5-1

ODE

$$\begin{cases} \dfrac{dy}{dt} = \bigcirc & t \in [a, b] \\ y(a) = y_0 \end{cases} \Rightarrow y(t) = ?$$

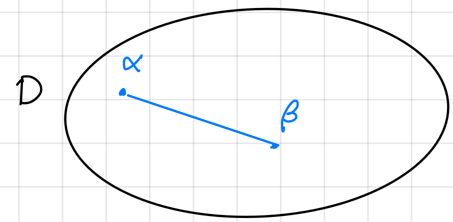
Def Lipschitz condition

$f(t,y)$: Lipschitz condition in y on $D \subset \mathbb{R}^2$

if $L > 0$. $\forall (t, y_1), (t, y_2) \in D$ — Lipschitz constant for f

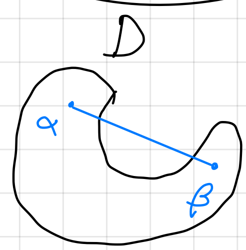
s.t. $|f(t, y_1) - f(t, y_2)| \le L |y_1 - y_2|$

Def



a set $D \subset \mathbb{R}^2$, D: convex

if $\forall \alpha = (t_1, y_1), \beta = (t_2, y_2), \alpha, \beta \in D$



① "$\alpha \rightsquigarrow \beta$" $\in D$

② $\forall \lambda \in [0,1]$ $\quad (1-\lambda)\alpha + \lambda\beta \in D$

$$\| $$

$$((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2) \in D$$

ex: $D = [a, b]$

$D = [a,b] \times [c,d]$ — D: convex

$D = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$

ex: $f(t,y) = t|y|$, $D = \{t \in [1,2], y \in [-3,4]\}$

$|f(t,y_1) - f(t,y_2)| = |t||y_1| - |t||y_2|$

$= |t|(|y_1| - |y_2|) \le |t| |y_1 - y_2| \le \boxed{2} |y_1 - y_2|$

L

$\therefore f$ satisfy the Lipschitz condition in y

Thm 1 $f: D \to \mathbb{R}$, D: convex

if $\exists L > 0$, s.t. $\boxed{\left|\frac{\partial f}{\partial y}(t,y)\right| \leq L, \quad \forall (t,y) \in D}$

$\Rightarrow$ f satisfy the Lipschitz condition in y with Lipschitz constent L

Thm 2

$D = \{ t \in [a,b], y \in \mathbb{R} \}$, $f(t,y)$: continuous on D

if f satisfy the Lipschitz condition on D in y

then $\begin{cases} y'(t) = f(t,y), & \forall t \in [a,b] \\ y(a) = \alpha \end{cases}$

has a unique solution $y(t)$ for $t \in [a,b]$

ex: $\begin{cases} y'(t) = \underline{1 + t\sin(ty)}, & t \in [0,2] \\ y(0) = 0 \end{cases}$

" $f(t,y)$

$\frac{\partial f}{\partial y} = t^2 \cos(ty)$, $\left|\frac{\partial f}{\partial y}(t,y)\right| = |t^2 \cos(ty)| \leq |t^2| \leq 4$

$\therefore$ by Thm 1, f satisfy Lipschitz condition on D in y with Lipschitz constent 4

$\therefore$ by Thm 2, $\exists!$ solution $y(t)$ in $t \in [0,2]$

△ fix-point

g : satisfy

$\Rightarrow \forall p_0 \in [p-\delta, p+\delta]$, let $p_n = g(p_{n-1})$, then $p_n \to p$ as $n \to \infty$

error : roud-off error

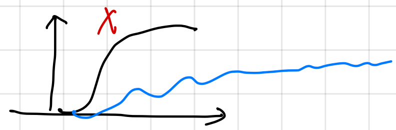
then $\begin{cases} y'(t) = f(t,y), & \forall\, t \in [a,b] \\ y(a) = \alpha \end{cases}$

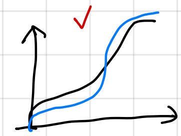
has a unique solution $y(t)$ for $t \in [a,b]$

then $\begin{cases} z'(t) = f(t,z) + \delta(t), & \forall\, t \in [a,b] \\ z(a) = \alpha + \delta_0 \end{cases}$

has a unique solution $z(t)$ for $t \in [a,b]$

hope: $|z(t) - y(t)| < \varepsilon$





Def well-posed problem

the problem $\begin{cases} y'(t) = f(t,y), & \forall\, t \in [a,b] \\ y(a) = \alpha \end{cases}$ is well-posed problem

