## EXERCISES

*In Exercises 1–8, decide whether or not the given set, together with the indicated operations of addition and scalar multiplication, is a (real) vector space.*

1. The set $\mathbb{R}^2$, with the usual addition but with scalar multiplication defined by $r[x, y] = [ry, rx]$.
2. The set $\mathbb{R}^2$, with the usual scalar multiplication but with addition defined by $[x, y] \tilde{+} [r, s] = [y + s, x + r]$.
3. The set $\mathbb{R}^2$, with addition defined by $[x, y] \tilde{+} [a, b] = [x + a + 1, y + b]$ and with scalar multiplication defined by $r[x, y] = [rx + r - 1, ry]$.
4. The set of all $2 \times 2$ matrices, with the usual scalar multiplication but with addition defined by $A \tilde{+} B = O$, the $2 \times 2$ zero matrix.
5. The set of all $2 \times 2$ matrices, with the usual addition but with scalar multiplication defined by $rA = O$, the $2 \times 2$ zero matrix.
6. The set $F$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, with scalar multiplication defined as in Example 7 but with addition defined by $(f \tilde{+} g)(x) = \max\{f(x), g(x)\}$.
7. The set $F$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, with scalar multiplication defined as in Example 7 but with addition defined by $(f \tilde{+} g)(x) = f(x) + 2g(x)$.
8. The set $F$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, with scalar multiplication defined as in Example 7 but with addition defined by $(f \tilde{+} g)(x) = 2f(x) + 2g(x)$.

*In Exercises 9–16, determine whether the given set is closed under the usual operations of addition and scalar multiplication, and is a (real) vector space.*

9. The set of all upper-triangular $n \times n$ matrices.
10. The set of all $2 \times 2$ matrices of the form
$$\begin{bmatrix} \times & 1 \\ 1 & \times \end{bmatrix},$$
where each $\times$ may be any scalar.
11. The set of all diagonal $n \times n$ matrices.
12. The set of all $3 \times 3$ matrices of the form
$$\begin{bmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{bmatrix},$$
where each $\times$ may be any scalar.
13. The set $\{0\}$ consisting only of the number 0.
14. The set $\mathbb{Q}$ of all rational numbers.
15. The set $\mathbb{C}$ of complex numbers; that is,
$$\mathbb{C} = \{a + b\sqrt{-1} \mid a, b \text{ in } \mathbb{R}\},$$
with the usual addition of complex numbers and with scalar multiplication defined in the usual way by $r(a + b\sqrt{-1}) = ra + rb\sqrt{-1}$ for any numbers $a$, $b$, and $r$ in $\mathbb{R}$.
16. The set $P_n$ of all polynomials in $x$, with real coefficients and of degree less than or equal to $n$, together with the zero polynomial.
17. Your answer to Exercise 3 should be that $\mathbb{R}^2$ with the given operations is a vector space.
    a. Describe the "zero vector" in this vector space.
    b. Explain why the relations $r[0, 0] = [r - 1, 0] \neq [0, 0]$ do not violate property (5) of Theorem 3.1.
18. Mark each of the following True or False.
    ___ a. Matrix multiplication is a vector-space operation on the set $M_{m \times n}$ of all $m \times n$ matrices.
    ___ b. Matrix multiplication is a vector-space operation on the set $M_n$ of all square $n \times n$ matrices.
    ___ c. Multiplication of any vector by the zero scalar always yields the zero vector.
    ___ d. Multiplication of a nonzero vector by a nonzero scalar never yields the zero vector.
    ___ e. No vector is its own additive inverse.

___ f. The zero vector is the only vector that is its own additive inverse.

___ g. Multiplication of two scalars is of no concern in the definition of a vector space.

___ h. One of the axioms for a vector space relates addition of scalars, multiplication of a vector by scalars, and addition of vectors.

___ i. Every vector space has at least two vectors.

___ j. Every vector space has at least one vector.

19. Prove property 2 of Theorem 3.1.

20. Prove property 3 of Theorem 3.1.

21. Prove property 5 of Theorem 3.1.

22. Prove property 6 of Theorem 3.1.

23. Let $V$ be a vector space. Prove that, if $\mathbf{v}$ is in $V$ and if $r$ is a scalar and if $r\mathbf{v} = \mathbf{0}$, then either $r = 0$ or $\mathbf{v} = \mathbf{0}$.

24. Let $V$ be a vector space and let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $V$. Prove that if $\mathbf{v}$ is not a scalar multiple of $\mathbf{w}$, then $\mathbf{v}$ is not a scalar multiple of $\mathbf{v} + \mathbf{w}$.

25. Let $V$ be a vector space, and let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $V$. Prove that there is a *unique* vector $\mathbf{x}$ in $V$ such that $\mathbf{x} + \mathbf{v} = \mathbf{w}$.

*Exercises 26–29 are based on the optional subsection on the universality of function spaces.*

26. Using the discussion of function spaces at the end of this section, explain how we can view the Euclidean vector space $\mathbb{R}^{mn}$ and the vector space $M_{m,n}$ of all $m \times n$ matrices as essentially the same vector space with just a different notation for the vectors.

27. Repeat Exercise 26 for the vector space $M_{2,6}$ of $2 \times 6$ matrices and the vector space $M_{3,4}$ of $3 \times 4$ matrices.

28. If you worked Exercise 16 correctly, you found that the polynomials in $x$ of degree at most $n$ do form a vector space $P_n$. Explain how $P_n$ and $\mathbb{R}^{n+1}$ can be viewed as essentially the same vector space, with just a different notation for the vectors.

29. Referring to the three preceding exercises, list the vector spaces $\mathbb{R}^{24}$, $\mathbb{R}^{25}$, $\mathbb{R}^{26}$, $P_{24}$, $P_{25}$, $P_{26}$, $M_{4,7}$, $M_{3,8}$, $M_{3,9}$, $M_{2,12}$, $M_{2,13}$, $M_{4,6}$, and $M_{5,5}$ in two or more columns in such a way that any two vector spaces listed in the same column can be viewed as the same vector space with just different notation for vectors, but two vector spaces that appear in different columns cannot be so viewed.

## 3.2 BASIC CONCEPTS OF VECTOR SPACES

We now extend the terminology we developed in Chapters 1 and 2 for the Euclidean spaces $\mathbb{R}^n$ to general vector spaces $V$. That is, we discuss linear combinations of vectors, the span of vectors, subspaces, dependent and independent vectors, bases, and dimension. Definitions of most of these concepts, and the theorems concerning them, can be lifted with minor changes from Chapters 1 and 2, replacing "in $\mathbb{R}^n$" wherever it occurs by "in a vector space $V$." Where we do make changes, the reasons for them are explained.

### Linear Combinations, Spans, and Subspaces

The major change from Chapter 1 is that our vector spaces may now be so large that they cannot be spanned by a *finite* number of vectors. Each Euclidean space $\mathbb{R}^n$ and each subspace of $\mathbb{R}^n$ can be spanned by a finite set of vectors, but this is not the case for a general vector space. For example, no