# 應數一線性代數 **2020** 春, 期末考

學號: sol., 姓名: ______

本次考試共有 10 頁 (包含封面)，有 14 題。如有缺頁或漏題，請立刻告知監考人員。

## 考試須知:

- 請在第一頁以及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______

1. (10 points) Find the projection matrix $P$ for the plane $W: 3x+2y+z=0$ in $\mathbb{R}^3$ and the find the projection $\vec{b}_w$ of $\vec{b}=[4,2,-1]$ on it.

   Answer: $\vec{b}_w = \frac{1}{14}\begin{bmatrix} 11 \\ -2 \\ -29 \end{bmatrix}$, $P = \frac{1}{14}\begin{bmatrix} 5 & -6 & -3 \\ -6 & 10 & -2 \\ -3 & -2 & 13 \end{bmatrix}$.

2. (10 points) Find the lease squares straight line fit to the four points (0,1) (1,3) (2,4) (3,4) and use it to approximate the fifth points (4, a).

   Answer: the line equation = $1.5 + x$, a= $5.5$.

3. (5 points) Find the coordinate vector of $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in $M_2$ relative to $\left( \overset{a}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \overset{+b}{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}, \overset{+c}{\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}}, \overset{+d}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \right)$

Answer: [3, 5, 1, 1]

$$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c & -c \\ 0 & 3c \end{bmatrix} + \begin{bmatrix} 0 & d \\ 0 & d \end{bmatrix}$$

$1 = 0 + 0 + c + 0$

$-2 =$

$3 =$

$4 =$

4. (10 points) Find the five $\underline{\text{fifth roots}}^{5^{th}}$ of $-243i$. (need not simplify)

A: $3\left(\cos\left(\frac{3\pi}{10} + \frac{2k\pi}{5}\right) + i \sin\left(\frac{3\pi}{10} + \frac{2k\pi}{5}\right)\right)$, $k = 0, 1, 2, 3, 4$

$-243i = r(\cos\theta + i\sin\theta)$

$\vec{v} = \sqrt[5]{r}\left(\cos\left(\frac{\theta}{5} + \frac{2\pi k}{5}\right) + i \sin\left(\frac{\theta}{5} + \frac{2\pi k}{5}\right)\right)$ $k = 0 \sim 4$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible $C$ such that $R_{B',B'} = C^{-1}R_{B,B}C$ for the given linear transformation $T$.
$T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as reflection of $\mathbb{R}^3$ through the plane $x+y-z=0$; $B = E$, $B' = ([1,0,1],[1,-1,0],[1,1,-1])$.

$C_{BB'}= \frac{1}{3}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $C_{BB'}= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $R_{B'B'}= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $R_{BB}= \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

Is C=$C_{BB'}$ or $C_{BB'}$? $C_{B'B}$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in x+y-z=0 \Rightarrow T(\vec{v}_1) = \vec{v}_1$

$T(\vec{v}_2) = \vec{v}_2$

normal of $x+y-z=0$ ; $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad T(\vec{v}_3) = -\vec{v}_3$

6. (5 points) Express $(\sqrt{3}+i)^6$ in the form $a+bi$ for $a, b$ are real numbers.

Answer: a=___$-64$___, b=___$0$___.

$\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$(\sqrt{3}+i)^6 = r^6(\cos 6\theta + i\sin 6\theta)$

7. (10 points) Using the Gram-Schmidt process to transform the basis $\{\underset{\vec{V}_1}{[1, i, 1-i]}, \underset{\vec{V}_2}{[1+i, 1-i, 1]}\}$ $[1,0,0], [0,1,0], [0,0,1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

Answer: the found orthogonal basis for $\mathbb{C}^3$ is $\{[1, i, 1-i], \underbrace{[3+3i, 5-5i, 2]}_{= \vec{V}_2 - \vec{V}_{2\vec{V}_1}}, [-12i, 4, 8+8i]\}$

G-S: $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$

8. (10 points) Find an unitary matrix $U$ and a diagonal matrix $D$ such that $D = U^{-1}AU$. Also find where

$$A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad U = \begin{array}{c} \begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \vec{g}_1 & \quad \vec{g}_2 & \quad \vec{g}_3 \end{matrix} \end{array}$$

① $|A - \lambda I| = \lambda(\lambda - 2)^2$

② $\boxed{\lambda = 0}$

$rref(A - 0I) \rightarrow A\vec{v}_1 = 0\vec{v}_1$

$\boxed{\lambda = 2}$

$rref(A - 2I) \Rightarrow sp(\vec{v}_2, \vec{v}_3)$

③ G-S. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\rightarrow \{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$

☆ $\because 0 \neq 2$

$\therefore \vec{v}_1 \perp sp(\vec{v}_2, \vec{v}_3)$

$\rightarrow \vec{g}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|}$

G-S: $\{\vec{v}_2, \vec{v}_3\}$

9. (10 points) Find a Jordan canonical form and a Jordan basis for the given matrix.

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$J = \begin{bmatrix} \underbrace{\boxed{i}}_{J_1} & & & & \\ & \underbrace{\boxed{i}}_{J_2} & & 0 & \\ & & \underbrace{\boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}}}_{J_3} & & \\ 0 & & & & \underbrace{\boxed{2}}_{J_4} \end{bmatrix}$$

$$\text{basis}: \left\{ \underset{b_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 2+i \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}}, \underset{b_3}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{b_4}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}}, \underset{b_5}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$$

$$(J-iI): \begin{cases} \vec{b}_1 \to \vec{0} \\ \vec{b}_2 \to \vec{0} \end{cases}$$

$$(J-2I): \quad \vec{b}_4 \to \vec{b}_3 \to \vec{0}, \qquad \vec{b}_5 \to \vec{0}$$

10. (10 points) Find a polynomial in $A$ that gives the zero matrix.

$$(A-iI)^2(A-2I)^1$$

$$\begin{bmatrix} \boxed{i} & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} \end{bmatrix}$$

11. (5 points) Prove or disprove the following: All $2 \times 2$ matrix with determinant 1 is an orthogonal matrix.

$\exists A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ s.t. $\det(A) = 1$, but $A$ is NOT orthogonal.

12. (10 points) Find all the possible $2 \times 2$ real matrix that is unitarily diagonalizable.

13. (5 points) Prove that for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $(\vec{u}^* \vec{v})^* = \overline{\vec{u}^* \vec{v}} = \vec{v}^* \vec{u} = \vec{u}^T \overline{\vec{v}}$

14. (10 points) Prove the following:
   (a) Show that every Hermitian matrix is normal.
   (b) Show that every unitary matrix is normal.
   (c) Show that, if $A^* = -A$, then $A$ is normal.

應數一線性代數期末考, 學號: ____________, 姓名: ____________, 以下由閱卷人員填寫

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Points: | 10 | 10 | 5 | 10 | 10 | 5 | 10 | 10 |
| Score: | | | | | | | | |

| Question: | 9 | 10 | 11 | 12 | 13 | 14 | | Total |
|---|---|---|---|---|---|---|---|---|
| Points: | 10 | 10 | 5 | 10 | 5 | 10 | | 120 |
| Score: | | | | | | | | |