應數一線性代數 2025 秋, 期末考 解答

學號: ________________________, 姓名: ________________________

本次考試共有 7 頁 (包含封面)，有 13 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 1-7 題為填空題。
- 8-13 題為計算證明題。請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ________________________

1. (10 points) Linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfy $T([1,3]) = [2,2,a]$, and $T([2,1]) = [3,b,6]$. If T is NOT one-to-one, then $a + b =$ **(1)** **7**

Solution：

Since $T : \mathbb{R}^2 \to \mathbb{R}^3$ is not one-to-one and the dimension of the domain is 2, the vectors $T([1,3])$ and $T([2,1])$ must be linearly dependent. This implies that the vectors $[2,2,a]^T$ and $[3,b,6]^T$ are proportional:

$$\frac{2}{3} = \frac{2}{b} = \frac{a}{6}$$

From $\frac{2}{3} = \frac{2}{b}$, we get $b = 3$. From $\frac{2}{3} = \frac{a}{6}$, we get $a = 4$. Thus, $a + b = 4 + 3 = 7$.

2. (10 points) Given B and the inverse matrix of B are below , then $a =$ **(2)** **0.8**

$$B = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & 5 & -1 \end{bmatrix}, \; A^{-1} = \begin{bmatrix} * & * & * \\ a & * & * \\ * & * & * \end{bmatrix}$$

Solution：

Using the formula for the inverse matrix $B^{-1} = \frac{1}{\det(B)}\text{adj}(B)$, the element a at position $(2,1)$ of B^{-1} is:

$$a = \frac{1}{\det(B)}(-1)^{1+2}\det(B_{12})$$

$$\det(B) = 5, \;\; \det(B_{12}) = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4 \implies a = \frac{1}{5}(-1)(-4) = 0.8$$

3. (10 points) Suppose that T is a linear transformation with standard matrix representation A, and that A is a 9×15 matrix such that the nullspace of A has dimension 5.

(a) The dimension of the range of T is **(3)** **10** . (b) The dimension of the kernel of T is **(4)** **5** .

Solution：

By the Rank-Nullity Theorem: $\text{rank}(T) + \text{nullity}(T) = \dim(\text{Domain})$. Given $\dim(\text{Domain}) = 15$ and $\text{nullity}(A) = 5$: (a) The dimension of the range of T is $\text{rank}(A) = 15 - 5 = 10$. (b) The dimension of the kernel of T is $\text{nullity}(A) = 5$.

4. (10 points) Find the area of the parallelogram(平行四邊形) in $\mathbb{R}^3$ determined by the vectors $[2,1,3]$ and $[4,-3,1]$. The area is **(5)** $10\sqrt{3}$.

Solution：

The area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ is the magnitude of their cross product $\|\mathbf{u} \times \mathbf{v}\|$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 4 & -3 & 1 \end{vmatrix} = \mathbf{i}(1+9) - \mathbf{j}(2-12) + \mathbf{k}(-6-4) = [10, 10, -10]$$

$$\text{Area} = \sqrt{10^2 + 10^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3}$$

5. (10 points) Let P_3 be the vector space of polynomials with degree at most 3 with real coefficients. The coordinate vector of $7x^3+3x^2-2x+3$ relative to the ordered basis $(x^2+x, x^3, x^3+x, 2x^2+1)$is **(6)** $[-3,6,1,3]^T$.

Solution：

Let the coordinate vector be $[c_1,c_2,c_3,c_4]^T$. We solve:

$$c_1(x^2+x)+c_2(x^3)+c_3(x^3+x)+c_4(2x^2+1)=7x^3+3x^2-2x+3$$

$$(c_2+c_3)x^3+(c_1+2c_4)x^2+(c_1+c_3)x+c_4=7x^3+3x^2-2x+3$$

The coordinate vector is $[-3,6,1,3]^T$.

