應數三數值分析 2020 秋, 期末考解答

學號: ______________________, 姓名: ______________________

本次考試共有 8 頁 (包含封面)，有 9 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 僅可以使用電子計算機，不可用手機替代。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。　敬，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。　遠，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳，亦不可使用允許範圍之外的物件。

1. (20 points) For $f(x) = e^{2x}$.

(a) Use three-point midpoint formula with $h = 0.1$ to approximate $f'(1.2)$ and calculate the exact error and the error bound.

Answer: $f'(1.2) =$ 22.193635, exact error = 0.147282, error bound = 0.179519

(b) Find a proper h such that the error bound will smaller than 0.0001.

Answer: h= 0.0026

$f'(x_0) = \dfrac{1}{2h}[-f(x_0-h)+f(x_0+h)] - \dfrac{h^2}{6}f^{(3)}(\xi_0)$, hence,

$$f'(1.2) \approx \frac{1}{2h}[-f(x_0-h)+f(x_0+h)] = \frac{1}{0.2}[-f(1.1)+f(1.3)] = 22.193635.$$

$f'(x) = 2e^{2x} \Rightarrow f'(1.2) = 22.04635$. Then the actual error $= |22.193635 - 22.046353| = 0.147282$

$f^{(3)}(x) = 8e^{2x} \Rightarrow M = 8e^{2\times 1.3} = 107.709904$.

Then the error bound $= \displaystyle\max_{1.1\le\xi_0\le 1.3} |\frac{h^2}{6}f^{(3)}(\xi_0)| = \frac{0.1^2}{6}M = 0.179519$.

The error bound $< 10^{-1}$, i.e. $\displaystyle\max_{1.2-h\le\xi_0\le x_0+h} |\frac{h^2}{6}f^{(3)}(\xi_0)| < 10^{-1}$. Therefore, $h > 0.0026$.

2. (15 points) The quadrature formula $\int_0^2 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1, and c_2.

Answer: $c_0 =$ 1/3, $c_1 =$ 4/3, $c_2 =$ 1/3

Let $f(x) = ax^2 + bx + c$.

$$\int_0^2 f(x)dx = \frac{8}{3}a + 2b + 2c$$

$$\begin{aligned} c_0 f(0) + c_1 f(1) + c_2 f(2) &= c_0 \times c + c_1 \times (a+b+c) + c_2 \times (4a+2b+c) \\ &= (c_1 + 4c_2)a + (c_1 + 2c_2)b + (c_0 + c_1 + c_2)c \end{aligned}$$

$$\begin{cases} c_1 + 4c_2 = 8/3 \\ c_1 + 2c_2 = 2 \\ c_0 + c_1 + c_2 = 2 \end{cases}$$

3. (15 points) Romberg integration is used to approximate $\int_0^1 f(x)dx$.
If $R_{11} = 4$ and $R_{22} = 5$, find $f(1/2)$=?

Answer: $f(1/2) =$ 5.5.

$$R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{4-1} = 5. \text{ Since } R_{1,1} = 4, R_{2,1} = \frac{19}{4}$$

$$R_{2,1} = \frac{1}{2}[R_{1,1} + h_1 \sum_{i=1}^{1} f(a + (2i-1)h_2)] = \frac{1}{2}(4 + f(\frac{1}{2})) \Rightarrow f(\frac{1}{2}) = 5.5$$

4. (20 points) Consider the definite integral $\int_1^3 (x+1)^4 + e^x dx$. It is desired to approximate it within 10^{-1} accuracy using the Composite Simposon's rule.

(a) Determine the possible smallest n value. Where n is the number of the subintervals.

Answer: n = 4.

Note that $f^{(4)}(x) = 24 + e^x$.

$$E = \left|\frac{b-a}{180}h^4 f^{(4)}(\xi)\right| = \left|\frac{3-1}{180}\left(\frac{2}{n}\right)^4 (24+e^{\xi})\right| \le \frac{2^5}{180n^4}\max_{1\le\xi\le3}|24+e^{\xi}| = \frac{2^5}{180n^4}|24+e^3|$$

$$\le 7.8374\frac{1}{n^4} \le 10^{-4} \Rightarrow n \ge 2.9754$$

Since n should be an even number, we have $n \ge 4$.

(b) Calculate the approximation by using the n value that you have found in (a).

Answer: 215.78977.

$n = 4, h = \frac{3-1}{4} = 0.5, x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3.$

$$\int_1^3 (x+1)^4 + e^x dx \approx \frac{0.5}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \approx 215.78977$$

(c) Evaluate the actual error, is it less than 10^{-1}.

Answer: actual error = $0.022522 < 10^{-1}$.

$$\int_1^3 (x+1)^4 + e^x dx = \frac{(x+1)^5}{5} + e^x \Big|_1^3 = 215.767255$$

actual error $= |215.767255 - 215.78977| = 0.022522 < 10^{-1}$

5. (15 points) Use **three point Gaussian quadrature** to approximate, $\int_0^1 xe^{x^2}dx$. You may use the table below. You do not need to simplify your answer.

Points	Weighting Function	Factors Arguments
2	c1 = 1.000000000 c2 = 1.000000000	x1 = -0.577350269 x2 = 0.577350269
3	c1 = 0.555555556 c2 = 0.888888889 c3 = 0.555555556	x1 = -0.774596669 x2 = 0.000000000 x3 = 0.774596669
4	c1 = 0.347854845 c2 = 0.652145155 c3 = 0.652145155 c4 = 0.347854845	x1 = -0.861136312 x2 = -0.339981044 x3 = 0.339981044 x4 = 0.8611363123

What is the highest order polynomial that this formula will provide an exact answer for?

