$\boxed{3\text{-}3}$

V : vector space , $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$: ordered basis for V

$\forall \vec{v} \in V$, $\exists\, r_1, r_2, \dots, r_n$: coefficient (scalar) s.t. $\vec{V} = r_1\vec{b}_1 + r_2\vec{b}_2 + \dots + r_n\vec{b}_n$

$\vec{v}_B = [r_1, r_2, \dots, r_n]$

Thm

1. $(\vec{v}+\vec{u})_B = \vec{v}_B + \vec{u}_B \qquad \forall \vec{u}, \vec{v} \in V$
2. $(r\vec{u})_B = r(\vec{u}_B) \qquad \forall r \in \mathbb{R}$

$\boxed{3\text{-}4}$

$T : V \longrightarrow V'$: linear transformation , V, V' : vector space

Def 1 if $\begin{cases} ① \ T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v}) & \forall \vec{u}, \vec{v} \in V \\ ② \ rT(\vec{u}) = T(r\vec{u}) & \forall r : \text{scalar} \end{cases}$

Def 2 if $rT(\vec{u}) + sT(\vec{v}) = T(r\vec{u} + s\vec{v}) \qquad \forall \vec{u}, \vec{v} \in V , \forall r, s : \text{scalar}$

☆ Kernel of T = $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$ → is the zero vector in V

☆ A_3. $\vec{0} + \vec{v} = \vec{v}$

Thm

$T(\vec{0}_V) = \vec{0}_{V'}$ ← is the zero vector in V'

↑ is the zero vector in V

Thm V, V': vector space

$T: V \longrightarrow V'$: linear transformation , Given $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$: basis for V

$\Rightarrow \forall \vec{v} \in V$, $T(\vec{v})$ is uniquely determined by $T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)$

↳ i.e. 給定 $T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)$ 的值後，$T(\vec{v})$ 的值也確定了

↳ T 就是唯一的一個 linear transformation

Thm

A linear transformation $\underbrace{T}_{V \to V'}$ is one-to-one iff $\ker(T) = \{\vec{0}_V\}$

↳ i.e. $\forall \vec{u} \neq \vec{v} \in V \Leftrightarrow T(\vec{u}) \neq T(\vec{v})$

Def. $T: V \longrightarrow V'$: invertible linear transformation

$\exists\, \tilde{T}: V' \to V$: linear transformation s.t. $(\tilde{T} \circ T)(\vec{v}) = \vec{v}$, $\forall\, \vec{v} \in V$

$(T \circ \tilde{T})(\vec{u}') = \vec{u}'$, $\forall\, \vec{u}' \in V'$

Denote T^{-1} ~~if A is the s.m.r. of T and A is invertible~~

Thm. $T: V \longrightarrow V'$: invertible linear transformation

iff T is one-to-one and onto V'

↳ if $T(\vec{v}_1) = T(\vec{v}_2)$ in $V' \Rightarrow \vec{v}_1 = \vec{v}_2$ $\forall\, \vec{v}' \in V'$, $\exists\, \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{v}'$

Cor $T: V \longrightarrow V'$, $\vec{v} \longmapsto T(\vec{v})$: invertible linear transformation $\Rightarrow T^{-1}: V' \longrightarrow V$, $T(\vec{v}) \longmapsto \vec{v}$

Def. $T: V \longrightarrow V'$: isomorphism

if T is invertible linear transformation (one-to-one and onto V')

Thm

$T : V \to \mathbb{R}^n$, where $\dim(V) = n$, $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$: ordered basis for V

$\vec{v} \longmapsto \vec{v}_B$

$\Rightarrow$ T is an invertible linear transformation (isomorphism)

B : ordered basis for V $\qquad$ B' : ordered basis for V'

$$
\begin{tikzcd}
V \arrow[rrr, "T"] \arrow[dddd] & & & V' \arrow[dddd] \\
 & \vec{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\vec{v}) \arrow[d, mapsto] & \\
 & \vec{v}_B \arrow[r, mapsto] & T(\vec{v})_{B'} & \\
 & & & \\
\mathbb{R}^n \arrow[rrr, "\tilde{T}"] & & & \mathbb{R}^m
\end{tikzcd}
$$

$$\vec{v}_B \longmapsto \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$

$$\tilde{T}(\vec{v}_B) = A\vec{v}_B$$

$\xrightarrow{\text{Def.}}$ A is the matrix representation of T relative to B , B'

$B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$ $\qquad\qquad$ B'

$$
\begin{tikzcd}
V \arrow[rr, "T"] \arrow[ddd] & & V' \arrow[ddd] \\
\vec{v} \arrow[rr, mapsto] \arrow[d, mapsto] & & T(\vec{v}) \arrow[d, mapsto] \\
\vec{v}_B \arrow[rr, mapsto] & & T(\vec{v})_{B'} \\
\mathbb{R}^n \arrow[rr, "\tilde{T}"] & & \mathbb{R}^m
\end{tikzcd}
$$

$$\vec{v}_B \overset{A}{\longmapsto} \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$

$$\tilde{T}(\vec{v}_B) = A\vec{v}_B$$

$$A = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \iff \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$\therefore A = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \cdots & T(\vec{b}_n)_{B'} \\ | & | & & | \end{bmatrix}$$

