4. The projection matrix P of a subspace W is idempotent and symmetric. Every symmetric idempotent matrix is the projection matrix for its column space.

EXERCISES

In Exercises 1–8, find the projection matrix for the given subspace, and find the projection of the indicated vector on the subspace.

1. $[1, 2, 1]$ on $\mathrm{sp}([2, 1, -1])$ in $\mathbb{R}^3$
2. $[1, 3, 4]$ on $\mathrm{sp}([1, -1, 2])$ in $\mathbb{R}^3$
3. $[2, -1, 3]$ on $\mathrm{sp}([2, 1, 1], [-1, 2, 1])$ in $\mathbb{R}^3$
4. $[1, 2, 1]$ on $\mathrm{sp}([3, 0, 1], [1, 1, 1])$ in $\mathbb{R}^3$
5. $[1, 3, 1]$ on the plane $x + y - 2z = 0$ in $\mathbb{R}^3$
6. $[4, 2, -1]$ on the plane $3x + 2y + z = 0$ in $\mathbb{R}^3$
7. $[1, 2, 1, 3]$ on $\mathrm{sp}([1, 2, 1, 1], [-1, 1, 0, -1])$ in $\mathbb{R}^4$
8. $[1, 1, 2, 1]$ on $\mathrm{sp}([1, 1, 1, 1], [1, -1, 1, -1], [-1, 1, 1, -1])$ in $\mathbb{R}^4$
9. Find the projection matrix for the x_1,x_2-plane in $\mathbb{R}^3$.
10. Find the projection matrix for the x_1,x_3-coordinate subspace of $\mathbb{R}^4$.
11. Find the projection matrix for the x_1,x_2,x_4-coordinate subspace of $\mathbb{R}^4$.
12. Show that boxed Eq. (3) of this section reduces to Eq. (1) of Section 6.1 for projecting $\mathbf{b}$ on $\mathrm{sp}(\mathbf{a})$.
13. Give a geometric argument indicating that every projection matrix is idempotent.
14. Let $\mathbf{a}$ be a unit column vector in $\mathbb{R}^n$. Show that $\mathbf{a}\mathbf{a}^T$ is the projection matrix for the subspace $\mathrm{sp}(\mathbf{a})$.
15. Mark each of the following True or False.

___ a. A subspace W of dimension k in $\mathbb{R}^n$ has associated with it a $k \times k$ projection matrix.

___ b. Every subspace W of $\mathbb{R}^n$ has associated with it an $n \times n$ projection matrix.

___ c. Projection of $\mathbb{R}^n$ on a subspace W is a linear transformation of $\mathbb{R}^n$ into itself.

___ d. Two different subspaces of $\mathbb{R}^n$ may have the same projection matrix.

___ e. Two different matrices may be projection matrices for the same subspace of $\mathbb{R}^n$.

___ f. Every projection matrix is symmetric.

___ g. Every symmetric matrix is a projection matrix.

___ h. An $n \times n$ symmetric matrix A is a projection matrix if and only if $A^2 = I$.

___ i. Every symmetric idempotent matrix is the projection matrix for its column space.

___ j. Every symmetric idempotent matrix is the projection matrix for its row space.

16. Show that the projection matrix $P = A(A^TA)^{-1}A^T$ given in Theorem 6.11 satisfies the following two conditions:
    a. $P^2 = P$,
    b. $P^T = P$.
17. What is the projection matrix for the subspace $\mathbb{R}^n$ of $\mathbb{R}^n$?
18. Let U be a subspace of W, which is a subspace of $\mathbb{R}^n$. Let P be the projection matrix for W, and let R be the projection matrix for U. Find PR and RP. [HINT: Argue geometrically.]
19. Let P be the projection matrix for a k-dimensional subspace of $\mathbb{R}^n$.
    a. Find all eigenvalues of P.
    b. Find the algebraic multiplicity and the geometric multiplicity of each eigenvalue found in part (a).
    c. Explain how we can deduce that P is diagonalizable, without using the fact that P is a symmetric matrix.
20. Show that every symmetric matrix whose only eigenvalues are 0 and 1 is a projection matrix.
21. Find all invertible projection matrices.

In Exercises 22–28, find the projection matrix for the subspace W having the given orthonormal basis. The vectors are given in row notation to save space in printing.

