Automatic cleaning of picks by matching predicted picks from HQ-89-14 model in a 10s interval.

Not a tectonic interpretation on my own: I am not able to do so

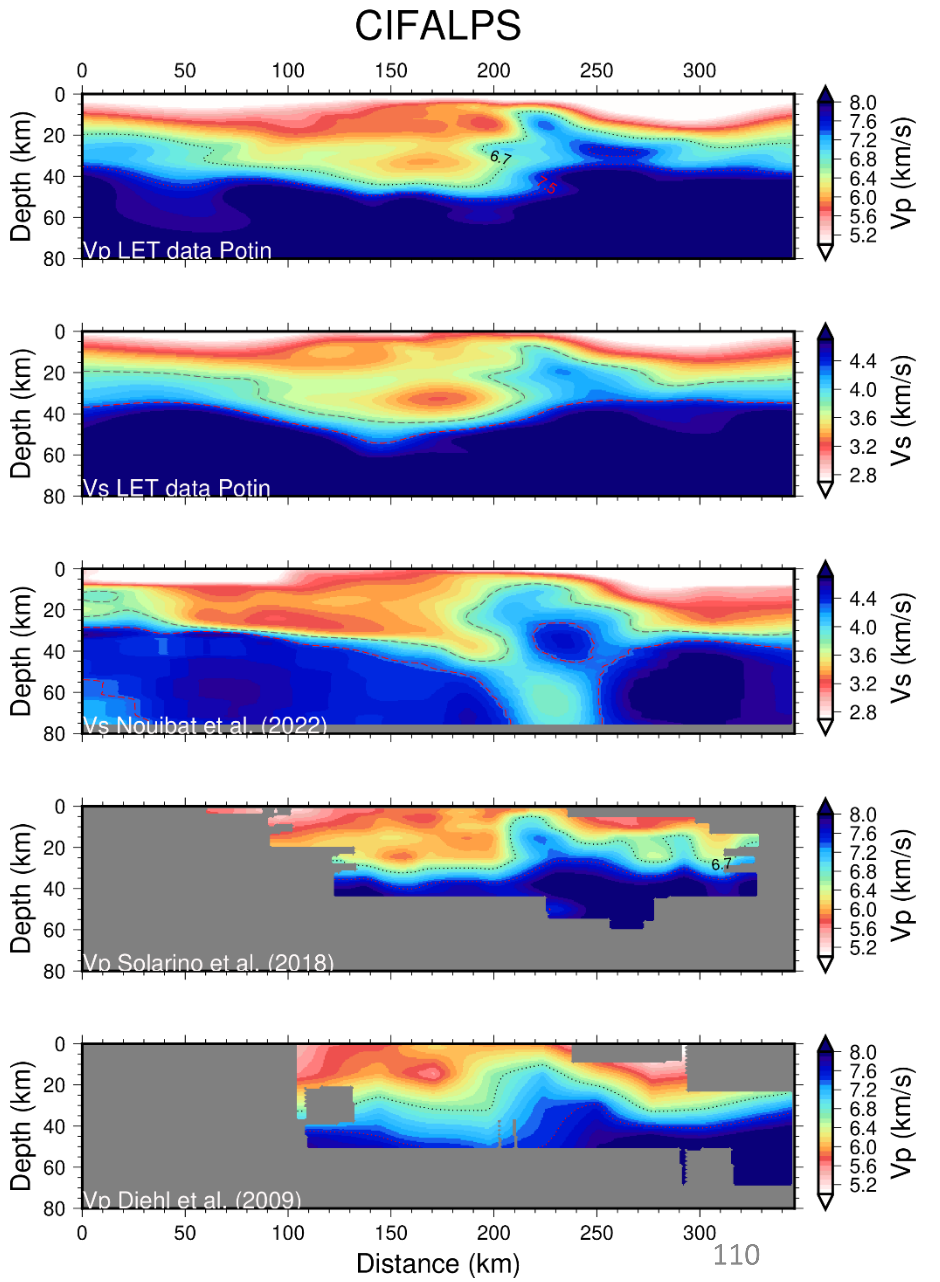
Such a target zone has been investigated by many groups using different methods

| Database | Selection | Events | Stations | P picks | S picks |
|---|---|---|---|---|---|
| POTIN-89-14 | All picks | 54 409 | 373 | 542 818 | 460 129 |
| HQ-89-14 | > 12 P & 6 S | 13 022 | 367 | 309 228 | 263 498 |
| SQ-89-14 | > 6 picks | 50 331 | 373 | 533 499 | 451 517 |
| WAlps-89-21 | All picks | 82 088 | 1043 | 952 317 | 670 786 |
| WA-89-21 | > 6 picks | 75 538 | 1043 | 936 977 | 661 522 |

# Western Alps: a target zone

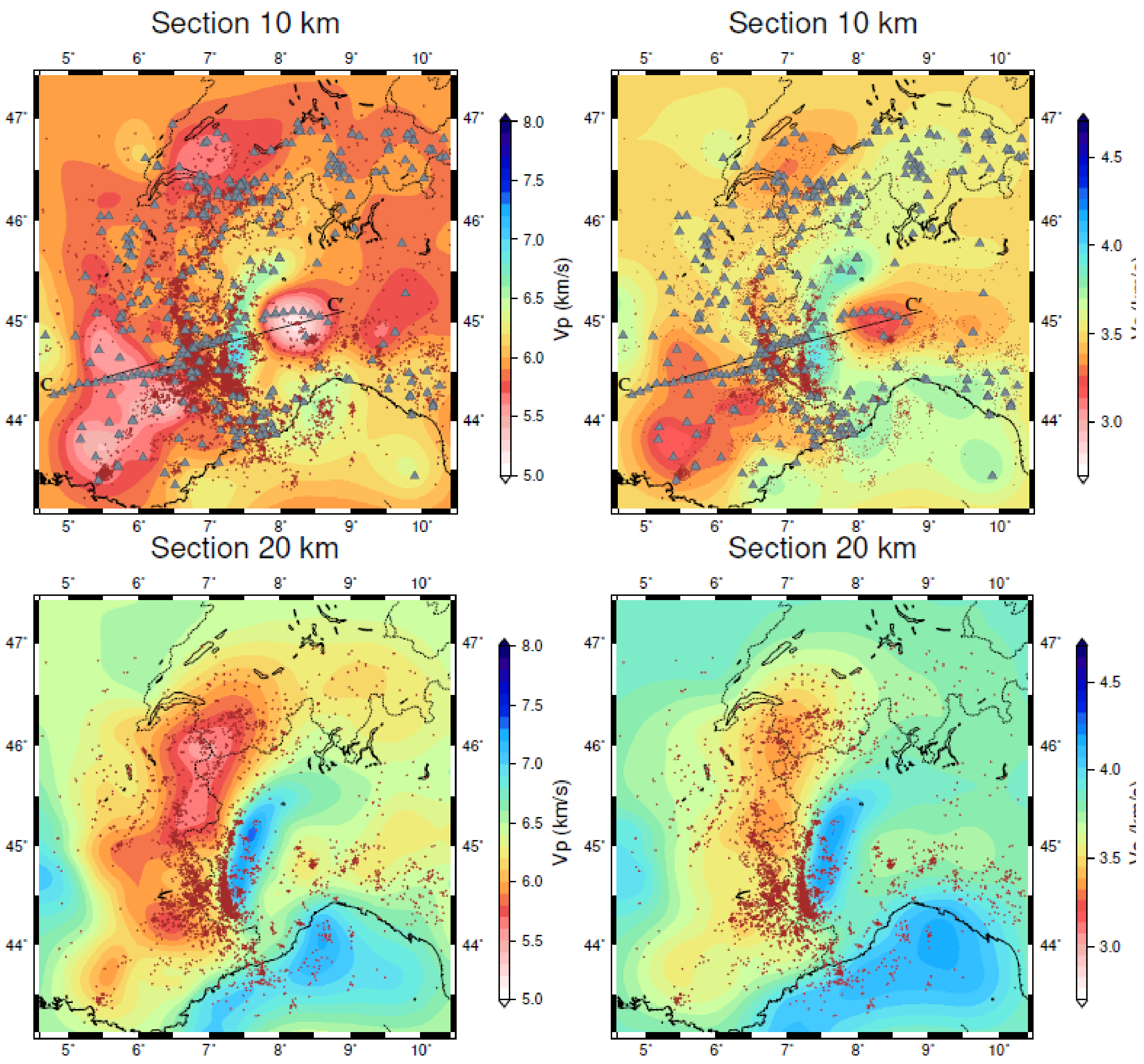
- Local earthquake tomography (Solarino et al., 1997; Paul et al., 2001; Diehl et al., 2009a,b; Solarino et al., 2018)
- Controlled source investigation (ECORS-CROP Deep Seismic Sounding Group, 1989; Nicolas et al., 1990; Thouvenot et al., 2007)
- TransD(?) ambient-noise tomography (Stehly et al., 2009; Lu et al., 2018; Zhao et al., 2020; Nouibat et al., 2022)
- Receiver-function approach (Zhao et al., 2015; Paul et al., 2022)
- Teleseismic traveltime tomography (Zhao et al., 2016; Paffrath et al., 2021)
- Teleseismic full waveform inversion (Beller et al., 2018)

Focus on methodological impacts
See the nice paper by Malusà et al (2021) for geodynamic interpretation

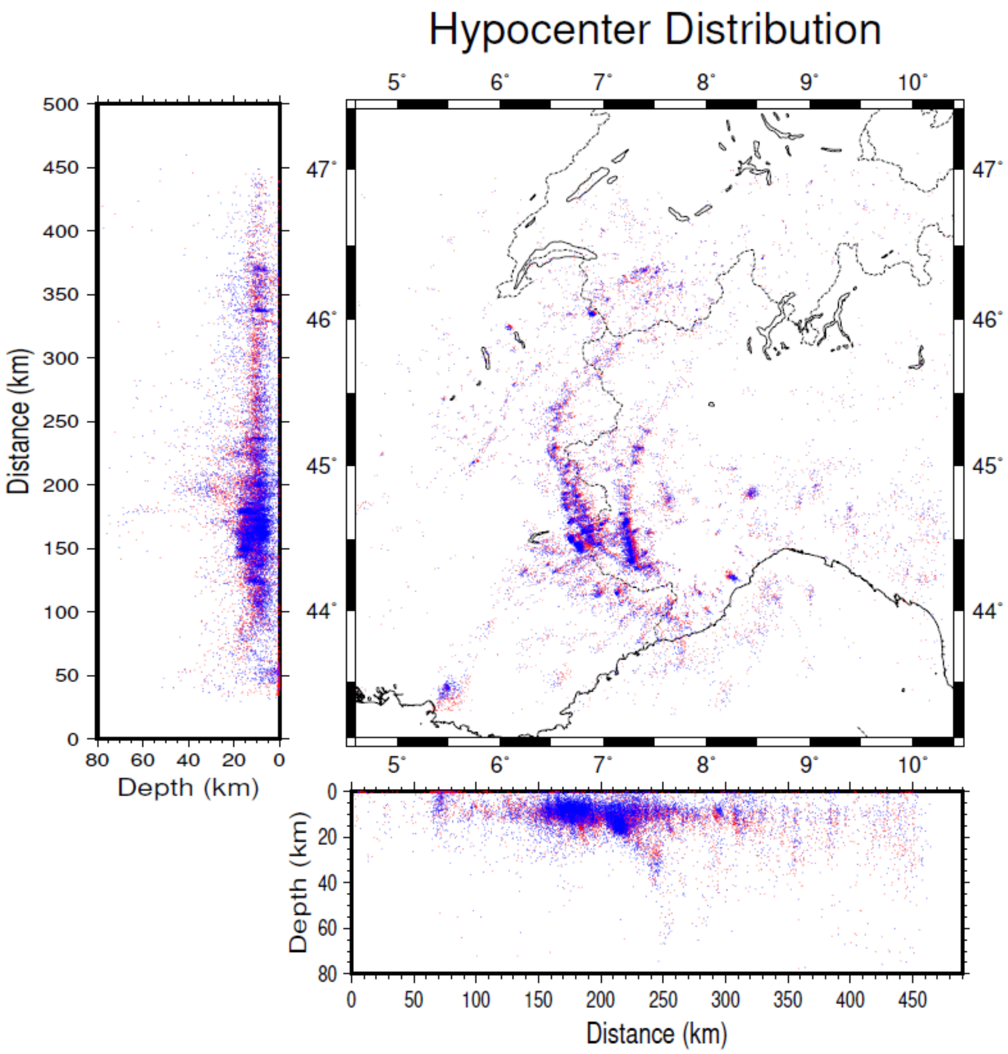



(from A. Paul)

# Vp & Vs models: penalty approach

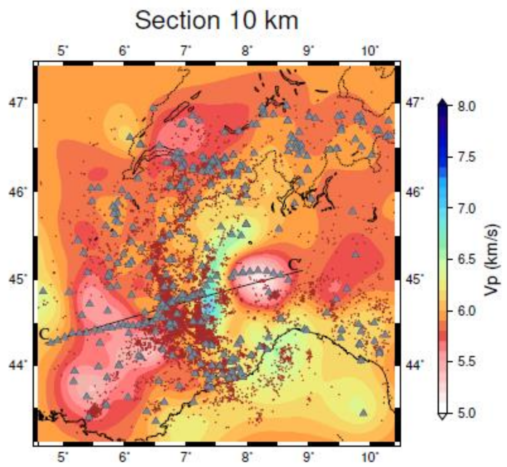


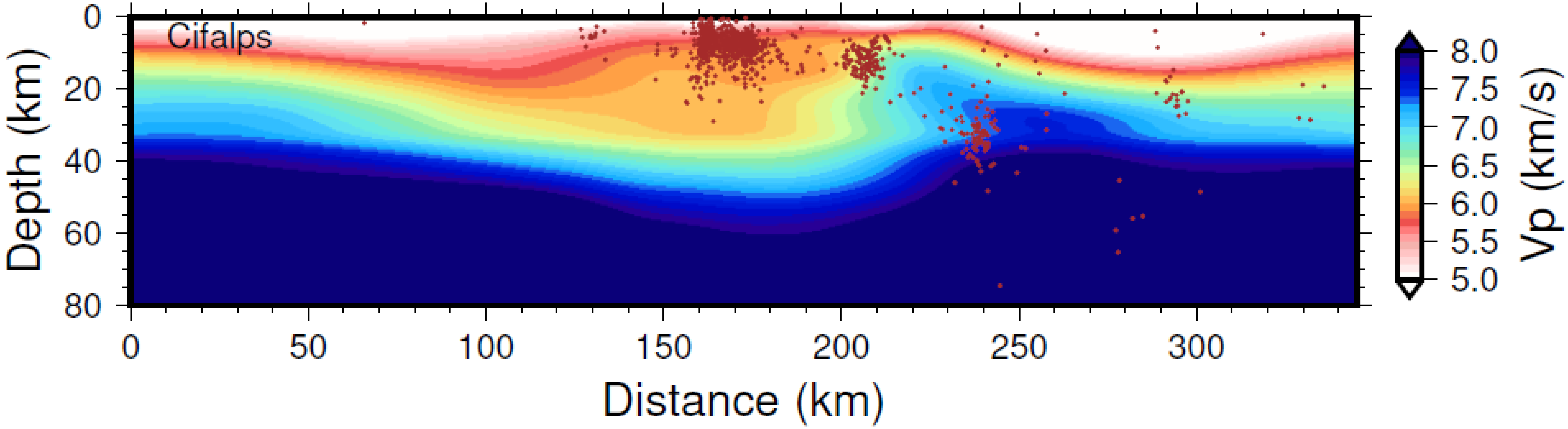

Upper layer sampling?

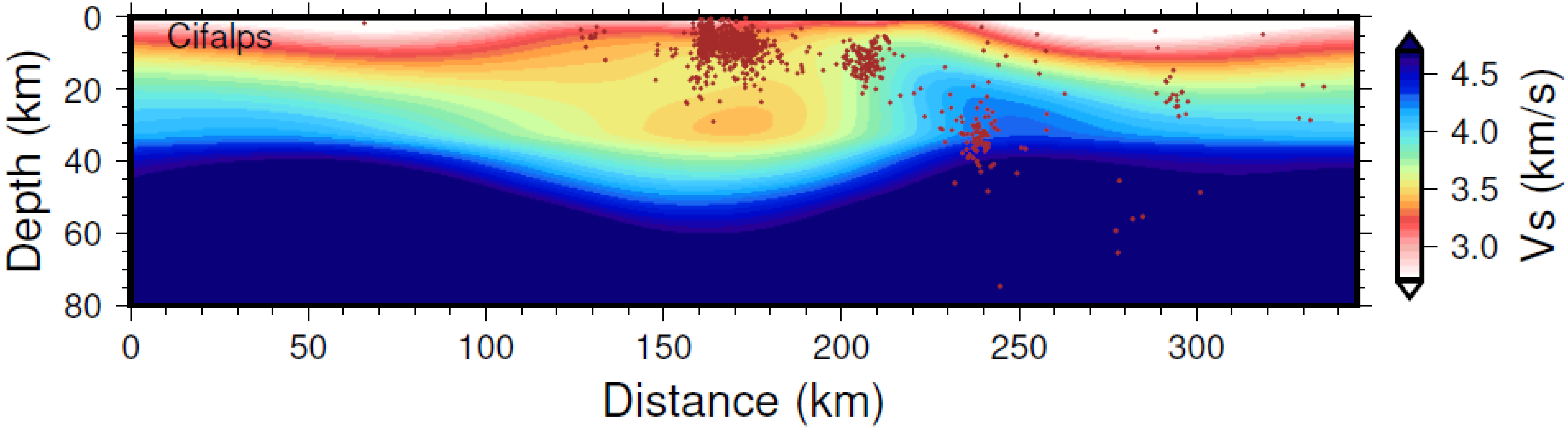



Initial location in red; Final location in blue
Shallower hypocenters in 3D final model

