in $\mathbb{R}$ v.s. ① A: $n\times n$ $\mathbb{R}$ matrix, hope $\exists C$: orthogonal, s.t. $C^TAC = D$

② C: orthogonal, 保長

Thm $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\vec{v} \longmapsto T(\vec{v}) = A\vec{v}$ $\Rightarrow \|\vec{v}\| = \|T(\vec{v})\|$ if A: orthogonal

in $\mathbb{C}$ v.s. ① A: $n\times n$ $\mathbb{C}$ matrix, hope $\exists U$: unitary, s.t. $U^*AU = D$

② U: unitary, 保長

Thm $T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$\vec{z} \longmapsto T(\vec{z}) = U\vec{z}$ $\Rightarrow \|\vec{z}\| = \|T(\vec{z})\|$ if U: unitary

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Thm $T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ if U: unitary, s.m.v. of T

$\vec{z} \longmapsto T(\vec{z}) = U\vec{z}$ $\Rightarrow \|\vec{z}\| = \|T(\vec{z})\|$

p.f. $\|\vec{z}\|^2 = \langle \vec{z}, \vec{z} \rangle = \vec{z}^*\vec{z}$

$$\|U\vec{z}\|^2 = (U\vec{z})^*(U\vec{z}) = \vec{z}^*\underbrace{U^*U}_{I}\vec{z} = \vec{z}^*\vec{z} = \|\vec{z}\|^2 \quad \therefore \|\vec{z}\| = \|U\vec{z}\|$$

Def.

A: symmetric if $A^T = A$

△ A: real symmetric $\Rightarrow C^TA\overset{\overset{\text{orthogonal}}{\uparrow}}{C} = D$

Def

H: hermitian if $H^* = H$

△ H: hermitian $\Rightarrow U^*H\overset{\overset{\text{unitary}}{\uparrow}}{U} = D$

△ in $\mathbb{R}^3$, $\vec{a}, \vec{b}$, 找 $\vec{c}$ s.t. $\vec{a} \perp \vec{c}$, $\vec{b} \perp \vec{c}$

pick $\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

check $\vec{a}\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$, $\vec{b}\cdot\vec{c} = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

△ in $\mathbb{C}^3$, $\vec{a}, \vec{b}$, 找 $\vec{c}$ s.t. $\langle \vec{a}, \vec{c} \rangle = 0 = \langle \vec{b}, \vec{c} \rangle$

pick $\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{vmatrix}$ check $\langle \vec{a}, \vec{c} \rangle = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{vmatrix} = 0$

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in ℝ v.s.

A : n×n real , $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq 0$, λ : eigenvalue , $\vec{v}$: eigenvector corr. to λ

①求 λ : sol $p(\lambda) = |A - \lambda I| = 0$

② 求 null$(A-\lambda_i I)$ 得 E_{λ_i}

in ℂ v.s.

A : n×n , $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq 0$, $\overset{\in \mathbb{C}}{\lambda}$: eigenvalue , $\overset{\in \mathbb{C}^n}{\vec{v}}$: eigenvector corr. to λ

↑ 有 n 個

①求 λ : sol $p(\lambda) = |A - \lambda I| = 0$

② 求 null$(A-\lambda_i I)$ 得 E_{λ_i}

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ex:

$$A = \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & i \\ 0 & 2-\lambda & 0 \\ -i & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)^2 - (2-\lambda)\, i\,(-i) = -\lambda(2-\lambda)^2$$

$\therefore \lambda = 0, 2, 2$

$\boxed{\lambda=0}$

eigenvector corr. to 0

$$[A - 0I] = \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow E_0 = sp\left(\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}\right) \leftarrow \text{dim } 1$$

$\boxed{\lambda=2}$

$$[A - 2I] = \begin{bmatrix} -1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow E_2 = sp\left(\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \leftarrow \text{dim } 2$$

$$\therefore C = \left[\begin{array}{ccc} -i & i & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}\right] , \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow D = C^{-1}AC$$

發現 $\vec{v}_1 \perp \vec{v}_2$, $\vec{v}_1 \perp \vec{v}_3$, $\vec{v}_2 \perp \vec{v}_3$, $\|\vec{v}_1\| = \|\vec{v}_2\| = \sqrt{2}$, $\|\vec{v}_3\| = 1$, $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$, $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$, $\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$

let $U = \begin{bmatrix} \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$: unitary s.t. $D = U^* A U$

$\boxed{\lambda=2}$

$$[A - 2I] = \begin{bmatrix} -1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{求 nullspace}$$

$$\begin{array}{ccc} & r & s \\ & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $y = r$, $z = s$

$x - iz = 0 \Rightarrow x - is = 0 \Rightarrow x = is$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} is \\ r \\ s \end{bmatrix} = s\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \therefore E_2 = \text{null}(A-2I) = sp\left(\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

in $\mathbb{R}$, $\mathbb{C}$

A: diagonalizable iff $\exists$ C: invertible, D: diag. s.t. $D = C^{-1}AC$

✡ $C_{n\times n}$: invertible iff $\text{rank}(C) = n$

linear comb. coef. 在 $\mathbb{C}$

iff {column vector of C} : basis for $\mathbb{R}^n$, $\mathbb{C}^n$

△ $AC = CD$, if C: invertible

$\Rightarrow$ iff $\{\underbrace{\text{column vector of } C}_{\text{eigenvectors for } A}\}$: basis for $\mathbb{C}^n$

