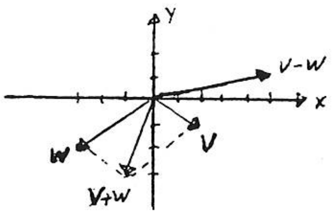
CHAPTER 1

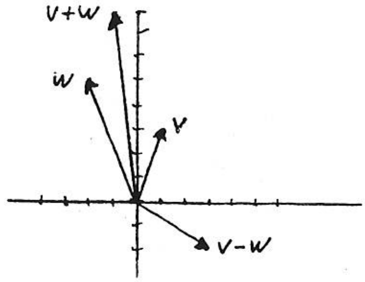
VECTORS, MATRICES, AND LINEAR SYSTEMS

SECTION 1.1 - Vectors in Euclidean Spaces

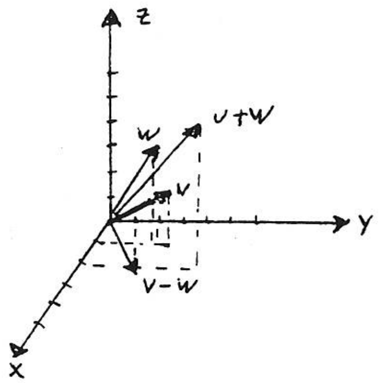
1. $\mathbf{v} + \mathbf{w} = [-1, -3]$
   $\mathbf{v} - \mathbf{w} = [5, 1]$



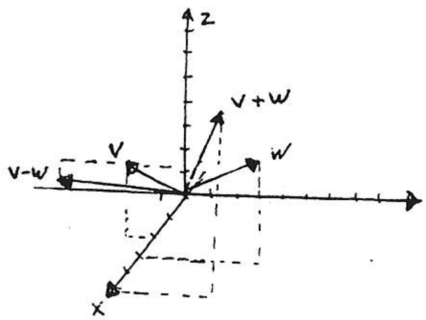
2. $\mathbf{v} + \mathbf{w} = [-1, 8]$
   $\mathbf{v} - \mathbf{w} = [3, -2]$



3. $\mathbf{v} + \mathbf{w} = 2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$
   $\mathbf{v} - \mathbf{w} = \mathbf{j} - 2\mathbf{k}$



4. $\mathbf{v} + \mathbf{w} = [5, 4, 7]$
   $\mathbf{v} - \mathbf{w} = [-1, -6, -1]$



5. $3\mathbf{u} - 2\mathbf{v} = 3[-1, 3, -2] - 2[4, 0, -1] =$
   $[-3, 9, -6] - [8, 0, -2] = [-11, 9, -4]$

6. $\mathbf{u} + 2(\mathbf{v} - 4\mathbf{w}) = [-1, 3, -2] + 2([4, 0, -1] - [-12, -4, 8]) =$
   $[-1, 3, -2] + 2[16, 4, -9] = [31, 11, -20]$

7. $\mathbf{u} + \mathbf{v} - \mathbf{w} = [-1, 3, -2] + [4, 0, -1] - [-3, -1, 2] =$
   $[-1, 3, -2] + [7, 1, -3] = [6, 4, -5]$

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8. $4(3\mathbf{u} + 2\mathbf{v} - 5\mathbf{w}) = 4([-3, 9, -6] + [8, 0, -2] - [-15, -5, 10])$
$= 4([-3, 9, -6] + [23, 5, -12]) = 4[20, 14, -18] =$
$[80, 56, -72]$

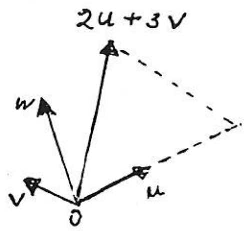
9. $\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} =$
$[1, 2, 1, 0] - [-4, 0, 2, 12] + [12, -20, 4, -8] =$
$[5, 2, -1, -12] + [12, -20, 4, -8] = [17, -18, 3, -20]$

10. $3\mathbf{u} + \mathbf{v} - \mathbf{w} = [3, 6, 3, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] =$
$[3, 6, 3, 0] + [-5, 5, 0, 8] = [-2, 11, 3, 8]$

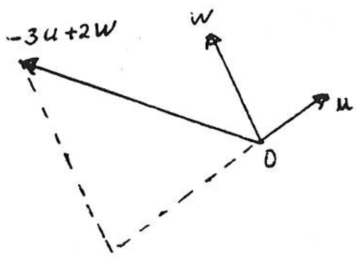
11. $4\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} =$
$[4, 8, 4, 0] - [-4, 0, 2, 12] + [12, -20, 4, -8] =$
$[8, 8, 2, -12] + [12, -20, 4, -8] = [20, -12, 6, -20]$

12. $-\mathbf{u} + 5\mathbf{v} + 3\mathbf{w} =$
$[-1, -2, -1, 0] + [-10, 0, 5, 30] + [9, -15, 3, -6] =$
$[-11, -2, 4, 30] + [9, -15, 3, -6] = [-2, -17, 7, 24]$

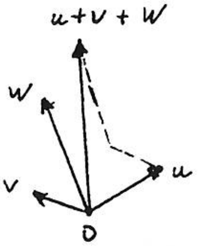
13.



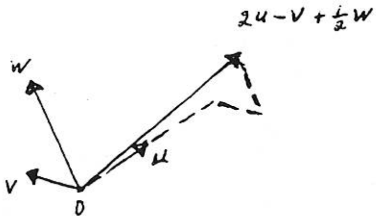
14.



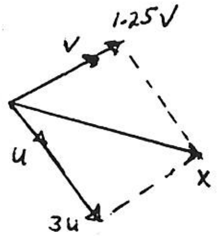
15.



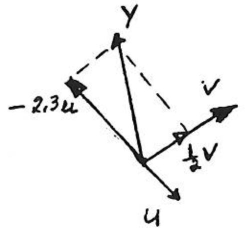
16.



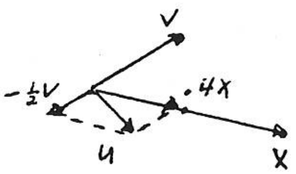
17. $r \approx 3,\ s \approx 1.25$



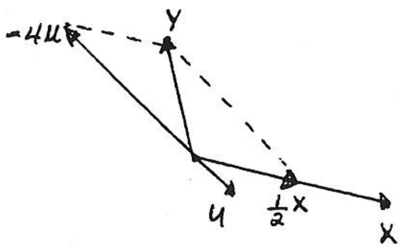
18. $r \approx -2.3,\ s \approx 0.5$



19. $r \approx .4,\ s \approx -0.5$



20. $r \approx -4,\ s \approx 0.5$



21. Looking at second components, we see that to have $[c, -3] = r[2, 6]$, we must have $6r = -3$, so $r = -1/2$. Thus $c = (-1/2)2 = -1$.

22. Looking at second components, we see that to have $[c^2, -4] = r[1, -2]$, we must have $-2r = -4$, so $r = 2$. Thus $c^2 = 2(1) = 2$, so $c^2 = 2$ and $c = \pm\sqrt{2}$.

23. If $[c, -c, 4] = r[-2, 2, 20]$, then $c = -2r$, $-c = 2r$, and $4r = 20$. Thus $r = 1/5$ and $c = -2/5$.

24. If $[c^2, c^3, c^4] = r[1, -2, 4]$, then $c^2 = r$, $c^3 = -2r$, and $c^4 = 4r$. From the first and third equations, we have $r^2 = 4r$, so $r = 0$ or $r = 4$, which gives $c = 0$ or $c = \pm 2$. The middle equation shows that $c \neq 2$. Thus $c = 0$ or $c = -2$.

However, $c = 0$ yields the zero vector for $[c^2, c^3, c^4]$, and parallel vectors are only defined for nonzero vectors. Thus $c = -2$.

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25. If $[13, -15] = r[1, 5] + s[3, c]$, then $c \neq 15$ since $[1, 5]$ and $[3, 15]$ are parallel, but are not parallel to $[13, -15]$. Any other choice for c allows $[13, -15]$ to form the diagonal of a parallelogram having adjacent edges formed by vectors $r[1, 5]$ and $s[3, c]$.

26. All c. Since $[-3, 5]$ and $[6, -11]$ are not parallel, every vector $[-1, c]$ forms the diagonal of a parallelogram with adjacent sides formed by vectors parallel to $[-3, 5]$ and to $[6, -11]$.

27. If $[1, c, -3] = r[1, 1, 0] + s[0, 1, 3]$, then $r = 1$, $c = r + s$, and $3s = -3$. Thus $s = -1$ and $c = 1 - 1 = 0$.

28. If $[1, c, c-1] = r[1, 2, 1] + s[3, 6, 3] = (r + 3s)[1, 2, 1]$, then $r + 3s = 1$, $c = 2(r + 3s)$, and $c - 1 = r + 3s$. The first and second equations yield $c = 2$, which does satisfy the third equation since $r + 3s = 1$. Thus $c = 2$.

29. If $[3, -2, c] = r[1, 2, -1] + s[0, 1, 3]$, then $3 = r$, $-2 = 2r + s$, and $c = -r + 3s$. The first two equations show that $s = -8$, and the last one then yields $c = -3 - 24 = -27$.

30. If $[c, -2c, c] = r[1, -1, 1] + s[0, 1, -3] + t[0, 0, 1]$, then $c = r$, $-2c = -r + s$, and $c = r - 3s + t$. From the first two equations, we obtain $-2c = -c + s$ so $s = -c$. The third equation then becomes $c = c + 3c + t$ so $t = -3c$. Thus we see that by taking $r = c$, $s = -c$, and $t = -3c$, we can obtain the vector $[c, -2c, c]$ for every value c.