# CifAlps cross-section



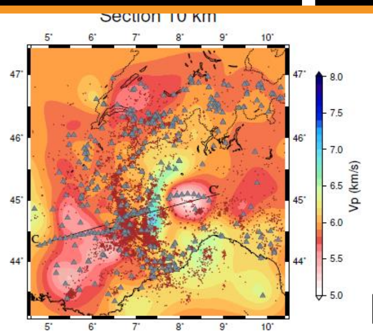



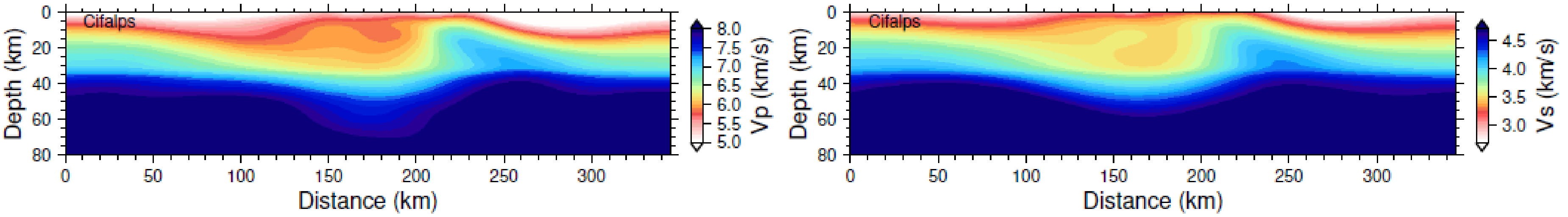

Known structural features such as the Ivrea body

See paper
by Malusà et al (2021)
for deeper understanding

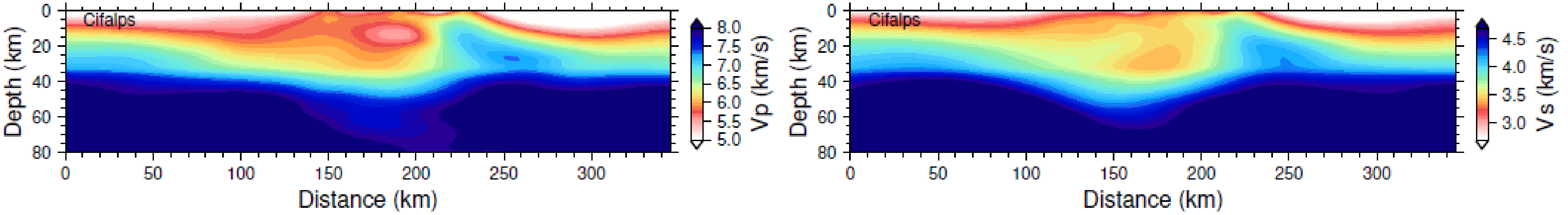
# CifAlps cross-section: data & initial models



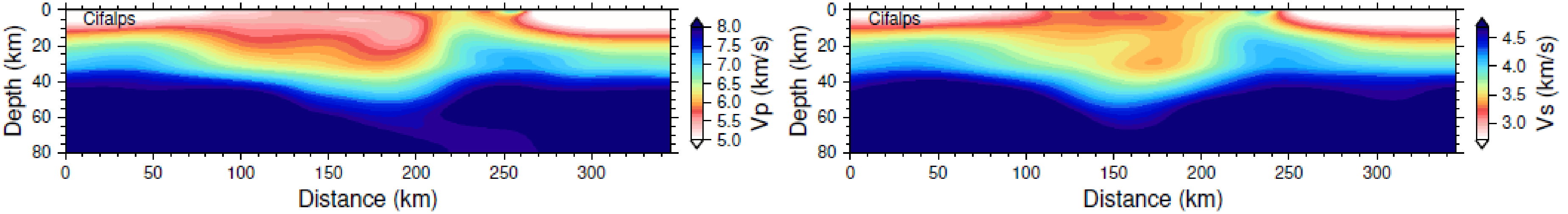

Full database
Standard scheme



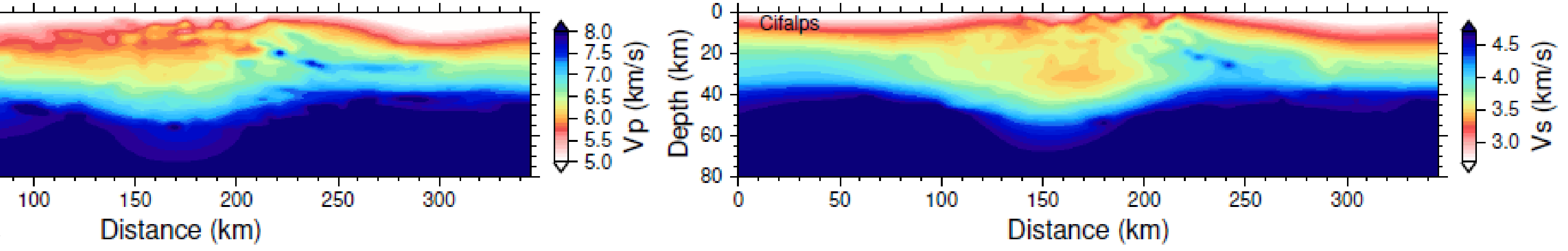

Selected Potin database
Standard scheme




ANT-inspired initial model
Standard scheme

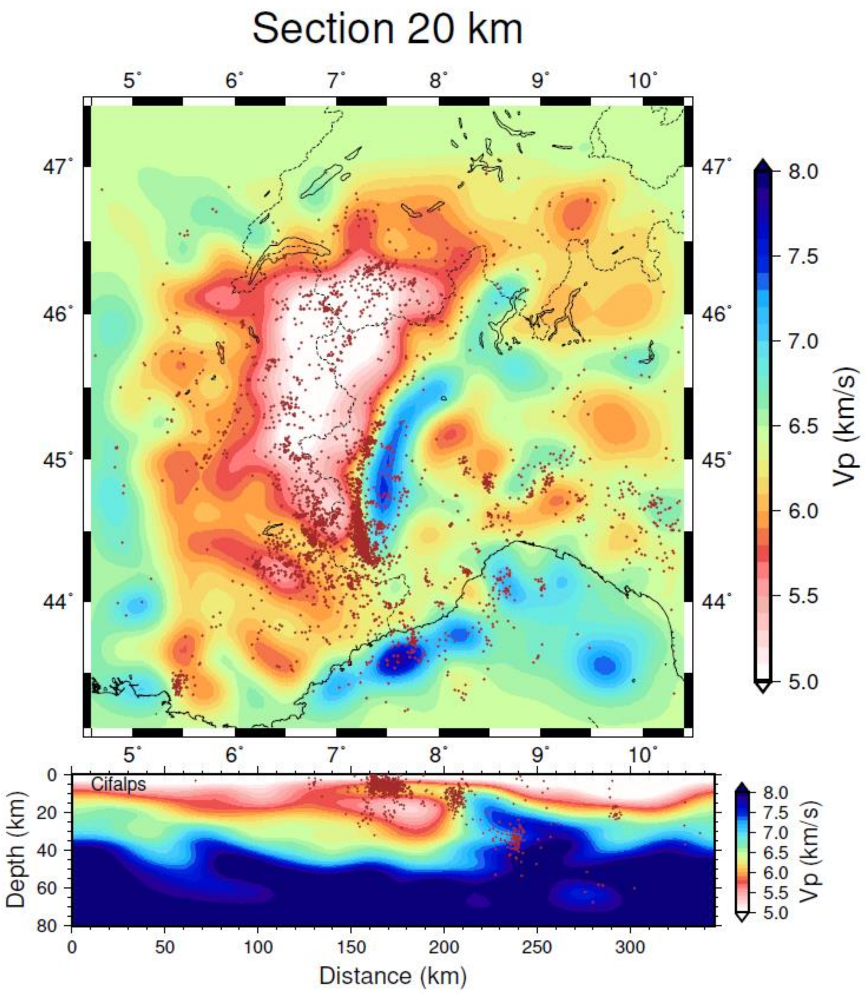



Only damped term
No smoothing term

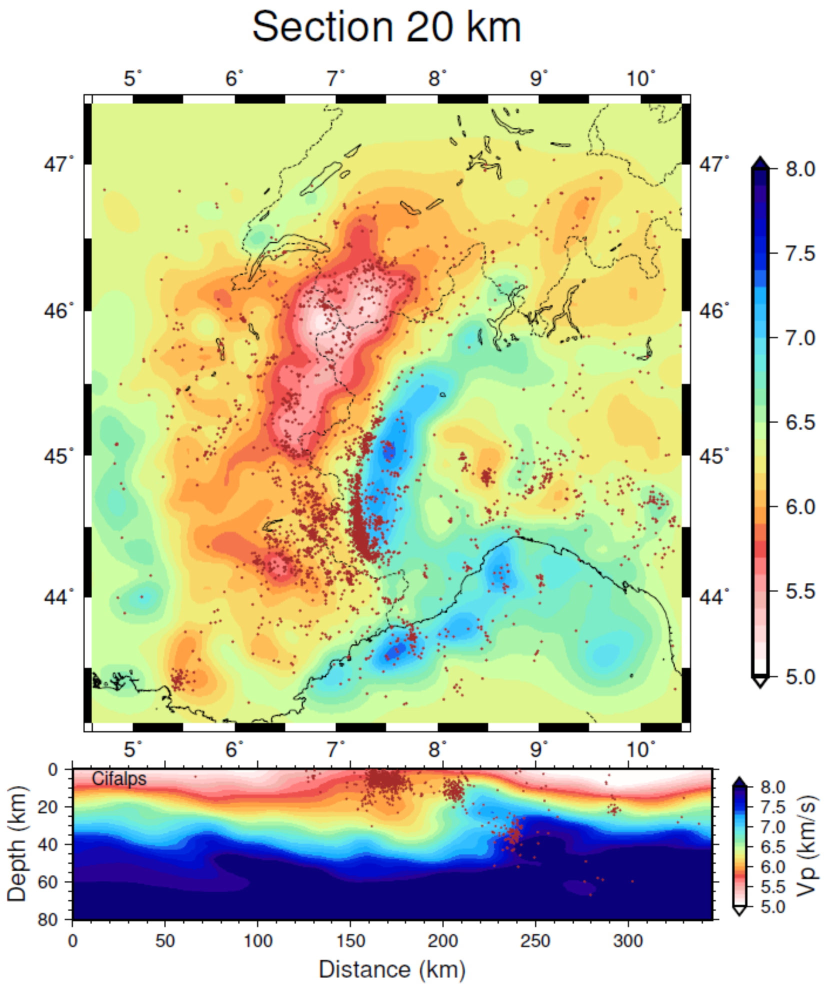



Never done in fact: > 1000 iterations

# Alternative model perturbation smoothing




Gaussian smoothing on model perturbation



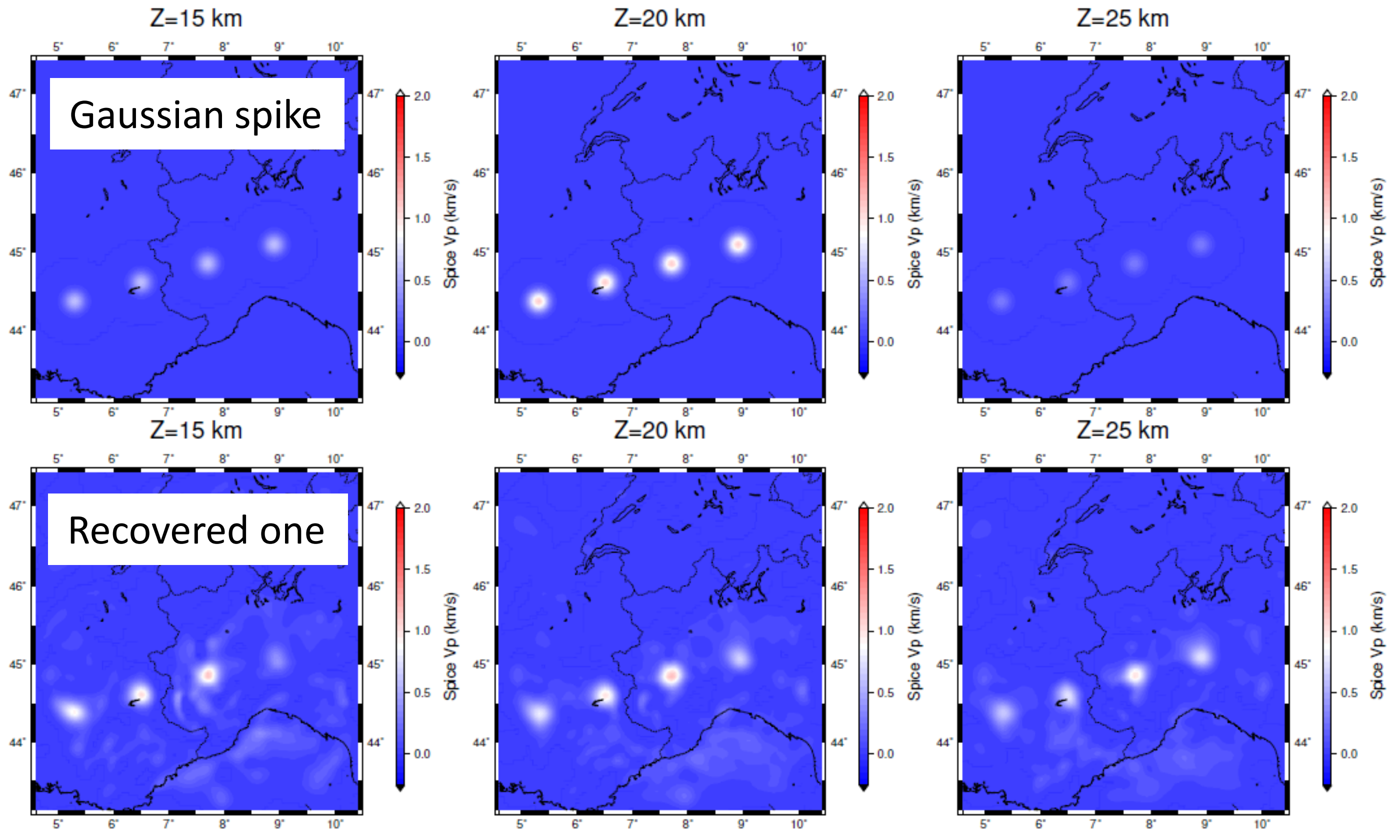

Total variation smoothing on model perturbation

# Spike analysis: local resolution # model unicity

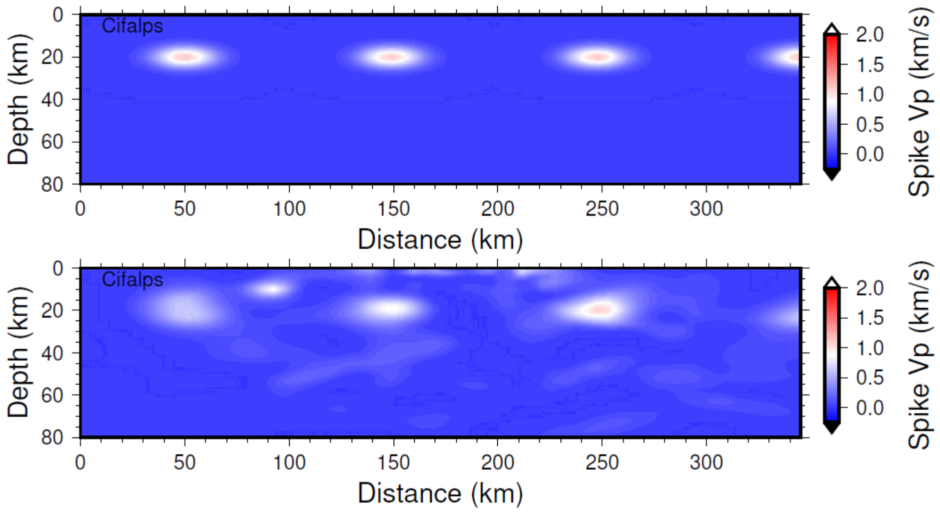
Different output models provide similar spike tests:
local resolution at the « optimal » model

Each spike reconstruction is independent: only the plotting combines them (#checkerboard test)

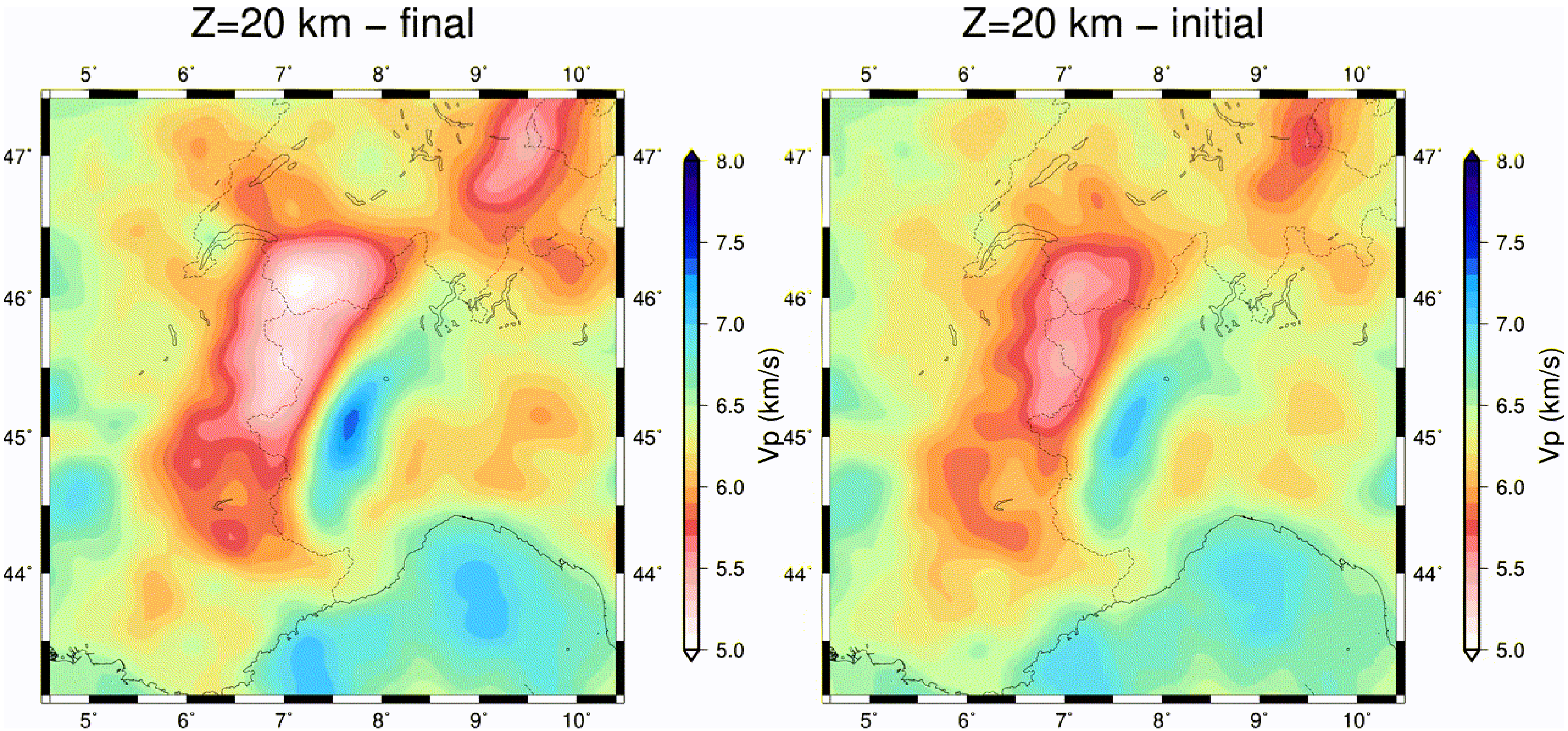
Horizontal section




Cifalps section



# Take-away message of ray approach




Traveltime tomography is agnostic to frequency content of seismic waves.

