partition is in row-echelon form. The solution is then found by back substitution.

8. The Gauss–Jordan method is similar to the Gauss method, except that pivots are adjusted to be 1 and zeros are created above as well as below the pivots.
9. A linear system $A\mathbf{x} = \mathbf{b}$ has no solutions if and only if, after $[A \mid \mathbf{b}]$ is row-reduced so that A is transformed into row-echelon form, there exists a row with only zero entries to the left of the partition but with a nonzero entry to the right of the partition. The linear system is then *inconsistent.*
10. If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system and if a row-echelon form H of A has at least one column containing no (nonzero) pivot, the system has an infinite number of solutions. The free variables corresponding to the columns containing no pivots can be assigned any values, and the reduced linear system can then be solved for the remaining variables.
11. An elementary matrix E is one obtained by applying a single elementary row operation to an identity matrix I. Multiplication of a matrix A on the left by E effects the same elementary row operation on A.

EXERCISES

In Exercises 1–6, reduce the matrix to (a) row-echelon form, and (b) reduced row-echelon form. Answers to (a) are not unique, so your answer may differ from the one at the back of the text.

1. $\begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 3 & -1 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 2 & -1 & 3 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 1 & 3 & 2 & -4 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 2 & 4 & -1 & 3 & 2 & -1 \end{bmatrix}$

In Exercises 7–12, describe all solutions of a linear system whose corresponding augmented matrix can be row-reduced to the given matrix. If requested, also give the indicated particular solution, if it exists.

7. $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 4 & 2 \end{array}\right]$, solution with $x_3 = 2$

8. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$

9. $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$,
solution with $x_3 = 3$, $x_4 = -2$

10. $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & 0 & -4 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$,
solution with $x_2 = 2$, $x_3 = 1$

11. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

12. $\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 13–20, find all solutions of the given linear system, using the Gauss method with back substitution.

13. $\begin{aligned} 2x - y &= 8 \\ 6x - 5y &= 32 \end{aligned}$

14. $\begin{aligned} 4x_1 - 3x_2 &= 10 \\ 8x_1 - x_2 &= 10 \end{aligned}$

15. $\begin{aligned} y + z &= 6 \\ 3x - y + z &= -7 \\ x + y - 3z &= -13 \end{aligned}$

16. $\begin{aligned} 2x + y - 3z &= 0 \\ 6x + 3y - 8z &= 0 \\ 2x - y + 5z &= -4 \end{aligned}$

17. $\begin{aligned} x_1 - 2x_2 &= 3 \\ 3x_1 - x_2 &= 14 \\ x_1 - 7x_2 &= -2 \end{aligned}$

18. $\begin{aligned} x_1 - 3x_2 + x_3 &= 2 \\ 3x_1 - 8x_2 + 2x_3 &= 5 \end{aligned}$

19. $\begin{aligned} x_1 + 4x_2 - 2x_3 &= 4 \\ 2x_1 + 7x_2 - x_3 &= -2 \\ 2x_1 + 9x_2 - 7x_3 &= 1 \end{aligned}$

20. $\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 &= 8 \\ 3x_1 - 7x_2 \qquad\quad + x_4 &= 0 \end{aligned}$

In Exercises 21–24, find all solutions of the linear system, using the Gauss–Jordan method.

21. $\begin{aligned} 3x_1 - 2x_2 &= -8 \\ 4x_1 + 5x_2 &= -3 \end{aligned}$

22. $\begin{aligned} 2x_1 + 8x_2 &= 16 \\ 5x_1 - 4x_2 &= -8 \end{aligned}$

23. $\begin{aligned} x_1 \qquad\quad - 2x_3 + x_4 &= 6 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 0 \\ 9x_1 - 3x_2 - x_3 - 7x_4 &= 4 \end{aligned}$

24. $\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 2 \\ 3x_1 + 6x_2 - 8x_3 - 2x_4 &= 1 \end{aligned}$

In Exercises 25–28, determine whether the vector **b** *is in the span of the vectors* $\mathbf{v}_i$.

25. $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix}$

26. $\mathbf{b} = \begin{bmatrix} 8 \\ 26 \\ 14 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ -12 \\ -9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$

27. $\mathbf{b} = \begin{bmatrix} 8 \\ 17 \\ -8 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -6 \\ 1 \\ -8 \end{bmatrix},$
$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -4 \end{bmatrix}$

28. $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 7 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ -2 \\ -8 \\ -9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix},$
$\mathbf{v}_4 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

29. Mark each of the following True or False.

___ a. Every linear system with the same number of equations as unknowns has a unique solution.

___ b. Every linear system with the same number of equations as unknowns has at least one solution.

___ c. A linear system with more equations than unknowns may have an infinite number of solutions.

___ d. A linear system with fewer equations than unknowns may have no solution.

___ e. Every matrix is row equivalent to a unique matrix in row-echelon form.

___ f. Every matrix is row equivalent to a unique matrix in reduced row-echelon form.

___ g. If $[A \mid \mathbf{b}]$ and $[B \mid \mathbf{c}]$ are row-equivalent partitioned matrices, the linear systems

$A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ have the same solution set.

