

3-3

$$(V, \oplus, \otimes)$$

$V$ : vector space,  $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$ : ordered basis for  $V$

$\forall \vec{v} \in V$ ,  $\exists r_1, r_2, \dots, r_n$ : coefficient (scalar) s.t.  $\vec{v} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \dots + r_n \vec{b}_n$

$$\vec{v}_B = [r_1, r_2, \dots, r_n]$$

Thm

$$1. (\vec{v} \oplus \vec{u})_B = \vec{v}_B + \vec{u}_B \quad \forall \vec{u}, \vec{v} \in V$$

$$2. (r \otimes \vec{u})_B = r(\vec{u}_B) \quad \forall r \in \mathbb{R}$$

3-4

$T: V \rightarrow V'$ : linear transformation,  $(V, \oplus, \otimes), (V', \oplus', \otimes')$ : vector space

Def 1 if  $\left\{ \begin{array}{l} \textcircled{1} T(\vec{u}) \oplus' T(\vec{v}) = T(\vec{u} \oplus \vec{v}) \\ \textcircled{2} r \otimes' T(\vec{u}) = T(r \otimes \vec{u}) \end{array} \right. \quad \forall \vec{u}, \vec{v} \in V$

$$\quad \quad \quad \forall r: \text{scalar}$$

Def 2 if  $(r \otimes' T(\vec{u})) \oplus' (s \otimes' T(\vec{v})) = T((r \otimes \vec{u}) \oplus (s \otimes \vec{v})) \quad \forall \vec{u}, \vec{v} \in V, \forall r, s: \text{scalar}$

★ kernel of  $T = \ker(T) = \{ \vec{v} \in V \mid T(\vec{v}) = \underline{\vec{0}} \}$  → is the zero vector in  $V'$   
✳  $A_3: \vec{0} \oplus \vec{v} = \vec{v}$

Thm

$T(\underline{\vec{0}}_V) = \underline{\vec{0}}_{V'}$ , → is the zero vector in  $V'$   
↑ is the zero vector in  $V$

Thm  $V, V'$ : vector space

$T: V \rightarrow V'$ : linear transformation, Given  $B = \{ \vec{b}_1, \vec{b}_2, \dots, \vec{b}_n \}$ : basis for  $V$

$\Rightarrow \forall \vec{v} \in V$ ,  $T(\vec{v})$  is uniquely determined by  $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$

↳ i.e. 给定  $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$  的值後,  $T(\vec{v})$  的值也確定

↳  $T$  就是唯一的一个 linear transformation

Thm

A linear transformation  $T \overset{V \rightarrow V'}{\text{is}} \underline{\text{one-to-one}}$  iff  $\ker(T) = \{ \underline{\vec{0}}_V \}$

↳ i.e.  $\forall \vec{u} \neq \vec{v} \in V \Leftrightarrow T(\vec{u}) \neq T(\vec{v})$

Def.  $T: V \rightarrow V'$  : invertible linear transformation

$\exists \tilde{T}: V' \rightarrow V$  : linear transformation s.t.  $(\tilde{T} \circ T)(\vec{v}) = \vec{v}$  ,  $\forall \vec{v} \in V$   
 $(T \circ \tilde{T})(\vec{u}') = \vec{u}'$  ,  $\forall \vec{u}' \in V'$

Denote  $T^{-1}$  ~~if  $A$  is the s.m.r. of  $T$  and  $A$  is invertible~~

Thm.  $T: V \rightarrow V'$  : invertible linear transformation

iff  $T$  is one-to-one and onto  $V'$

$\hookrightarrow$  if  $T(\vec{v}_1) = T(\vec{v}_2)$  in  $V' \Rightarrow \vec{v}_1 = \vec{v}_2$

$\forall \vec{v}' \in V'$  ,  $\exists \vec{v} \in V$  s.t.  $T(\vec{v}) = \vec{v}'$

Cor  $T: V \rightarrow V'$  : invertible linear transformation  $\Rightarrow T^{-1}: V' \rightarrow V$   
 $\vec{v} \mapsto T(\vec{v})$   $T(\vec{v}) \mapsto \vec{v}$