*Avoid* over-interpretation of images
Support your interpretation with other reconstructions

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# Eikonal-based tomography: model gradient

Delayed eikonal-based tomography
based on model gradient building

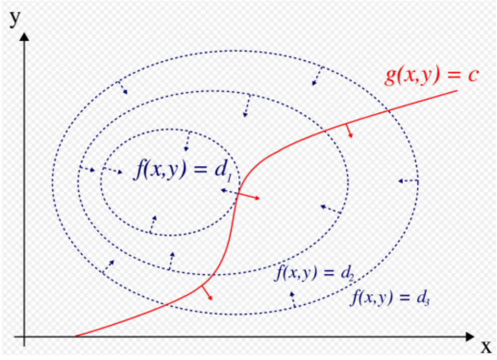
Differential geometry: no need to introduce Lagrangian multipliers but such introduction leads to error-free manipulation

(see also Menke, 2012)

$$\min_{s} \frac{1}{N_{s,r}} \sum_{s,r} \frac{1}{2} (T_{obs} - \mathcal{R}(T(s)))^t \, (T_{obs} - \mathcal{R}(T(s)))$$

subject to $|\nabla T(x,y,z)| = s(x,y,z)$ or $\mathcal{H}(x, s, \nabla T_e)$=0

$s(x,y,z)$ is the slowness




Lagrangian multipliers approach (https://en.wikipedia.org/wiki/Lagrange_multiplier)

For the case of only one constraint and only two choice variables (as exemplified in Figure 1), consider the optimization problem

maximize $f(x, y)$
subject to $g(x, y) = 0$.

(Sometimes an additive constant is shown separately rather than being included in $g$, in which case the constraint is written $g(x, y) = c$, as in Figure 1.) We assume that both $f$ and $g$ have continuous first partial derivatives. We introduce a new variable ($\lambda$) called a **Lagrange multiplier** and study the **Lagrange function** (or **Lagrangian** or **Lagrangian expression**) defined by

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y),$$

where the $\lambda$ term may be either added or subtracted. If $f(x_0, y_0)$ is a maximum of $f(x, y)$ for the original constrained problem, then there exists $\lambda_0$ such that $(x_0, y_0, \lambda_0)$ is a stationary point for the Lagrange function (stationary points are those points where the partial derivatives of $\mathcal{L}$ are zero). However, not all stationary points yield a solution of the original problem. Thus, the method of Lagrange multipliers yields a necessary condition for optimality in constrained problems.[3][4][5][6][7] Sufficient conditions for a minimum or maximum also exist, but if a particular candidate solution satisfies the sufficient conditions, it is only guaranteed that that solution is the best one *locally* – that is, it is better than any permissible nearby points. The *global* optimum can be found by comparing the values of the original objective function at the points satisfying the necessary and locally sufficient conditions.

The method of Lagrange multipliers relies on the intuition that at a maximum, $f(x, y)$ cannot be increasing in the direction of any neighboring point where $g = 0$. If it were, we could walk along $g = 0$ to get higher, meaning that the starting point wasn't actually the maximum.

wikipedia

$$\mathcal{L}(s,T,\lambda) = \underbrace{\frac{1}{N_{e,r}} \sum_{e,r} \frac{1}{2} (T_{obs} - \mathcal{R}[T_e])^t \, (T_{obs} - \mathcal{R}[T_e])} - \frac{1}{2} \sum_{e} \int_{\Omega} \lambda_e \, \mathcal{H}(x, s, \nabla T_e) dx$$ converted in a minimisation problem

# Lagrangian formulation

$$\mathcal{L}(s,T,\lambda) = \underbrace{\frac{1}{N_{e,r}}\sum_{e,r}\frac{1}{2}(T_{obs} - \mathcal{R}[T_e])^t W_d^t W_d (T_{obs} - \mathcal{R}[T_e])}_{\text{Data misfit}} - \frac{1}{2}\sum_{e}\int_{\Omega} \lambda_e\, \mathcal{H}(x,s,\nabla T_e) dx$$

Equation-based constraint

where the operator $\mathcal{R}$ extracts synthetic times at receiver positions, fields $T(x)$, $\lambda(x)$ for one event and model parameter $s(x)$ are independent quantities (they are connected at the minimisation point, called realization point). Data are weighted by a matrix $W_d$

$\mathcal{L}(s,T,\lambda) = \mathcal{L}_1(s,T) + \mathcal{L}_2(s,T,\lambda)$:  $\mathcal{L}_1$ has the same value as the misfit function $\mathcal{C}(s)$

The optimal solution will be given by $\mathrm{d}\mathcal{L} = 0$

$$\begin{cases} \dfrac{\partial \mathcal{L}^t}{\partial s}.\delta s = 0 \\ \dfrac{\partial \mathcal{L}^t}{\partial T}.\delta T = 0 \\ \dfrac{\partial \mathcal{L}^t}{\partial \lambda}.\delta \lambda = 0 \end{cases}$$

necessary conditions: KKT conditions (Karush-Kuhn-Tucker)

first-order optimality conditions in a full space search $(s,T,\lambda)$ ☹

# Reduced-model approach

If $\frac{\partial \mathcal{L}^t}{\partial \lambda} . \delta\lambda = 0 \rightarrow \mathcal{H}(\nabla \mathrm{T}) = 0 \rightarrow T^*$ — Eikonal

and $\frac{\partial \mathcal{L}^t}{\partial T} . \delta T = 0 \rightarrow \frac{\partial \mathcal{L}_1^t}{\partial T} . \delta T + \frac{\partial \mathcal{L}_2^t}{\partial T} . \delta T = 0 \rightarrow \lambda^*,$ — Adjoint

then $d\mathcal{L} = \frac{\partial \mathcal{L}^t\,(s, T^*(s), \lambda^*(s))}{\partial s} . \delta s = \frac{\partial \mathcal{C}}{\partial s}(s). \delta s$ — Gradient

# Reduced-model approach

If $\frac{\partial \mathcal{L}^t}{\partial \lambda}.\delta\lambda = 0 \rightarrow \mathcal{H}(\nabla \mathrm{T}) = 0 \rightarrow T^*$ — Eikonal

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then $d\mathcal{L} = \frac{\partial \mathcal{L}^t\,(s,T^*(s),\lambda^*(s))}{\partial s}.\delta s = \frac{\partial \mathcal{C}}{\partial s}(s).\delta s$ — Gradient

For one source: $\mathcal{H}\big(\nabla(\mathrm{T}+\delta T)\big) = \mathcal{H}(\nabla T) + \frac{\partial \mathcal{H}}{\partial \nabla T}.\nabla\delta T$

$$\Big(\frac{\partial \mathcal{H}}{\partial \nabla_x T}, \frac{\partial \mathcal{H}}{\partial \nabla_y T}, \frac{\partial \mathcal{H}}{\partial \nabla_z T}\Big)$$

$\frac{\partial \mathcal{L}_2^t}{\partial T}.\delta T$ ?

$$\frac{\partial \mathcal{L}_2^t}{\partial T}.\delta T = -\frac{1}{2}\int_\Omega \lambda \frac{\partial \mathcal{H}}{\partial \nabla T}.\nabla\delta T dx$$

Divergence theorem: $\nabla.(\delta T\lambda\frac{\partial \mathcal{H}}{\partial \nabla T})=\delta T\ \nabla.\left(\lambda\frac{\partial \mathcal{H}}{\partial \nabla T}\right) + \nabla\delta T.\lambda\frac{\partial \mathcal{H}}{\partial \nabla T}$

$$\frac{\partial \mathcal{L}_2^t}{\partial T}.\delta T = -\frac{1}{2}\int_\Omega \nabla.(\delta T\lambda\frac{\partial \mathcal{H}}{\partial \nabla T})dx + \frac{1}{2}\int_\Omega \delta T\nabla.(\lambda\frac{\partial \mathcal{H}}{\partial \nabla T})dx$$

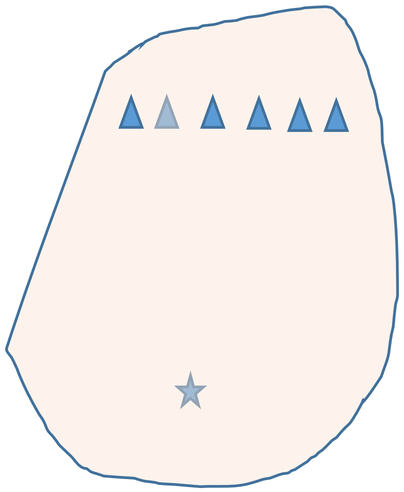
$$\frac{\partial \mathcal{L}_2^t}{\partial T}.\delta T = -\frac{1}{2}\int_{\partial\Omega} \delta T\lambda\frac{\partial \mathcal{H}}{\partial \nabla T}dl + \frac{1}{2}\int_\Omega \delta T\nabla.(\lambda\frac{\partial \mathcal{H}}{\partial \nabla T})dx$$

Differential manipulation

# Adjoint PDE: transport equation

$$\frac{\partial \mathcal{L}^t}{\partial T}.\delta T = \frac{1}{N_{e,r}}\sum_{e,r}(T_{obs} - \mathcal{R}[T_e(s)])^t W_d^t\,.\delta T - \cancel{\frac{1}{2}\sum_{e}\int_{\partial\Omega}\delta T \lambda_e \frac{\partial \mathcal{H}}{\partial \nabla T} dl} + \frac{1}{2}\int_{\Omega}\delta T \nabla.\left(\lambda_e \frac{\partial \mathcal{H}}{\partial \nabla T}\right) dx = 0$$

No restriction if Dirichlet condition ($\lambda = 0$ over $\partial\Omega$)



For one event: $\int_{\Omega}(T_{obs} - \mathcal{R}[T(s)])^t W_d^t\,\delta(x - x_r)\delta T dx$

$$\int_{\Omega}\delta T\left[\nabla.\left(\lambda \frac{\partial \mathcal{H}}{\partial \nabla T}\right) + (T_{obs} - \mathcal{R}[T(s)])^t W_d^t\,\delta(x - x_r)\right] dx = 0 \quad \forall\, \delta T$$

Transport equation (similar to the one for the amplitude)

# Adjoint PDE: transport equation

$$\mathcal{L}(s,T,\lambda) = \underbrace{\frac{1}{N_{e,r}} \sum_{e,r} \frac{1}{2} (T_{obs} - \mathcal{R}[T_e])^t W_d^t W_d (T_{obs} - \mathcal{R}[T_e])} - \frac{1}{2} \sum_e \int_\Omega \lambda_e \, \mathcal{H}(x,s,\nabla T_e) dx$$

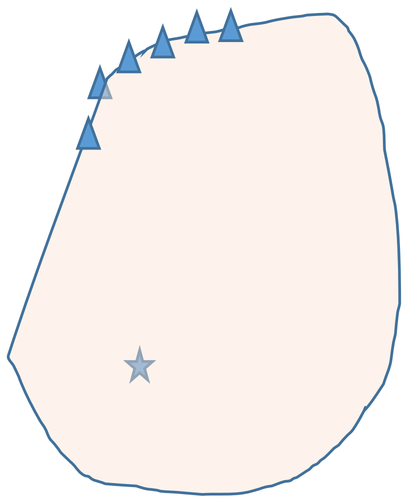
$$\left[ \nabla . \left( \lambda(x) \frac{\partial \mathcal{H}}{\partial \nabla T} \right) = - \sum_r (T_{obs} - \mathcal{R}[T(s)])^t W_d^t \, \delta(x - x_r) \right]$$

Reduced-parameter $\lambda(x)$

Linear PDE for adjoint quantity $\lambda(x)$ for one event but all associated receivers

Adjoint/Lagrangian approach: a simple way for differential geometry …

# Adjoint PDE: boundary conditions



(Leung & Qian, 2006; Taillandier et al, 2009;  Waheed et al, 2016)

$$\nabla . \left( \lambda(x) \frac{\partial \mathcal{H}}{\partial \nabla T} \right) = 0$$

$$\lambda(x) \frac{\partial \mathcal{H}}{\partial \nabla T}\bigg|_{\partial \Omega} = \sum_e (T_{obs} - \mathcal{R}[T(s)])^t C_d^t$$

Continuity of boundary conditions?

RHS injection (even if singular – point dirac injection) is easier than boundary conditions (missing values)

(limited aperture of receiver distribution)

especially when considering other observables (Tavakoli F. et al, 2017)

# Model gradient estimation

*independent variables* $s, T, \lambda$

$$\mathcal{L}(s,T,\lambda) = \frac{1}{N_{e,r}} \sum_{e,r} \frac{1}{2} (T_{obs} - \mathcal{R}[T])^t W_d^T W_d (T_{obs} - \mathcal{R}[T]) - \frac{1}{2} \sum_e \int_\Omega \lambda\, \mathcal{H}(x, \nabla T, s) dx$$

- Isotropic model: $\mathcal{H}(x, \nabla T, s) = |\nabla T|^2 - s(x)^2$ for one event

$$\frac{\partial \mathcal{C}}{\partial s}(s) = \left.\frac{\partial \mathcal{L}(s,T,\lambda)}{\partial s}\right|_{s,T^*,\lambda^*} = \sum_e \int_\Omega \lambda^*(x) s(x) dx$$

$T^*$ *solution of the Eikonal equation*

$\lambda^*$ *solution of the transport equation* (connected to $T^*$)

Model slowness $s(x)$?

# Model gradient estimation (anisotropy)

$$\mathcal{L}(s,T,\lambda) = \frac{1}{N_{e,r}} \sum_{e,r} \frac{1}{2} (T_{obs} - \mathcal{R}[T])^t W_d^T W_d (T_{obs} - \mathcal{R}[T]) - \frac{1}{2} \sum_e \int_\Omega \lambda \, \mathcal{H}(x, \nabla T, s) dx$$

- Anisotropic model: $m(v_v, \varepsilon, \delta)$ – $A(x,m), B(x,m), E(x,m)$ *closed-form expression*

$$\rightarrow \mathcal{H}(x, \nabla T) = A(x, v_v, \varepsilon, \delta)(\nabla_x T)^2 + B(x, v_v, \varepsilon, \delta)(\nabla_z T)^2 + E(x, v_v, \varepsilon, \delta)(\nabla_x T)^2 (\nabla_z T)^2 - 1 = 0$$

(Alkhalifah, 2003; Waheed, 2014; Tavakoli F., 2017; Le Bouteiller, 2018)

$$\frac{\partial \mathcal{C}}{\partial m}(v_v, \varepsilon, \delta) = \left. \frac{\partial \mathcal{L}(v_v, \varepsilon, \delta, T, \lambda)}{\partial m} \right|_{s,T^*,\lambda^*} = \sum_e \int_\Omega \lambda^*(x) \frac{\partial \mathcal{H}(\nabla T^*(s))}{\partial m} dx$$

$$\frac{\partial \mathcal{H}(\nabla T(s))}{\partial v_v} ?$$

analytical expression easy to compute from $A(x,m), B(x,m), E(x,m)$

# Adjoint PDE from reverse differentiation

$$\left[\nabla.\left(\lambda(x)\frac{\partial \mathcal{H}}{\partial \nabla T}\right) = -\sum_r (T_{obs} - \mathcal{R}[T(s)])^t W_d^t\, \delta(x - x_r)\right]$$

Adjoint/Lagrangian approach: a simple way for differential geometry …

Connection with the so-called reverse differentiation
related to the chain rule for derivatives

Reverse differentiation fashionable concept especially for AI applications
with automatic differentiation approaches

(see illustrations on internet by J.-M. Mirebeau with notebooks)

# Reverse differentiation

## What is reverse differentiation?

*slowness* $u = {}^{1}/_{c}$

**Forward differentiation** (more familiar concept): FD

$$T(x, u + \varepsilon\delta u) = T(x, u) + \varepsilon\, \mu(x, u)$$

linear approximation $\mu$ (tangent information) for the time perturbation

**Reverse differentiation** (maybe less familiar concept): RD

$$T(x, u + \varepsilon\delta u) = T(x, u) + \varepsilon \int_{\Omega} \varrho(x, u)\delta u$$

Linear approximation $\varrho$ (tangent information) for the slowness perturbation over $\Omega$

Numerical **consistent** relation between time perturbation and slowness perturbation! $\mu(x, u) = \int_{\Omega} \varrho(x, u)\delta u$

# FD & RD perturbation versus ray perturbation

**FD:** **slowness perturbation -> time perturbation**

For a given point $x$, $\delta u(x) \rightarrow \delta T(.)$ everywhere

**RD:** **time perturbation -> slowness perturbation**

For a given point $x$, $\delta T(x) \rightarrow \delta u(.)$ everywhere

i.e. time residual at a given station

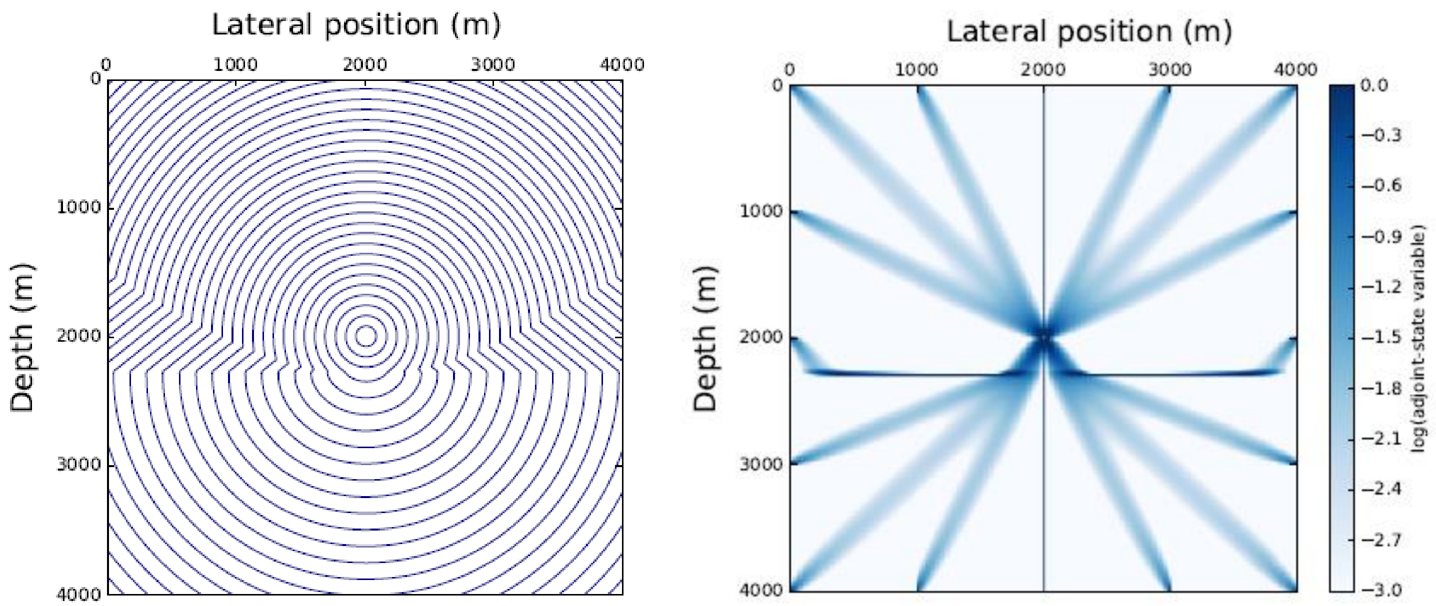
## Ray linear relation

$$T(x, u + \varepsilon \delta u) = T(x, u) + \varepsilon \int_{source}^{receiver} \delta u \, dl$$

