$[1, 3, 3, 1]$, $[0, 1, 2, 1]$, $[0, 0, 1, 1]$, and $[0, 0, 0, 1]$. Reducing the matrix corresponding to the associated linear system, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 2 & 1 & 0 & -4 \\ 0 & 1 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Thus the required coordinate vector is $p(x)_{B'} = [1, -2, 0, 0]$, and so

$$x^3 + x^2 - x - 1 = (x + 1)^3 - 2(x + 1)^2.$$ ■

Linear algebra is not the only tool that can be used to solve the problem in Example 5. Exercise 13 suggests a polynomial algebra solution, and Exercise 16 describes a calculus solution.

SUMMARY

Let V be a vector space with basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$.

1. $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ is an ordered basis; the vectors are regarded as being in a specified order in this n-tuple notation.
2. Each vector $\mathbf{v}$ in V has a unique expression as a linear combination:
$$\mathbf{v} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n.$$
3. The vector $\mathbf{v}_B = [r_1, r_2, \ldots, r_n]$ for the uniquely determined scalars r_i in the preceding equation (summary item 2) is the coordinate vector of $\mathbf{v}$ relative to B.
4. The vector space V can be coordinatized, using summary item 3, so that V is *isomorphic* to $\mathbb{R}^n$.

EXERCISES

In Exercises 1–10, find the coordinate vector of the given vector relative to the indicated ordered basis.

1. $[-1, 1]$ in $\mathbb{R}^2$ relative to $([0, 1], [1, 0])$
2. $[-2, 4]$ in $\mathbb{R}^2$ relative to $([0, -2], [-\frac{1}{2}, 0])$
3. $[4, 6, 2]$ in $\mathbb{R}^3$ relative to $([2, 0, 0], [0, 1, 1], [0, 0, 1])$
4. $[4, -2, 1]$ in $\mathbb{R}^3$ relative to $([0, 1, 1], [2, 0, 0], [0, 3, 0])$
5. $[3, 13, -1]$ in $\mathbb{R}^3$ relative to $([1, 3, -2], [4, 1, 3], [-1, 2, 0])$
6. $[9, 6, 11, 0]$ in $\mathbb{R}^4$ relative to $([1, 0, 1, 0], [2, 1, 1, -1], [0, 1, 1, -1], [2, 1, 3, 1])$
7. $x^3 + x^2 - 2x + 4$ in P_3 relative to $(1, x^2, x, x^3)$
8. $x^3 + 3x^2 - 4x + 2$ in P_3 relative to $(x, x^2 - 1, x^3, 2x^2)$
9. $x + x^4$ in P_4 relative to $(1, 2x - 1, x^3 + x^4, 2x^3, x^2 + 2)$
10. $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in M_2 relative to
$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$$
11. Find the coordinate vector of the polynomial $x^3 - 4x^2 + 3x + 7$ relative to the ordered basis $B' = ((x - 2)^3, (x - 2)^2, (x - 2), 1)$ of

the vector space P_3 of polynomials of degree at most 3. Use the method illustrated in Example 5.

12. Find the coordinate vector of the polynomial $4x^3 - 9x^2 + x$ relative to the ordered basis $B' = ((x-1)^3, (x-1)^2, (x-1), 1)$ of the vector space P_3 of polynomials of degree at most 3. Use the method illustrated in Example 5.

13. Example 5 showed how to use *linear algebra* to rewrite the polynomial $p(x) = x^3 + x^2 - x - 1$ in powers of $x + 1$ rather than in powers of x. This exercise indicates a *polynomial algebra* solution to this problem. Replace x in $p(x)$ by $[(x + 1) - 1]$, and expand using the binomial theorem, keeping the $(x + 1)$ intact. Check your answer with that in Example 5.

14. Repeat Exercise 11 using the polynomial algebra method indicated in Exercise 13.

15. Repeat Exercise 12 using the polynomial algebra method indicated in Exercise 13.

16. Example 5 showed how to use *linear algebra* to rewrite the polynomial $p(x) = x^3 + x^2 - x - 1$ in powers of $x + 1$ rather than in powers of x. This exercise indicates a *calculus* solution to this problem. Form the equation

$$x^3 + x^2 - x - 1 = b_3(x+1)^3 + b_2(x+1)^2 + b_1(x+1) + b_0.$$

Find b_0 by substituting $x = -1$ in this equation. Then equate the derivatives of both sides, and substitute $x = -1$ to find b_1. Continue differentiating both sides and substituting $x = -1$ to find b_2 and b_3. Check your answer with that in Example 5.

17. Repeat Exercise 11 using the calculus method indicated in Exercise 16.

18. Repeat Exercise 12 using the calculus method indicated in Exercise 16.

19. a. Prove that $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$ is an independent set of functions in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

b. Find a basis for the subspace of F generated by the functions

$$f_1(x) = 1 - 2\sin x + 4\cos x - \sin 2x - 3\cos 2x,$$

$$f_2(x) = 2 - 3\sin x - \cos x + 4\sin 2x + 5\cos 2x$$

$$f_3(x) = 5 - 8\sin x + 2\cos x + 7\sin 2x + 7\cos 2x$$

$$f_4(x) = -1 + 14\cos x - 11\sin 2x - 19\cos 2x$$

20. Prove that for every positive integer n and every $a \in \mathbb{R}$, the set

$$\{(x-a)^n, (x-a)^{n-1}, \ldots, (x-a)^2, x-a, 1\}$$

is a basis for the vector space P_n of polynomials of degree at most n.

21. Find the polynomial in P_2 whose coordinate vector relative to the ordered basis $B = (x + x^2, x - x^2, 1 + x)$ is $[3, 1, 2]$.

22. Let V be a nonzero finite-dimensional vector space. Mark each of the following True or False.

___ **a.** The vector space V is isomorphic to $\mathbb{R}^n$ for some positive integer n.

___ **b.** There is a unique coordinate vector associated with each vector $\mathbf{v} \in V$.

___ **c.** There is a unique coordinate vector associated with each vector $\mathbf{v} \in V$ relative to a basis for V.

___ **d.** There is a unique coordinate vector associated with each vector $\mathbf{v} \in V$ relative to an ordered basis for V.

___ **e.** Distinct vectors in V have distinct coordinate vectors relative to the same ordered basis B for V.

___ **f.** The same vector in V cannot have the same coordinate vector relative to different ordered bases for V.

___ **g.** There are six possible ordered bases for $\mathbb{R}^3$.

___ **h.** There are six possible ordered bases for $\mathbb{R}^3$, consisting of the standard unit coordinate vectors in $\mathbb{R}^3$.

___ **i.** A reordering of elements in an ordered basis for V corresponds to a similar reordering of components in coordinate vectors with respect to the basis.

___ **j.** Addition and multiplication by scalars in V can be computed in terms of coordinate vectors with respect to any fixed ordered basis for V.