V and $\mathbb{R}^n$ are isomorphic, we see (as in Example 2) that this matrix is also the change-of-coordinates matrix from B to B'.

There are times when it is feasible to use Eq. (9) to find the change-of-coordinates matrix—just noticing how the vector $\mathbf{b}_j$ in B can be expressed as a linear combination of vectors in B' to determine the jth column vector of $C_{B,B'}$ rather than actually coordinatizing V and working within $\mathbb{R}^n$. The next example illustrates this.

EXAMPLE 3 Use Eq. (9) to find the change-of-coordinates matrix $C_{B,B'}$ from the basis $B = (x^2 - 1, x^2 + 1, x^2 + 2x + 1)$ to the basis $B' = (x^2, x, 1)$ in the vector space P_2 of polynomials of degree at most 2.

SOLUTION Relative to $B' = (x^2, x, 1)$, we see at once that the coordinate vectors of $x^2 - 1$, $x^2 + 1$, and $x^2 + 2x + 1$ are $[1, 0, -1]$, $[1, 0, 1]$, and $[1, 2, 1]$, respectively. Changing these to column vectors, we obtain

$$C_{B,B'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$ ▪

SUMMARY

Let V be a vector space with ordered basis $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$.

1. Each vector $\mathbf{v}$ in V has a unique expression as a linear combination

$$\mathbf{v} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_n\mathbf{b}_n.$$

The vector $\mathbf{v}_B = [r_1, r_2, \ldots, r_n]$ is the *coordinate vector* of $\mathbf{v}$ relative to B. Associating with each vector $\mathbf{v}$ its coordinate vector $\mathbf{v}_B$ *coordinatizes* V, so that V is *isomorphic* to $\mathbb{R}^n$.

2. Let B and B' be ordered bases of V. There exists a unique $n \times n$ matrix $C_{B,B'}$ such that $C_{B,B'}\mathbf{v}_B = \mathbf{v}_{B'}$ for all vectors $\mathbf{v}$ in V. This is the *change-of-coordinates* matrix from B to B'. It can be computed by coordinatizing V and then applying the boxed procedure on page 391. Alternatively, it can be computed using Eq. (9).

EXERCISES

Exercises 1–7 are a review of Section 3.3. In Exercises 1–6, find the coordinate vector of the given vector relative to the given ordered basis.

1. $\cos 2x$ in $\mathrm{sp}(\sin^2 x, \cos^2 x)$ relative to $(\sin^2 x, \cos^2 x)$
2. $x^3 + x^2 - 2x + 4$ in P_3 relative to $(1, x^2, x, x^3)$.
3. $x^3 + 3x^2 - 4x + 2$ in P_3 relative to $(x, x^2 - 1, x^3, 2x^2)$
4. $x + x^4$ in P_4 relative to $(1, 2x - 1, x^3 + x^4, 2x^3, x^2 + 2)$

5. $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in M_2 relative to

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

6. $\sinh x$ in $\text{sp}(e^x, e^{-x})$ relative to (e^x, e^{-x})

7. Find the polynomial in P_2 whose coordinate vector relative to the ordered basis $B = (x + x^2, x - x^2, 1 + x)$ is $[3, 1, 2]$.

8. Let B be an ordered basis for $\mathbb{R}^3$. If

$$C_{E,B} = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix},$$

find the coordinate vector $\mathbf{v}_B$.

9. Let V be a vector space with ordered bases B and B'. If

$$C_{B,B'} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$\mathbf{v} = 3\mathbf{b}_1 - 2\mathbf{b}_2 + \mathbf{b}_3,$$

find the coordinate vector $\mathbf{v}_{B'}$.

In Exercises 10–14, find the change-of-coordinates matrix (a) from B to B', and (b) from B' to B. Verify that these matrices are inverses of each other.

10. $B = ([1, 1], [1, 0])$ and $B' = ([0, 1], [1, 1])$ in $\mathbb{R}^2$

11. $B = ([2, 3, 1], [1, 2, 0], [2, 0, 3])$ and $B' = ([1, 0, 0], [0, 1, 0], [0, 0, 1])$ in $\mathbb{R}^3$

12. $B = ([1, 0, 1], [1, 1, 0], [0, 1, 1])$ and $B' = ([0, 1, 1], [1, 1, 0], [1, 0, 1])$ in $\mathbb{R}^3$

13. $B = ([1, 1, 1, 1], [1, 1, 1, 0], [1, 1, 0, 0], [1, 0, 0, 0])$ and the standard ordered basis $B' = E$ for $\mathbb{R}^4$

14. $B = (\sinh x, \cosh x)$ and $B' = (e^x, e^{-x})$ in $\text{sp}(\sinh x, \cosh x)$

15. Find the change-of-coordinates matrix from B' to B for the bases $B = (x^2, x, 1)$ and $B' = (x^2 - x, 2x^2 - 2x + 1, x^2 - 2x)$ of P_2 in Example 2. Verify that this matrix is the inverse of the change-of-coordinates matrix from B to B' found in that example.

16. Proceeding as in Example 2, find the change-of-coordinates matrix from $B = (x^3, x^2, x, 1)$ to $B' = (x^3 - x^2, x^2 - x, x - 1, x^3 + 1)$ in the vector space P_3 of polynomials of degree at most 3.