Locally, the ray sensitivity is the length of the ray inside the cell of the grid

# Eikonal sensitivity kernel (SK)

(Le Bouteiller, 2019)

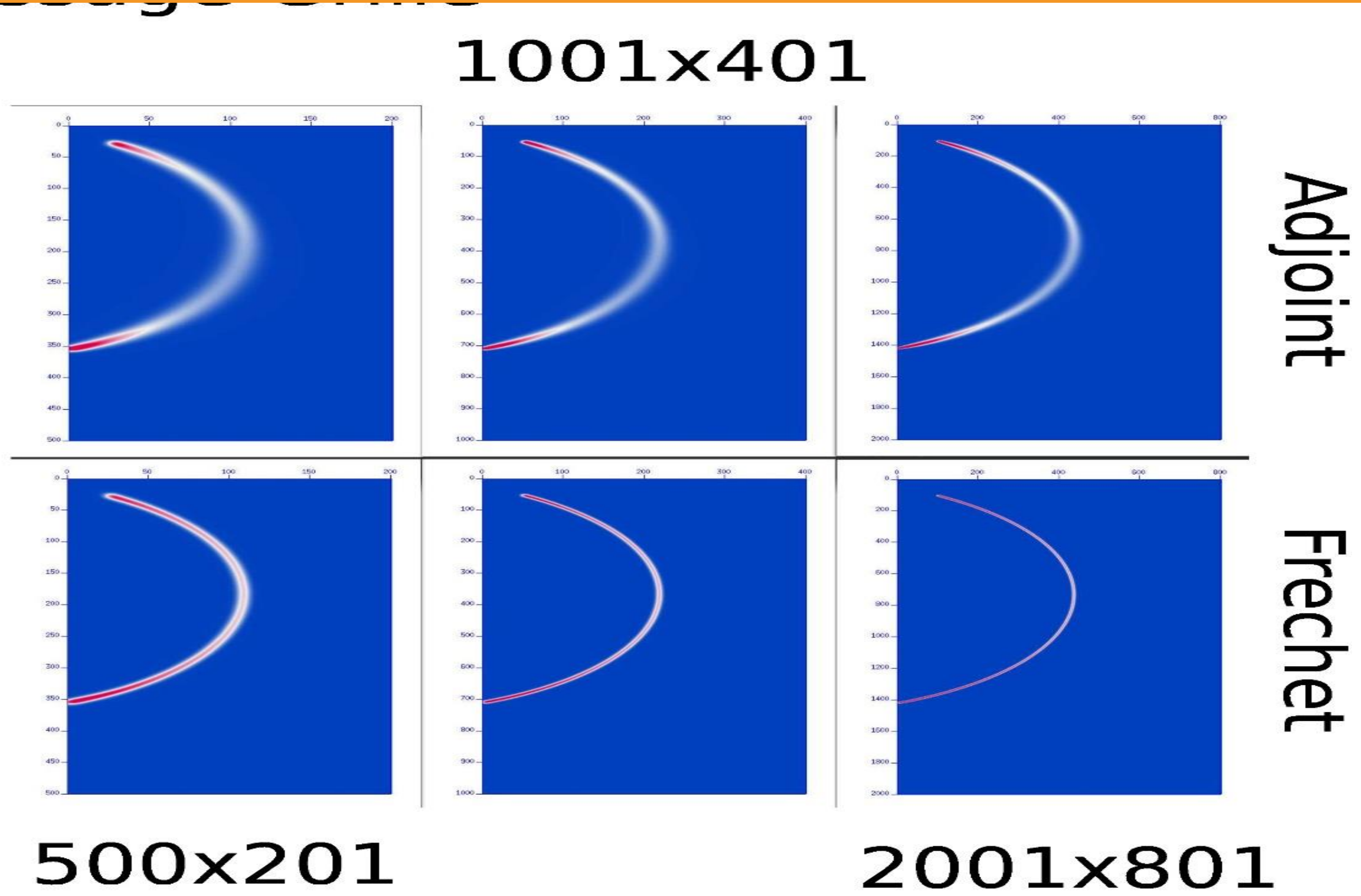



RD:
SK field

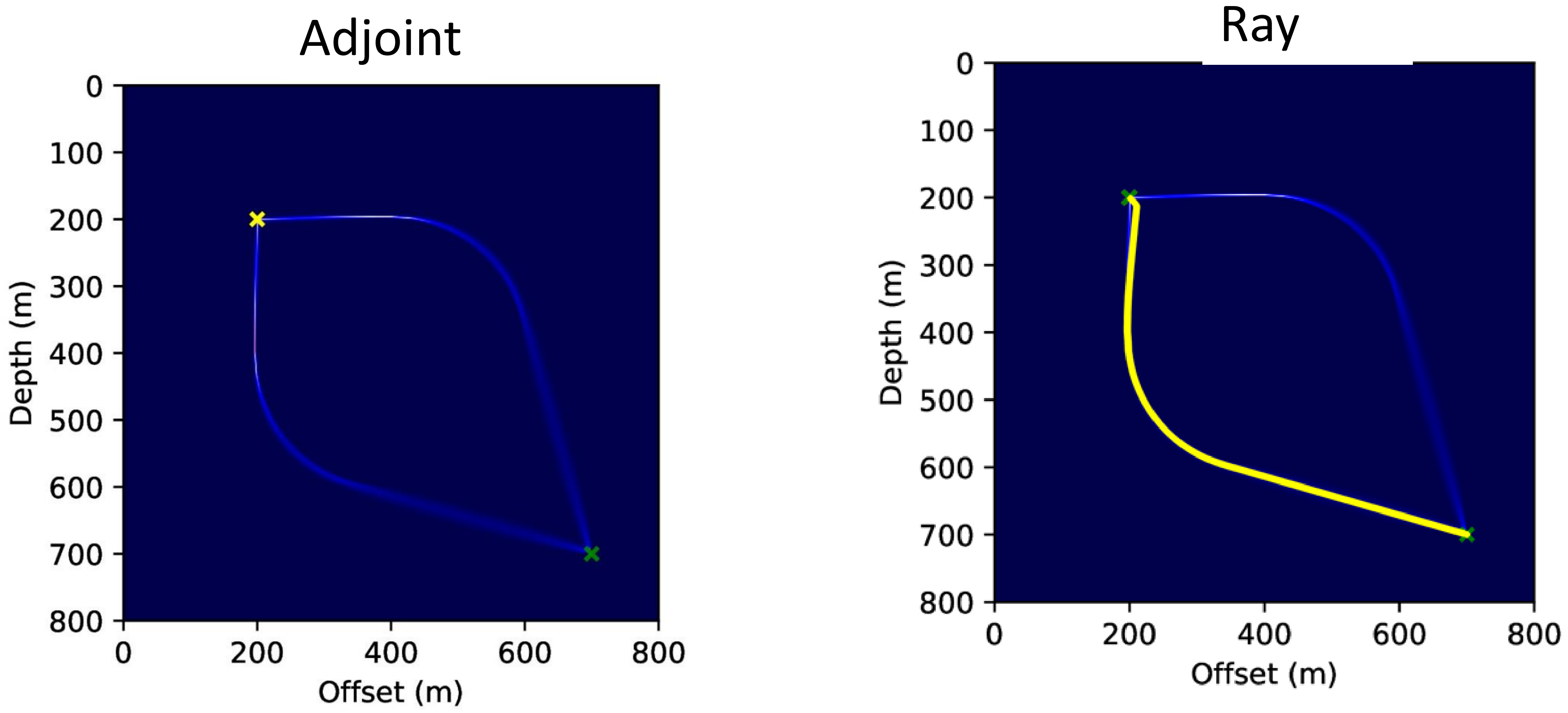
**Reverse differentiation** gives Eikonal sensitivity kernels:
where to insert velocity anomalies to match time data (sum of kernels over receivers) (Taillandier, 2009; Lelièvre et al, 2011; Tavakoli F. et al, 2017, 2019; Sambolian et al, 2019, 2021; Tong, 2021)

Sensitivity kernel defines zones of velocity perturbation affecting the time/phase at the receiver (agnostic to the frequency content of waves!)

# RD linear PDEs



# Cut-locus points




Cut-locus points: points where two rays provide same time:
Mathematical curiosity helping to understand the spreading of adjoint solution
(illustration of  Fermat principle misleading interpretation)

# Explicit or automatic differentiation

Differential geometry: no need to introduce Lagrangian multipliers but such introduction leads to error-free manipulation

However, automatic differentiation based on the *(reverse)* chain rule might ease the manipulation!

# FD & RD linear PDEs

FD: **raw implementation** (linear PDE)

$$\nabla\mu\,(x).\nabla T(x) = u(x)\,\delta u(x)$$

*Linearized Eikonal equation* (Aldrige, 1994; Fomel, 2001; Franklin & Harris, 2001; Alkhalifah, 2002; Alkhalifah & Fomel, 2010; Li & Fomel, 2013)

FD: automatic differentiation (AD) implementation

Reverse chain rule (intermediate results to be saved) (Mirebeau & Dreo, 2017)

RD: **raw implementation** (linear PDE)

$$\begin{gathered}\nabla.[\varrho\,(x)\nabla T(x)] = 0 \qquad in\ \ \Omega \\ s.t. \\ \varrho\,(x)[n(x).\nabla T(x)] = \delta T\ \text{ on } \delta\Omega\end{gathered}$$

*Adjoint transport equation* (Leung & Qian, 2006; Taillandier, 2009; Tavakoli F et al, 2015, Tong, 2021)

RD: automatic differentiation (AD) implementation

Reverse chain rule (intermediate results to be saved) (Mirebeau & Portegies, 2019)

# On the use of FD & RD PDEs solvers

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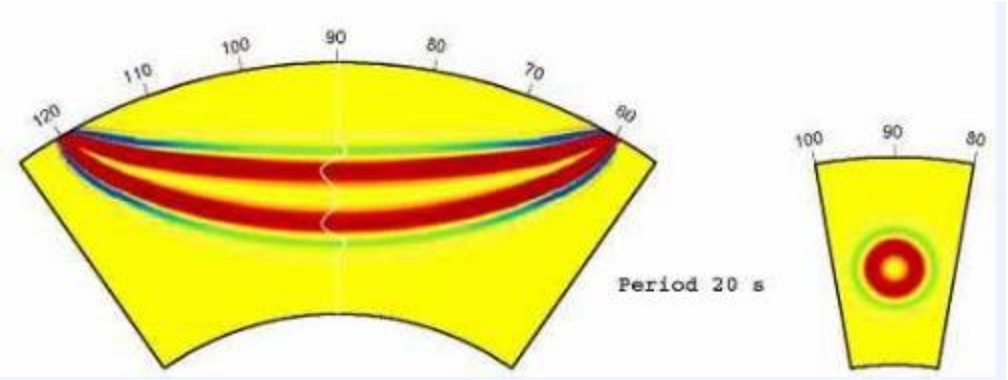
**First-arrival traveltime tomography**: we need

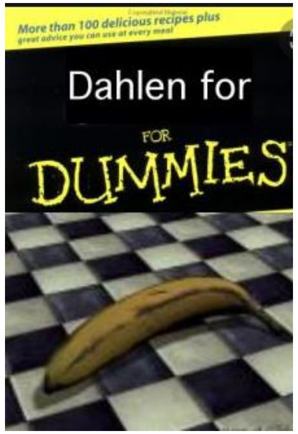
- ➢ Eikonal solver (time; non-linear PDE)
- ➢ RD solver (time SK; linear PDE) Péclet number is infinite

Computer complexity (non-linear, singularities; CPU, memory)…

# Banana-Donut debate

https://www.geoazur.fr/GLOBALSEIS/nolet/BDdiscussion.html




(Dahlen et al, 2000; Dahlen & Nolet, 2005; Nolet, 2008)

Phase/time tomography based on viscous solution does not provide the so-called BD shape promoted by Dahlen & Nolet … !!!

Such shape is related to **ray concept** (and not to the **time/phase concept** for which the zero-sensitivity along the ray does not exist)

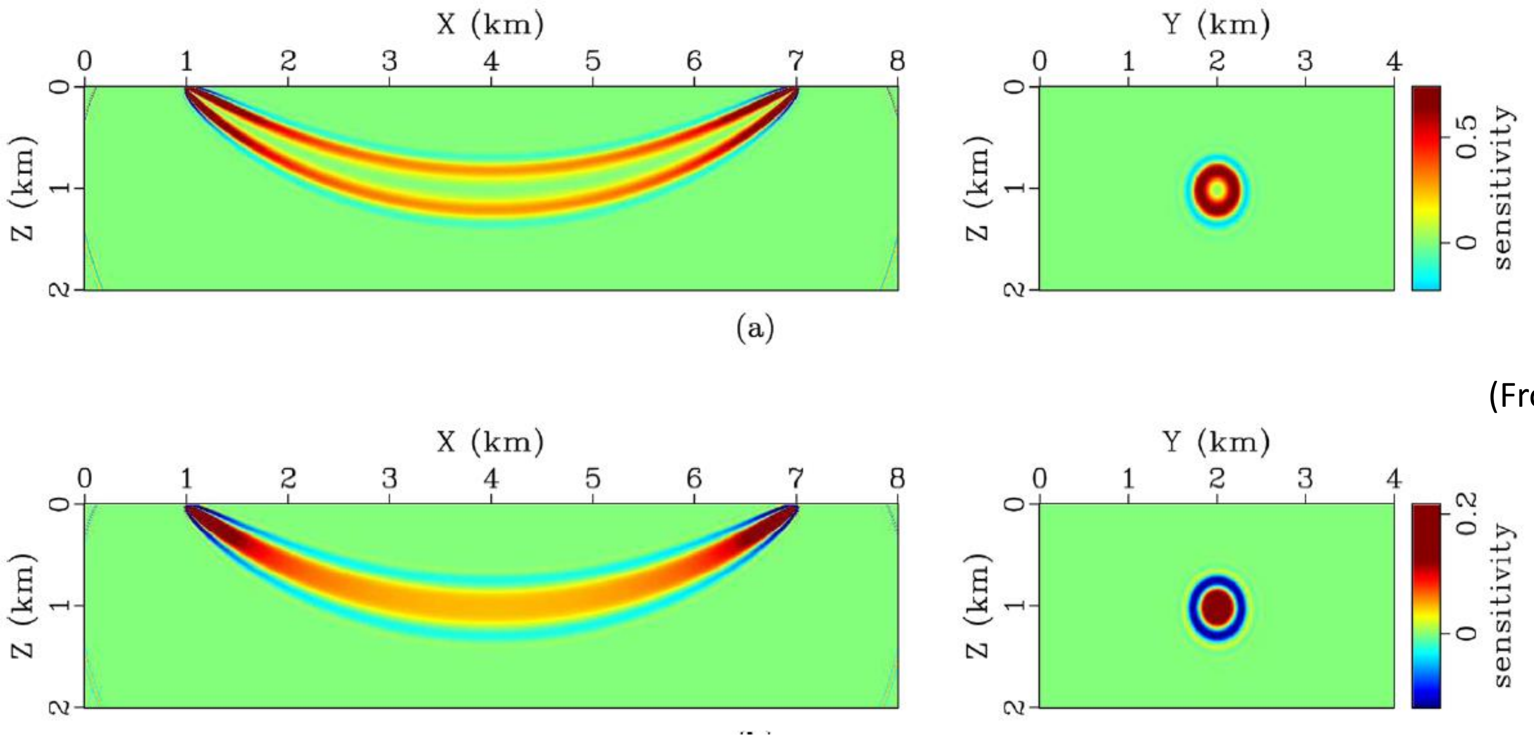
Many authors have tried to understand better this theoretical contradiction …

When considering phase information

**DIFFRACTION EFFECT NOT INCLUDED IN RAY APPROACH**
**DIFFRACTION EFFECT CAN BE INCLUDED IN THE SO-CALLED EIKONAL APPROACH**

RAY SK

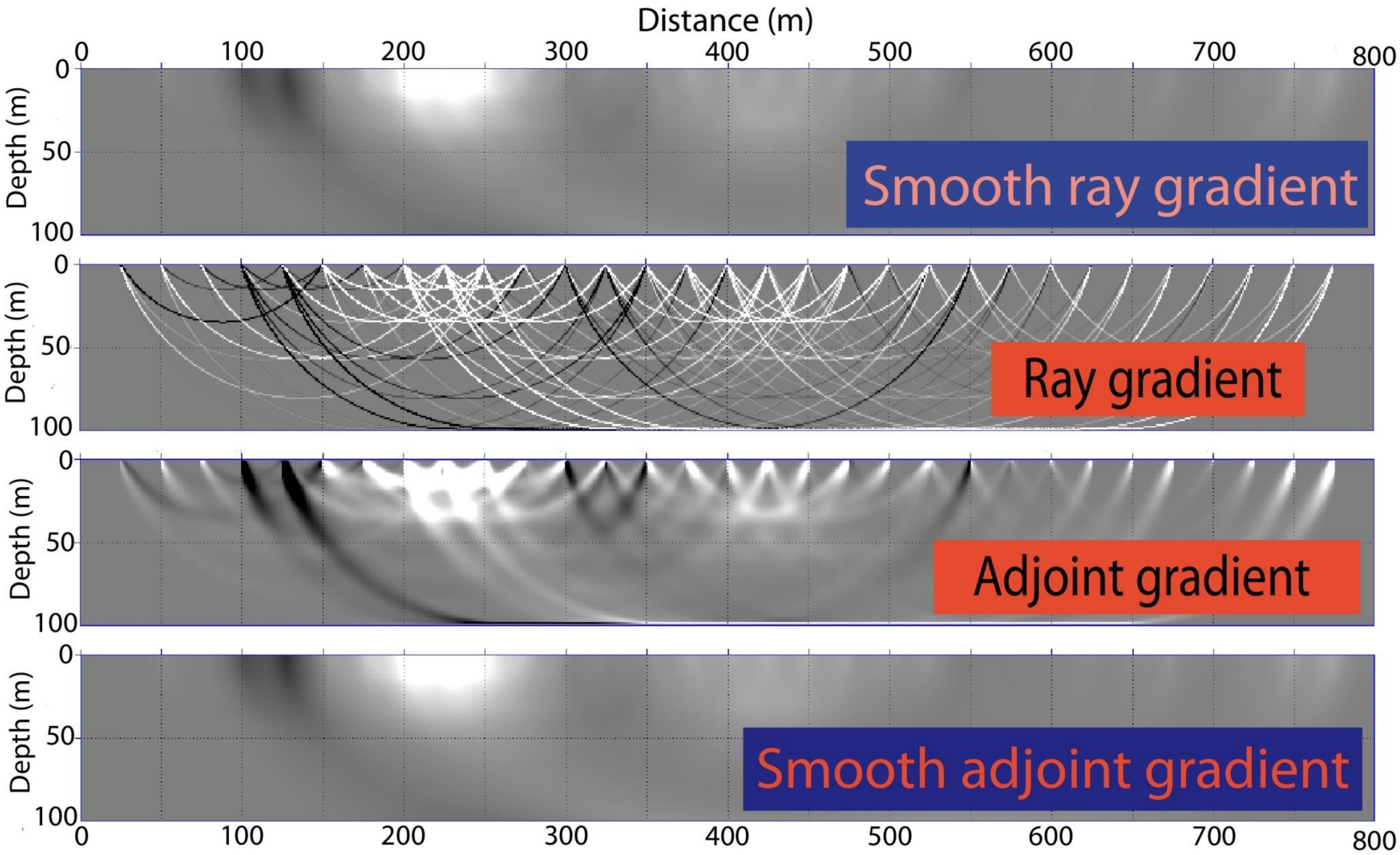
EIK. SK




(From Djebbi & Alkhalifa, 2014)

Outcome of this debate: is it really important for applications?

