

Section 9-3

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Prove that every 2×2 real matrix that is unitarily diagonalizable has one of the following forms $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ -b & d \end{bmatrix}$ for $a, b, d \in \mathbb{R}$

解答：

Every 2×2 real matrix A can be written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is unitarily diagonalizable, A is normal, i.e. $A^*A = AA^*$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \\ \begin{bmatrix} a\bar{a} + c\bar{c} & \bar{a}b + \bar{c}d \\ a\bar{b} + c\bar{d} & b\bar{b} + d\bar{d} \end{bmatrix} = \begin{bmatrix} a\bar{a} + b\bar{b} & a\bar{c} + b\bar{d} \\ \bar{a}c + \bar{b}d & c\bar{c} + d\bar{d} \end{bmatrix}$$

Hence: (notice that a, b, c, d are real.)

$$(1). a\bar{a} + c\bar{c} = a\bar{a} + b\bar{b} \implies a^2 + c^2 = a^2 + b^2$$

$$(2). \bar{a}b + \bar{c}d = a\bar{c} + b\bar{d} \implies ab + cd = ac + bd$$

$$(3). a\bar{b} + c\bar{d} = \bar{a}c + \bar{b}d \implies ab + cd = ac + bd$$

$$(4). b\bar{b} + d\bar{d} = c\bar{c} + d\bar{d} \implies b^2 + d^2 = c^2 + d^2$$

by (1) and (4), we have $b = c$ or $b = -c$. And (2)(3) holds for both cases.

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Prove that an $n \times n$ matrix A is unitarily diagonalizable if and only if $\|A\vec{v}\| = \|A^*\vec{v}\|$ for all $\vec{v} \in \mathbb{C}^n$.

解答：

If A is unitarily diagonalizable, then there exists a unitary U and a diagonal matrix D such that $A = UDU^*$. $\|A\vec{v}\|^2 = (A\vec{v})^*(A\vec{v}) = \vec{v}^*A^*A\vec{v} = \vec{v}^*(UDU^*)^*(UDU^*)\vec{v} = \vec{v}^*UD^*U^*UDU^*\vec{v} = \vec{v}^*UD^*DU^*\vec{v} = \vec{v}^*UDD^*U^*\vec{v} = \vec{v}^*UDU^*UD^*U^*\vec{v} = \vec{v}^*(UD^*U^*)^*(UD^*U^*)\vec{v} = \|A^*\vec{v}\|^2$

Since all norms are real and positive, $\|A\vec{v}\| = \|A^*\vec{v}\|$.

If $\|A\vec{v}\| = \|A^*\vec{v}\|$, we have $\|A\vec{v}\|^2 = (A\vec{v})^*(A\vec{v}) = \vec{v}^*A^*A\vec{v} = \vec{v}^*AA^*\vec{v} = \|A^*\vec{v}\|^2$. That is $\vec{v}^*A^*A\vec{v} = \vec{v}^*AA^*\vec{v}$ for all $\vec{v} \in \mathbb{C}^n$. We have $A^*A = AA^*$. By theorem 9.7, A is unitarily diagonalizable.