8. The range of T is the subspace $\{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\}$ of V', and the kernel $\ker(T)$ is the subspace $\{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}'\}$ of V.
9. T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$. If T is one-to-one and has as its range all of V', then T^{-1}: $V' \to V$ is well-defined and is a linear transformation. In this case, both T and T^{-1} are isomorphisms.
10. If $T: V \to V'$ is a linear transformation and $T(\mathbf{p}) = \mathbf{b}$, then the solution set of $T(\mathbf{x}) = \mathbf{b}$ is $\{\mathbf{p} + \mathbf{h} \mid \mathbf{h} \in \ker(T)\}$.
11. Let $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ and $B' = (\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_m)$ be ordered bases for V and V', respectively. The *matrix representation of T relative to B,B'* is the $m \times n$ matrix A having $T(\mathbf{b}_j)_{B'}$ as its jth column vector. We have $T(\mathbf{v})_{B'} = A(\mathbf{v}_B)$ for all $\mathbf{v} \in V$.

EXERCISES

In Exercises 1–5, let F be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. Determine whether the given function T is a linear transformation. If it is a linear transformation, describe the kernel of T and determine whether the transformation is invertible.

1. $T: F \to \mathbb{R}$ defined by $T(f) = f(-4)$
2. $T: F \to \mathbb{R}$ defined by $T(f) = f(5)^2$
3. $T: F \to F$ defined by $T(f) = f + f$
4. $T: F \to F$ defined by $T(f) = f + 3$, where 3 is the constant function with value 3 for all $x \in \mathbb{R}$.
5. $T: F \to F$ defined by $T(f) = -f$
6. Let $C_{0,2}$ be the space of continuous functions mapping the interval $0 \le x \le 2$ into $\mathbb{R}$. Let $T: C_{0,2} \to \mathbb{R}$ be defined by $T(f) = \int_0^2 f(x)\,dx$. See Example 3. If possible, give three different functions in $\ker(T)$.
7. Let C be the space of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$, and let $T: C \to C$ be defined by $T(f) = \int_1^x f(t)\,dt$. See Example 4. If possible, give three different functions in $\ker(T)$.
8. Let F be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, and let $T: F \to F$ be a linear transformation such that $T(e^{2x}) = x^2$, $T(e^{3x}) = \sin x$, and $T(1) = \cos 5x$. Find the following, if it is determined by this data.
   a. $T(e^{5x})$
   b. $T(3 + 5e^{3x})$
   c. $T(3e^{4x})$
   d. $T\left(\dfrac{e^{4x} + 2e^{5x}}{e^{2x}}\right)$
9. Note that one solution of the differential equation $y'' - 4y = \sin x$ is $y = -\frac{1}{5}\sin x$. Use summary item 10 and Illustration 1 to describe all solutions of this equation.

Let D_∞ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T: D_\infty \to D_\infty$ of the form $T(f) = a_n f^{(n)} + \cdots + a_1 f' + a_0 f$ where $a_n \neq 0$ is an n-dimensional subspace of D_∞.

In Exercises 10–15, use the preceding information, summary item 10, and your knowledge of calculus to find all solutions in D_∞ of the given differential equation. See Illustration 1 in the text.

10. $y' = \sin 2x$
11. $y'' = -\cos x$
12. $y' - y = x$
13. $y'' + 4y = x^2$
14. $y'' + y' = 3e^x$
15. $y^{(3)} - 2y'' = x$
16. Let V and V' be vector spaces having ordered bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ and $B' = (\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3, \mathbf{b}'_4)$, respectively. Let $T: V \to V'$ be a linear transformation such that
$$\begin{aligned} T(\mathbf{b}_1) &= 3\mathbf{b}'_1 + \mathbf{b}'_2 + 4\mathbf{b}'_3 - \mathbf{b}'_4 \\ T(\mathbf{b}_2) &= \mathbf{b}'_1 + 2\mathbf{b}'_2 - \mathbf{b}'_3 + 2\mathbf{b}'_4 \\ T(\mathbf{b}_3) &= -2\mathbf{b}'_1 - \mathbf{b}'_2 + 2\mathbf{b}'_3. \end{aligned}$$
Find the matrix representation A of T relative to B,B'.