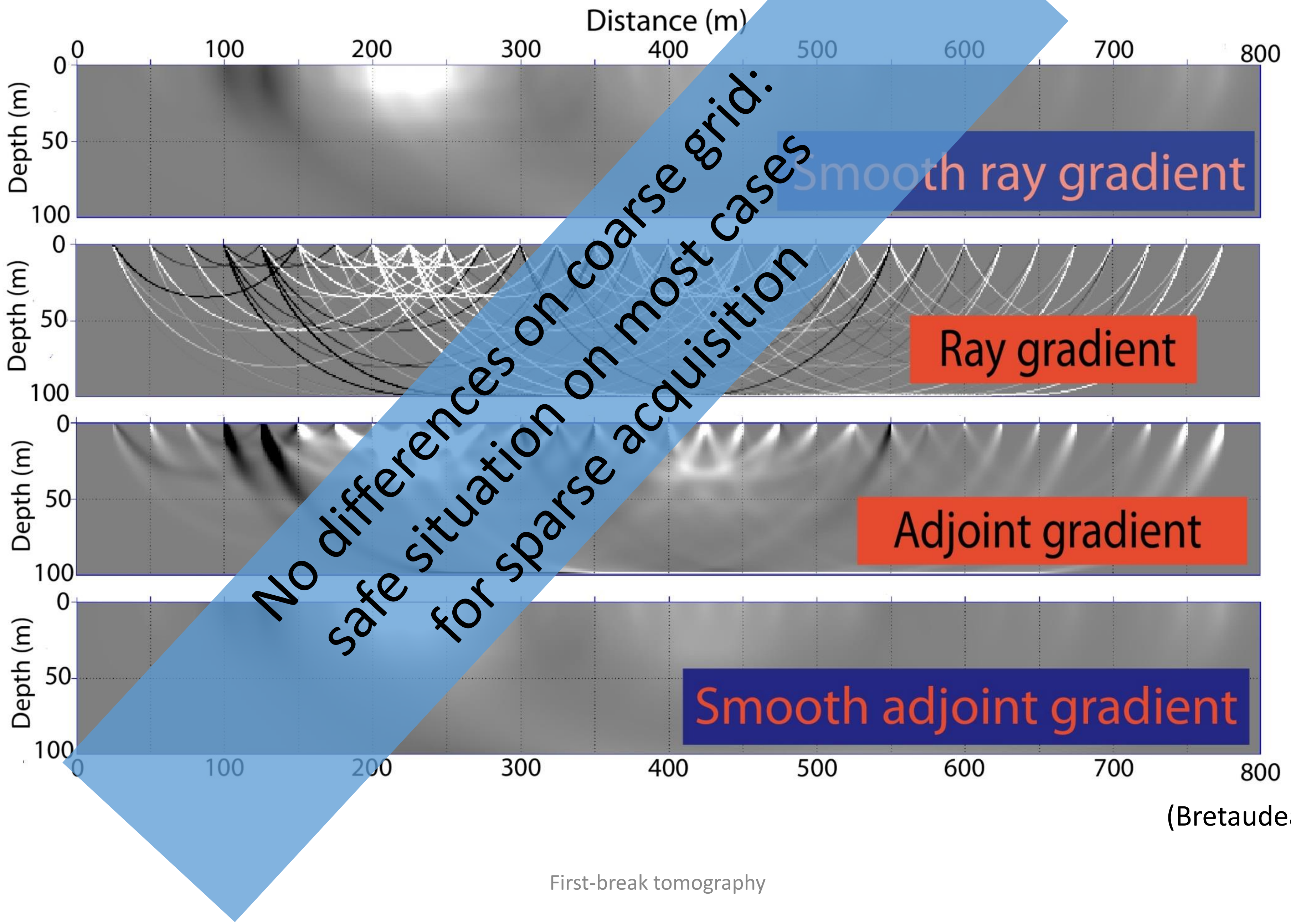
# Projection of gradients on discrete model




Eikonal+Adjoint

(Bretaudeau et al, 2014)

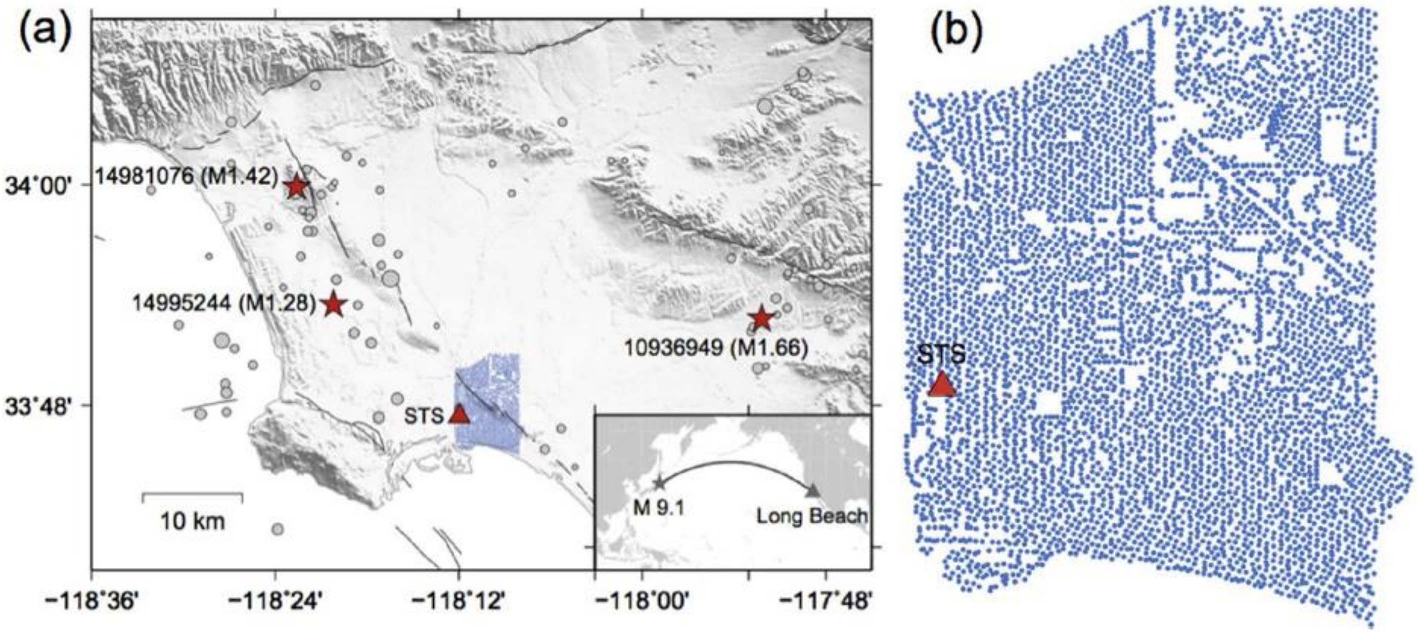
# Projection of gradients on discrete model




(Bretaudeau et al, 2014)

# Ray SK versus Eikonal SK?

Does Eikonal-based tomography provide improved images than ray-based tomography?




How to take benefit of such data density?

My negative answer (2014) has moved to a more positive answer (2019), especially when considering drastic increase of the acquisition density

# Is it still true with array densification?

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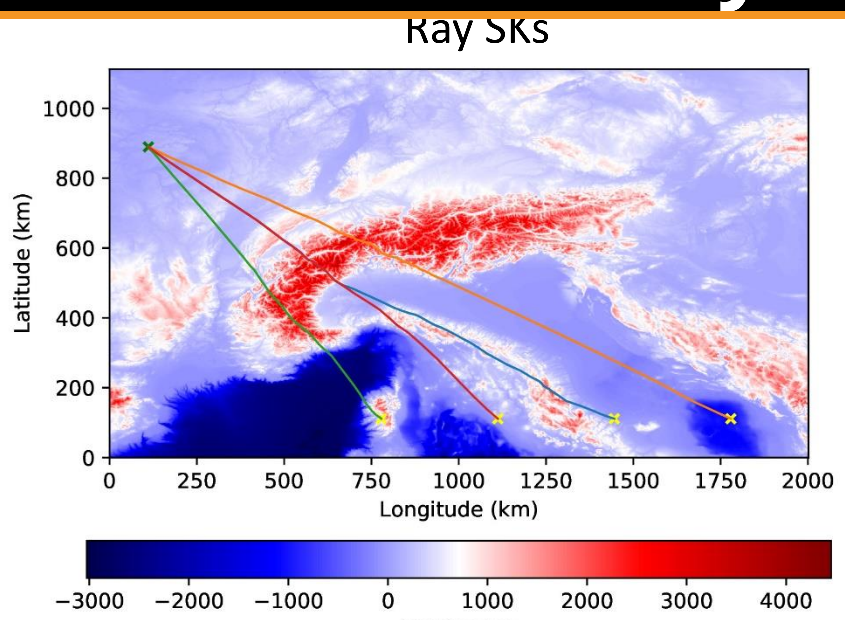
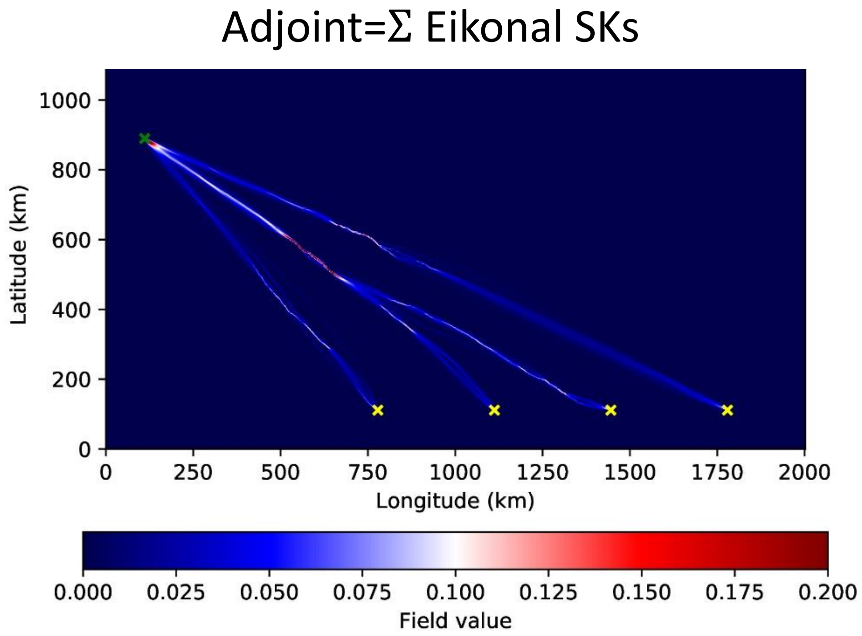


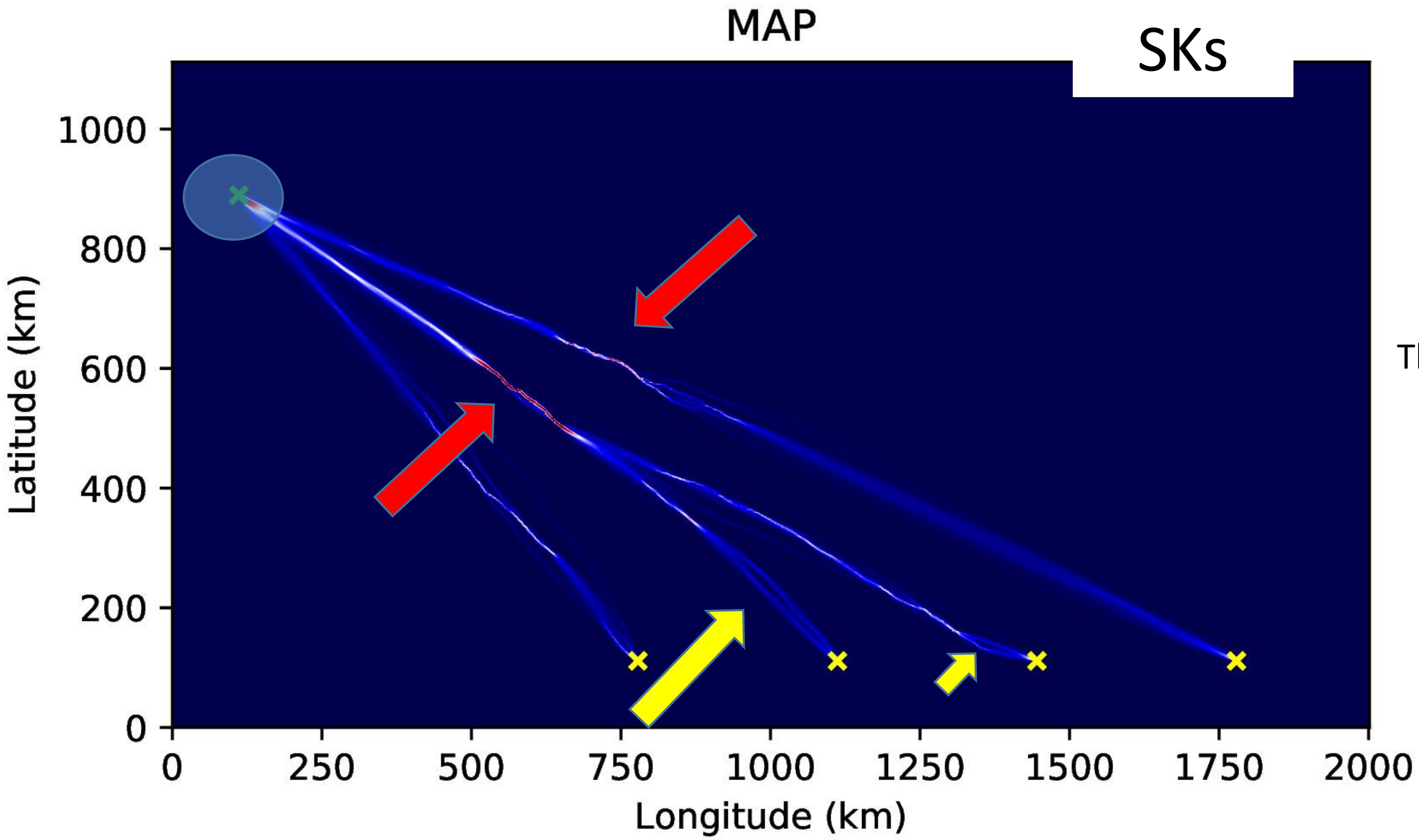

Toy example:
topography taken as velocity perturbation
(mimicking HF pattern)

Rays: same value (ray length) of the sensitivity (velocity impacts only the ray path itself)

Eikonal SK exhibits more complex pattern: variable values along the trajectory !!!

SKs: different values with possible different paths before reaching the station

# Eikonal SK: modulation & multi-paths




modulation

They do not exist in ray approach

multi-paths

# Delayed Eikonal-based algorithm

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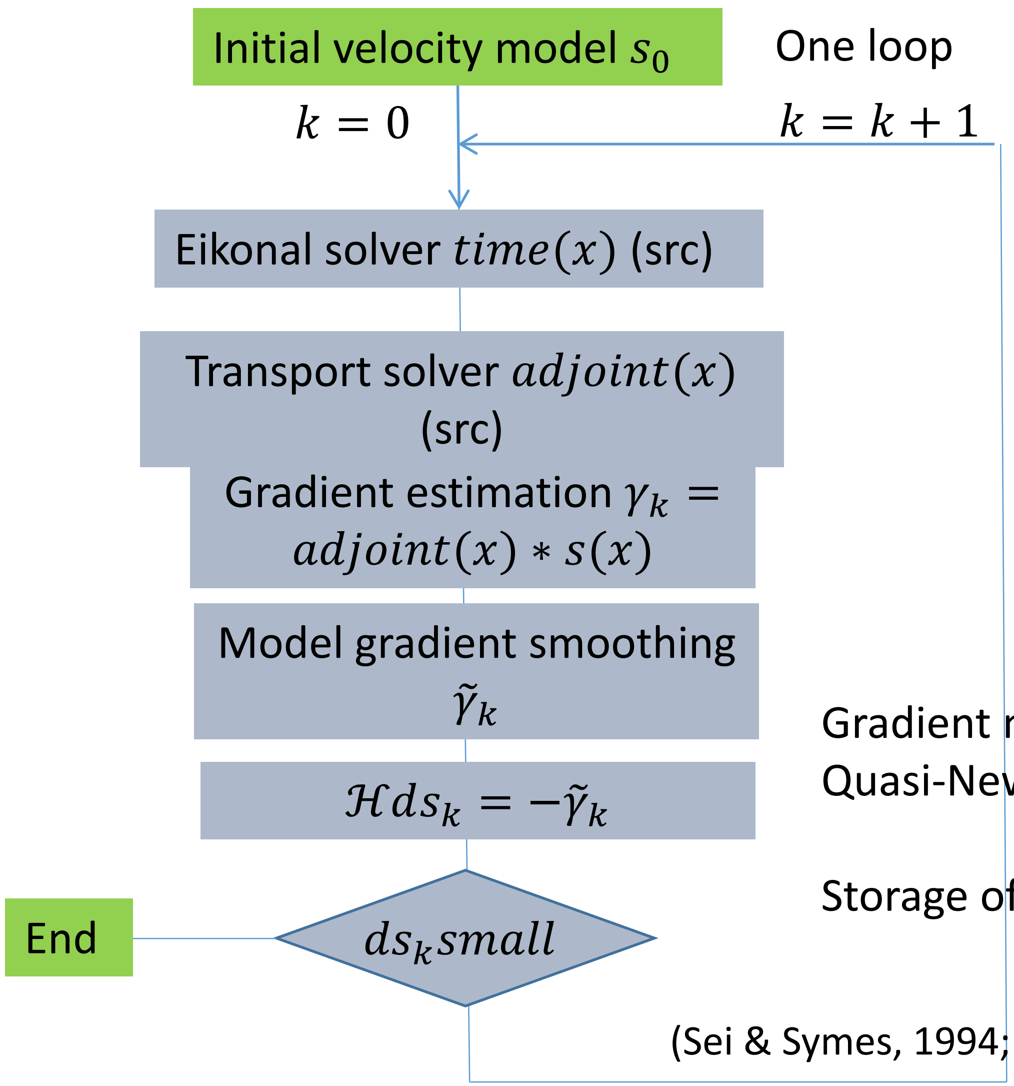
## Linearized Inverse Problem

### One-loop procedure




Gradient methods/
Quasi-Newton methods such as l-BFGS method

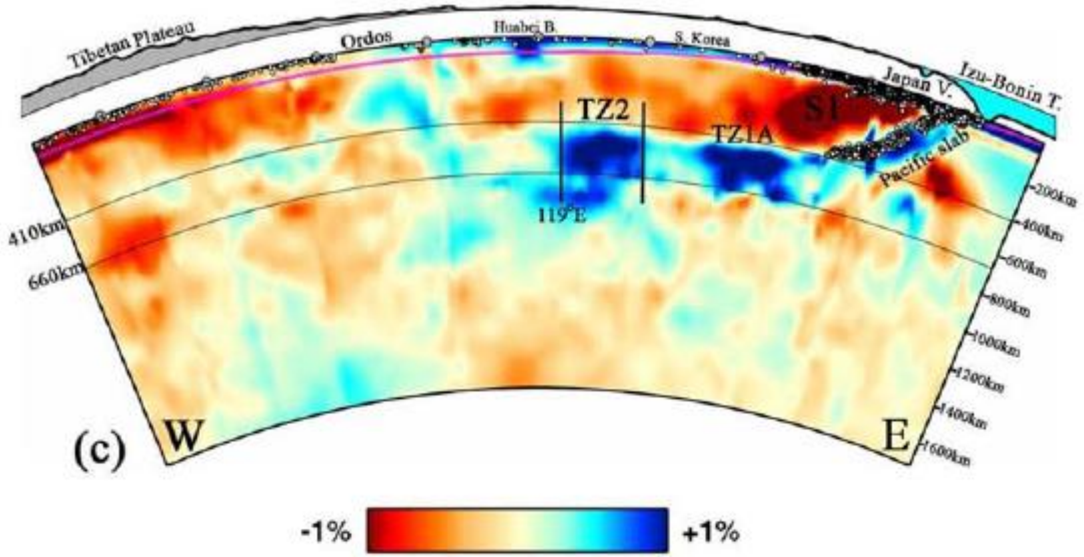
Storage of previous gradients and models

(Sei & Symes, 1994; Leung & Qian, 2006; Taillandier et al, 2009)

# DRT versus DET: spatial difference

## Delayed ray-based Tomography

$$\delta \mathrm{T}(\mathrm{s},\mathrm{r}) = \int u(x(l))dl = \iiint u(x)\delta\big(x - x(l)\big)K(s,r,x)dv$$




Still DRT provides impressive images while we do believe that DET would provide better images in the future, thanks to the densification of the available data.