___ **h.** A linear system with a square coefficient matrix A has a unique solution if and only if A is row equivalent to the identity matrix.

___ **i.** A linear system with coefficient matrix A has an infinite number of solutions if and only if A can be row-reduced to an echelon matrix that includes some column containing no pivot.

___ **j.** A consistent linear system with coefficient matrix A has an infinite number of solutions if and only if A can be row-reduced to an echelon matrix that includes some column containing no pivot.

In Exercises 30–37, describe all possible values for the unknowns x_i so that the matrix equation is valid.

30. $2[x_1 \quad x_2] - [4 \quad 7] = [-2 \quad 11]$

31. $4[x_1 \quad x_2] + 2[x_1 \quad 3] = [-6 \quad 18]$

32. $[x_1 \quad x_2]\begin{bmatrix} 1 \\ -3 \end{bmatrix} = [2]$

33. $[x_1 \quad x_2]\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = [0 \quad -14]$

34. $\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$

35. $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$

36. $[x_1 \quad x_2]\begin{bmatrix} 3 & 0 & 4 \\ 2 & 1 & -1 \end{bmatrix} = [3 \quad 3 \quad -7]$

37. $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

38. Determine all values of the b_i that make the linear system

$$\begin{aligned} x_1 + 2x_2 &= b_1 \\ 3x_1 + 6x_2 &= b_2 \end{aligned}$$

consistent.

39. Determine all values b_1 and b_2 such that $\mathbf{b} = [b_1, b_2]$ is a linear combination of $\mathbf{v}_1 = [1, 3]$ and $\mathbf{v}_2 = [5, -1]$.

40. Determine all values of the b_i that make the linear system

$$\begin{aligned} x_1 + x_2 - x_3 &= b_1 \\ 2x_2 + x_3 &= b_2 \\ x_2 - x_3 &= b_3 \end{aligned}$$

consistent.

41. Determine all values b_1, b_2, and b_3 such that $\mathbf{b} = [b_1, b_2, b_3]$ lies in the span of $\mathbf{v}_1 = [1, 1, 0]$, $\mathbf{v}_2 = [3, -1, 4]$, and $\mathbf{v}_3 = [-1, 2, -3]$.

42. Find an elementary matrix E such that

$$E\begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 3 & 4 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & 2 & -11 \end{bmatrix}.$$

43. Find an elementary matrix E such that

$$E\begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 3 & 4 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 4 & 9 \\ 3 & 4 & 5 & 1 \end{bmatrix}.$$

44. Find a matrix C such that

$$C\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -6 \end{bmatrix}.$$

45. Find a matrix C such that

$$C\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 1 & 2 \end{bmatrix}.$$

In Exercises 46–51, let A be a 4×4 matrix. Find a matrix C such that the result of applying the given sequence of elementary row operations to A can also be found by computing the product CA.

46. Interchange row 1 and row 2.

47. Interchange row 1 and row 3; multiply row 3 by 4.

48. Multiply row 1 by 5; interchange rows 2 and 3; add 2 times row 3 to row 4.

49. Add 4 times row 2 to row 4; multiply row 4 by -3; add 5 times row 4 to row 1.

50. Interchange rows 1 and 4; add 6 times row 2 to row 1; add -3 times row 1 to row 3; add -2 times row 4 to row 2.

51. Add 3 times row 2 to row 4; add -2 times row 4 to row 3; add 5 times row 3 to row 1; add -4 times row 1 to row 2.

Exercise 24 in Section 1.3 is useful for the next three exercises.

52. Prove Theorem 1.8 for the row-interchange operation.
53. Prove Theorem 1.8 for the row-scaling operation.
54. Prove Theorem 1.8 for the row-addition operation.
55. Prove that row equivalence $\sim$ is an equivalence relation by verifying the following for $m \times n$ matrices A, B, and C.
    a. $A \sim A$. (Reflexive Property)
    b. If $A \sim B$, then $B \sim A$. (Symmetric Property)
    c. If $A \sim B$ and $B \sim C$, then $A \sim C$. (Transitive Property)
56. Find a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, -4)$, $(-1, 0)$, and $(2, 3)$.
57. Find a, b, c, and d such that the quartic curve $y = ax^4 + bx^3 + cx^2 + d$ passes through $(1, 2)$, $(-1, 6)$, $(-2, 38)$, and $(2, 6)$.
58. Let A be an $m \times n$ matrix, and let $\mathbf{c}$ be a column vector such that $A\mathbf{x} = \mathbf{c}$ has a unique solution.
    a. Prove that $m \geq n$.
    b. If $m = n$, must the system $A\mathbf{x} = \mathbf{b}$ be consistent for every choice of $\mathbf{b}$?
    c. Answer part (b) for the case where $m > n$.