$\because$ for eigenvalue λ, $\dim(E_\lambda) \le$ alg. multi. of λ

and $\Sigma \dim(E_\lambda) = n$

$\therefore$ for each eigenvalue λ, $\dim(E_\lambda) =$ geo. multi of λ = alg. multi of λ

△ $\boxed{\text{geo. multi of } \lambda = \text{alg. multi of } \lambda \Rightarrow A: \text{diagonalizable}}$

---

ex:

$$A = \begin{bmatrix} i & c & 1 \\ 0 & i & 2i \\ 0 & 0 & 1 \end{bmatrix}$$, A: diagonalizable, 求 c.

sol

$$|A - \lambda I| = \begin{vmatrix} i-\lambda & c & 1 \\ 0 & i-\lambda & 2i \\ 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - i)^2 \Rightarrow \lambda = 1, \underbrace{i, i}_{\text{alg multi. } 2}$$

$\boxed{\lambda = i}$

$$[A - iI] = \begin{bmatrix} 0 & c & 1 \\ 0 & 0 & 2i \\ 0 & 0 & 1-i \end{bmatrix} \sim \begin{bmatrix} \overset{\downarrow}{0} & c & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dim(E_i) = \begin{cases} 1 & \text{if } c \neq 0 \\ \textcircled{2} & \text{if } c = 0 \end{cases}$$

$\therefore$ A : diag-able iff $c = 0$

Schur's Lemma

A : $n \times n$ (complex) $\Rightarrow$ $\exists\, U$: unitary s.t. $U^* A U$: $\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$

Thm

A : hermitian $\Rightarrow$ $\exists\, U$: unitary s.t. $U^* A U$: diag.

Moreover, all eigenvalues of A are real.

pf. of Thm

① by Schur's Lemma, $\exists\, U$: unitary s.t. $U^* A U = R$: $\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$

② $$\underbrace{(U^* A U)}_{R}{}^* = U^* A^* (U^*)^* = U^* \underbrace{A^*}_{A:\ \text{hermitian, i.e. } A^* = A} U = \underbrace{U^* A U} = R$$

$$\therefore R^* = R \quad , \quad R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ 0 & & \ddots & r_{nn} \end{bmatrix} = R^* = \begin{bmatrix} \bar{r}_{11} & & 0 \\ \bar{r}_{12} & \bar{r}_{22} & \\ \vdots & & \ddots \\ \underbrace{\bar{r}_{1n} \ \cdots}_{0} & & \bar{r}_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & & 0 \\ & r_{22} & \\ 0 & & \ddots \ r_{nn} \end{bmatrix}$$

$\forall\, r_{ii} \in \mathbb{R}$

$\therefore$ $U^* A U = R$: real diag.

Cor.

A : real symmetric $\Rightarrow$ $\exists\, C$: orthogonal s.t. $C^T A C = D$: diag.

pf.

$\because$ A : real symmetric $\therefore$ A : hermitian $\therefore$ all eigenvalue are real

$\therefore$ $A - \lambda_i I$: real matrix $\sim$ $\begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix}$: real $\therefore$ E_{λ_i} : $\overbrace{\text{has basis}}^{\text{orthornormal}}$ in $\mathbb{R}^n$

let $C = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$, $\{\vec{v}_i\}$: basis for each E_{λ_j} $\underbrace{,}_{\text{by next thm}}$ then C : orthogonal.

Thm.

H : hermitian has $\begin{cases} H\vec{v}_1 = \lambda_1 \vec{v}_1 \\ H\vec{v}_2 = \lambda_2 \vec{v}_2 \end{cases}$, $\vec{v}_1 \neq 0$, $\vec{v}_2 \neq 0$, $\lambda_1 \neq \lambda_2$

$\Rightarrow \vec{v}_1 \perp \vec{v}_2$

pf.

$$\lambda_2 \langle \vec{V}_1 , \vec{V}_2 \rangle = \lambda_2 \vec{V}_1^{*} \vec{V}_2 = \vec{V}_1^{*} (\lambda_2 \vec{V}_2) = \vec{V}_1^{*} (H \vec{V}_2) = \vec{V}_1^{*} H \vec{V}_2$$

$\|$ ← H: hermitian

$$\lambda_1 \langle \vec{V}_1 , \vec{V}_2 \rangle = \overline{\lambda}_1 \vec{V}_1^{*} \vec{V}_2 = (\lambda_1 \vec{V}_1)^{*} \vec{V}_2 = (H \vec{V}_1)^{*} \vec{V}_2 = \vec{V}_1^{*} H^{*} \vec{V}_2$$

↑ all eigenvalues for hermitian are real

$$\therefore \lambda_2 \langle \vec{V}_1 , \vec{V}_2 \rangle = \lambda_1 \langle \vec{V}_1 , \vec{V}_2 \rangle \qquad \therefore \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \langle \vec{V}_1 , \vec{V}_2 \rangle = 0$$

$$\therefore \langle \vec{V}_1 , \vec{V}_2 \rangle = 0 \quad , \text{i.e. } \vec{V}_1 \perp \vec{V}_2$$