☆ $\tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$

Def. $T: V \to V'$: linear transformation

$\exists\, A$ is the matrix representation of T relative to B, B'

s.t. $\forall\, \vec{v} \in V$, $T(\vec{v})_{B'} = A\vec{v}_B$

$\boxed{7\text{-}2}$

$B = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n)$ $\qquad\qquad\qquad$ B'

$$
\begin{tikzcd}
V \arrow[rrr, "T"] \arrow[ddd] & \vec{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\vec{v}) \arrow[d, mapsto] & V' \arrow[ddd] \\
 & \vec{v}_B \arrow[r, mapsto] & T(\vec{v})_{B'} & \\
 & & & \\
\mathbb{R}^n \arrow[rrr, "\tilde{T}"] & & & \mathbb{R}^m \\
\vec{v}_B \arrow[rrr, mapsto, "R_{BB'}"] & & & \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}
\end{tikzcd}
$$

$$\overset{\shortparallel}{R_{BB'}\vec{v}_B}$$

★ $\tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$

$$R_{B,B'} = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \iff \vec{v} = \vec{b}_j$$

$$\therefore\ \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$R_{B,B'} = \begin{bmatrix} | & | & & | \\ \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \cdots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \cdots & T(\vec{b}_n)_{B'} \\ | & | & & | \end{bmatrix}$$

Def. $T: V \to V'$: linear transformation

$\exists\ R_{B,B'}$ is the matrix representation of T relative to B, B'

s.t. $\forall\ \vec{v} \in V$, $T(\vec{v})_{B'} = R_{B,B'}\vec{v}_B$

$\boxed{3\text{-}5}$ Inner Product Space

$\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall r, s \in \mathbb{R}$

$A_0 : \vec{v} \oplus \vec{u} \in V$ $\quad S_0 : r \otimes \vec{v} \in V$

$A_1 : (\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$ $\quad S_1 : r \otimes (\vec{u} \oplus \vec{v}) = r \otimes \vec{u} \oplus r \otimes \vec{v}$

$A_2 : \vec{u} \oplus \vec{w} = \vec{w} \oplus \vec{u}$ $\quad S_2 : (r+s) \otimes \vec{v} = (r \otimes \vec{v}) \oplus (s \otimes \vec{v})$

$A_3 : \vec{0} \oplus \vec{v} = \vec{v}$ $\quad S_3 : r \otimes (s \otimes \vec{v}) = (rs) \otimes \vec{v}$

$A_4 : \vec{v} \oplus (-\vec{v}) = (-\vec{v}) \oplus \vec{v} = \vec{0}$ $\quad S_4 : 1 \otimes \vec{v} = \vec{v}$

Recall $(V, \oplus, \otimes)$ is a vector space if $A_0 \sim A_4$, $S_0 \sim S_4$ holds

$\langle , \rangle : V \times V \longrightarrow \mathbb{R}$

$\vec{u}, \vec{v} \longmapsto \langle \vec{u}, \vec{v} \rangle$

Def. $(V, \oplus, \otimes, \langle , \rangle)$ is an inner product space if $(V, \oplus, \otimes)$ is a vector space and $D_1 \sim D_4$ holds

$\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall r, s$: scalar

$D_1 : \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

+ in $\mathbb{R}$

$D_2 : \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

$D_3 : r \langle \vec{u}, \vec{v} \rangle = \langle r \otimes \vec{u}, \vec{v} \rangle = \langle \vec{u}, r \otimes \vec{v} \rangle$

$D_4 : \langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}_V$

Def. $(V, \oplus, \otimes, <,>)$ is an inner product space
$\forall \vec{v} \in V$, the magnitude or the norm of $\vec{v}$ is $\|\vec{v}\| = \sqrt{<\vec{v}, \vec{v}>}$

Prop. $\forall \vec{u}, \vec{v} \in V$, the angle between $\vec{u}$ and $\vec{v}$ is $\dfrac{<\vec{u}, \vec{v}>}{\|\vec{u}\| \times \|\vec{v}\|}$

Def. $\forall \vec{u}, \vec{v} \in V$, $\vec{u}, \vec{v}$ are orthogonal if $<\vec{u}, \vec{v}> = 0$

Thm (Schwarz Inequality) : $|<\vec{u}, \vec{v}>| \leq \|\vec{u}\| \times \|\vec{v}\|$

Thm (Triangle Inequality) : $\|\vec{u} \oplus \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Recall $(P, \oplus, \otimes)$ is a vector space, where P is the set of all polynomials with real coefficient.

$\oplus, \otimes$ are normal operator for polynomials

ex: $(P^{[0,1]}, \oplus, \otimes, \langle , \rangle)$ is an inner product space

, where $P^{[0,1]}$ is the set of all polynomials with real coefficient and domain $0 \le x \le 1$

$\oplus, \otimes$ are normal operator for polynomials

$\langle f(x), g(x) \rangle = \int_0^1 f(x) g(x) \, dx$, $\forall f(x), g(x) \in P^{[0,1]}$

ex: $(P^{[a,b]}, \oplus, \otimes, \langle , \rangle_w)$ is an inner product space

, where $P^{[0,1]}$ is the set of all polynomials with real coefficient and domain $a \le x \le b$

$\oplus, \otimes$ are normal operator for polynomials

$\langle f(x), g(x) \rangle_w = \int_a^b f(x) g(x) w(x) \, dx$, $\forall f(x), g(x) \in P^{[a,b]}$