In Exercises 17–19, let V and V′ be vector spaces with ordered bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ and $B' = (\mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_3', \mathbf{b}_4')$, respectively, and let $T: V \to V'$ be the linear transformation having the given matrix A as matrix representation relative to B,B′. Find T(v) for the given vector v.

17. $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, $\mathbf{v} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$

18. A as in Exercise 17, $\mathbf{v} = 3\mathbf{b}_3 - \mathbf{b}_1$

19. $A = \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathbf{v} = 6\mathbf{b}_1 - 4\mathbf{b}_2 + \mathbf{b}_3$

In Exercises 20–33, we consider A to be the matrix representation of the indicated linear transformation T (you may assume it is linear) relative to the indicated ordered bases B and B′. Starting with Exercise 21, we let D denote differentiating once, D^2 differentiating twice, and so on.

20. Let V and V' be vector spaces with ordered bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ and $B' = (\mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_3')$, respectively. Let $T: V \to V'$ be the linear transformation such that

$$\begin{aligned} T(\mathbf{b}_1) &= \mathbf{b}_1' + 2\mathbf{b}_2' - 3\mathbf{b}_3' \\ T(\mathbf{b}_2) &= 3\mathbf{b}_1' + 5\mathbf{b}_2' + 2\mathbf{b}_3' \\ T(\mathbf{b}_3) &= -2\mathbf{b}_1' - 3\mathbf{b}_2' - 4\mathbf{b}_3'. \end{aligned}$$

   a. Find the matrix A.
   b. Use A to find $T(\mathbf{v})_{B'}$, if $\mathbf{v}_B = [2, -5, 1]$.
   c. Show that T is invertible, and find the matrix representation of T^{-1} relative to B',B.
   d. Find $T^{-1}(\mathbf{v}')_B$ if $\mathbf{v}'_{B'} = [-1, 1, 3]$.
   e. Express $T^{-1}(\mathbf{b}_1')$, $T^{-1}(\mathbf{b}_2')$, and $T^{-1}(\mathbf{b}_3')$ as linear combinations of the vectors in B.

21. Let $T: P_3 \to P_3$ be defined by $T(p(x)) = D(p(x))$, the derivative of $p(x)$. Let the ordered bases for P_3 be $B = B' = (x^3, x^2, x, 1)$.
   a. Find the matrix A.
   b. Use A to find the derivative of $4x^3 - 5x^2 + 10x - 13$.
   c. Noting that $T \circ T = D^2$, find the second derivative of $-5x^3 + 8x^2 - 3x + 4$ by multiplying a column vector by an appropriate matrix.

22. Let $T: P_3 \to P_3$ be defined by $T(p(x)) = x\,D(p(x))$ and let the ordered bases B and B' be as in Exercise 21.
   a. Find the matrix representation A relative to B,B'.
   b. Working with the matrix A and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.
   c. The transformation T can be decomposed into $T = T_2 \circ T_1$, where $T_1: P_3 \to P_2$ is defined by $T_1(p(x)) = D(p(x))$ and $T_2: P_2 \to P_3$ is defined by $T_2(p(x)) = xp(x)$. Find the matrix representations of T_1 and T_2 using the ordered bases B of P_3 and $B'' = (x^2, x, 1)$ of P_2. Now multiply these matrix representations for T_1 and T_2 to obtain a 4×4 matrix, and compare with the matrix A. What do you notice?
   d. Multiply the two matrices found in part (c) to obtain a 3×3 matrix. Let $T_3: P_2 \to P_2$ be the linear transformation having this matrix as matrix representation relative to B'',B'' for the ordered basis B'' of part (c). Find $T_3(a_2x^2 + a_1x + a_0)$. How is T_3 related to T_1 and T_2?

23. Let V be the subspace $\text{sp}(x^2e^x, xe^x, e^x)$ of the vector space of all differentiable functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let $T: V \to V$ be the linear transformation of V into itself given by taking second derivatives, so $T = D^2$, and let $B = B' = (x^2e^x, xe^x, e^x)$. Find the matrix A by
   a. following the procedure in summary item 11, and
   b. finding and then squaring the matrix A_1 that represents the transformation D corresponding to taking first derivatives.