6. (10 points) Suppose that C is a 6×6 matrix with determinant 4. The $\det(7C^{-1})$ is **(7)** $\frac{7^6}{4}$.

Solution：

Using the property $\det(kA)=k^n\det(A)$ for an $n\times n$ matrix, where $n=6$:

$$\det(7C^{-1})=7^6\cdot\det(C^{-1})=7^6\cdot\frac{1}{\det(C)}=\frac{7^6}{4}$$

7. (10 points)

$$D=\begin{bmatrix}2&4&-2&0&5&-1&9\\1&2&-1&3&1&0&1\\1&2&1&0&3&1&2\\0&5&0&0&1&0&0\\1&2&-1&0&4&0&3\\0&2&0&0&0&0&0\\0&1&0&0&0&0&5\end{bmatrix}$$, The determinant of D is **(8)** **-60** .

Solution：

We compute the determinant of D by performing cofactor expansion on columns/rows with the most zeros.

1. Expand along the 4th column: $\det(D)=3\cdot(-1)^{2+4}\cdot\det(D_{24})$.

2. Expand the remaining 6×6 along the 5th row (only element is 2 at position (5,2)):

$$\det(D)=3\cdot[(-1)^{5+2}\cdot 2\cdot\det(\begin{bmatrix}2&-2&5&-1&9\\1&1&3&1&2\\0&0&1&0&0\\1&-1&4&0&3\\0&0&0&0&5\end{bmatrix})$$

3. Continuing expansion along the 3rd row for 1, then the 4th row for 5:

$$\det(D)=-6\cdot[1\cdot 5\cdot\begin{vmatrix}2&-2&-1\\1&1&1\\1&-1&0\end{vmatrix}]$$

4. Evaluating the 3×3 determinant: 2.

Final result: $\det(D)=-30\times 2=-60$.

8. (10 points) Let P_3 be the vector space of polynomials with degree at most 3 with real coefficients. $T : P_3 \to P_3$ be defined by $T(p(x)) = 2p(x) - 3\frac{d}{dx}p(x)$

(a) Prove that T is a linear transformation.

(b) Let the ordered basis for P_3 is $B = (1, x+1, x^2, x^3 - 1)$. Fine the matrix representation A of T relative to the ordered bases B.

Answer: (b) $A = \begin{bmatrix} 2 & -3 & 6 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution：

Similar with 3-4 exaqmple 9.

(a) Proof that T is a linear transformation: To show that T is a linear transformation, we must verify the properties of linear combination. Let $p(x), q(x) \in P_3$ and $r, s \in \mathbb{R}$.

$$\begin{aligned} T(rp(x) + sq(x)) &= 2(rp(x) + sq(x)) - 3\frac{d}{dx}(rp(x) + sq(x)) \\ &= 2rp(x) + 2sq(x) - 3\left(r\frac{d}{dx}p(x) + s\frac{d}{dx}q(x)\right) \\ &= r\left(2p(x) - 3\frac{d}{dx}p(x)\right) + s\left(2q(x) - 3\frac{d}{dx}q(x)\right) \\ &= rT(p(x)) + sT(q(x)) \end{aligned}$$

Since the property holds, T is a linear transformation.

(b) Matrix representation A relative to the basis $B = (1, x+1, x^2, x^3 - 1)$: To find the matrix $[T]_B$, we apply T to each basis vector and express the result as a linear combination of the basis B.