(Li & van der Hilst, 2010)

## Delayed Eikonal-based Tomography?

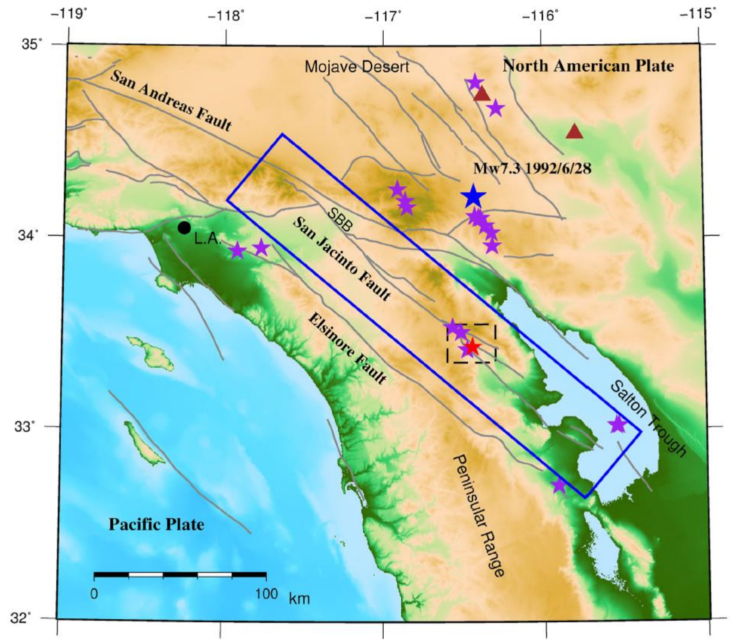
$$\delta \mathrm{T}(s,r) = \iiint u(x)\, K(s,r,x)dv$$

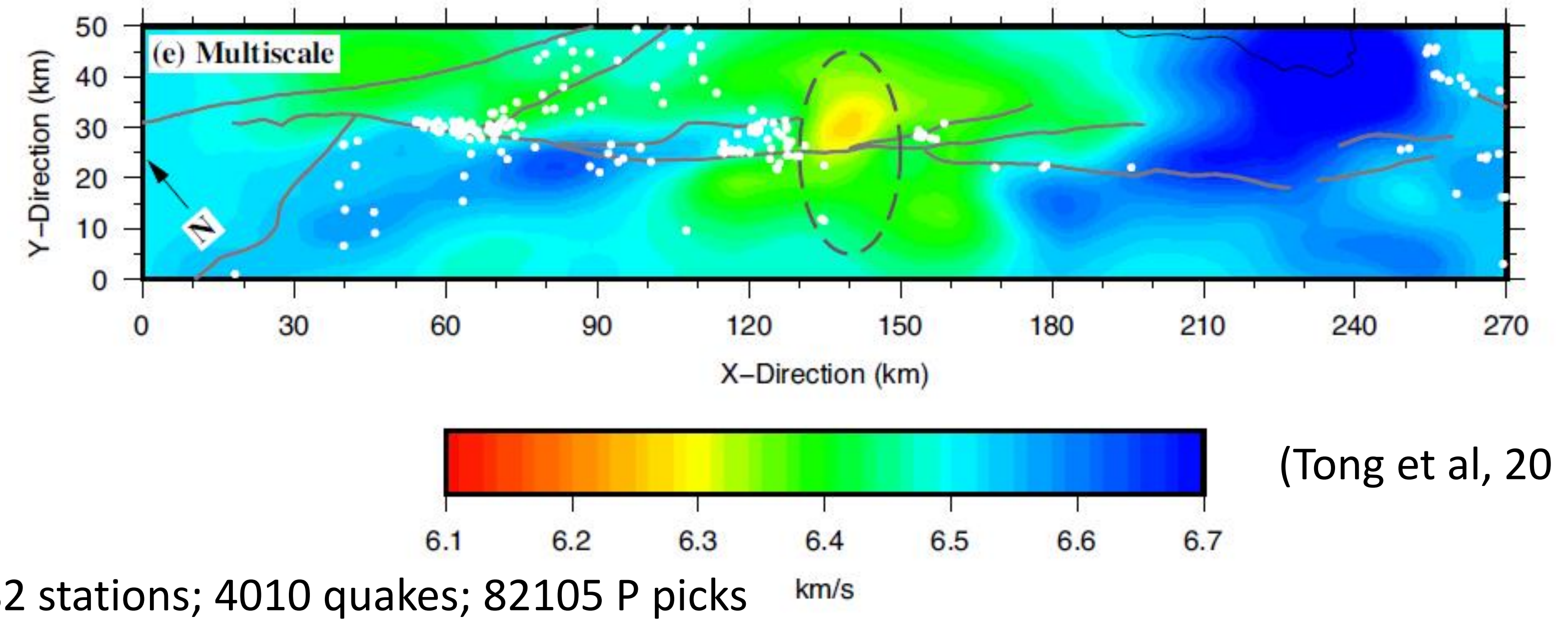
Volumic $K(s,r,x)$ still frequency-independent

# Delayed Eikonal-based Tomography?

$$\delta T(s,r) = \iiint u(x)\, K(s,r,x) dv$$

Volumic $K(s,r,x)$ still frequency-independent




(Tong et al, 2019)

182 stations; 4010 quakes; 82105 P picks

DAS perspective: dense kinematic observables?

# DRT versus DET: winner?

## Delayed ray-based Tomography

$$\delta \mathrm{T}(\mathrm{s},\mathrm{r}) = \int u(x(l))dl = \iiint u(x)\delta\big(x - x(l)\big)K(s,r,x)dv$$

## Delayed Eikonal-based Tomography?

$$\delta \mathrm{T}(s,r) = \iiint u(x)\, K(s,r,x)dv$$

Volumic $K(s,r,x)$ still frequency-independent

Delayed Eikonal-based tomography has the same computational complexity than Delayed Ray-based tomography (asymptotic framework versus high-frequency asymptotic framework).

Both are agnostic to frequency content of seismic waves … **blue-sky information.**

These approaches are needed in order to start wave-equation tomography which is sensitive to frequency content, but which is subjected to cycle-skipping issues. Moreover, wave-equation tomography requires significant computer resources because the wave (elastodynamic) equation has to be solved.

# Outline on first-arrival traveltime tomography

- Images at very different scales
- Waves and Phases: various concepts
- Few points on first-break ray-based tomography
- Illustration on 30-years Western Alps tomography
- First-break eikonal-based tomography
- First-break wave-equation-based tomography
- Hypocenter-velocity joint inversion
- Conclusion

# Wave-equation tomography

## Wave Equation Tomography (WET)

(Woodward, 1989; Luo & Schuster, 1991; Woodward, 1992, Van Leeuwen & Mulder, 2010)

Somehow immune to amplitude effects

- Time windowing strategy for waveform extraction (Maggi et al., 2009)

- Cross-correlation between observed and synthetic waveforms for time shift evaluation (obtained through wave equation solvers: expensive!)

Time delay will be frequency-dependent: for example, 0.6 s time shift between P time in band [2-0.5 Hz] and P time in band (0.1-0.03 Hz) …

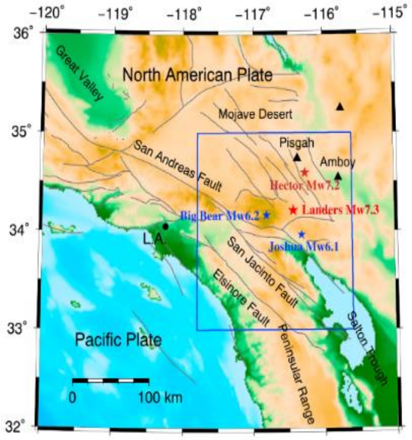
- Starting model should be phase-compatible (could be cycle-skipped)

Interference between phases in the same window – non-linear effects (Nolet, 2008)

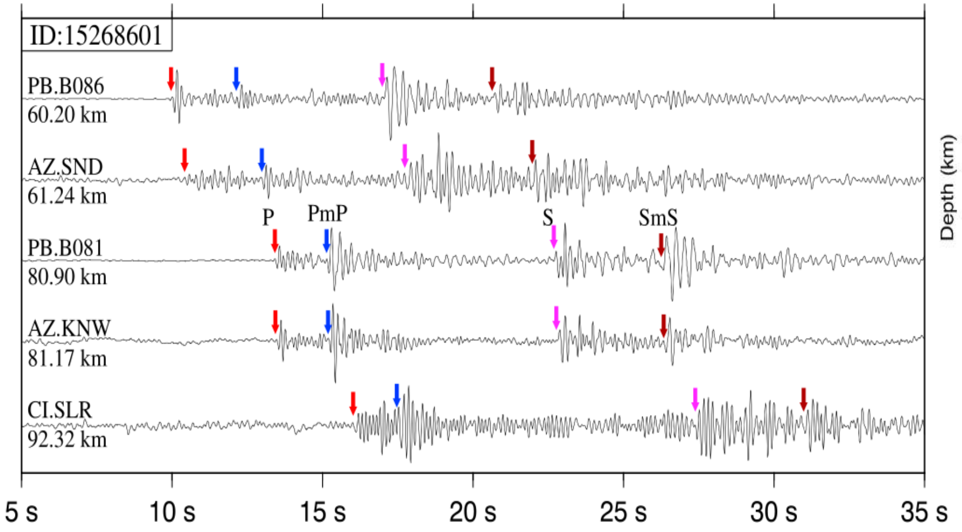
Model « see » the interference and not each phase … in a dynamic way (updated phase delays)

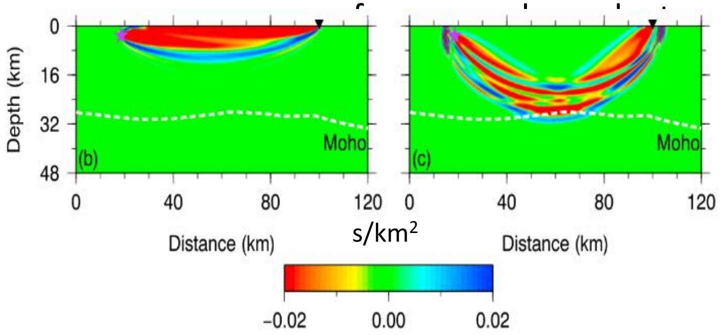
## Wave Equation Tomography (WET)

(more and more used tool in seismology)




(Huang et al, 2016)




Time-domain cross-correlation inside a given window: « traveltime » sensitive to frequency

- Sensitivity kernel from Wave Equation (WE) (Tape et al., 2009, 2010; …)

- Sensitivity kernel from Ray Theory (RT) (Dahlen et al, 2000; Dahlen & Nolet, 2005)

Sensitive to both amplitudes

Sensitive to synthetic amplitudes

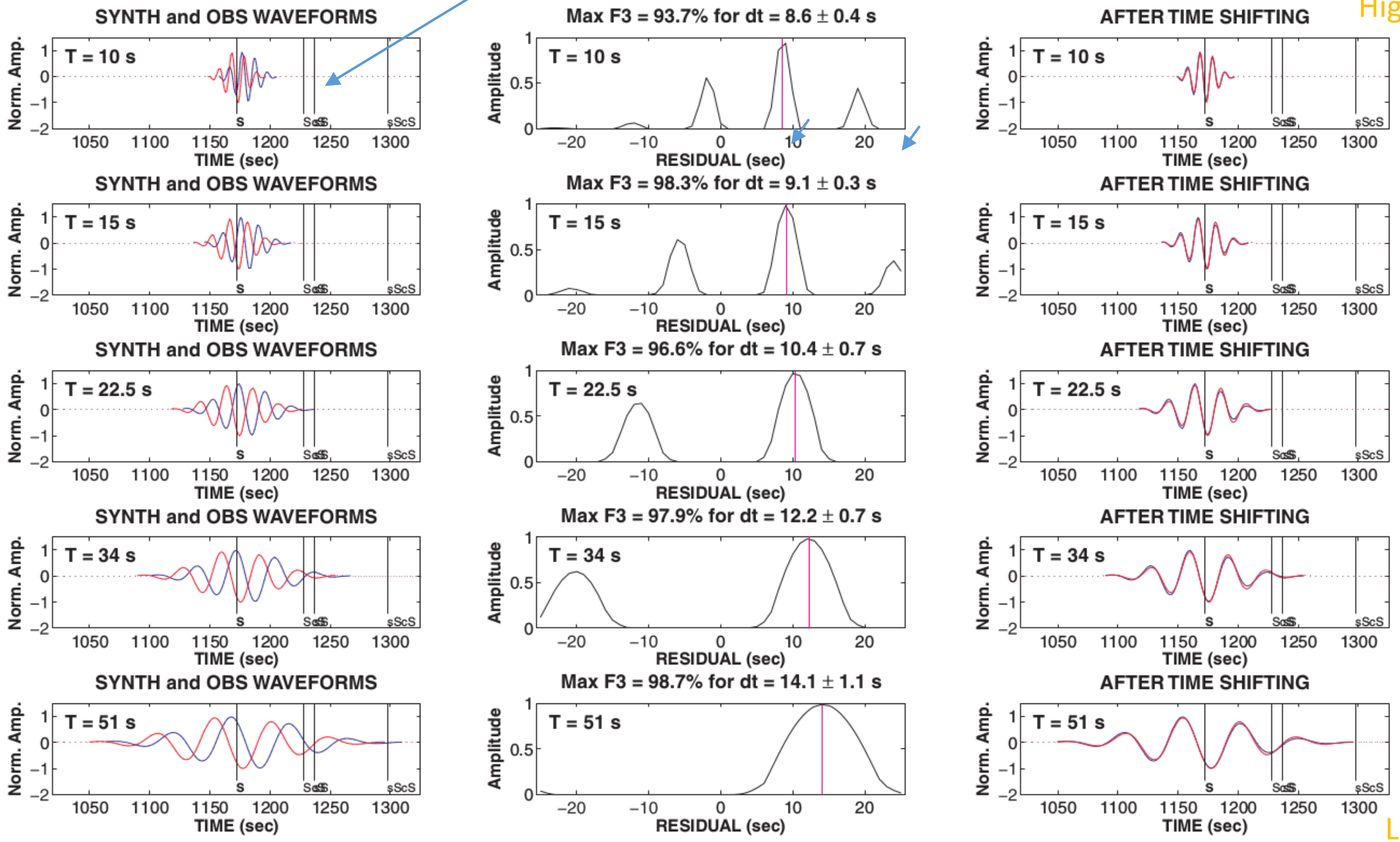
Consistency between finite-frequency data and ray concept?

