4. Arithmetic computations in the set $\mathbb{C}$ of complex numbers, including division and extraction of roots, can be accomplished as illustrated in this section and can be represented geometrically as shown in the figures of this section.

EXERCISES

1. Find the sum $z + w$ and the product zw if

a. $z = 1 + 2i$, $w = 3 - i$,

b. $z = 3 + i$, $w = i$.

2. Find the sum $z + w$ and the product zw if

a. $z = 2 + 3i$, $w = 5 - i$,

b. $z = 1 + 2i$, $w = 2 - i$.

3. Find $|z|$ and $\bar{z}$, and verify that $z\bar{z} = |z|^2$, if

a. $z = 3 + 2i$,

b. $z = 4 - i$.

4. Find $|z|$ and $\bar{z}$, and verify that $z\bar{z} = |z|^2$, if

a. $z = 2 + i$,

b. $z = 3 - 4i$.

5. Show that z is a real number if and only if $z = \bar{z}$.

6. Express z^{-1} in the form $a + bi$, for a and b real numbers, if

a. $z = -1 + i$,

b. $z = 3 + 4i$.

7. Express z/w in the form $a + bi$, for a and b real numbers, if

a. $z = 1 + 2i$, $w = 1 + i$,

b. $z = 3 + i$, $w = 3 + 4i$.

8. Find the modulus and principal argument for

a. $\sqrt{3} - i$,

b. $-\sqrt{3} - i$.

9. Find the modulus and principal argument for

a. $-2 + 2i$,

b. $-2 - 2i$.

10. Express $(\sqrt{3} + i)^6$ in the form $a + bi$ for a and b real numbers. [HINT: Write the given number in polar form.]

11. Express $(1 + i)^8$ in the form $a + bi$ for a and b real numbers. [HINT: Write the given number in polar form.]

12. Prove properties 1, 2, and 5 of Theorem 9.1.

13. Prove property 4 of Theorem 9.1.

14. Illustrate Eqs. (5) in the text for $z_1 = \sqrt{3} + i$ and $z_2 = -1 + \sqrt{3}i$.

15. Illustrate Eqs. (5) in the text for $z_1 = 2 + 2i$ and $z_2 = 1 + \sqrt{3}i$.

16. If $z^8 = 16$, find $|z|$.

17. Mark each of the following True or False.

___ a. The existence of complex numbers is more doubtful than the existence of real numbers.

___ b. Pencil-and-paper computations with complex numbers are more cumbersome than with real numbers.

___ c. The square of every complex number is a positive real number.

___ d. Every complex number has two distinct square roots in $\mathbb{C}$.

___ e. Every nonzero complex number has two distinct square roots in $\mathbb{C}$.

___ f. The Fundamental Theorem of Algebra asserts that the algebraic operations of addition, subtraction, multiplication, and division are possible with any two complex numbers, as long as we do not divide by zero.

___ g. The product of two complex numbers cannot be a real number unless both numbers are themselves real or unless both are of the form bi, where b is a real number.

___ h. If $(a + bi)^3 = 8$, then $a^2 + b^2 = 4$.

___ i. If $\text{Arg}(z) = 3\pi/4$ and $\text{Arg}(w) = -\pi/2$, then $\text{Arg}(z/w) = 5\pi/4$.

___ j. If $z + \bar{z} = 2z$, then z is a real number.

18. Find the three cube roots of 8.

19. Find the four fourth roots of -16.

20. Find the three cube roots of -27.

21. Find the four fourth roots of 1.

22. Find the six sixth roots of 1.

23. Find the eight eighth roots of 256.

24. A **primitive nth root of unity** is a complex number z such that $z^n = 1$ but $z^m \neq 1$ for $m < n$.
    a. Give a formula for one primitive nth root of unity.
    b. Find the primitive fourth roots of unity.
    c. How many primitive eighth roots of unity are there? [HINT: Argue geometrically in terms of polar forms.]

25. Let $z, w \in \mathbb{C}$. Show that $|z + w| \leq |z| + |w|$. [HINT: Remember that $\mathbb{C}$ is a real vector space of dimension 2, naturally isomorphic to $\mathbb{R}^2$.]

26. Show that the nth roots of $z \in \mathbb{C}$ can be represented geometrically as n equally spaced points on the circle $x^2 + y^2 = |z|^2$.

27. Show that the infinite list of values $\phi = \frac{\theta}{n} \pm \frac{2k\pi}{n}$ for $k = 0, 1, 2, \ldots$ yields just n distinct complex numbers $w = s(\cos\phi + i\sin\phi)$ of modulus s.

28. In calculus it is shown that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.$$

    a. Proceeding formally, show that $e^{i\theta} = \cos\theta + i\sin\theta$. (This is **Euler's formula.**)
    b. Show that every complex number z may be written in the form $z = re^{i\theta}$, where $r = |z|$ and $\theta = \mathrm{Arg}(z)$.
    c. Assuming that the usual laws of exponents hold for exponents that are complex numbers, use part b to derive again Eqs. (4) and (5) describing multiplication and division of complex numbers in polar form.

9.2 MATRICES AND VECTOR SPACES WITH COMPLEX SCALARS

Complex Matrices and Linear Systems

Both the real number system $\mathbb{R}$ and the complex number system $\mathbb{C}$ are algebraic structures known as *fields*. In a field, we can add any two elements and multiply any two elements to produce an element of the field. Addition and multiplication are commutative and associative operations, and multiplication is distributive over addition. The field contains an additive identity 0 and a multiplicative identity 1. Every element c in the field has an additive inverse $-c$ in the field—that is, an element that, when added to c, produces the additive identity. Similarly, every nonzero element d in the field has a multiplicative inverse $1/d$ in the field.

The part of our work in Chapters 1, 2, and 3 that rests only on the field axioms of $\mathbb{R}$ is equally valid if we allow complex scalars. In particular, we can work with *complex matrices*—that is, with matrices having complex entries: adding matrices of the same size, multiplying matrices of appropriate sizes, and multiplying a matrix by a complex scalar. We can solve linear systems by using the same Gauss or Gauss–Jordan methods that we used in Chapter 1. All of our work in Chapter 1, Sections 3 through 6, makes perfectly good sense when applied to complex scalars. Pencil-and-paper computations are more tedious, however.