31. $[4, 2] - [-1, 3] = [5, -1]$

32. $[4, -2, -6] - [-3, 2, 5] = [7, -4, -11]$

33. $[3, -2, 1, 7] - [2, 1, 5, -6] = [1, -3, -4, 13]$

34. $[-5, -4, -3, -2, -1] - [1, 2, 3, 4, 5] = [-6, -6, -6, -6, -6]$

35. $x\begin{bmatrix}3\\1\\2\end{bmatrix} + y\begin{bmatrix}-2\\-1\\1\end{bmatrix} + z\begin{bmatrix}4\\-3\\-5\end{bmatrix} = \begin{bmatrix}10\\0\\-3\end{bmatrix}$

36. $x_1\begin{bmatrix}1\\3\end{bmatrix} + x_1\begin{bmatrix}-3\\0\end{bmatrix} + x_3\begin{bmatrix}2\\-4\end{bmatrix} + x_4\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}-6\\12\end{bmatrix}$

37. a)
$$\begin{aligned} -3p \phantom{{}-2q} &- 4r + 6s = 8 \\ 4p - 2q &+ 3r \phantom{{}+6s} = -3 \\ 6p + 5q &- 2r + 7s = 1 \end{aligned}$$

b)
$$p\begin{bmatrix}-3\\4\\6\end{bmatrix} + q\begin{bmatrix}0\\-2\\5\end{bmatrix} + r\begin{bmatrix}-4\\3\\-2\end{bmatrix} + s\begin{bmatrix}6\\0\\7\end{bmatrix} = \begin{bmatrix}8\\-3\\1\end{bmatrix}$$

38.
$$\begin{aligned} -2r_1 + 5r_2 + 16r_3 &= 5 \\ 3r_1 + 13r_2 \phantom{{}+16r_3} &= -8 \\ -4r_2 - 9r_3 &= 11 \end{aligned}$$

39. F T F F T F T F F T

40. a)
$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [u_1 + v_1, \cdots, u_n + v_n] + [w_1, \cdots, w_n] \\ &= [(u_1 + v_1) + w_1, \cdots, (u_n + v_n) + w_n] \\ &= [u_1 + (v_1 + w_1), \cdots, u_n + (v_n + w_n)] \\ &= [u_1, \cdots, u_n] + [v_1 + w_1, \cdots, v_n + w_n] \\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) \end{aligned}$$

b)
$$\begin{aligned} \mathbf{0} + \mathbf{v} &= [0, \cdots, 0] + [v_1, \cdots, v_n] \\ &= [0 + v_1, \cdots, 0 + v_n] = [v_1, \cdots, v_n] = \mathbf{v} \end{aligned}$$

c)
$$\begin{aligned} \mathbf{v} + (-\mathbf{v}) &= [v_1, \cdots, v_n] + [-v_1, \cdots, -v_n] \\ &= [v_1 - v_1, \cdots, v_n - v_n] = [0, \cdots, 0] = \mathbf{0} \end{aligned}$$

41. a)
$$\begin{aligned} \mathrm{r}(\mathbf{v} + \mathbf{w}) &= r([v_1, \cdots, v_n] + [w_1, \cdots, w_n]) \\ &= r[v_1 + w_1, \cdots, v_n + w_n] \\ &= [r(v_1 + w_1), \cdots, r(v_n + w_n)] \\ &= [rv_1 + rw_1, \cdots, rv_n + rw_n] \\ &= [rv_1, \cdots, rv_n] + [rw_1, \cdots, rw_n] \\ &= r[v_1, \cdots, v_n] + r[w_1, \cdots, w_n] \\ &= \mathrm{r}\mathbf{v} + \mathrm{r}\mathbf{w} \end{aligned}$$

b)
$$\begin{aligned} r(s\mathbf{v}) &= r(s[v_1, \cdots, v_n]) = r[sv_1, \cdots, sv_n] \\ &= [r(sv_1), \cdots, r(sv_n)] = [(rs)v_1, \cdots, (rs)v_n] \\ &= rs[v_1, \cdots, v_n] = (rs)\mathbf{v} \end{aligned}$$

c) $1\mathbf{v} = 1[v_1, \cdots, v_n] = [1v_1, \cdots, 1v_n] = [v_1, \cdots, v_n] = \mathbf{v}$

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42. $\left.\begin{aligned} r - 2s &= b_1 \\ 3r + 5s &= b_2 \end{aligned}\right]$ Adding -3 times the first equation to the second equation yields $11s = b_2 - 3b_1$. Thus $s = \frac{1}{11}(b_2 - 3b_1)$. Substituting into the first equation yields $r = b_1 + \frac{2}{11}(b_2 - 3b_1)$ $= \frac{1}{11}(5b_1 + 2b_2)$. Substitution into the equations shows that these values of r and s yield a solution for all $b_1, b_2 \in \mathbb{R}$.

M1. [-58, 79, -36, -190] **M2.** [19, -7, 3, 73]

M3. Error using + since **a** and **u** have different numbers of components.

M4. a) [2.1667, -0.2333, 3.8095]
b) [2.16666666666667, -0.23333333333333, 3.80952380952381]
c) [13/6, -7/30, 80/21]

M5. a) [-2.4524, 4.4500, -11.3810]
b) [-2.45238095238095, 4.45000000000000, -11.38095238095238]
c) [-103/42, 89/20, -293/21]

M6. a) [1.0952, -3.9048, -1.333, 1.0476]
b) [1.09523809523810, -3.90476190476190, -1.33333333333333, 1.04761904761905]
c) [23/21, -82/21, -4/3, 22/21]

M7. a) [0.0357, 0.0075, 0.1962]
b) [0.03571428571429, 0.00750000000000, 0.19619047619048]
c) [1/28, 3/400, 103/525]

M8. $\begin{bmatrix} -37 \\ 59 \\ -10 \\ -110 \end{bmatrix}$. They are the same.

M9. Error using + since **u** is a row vector and $\mathbf{u}^T$ is a column vector.

M10. [-1/6, 8/3, 11/2, 25/3, 67/6, 14]

SECTION 1.2 - The Norm and the Dot Product

1. $\|-\mathbf{u}\| = \|[1, -3, -4]\| = \sqrt{1^2 + (-3)^2 + (-4)^2} = \sqrt{26}$

2. $\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$

3. $\|\mathbf{u} + \mathbf{v}\| = \|[-1, 3, 4] + [2, 1, -1]\| = \|[1, 4, 3]\| =$
$\sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$

4. $\|\mathbf{v} - 2\mathbf{u}\| = \|[2, 1, -1] - 2[-1, 3, 4]\| =$
$\|[2, 1, -1] - [-2, 6, 8]\| = \|[4, -5, -9]\| =$
$\sqrt{(4)^2 + (-5)^2 + (-9)^2} = \sqrt{122}$

5. $\|3\mathbf{u} - \mathbf{v} + 2\mathbf{w}\| = \|3[-1, 3, 4] - [2, 1, -1] + 2[-2, -1, 3]\|$
$= \|[-3, 9, 12] - [2, 1, -1] + [-4, -2, 6]\| = \|[-9, 6, 19]\|$
$= \sqrt{(-9)^2 + 6^2 + (19)^2} = \sqrt{478}$

6. $\left\|\frac{4}{5}\mathbf{w}\right\| = \frac{4}{5}\|[-2, -1, 3]\| = \frac{4}{5}(\sqrt{4 + 1 + 9}) = \frac{4}{5}(\sqrt{14})$

7. $\|\mathbf{u}\| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{1 + 9 + 16} = \sqrt{26}$;
$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{26}}[-1, 3, 4] = [\frac{-1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}]$

8. $\|\mathbf{w}\| = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$;
$\frac{-1}{\|\mathbf{w}\|}\mathbf{w} = \frac{-1}{\sqrt{14}}[-2, -1, 3] = [\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}]$

9. $\mathbf{u} \cdot \mathbf{v} = (-1)(2) + (3)(1) + (4)(-1) = -2 + 3 - 4 = -3$

10. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = [-1, 3, 4] \cdot ([2, 1, -1] + [-2, -1, 3])$
$= [-1, 3, 4] \cdot [0, 0, 2] = 0 + 0 + 8 = 8$

11. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = ([-1, 3, 4] + [2, 1, -1]) \cdot [-2, -1, 3]$
$= [1, 4, 3] \cdot [-2, -1, 3] = -2 - 4 + 9 = 3$

12. $\cos\theta = \dfrac{[-1, 3, 4] \cdot [2, 1, -1]}{\sqrt{1 + 9 + 16}\,\sqrt{4 + 1 + 1}} = \dfrac{-3}{\sqrt{156}} \approx -0.2401922$;
$\theta = \cos^{-1}(\frac{-3}{\sqrt{156}}) \approx 103.9°$

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13. $\cos\theta = \dfrac{[-1, 3, 4]\cdot[-2, -1, 3]}{\sqrt{1 + 9 + 16}\sqrt{4 + 1 + 9}} = \dfrac{11}{\sqrt{364}} \approx -0.5765567;$

$\theta = \cos^{-1}(\dfrac{11}{\sqrt{364}}) \approx 54.8°$

14. $[-1, 3, 4]\cdot[x, -3, 5] = -x - 9 + 20 = 0$, $-x + 11 = 0$, $x = 11$

15. $[-1, 3, 4]\cdot[-3, y, 10] = 3 + 3y + 40 = 0$, $3y + 43 = 0$, $y = -43/3$

16. $[x, y, z]\cdot[-1, 3, 4] = -x + 3y + 4z = 0$,
$[x, y, z]\cdot[2, 1, -1] = 2x + y - z = 0$.
Adding 2 times the first equation to the second, we obtain $7y + 7z = 0$, so $y = -z$. Replacing y by $-z$ in either the first or second equation yields $x = z$. Thus if we let $z = r$, we see that every vector of the form $[r, -r, r]$ for $r \neq 0$ is a solution. One answer, taking $r = 1$, is $[1, -1, 1]$.

17. $[x, y, z]\cdot[-1, 3, 4] = -x + 3y + 4z = 0$,
$[x, y, z]\cdot[-2, -1, 3] = -2x - y + 3z = 0$.
Adding -2 times the first equation to the second, we obtain $-7y - 5z = 0$, so $y = -\frac{5}{7}z$. Relacing y by $-\frac{5}{7}z$ in the first or second equation yields $x = \frac{13}{7}z$. Thus if we let $z = 7r$, we see that every vector of the form $[13r, -5r, 7r]$ for $r \neq 0$ is a solution. One answer, taking $r = 1$, is $[13, -5, 7]$.