17. Find the change-of-coordinates matrix from B' to B for the vector space and ordered bases given in Exercise 16. Show that this matrix is the inverse of the matrix found in Exercise 16.

In Exercises 18–21, use Eq. (9) in the text to find the change-of-coordinates matrix from B to B' for the given vector space V with ordered bases B and B'.

18. $V = P_3$, B is the basis in Exercise 3, $B' = (x_3, x_2, x, 1)$

19. $V = P_3$, $B = (x^3 + x^2 + 1, 2x^3 - x^2 + x + 1, x^3 - x + 1, x^2 + x + 1)$, $B' = (x^3, x^2, x, 1)$

20. V is the space M_2 of all 2×2 matrices, B is the basis in Exercise 5,

$$B' = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

21. V is the subspace $\text{sp}(\sin^2 x, \cos^2 x)$ of the vector space F of all real-valued functions with domain $\mathbb{R}$, $B = (\cos 2x, 1)$, $B' = (\sin^2 x, \cos^2 x)$

22. Find the change-of-coordinates matrix from B' to B for the vector space V and ordered bases B and B' of Exercise 21.

23. Let B and B' be ordered bases for $\mathbb{R}^n$. Mark each of the following True or False.

___ a. Every change-of-coordinates matrix is square.

___ b. Every $n \times n$ matrix is a change-of-coordinates matrix relative to some bases of $\mathbb{R}^n$.

___ c. If B and B' are orthonormal bases, then $C_{B,B'}$ is an orthogonal matrix.

___ d. If $C_{B,B'}$ is an orthogonal matrix, then B and B' are orthonormal bases.

___ e. If $C_{B,B'}$ is an orthogonal matrix and B is an orthonormal basis, then B' is an orthonormal basis.

___ f. For all choices of B and B', we have $\det(C_{B,B'}) = 1$.

___ g. For all choices of B and B', we have $\det(C_{B,B'}) \neq 0$.

___ h. $\det(C_{B,B'}) = 1$ if and only if $B = B'$.

___ i. $C_{B,B'} = I$ if and only if $B = B'$.

___ j. Every invertible $n \times n$ matrix is the change-of-coordinates matrix $C_{B,B'}$ for some ordered bases B and B' of $\mathbb{R}^n$.

24. For ordered bases B and B' in $\mathbb{R}^n$, explain how the change-of-coordinates matrix from B to B' is related to the change-of-coordinates matrices from B to E and from E to B'.

25. Let B, B', and B'' be ordered bases for $\mathbb{R}^n$. Find the change-of-coordinates matrix from B to B'' in terms of $C_{B,B'}$ and $C_{B',B''}$. [Hint: For a vector $\mathbf{v}$ in $\mathbb{R}^n$, what matrix times $\mathbf{v}_B$ gives $\mathbf{v}_{B''}$?]

7.2 MATRIX REPRESENTATIONS AND SIMILARITY

Let V and V' be vector spaces with ordered bases $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ and $B' = (\mathbf{b}_1', \mathbf{b}_2', \ldots, \mathbf{b}_m')$, respectively. If $T: V \to V'$ is a linear transformation, then Theorem 3.10 shows that the *matrix representation of* T relative to B, B', which we now denote as $R_{B,B'}$ is given by

$$R_{B,B'} = \begin{bmatrix} | & | & & | \\ T(\mathbf{b}_1)_{B'} & T(\mathbf{b}_2)_{B'} & \cdots & T(\mathbf{b}_n)_{B'} \\ | & | & & | \end{bmatrix}, \tag{1}$$

where $T(\mathbf{b}_j)_{B'}$ is the coordinate vector of $T(\mathbf{b}_j)$ relative to B'. Furthermore, $R_{B,B'}$ is the *unique* matrix satisfying

$$T(\mathbf{v})_{B'} = R_{B,B'}\mathbf{v}_B \quad \text{for all } \mathbf{v} \text{ in } V. \tag{2}$$

Let us denote by $M_{T(B)}$ the $m \times n$ matrix whose jth column vector is $T(\mathbf{b}_j)$. From Eq. (1), we see that we can compute $R_{B,B'}$ by row-reducing the augmented matrix $[M_{B'} \mid M_{T(B)}]$ to obtain $[I \mid R_{B,B'}]$. The discussion surrounding Theorem 3.10 shows how computations involving $T: V \to V'$ can be carried out in $\mathbb{R}^n$ and $\mathbb{R}^m$ using $R_{B,B'}$ and coordinates of vectors relative to the basis B of V and B' of V', in view of the isomorphisms of V with $\mathbb{R}^n$ and of V' with $\mathbb{R}^m$ that these coordinatizations provide.

It is the purpose of this section to study the effect that choosing different bases for coordinatization has on the matrix representations of a linear transformation. For simplicity, we shall derive our results in terms of the vector spaces $\mathbb{R}^n$. They can then be carried over to other finite-dimensional vector spaces using coordinatization isomorphisms.

The Multiplicative Property of Matrix Representations

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $T': \mathbb{R}^m \to \mathbb{R}^s$ be linear transformations. Section 2.3 showed that composition of linear transformations corresponds to multiplication