22. $W = \mathrm{sp}(\mathbf{a}_1, \mathbf{a}_2)$ in $\mathbb{R}^3$, where

$$\mathbf{a}_1 = [1/\sqrt{2}, 0, -1/\sqrt{2}] \text{ and}$$
$$\mathbf{a}_2 = [1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]$$

23. $W = \mathrm{sp}(\mathbf{a}_1, \mathbf{a}_2)$ in $\mathbb{R}^3$, where
$\mathbf{a}_1 = [\frac{3}{5}, \frac{4}{5}, 0]$ and $\mathbf{a}_2 = [0, 0, 1]$

24. $W = \text{sp}(\mathbf{a}_1, \mathbf{a}_2)$ in $\mathbb{R}^4$, where
$\mathbf{a}_1 = \left[3/(5\sqrt{2}), 4/(5\sqrt{2}), \frac{1}{2}, -\frac{1}{2}\right]$ and
$\mathbf{a}_2 = \left[4/(5\sqrt{2}), -3/(5\sqrt{2}), \frac{1}{2}, \frac{1}{2}\right]$
25. $W = \text{sp}(\mathbf{a}_1, \mathbf{a}_2)$ in $\mathbb{R}^4$, where $\mathbf{a}_1 = \left[\frac{2}{7}, 0, \frac{3}{7}, -\frac{6}{7}\right]$ and $\mathbf{a}_2 = \left[-\frac{3}{7}, \frac{6}{7}, \frac{2}{7}, 0\right]$
26. $W = \text{sp}(\mathbf{a}_1, \mathbf{a}_2)$ in $\mathbb{R}^4$, where
$\mathbf{a}_1 = \left[0, -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right]$ and $\mathbf{a}_2 = \left[\frac{2}{3}, 0, -\frac{1}{3}, \frac{2}{3}\right]$
27. $W = \text{sp}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ in $\mathbb{R}^4$, where
$\mathbf{a}_1 = [1/\sqrt{3}, 0, 1/\sqrt{3}, 1/\sqrt{3}]$,
$\mathbf{a}_2 = [1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}, 0]$, and
$\mathbf{a}_3 = [1/\sqrt{3}, -1/\sqrt{3}, 0, -1/\sqrt{3}]$
28. $W = \text{sp}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ in $\mathbb{R}^4$, where
$\mathbf{a}_1 = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$, $\mathbf{a}_2 = \left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]$, and
$\mathbf{a}_3 = \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right]$

In Exercises 29–32, find the projection of $\mathbf{b}$ *on* W.

29. The subspace W in Exercise 22; $\mathbf{b} = [6, -12, -6]$
30. The subspace W in Exercise 23; $\mathbf{b} = [20, -15, 5]$
31. The subspace W in Exercise 26; $\mathbf{b} = [9, 0, -9, 18]$
32. The subspace W in Exercise 28; $\mathbf{b} = [4, -12, -4, 0]$
33. Let W be a subspace of $\mathbb{R}^n$, and let P be the projection matrix for W. **Reflection of $\mathbb{R}^n$ in W** is the mapping of $\mathbb{R}^n$ into itself that carries each vector $\mathbf{b}$ in $\mathbb{R}^n$ into its reflection $\mathbf{b}_r$ according to the following geometric description:

    Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on W. Starting at the tip of $\mathbf{b}$, travel in a straight line to the tip of $\mathbf{p}$, and then continue in the same direction an equal distance to arrive at $\mathbf{b}_r$. (See Figure 6.11.)

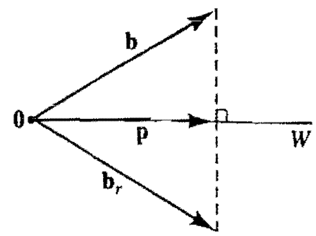

FIGURE 6.11
Reflection of $\mathbb{R}^n$ through W.

    Show that $\mathbf{b}_r = (2P - I)\mathbf{b}$. (Notice that, because reflection can be accomplished by matrix multiplication, this reflection must be a linear transformation of $\mathbb{R}^n$ into itself.)

The formula $A(A^TA)^{-1}A^T$ for a projection matrix can be tedious to compute using pencil and paper, but the routine MATCOMP in LINTEK, or MATLAB, can do it easily. In Exercises 34–38, use MATCOMP or MATLAB to find the indicated vector projections.

34. The projections in $\mathbb{R}^6$ of $[-1, 2, 3, 1, 6, 2]$ and $[2, 0, 3, -1, 4, 5]$ on $\text{sp}([1, -2, 3, 1, 4, 0])$
35. The projections in $\mathbb{R}^3$ of $[1, -1, 4]$, $[3, 3, -1]$, and $[-2, 4, 7]$ on $\text{sp}([1, 3, -4], [2, 0, 3])$
36. The projections in $\mathbb{R}^4$ of $[-1, 3, 2, 0]$ and $[4, -1, 1, 5]$ on $\text{sp}([0, 1, 2, 1], [-1, 2, 1, 4])$
37. The projections in $\mathbb{R}^4$ of $[2, 1, 0, 3]$, $[1, 1, -1, 2]$, and $[4, 3, 1, 3]$ on $\text{sp}([1, 0, -1, 0], [1, 2, -1, 4], [2, 1, 3, -1])$
38. The projections in $\mathbb{R}^5$ of $[2, 1, -3, 2, 4]$ and $[1, -4, 0, 1, 5]$ on $\text{sp}([3, 1, 4, 0, 1], [2, 1, 3, -5, 1])$

6.5 THE METHOD OF LEAST SQUARES

The Nature of the Problem

In this section we apply our work on projections to problems of data analysis. Suppose that data measurements of the form (a_i, b_i) are obtained from

测量. 衡量