18. $\|[42, 14]\| = 14\|[3, 1]\| = 14\sqrt{10}$

19. $\|[10, 20, 25, -15]\| = 5\|[2, 4, 5, -3]\| = 5\sqrt{4 + 16 + 25 + 9}$
$= 5\sqrt{54} = 15\sqrt{6}$

20. $[14, 21, 28]\cdot[4, 8, 20] = (7[2, 3, 4])\cdot(4[1, 2, 5])$
$= 28([2, 3, 4]\cdot[1, 2, 5]) = 28(2 + 6 + 20) = 28^2$
$= (30 - 2)^2 = 900 - 120 + 4 = 784$

21. $[12, -36, 24]\cdot[25, 30, 10] = (12[1, -3, 2])\cdot(5[5, 6, 2]) =$
$60([1, -3, 2]\cdot[5, 6, 2]) = 60(5 - 18 + 4) = (60)(-9) = -540$

22. $\cos\theta = \dfrac{[1, -1, 2, 3, 0, 4]\cdot[7, 0, 1, 3, 2, 4]}{\sqrt{1 + 1 + 4 + 9 + 0 + 16}\sqrt{49 + 0 + 1 + 9 + 4 + 16}}$

$= \dfrac{34}{\sqrt{2449}} \approx 0.687044$, so $\theta = \cos^{-1}(\dfrac{34}{\sqrt{2449}}) \approx 46.6°$.

23. $[4, 1, -1] - [2, 0, 4] = [2, 1, -5]$,
$[6, 7, 7] - [2, 0, 4] = [4, 7, 3]$,
$[2, 1, -5]\cdot[4, 7, 3] = 8 + 7 - 15 = 0$. Since the vectors are perpendicular, the triangle formed by the three points is a right triangle.

24. The angle θ between unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfies the relation $\mathbf{u}_1\cdot\mathbf{u}_2 = \|\mathbf{u}_1\|\|\mathbf{u}_2\|(\cos\theta) = \cos\theta$. Thus $\theta = \arccos(\mathbf{u}_1\cdot\mathbf{u}_2)$.

25. $[-1, 4]\cdot[8, 2] = -8 + 8 = 0$; perpendicular

26. $[-2, -1]\cdot[5, 2] = -10 - 2 = -12 \neq 0$; not perpendicular
If $r[-2, -1] = [5, 2]$, then $-2r = 5$ so $r = -5/2$, and also $-r = 2$ so $r = -2$. Since $-5/2 \neq -2$, the vectors are not parallel either.

27. $-3[3, 2, 1] = [-9, -6, -3]$; parallel with opposite direction

28. $[2, 1, 4, -1]\cdot[0, 1, 2, 4] = 0 + 1 + 8 - 4 = 5 \neq 0$; not perpendicular. If $r[2, 1, 4, -1] = [0, 1, 2, 4]$, then $2r = 0$ so $r = 0$, and also $r = 1$. Since $0 \neq 1$, the vectors are not parallel either.

29. $[10, 4, -1, 8]\cdot[-5, -2, 3, -4] = -50 - 8 - 3 - 32 = -93 \neq 0$; not perpendicular. If $r[10, 4, -1, 8] = [-5, -2, 3, -4]$, then $10r = -5$ so $r = -1/2$. This value of r also satisfies $4r = -2$. But equating 3rd components yields $-r = 3$ so $r = -3$. Since $-1/2 \neq -3$, the vectors are not parallel either.

30. $[4, 1, 2, 1, 6]\cdot[8, 2, 4, 2, 3] > 0$; not perpendicular
If $r[4, 1, 2, 1, 6] = [8, 2, 4, 2, 3]$, then $4r = 8$ so $r = 2$. This value of r also satisfies $r = 2$, $2r = 4$, and $r = 2$. But equating the last components yields $6r = 3$ so $r = 1/2$. Since $2 \neq 1/2$, the vectors are not parallel either.

31. The vector $\mathbf{v} - \mathbf{w}$, when translated to start at $(w_1, w_2, \cdots, w_n)$, reaches to the tip of the vector $\mathbf{v}$, so its length is the distance from $(w_1, w_2, \cdots, w_n)$ to $(v_1, v_2, \cdots, v_n)$.

32. $\|[-1, 4, 2] - [0, 8, 1]\| = \|[-1, -4, 1]\| = \sqrt{(-1)^2 + (-4)^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$

33. $\|[2, 1, 3] - [4, 1, -2]\| = \|[-2, -2, 5]\| = \sqrt{4 + 4 + 25} = \sqrt{33}$

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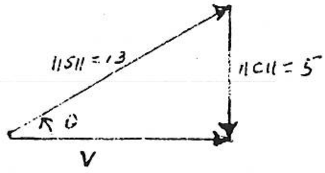
34. $\|[3, 1, 2, 4] - [-1, 2, 1, 2]\| = \|[4, -1, 1, 2]\| =$
$\sqrt{16 + 1 + 1 + 4} = \sqrt{22}$

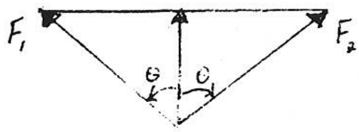
35. $\|[-1, 2, 1, 4, 7, -3] - [2, 1, -3, 5, 4, 5]\| =$

$\|[-3, 1, 4, -1, 3, -8]\| = \sqrt{9 + 1 + 16 + 1 + 9 + 64} = \sqrt{100}$
$= 10$

36. From the figure, we see that
$\theta = \sin^{-1}(\frac{5}{13}) \approx 22.62°$. Now
$90° - 22.62° = 67.38°$, so we should steer in direction 067.38.



37. Let T be the tension. Referring to the figure, we see that

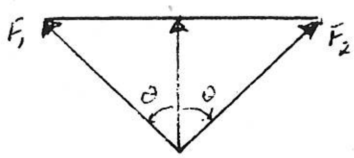


$T = \|\mathbf{F}_1\| = \|\mathbf{F}_2\|$,
$\mathbf{F}_1 = (-T \sin \theta)\mathbf{i} + (T \cos \theta)\mathbf{j}$,
$\mathbf{F}_2 = (T \sin \theta)\mathbf{i} + T(\cos \theta)\mathbf{j}$.

Thus $100 = \|\mathbf{F}_1 + \mathbf{F}_2\| = \|(2T \cos \theta)\mathbf{j}\| = 2T(\frac{\sqrt{3}}{2}) = \sqrt{3}T$.

Therefore $T = 100/\sqrt{3}$ lb.

38. Let T be the tension. Referring to the figure, we see that



$T = \|\mathbf{F}_1\| = \|\mathbf{F}_2\|$,
$\mathbf{F}_1 = (-T \sin \theta)\mathbf{i} + (T \cos \theta)\mathbf{j}$,
$\mathbf{F}_2 = (T \sin \theta)\mathbf{i} + T(\cos \theta)\mathbf{j}$.

a) $100 = \|\mathbf{F}_1 + \mathbf{F}_2\| = \|(2T \cos \theta)\mathbf{j}\| = 2T|\cos \theta|$. Therefore $T = 50/|\cos \theta|$ lb. (Of course, $\cos \theta$ is actually positive for this problem.)
b) Using part (a), we have $T = 50/(\cos 0) = 50/1 = 50$ lb.
c) The rope will break since as $\theta \longrightarrow 90°$, we have $(\cos \theta) \longrightarrow 0$, so $T \longrightarrow \infty$ lb.

39. a) Referring to Fig. 1.29, we see that $(\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}$ is a unit vector along the right-hand rope. Thus $\mathbf{F}_1 = (T_1 \sin \theta_1)\mathbf{i} + (T_1 \cos \theta_1)\mathbf{j}$.
b) By an analogous argument $\mathbf{F}_2 = -(T_2 \sin \theta_2)\mathbf{i} + T_2(\cos \theta_2)\mathbf{j}$.

c) $T_1 \sin \theta_1 - T_2 \sin \theta_2 = 0, \quad T_1 \cos \theta_1 + T_2 \cos \theta_2 = 100$

d) The equations in part (c) become

$$(1/\sqrt{2})T_1 - (1/2)T_2 = 0$$

$$(1/\sqrt{2})T_1 + (\sqrt{3}/2)T_2 = 100.$$ Subtracting, we obtain

$$-\left(\frac{\sqrt{3}+1}{2}\right)T_2 = -100, \quad \text{so}$$

$$T_2 = \frac{200}{\sqrt{3}+1}\ \text{lb} \quad \text{and} \quad T_1 = \sqrt{2}\left(\frac{1}{2}\ T_2\right) = \frac{100\sqrt{2}}{\sqrt{3}+1}\ \text{lb}.$$

40. T T F F T F T F F F

41. a) We have $\|\mathbf{v}\| = \|[v_1, v_2, \cdots, v_n]\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ which is nonnegative, and equals 0 if and only if each $v_i = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$.

b) We have $\|r\mathbf{v}\| = \|r[v_1, v_2, \cdots, v_n]\| =$

$$\|[rv_1, rv_2, \cdots, rv_n]\| = \sqrt{r^2v_1^2 + r^2v_2^2 + \cdots + r^2v_n^2} =$$

$$|r|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = |r|\,\|\mathbf{v}\|.$$

42. a) We have $\mathbf{v}\cdot\mathbf{w} = [v_1, v_2, \cdots, v_n]\cdot[w_1, w_2, \cdots, w_n]$
$= v_1w_1 + v_2w_2 + \cdots + v_nw_n = w_1v_1 + w_2v_2 + \cdots + w_nv_n$
$= [w_1, w_2, \cdots, w_n]\cdot[v_1, v_2, \cdots, v_n] = \mathbf{w}\cdot\mathbf{v}.$

b) We have $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$
$= [u_1, u_2, \cdots, u_n]\cdot[v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n]$
$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n)$
$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n$
$= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n)$
$= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.$

c) We have $r(\mathbf{v}\cdot\mathbf{w}) = r([v_1, v_2, \cdots, v_n]\cdot[w_1, w_2, \cdots, w_n]) =$
$r(v_1w_1 + v_2w_2 + \cdots + v_nw_n) = r(w_1v_1 + w_2v_2 + \cdots + w_nv_n)$
$= (rv_1)w_1 + (rv_2)w_2 + \cdots + (rv_n)w_n = (r\mathbf{v})\cdot\mathbf{w}$
$= v_1(rw_1) + v_2(rw_2) + \cdots + v_n(rw_n) = \mathbf{v}\cdot(r\mathbf{w}).$

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43. Now
$$\begin{aligned}(\mathbf{v} - \mathbf{w})\cdot(\mathbf{v} + \mathbf{w}) &= \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{w}\\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2.\end{aligned}$$
Thus $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are perpendicular if and only if $(\mathbf{v} - \mathbf{w})\cdot(\mathbf{v} + \mathbf{w}) = 0$, which occurs if and only if $\|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 0$, which happens if and only if $\|\mathbf{v}\| = \|\mathbf{w}\|$.

