Homework 3 Solution

2-2

題號： 9, 14(a)(b)

2-2 #9

Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(\frac{x}{2})$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2} . Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

Answer:

Since $g'(x) = \frac{1}{4}\cos x^2$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \leq g(x) \leq g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \leq \frac{1}{4}$, for $0 \leq x \leq 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n$$

For the bound to be less than 0.1, we need $n \geq 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

2-2 #14

For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5} , and perform the calculations.

(a) $2 + \sin x - x = 0$ use $[2, 3]$

(b) $x^3 - 2x - 5 = 0$ use $[2, 3]$

Answer:

The inequalities in Corollary 2.4 give $|p_n - p| < k^n \max(p_0 - a, b - p_0)$.

We want $k^n \max(p_0 - a, b - p_0) < 10^{-5}$, so we need $n > \dfrac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}$.

(a) Using $g(x) = 2 + \sin x$ we have $k = 0.9899924966$, so that with $p_0 = 2$ we have $n > \dfrac{\ln(0.00001)}{\ln k} = 1144.663221$. However, our tolerance is met with $p_{63} = 2.5541998$.

(b) Using $g(x) = \sqrt[3]{2x + 5}$ we have $k = 0.1540802832$, so that with $p_0 = 2$ we have $n > \dfrac{\ln(0.00001)}{\ln k} = 6.155718005$. However, our tolerance is met with $p_6 = 2.0945503$.

2-3

題號： 13, 17, 18, 19

2-3 #13
The fourth-degree polynomial $f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$ has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the

(b) Secant method

(c) Newton' s method

Use the endpoints of each interval as the initial approximations in part (a) and (b) and the midpoints as the initial approximation in part (c).

Answer:

(b) For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$

(c) For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

2-3 #17
The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.

(a) Determine, within 10^{-6} , the only negative zero.

(b) Determine, within 10^{-6} , the four smallest positive zeros.

(c) Determine a reasonable initial approximation to find the nth smallest positive zero of f . [Hint: Sketch an approximate graph of f.]

(d) Use part (c) to determine, within 10^{-6} , the 25^{th} smallest positive zero of f.

Answer:

For $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$, we have the following roots.

(a) For $p_0 = -0.5$, we have $p_3 = 0.4341431$.

(b) For $p_0 = 0.5$, we have $p_3 = 0.4506567$.
For $p_0 = 1.5$, we have $p_3 = 1.7447381$.
For $p_0 = 2.5$, we have $p_5 = 2.2383198$.
For $p_0 = 3.5$, we have $p_4 = 3.7090412$.

(c) The initial approximation $n - 0.5$ is quite reasonable.

(d) For $p_0 = 24.5$, we have $p_2 = 24.4998870$.

2-3 #18
Use Newton' s method to solve the equation $0 = \frac{1}{2} + \frac{1}{4}x^2 - x \sin x - \frac{1}{2} \cos 2x$, with $p_0 = \frac{\pi}{2}$. Iterate using Newton' s method until an accuracy of 10^{-5} is obtained. Explain why the result

seems unusual for Newton' s method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

Answer:

Newton' s method gives $p_{15} = 1.895488$, for $p_0 = \frac{\pi}{2}$; and $p_{19} = 1.895489$, for $p_0 = 5\pi$. The sequence does not converge in 200 iterations for $p_0 = 10\pi$. The results do not indicate the fast convergence usually associated with Newton' s method.

2-3 #19

Use Newton' s method to approximate, to within 10^{-4} , the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$. [Hint: Minimize $[d(x)]^2$, where $d(x)$ represents the distance from (x, x^2) to $(1, 0)$.]

Answer:

For $p_0 = 1$, we have $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.