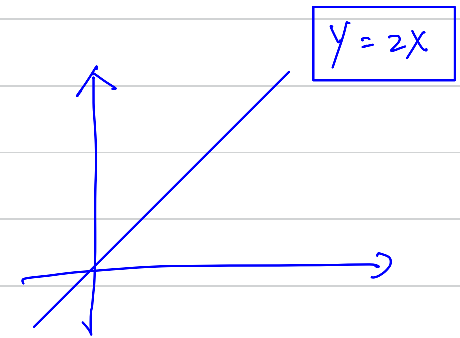
2-3

$f(x) = 2x$



map $f: \underset{\text{domain}}{X} \longrightarrow \underset{\text{codomain}}{Y}$

$\cup\!\mathrm{I}$

$f(X)$ → range

ex:

$f: \{1, 2, 3, \ldots, 10\} \longrightarrow \mathbb{N}$ ← 自然數

$x \longmapsto f(x) = 2x$

① $\{2, 4, 6, \ldots, 20\}$

② $\{1, 2, 3, \ldots, 20\}$

③ $\mathbb{N}$

e.g. $f(x) = \dfrac{x-3}{x-2}$ $\quad x \neq 2$

定義域 domain : $\{1, 2, \ldots, 10\}$

2? 應變域 codomain $= \mathbb{N}$

值域 range $= \{2, 4, 6, 8, \ldots, 20\} = \{f(x) \mid x \in X\}$

$\boxed{\text{Well-defined}}$

① f 在 X 上可以送出去

② $f(X) \subseteq Y$

Def.

T (map) $: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **linear transformation**

if

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, $\forall\, \vec{v}, \vec{u} \in \mathbb{R}^n$

Preservation of vector addition

② $T(r\vec{v}) = rT(\vec{v})$, $\forall\, r \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$

Preservation of scalar multiplication

P.S. $T(\vec{0}) = T(0 \cdot \vec{0}) = 0\,T(\vec{0}) = \vec{0}$

or

if $T(r\vec{u}+s\vec{v}) = rT(\vec{u}) + sT(\vec{v})$, $\forall\, r, s \in \mathbb{R}$, $\vec{u}, \vec{v} \in \mathbb{R}^n$

Preservation of linear combination

Prop.

$T: \mathbb{R}^n \to \mathbb{R}^m$: linear trans

if W : subspace of $\mathbb{R}^n$ then $\underset{\{T(\vec{u}) \mid \vec{u} \in W\}}{T(W)}$: subspace of $\mathbb{R}^m$

Recall

W : subspace of $\mathbb{R}^m$

if ① W : subset of $\mathbb{R}^m$

② $\vec{u}+\vec{v} \in W$, $\forall \vec{u}, \vec{v} \in W$

③ $r\vec{v} \in W$, $\forall r \in \mathbb{R}$

p.f.

(i) $\because$ range in condomain $\therefore$ $T(W)$ in $\mathbb{R}^m$

(ii) $\forall \vec{p}, \vec{q} \in T(W)$, $r \in \mathbb{R}$, claim: $\begin{cases} ① \ \vec{p}+\vec{q} \in T(W) \\ ② \ r\vec{p} \in T(W) \end{cases}$

☆ $\vec{u}, \vec{v}$ 不一定唯一

maybe $T(\vec{\alpha}) = \vec{p}$, $T(\vec{\beta}) = \vec{p}$

$\because \vec{p}, \vec{q} \in T(W)$ $\therefore \exists \vec{u}, \vec{v} \in W$ s.t. $T(\vec{u}) = \vec{p}$, $T(\vec{v}) = \vec{q}$

$\vec{p}+\vec{q} = \overbrace{T(\vec{u}) + T(\vec{v}) = T(\underbrace{\vec{u}+\vec{v}}_{\in W})}^{T:\ \text{linear trans.}} \in T(W)$ $\left(\begin{array}{l} \because \vec{u}, \vec{v} \in W,\ W: \text{subspace of } \mathbb{R}^n \\ \therefore \vec{u}+\vec{v} \in W \end{array} \right)$

(iii) $r\vec{p} = \underbrace{r\,T(\vec{u}) = T(\overbrace{r\vec{u}}^{\in W})}_{T:\ \text{linear trans.}} \in T(W)$ #

ex:

given $A_{m\times n}$, define $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \longmapsto T(\vec{x}) = A\vec{x}$

$\begin{array}{cc} m\times n & n\times 1 \\ (A & \vec{x}) \end{array} \rightarrow m\times 1$

check T is an linear trans

① $\underset{T(\vec{x})+T(\vec{y})}{\underset{\shortparallel}{A\vec{x}+A\vec{y}}} = \underset{T(\vec{x}+\vec{y})}{\underset{\shortparallel}{A(\vec{x}+\vec{y})}}$

② $\underset{T(r\vec{x})}{\underset{\shortparallel}{A(r\vec{x})}} = \underset{rT(\vec{x})}{\underset{\shortparallel}{rA\vec{x}}}$

ex:

$T : \mathbb{R} \rightarrow \mathbb{R}$: NOT linear trans.

$x \mapsto \sin(x)$

check : $\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) \overset{?}{\neq} \sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)$

$x=\frac{\pi}{4}$, $y=\frac{\pi}{4}$

Thm

$T: \mathbb{R}^n \to \mathbb{R}^m$: linear trans , $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$: basis for $\mathbb{R}^n$

$\forall \vec{p} \in T(\mathbb{R}^n)$ can be expressed by $\mathcal{B}' = \{T(\vec{b}_1), \ldots, T(\vec{b}_n)\}$ $\leftarrow sp(T(\vec{b}_1), \ldots, T(\vec{b}_n)) = T(\mathbb{R}^n)$

p.f.

$\because \vec{p} \in T(\mathbb{R}^n) \quad \therefore \exists \vec{v}$ s.t. $\vec{p} = T(\vec{v})$

$\because \mathcal{B}$: basis for $\mathbb{R}^n$

$\therefore \exists! \, r_1, \ldots, r_n \in \mathbb{R}$ s.t. $r_1\vec{b}_1 + \ldots + r_n\vec{b}_n = \vec{v}$

$$
\begin{aligned}
\therefore \vec{p} &= T(\vec{v}) \\
&= T(r_1\vec{b}_1 + \ldots + r_n\vec{b}_n) \\
&\vdots \\
&= T(r_1\vec{b}_1 + r_2\vec{b}_2 + r_3\vec{b}_3) + T(r_4\vec{b}_4) + \ldots + T(r_n\vec{b}_n) \\
&= T(r_1\vec{b}_1 + r_2\vec{b}_2) + T(r_3\vec{b}_3) + \ldots + T(r_n\vec{b}_n) \\
&= T(r_1\vec{b}_1) + T(r_2\vec{b}_2) + T(r_3\vec{b}_3) + \ldots + T(r_n\vec{b}_n) \\
&= r_1 T(\vec{b}_1) + \ldots + r_n T(\vec{b}_n)
\end{aligned}
$$