44. We know that $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, so $\mathbf{w}\cdot\mathbf{u} = 0$ and $\mathbf{w}\cdot\mathbf{v} = 0$. Then
$$\begin{aligned}\mathbf{w}\cdot(r\mathbf{u} + s\mathbf{v}) &= \mathbf{w}\cdot(r\mathbf{u}) + \mathbf{w}\cdot(s\mathbf{v})\\ &= r(\mathbf{w}\cdot\mathbf{u}) + s(\mathbf{w}\cdot\mathbf{v})\\ &= r(0) + s(0)\\ &= 0.\end{aligned}$$
Therefore $\mathbf{w}$ is perpendicular to $r\mathbf{u} + s\mathbf{v}$.

45. Assume that the rhombus has one corner at the origin. Let $\mathbf{v}$ and $\mathbf{w}$ be vectors along the sides of the rhombus (emanating from the origin). Since $\|\mathbf{v}\| = \|\mathbf{w}\|$, Exercise 43 shows that $\mathbf{v} - \mathbf{w}$ is perpendicular to $\mathbf{v} + \mathbf{w}$. Since one diagonal of the rhombus is $\mathbf{v} + \mathbf{w}$ and the other is parallel to $\mathbf{v} - \mathbf{w}$, these diagonals are perpendicular.

46. Figure 1.30 suggests that $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v} - \mathbf{w}\|$. Using $\mathbf{v}\cdot\mathbf{w} = 0$, we proceed to prove this.

Now
$$\begin{aligned}\|\mathbf{v} + \mathbf{w}\|^2 = (\mathbf{v} + \mathbf{w})\cdot(\mathbf{v} + \mathbf{w}) &= \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}\\ &= \mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.\end{aligned}$$
Similarly, $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

Therefore $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$, so $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v} - \mathbf{w}\|$.

Let P be the midpoint of the hypotenuse in Fig. 1.30. The distance from P to the vertex at the right angle is $\frac{1}{2}\|\mathbf{v} + \mathbf{w}\|$; the distance from P to the other two vertices is $\frac{1}{2}\|\mathbf{v} - \mathbf{w}\|$. Therefore these distances are equal.

M1. $\|\mathbf{a}\| \approx 6.3246$; the two results agree.

M2. $\|\mathbf{b}\| \approx 7.4162$

M3. $\|\mathbf{u}\| \approx 1.8251$

M4. a) $\mathbf{v}\cdot\mathbf{w} \approx 26.8074$ b) $\mathbf{v}\cdot\mathbf{w} \approx \dfrac{12385}{462}$

M5. a) 485.1449 b) Not found by MATLAB.

M6. angle $\approx$ 1.1999 radians

M7. angle $\approx$ 0.9499 radians

M8. angle $\approx 114.4194°$

M9. angle $\approx 147.4283°$

SECTION 1.3 - Matrices and Their Algebra

1. $3A = \begin{bmatrix} -6 & 3 & 9 \\ 12 & 0 & -3 \end{bmatrix}$ 2. $0B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 3. $A + B = \begin{bmatrix} 2 & 2 & 1 \\ 9 & -1 & 2 \end{bmatrix}$

4. Impossible   5. $C - D = \begin{bmatrix} 6 & -3 \\ -3 & 1 \\ -2 & 5 \end{bmatrix}$

6. $4A - 2B = \begin{bmatrix} -8 & 4 & 12 \\ 16 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 8 & 2 & -4 \\ 10 & -2 & 6 \end{bmatrix} = \begin{bmatrix} -16 & 2 & 16 \\ 6 & 2 & -10 \end{bmatrix}$

7. Impossible   8. Impossible

9. $(2A)(5C) = 10AC = 10\begin{bmatrix} -13 & 14 \\ 11 & -6 \end{bmatrix} = \begin{bmatrix} -130 & 140 \\ 110 & -60 \end{bmatrix}$

10. $(5D)(4B) = 20DB = 20\begin{bmatrix} -6 & -6 & 14 \\ 37 & -2 & 9 \\ -19 & 2 & -7 \end{bmatrix} = \begin{bmatrix} -120 & -120 & 280 \\ 740 & -40 & 180 \\ -380 & 40 & -140 \end{bmatrix}$

11. Impossible   12. $AC = \begin{bmatrix} -13 & 14 \\ 11 & -6 \end{bmatrix}$; $(AC)^2 = \begin{bmatrix} 323 & -266 \\ -209 & 190 \end{bmatrix}$

13. $(2A - B)D = \begin{bmatrix} -8 & 1 & 8 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 27 & -35 \\ -4 & 26 \end{bmatrix}$

14. $(AD)B = \begin{bmatrix} 8 & -8 \\ -15 & 11 \end{bmatrix}\begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 16 & -40 \\ -5 & -26 & 63 \end{bmatrix}$

15. $(A^T)A = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 20 & -2 & -10 \\ -2 & 1 & 3 \\ -10 & 3 & 10 \end{bmatrix}$

16. $BC = \begin{bmatrix} 14 & -2 \\ 1 & -5 \end{bmatrix}$, $CB = \begin{bmatrix} 3 & 3 & -7 \\ 30 & -6 & 18 \\ -2 & -5 & 12 \end{bmatrix}$

17. a) $A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $A^7 = \begin{bmatrix} 128 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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18. a) $A^2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

b) $A^7 = (A^2)^3 A = \begin{bmatrix} -8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 128 & 0 \\ -16 & 0 & 0 \end{bmatrix}$

19. $\mathbf{xy} = [-2\ \ 3\ \ -1] \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = [-14]$, $\mathbf{yx} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} [-2\ \ 3\ \ -1] = \begin{bmatrix} -8 & 12 & -4 \\ 2 & -3 & 1 \\ -6 & 9 & -3 \end{bmatrix}$

20. $\begin{bmatrix} 1 & -1 & 2 & 5 \\ -1 & 4 & -7 & 8 \\ 2 & -7 & -1 & 6 \\ 5 & 8 & 6 & 3 \end{bmatrix}$

21. T F F T F T T T F T (e & i, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$)

22. a) Let $\mathbf{x}$ be a $1 \times n$ row vector. If $\mathbf{x}A$ is defined, then A is $n \times m$ for some m, and $\mathbf{x}A$ is then a $1 \times m$ row vector.

b) Similar proof to (a), or we can note that $(A\mathbf{y})^T = \mathbf{y}^T A^T$ and appeal to part (a).

23. Equation (7) indicates that for column vectors $\mathbf{a}$ and $\mathbf{b}$ with n components, we have $\mathbf{a} \cdot \mathbf{b} = \mathbf{b}^T \mathbf{a}$. Thus for A, $\mathbf{b}$, and $\mathbf{c}$ as given in this exercise, we have $(A\mathbf{b}) \cdot (A\mathbf{c}) = (A\mathbf{b})^T (A\mathbf{c})$ $= \mathbf{b}^T A^T A\mathbf{c}$. An alternate answer is $\mathbf{c}^T A^T A\mathbf{b}$.

24. Now $\mathbf{c}A = ((\mathbf{c}A)^T)^T = (A^T \mathbf{c}^T)^T$ and we know that $A^T \mathbf{c}^T$ is a linear column of the column vectors of A^T where the coefficient of the jth column vector is c_j. Since the transpose of a linear combination of column vectors is a linear combination of row vectors with the same coefficients, we see that $\mathbf{c}A = (A^T \mathbf{c}^T)^T$ is the linear combination of the row vectors of A, where the coefficient of the ith row vector is c_i.

25. The (i,j)th entry in $A + B$ is $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, which is the (i,j)th entry in $B + A$. Since $A + B$ and $B + A$ have the same size, they are equal.

26. The (i,j)th entry in $(A + B) + C$ is $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$, which is the (i,j)th entry in $A + (B + C)$. Since $(A + B) + C$ and $A + (B + C)$ have the same size, they are equal.

27. The (i,j)th entry in $(r + s)A$ is $(r + s)a_{ij} = ra_{ij} + sa_{ij}$, which is the (i,j)th entry in $rA + sA$. Since $(r + s)A$ and $rA + sA$ have the same size, they are equal.

28. The (i,j)th entry in $(rs)A$ is $(rs)a_{ij} = r(sa_{ij})$, which is the (i,j)th entry in $r(sA)$. Since $(rs)A$ and $r(sA)$ have the same size, they are equal.

29. Using D2 of Theorem 1.3, the (i,j)th entry in $A(B + C)$ is
(ith row of A)•(jth column of $B + C$) =
(ith row of A)•(jth col of B) + (ith row of A)•(jth col of C)
which is the (i,j)th entry in $AB + 4AC$. Since $A(B + C)$ and $AB + AC$ have the same size, they are equal.

30. If $A = [a_{ij}]$ is an $m \times n$ matrix, then A^T is the $n \times m$ matrix whose (i,j)th entry is a_{ji} and $(A^T)^T$ is the $m \times n$ matrix matrix whose (i,j)th entry is again a_{ij}. Thus $(A^T)^T = A$.

31. The (i,j)th entry of $(A + B)^T$ is $a_{ji} + b_{ji}$, which is the (i,j)th entry of $A^T + B^T$. Since $(A + B)^T$ and $A^T + B^T$ have the same size, they are equal.

32. The (i,j)th entry of $(AB)^T$ is the (j,i)th entry in AB, which is

$$\begin{aligned} & (j\text{th row of } A)\bullet(i\text{th column of } B) \\ &= (i\text{th column of } B)\bullet(j\text{th row of } A) \qquad \text{(Theorem 1.3)} \\ &= (i\text{th row of } B^T)\bullet(j\text{th column of } A^T), \end{aligned}$$

which is the (i,j)th entry of B^TA^T. Since $(AB)^T$ and B^TA^T have the same size, they are equal.

