第四組

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第一題

1. Let N= {1, 2, 3, . . .} be the set of positive integers. Find all functions f, defined onNand taking values inN, such that $(n-1)^2 < f(n)f\big(f(n)\big) < n^2+n$ for every positive integer n.

翻譯:

令N={1,2,3,...}是正整數所組成的集合，求找到所有函數$f: \mathrm{N} \to \mathrm{N}$，使得對於任意正整數 n 都有

$$(n-1)^2 < f(n)f\big(f(n)\big) < n^2+n.$$

Solution.

The only such function is $f(n) = n$. Assume that f satisfies the given condition. It will be shown by induction that $f(n) = n$ for all n ∈ N. Substituting n = 1 yields that $0 < f(1)f\big(f(1)\big) < 2$ which implies the base case $f(1) = 1$. Now assume that $f(k) = k$ for all $\mathrm{k} < \mathrm{n}$ and assume for contradiction that $f(n) = n$. On the one hand, if $f(n) \le n-1$ then$f\big(f(n)\big)$= $f(n)$ and $f(n)\ f\big(f(n)\big)$ = $f(n)^2 \le (n-1)^2$ which is a contradiction. On the other hand, if $f(n) \ge n+1$ then there are several ways to proceed.
Assume $f(n) = M \ge n+1$. Then$(\mathrm{n}+1)f(M) \le\ f(n)f\big(f(n)\big) < n^2+n$. Therefore $f(M) < n$, and hence $f(M)f\big(f(M)\big) = f(M)^2 < n^2 \le (M-1)^2$, which is a contradiction. This completes the induction.

講解:

唯一的函數$f(n)=n$

假設f滿足給定的條件，並歸納$f(n)=n$ 對所有 $n\in \mathrm{N}$

當$n=1$代入 得出

$$0<f(1)f\big(f(1)\big)<2$$

這意味著$f(1)=1$

現在，假設$f(k)=k$ 對所有$k<n$ 且假定與$f(n)=n$矛盾

另一方面，如果$f(n)\le n-1$，則$f(n)f\big(f(n)\big)=f(n)^2\le (n-1)^2$與之矛盾

另一方面，如果$f(n)\ge n+1$，則

假設$f(n)=M\ge n+1$，則$(\mathrm{n}+1)f(M)\le\ f(n)f\big(f(n)\big)<n^2+n=n(n+1)$

因此

$$f(M)<n$$

而且

$$f(M)f\big(f(M)\big)=f(M)^2<n^2\le (M-1)^2\ (\ni\in)$$

這樣矛盾，這樣就完成了歸納

第二題

2. Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that AB · AC + BC · BA + CA · CB AH · AD + BH · BE + CH · CF ≤ 2 .

2.讓三角形ABC是一個銳角三角形，他的三個高分別為AD、BE、CF。讓H是三角形的垂心(三高的交點)，請證明

$$\frac{\mathrm{AB}\cdot\mathrm{AC}+\mathrm{BC}\cdot\mathrm{BA}+\mathrm{CA}\cdot\mathrm{CB}}{\mathrm{AH}\cdot AD+BH\cdot BE+CH\cdot CF}\leq 2$$

第三題

3. On a $(4n + 2) \times (4n + 2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

在$(4n + 2) \times (4n + 2)$的方形網格內，一隻烏龜可經由內部小方型的共用邊移到另一個方形，這隻烏龜從最角落開始，每個位置只能走一次，再走回起點，試問n裡面哪一個最大的正整數k可以使得每一列每一行的邊都經過至少k次?

第四題

4. Let ABC be an acute-angled triangle with circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that BP · CQ = AP · AQ. Prove that Γ is tangent to the circumcircle of triangle BOC.

假設三角形ABC是一個銳角三角形有一個外心O，Γ是一個圓心與點A等高的圓，並通過頂點A還有AB上一點P及AC上一點Q，假設BP · CQ = AP · AQ，證明Γ跟三角形BOC的外接圓相切

第五題

5. Let p be a prime number for which p−1 2 is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most 1 + √ 2p positive integers n such that n < p and p divides a n + b n + c

5.讓p屬於質數為了讓每個 (p−1) /2 也屬於質數
讓a, b, c 是整數且不能整除於p
證明最多有一個1 + √ 2p正整數n使得n<p和p整除
a^n+b^n+c^n

相似題:

1.Let N= {1, 2, 3, ...} be the set of positive integers. Find all functions f, defined onNand taking values inN, such that $1-\frac{1}{n}<f(n)f\big(f(n)\big)<2^{n+1}$ for every positive integer n.