SUMMARY

1. A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(r\mathbf{u}) = rT(\mathbf{u})$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars r.
2. If A is an $m \times n$ matrix, then the function $T_A: \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ is a linear transformation.
3. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined by $T(\mathbf{b}_1)$, $T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)$ for any basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$.
4. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ matrix whose jth column vector is $T(\mathbf{e}_j)$. Then $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$; the matrix A is the standard matrix representation of T. The kernel of T is the nullspace of A, and the range of T is the column space of A.
5. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $T': \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations with standard matrix representations A and B, respectively. The composition $T' \circ T$ of the two transformations is a linear transformation, and its standard matrix representation is BA.
6. If $\mathbf{y} = T(\mathbf{x}) = A(\mathbf{x})$ where A is an invertible $n \times n$ matrix, then T is invertible and the transformation T^{-1} defined by $T^{-1}(\mathbf{y}) = A^{-1}\mathbf{y}$ is the inverse of T. Both $T^{-1} \circ T$ and $T \circ T^{-1}$ are the identity transformation of $\mathbb{R}^n$.

EXERCISES

1. Is $T([x_1, x_2, x_3]) = [x_1 + x_2, x_1 - 3x_2]$ a linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^2$? Why or why not?
2. Is $T([x_1, x_2, x_3]) = [0, 0, 0, 0]$ a linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^4$? Why or why not?
3. Is $T([x_1, x_2, x_3]) = [1, 1, 1, 1]$ a linear transformation of $\mathbb{R}^3$ into $\mathbb{R}^4$? Why or why not?
4. Is $T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$ a linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^3$? Why or why not?

In Exercises 5–12, assume that T is a linear transformation. Refer to Example 7 for Exercises 9–12, if necessary.

5. If $T([1, 0]) = [3, -1]$ and $T([0, 1]) = [-2, 5]$, find $T([4, -6])$.
6. If $T([-1, 0]) = [2, 3]$ and $T([0, 1]) = [5, 1]$, find $T([-3, -5])$.
7. If $T([1, 0, 0]) = [3, 1, 2]$, $T([0, 1, 0]) = [2, -1, 4]$, and $T([0, 0, 1]) = [6, 0, 1]$, find $T([2, -5, 1])$.
8. If $T([1, 0, 0]) = [-3, 1]$, $T([0, 1, 0]) = [4, -1]$, and $T([0, -1, 1]) = [3, -5]$, find $T([-1, 4, 2])$.
9. If $T([-1, 2]) = [1, 0, 0]$ and $T([2, 1]) = [0, 1, 2]$, find $T([0, 10])$.
10. If $T([-1, 1]) = [2, 1, 4]$ and $T([1, 1]) = [-6, 3, 2]$, find $T([x, y])$.
11. If $T([1, 2, -3]) = [1, 0, 4, 2]$, $T([3, 5, 2]) = [-8, 3, 0, 1]$, and $T([-2, -3, -4]) = [0, 2, -1, 0]$, find $T([5, -1, 4])$. [*Computational aid: See* Example 4 in Section 1.5.]
12. If $T([2, 3, 0]) = 8$, $T([1, 2, -1]) = -5$, and $T([4, 5, 1]) = 17$, find $T([-3, 11, -4])$. [*Computational aid: See the answer to* Exercise 7 in Section 1.5.]

In Exercises 13–18, the given formula defines a linear transformation. Give its standard matrix representation.

13. $T([x_1, x_2]) = [x_1 + x_2, x_1 - 3x_2]$
14. $T([x_1, x_2]) = [2x_1 - x_2, x_1 + x_2, x_1 + 3x_2]$
15. $T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$
16. $T([x_1, x_2, x_3]) = [2x_1 + x_2 + x_3, x_1 + x_2 + 3x_3]$
17. $T([x_1, x_2, x_3]) = [x_1 - x_2 + 3x_3, x_1 + x_2 + x_3, x_1]$
18. $T([x_1, x_2, x_3]) = x_1 + x_2 + x_3$
19. If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T([x_1, x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T': \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T'([x_1, x_2, x_3]) = [x_1 - x_2 + x_3, x_1 + x_2]$, find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T' \circ T)([x_1, x_2])$.
20. Referring to Exercise 19, find the standard matrix representation for the linear transformation $T \circ T'$ that carries $\mathbb{R}^3$ into $\mathbb{R}^3$. Find a formula for $(T \circ T')([x_1, x_2, x_3])$.

In Exercises 21–28, determine whether the indicated linear transformation T is invertible. If it is, find a formula for $T^{-1}(x)$ in row notation. If it is not, explain why it is not.

21. The transformation in Exercise 13.
22. The transformation in Exercise 14.
23. The transformation in Exercise 15.
24. The transformation in Exercise 16.
25. The transformation in Exercise 17.
26. The transformation in Exercise 18.
27. The transformation in Exercise 19.
28. The transformation in Exercise 20.
29. Mark each of the following True or False.