33. Let $A = [a_{ij}]$ be an $m \times n$ matrix, $B = [b_{jk}]$ be an $n \times r$ matrix, and $C = [c_{kq}]$ be an $r \times s$ matrix. Then the ith row of AB is

$$\left(\sum_{j=1}^{n} a_{ij}b_{j1},\ \sum_{j=1}^{n} a_{ij}b_{j2},\ \ldots,\ \sum_{j=1}^{n} a_{ij}b_{jr}\right)$$

so the (i,q)th entry in $(AB)C$ is

$$(\sum_{j=1}^{n} a_{ij}b_{j1})c_{1q} + (\sum_{j=1}^{n} a_{ij}b_{j2})c_{2q} + \cdots + (\sum_{j=1}^{n} a_{ij}b_{jr})c_{rq}$$

$$= \sum_{k=1}^{r}\left[\sum_{j=1}^{n}(a_{ij}b_{jk}c_{kq})\right].$$

Also, the qth column of BC has components

$$\sum_{k=1}^{r} b_{1k}c_{kq},\ \sum_{k=1}^{r} b_{2k}c_{kq},\ \ldots,\ \sum_{k=1}^{r} b_{nk}c_{kq}$$

so the (i,q)th entry in $A(BC)$ is

$$a_{i1}(\sum_{k=1}^{r} b_{1k}c_{kq}) + a_{i2}(\sum_{k=1}^{r} b_{2k}c_{kq}) + \cdots + a_{in}(\sum_{k=1}^{r} b_{nk}c_{kq})$$

$$= \sum_{j=1}^{n} a_{ij}(\sum_{k=1}^{r} b_{jk}c_{kq}) = \sum_{j=1}^{n}\left[\sum_{k=1}^{r}(a_{ij}b_{jk}c_{kq})\right]$$

$$= \sum_{k=1}^{r}\left[\sum_{j=1}^{n}(a_{ij}b_{jk}c_{kq})\right].$$

Since $(AB)C$ and $A(BC)$ are both $m \times s$ matrices with the same (i,q)th entry, they must be equal.

34. Let $A = [a_{ij}]$ be $m \times n$ and let $B = [b_{jk}]$ be $n \times s$. The (i,k)th entry in $(rA)B$ is $\sum_{j=1}^{n}(ra_{ij})b_{jk} = r\sum_{j=1}^{n} a_{ij}b_{jk}$. The (i,k)th entry in $A(rB)$ is $\sum_{j=1}^{n} a_{ij}(rb_{jk}) = r\sum_{j=1}^{n} a_{ij}b_{jk}$. The (i,k)th entry in $r(AB)$ is $r\sum_{j=1}^{n} a_{ij}b_{jk}$. Since $(rA)B$, $A(rB)$, and $r(AB)$ all have the same size, they are equal.

35. a) $n \times m$ b) $n \times n$ c) $m \times m$

36. Because $\mathbf{v}$ is $n \times 1$ and $\mathbf{w}^T$ is $1 \times n$, we see that $\mathbf{v}\mathbf{w}^T$ is $n \times n$. We have $(\mathbf{v}\mathbf{w}^T)^T = (\mathbf{w}^T)^T\mathbf{v}^T = \mathbf{w}\mathbf{v}^T$. Similarly, $(\mathbf{w}\mathbf{v}^T) = \mathbf{v}\mathbf{w}^T$, so each is the transpose of the other.

37. Since $h_{ji} = 1/(j + i - 1) = 1/(i + j - 1) = h_{ij}$, we see that H is a symmetric matrix.

38. Since $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$, we see that $A + A^T$ is symmetric.

39. Since $(AA^T)^T = (A^T)^T A^T = AA^T$, we see that AA^T is symmetric.

40. a) $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$ and $(A^3)^T = (AA^2)^T = (A^2)^T A^T$ $= (A^T)^2 A^T = (A^T)^3$.

b) *Statement:* $(A^n)^T = (A^T)^n$ for $n \geq 1$.

Proof: The statement becomes $A^T = A^T$ for $n = 1$. Let $k \geq 1$ and assume (inductive hypothesis) that the statement holds for $n = k$. Then $(A^{k+1})^T = (AA^k)^T = (A^k)^T A^T$. Using the inductive hypothesis, we have $(A^k)^T A^T = (A^T)^k A^T$ $= (A^T)^{k+1}$ which completes the inductive proof.

41. a) The jth entry in the column vector $A\mathbf{e}_j$ is $[a_{i1}, a_{i2}, \cdots, a_{in}]\cdot\mathbf{e}_j = a_{ij}$. Therefore $A\mathbf{e}_j$ is the jth column vector of A.

b) i) We have $A\mathbf{e}_j = \mathbf{0}$ for each $j = 1, 2, \cdots, n$ so by part (a), the jth column of A is the zero vector for each j. That is, $A = O$.

ii) $A\mathbf{x} = B\mathbf{x}$ for each $\mathbf{x}$
if and only if $(A - B)\mathbf{x} = \mathbf{0}$ for each $\mathbf{x}$
if and only if $A - B = O$ by part (i)
if and only if $A = B$.

42. $(A + B)^2 = A^2 + 2AB + B^2$ if and only if $AB = BA$. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $AB \neq BA$,

$$(A + B)^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

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43. $(A + B)(A - B) = A^2 - B^2$ if and only if $AB = BA$. Let

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $AB \neq BA$,

$(A + B)(A - B) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and

$A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

44. Consider the decomposition $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$. Now $B = \frac{1}{2}(A + A^T)$ is symmetric by Exercise 38. Letting $C = \frac{1}{2}(A - A^T)$, we find that $C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T)$ $= \frac{1}{2}(A^T - A) = -C$, so C is skew symmetric. This decomposition $A = B + C$ thus expresses A as the sum of a symmetric matrix and a skew symmetric matrix.

For uniqueness, suppose that $A = B_1 + C_1$ is another such decomposition. Then $A^T = (B_1 + C_1)^T = B_1{}^T + C_1{}^T = B_1 - C_1$. It follows that $A + A^T = 2B_1$ so $B_1 = \frac{1}{2}(A + A^T) = B$. Similarly, $A - A^T = 2C_1$ and $C_1 = \frac{1}{2}(A - A^T) = C$.

45. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ r & 0 & r \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} = \begin{bmatrix} 2 & 0 & r \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

No value of r causes these products to be equal.

46. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & r & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & r & 0 \\ 2 & 0 & 2 \end{bmatrix}$

These products are equal for all values of r.

47. $\begin{bmatrix} 98 & 736 & 1418 & 1619 & 2641 \\ 33 & 26346 & 2896 & 6634 & -5379 \\ -3471 & 18798 & 1132 & 2989 & -3579 \\ -1148 & 24240 & 4674 & 4288 & -2478 \\ -3046 & 15972 & 2980 & 3350 & -2356 \end{bmatrix}$

48. $\begin{bmatrix} 588 & -398 & -266 & 494 \\ 646 & 1038 & -55 & 290 \\ -581 & 2175 & 953 & -502 \\ -676 & 2762 & 2150 & -7 \\ 248 & 1424 & -602 & 105 \end{bmatrix}$

49. $\begin{bmatrix} 19616 & 32254 & 6142 & 39668 & 5324 \\ 118998 & 304974 & 237009 & 436421 & 251645 \\ 119592 & 253590 & 184626 & 348650 & 232514 \\ 189786 & 351337 & 166597 & 342753 & 299707 \\ 114414 & 240371 & 177647 & 353409 & 204283 \end{bmatrix}$

50. Undefined

51. $\begin{bmatrix} 68 & 42 & 86 & 28 & 60 \\ -18 & 534 & -246 & 112 & -176 \\ 32 & -132 & 204 & 32 & -54 \\ -206 & -28 & -232 & 56 & 380 \end{bmatrix}$

52. $\begin{bmatrix} 36499 & 30055 & -6890 & 10010 & 50800 \\ 30055 & 32653 & -11723 & 3622 & 44680 \\ -6890 & -11723 & 17179 & 14629 & -15569 \\ 10010 & 3622 & 14629 & 22237 & 10107 \\ 50800 & 44680 & -15569 & 10107 & 75753 \end{bmatrix}$

53. $\begin{bmatrix} -29030 & -39510 & 1738 & -21360 & 3766 \\ 4917 & -337893 & -31923 & -79832 & 61663 \\ 13044 & -197160 & -44544 & -41558 & 22684 \\ -19716 & -293988 & -54806 & -78182 & 50545 \\ 5876 & -181424 & -25374 & -51704 & 5279 \end{bmatrix}$

54. $\begin{bmatrix} 26918 & -3837 & -13284 & 14372 & -6772 \\ 41446 & 353741 & 209672 & 310468 & 194032 \\ 934 & 34811 & 44048 & 53254 & 27054 \\ 20416 & 82936 & 44566 & 82892 & 52440 \\ -2966 & -64951 & -27916 & -52777 & -9719 \end{bmatrix}$

M1. a) 32 b) 14629 c) -41558

M2. We obtained -1, which is the 8th entry in B, starting the numbering at the upper left-hand corner and proceeding down consecutive columns

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M3. $\begin{bmatrix} 141 & -30 & -107 \\ -30 & 50 & 18 \\ -107 & 18 & 189 \end{bmatrix}$ M4. -79 M5. -117 M6. 2.16

M7. 1/2 M8. G = 8*F - 4*ones(10,10)

SECTION 1.4 - Solving Systems of Linear Equations

1. a) $\begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 3 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ b) $\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. a) $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 7 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 7 \end{bmatrix}$ b) $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. a) $\begin{bmatrix} 0 & 2 & -1 & 3 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix} \sim$

$\sim \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b) $\sim \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. a) $\begin{bmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 1 & 3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix}$ b) $\sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. a) $\begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -2 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}$ b) $\sim \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

6. a) $\begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 2 & 4 & -1 & 3 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 3 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$

b) $\sim \begin{bmatrix} 2 & 4 & 0 & 5 & 1 & 3 \\ 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$

7. $\mathbf{x} = \begin{bmatrix} 5-6r \\ 2-4r \\ r \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 2 \end{bmatrix}$ 8. $\mathbf{x} = \begin{bmatrix} -7 \\ -5 \\ 2 \end{bmatrix}$ 9. $\mathbf{x} = \begin{bmatrix} 1-2r \\ -2-r-3s \\ r \\ s \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 3 \\ -2 \end{bmatrix}$

10. $\mathbf{x} = \begin{bmatrix} -4-r-3s \\ r \\ s \\ s \\ -2 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$ 11. $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -5 \\ 2 \end{bmatrix}$ 12. Inconsistent