24. Let $T: P_3 \to P_2$ be defined by $T(p(x)) = p'(2x + 1)$, where $p'(x) = D(p(x))$, and let $B = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1)$.
   a. Find the matrix A.
   b. Use A to compute $T(4x^3 - 5x^2 + 4x - 7)$.

25. Let $V = \text{sp}(\sin^2 x, \cos^2 x)$ and let $T: V \to V$ be defined by taking second derivatives.

Taking $B = B' = (\sin^2 x, \cos^2 x)$, find A in two ways.

a. Compute A as described in summary item 11.

b. Find the space W spanned by the first derivatives of the vectors in B, choose an ordered basis for W, and compute A as a product of the two matrices representing the differentiation map from V into W followed by the differentiation map from W into V.

26. Let $T: P_3 \to P_3$ be the linear transformation defined by $T(p(x)) = D^2(p(x)) - 4D(p(x)) + p(x)$. Find the matrix representation A of T, where $B = (x, 1 + x, x + x^2, x^3)$.

27. Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ be the subspace of the vector space of all real-valued functions with domain $\mathbb{R}$, and let $B = (e^{2x}, e^{4x}, e^{8x})$. Find the matrix representation A relative to B, B of the linear transformation $T: W \to W$ defined by $T(f) = D^2(f) + 2D(f) + f$.

28. For W and B in Exercise 27, find the matrix representation A of the linear transformation $T: W \to W$ defined by $T(f) = \int_{-\infty}^{x} f(t)\, dt$.

29. For W and B in Exercise 27, find $T(ae^{2x} + be^{4x} + ce^{8x})$ for the linear transformation T whose matrix representation relative to B,B is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

30. Repeat Exercise 29, given that

$$A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

31. Let W be the subspace $\text{sp}(\sin 2x, \cos 2x)$ of the vector space of all real-valued functions with domain $\mathbb{R}$, and let $B = (\sin 2x, \cos 2x)$. Find the matrix representation A relative to B, B for the linear transformation $T: W \to W$ defined by $T(f) = D^2(f) + 2D(f) + f$.

32. For W and B in Exercise 31, find $T(a \sin 2x + b \cos 2x)$ for $T: W \to W$ whose matrix representation is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

33. For W and B in Exercise 31, find $T(a \sin 2x + b \cos 2x)$ for $T: W \to W$ whose matrix representation is

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

34. Let V and V' be vector spaces. Mark each of the following True or False.

___ a. A linear transformation of vector spaces preserves the vector-space operations.

___ b. Every function mapping V into V' relates the algebraic structure of V to that of V'.

___ c. A linear transformation $T: V \to V'$ carries the zero vector of V into the zero vector of V'.

___ d. A linear transformation $T: V \to V'$ carries a pair $\mathbf{v}, -\mathbf{v}$ in V into a pair $\mathbf{v}', -\mathbf{v}'$ in V'.

___ e. For every vector $\mathbf{b}'$ in V', the function $T_{\mathbf{b}'}: V \to V'$ defined by $T_{\mathbf{b}'}(\mathbf{v}) = \mathbf{b}'$ for all $\mathbf{v}$ in V is a linear transformation.

___ f. The function $T_{\mathbf{0}'}: V \to V'$ defined by $T_{\mathbf{0}'}(\mathbf{v}) = \mathbf{0}'$, the zero vector of V', for all $\mathbf{v}$ in V is a linear transformation.

___ g. The vector space P_{10} of polynomials of degree ≤ 10 is isomorphic to $\mathbb{R}^{10}$.

___ h. There is exactly one isomorphism $T: P_{10} \to \mathbb{R}^{11}$.

___ i. Let V and V' be vector spaces of dimensions n and m, respectively. A linear transformation $T: V \to V'$ is invertible if and only if $m = n$.

___ j. If T in part (i) is an invertible transformation, then $m = n$.

35. Prove that the two conditions in Definition 3.9 for a linear transformation are equivalent to the single condition in Eq. (3).

