應數一線性代數 2021 春, 期末考SOLUTION

考試時間：2021/06/24, 09:10 - 12:00,

收卷截止時間：12:10

考卷繳交位置：Google Classroom

考試須知:

- 本次是開書限時考，禁止交談討論。開書範圍是你事先準備的紙本/電子檔的作業或筆記，我課程網頁上提供的資訊，紙本/電子檔課本。
- 需要開鏡頭麥克風。鏡頭需要看得到你的身邊，你在作答的紙面，還有你在使用的電子資源的畫面（例如電腦螢幕或平板螢幕)。我不需要直接閱讀螢幕內容，我只要看看畫面的形狀色塊，確定你在看什麼就好。
- 請將紙面答案卷掃成一份 pdf 檔，畫面請清晰並且轉正。第一頁左上寫明姓名學號，每一題前面註明題號，頁面請按照題號順序編排不要跳號。
- 注意事先準備充足的紙張。考試途中不能向外求助更多的計算紙。

1. (5 points) Find the coordinate vector of x^3+3x^2-4x+3 in P_3 relative to $(x^3-x^2, x^2-x, x-1, x^3+1)$

$$[-0.5, 2.5, -1.5, 1.5] = \left[-\frac{1}{2}, \frac{5}{2}, -\frac{3}{2}, \frac{3}{2}\right] = \frac{1}{2}[-1, 5, -3, 3]$$

2. (5 points) Express $(\sqrt{3}i-1)^8$ in the form $a+bi$ for a, b are real numbers. Find a, b.

$$2^8(-\frac{1}{2}-\frac{\sqrt{3}}{2}i) = -128-128\sqrt{3}i$$

3. (10 points) Find the six sixth roots of $-8i$. (need not simplify)

$$\sqrt{2}\left(\cos\left(45^\circ+60k^\circ\right)+i\sin\left(45^\circ+60k^\circ\right)\right) \text{ or } \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{k\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{k\pi}{3}\right)\right), \text{ for } i = 0, 1, 2, 3, 4, 5$$

4. (10 points) Find a vector perpendicular to both $[0, i, 1+i]$ and $[1+i, 1-i, 1]$ in $\mathbb{C}^3$.

$$[-2-i, -2i, 1+i]$$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for linear transformation $T: P_2 \to P_2$ defined by $T(p(x)) = p(x+1)+p(x)$, $B = (x^2, x, 1)$, $B' = (x^2+1, x+1, 2)$.

$$R_{B,B}\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}, R_{B',B'} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -0.5 & 0.5 & 2 \end{bmatrix}, C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, C_{B,B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -0.5 & 0.5 \end{bmatrix}$$

6. (10 points) Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$, where

$$A = \begin{bmatrix} 3 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = U^{-1}AU = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2}i & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2}i & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}i & \frac{\sqrt{2}}{2}i \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 2i & 0 \\ 5 & 0 & -1 & 0 & 2i \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2i & 1 & 0 \\ 0 & 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & 0 & 2i \end{bmatrix},\ \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1-2i \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix},\ \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},\ \vec{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1-2i \\ 0 \end{bmatrix}$$

$$A\vec{v}_1 = \vec{v}_1,\ A\vec{v}_2 = \vec{v}_2,\ A\vec{v}_3 = 2i(\vec{v}_3),\ A\vec{v}_4 = 2i(\vec{v}_4) + \vec{v}_3,\ A\vec{v}_5 = 2i(\vec{v}_5)$$

8. (10 points) Answer the following question.

1. Find the eigenvalues of the given Matrix J.
2. Give the rank and nullity of $(J-\lambda)^k$ for each eigenvalue λ of J and for every positive integer k.
3. Draw schemata of the strings of vectors in the standard basis arising from the Jordan blocks in J.
4. For each standard basis vector $\vec{e}_k$, express $J\vec{e}_k$ as a linear combination of vectors in the standard basis.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix} = J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & & \mathbf{0} \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & \\ & & \boxed{\begin{matrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{matrix}} & & \\ \mathbf{0} & & & \boxed{2} & \\ & & & & \boxed{i} \end{bmatrix}$$

1. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_8 = 2,\ \lambda_5 = \lambda_6 = \lambda_7 = \lambda_9 = i$
2. $(J-2I)$ has rank 6 and nullity 3,
   $(J-2I)^k$ has rank 4 and nullity 5, for $k \geq 2$,
   $(J-iI)$ has rank 7 and nullity 2,
   $(J-iI)^2$ has rank 6 and nullity 3,
   $(J-iI)^k$ has rank 5 and nullity 4 for $k \geq 3$,
3. The strings are: $(J-2I): \begin{cases} \vec{e}_2 \to \vec{e}_1 \to 0 \\ \vec{e}_4 \to \vec{e}_3 \to 0 \\ \qquad \vec{e}_8 \to 0 \end{cases},\ (J-iI): \begin{cases} \vec{e}_7 \to \vec{e}_6 \to \vec{e}_5 \to 0 \\ \qquad\qquad\quad \vec{e}_9 \to 0 \end{cases}$
4. $\begin{cases} J\vec{e}_1 = 2\vec{e}_1, \\ J\vec{e}_2 = 2\vec{e}_2 + \vec{e}_1, \end{cases},\ \begin{cases} J\vec{e}_3 = 2\vec{e}_3, \\ J\vec{e}_4 = 2\vec{e}_4 + \vec{e}_3, \end{cases},\ \begin{cases} J\vec{e}_5 = i\vec{e}_5 \\ J\vec{e}_6 = i\vec{e}_6 + \vec{e}_5, \\ J\vec{e}_7 = i\vec{e}_7 + \vec{e}_6, \end{cases},\ \{J\vec{e}_8 = 2\vec{e}_8,\ \{J\vec{e}_9 = i\vec{e}_9$

9. (5 points) Prove or disprove the following: For a square matrix A, we have $\det(A^*) = \det(A)$

$$A = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix},\ det(A) = i \neq det(A) = -i$$

10. (5 points) Prove or disprove the following: If U is unitary, the $\overline{U}$ also an unitary matrix.

    Let U is an unitary matrix, i.e. $U^*U = I = UU^*$. $(\overline{U})^*\overline{U} = \overline{\overline{U}}^T\,\overline{U} = U^T\overline{U} = \overline{U^*U} = \overline{I} = I$. Similarly, $\overline{U}(\overline{U})^* = I$. Hence $\overline{U}$ is a unitary matrix.

11. (10 points) Find all the possible $a, b, z \in \mathbb{C}$ such that matrix $\begin{bmatrix} z & a \\ b & z \end{bmatrix}$ is unitarily diagonalizable.

$$M = \begin{bmatrix} z & a \\ b & z \end{bmatrix}$$

$$MM^* = \begin{bmatrix} z & a \\ b & z \end{bmatrix}\begin{bmatrix} \overline{z} & \overline{b} \\ \overline{a} & \overline{z} \end{bmatrix} = \begin{bmatrix} |z|^2 + |a|^2 & \overline{b}z + a\overline{z} \\ b\overline{z} + \overline{a}z & |b|^2 + |z|^2 \end{bmatrix} = M^*M = \begin{bmatrix} \overline{z} & \overline{b} \\ \overline{a} & \overline{z} \end{bmatrix}\begin{bmatrix} z & a \\ b & z \end{bmatrix} = \begin{bmatrix} |z|^2 + |b|^2 & \overline{b}z + a\overline{z} \\ b\overline{z} + \overline{a}z & |a|^2 + |z|^2 \end{bmatrix}$$

    M is unitarily diagonalizable if $|a| = |b|$

12. (10 points) Show that the n^{th} roots of $z \in \mathbb{C}$ can be represented geometrically as n equally spaced points on the circle $x^2 + y^2 = |z|^2$.

    The n^{th} roots of $z = r(\cos\theta + i\sin\theta)$ are given by

$$r^{\frac{1}{n}}(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)), \quad \text{for } k = 0, 1, 2, ..., n-1.$$

    Since the points $\left\{\frac{\theta+2k\pi}{n} \mid k = 0, 1, 2, ..., n-1\right\}$ divide the interval $[0, 2\pi]$ into n subintervals of equal width $\frac{2\pi}{n}$ we see that the n^{th} roots of z are equally spaced points on the circle $x^2 + y^2 = (r^{\frac{1}{n}})^2 = (\sqrt[n]{|z|})^2$.