13. $\left[\begin{array}{cc|c} 2 & -1 & 8 \\ 6 & -5 & 32 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 8 \\ 0 & -2 & 8 \end{array}\right]$ $\begin{aligned} 2x - y &= 8 \\ -2y &= 8 \end{aligned}$ $\begin{aligned} x &= 2 \\ y &= -4 \end{aligned}$

14. $\left[\begin{array}{cc|c} 4 & -3 & 10 \\ 8 & -1 & 10 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -3 & 10 \\ 0 & 5 & -10 \end{array}\right]$ $\begin{aligned} 4x_1 - 3x_2 &= 10 \\ 5x_2 &= -10 \end{aligned}$ $\begin{aligned} x_1 &= 1 \\ x_2 &= -2 \end{aligned}$

15. $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 3 & -1 & 1 & -7 \\ 1 & 1 & -3 & -13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -13 \\ 3 & -1 & 1 & -7 \\ 0 & 1 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -13 \\ 0 & -4 & 10 & 32 \\ 0 & 1 & 1 & 6 \end{array}\right] \sim$

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -13 \\ 0 & 1 & 1 & 6 \\ 0 & -4 & 10 & 32 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 14 & 56 \end{array}\right]$ $\begin{aligned} x + y - 3z &= -13 \\ y + z &= 6 \\ 14z &= 56 \end{aligned}$ $\begin{aligned} x &= -3 \\ y &= 2 \\ z &= 4 \end{aligned}$

16. $\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 6 & 3 & -8 & 0 \\ 2 & -1 & 5 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 8 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{aligned} 2x + y - 3z &= 0 \\ y - 4z &= 2 \\ z &= 0 \end{aligned}$ $\begin{aligned} x &= -1 \\ y &= 2 \\ z &= 0 \end{aligned}$

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17. $\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 14 \\ 1 & -7 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} x_1 - 2x_2 &= 3 \\ x_2 &= 1 \end{aligned}$ $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 3 & -8 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$ $\begin{aligned} x_1 - 3x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \end{aligned}$ $\mathbf{x} = \begin{bmatrix} -1+2s \\ -1+s \\ s \end{bmatrix}$

19. $\begin{bmatrix} 1 & 4 & -2 & 4 \\ 2 & 7 & -1 & -2 \\ 2 & 9 & -7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 & 4 \\ 0 & -1 & 3 & -10 \\ 0 & 1 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 & 4 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 0 & -17 \end{bmatrix}$ Inconsistent

20. $\begin{bmatrix} 1 & -3 & 2 & -1 & 8 \\ 3 & -7 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -1 & 8 \\ 0 & 2 & -6 & 4 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -1 & 8 \\ 0 & 1 & -3 & 2 & -12 \end{bmatrix}$

$x_3 = r,\ x_4 = s$

$x_2 - 3x_3 + 2x_4 = -12,\ x_2 = -12 + 3r - 2s$

$$\begin{aligned} x_1 = 3x_2 - 2x_3 + x_4 + 8 &= 3(-12 + 3r - 2s) - 2r + s + 8 \\ &= -28 + 7r - 5s \end{aligned}$$

21. $\begin{bmatrix} 3 & -2 & -8 \\ 4 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & -8 \\ 1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 5 \\ 0 & -23 & -23 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 8 & 16 \\ 5 & -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 8 \\ 0 & -24 & -48 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 2 & -1 & 1 & -3 & 0 \\ 9 & -3 & -1 & -7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & -1 & 5 & -5 & -12 \\ 0 & -3 & 17 & -16 & -50 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & 1 & -5 & 5 & 12 \\ 0 & 0 & 2 & -1 & -14 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & 0 & 0 & -8 \\ 0 & 1 & 0 & 5/2 & -23 \\ 0 & 0 & 1 & -1/2 & -7 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -8 \\ -23-5s \\ -7+s \\ 2s \end{bmatrix}$

24. $\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 3 & 6 & -8 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -14 & -13 \\ 0 & 0 & 1 & -5 & -5 \end{bmatrix}$

$x_1 = -13 - 2r + 14s,\ x_2 = r,\ x_3 = -5 + 5s,\ x_4 = s$

25. $\begin{bmatrix} 0 & 1 & -3 & 3 \\ 2 & 4 & -1 & 5 \\ 4 & -2 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 0 & -10 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & -23 & 23 \end{bmatrix}$ which shows

that $\begin{bmatrix}3\\5\\3\end{bmatrix}$ is in the span of $\begin{bmatrix}0\\2\\4\end{bmatrix}$, $\begin{bmatrix}1\\4\\-2\end{bmatrix}$, and $\begin{bmatrix}-3\\-1\\5\end{bmatrix}$.

26. $\left[\begin{array}{rrr|r}1 & -4 & 1 & 8\\3 & -12 & 5 & 26\\2 & -9 & -1 & 14\end{array}\right] \sim \left[\begin{array}{rrr|r}1 & -4 & 1 & 8\\0 & 0 & 2 & 2\\0 & -1 & -3 & -2\end{array}\right] \sim \left[\begin{array}{rrr|r}1 & -4 & 1 & 8\\0 & 1 & 3 & 2\\0 & 0 & 2 & 2\end{array}\right]$ which shows

that $\begin{bmatrix}8\\26\\14\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\3\\2\end{bmatrix}$, $\begin{bmatrix}-4\\-12\\-9\end{bmatrix}$, and $\begin{bmatrix}1\\5\\-1\end{bmatrix}$.

27. $\left[\begin{array}{rrrr|r}1 & 2 & -3 & 0 & 8\\2 & 5 & -6 & 0 & 17\\-1 & -2 & 1 & -1 & -8\\0 & 5 & -8 & -4 & 3\end{array}\right] \sim \left[\begin{array}{rrrr|r}1 & 2 & -3 & 0 & 8\\0 & 1 & 0 & 0 & 1\\0 & 0 & -2 & -1 & 0\\0 & 5 & -8 & -4 & 3\end{array}\right] \sim \left[\begin{array}{rrrr|r}1 & 2 & -3 & 0 & 8\\0 & 1 & 0 & 0 & 1\\0 & 0 & -2 & -1 & 0\\0 & 0 & -8 & -4 & -2\end{array}\right] \sim$

$\left[\begin{array}{rrrr|r}1 & 2 & -3 & 0 & 8\\0 & 1 & 0 & 0 & 1\\0 & 0 & -2 & -1 & 0\\0 & 0 & 0 & 0 & -2\end{array}\right]$ which shows that $\begin{bmatrix}8\\17\\-8\\3\end{bmatrix}$ is not in the span of

$\begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$, $\begin{bmatrix}2\\5\\-2\\5\end{bmatrix}$, $\begin{bmatrix}-3\\-6\\1\\-8\end{bmatrix}$, and $\begin{bmatrix}0\\0\\-1\\-4\end{bmatrix}$.

28. $\left[\begin{array}{rrrr|r}1 & -3 & 1 & 2 & 2\\1 & -2 & 2 & 4 & -1\\2 & -8 & -1 & 0 & 3\\3 & -9 & 4 & 0 & 7\end{array}\right] \sim \left[\begin{array}{rrrr|r}1 & -3 & 1 & 2 & 2\\0 & 1 & 1 & 2 & -3\\0 & -2 & -3 & -4 & -1\\0 & 0 & 1 & -6 & 1\end{array}\right] \sim \left[\begin{array}{rrrr|r}1 & -3 & 2 & 3 & 2\\0 & 1 & 1 & 2 & -3\\0 & 0 & -1 & 0 & -7\\0 & 0 & 1 & -6 & 1\end{array}\right] \sim$

$\left[\begin{array}{rrrr|r}1 & -3 & 2 & 3 & 2\\0 & 1 & 1 & 2 & -3\\0 & 0 & 1 & 0 & 7\\0 & 0 & 0 & -6 & -6\end{array}\right]$ which shows that $\begin{bmatrix}2\\-1\\3\\7\end{bmatrix}$ is in the span of

$\begin{bmatrix}1\\1\\2\\3\end{bmatrix}$, $\begin{bmatrix}-3\\-2\\-8\\-9\end{bmatrix}$, $\begin{bmatrix}1\\2\\-1\\4\end{bmatrix}$, and $\begin{bmatrix}2\\4\\0\\0\end{bmatrix}$.

29. F F T T F T T T F T

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30. $[2x_1,\ 2x_2] = [2,\ 18];\quad x_1 = 1,\ x_2 = 9$

31. $[4x_1 + 2x_1,\ 4x_2 + 6] = [-6,\ 18];\quad x_1 = -1,\ x_2 = 3$

32. $x_1 - 3x_2 = 2;\quad x_1 = 3r + 2,\ x_2 = r$

33. $\begin{array}{rcr} 3x_1 + 2x_2 &=& 0 \\ -x_1 + 4x_2 &=& -14 \end{array};\quad \left[\begin{array}{rr|r} -1 & 4 & -14 \\ 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} -1 & 4 & -14 \\ 0 & 14 & -42 \end{array}\right];\quad \begin{array}{rcr} x_1 &=& 2 \\ x_2 &=& -3 \end{array}$

34. $\left[\begin{array}{rr|r} 1 & -5 & 13 \\ 3 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -5 & 13 \\ 0 & 17 & -34 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right];\quad x_1 = 3,\ x_2 = -2$

35. $\begin{bmatrix} x_1 + x_2 & x_1 \\ x_3 + x_4 & x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix};\quad x_1 = 1,\ x_2 = -1,\ x_3 = 1,\ x_4 = 2$

36. $[3x_1 + 2x_2,\ x_2,\ 4x_1 - x_2] = [3,\ 3,\ -7];\quad x_1 = -1,\ x_2 = 3$

37. $\begin{bmatrix} 3x_1 + 2x_2 & 5x_1 + 3x_2 \\ 3x_3 + 2x_4 & 5x_3 + 3x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\left[\begin{array}{rr|r} 3 & 2 & 1 \\ 5 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2/3 & 1/3 \\ 0 & -1/3 & -5/3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array}\right];\quad x_1 = -3,\ x_2 = 5$

$\left[\begin{array}{rr|r} 3 & 2 & 0 \\ 5 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2/3 & 0 \\ 0 & -1/3 & 1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right];\quad x_3 = 2,\ x_4 = -3$

38. $\left[\begin{array}{rr|r} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right] \sim \left[\begin{array}{rr|c} 1 & 2 & b_1 \\ 0 & 0 & -3b_1 + b_2 \end{array}\right].$ The system is consistent if $-3b_1 + b_2 = 0$. Thus we must have $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s \\ 3s \end{bmatrix}$ where s is any real number.

