4. A square matrix A is unitarily diagonalizable if and only if it is normal, so that $AA^* = A^*A$.
5. Schur's lemma states that every square matrix is unitarily equivalent to an upper-triangular matrix.

EXERCISES

In Exercises 1–12, find a unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$ for the given matrix A.

1. $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$
4. $A = \begin{bmatrix} 9 & 3-i \\ 3+i & 0 \end{bmatrix}$
5. $A = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
7. $A = \begin{bmatrix} 1 & 2-2i & 0 \\ 2+2i & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
8. $A = \begin{bmatrix} 0 & 0 & 1+2i \\ 0 & 5 & 0 \\ 1-2i & 0 & 4 \end{bmatrix}$
9. $A = \begin{bmatrix} 1 & 0 & 2+2i \\ 0 & -3 & 0 \\ 2-2i & 0 & -1 \end{bmatrix}$
10. $A = \begin{bmatrix} 2 & 0 & 1-i \\ 0 & 3 & 0 \\ 1+i & 0 & 1 \end{bmatrix}$
11. $A = \begin{bmatrix} 3 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix}$
12. $A = \begin{bmatrix} -3 & 5i & 1+i \\ -5i & 3 & 0 \\ 1-i & 0 & 3 \end{bmatrix}$
13. Find all $a \in \mathbb{C}$ such that the matrix $\begin{bmatrix} i & 4 \\ a & i \end{bmatrix}$ is unitarily diagonalizable.
14. Find all $a, b \in \mathbb{C}$ such that the matrix $\begin{bmatrix} i & a \\ b & i \end{bmatrix}$ is unitarily diagonalizable.
15. Find all $a \in \mathbb{C}$ such that the matrix $\begin{bmatrix} i & a \\ 1 & 3i \end{bmatrix}$ is unitarily diagonalizable.
16. Find all $a, b \in \mathbb{C}$ such that the matrix $\begin{bmatrix} a & -i \\ i & b \end{bmatrix}$ is unitarily diagonalizable.
17. Prove that every 2×2 real matrix that is unitarily diagonalizable has one of the following forms: $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, for $a, b, d \in \mathbb{R}$.
18. Determine whether the matrix $\begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix}$ is unitarily diagonalizable.
19. Mark each of the following True or False.

___ a. Every square matrix is unitarily equivalent to a diagonal matrix.
___ b. Every square matrix is unitarily equivalent to an upper-triangular matrix.
___ c. Every Hermitian matrix is unitarily equivalent to a diagonal matrix.
___ d. Every unitarily diagonalizable matrix is Hermitian.
___ e. Every real symmetric matrix is Hermitian.
___ f. Every diagonalizable matrix is normal.
___ g. Every unitarily diagonalizable matrix is normal.
___ h. Every real symmetric matrix is normal.
___ i. Every square matrix is diagonalizable, although perhaps not by a unitary matrix.
___ j. Every square matrix with eigenvalues of algebraic multiplicity 1 is diagonalizable by a unitary matrix.

20. Prove that the eigenvalues of a Hermitian matrix are real, without using Theorem 9.5 or Schur's lemma. [HINT: Let $A\mathbf{v} = \lambda\mathbf{v}$, and use the fact that $\mathbf{v}^*A\mathbf{v} = \mathbf{v}^*A^*\mathbf{v}$.]

21. Argue directly from Theorem 9.5 that eigenvectors from different eigenspaces of a Hermitian matrix are orthogonal.

22. Suppose that A is an $n \times n$ matrix such that $A^* = -A$. Show that
    a. A has eigenvalues of the form ri, where $r \in \mathbb{R}$,
    b. A is diagonalizable by a unitary matrix. [HINT FOR BOTH PARTS: Work with iA.]

23. Prove that an $n \times n$ matrix A is unitarily diagonalizable if and only if $\|A\mathbf{v}\| = \|A^*\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{C}^n$.

24. Prove that a normal matrix is Hermitian if and only if all its eigenvalues are in $\mathbb{R}$.

25. a. Prove that a diagonal matrix is normal.
    b. Prove that, if A is normal and B is unitarily equivalent to A, then B is normal.
    c. Deduce from parts a and b that a unitarily diagonalizable matrix is normal.

26. a. Prove that every normal matrix A is unitarily equivalent to a normal upper-triangular matrix B. (Use Schur's lemma and part b of Exercise 25.)
    b. Prove that an $n \times n$ normal upper-triangular matrix B must be diagonal. [HINT: Let $C = B^*B = BB^*$. Equating the computations of c_{11} from B^*B and from BB^*, show that $b_{1j} = 0$ for $1 < j \leq n$. Then equate the computations of c_{22} from B^*B and from BB^* to show that $b_{2j} = 0$ for $2 < j \leq n$. Continue this process to show that B is lower triangular.]
    c. Deduce from parts a and b that a normal matrix is unitarily diagonalizable.

In Exercises 27 and 28, use the command **[U, D] = eig(A)** *in MATLAB to work the indicated exercise. If your MATLAB answer U for the unitary matrix differs from the U that we found using pencil and paper and put in the answers at the end of our text, explain how you can get from one answer to the other.*

27. Exercise 9

28. Exercise 11

9.4 JORDAN CANONICAL FORM

Jordan Blocks

We have spent considerable time on diagonalization of matrices. The preceding section was concerned primarily with unitary diagonalization. As we have seen, diagonal matrices are easily handled. Unfortunately, not every $n \times n$ matrix A can be diagonalized, because we cannot always find a basis for $\mathbb{C}^n$ consisting of eigenvectors of A. We remind you of this with an example that is well worth studying.

EXAMPLE 1 Show that the matrix

$$J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is not diagonalizable.