1. $T(1) = 2(1) - 3(0) = \underline{2} = \mathbf{2}(1) + 0(x+1) + 0(x^2) + 0(x^3 - 1)$
2. $T(x+1) = 2(x+1) - 3(1) = \underline{2x - 1} = 2(x+1) - 3 = -\mathbf{3}(1) + \mathbf{2}(x+1) + 0(x^2) + 0(x^3 - 1)$
3. $T(x^2) = 2(x^2) - 3(2x) = \underline{2x^2 - 6x} = \mathbf{6}(1) + -\mathbf{6}(x+1) + \mathbf{2}(x^2) + 0(x^3 - 1)$
4. $T(x^3 - 1) = 2(x^3 - 1) - 3(3x^2) = \underline{2x^3 - 9x^2 - 2} = \mathbf{0}(1) + \mathbf{0}(x+1) + -\mathbf{9}(x^2) + \mathbf{2}(x^3 - 1)$

Placing the coordinates of these images into the columns of matrix A:

$$A = [T]_B = \begin{bmatrix} 2 & -3 & 6 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

9. (10 points) Let F is the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and $S = \{sin(-x), 1, sin(x), sin(2x)\}$.

Is S linear independent in F? (Yes / $\boxed{No}$) . If not, find a basis of $sp(S)$ $\{1, \sin(x), \sin(2x)\}$.

Solution：

注意一下，這邊是計算證明題，需要有解釋，不然沒分。然後第二部分答案不唯一。

Is S linearly independent in F? No.

The set $S = \{\sin(-x), 1, \sin(x), \sin(2x)\}$. Recall the trigonometric identity for odd functions: $\sin(-x) = -\sin(x)$. This means we can write a non-trivial linear combination that equals the zero function:

$$1 \cdot \sin(-x) + 0 \cdot 1 + 1 \cdot \sin(x) + 0 \cdot \sin(2x) = -\sin(x) + \sin(x) = 0$$

Since there exists a set of coefficients (not all zero) such that the linear combination is zero, the set S is **linearly dependent**.

Find a basis of span(S):

To find a basis, we remove the redundant vector(s) that can be expressed as a linear combination of others.

1. We observed that $\sin(-x) = -\sin(x)$, so $\sin(-x)$ is in the span of $\{\sin(x)\}$.
2. The remaining set is $\{1, \sin(x), \sin(2x)\}$.
3. We check if these are linearly independent. Consider $c_1(1) + c_2 \sin(x) + c_3 \sin(2x) = 0$ for all $x \in \mathbb{R}$.
   - Let $x = 0$: $c_1(1) + 0 + 0 = 0 \implies c_1 = 0$.
   - Let $x = \pi/2$: $c_2 \sin(\pi/2) + c_3 \sin(\pi) = 0 \implies c_2(1) + 0 = 0 \implies c_2 = 0$.
   - Let $x = \pi/4$: $c_3 \sin(\pi/2) = 0 \implies c_3(1) = 0 \implies c_3 = 0$.

Since $c_1 = c_2 = c_3 = 0$ is the only solution, the set $\{1, \sin(x), \sin(2x)\}$ is linearly independent and spans the same space as S.

Basis of span(S): $\{1, \sin(x), \sin(2x)\}$

10. (10 points) (a) Build a linear transformation that is one-to-one but not onto.

Solution：

Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

- **One-to-one:** The kernel is found by setting $T(\mathbf{x}) = \mathbf{0}$, which implies $x_1 = 0$ and $x_2 = 0$. Since $\ker(T) = \{\mathbf{0}\}$, T is one-to-one.
- **Not onto:** The range of T is the xy-plane in $\mathbb{R}^3$. Any vector with a non-zero z-component (e.g., $[0, 0, 1]^T$) has no preimage. Since $\text{rank}(T) = 2 < \dim(\mathbb{R}^3)$, it is not onto.

(b) Build a linear transformation that is onto but not one-to-one.

Solution：

Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **Onto:** For any vector $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$, we can choose $\mathbf{x} = [y_1, y_2, 0]^T$ such that $T(\mathbf{x}) = \mathbf{y}$. Since the range is all of $\mathbb{R}^2$, it is onto.
- **Not one-to-one:** The kernel consists of all vectors where $x_1 = 0$ and $x_2 = 0$, specifically $\ker(T) = \{[0, 0, x_3]^T \mid x_3 \in \mathbb{R}\}$. Since the kernel contains non-zero vectors, T is not one-to-one.