36. Let V, V', and V'' be vector spaces, and let $T: V \to V'$ and $T': V' \to V''$ be linear transformations. Prove that the composite function $(T' \circ T): V \to V''$ defined by $(T' \circ T)(\mathbf{v}) = T'(T(\mathbf{v}))$ for each $\mathbf{v}$ in V is again a linear transformation.

37. Prove that, if T and T' are invertible linear transformations of vector spaces such that $T' \circ T$ is defined, $T' \circ T$ is also invertible.

38. State conditions for an $m \times n$ matrix A that are equivalent to the condition that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$ is an isomorphism.

39. Let $\mathbf{v}$ and $\mathbf{w}$ be independent vectors in V, and let $T: V \to V'$ be a one-to-one linear transformation of V into V'. Prove that $T(\mathbf{v})$ and $T(\mathbf{w})$ are independent vectors in V'.

40. Let V and V' be vector spaces, let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for V, and let $\mathbf{c}_1', \mathbf{c}_2', \ldots, \mathbf{c}_n' \in V'$. Prove that there exists a linear transformation $T: V \to V'$ such that $T(\mathbf{b}_i) = \mathbf{c}_i'$ for $i = 1, 2, \ldots, n$.

41. State and prove a generalization of Exercise 40 for any vector spaces V and V', where V has a basis B.

42. If the matrix representation of $T: \mathbb{R}^n \to \mathbb{R}^n$ relative to B,B is a diagonal matrix, describe the effect of T on the basis vectors in B.

Exercises 43 and 44 show that the set $L(V, V')$ of all linear transformations mapping a vector space V into a vector space V' is a subspace of the vector space of all functions mapping V into V'. (See summary item 5 in Section 3.1.)

43. Let T_1 and T_2 be in $L(V, V')$, and let $(T_1 + T_2): V \to V'$ be defined by

$$(T_1 + T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$$

for each vector $\mathbf{v}$ in V. Prove that $T_1 + T_2$ is again a linear transformation of V into V'.

44. Let T be in $L(V, V')$, let r be any scalar in $\mathbb{R}$, and let $rT: V \to V'$ be defined by

$$(rT)(\mathbf{v}) = r(T(\mathbf{v}))$$

for each vector $\mathbf{v}$ in V. Prove that rT is again a linear transformation of V into V'.

45. If V and V' are the finite-dimensional spaces in Exercises 43 and 44 and have ordered bases B and B', respectively, describe the matrix representations of $T_1 + T_2$ in Exercise 43 and rT in Exercise 44 in terms of the matrix representations of T_1, T_2, and T relative to B,B'.

46. Prove that if $T: V \to V'$ is a linear transformation and $T(\mathbf{p}) = \mathbf{b}$, then the solution set of $T(x) = \mathbf{b}$ is $\{\mathbf{p} + \mathbf{h} \mid \mathbf{h} \in \ker(T)\}$.

47. Prove that, for any five linear transformations T_1, T_2, T_3, T_4, T_5 mapping $\mathbb{R}^2$ into $\mathbb{R}^2$, there exist scalars c_1, c_2, c_3, c_4, c_5 (not all of which are zero) such that $T = c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4 + c_5T_5$ has the property that $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

48. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that, if $T(T(\mathbf{x})) = T(\mathbf{x}) + T(\mathbf{x}) + 3\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, then T is a one-to-one mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$.

49. Let V and V' be vector spaces having the same finite dimension, and let $T: V \to V'$ be a linear transformation. Prove that T is one-to-one if and only if range$(T) = V'$. [HINT: Use Exercise 36 in Section 3.2.]

50. Give an example of a vector space V and a linear transformation $T: V \to V$ such that T is one-to-one but range$(T) \neq V$. [HINT: By Exercise 49, what must be true of the dimension of V?]

51. Repeat Exercise 50, but this time make range$(T) = V$ for a transformation T that is not one-to-one.

3.5 INNER-PRODUCT SPACES (Optional)

In Section 1.2, we introduced the concepts of the length of a vector and the angle between vectors in $\mathbb{R}^n$. *Length* and *angle* are defined and computed in $\mathbb{R}^n$ using the dot product of vectors. In this section, we discuss these notions for more general vector spaces. We start by recalling the properties of the dot product in $\mathbb{R}^n$, listed in Theorem 1.3, Section 1.2.