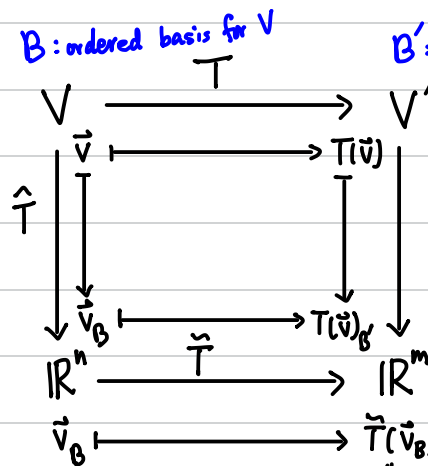
Def.  $T: V \rightarrow V'$  : isomorphism

if  $T$  is invertible linear transformation (one-to-one and onto  $V'$ )

Thm

$\hat{T}: V \rightarrow \mathbb{R}^n$ , where  $\dim(V) = n$ ,  $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$ : ordered basis for  $V$   
 $\vec{v} \mapsto \vec{v}_B$

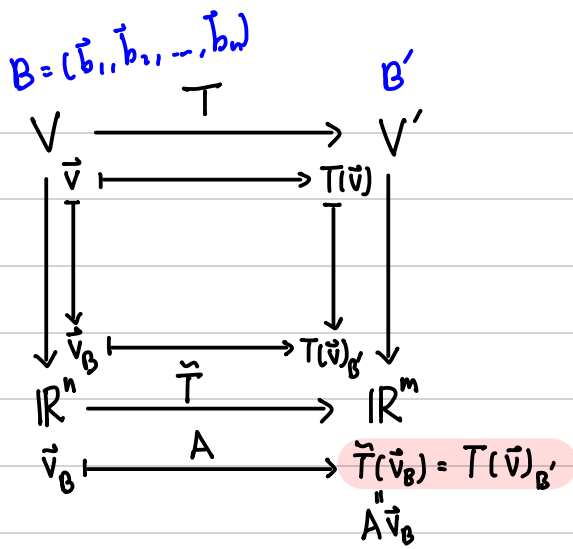
$\Rightarrow \hat{T}$  is an invertible linear transformation (isomorphism)



$T: V \rightarrow V'$  : linear transformation

$A \vec{v}_B \xrightarrow{\text{Def.}}$

$A$  is the matrix representation of  $T$  relative to  $B, B'$   
 or  $A$  is the s.m.r. of  $\tilde{T}$



$$A = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \Leftrightarrow \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$\star \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$

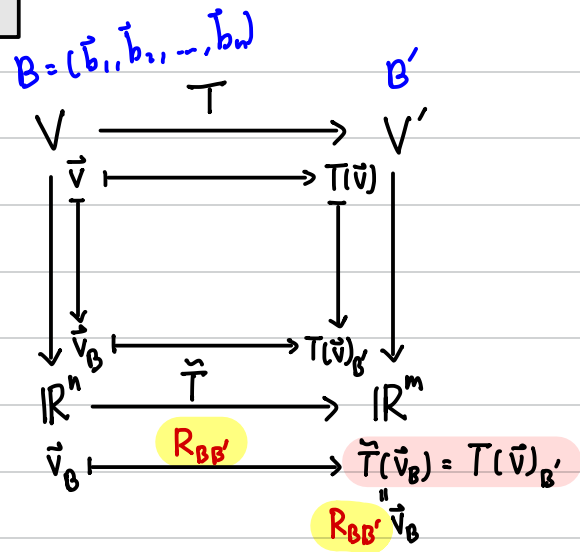
$$\therefore A = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \dots & T(\vec{b}_n)_{B'} \\ | & | & & | \end{bmatrix}$$

Def.  $T: V \rightarrow V'$  : linear transformation

$\exists$   $A$  is the matrix representation of  $T$  relative to  $B, B'$

s.t.  $\forall \vec{v} \in V, T(\vec{v})_{B'} = A \vec{v}_B$

7-2



$$R_{B,B'} = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \Leftrightarrow \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$\star \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$

$$R_{B,B'} = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \end{bmatrix} = \begin{bmatrix} T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \dots & T(\vec{b}_n)_{B'} \end{bmatrix}$$

Def.  $T: V \rightarrow V'$  : linear transformation

$\exists R_{B,B'}$  is the matrix representation of  $T$  relative to  $B, B'$

s.t.  $\forall \vec{v} \in V, T(\vec{v})_{B'} = R_{B,B'} \vec{v}_B$

### 3-5 Inner Product Space

Recall  $(V, \oplus, \otimes)$  is a vector space  
if  $A_0 \sim A_4$ ,  $S_0 \sim S_4$  holds