① the unique solution $y(t)$ exist for $t \in [a,b]$

② $\exists\, \varepsilon_0 > 0$, $K > 0$

s.t. $\forall\, 0 < \varepsilon < \varepsilon_0$, $\forall\, \delta(t)$: contin, with $|\delta(t)| < \varepsilon$, $\forall\, t \in [a,b]$

then $\begin{cases} \underbrace{z'(t) = f(t,z) + \delta(t)}_{\text{perturbed problem}}, & \forall\, t \in [a,b] \\ z(a) = \alpha + \underbrace{\delta_0}_{\text{round-off error}} \end{cases}$ has a unique solution $z(t)$

s.t. $|z(t) - y(t)| < K\varepsilon \quad \forall\, t \in [a,b]$

Thm

$D = \{ t \in [a,b],\ y \in \mathbb{R} \}$

f: ① continuous on D, ② Lipschitz condition on D in y → use Thm1

$\Rightarrow \begin{cases} y'(t) = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$ is well-posed problem ex:

$$\begin{cases} y'(t) = \underbrace{y - t^2 + 1}_{=f(t,y)} , & t \in [0,2] \\ y(0) = 0.5 \end{cases} , \quad D = \{ t \in [0,2] , y \in \mathbb{R} \}$$

sol.

① $\left| \frac{\partial f}{\partial y} \right| = |1| = 1 = L$ $\quad \therefore$ f satisfy Lipschitz condition on D in y

② f: continuous on D

$\Rightarrow$ well-posed.

▲

$$\begin{cases} z'(t) = \underbrace{z - t^2 + 1 + \delta}_{=g(t)} \\ z(0) = 0.5 + \delta_0 \end{cases} , \quad \left| \frac{\partial g}{\partial y} \right| = |1| \quad \therefore \text{has unique solution } z(t)$$

$$\because \begin{cases} y(t) = (t+1)^2 - 0.5 e^t \\ z(t) = (t+1)^2 + (\delta + \delta_0 - 0.5) e^t - \delta \end{cases}$$

$$\therefore |y(t) - z(t)| = |(\delta + \delta_0) e^t - \delta| \le |(\delta + \delta_0)| |e^t| + |\delta| \le |\delta + \delta_0| e^2 + |\delta|$$

$$\boxed{\because |\delta| < \varepsilon , \ |\delta_0| < |\delta| < \varepsilon} \longrightarrow \le \underbrace{(2e^2 + 1)}_{K} \varepsilon$$

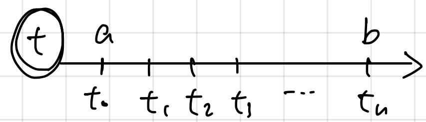
5-2 Euler's Method

$$\begin{cases} y'(t) = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$ is well-posed problem

$\Rightarrow$ has uniques solution $y(t)$

Taylor: $y(t) = y(x_0) + (t-x_0)\,y'(x_0) + \dfrac{(t-x_0)^2}{2!}\,y''(\xi)$

$\underset{t_{i+1}}{\uparrow}$ $\underset{t_i}{\uparrow}$



$$y(t_{i+1}) = y(t_i) + (t_{i+1}-t_i)\,y'(t_i) + \boxed{\frac{(t_{i+1}-t_i)^2}{2!}\,y''(\xi_i)}$$ 省略

$$\boxed{y(t_{i+1})} \approx y(t_i) + (t_{i+1}-t_i)\,y'(t_i)$$

$$= y(t_i) + (t_{i+1}-t_i)\,f(t_i, y(t_i))$$

Euler's Method

① let $w_0 = \alpha$, $t_i = a + ih$ s.t. $t_0 = a$, $t_n = b$. $\left(h = \dfrac{b-a}{n}\right)$

② $w_{i+1} = w_i + h\,f(t_i, w_i)$

$h \downarrow \Rightarrow w_i \to y(t_i)$

$\Rightarrow w_i \approx y(t_i)$