Error bound for Euler's

f: contin , $y' = f$
$\Rightarrow y \in C^2([a.b])$

Thm.

$D = \{ (t.y) \mid t \in [a.b],\ y \in \mathbb{R} \}$

f: contin and Lipschitz condition on D in y
with Lipschitz constant L

$\exists M$: constant s.t. $|y''(t)| \le M$. $\forall t \in [a.b]$

then we have $y(t)$ as the unique sol. for

$$\begin{cases} y'(t) = f(t.y) , & t \in [a.b] \\ y(a) = \alpha \end{cases}$$

Moreover, use Euler's to get the apporxi. sol. $w_0, w_1, \ldots, w_N$

$$\Rightarrow \underbrace{|y(t_i) - w_i|}_{= O(h)} \le \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right), \quad h = \frac{b-a}{N}$$

$O(h)$, i.e. linearly dep. on h

pf.

We have $w_{i+1} = w_i + h f(t_i, w_i)$

$$y_{i+1} = y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

$$\therefore y_{i+1} - w_{i+1} = y_i - w_i + h[f(t_i, y_i) - f(t_i, w_i)] + \frac{h^2}{2} y''(\xi_i)$$

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i| + h \underbrace{|f(t_i, y_i) - f(t_i, w_i)|}_{\le L|y_i - w_i|\ (\because \text{Lipschitz})} + \frac{h^2}{2} \underbrace{|y''(\xi_i)|}_{\le M}$$

$$\le (1 + hL)|y_i - w_i| + \frac{h^2 M}{2}$$

Claim: if $\{a_i\}_{i=1}^{\infty}$: seq, with $a_{i+1} = (1+s)a_i + t$, $a_0 \geq \frac{-t}{s}$

$\Rightarrow$ $a_{i+1} \leq e^{(1+i)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$

p.f of claim

$$
\begin{aligned}
a_{i+1} &\leq (1+s)a_i + t \\
&\leq (1+s)[(1+s)a_{i-1} + t] + t = (1+s)^2 a_{i-1} + [1+(1+s)]t \\
&\leq (1+s)^2[(1+s)a_{i-2} + t] + [1+(1+s)]t = (1+s)^3 a_{i-2} \\
&\qquad + [1+(1+s)+(1+s)^2]t \\
&\ \ \vdots \\
&\leq (1+s)^{i+1}a_0 + [1+(1+s)+(1+s)^2+\dots+(1+s)^i]t \\
&= (1+s)^{i+1}a_0 + \boxed{\frac{1-(1+s)^{i+1}}{1-(1+s)}}\,t \rightarrow \frac{1}{s}\left[(1+s)^{i+1}-1\right] \\
&= (1+s)^{i+1}a_0 + \frac{(1+s)^{i+1}-1}{s}\,t = (1+s)^{i+1}\left[a_0 + \frac{t}{s}\right] - \frac{t}{s}
\end{aligned}
$$

$\because$ $e^x = 1 + x + \frac{1}{2}x^2 e^{\xi}$ $\therefore$ $0 \leq (1+x) \leq e^x$ $\therefore$ $0 \leq (1+x)^m \leq e^{mx}$

$$= e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

by Claim, $t_{i+1} - t_0 = t_{i+1} - a$

$$\therefore |y_{i+1} - w_{i+1}| \leq e^{\overbrace{(i+1)h}^{=\,t_{i+1}-t_0} L}\left(\overbrace{|y_0 - w_0|}^{=\,0} + \frac{h^2 M}{2hL}\right) - \frac{h^2 M}{2hL}$$

$$\leq \frac{hM}{2L}\left(e^{(t_i - a)} - 1\right) \quad \#$$

ex: $\begin{cases} y' = y - t^2 + 1, & t \in [0,2] \\ y(0) = 0.5 \end{cases}$

$\Rightarrow$ exact sol. $y(t) = (1+t)^2 - 0.5e^t$

$\therefore\ y''(t) = 2 - 0.5e^t \quad , t \in [0,2]$

$\therefore\ |y''(t)| \le M = 0.5e^2 - 2$ 求 error bound.

Euler's Method	round-off error
$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf(t_i, w_i) \end{cases}$	$u_0 = \alpha + \delta_0$ $u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1}$

$\triangle\ |y(t_i) - w_i| \le \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right)$, $h = \frac{b-a}{N}$ $h \searrow$, error $\searrow$

$\triangle\ |y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i-a)} - 1\right] + |\delta_0| e^{L(t_i - a)}$

let $E(h) = \frac{hM}{2} + \frac{\delta}{h}$, 求 $\min E(h)$

$E'(h) = \frac{M}{2} - \frac{\delta}{h^2} = 0 \Rightarrow h = \sqrt{\frac{2\delta}{M}}$ $\leftarrow$ min error

$\boxed{5\text{-}3}$

$$y(t) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{3!}y^{(3)}(x_0) + \dots$$

$$+ \frac{h^n}{n!}y^{(n)}(x_0) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi)$$

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$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \frac{h^2}{2}y''(t_i) + \dots + \frac{h^n}{n!}y^{(n)}(t_i) + \boxed{\frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi)}$$

Q: $y(t)$ 不知, $y''(t) = ?$

$$y'(t) = \frac{d}{dt}y(t) = f(t, y) = y - t^2 + 1$$

$$y''(t) = \frac{d}{dt}f(t, y(t)) = \frac{d}{dt}(y(t) - t^2 + 1)$$

$$= y'(t) - 2t = f(t, y) - 2t$$

$$= y - t^2 - 2t + 1$$

Euler's method with order n.

① $w_0 = \alpha$

② $w_{i+1} = w_i + h\,T_n(t_i, w_i)$

where $T_n(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}\frac{d}{dt}f(t_i, w_i) + \dots + \frac{h^{n-1}}{n!}\left(\frac{d}{dt}\right)^{n-1}f(t_i, w_i)$

ex:
$$\begin{cases} y'(t) = y - t^2 + 1, & 0 \le t \le 2 \\ y(0) = 0.5 \end{cases}$$

use Euler's method with order 4.

$$f(t,y) = y - t^2 + 1$$

$$\frac{d}{dt} f(t,y) = y' - 2t = y - t^2 - 2t + 1$$

$$\frac{d^2}{dt} f(t,y) = y' - 2t - 2 = y - t^2 - 2t - 1$$

$$\frac{d^3}{dt} f(t,y) = y' - 2t - 2 = y - t^2 - 2t - 1$$

$$\therefore T_4(t_i, w_i) = (w_i - t_i^2 + 1) + \frac{h}{2}[w_i - t_i^2 - 2t_i + 1]$$
$$+ \frac{h^2}{6}[w_i - t_i^2 - 2t_i - 1] + \frac{h^3}{24}[w_i - t_i^2 - 2t_i - 1]$$

① $w_0 = \alpha$

② $w_{i+1} = w_i + h\, T_4(t_i, w_i)$