39. $\left[\begin{array}{rr|r} 1 & 5 & b_1 \\ 3 & -1 & b_2 \end{array}\right] \sim \left[\begin{array}{rr|c} 1 & 5 & b_1 \\ 0 & -16 & -3b_1 + b_2 \end{array}\right].$ The system is consistent for all values of b_1 and b_2, so every vector $\mathbf{b}$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

40. $\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 1 & -1 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & -1 & b_3 \\ 0 & 0 & 3 & -2b_3 + b_2 \end{array}\right]$. The system is consistent for all values of b_1 and b_2.

41. $\left[\begin{array}{ccc|c} 1 & 3 & -1 & b_1 \\ 1 & -1 & 2 & b_2 \\ 0 & 4 & -3 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & b_1 \\ 0 & -4 & 3 & b_2 - b_1 \\ 0 & 4 & -3 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & b_1 \\ 0 & 4 & -3 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{array}\right]$.

The system is consistent if $b_3 + b_2 - b_1 = 0$, so only vectors of the form $\mathbf{b} = [r + s, r, s]$ for $r, s \in \mathbb{R}$ are in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

42. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ 43. $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 44. $C = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

45. $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

In Exercises 46 through 51, start with the 4 × 4 identity matrix and perform the given operations in sequence to obtain the matrix shown as answer.

46. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 47. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 48. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

49. $\begin{bmatrix} 1 & -60 & 0 & -15 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -12 & 0 & -3 \end{bmatrix}$ 50. $\begin{bmatrix} 0 & 6 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & -18 & 1 & -3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 51. $\begin{bmatrix} 2 & -30 & 5 & -10 \\ -4 & 121 & -20 & 40 \\ 0 & -6 & 1 & -2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

52. Suppose the ith and jth rows of an $m \times n$ matrix A are exchanged resulting in a matrix B. Let the ith and jth rows of the $m \times m$ identity matrix I be exchanged resulting in the elementary matrix E. Let $C = EA$. Then multiplication of A

on the left by the ith row vector of E is the same as multiplication of A on the left by the jth row vector of I, so the ith row vector of C is the jth row vector of A. By a similar argument, the jth row vector of C is the ith row vector of A. The other rows of C are of course the same as the corresponding rows of A. Thus $B = C$.

53. Let A be an $m \times n$ matrix, and let $\mathbf{e}_i$ be the ith row vector of the $m \times m$ identity matrix I. Note that $(r\mathbf{e}_i)A = r(\mathbf{e}_iA) = r(i\text{th row vector of } A)$. Thus if the ith row vector of I is multiplied by r to produce a matrix E, then multiplication of A on the left by E multiplies the ith row vector of A by r, while the other rows are left unchanged.

54. Let A be an $m \times n$ matrix and let $\mathbf{e}_i$ and $\mathbf{e}_j$ be the ith and jth row vectors, respectively, of the $m \times m$ identity matrix I. Let E be the elementary matrix obtained from I by adding r times the i th row vector to the j th row vector. Note that

$$\begin{aligned}(r\mathbf{e}_i + \mathbf{e}_j)A &= (r\mathbf{e}_i)A + \mathbf{e}_jA = r(\mathbf{e}_iA) + \mathbf{e}_j(A\ ) \\ &= r(i\text{th row of A}) + (j\text{th row of } A).\end{aligned}$$

It follows that multiplication of A on the left by E replaces the jth row vector of A by itself plus r times the ith row vector, and leaves the other rows of A alone.

55. a) The empty sequence of elementary row operations applied to A produces A, so $A \sim A$.
    b) Suppose $A \sim B$, so that A is row equivalent to B. Then there is a sequence of elementary row operations that can be performed successively on A to obtain B. Performing the inverse of each of these elementary row operations sequentially and in reverse order on B will produce A. Thus $B \sim A$.
    c) Suppose that $A \sim B$ and $B \sim C$. Then there are two sequences of elementary row operations, the first of which will reduce A to B and the second of which will reduce B to C. Thus the first sequence followed by the second sequence, when performed on A, will reduce A to C, so that $A \sim C$.

56. Substituting the points in the equation, we obtain $a + b + c = -4$, $a - b + c = 0$, and $4a + 2b + c = 3$.

$$\begin{bmatrix} 1 & 1 & 1 & | & -4 \\ 1 & -1 & 1 & | & 0 \\ 4 & 2 & 1 & | & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & | & -4 \\ 0 & -2 & 0 & | & 4 \\ 0 & -2 & -3 & | & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & | & -2 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & -3 & | & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -5 \end{bmatrix}$$

so $a = 3$, $b = -2$, and $c = -5$, giving $y = 3x^2 - 2x - 5$.

57. Substituting the points in the equation, we obtain $a + b + c + d = 2$, $a - b + c + d = 6$, $16a - 8b + 4c + d = 38$, $16a + 8b + 4c + d = 6$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 2 \\ 1 & -1 & 1 & 1 & | & 6 \\ 16 & -8 & 4 & 1 & | & 38 \\ 16 & 8 & 4 & 1 & | & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & | & 2 \\ 0 & -2 & 0 & 0 & | & 4 \\ 0 & -24 & -12 & -15 & | & 6 \\ 0 & -8 & -12 & -15 & | & -26 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & | & 4 \\ 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & -12 & -15 & | & -42 \\ 0 & 0 & -12 & -15 & | & -42 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/4 & | & 1/2 \\ 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 4 & 5 & | & 14 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/2 + s/4 \\ -2 \\ (14-5s)/4 \\ s \end{bmatrix}; \quad \text{let } s = 2, \quad \begin{matrix} a = 1 \\ b = -2 \\ c = 1 \\ d = 2 \end{matrix}, \quad y = x^4 - 2x^3 + x^2 + 2.$$

58. a) If $[A \mid \mathbf{c}] \sim [H \mid \mathbf{d}]$ where H has echelon form, then each column of H must have a pivot so there are at least as many rows as columns in H, that is, $m \geq n$.

b) If $m = n$, then $A \sim I$, the $n \times n$ identity matrix, since $A\mathbf{x} = \mathbf{c}$ has a *unique* solution. Thus $[A \mid \mathbf{b}] \sim [I \mid \mathbf{d}']$ and $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{d}'$.

c) If $m > n$, then the final row of H in part (a) has all zero entries. Let $\mathbf{e}_n$ be the final column of the $n \times n$ identity matrix. Reversing the elementary row operations that reduce A to H, we see that there exists a vector $\mathbf{b}$ such that $[H \mid \mathbf{e}_n] \sim [A \mid \mathbf{b}]$, and the system $A\mathbf{x} = \mathbf{b}$ is therefore inconsistent.

59. $\mathbf{x} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

60. Inconsistent

61. $\mathbf{x} \approx \begin{bmatrix} -1.2857 \\ 3.1429 \\ 1.2857 \end{bmatrix}$

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62. $\mathbf{x} = \begin{bmatrix} 1 - 11s \\ 3 - 7s \\ s \end{bmatrix}$ 63. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ 64. $\mathbf{x} \approx \begin{bmatrix} 0.0345 \\ -0.5370 \\ -1.6240 \\ 0.1222 \\ 0.8188 \end{bmatrix}$

65. $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$

66. $\mathbf{x} \approx \begin{bmatrix} -1.2857 \\ 3.1429 \\ 1.2857 \end{bmatrix}$ 67. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ 68. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$

M1-M9. See the solutions to Exercises 21, 23, 60, 61, 62, 6, 24, 63, and 64 respectively.

SECTION 1.5 - Inverses of Square Matrices

1. a) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right]$, $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

   b) The matrix A itself is an elementary matrix.

2. a) $\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 4/3 & -1 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right]$,

   $A^{-1} = \begin{bmatrix} 4/3 & -1 \\ -1/2 & 1/2 \end{bmatrix}$

   b) $E_1 = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

   $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

3. $\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & 0 & -4/3 & 1 \end{array}\right]$. Not invertible

4. a) $\left[\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 8 & 9 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -9/2 & 7/2 \\ 0 & 1 & 4 & -3 \end{array}\right]$,

$A^{-1} = \begin{bmatrix} -9/2 & 7/2 \\ 4 & -3 \end{bmatrix}$

b) $E_1 = \begin{bmatrix} 1/6 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -8 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & -7/6 \\ 0 & 1 \end{bmatrix}$,

$A = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 7/6 \\ 0 & 1 \end{bmatrix}$

5. a) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$, $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

b) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. a) $\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -3 & 1 & -7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 5 & -2 & 1 & 0 \\ 0 & 4 & -10 & 3 & 0 & 1 \end{array}\right] \sim$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & 1/2 & 0 \\ 0 & 1 & -5/1 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{array}\right]$. Not invertible

7. a) $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1 & -3/2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$, $A^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$

b) $E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

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$$E_4 = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

8. a) $$\left[\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 2 & -3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 12 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 12 & 0 & 0 & 1 \\ 2 & -3 & 5 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 12 & 0 & 0 & 1 \\ 0 & -3 & -19 & 0 & 1 & -2 \\ 0 & 2 & 13 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 12 & 0 & 0 & 1 \\ 0 & 1 & 19/3 & 0 & -1/3 & 2/3 \\ 0 & 0 & 1/3 & 1 & 2/3 & -1/3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -36 & -24 & 13 \\ 0 & 1 & 0 & -19 & -13 & 7 \\ 0 & 0 & 1 & 3 & 2 & -1 \end{array}\right],\quad A^{-1} = \begin{bmatrix} -36 & -24 & 13 \\ -19 & -13 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

b) $$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix},\ E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -19/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\cdot$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 19/3 \\ 0 & 0 & 1 \end{bmatrix}$$

9. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/5 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 3 & -4 \end{bmatrix} \sim$

(now add row 4 to row 3)

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus the span of the column vectors of A is all of $\mathbb{R}^4$.

12. $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 5 & -2 \end{bmatrix} \sim$

$$\begin{bmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that A is not invertible, so the span of its column vectors is not all of $\mathbb{R}^4$.