Born approximation (Lippmann & Schwinger, 1950; Dahlen & Tromp, 1998; Zhao et al., 2000; Zhao & Chevrot, 2011a,b)

# Dynamic phase measurement: Xcorrelation

« Tagging »

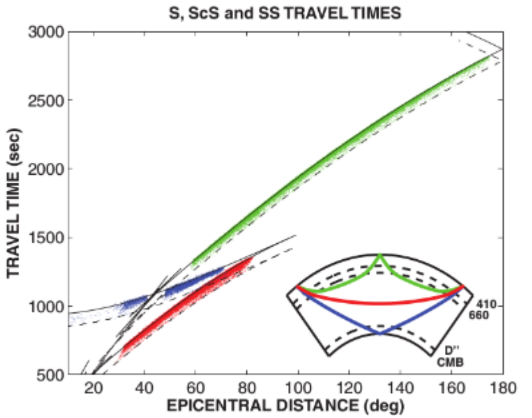
Interference btw ScS and sS

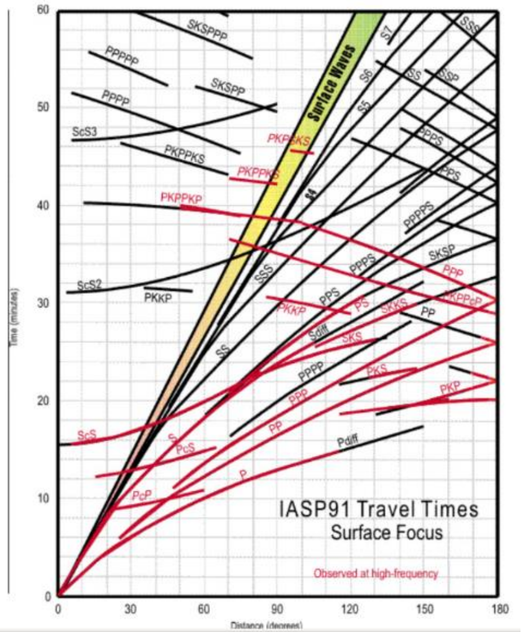



High frequency

Instantaneous phase

Low frequency

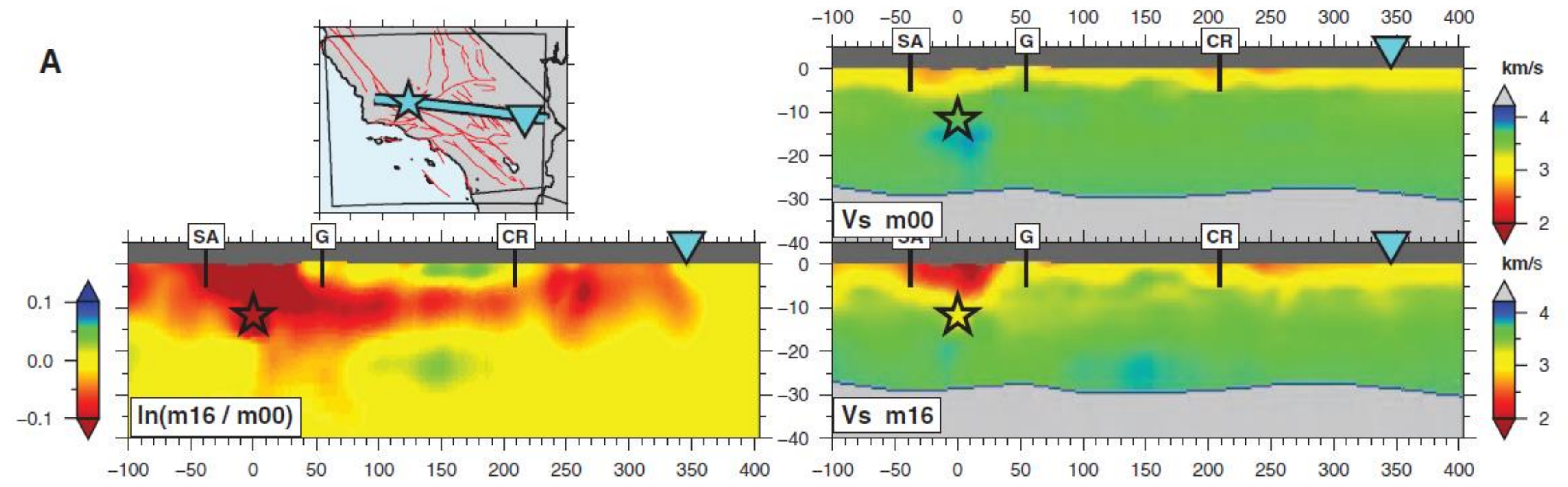



Zaroli et al (2010)

~400 000 phases S, SS, ScS « nearly » automatically

# Wave-equation tomography




SCEC integrated initial model

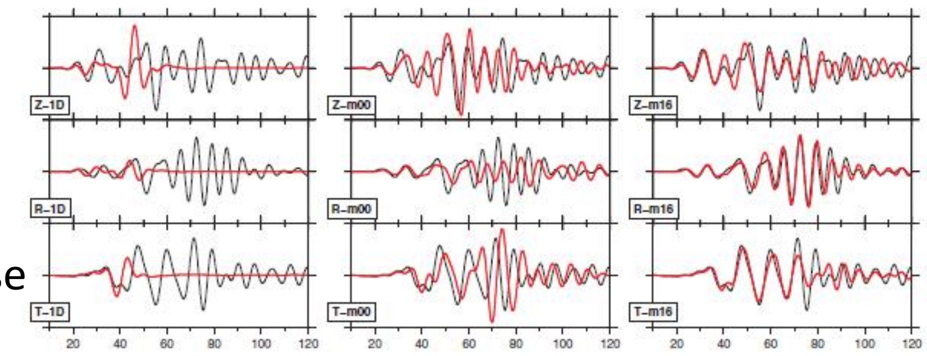
New model at iteration 16

Earthquake database

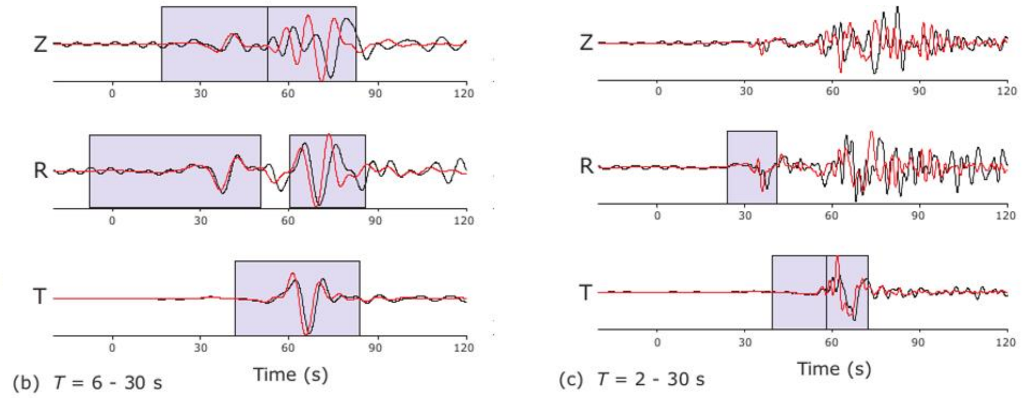
vertical

radial

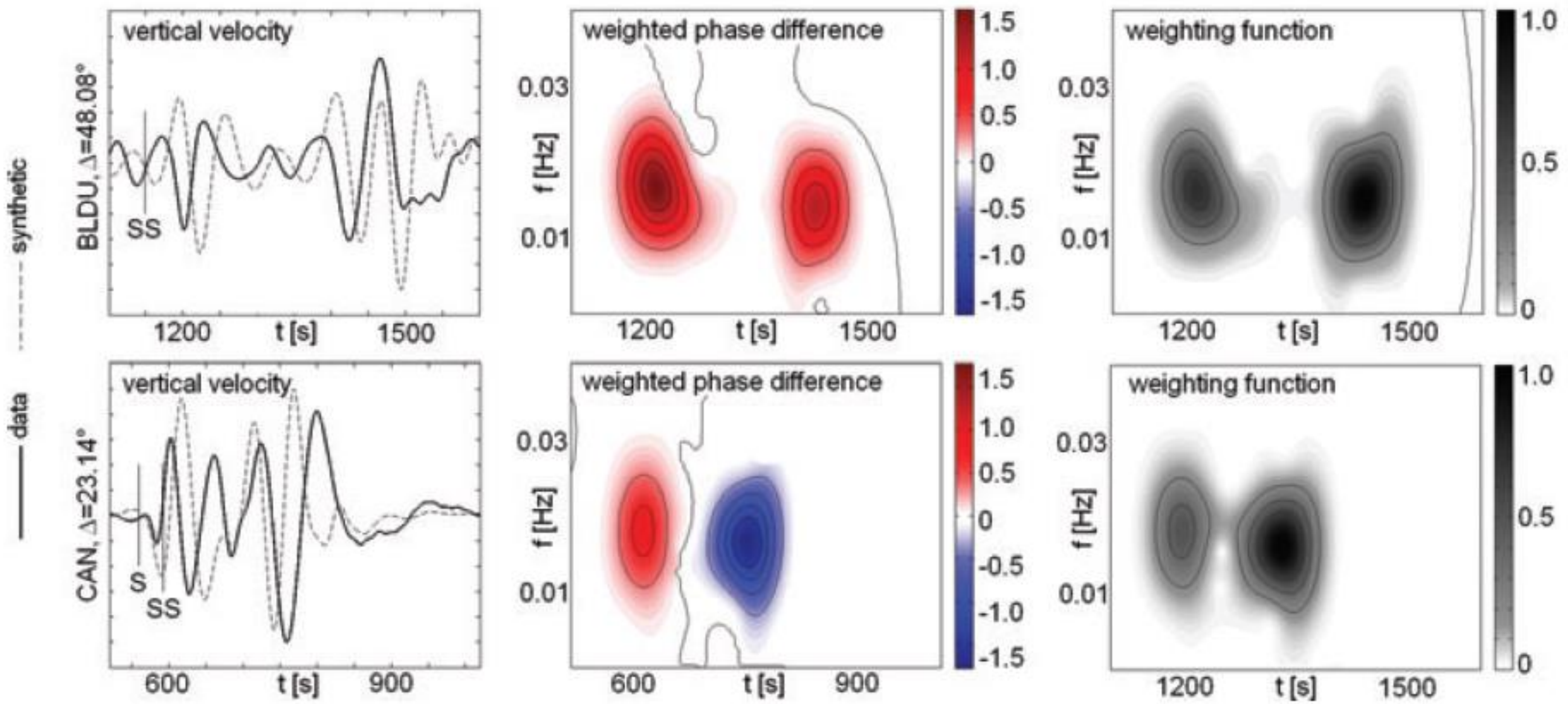
transverse




Phase delay through cross-correlation picks between observed and synthetic waveforms



# WET: instantaneous phase misfit




(Fichtner et al., 2008; Fichtner et al., 2009)

Phases are not picked as for DRT or DET (unless dynamic wrapping)!

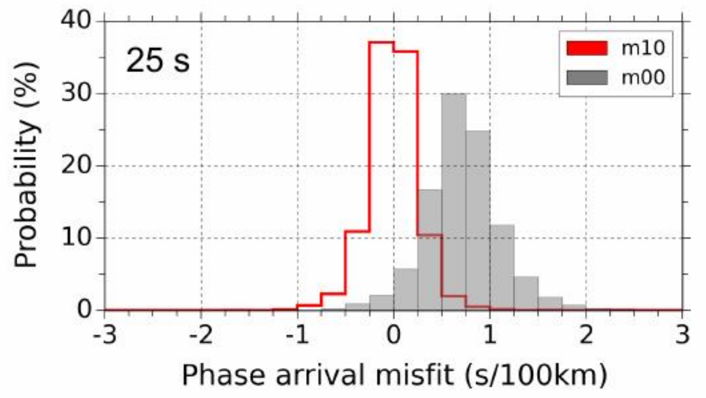
Phase delays are the extracted observables highly correlated to current model (Nolet, 2008)

# Wave-equation tomography (ambient noise)

Real example for Europe using phases cross-correlation and elastodynamic wave propagation

Lu et al (2018)

From ambient noise analysis

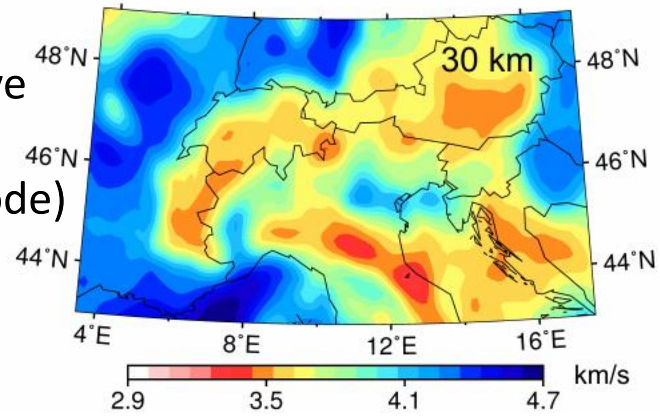



Seismology ☺

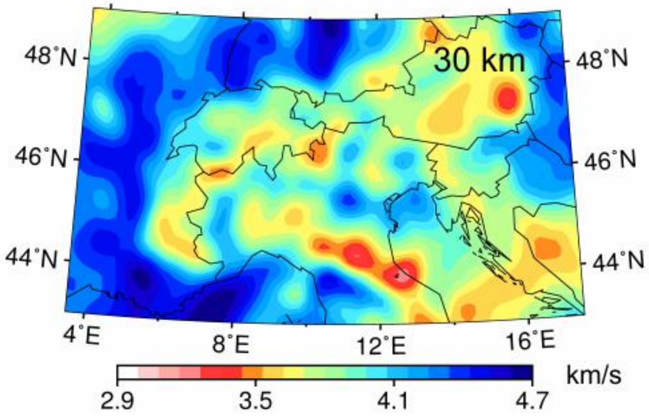
Which phases in WET
When doing ambient noise?

Dispersion curve analysis (fundamental mode)

Initial phase tomography




Wave equation tomography



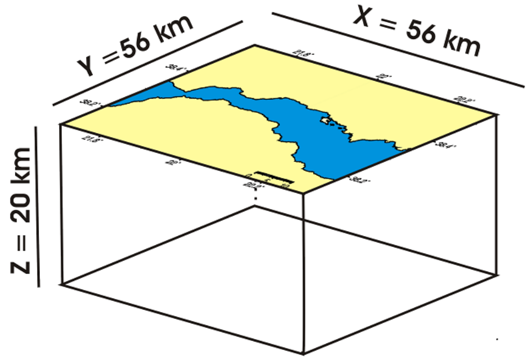

From active acquisitions

Seismic ☹/☺

# Outline on first-arrival traveltime tomography

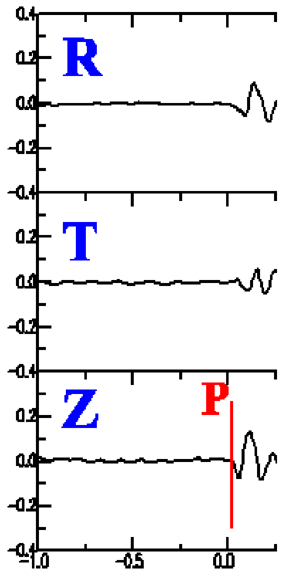
- Images at very different scales
- Waves and Phases: various concepts
- Few points on first-break ray-based tomography
- Illustration on 30-years Western Alps tomography
- First-break eikonal-based tomography
- First-break wave-equation-based tomography
- Hypocenter-velocity joint inversion
- Conclusion

# Hypocenter-velocity inversion



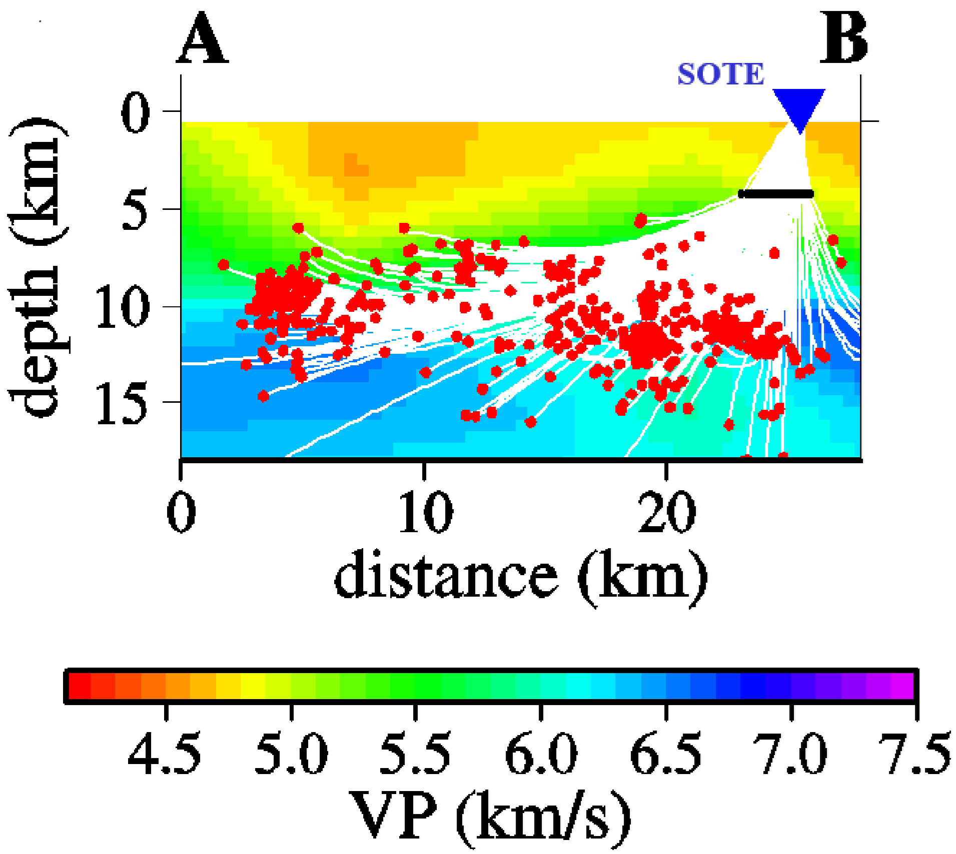

Target zone

Data: picked times




on seismograms

Latorre et al. (2004)




- Reconstruction at the local scale

Hypocenter parameters: $\mathrm{h} = (x_1^h, x_2^h, x_3^h, \tau_0^h)$

Velocity structure: $v(x_1, x_2, x_3)$ (slowness $u$)

- Input time data (~ 40 stations in 90's)

10000-50000 (P and/or S) picked times

- Discretization for local targets

1000-5000 events: few 1000 unknowns

Velocity/slowness values on grid 10x10x10

5.6 km x 5.6 km x 2 km: few 1000 unknowns

Modern seismic network
500-1000 stations are standard
worth revisiting such an inversion

# Multi-parametric inversion

Travel time inversion is a non-linear problem which is solved through linearized system

$$r_k^i = (t^{obs} - t^{syn}) = \sum_{l=1}^{3} \frac{\partial T_k^i}{\partial x_l^i} \delta x_l^i + \delta \tau_0^i + \int_i^k \delta u \, dl$$

where time residual $r_k^i$ from event $i$ at station $k$

and integral $\int_i^k \delta s dl$ is computed along the ray from event to station

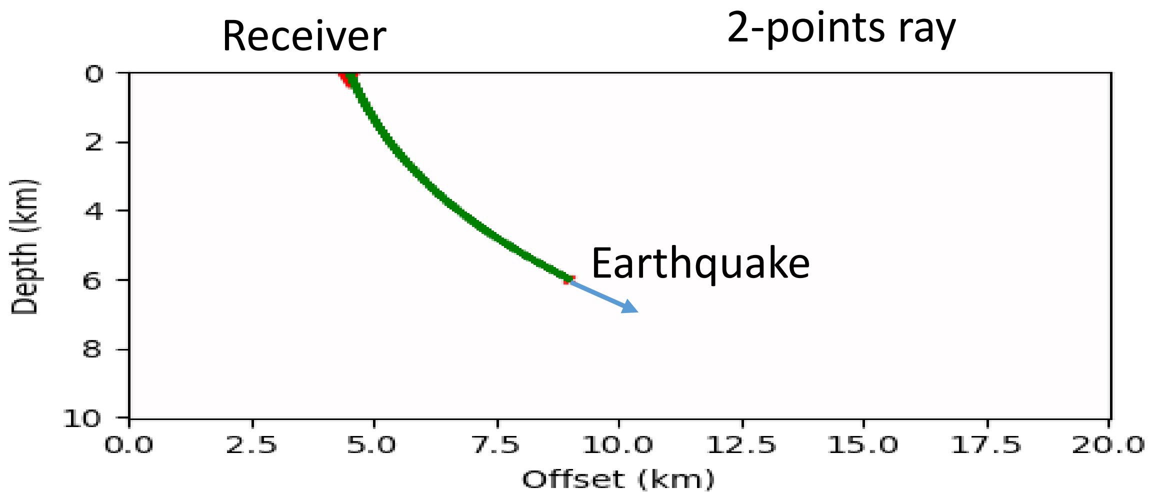
Discrete formulation of the slowness $u$

$$r_k^i = \sum_{l=1}^{3} \frac{\partial T_k^i}{\partial x_l^i} \delta x_l^i + \delta \tau_0^i + \sum_{j=1}^{m} \delta u_j \frac{\partial T_k^i}{\partial u_l^{i,k}}$$

Wanted parameters $\delta x_l^i, \delta \tau_0^i, \delta u_j$

Linear system at each hypocenter/slowness update

$$\frac{\partial T}{\partial h} dh + \frac{\partial T}{\partial u} du = r$$



# Multi-parametric inversion

SEISCOPE

## Travel time inversion is a non-linear problem solved through linearized system

Linear system at each hypocenter/slowness update

$$\frac{\partial T}{\partial h} dh + \frac{\partial T}{\partial u} du = r$$

All unknowns have to be inverted all at once

☹ Multi-parameter inversion
(cross-talk difficult to mitigate)

One can try an alternate inversion by locating first quakes in the current model and then by inverting slowness while keeping events fixed.