Answer: the highest order = ____5____,

approximate = ________________________________.

Since we use 3 points, $2 \times 3 - 1 = 5$.

from [1, 0] to [-1, 1], we let $t = 2x - 1$, then $dt = 2dx$. Let

$$g(t) = \left(\frac{t+1}{2}\right) e^{(\frac{t+1}{2})^2}$$

$$\int_0^1 xe^{x^2}dx = \int_{-1}^1 \frac{1}{2}g(t)dt \approx c_1 g(x_1) + c_2 g(x_2) + c_3 g(x_3),$$

where

c1 = 0.555555556	x1 = -0.774596669
c2 = 0.888888889	x2 = 0.000000000
c3 = 0.555555556	x3 = 0.774596669

6. (15 points) The following data give approximations to the integral $M = \int_0^\pi \sin(x)dx$
$N_1(h) = 1.570796, N_1(h/2) = 1.896119, N_1(h/4) = 1.974232, N_1(h/8) = 1.993570.$
Assuming $M = N_1(h) + K_1h^2 + K_2h^4 + K_3h^6 + K_4h^8 + O(h^10)$, construct an extrapolation table to determine $N_4(h)$.

Answer: $N_4(h) =$ 1.99999, and draw the table below.

Since the approximation formula with truncation error contain only the even powers of h. We have

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{\left[N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)\right]}{4^{j-1} - 1}$$

$N_1(h) = 1.570796$			
$N_1(\frac{h}{2}) = 1.896119$	$N_2(h) = 2.00456$		
$N_1(\frac{h}{4}) = 1.974232$	$N_2(\frac{h}{2}) = 2.00026$	$N_3(h) = 1.99977$	
$N_1(\frac{h}{8}) = 1.993570$	$N_2(\frac{h}{2}) = 2.00001$	$N_3(\frac{h}{2}) = 1.99999$	$N_4(h) = 1.99999$

第二部份，額外加分題

7. (15 points) Use the Newton's Method to find a solution within $\epsilon = 10^{-4}$ for the function $f(x) = x - \cos(x) = 0$ where $0 \le x \le \frac{\pi}{2}$, starting with $p_0 = 0$

the Newton's Method is

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \text{ for } n \ge 1.$$

$p_1 = 0$

$p_2 = 0.7503638$

$p_3 = 0.7391128$

$p_4 = 0.739085$

8. (15 points) Let $f(x) = \sin(e^x - 2)$. Find the Hermite polynomial that agrees with the function and its derivative at the points $x_0 = 0.8, x_1 = 1.0$. Then use your function to approximate $f(0.9)$. **Answer:** $f(0.9) =$ 0.44392477

the Hermite polynomial = $0.2236 + 2.169(x-0.8) + 0.0155(x-0.8)^2 - 3.2177(x-0.8)^2(x-1)$.

$\sin(e^{0.8}-2)$=0.22363362			
$\sin(e^{0.8}-2)$=0.22363362	2.16917528		
$\sin(e^{1.0}-2)$=0.65809197	2.17229172	0.01558224	
$\sin(e^{1.0}-2)$=0.65809197	2.04669647	-0.62797625	-3.21779244

$$0.2236 + 2.169(0.9-0.8) + 0.0155(0.9-0.8)^2 - 3.2177(0.9-0.8)^2(0.9-1) \approx 0.44392477$$

9. (20 points) The iterative method to solve $f(x) = 0$, given by the fixed-point method $g(x) = x$, where

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} - \frac{f''(p_{n-1})}{2f'(p_{n-1})}\left[\frac{f(p_{n-1})}{f'(p_{n-1})}\right]^2 \quad \text{for } n = 1, 2, 3, ...,$$

has $g'(p) = g''(p) = 0$. This will generally yield cubic ($\alpha = 3$) convergence.

Expanding $g(x)$ in Taylor polynomial for $x \in [p-\delta, p+\delta]$ gives

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(\xi)}{3!}(x-p)^3$$

where ξ lies between x and p. The problem gave $g'(p) = g''(p) = 0$ imply that

$$g(x) = p + \frac{g'''(\xi)}{6}(x-p)^3$$

In particular, when $x = p_n$,

$$p_{n+1} = g(p_n) = p + \frac{g'''(\xi_n)}{6}(p_n - p)^3$$

with ξ_n lies between p_n and p. Thus $p_{n+1} - p = \frac{g'''(\xi_n)}{6}(p_n - p)^3$ Since

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)}\left[\frac{f(x)}{f'(x)}\right]^2 \quad \text{and } f(x) = e^x - x - 1$$

, we have $|g'(x)| \le k < 1$ on $[p-\delta, p+\delta]$ and g maps $[p-\delta, p+\delta]$ into itself, it follows from the Fixed-Point Theorem that $\{p_n\}_{n=0}^{\infty}$ converges to p. But ξ_n is between p and p_n for each n, so $\{\xi_n\}_{n=0}^{\infty}$ also converges to p, and we have $\lim_{n\to\infty} g'''(\xi_n) = g'''(p)$ Thus

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n - p|^{\alpha}} = \lim_{n\to\infty}\frac{|g'''(\xi_n)||p_n - p|^{3-\alpha}}{6} = \frac{|g'''(p)| \times 0}{6} = 0, \text{ for } \alpha = 1, 2$$

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n - p|^{3}} = \lim_{n\to\infty}\frac{|g'''(\xi_n)|}{6} = \frac{|g'''(p)|}{6}$$

Hence, if $g'''(p) \neq 0$, fixed-point iteration exhibits cubic convergence with asymptotic error constant $|g'''(p)|$.

學號: ______________________, 姓名: ______________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	20	15	15	20	15	15	15	15	20	150
Score:										