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V &\longrightarrow \mathbb{R} \\ \uparrow \\ \vec{u}, \vec{v} &\longmapsto \langle \vec{u}, \vec{v} \rangle \end{aligned}$$

Def.  $(V, \oplus, \otimes, \langle \cdot, \cdot \rangle)$  is an inner product space  
if  $(V, \oplus, \otimes)$  is a vector space and  $D_1 \sim D_4$  holds

$$\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall r, s \in \mathbb{R}$$

$$A_0 : \vec{v} \oplus \vec{u} \in V$$

$$S_0 : r \otimes \vec{v} \in V$$

$$A_1 : (\vec{u} \oplus \vec{v}) \otimes \vec{w} = \vec{u} \otimes \vec{w} \oplus (\vec{v} \otimes \vec{w})$$

$$S_1 : r \otimes (\vec{u} \oplus \vec{v}) = r \otimes \vec{u} \oplus r \otimes \vec{v}$$

$$A_2 : \vec{u} \oplus \vec{w} = \vec{w} \oplus \vec{u}$$

$$S_2 : (r+s) \otimes \vec{v} = (r \otimes \vec{v}) \oplus (s \otimes \vec{v})$$

$$A_3 : \vec{0} \oplus \vec{v} = \vec{v}$$

$$S_3 : r \otimes (s \otimes \vec{v}) = (rs) \otimes \vec{v}$$

$$A_4 : \vec{v} \oplus (-\vec{v}) = (-\vec{v}) \otimes \vec{v} = \vec{0}$$

$$S_4 : 1 \otimes \vec{v} = \vec{v}$$

$$\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall r, s : \text{scalar}$$

$$D_1 : \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$D_2 : \langle \vec{u}, \vec{v} \oplus \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \text{+ in } \mathbb{R}$$

$$D_3 : r \langle \vec{u}, \vec{v} \rangle = \langle r \otimes \vec{u}, \vec{v} \rangle = \langle \vec{u}, r \otimes \vec{v} \rangle$$

$$D_4 : \langle \vec{u}, \vec{u} \rangle \geq 0 \text{ and } \langle \vec{u}, \vec{u} \rangle = 0 \text{ iff } \vec{u} = \vec{0}_V$$

Def.  $(V, \oplus, \otimes, \langle, \rangle)$  is an inner product space

$\forall \vec{v} \in V$ , the magnitude or the norm of  $\vec{v}$  is  $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

Prop.  $\forall \vec{u}, \vec{v} \in V$ , the angle between  $\vec{u}$  and  $\vec{v}$  is  $\cos^{-1} \left( \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| * \|\vec{v}\|} \right)$

Def.  $\forall \vec{u}, \vec{v} \in V$ ,  $\vec{u}, \vec{v}$  are orthogonal if  $\langle \vec{u}, \vec{v} \rangle = 0$

Thm (Schwarz Inequality) :  $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| * \|\vec{v}\|$

Thm (Triangle Inequality) :  $\|\vec{u} \oplus \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$



Recall  $(P, \oplus, \otimes)$  is a vector space, where  $P$  is the set of all polynomials with real coefficient.  
 $\oplus, \otimes$  are normal operator for polynomials

ex:  $(P^{[0,1]}, \oplus, \otimes, \langle \cdot, \cdot \rangle)$  is an inner product space  
, where  $P^{[0,1]}$  is the set of all polynomials with real coefficient and domain  $0 \leq x \leq 1$   
 $\oplus, \otimes$  are normal operator for polynomials  
 $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)dx, \quad \forall f(x), g(x) \in P^{[0,1]}$

ex:  $(P^{[a,b]}, \oplus, \otimes, \langle \cdot, \cdot \rangle_w)$  is an inner product space  
, where  $P^{[a,b]}$  is the set of all polynomials with real coefficient and domain  $a \leq x \leq b$   
 $\oplus, \otimes$  are normal operator for polynomials  
 $\langle f(x), g(x) \rangle_w = \int_a^b f(x)g(x)w(x)dx, \quad \forall f(x), g(x) \in P^{[a,b]}$