As shown by Thurber (1992) and Roecker et al. (2006), slowness perturbation will minimize any changes in hypocenter locations.

Annealing (SVD) strategies proposed by Pavlis & Booker (1980), Spencer & Gubbins (1980) or Rodi et al (1981) do not behave correctly.

Scaling & normalization are required: many numerical trials are needed

# Three possible strategies

- Alternating strategy (Stork a Clayton, 1986)

$$\underset{h}{\operatorname{argmin}} \left\| d(u,h) - d^{obs} \right\|_2^2 \quad \fallingdotseq \quad \underset{u}{\operatorname{argmin}} \left\| d(u,h) - d^{obs} \right\|_2^2$$

Strong cross-talk

- Joint inversion (Bishop et al., 1985)

$$\underset{u,h}{\operatorname{argmin}} \left\| d(u,h) - d^{obs} \right\|_2^2$$

Scaling model parameter and Fréchet derivative

- Variable projection (Golub & Pereyra, 2003)

Consistency (Chauris et al., 2002; Guillaume et al., 2008)

$$\underset{u}{\operatorname{argmin}} \left\| d(u,h(u)) - d^{obs} \right\|_2^2$$

Subproblem to be performed

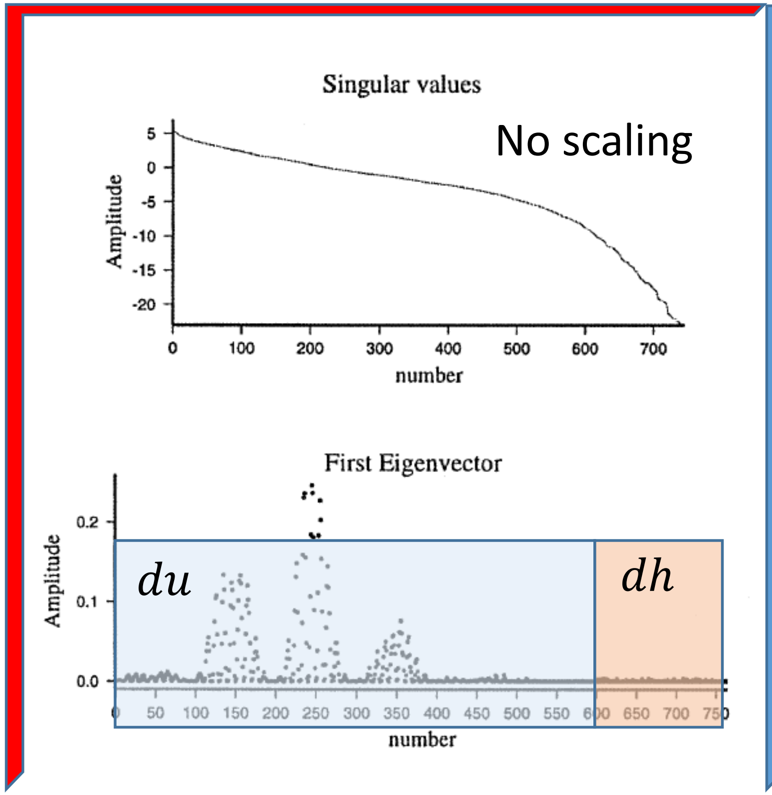
# Different sensitivities

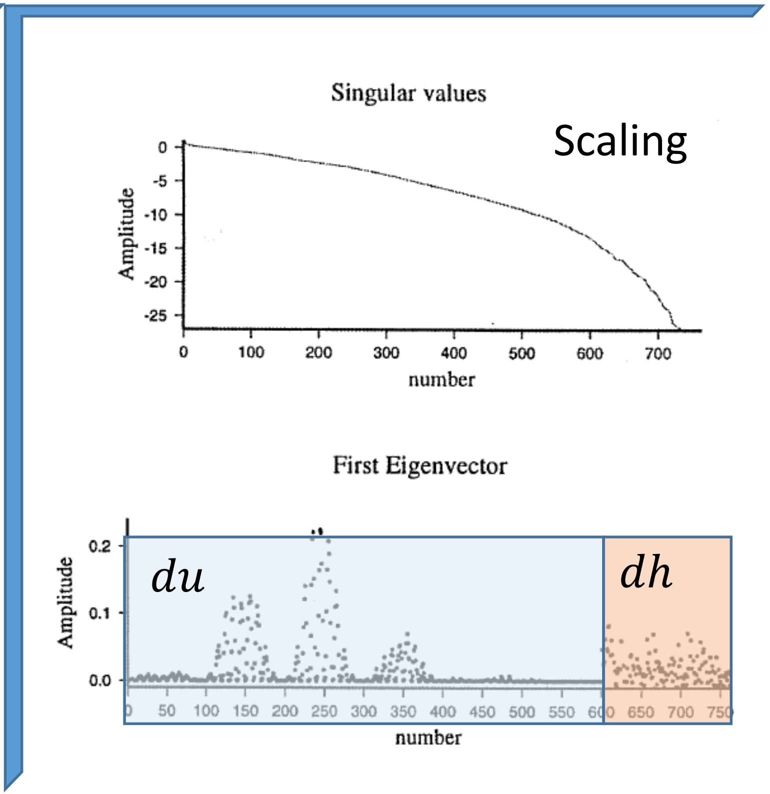
$$\frac{\partial T}{\partial u} du + \frac{\partial T}{\partial h} dh = r = J \begin{bmatrix} du \\ dh \end{bmatrix}$$

SVD analysis of Jacobian $J$
(Lemeur et al, 1997)

All softwares include some kind of scaling/normalization between these # classes of parameters …

Aside potential slow convergence, experience is required from users




Could we do it better?
Yes, we can …

$$\left(\frac{\partial T}{\partial u} + \frac{\partial T}{\partial h}\frac{\partial h}{\partial u}\right) du = r = \frac{dT}{du} du$$

Single class: slowness values

# One hypocenter hypothesis?

Does it make sense to locate a pointsize quake in an inaccurate velocity structure?

Do split the quake database into few smaller databases enabling the location of virtual hypocenters: instead of one hypocenter for an earthquake, we have as many as the number of database subsets.

Updating the velocity structure such as these virtual hypocenters move to an unique position.

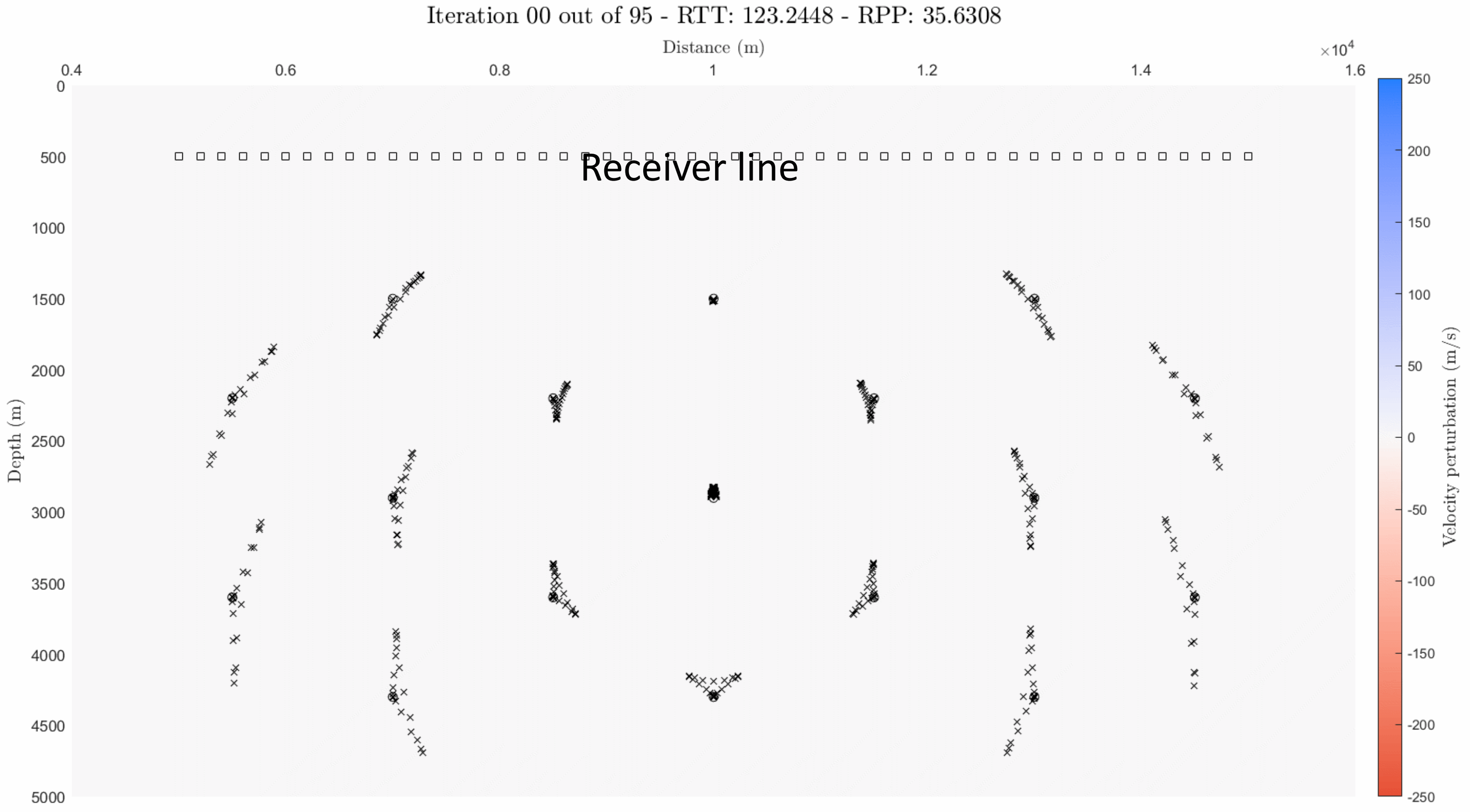
Question: is it a feasable strategy?

Maybe when considering similar concept from slope tomography **(Sambolian et al., 2021)**

# Simple example: event focusing

Circular velocity anomaly (250 m/s)

Earthquakes are recorded at all stations




In principle, it may work!!!

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# Take-away message

Method-oriented message

- Delayed ray-based and eikonal-based tomography **agnostic to frequency content**: user must drive these tools with this external information to be designed
  - Both approaches **same computational complexity**
  - Useful for interpretation **(trajectories)**
  - Hypocenter-velocity inversion
  - Useful for **uncertainty quantification after WET**

- Delayed wave-equation-based tomography is **sensitive to frequency content** and need a **fair prediction** of the synthetic waveform for doing cross-correlation between windowed traces
  - **Significant increase** in computational complexity
  - Hypocenter-velocity wave-equation based tomography **missing!**

# Take-away message

<u>Application-oriented message</u>

Joint hypocenter-velocity inversion requires **adhoc sensitivity balance** between velocity parameters and hypocenter parameters: no consistent approach exists as far as I know.

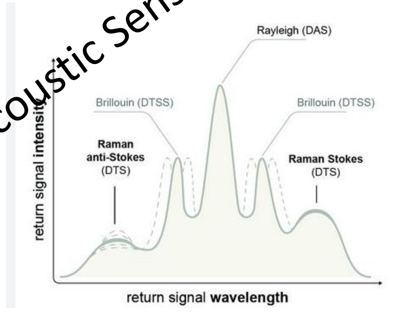
**Initial model** (and initial earthquake locations to lesser extent) design important!

**Grid/Laplacian smoothing** and **model gradient smoothing** play different roles

Increasing the **acquisition density** may sample better the near-surface zone

DAS is expected to be a game changer … in the near future???

Distributed Acoustic Sensing



# End …

Questions?

*Thank you very much*
*For your attention*


Many thanks to sponsors of the SEISCOPE consortium

# Fermat principle to Eikonal equation

First-arrival traveltimes follow Fermat principle of minimum time along any trajectory connecting the starting point and the end point.
The related variational problem can be written

$$\delta \int u(l)dl = 0$$

where $u(l)$ is the slowness at a given point and the curvilinear coordinate at this point is given by the quantity $l$.

The Eikonal equation can be obtained from the wave equation. It can also be thought of as the Hamilton-Jacobi equation of the above variational problem

(Kalaba, 1961 (isotropic case); Brandstatter, 1974 (anisotropic case)).

following slides are inspired from paper by Lakshminarayanan and Varadharajan (1997)

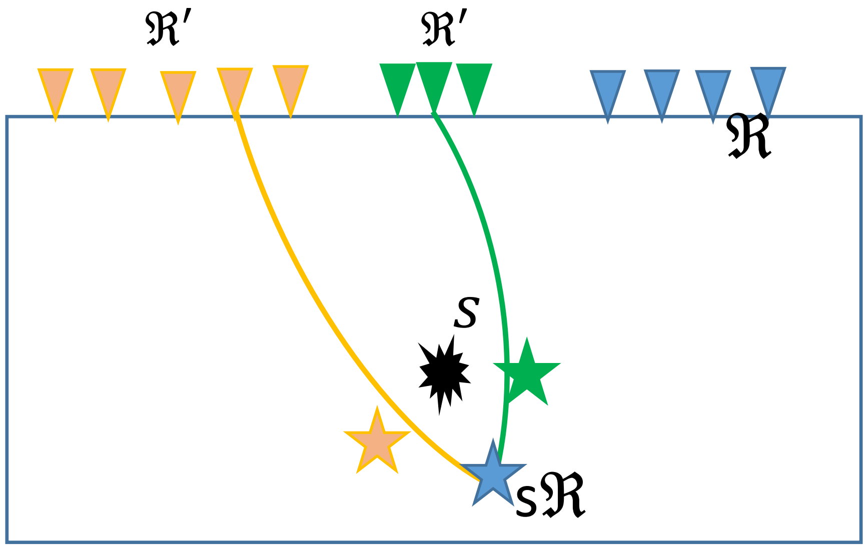
# Virtual location parameters

For an event, many picks when dense acquisition

Subdivision of this dataset into subsets which induces a cosubset for each subset

Locate virtual hypocenter for each subset

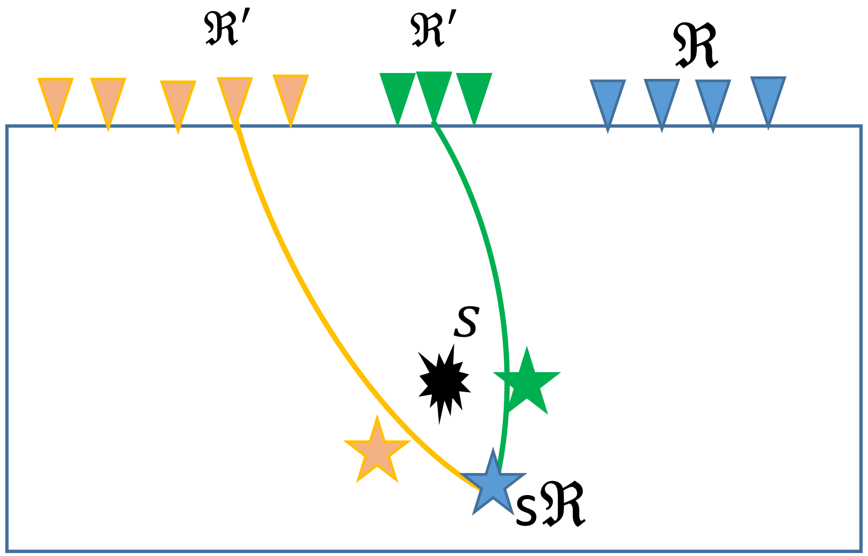
Evaluate residues at cosubset stations




True location S in black
Three virtual locations (blue, green, orange)
Residues of virtual blue location at blue and green stations (cosubset)

# Adjoint variables of location parameters

Projection of location parameters
into adjoint variables
through misfit expression




Adjoint variables ($\alpha_{sr}$) of the blue subset can be computed from residues at cosubset stations

$$\delta t_{sr,r'} = t_{sr,r'}(u) - t_{s,r'}^{obs}$$

$$\alpha_{sr}\partial_{x_{sr}} t_r(x_{sr}, z_{sr}) = \sum_{\mathfrak{R}', \mathfrak{R}' \neq \mathfrak{R}} \delta t_{sr,\mathfrak{R}'}\, \partial_{x_{sr}} t_{\mathfrak{R}'}(x_{sr}, z_{sr})$$

# Eikonal PDE and adjoint PDE

Solving a non-linear PDE ☹

and a linear PDE ☺

(Isotropic case)

- Hamilton-Jacobi equation $\mathcal{H}(x, z, \nabla t_r(\mathrm{x},\mathrm{z})) = \frac{1}{2}(\nabla t_r(x,z)^2 - u^2(x,z))$

  *Forward mode*

- Transport equation $\nabla.(\lambda_r(x,z)\nabla t_r(\mathrm{x},\mathrm{z})) = \sum_s \sum_{r'=1,r'\neq r}^{N_r^S} (t_{sr',r} - t_{s,r}^{obs}) + \sum_s \alpha_{sr}$

  *Reverse mode*

time residues

Indirect contribution of location

missing term
when keeping earthquakes fixed

Two possible algorithms for each receiver

- Fast Marching Method (Vidale, 1988; Tsitsiklis, 1995)

Or

- Fast Sweeping Method (Tsai et al, 2003; Zhao, 2005) used by Sambolian et al (2019,2020)

$$r = \left(\frac{\partial T}{\partial u} + \frac{\partial T}{\partial h}\frac{\partial h}{\partial u}\right) du$$

# Misfit gradient wrt slowness

Misfit gradient

$$\gamma_u(x,z) = \nabla_u \sum_s \phi_s(u)$$

$$\gamma_u(x,z) = \frac{1}{2}\sum_r u(x,z)\lambda_r(x,z)$$

Contribution from misfit

$$\sum_s \sum_{r'=1, r' \neq r}^{N_r^S} \left(t_{sr',r} - t_{s,r}^{obs}\right)$$

For each receiver r (1,2,3 ... )

Contribution from location

$$\sum_s \alpha_{sr}$$

In the alternate strategy of the standard formulation, the contribution of location is missing

Three receivers example