* 這兩小題在回答時，記得也要告知 domain 跟 codomain。然後 domain, codomain 本身要是 vector space。

** 另外 mapping 本身也要記得有 linear transformation 的性質（由 2-3 example 3 知道，用矩陣構造不需驗）。

11. (10 points) Determine the set S_1 of all functions f such that $f(0) = 0$ is a subspace in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Answer: Is S_1 a subspace of F? (Yes / $\boxed{No}$)

Solution：

1. **Closed under Addition:** Let $f(x), g(x) \in S_1$. By definition, $f(0) = 0$ and $g(0) = 0$. Consider the sum function $(f \oplus g)$:
$$(f \oplus g)(0) = f(0) + g(0) = 0 + 0 = 0 \implies (f \oplus g) \in S_1$$
2. **Closed under Scalar Multiplication:** Let $f(x) \in S_1$ and $r \in \mathbb{R}$. By definition, $f(0) = 0$. Consider the scalar product function $(r \otimes f)$:
$$(r \otimes f)(0) = r \cdot f(0) = r \cdot 0 = 0 \implies (r \otimes f) \in S_1$$

Thus S_1 is a subspace of F.

12. (10 points) Consider the set $\mathbb{R}^2$, with the addition defined by $[x,y] \oplus [a,b] = [x+a+2, y+b]$, and with scalar multiplication defined by $r \otimes [x,y] = [r(x+2)-2, ry]$.

   a. Is this set a vector space? ($\boxed{Yes}$ / No)
   Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. If the set is a vector space, then find the zero vector and the additive inverse (加法反元素) in this vector space. *Hint:* The zero vector may NOT be the vector $[0,0]$.

   Answer: the zero vector is **[-2, 0]**, for any vectors [x,y], the -[x,y] is **[-x-4, -y]**

Solution：

這題的零向量跟加法反元素的部分，照定義的做法是這樣：

1. **Find the Zero Vector $\vec{0}$:** Let $\vec{0} = [e_1, e_2]$. We require $[x,y] \oplus [e_1, e_2] = [x,y]$:

$$[x + e_1 + 2, y + e_2] = [x,y]$$

   Comparing components: $x + e_1 + 2 = x \implies e_1 = -2$ $y + e_2 = y \implies e_2 = 0$

   Zero vector: $[-2, 0]$

2. **Find the Additive Inverse $-[x,y]$:** Let $-[x,y] = [x', y']$. We require $[x,y] \oplus [x', y'] = \vec{0} = [-2, 0]$:

$$[x + x' + 2, y + y'] = [-2, 0]$$

   Comparing components: $x + x' + 2 = -2 \implies x' = -x - 4$ $y + y' = 0 \implies y' = -y$

   Additive inverse: $[-x-4, -y]$

** 我在題目裡特別給提示了『零向量不是 [0,0]』，還是有一堆人踩中這個點 ! !

或者是用簡單的做法：

By the Theorem 3.1, if $(V, \oplus, \otimes)$ is a vector space, then

1. **Find the Zero Vector $\vec{0}$:**
   We require $\vec{0} = 0 \otimes [x,y] \implies \vec{0} = 0 \otimes [x,y] = [0(x+2)-2, 0y] = [-2, 0]$.

2. **Find the Additive Inverse $-[x,y]$:**
   We require $-[x,y] = (-1) \otimes [x,y] \implies -[x,y] = (-1) \otimes [x,y] = [-(x+2)-2, -y] = [-x-4, -y]$.

接著驗證十點條件，都對的話，就是 vector space!

Verification of Vector Space Axioms: Let $\vec{u} = [x,y]$, $\vec{v} = [a,b]$, $\vec{w} = [c,d] \in \mathbb{R}^2$ and $r, s \in \mathbb{R}$.