___ a. Every linear transformation is a function.

___ b. Every function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ is a linear transformation.

___ c. Composition of linear transformations corresponds to multiplication of their standard matrix representations.

___ d. Function composition is associative.

___ e. An invertible linear transformation mapping $\mathbb{R}^n$ into itself has a unique inverse.

___ f. The same matrix may be the standard matrix representation for several different linear transformations.

___ g. A linear transformation having an $m \times n$ matrix as standard matrix representation maps $\mathbb{R}^n$ into $\mathbb{R}^m$.

___ h. If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for some standard basis vector $\mathbf{e}_i$ of $\mathbb{R}^n$.

___ i. If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for all standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

___ j. If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and T and T' are linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then $T(\mathbf{x}) = T'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $T(\mathbf{b}_i) = T'(\mathbf{b}_i)$ for $i = 1, 2, \ldots, n$.

30. Verify that $T^{-1}(T(\mathbf{x})) = \mathbf{x}$ for the linear transformation T in Example 9 of the text.
31. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $T': \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Prove directly from Definition 2.3 that $(T' \circ T)\ \mathbb{R}^n \to \mathbb{R}^k$ is also a linear transformation.
32. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove from Definition 2.3 that $T(r\mathbf{u} + s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars r and s.

Exercise 33 shows that the reduced row-echelon form of a matrix is unique.

33. Let A be an $m \times n$ matrix with row-echelon form H, and let V be the row space of A (and thus of H). Let $W_k = \mathrm{sp}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k)$ be the subspace of $\mathbb{R}^n$ generated by the first k rows of the $n \times n$ identity matrix. Consider $T_k: V \to W_k$ defined by

$$T_k([x_1, x_2, \ldots, x_n]) = [x_1, x_2, \ldots, x_k, 0, \ldots, 0].$$

a. Show that T_k is a linear transformation of V into W_k and that $T_k[V] = \{T_k(\mathbf{v}) \mid \mathbf{v} \text{ in } V\}$ is a subspace of W_k.

b. If $T_k[V]$ has dimension d_k, show that, for each $j < n$, we have either $d_{j+1} = d_j$ or $d_{j+1} = d_j + 1$.
c. Assume that A has four columns. Referring to part (b), suppose that $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$. Find the number of pivots in H, and give the location of each.
d. Repeat part (c) for the case where A has six columns and $d_1 = 1$, $d_2 = d_3 = d_4 = 2$, and $d_5 = d_6 = 3$.
e. Argue that, for any matrix A, the number of pivots and the location of each pivot in any row-echelon form of A is always the same.
f. Show that the reduced row-echelon form of a matrix A is unique. [HINT: Consider the nature of the basis for the row space of A given by the nonzero rows of H.]

34. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let U be a subspace of $\mathbb{R}^m$. Prove that the inverse image $T^{-1}[U]$ is a subspace of $\mathbb{R}^n$.

In Exercises 35–38, let T_1, T_2, T_3, and T_4 be linear transformations whose standard matrix representations are

$$A = \begin{bmatrix} -4 & 5 & 7 \\ 2 & 4 & 5 \\ 1 & 8 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix},$$

$$C = \begin{bmatrix} -5 & 3 & -6 \\ 11 & 7 & -1 \end{bmatrix}, \quad \text{and}$$

$$D = \begin{bmatrix} 3 & -4 & 1 \\ -2 & 5 & 0 \end{bmatrix},$$

respectively. Use LINTEK or MATLAB to compute the indicated quantity, if it is defined. Load data files for the matrices if the data files are available.

35. $(T_1 \circ T_2 \circ T_4)([1, 2, 1])$

36. $(T_3 \circ T_1^{-1} \circ T_2)([0, -1])$

37. $(T_4 \circ (T_2 \circ T_3)^{-1} \circ T_2)([-1, 0])$

38. $(T_2 \circ (T_3 \circ T_2)^{-1} \circ T_4)([-1, 0, 1])$

39. Work with Topic 4 of the LINTEK routine VECTGRPH until you can consistently achieve a score of at least 80%.

40. Work with Topic 5 of the LINTEK routine VECTGRPH until you can regularly attain a score of at least 82%.

2.4 LINEAR TRANSFORMATIONS OF THE PLANE (OPTIONAL)

From the preceding section, we know that every linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\mathbf{x}) = T_A(\mathbf{x}) = A\mathbf{x}$, where A is some 2×2 matrix. Different 2×2 matrices A give different transformations because $T(\mathbf{e}_1) = A\mathbf{e}_1$ is the first column vector of A and $T(\mathbf{e}_2) = A\mathbf{e}_2$ is the second column vector. The entire plane is mapped onto the column space of the matrix A. In this section we discuss these linear transformations of the plane $\mathbb{R}^2$ into itself, where we can draw reasonable pictures. We will use the familiar x,y-notation for coordinates in the plane.

The Collapsing (Noninvertible) Transformations

For a 2×2 matrix A to be noninvertible, it must have rank 0 or 1. If $\text{rank}(A) = 0$, then A is the zero matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$