Week 17

ex

A : eigen: $\lambda, \lambda, \lambda, 2, 2$

$(A-\lambda I)$: $\vec{b}_2 \to \vec{b}_1 \to \vec{0}$

$\vec{b}_5 \to \vec{b}_4 \to \vec{b}_3 \to \vec{0}$

$(A-2I)$: $\vec{b}_7 \to \vec{b}_6 \to \vec{0}$

$\vec{b}_8 \to \vec{0}$

$\Rightarrow$ $null(A-\lambda I) = sp(\vec{b}_1, \vec{b}_3)$

$null((A-\lambda I)^2) = sp(\vec{b}_1, \vec{b}_3, \vec{b}_2, \vec{b}_4)$

$null((A-\lambda I)^3) = sp(\vec{b}_1, \vec{b}_3, \vec{b}_2, \vec{b}_4, \vec{b}_5)$

① pick $\vec{v} \in null((A-\lambda I)^3) \setminus null((A-\lambda I)^2)$

let $\vec{v} = \vec{b}_5$, $\vec{b}_4 = (A-\lambda I)\vec{b}_5$, $\vec{b}_3 = (A-\lambda I)\vec{b}_4$

pick $\vec{u} \in null((A-\lambda I)^2) \setminus null(A-\lambda I) \setminus sp(\vec{b}_5, \vec{b}_4, \vec{b}_3)$

let $\vec{u} = \vec{b}_2$, $\vec{b}_1 = (A-\lambda I)\vec{b}_2$

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ex:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

, 求 Jordan form , Jordan basis

$|A-\lambda I| = (-\lambda)^6 \Rightarrow \lambda : 0$

$$(A-0I)^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (A-0I)^2 = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (A-0I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nullity 3

$null(A-0I) = sp(\vec{e}_1, \vec{e}_3, \vec{e}_5)$

nullity 5

$null((A-0I)^2) = sp(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_5, \vec{e}_6)$

nullity 6

$null((A-0I)^3) = sp(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_6)$

①

$\because$ $(A-0I)$: nullity 3

$\therefore$ $\vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 \to \vec{0}$

$\vec{b}_5 \to \vec{b}_4 \to \vec{0}$

$\vec{b}_6 \to \vec{0}$

$null((A-0I)^3)$

$null((A-0I)^2)$

$null(A-0I)$

②

(i) pick $\vec{b}_3 = \vec{e}_4$

(ii) $\vec{b}_2 = (A - 0I)\,\vec{e}_4$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2\vec{e}_2$$

(iii) $\vec{b}_1 = (A - 0I)\,\vec{b}_2$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

$\therefore\ \vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 \to \vec{0}$

$\Rightarrow\ \vec{e}_4 \to 2\vec{e}_2 \to 2\vec{e}_1 \to \vec{0}$

③ $\text{null}((A-0I)^2) \setminus \text{null}(A-0I) \setminus sp(\vec{e}_4, 2\vec{e}_2, 2\vec{e}_1)$

$= sp(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_5, \vec{e}_6) \setminus sp(\vec{e}_1, \vec{e}_3, \vec{e}_5) \setminus sp(\vec{e}_4, 2\vec{e}_2, 2\vec{e}_1)$

$= sp(\vec{e}_6)$

(i) pick $\vec{b}_5 = \vec{e}_6$

$\vec{b}_4 = (A - 0I)\,\vec{b}_5$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_3$$

$\therefore\ \vec{b}_5 \to \vec{b}_4 \to \vec{0}$

$\Rightarrow\ \vec{e}_6 \to \vec{e}_3 \to \vec{0}$

④ $\text{null}(A-0I) \setminus sp(\vec{e}_4, 2\vec{e}_2, 2\vec{e}_1) \setminus sp(\vec{e}_6, \vec{e}_3)$

$= sp(\vec{e}_1, \vec{e}_3, \vec{e}_5) \setminus sp(\vec{e}_4, 2\vec{e}_2, 2\vec{e}_1) \setminus sp(\vec{e}_6, \vec{e}_3)$

$= sp(\vec{e}_5)$

pick $\vec{b}_6 = \vec{e}_5$

$\therefore\ (A - 0I):$

$$\begin{array}{l} \overset{\vec{b}_3}{\vec{e}_4} \to \overset{\vec{b}_2}{2\vec{e}_2} \to \overset{\vec{b}_1}{2\vec{e}_1} \to \vec{0} \\ \qquad\quad\ \ \overset{\vec{b}_5}{\vec{e}_6} \to \overset{\vec{b}_4}{\vec{e}_3} \to \vec{0} \\ \qquad\qquad\qquad\ \ \overset{\vec{b}_6}{\vec{e}_5} \to \vec{0} \end{array}$$

$$A \sim J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Jordan basis : $\{2\vec{e}_1, 2\vec{e}_2, \vec{e}_4, \vec{e}_3, \vec{e}_6, \vec{e}_5\}$

ex:

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 2 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

① $|A-\lambda I| = -(\lambda+2)^3 \quad \therefore \lambda = -2$

$$(A+2I) = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ nullity 1

$\text{null}(A+2I) = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$$(A+2I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ nullity 2

$\text{null}((A+2I)^2) = sp\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$$(A+2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ nullity 3

$\text{null}((A+2I)^3) = sp\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

$\therefore \vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 \to \vec{0}$

$\because \vec{b}_3 \in sp\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \setminus sp\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$\therefore$ pick $\vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{b}_2 = (A+2I)\vec{b}_3 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{b}_1 = (A+2I)\vec{b}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore \vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 \to \vec{0}$

$$\Rightarrow \boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \to \vec{0}}$$

Jordan basis

$$\boxed{J = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}}$$

Jordan basis : $\left\{ \underset{\vec{b}_1}{\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{b}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

order matters!!