- **Closure under Addition:** $\vec{u} \oplus \vec{v} = [x+a+2, y+b]$ in $\mathbb{R}^2$.
- **Closure under Scalar Multiplication:** $r \otimes \vec{u} = [r(x+2)-2, ry]$ in $\mathbb{R}^2$.
- **A1 (Associativity of $\oplus$):**
  $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = [x+a+2, y+b] \oplus [c,d] = [(x+a+2)+c+2, y+b+d] = [x+a+c+4, y+b+d]$.
  $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = [x,y] \oplus [a+c+2, b+d] = [x+(a+c+2)+2, y+b+d] = [x+a+c+4, y+b+d]$. (Holds)
- **A2 (Commutativity of $\oplus$):** $\vec{u} \oplus \vec{v} = [x+a+2, y+b] = [a+x+2, b+y] = \vec{v} \oplus \vec{u}$. (Holds)

- **A3 (Identity Element):** There exists $\vec{0} \oplus \vec{u} = [-2, 0] \oplus [x, y] = [(-2) + x + 2, 0 + y] = [x, y] = \vec{u}$. (Holds)
- **A4 (Inverse Element):** For each $\vec{u}$, there exists $-\vec{u} = [-x - 4, -y]$ such that $\vec{u} \oplus (-\vec{u}) = \vec{0}$.
  $\vec{u} \oplus (-\vec{u}) = [x, y] \oplus [-x - 4, -y] = [x + (-x - 4) + 2, y + (-y)] = [-2, 0] = \vec{0}$. (Holds)
- **S1 (Scalar Distributivity over Vector Addition):**
  $r \otimes (\vec{u} \oplus \vec{v}) = r \otimes [x + a + 2, y + b] = [r(x + a + 2 + 2) - 2, r(y + b)] = [r(x + a + 4) - 2, r(y + b)]$.
  $(r \otimes \vec{u}) \oplus (r \otimes \vec{v}) = [r(x + 2) - 2, ry] \oplus [r(a + 2) - 2, rb] = [(r(x + 2) - 2) + (r(a + 2) - 2) + 2, ry + rb] = [r(x + 2 + a + 2) - 2, r(y + b)]$. (Holds)
- **S2 (Vector Distributivity over Scalar Addition):**
  $(r + s) \otimes \vec{u} = [(r + s)(x + 2) - 2, (r + s)y] = [r(x + 2) + s(x + 2) - 2, ry + sy]$.
  $(r \otimes \vec{u}) \oplus (s \otimes \vec{u}) = [r(x + 2) - 2, ry] \oplus [s(x + 2) - 2, sy] = [(r(x + 2) - 2) + (s(x + 2) - 2) + 2, ry + sy] = [r(x + 2) + s(x + 2) - 2, ry + sy]$. (Holds)
- **S3 (Associativity of Scalar Multiplication):**
  $r \otimes (s \otimes \vec{u}) = r \otimes [s(x + 2) - 2, sy] = [r(s(x + 2) - 2 + 2) - 2, r(sy)] = [rs(x + 2) - 2, rsy]$.
  $(rs) \otimes \vec{u} = [rs(x + 2) - 2, rsy]$. (Holds)
- **S4 (Scalar Identity):** $1 \otimes \vec{u} = [1(x + 2) - 2, 1(y)] = [x, y]$. (Holds)

13. (10 points) Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$. Show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Solution：

Section 4-1 problem 59. 用定義驗證

本題特別要注意，驗證之前要先定義 $\vec{a}, \vec{b}, \vec{c}$ 的各 components 為何。

另外是向量的符號要用對。習慣上，(a_1, a_2, a_3) 是指一個點，$[a_1, a_2, a_3]$ 才是一個向量。

學號: ______________, 姓名: ______________,

(1) 7	(2) 0.8	(3) 10	(4) 5
(5) $10\sqrt{3}$	(6) $[-3, 6, 1, 3]^T$	(7) $\frac{7^6}{4} = \frac{117649}{4}$	(8) -60

以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	10	130
Score:														