A problem we meet when reducing a matrix with the aid of a computer involves determining when a computed entry should be 0. The computer might give an entry as 0.00000001, because of roundoff error, when it should be 0. If the computer uses this entry as a pivot in a future step, the result is chaotic! For this reason, it is common practice to program the computer to replace all sufficiently small computed entries with 0, where the meaning of "sufficiently small" must be specified in terms of the size of the nonzero entries in the original matrix. The routine YUREDUCE in LINTEK provides drill on the steps involved in reducing a matrix without requiring burdensome computation. The program computes the smallest nonzero coefficient magnitude m and asks the user to enter a number r (for ratio); all computed entries of magnitude less than rm produced during reduction of the coefficient matrix will be set equal to zero. In Exercises 59–64, use the routine YUREDUCE, specifying r = 0.0001, to solve the linear system.

59. $$\begin{aligned} 3x_1 - x_2 &= -10 \\ 7x_1 + 2x_2 &= 7 \\ 2x_1 - 5x_2 &= -37 \end{aligned}$$

60. $$\begin{aligned} 5x_1 - 2x_2 &= 11 \\ 8x_1 + x_2 &= 3 \\ 6x_1 - 5x_2 &= -4 \end{aligned}$$

61. $$\begin{aligned} 7x_1 - 2x_2 + x_3 &= -14 \\ -4x_1 + 5x_2 - 3x_3 &= 17 \\ 5x_1 - x_2 + 2x_3 &= -7 \end{aligned}$$

62. $$\begin{aligned} -3x_1 + 5x_2 + 2x_3 &= 12 \\ 5x_1 - 7x_2 + 6x_3 &= -16 \\ 11x_1 - 17x_2 + 2x_3 &= -40 \end{aligned}$$

63. $$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + 2x_5 &= 1 \\ 2x_1 + x_2 - 4x_3 - x_4 + 5x_5 &= 16 \\ 8x_1 - x_2 + 3x_3 - x_4 - x_5 &= 1 \\ 4x_1 - 2x_2 + 3x_3 - 8x_4 + 2x_5 &= -5 \\ 5x_1 + 3x_2 - 4x_3 + 7x_4 - 6x_5 &= 7 \end{aligned}$$

64. $$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + 2x_5 &= 1 \\ 2x_1 + x_2 - 4x_3 - x_4 + 5x_5 &= 10 \\ 8x_1 - x_2 + 3x_3 - x_4 - x_5 &= -5 \\ 4x_1 - 2x_2 + 3x_3 - 8x_4 + 2x_5 &= -3 \\ 5x_1 + 3x_2 - 4x_3 + 7x_4 - 6x_5 &= 1 \end{aligned}$$

The routine MATCOMP in LINTEK can also be used to find the solutions of a linear system. MATCOMP will bring the left portion of the augmented matrix to reduced row-echelon form and display the result on the screen. The user can *then find the solutions. Use MATCOMP in the remaining exercises.*

65. Find the reduced row-echelon form of the matrix in Exercise 6, by taking it as a coefficient matrix for zero systems.

66. Solve the linear system in Exercise 61.

67. Solve the linear system in Exercise 62.

68. Solve the linear system in Exercise 63.

MATLAB

When reducing a matrix X to reduced row-echelon form, we may need to swap row i with row k. This can be done in MATLAB using the command

X([i k],:) = X([k i],:).

If we wish to multiply the ith row by the reciprocal of x_{ij} to create a pivot 1 in the ith row and jth column, we can give the command

X(i,:) = X(i,:)/X(i,j).

When we have made pivots 1 and wish to make the entry in row k, column j equal to zero using the pivot in row i, column j, we always multiply row i by the negative of the entry that we wish to make zero, and add the result to row k. In MATLAB, this has the form

X(k,:) = X(k,:) − X(k, j)*X(i,:).

Access MATLAB and enter the lines

```
X = ones(4); i = 1; j = 2; k = 3;
X([i k],:) = X([k i],:)
X(i,:) = X(i,:)/X(i,j)
X(k,:) = X(k,:) - X(k, j)*X(i,:),
```

which you can then access using the up-arrow key and edit repeatedly to row-reduce a matrix X. MATLAB will not show a partition in X—you have to supply the partition mentally. If your installation contains the data files for our text, enter **fbc1s4** *now. We will be asking you to work with some of the augmented matrices used in the exercises for this section. In our data file, the augmented matrix for Exercise 63 is called E63, etc. Solve the indicated system by setting X equal to the appropriate matrix and reducing it using the up-arrow key and editing repeatedly the three basic commands above. In MATLAB, only the commands executed most recently can be accessed by using the up-arrow key. To avoid losing the command to interchange rows, which is seldom necessary, execute it at least once in each exercise even if it is not needed. (Interchanging the same rows twice leaves a matrix unchanged.) Solve the indicated exercises listed below.*

M1. Exercise 21

M2. Exercise 23

M3. Exercise 60

M4. Exercise 61

M5. Exercise 62