

Section 1.3 Matrices and Their Algebra

32. Method 1: The $(i, j)^{th}$ entry of $(AB)^T$ is the $(j, i)^{th}$ entry in AB , which is

$$\begin{aligned} & (j^{th} \text{ row of } A) \cdot (i^{th} \text{ column of } B) \\ &= (i^{th} \text{ column of } B) \cdot (j^{th} \text{ row of } A) \\ &= (i^{th} \text{ row of } B^T) \cdot (j^{th} \text{ column of } A^T) \end{aligned}$$

which is the $(i, j)^{th}$ entry of $B^T A^T$. Since $(AB)^T$ and $B^T A^T$ have the same size, they are equal.

32. Method 2:

Let

$$A = [a_{ij}]_{m \times n}, \quad B = [b_{ij}]_{n \times s}, \quad AB = C = [c_{ij}]_{m \times s}.$$

Then

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

So

$$(AB)^T = C^T = [c'_{ij}], \quad \text{where } c'_{ij} = c_{ji} = \sum_{k=1}^n a_{jk} b_{ki}.$$

Now, consider

$$A^T = [a'_{ij}]_{n \times m}, \quad B^T = [b'_{ij}]_{s \times n}, \quad B^T A^T = D = [d_{ij}]_{s \times m}.$$

Here

$$a'_{ij} = a_{ji}, \quad b'_{ij} = b_{ji}.$$

Thus

$$d_{ij} = \sum_{k=1}^n b'_{ik} a'_{kj} = \sum_{k=1}^n b_{ki} a_{jk} = \sum_{k=1}^n a_{jk} b_{ki} = c'_{ij}.$$

Therefore,

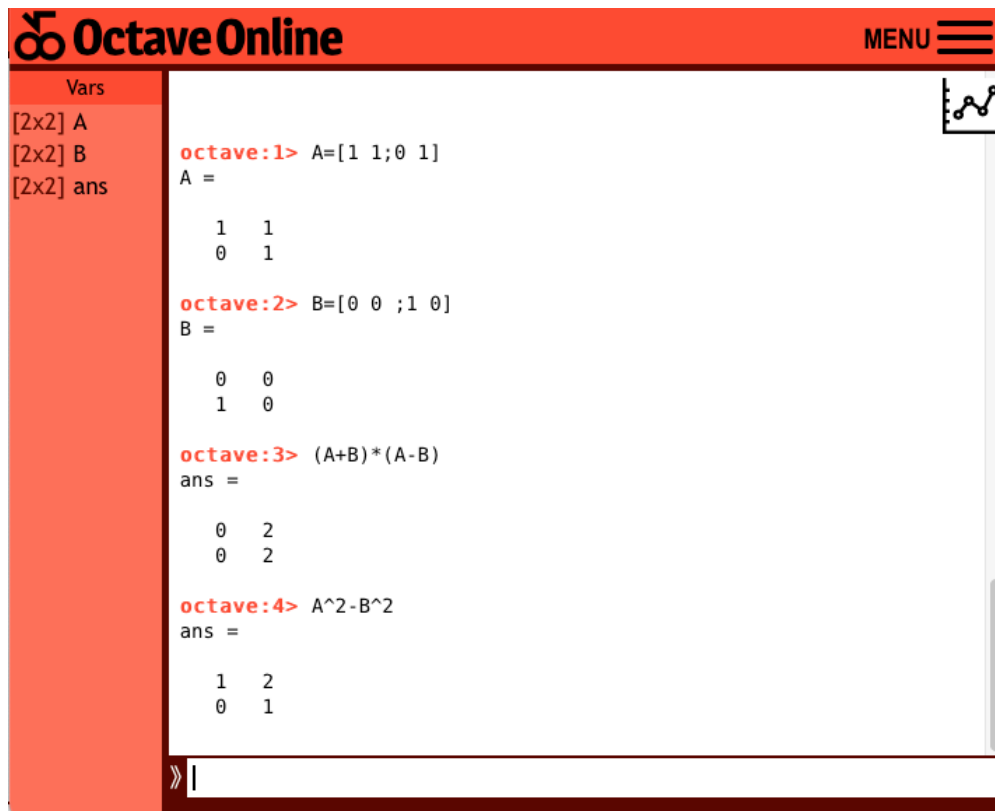
$$C^T = D \quad \Rightarrow \quad (AB)^T = B^T A^T.$$

43.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ (A+B)(A-B) &= \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \quad A^2 - B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Since $(A+B)(A-B) = (A^2-B^2)+(BA-AB)$, we know that $(A+B)(A-B) = (A^2-B^2)$ only if $BA-AB=0$. Therefore, the state holds only under the conditions that A, B are commutative.

p.s. You can using <https://octave-online.net> to check your example as below:



The screenshot shows the Octave Online interface. On the left, a sidebar lists variables: [2x2] A, [2x2] B, and [2x2] ans. The main area displays the following commands and results:

```
octave:1> A=[1 1;0 1]
A =
    1    1
    0    1

octave:2> B=[0 0 ;1 0]
B =
    0    0
    1    0

octave:3> (A+B)*(A-B)
ans =
    0    2
    0    2

octave:4> A^2-B^2
ans =
    1    2
    0    1
```

45. ** 課本答案錯了喔！

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ r & 0 & r \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} = \begin{bmatrix} 2 & 0 & r \\ 0 & 1 & 0 \\ 2 & 0 & r \end{bmatrix}$$

Therefore, $AB = BA$ if and only if $r = 2$.