13. a) $\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 5 & -7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -3/2 & 1/2 & 0 \\ 0 & 1/2 & -5/2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -7 & 3 \\ 0 & 1 & -5 & 2 \end{array}\right]$,

$$A^{-1} = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}$$

b) $\begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -37 \\ -26 \end{bmatrix}$

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14. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$$

15. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -7r + 5s + 3t \\ 3r - 2s - 2t \\ 3r - 2s - t \end{bmatrix}$$

16. $$C = A^{-1}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 4 \\ 12 & 11 \end{bmatrix}$$

17. $$C = A^{-1}\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}A^{-1} =$$

$$\begin{bmatrix} 2 & 6 & 11 \\ -1 & 7 & 10 \\ 11 & 8 & 22 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 46 & 33 & 30 \\ 39 & 29 & 26 \\ 99 & 68 & 63 \end{bmatrix}$$

18. Since $2A = 2(IA) = 2(AI) = A(2I)$, we see that we can take $B = 2I$, where I is the 3×3 identity matrix.

19. Using the solution to the preceding exercise, we see that $A^2 + 2A = A^2 + A(2I) = A(A + 2I)$. Thus we can take

$$B = A + 2I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 3 & 4 \end{bmatrix}.$$

20. Since $\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & r & 3 \\ 0 & 0 & 0 \end{bmatrix}$, the given matrix is not invertible for any value r.

21. Since $\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & r-2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & r \end{bmatrix}$, the matrix is invertible for any value of r except $r = 0$.

22. If A and B are row equivalent, then there is a sequence $E_1, E_2, \cdots, E_k$ of elementary $m \times m$ matrices such that $B = E_kE_{k-1}\cdots E_2E_1A$. Thus $C = E_kE_{k-1}\cdots E_2E_1$ is the required invertible matrix.
Conversely, if $B = CA$ and C is invertible, then C may be expressed as a product of elementary matrices as shown in the text. Therefore, B and A are row equivalent.

23. T T T F T T T F F F [e & f: Note A^k invertible implies that for some B, $A^kB = I$ so $A(A^{k-1}B) = I$ and A is invertible.]

24. Assume that $AA^{-1} = A^{-1}A = I$. Then $(AA^{-1})^T = (A^{-1}A)^T = I$, that is, $(A^{-1})^TA^T = A^T(A^{-1})^T = I$. Therefore A^T is invertible with inverse $(A^{-1})^T$, that is, $(A^T)^{-1} = (A^{-1})^T$.

25. a) No. Consider $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
b) Yes. $A + A = 2A = (2I)A$ is a product of invertible matrices and so is itself invertible.

26. Let B be a matrix such that $A^2B = BA^2 = I$. Then $A(AB) = (BA)A = I$ so A is invertible and $A^{-1} = AB = BA$.

27. a) Note that $A(A^{-1}B) = (AA^{-1})B = IB = B$, so $X = A^{-1}B$ is a solution. To show uniqueness, suppose $AX = B$. Then $A^{-1}(AX) = A^{-1}B$, $(A^{-1}A)X = A^{-1}B$, $IX = A^{-1}B$, $X = A^{-1}B$ so this is the only solution.
b) Let $E_1, E_2, \cdots, E_k$ be elementary matrices that reduce $[A \mid B]$ to $[I \mid X]$, and let $C = E_kE_{k-1}\cdots E_2E_1$. Then $CA = I$ and $CB = X$. Thus $C = A^{-1}$ and so $X = A^{-1}B$.

28. In matrix notation, we never write $\frac{1}{A}$, but rather write A^{-1}. If we try to use quotient notation, we would not know whether $\frac{A}{B}$ means $B^{-1}A$ or AB^{-1}. Accordingly, we expect the desired analogous equality contains $(A + B)$ times factors A^{-1} and B^{-1} on the left or the right or one on each side. We find that $A^{-1}(A + B)B^{-1} = A^{-1}AB^{-1} + A^{-1}BB^{-1} = B^{-1} + A^{-1} = A^{-1} + B^{-1}$.

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29. a) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

b) If $A^r = O$ and A is invertible, then $A = A^r(A^{-1})^{r-1} = O(A^{-1})^{r-1} = O$, that is, $A = O$, which is a contradiction since A is invertible.

30. a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) If $AA = A$ and A is invertible, then $A^{-1}(AA) = A^{-1}A$, $(A^{-1}A)A = I$, $IA = I$, so $A = I$.

31. $\begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2$ has the form $\begin{bmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Squaring again we obtain the form $\begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore the 6th power of the original matrix is the zero matrix. (In fact, so is the 5th power.)

32. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

33. I and $-I$

34. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

35. a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d/h & -b/h \\ -c/h & a/h \end{bmatrix} = \begin{bmatrix} \dfrac{ad-bc}{h} & 0 \\ 0 & \dfrac{ad-bc}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) If $h \neq 0$, then part (a) shows that A is invertible. Assume A is invertible. Then we cannot have $a = c = 0$. Without loss in generality, we assume $a \neq 0$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} 1 & b/a \\ 0 & d-(bc/a) \end{bmatrix} = \begin{bmatrix} 1 & b/a \\ 0 & (ad-bc)/a \end{bmatrix}$ which must be reducible to the identity matrix. Thus we must have $ad - bc \neq 0$.

36. a) Let E be obtained from I by interchanging the ith and jth rows of I. Then E can also be obtained by interchanging the ith and jth columns of I.
b) Let E be obtained from I by multiplying the ith row of I by r. Then E can also be obtained by multiplying the ith column of I by r.
c) Let E be obtained from I by adding r times the jth row of I to the ith row of I. Then E can also be obtained by adding r times the ith column of I to the jth column of I.

37. The matrix AE can be obtained from A by applying to A the elementary column operation represented by E (see Exercise 36). Note $(AE)^T = E^TA^T$ affects the rows of A^T which are the columns of A.

38. We note that $(EA)^{-1} = A^{-1}E^{-1}$ and use Exercises 36 and 37.
a) If rows i and j of A are interchanged resulting in a matrix $B = EA$, then B^{-1} can be obtained by interchanging columns i and j of A^{-1}.
b) If the ith row of A is multiplied by r resulting in a matrix $B = EA$, then B^{-1} can be obtained by multiplying the the ith column of A^{-1} by $1/r$.
c) Let $B = EA$ be the matrix obtained from A by adding r times the ith row to the jth row of A. Then B^{-1} can be obtained from A^{-1} by adding $-r$ times the jth column to the ith column.

39. $\begin{bmatrix} 0.355 & -0.0645 & 0.161 \\ -0.129 & 0.387 & 0.0323 \\ -0.0968 & 0.290 & -0.226 \end{bmatrix}$ **40.** $\begin{bmatrix} -0.588 & 0.312 & -0.0854 \\ 0.487 & -0.241 & 0.131 \\ -0.166 & 0.216 & -0.0754 \end{bmatrix}$

41. $\begin{bmatrix} 0.0275 & -0.0296 & -0.0269 & 0.0263 \\ 0.168 & -0.0947 & 0.0462 & -0.0757 \\ 0.395 & -0.138 & -0.00769 & -0.0771 \\ -0.0180 & 0.0947 & 0.00385 & 0.0257 \end{bmatrix}$

42. The matrix is not invertible. **43.** (See Exercise 9.)

44. The matrix is not invertible. **45.** (See Exercise 41.)

46. $\begin{bmatrix} -0.588 & 0.312 & -0.0854 \\ 0.487 & -0.241 & 0.131 \\ -0.166 & 0.216 & -0.0754 \end{bmatrix}$

47. $\begin{bmatrix} 0.291 & 0.199 & 0.0419 & -0.00828 & -0.272 \\ -0.0564 & 0.159 & 0.148 & -0.0737 & -0.0699 \\ 0.0276 & 0.145 & -0.00841 & -0.0302 & -0.0250 \\ -0.0467 & 0.122 & -0.029 & 0.133 & -0.00841 \\ 0.0116 & -0.128 & -0.0470 & -0.0417 & 0.178 \end{bmatrix}$

48. $\begin{bmatrix} 2 & -1 & 0 & 1 & 6 \\ 3 & -1 & 2 & 4 & 6 \\ 0 & 1 & 3 & 4 & 8 \\ -1 & 1 & 1 & 1 & 8 \\ 3 & 1 & 4 & -11 & 10 \end{bmatrix}$

M1. 0.001783 **M2.** -16.6166

M3. 0.4397 **M4.** 0.09985 **M5.** -418.07 **M6.** 23020

M7. -0.001071 **M8.** -171.01

SECTION 1.6 - Homogeneous Systems, Subspaces, and Bases

1. Let $L = \{[r, -r] \mid r \in \mathbb{R}\}$ which is a nonempty subset of $\mathbb{R}^2$. Let $[a, -a]$, $[b, -b] \in L$. Then

$$[a, -a] + [b, -b] = [a + b, -a - b] = [a + b, -(a + b)]$$

which has the form $[r, -r]$ and thus is in L . Also

$$r[a, -a] = [ra, -(ra)]$$

which is in L. Thus L is a subspace of $\mathbb{R}^2$.

2. Let $L = \{[x, x + 1] \mid x \in L\}$ which is a nonempty subset of $\mathbb{R}^2$. Clearly $[0, 0]$ is not in L. Thus L is not a subspace. The only lines in $\mathbb{R}^2$ that are subspaces are those that contain the origin.

3. Let $S = \{[n, m] \mid n$ and m are integers$\} \subseteq \mathbb{R}^2$. Since $[1, 1] \in S$ but $\frac{1}{2}[1, 1] = [\frac{1}{2}, \frac{1}{2}]$ is not in S, we see that S is not closed under scalar multiplication, so it cannot be a subspace of $\mathbb{R}^2$.

4. Let $S = \{[x, y] \mid x, y \in \mathbb{R}$ and $x, y \geq 0\}$. Since $[1, 1] \in S$ but $(-1)[1, 1] = [-1, -1]$ is not in S, we see that S is not closed under scalar multiplication, so it cannot be a subspace of $\mathbb{R}^2$.

5. Let $P = \{[x, y, z] \mid x, y \in \mathbb{R} \text{ and } z = 3x + 2\}$. Clearly $[0, 0, 0]$ is not in P, so P is not a subspace of $\mathbb{R}^3$. The only planes in $\mathbb{R}^3$ that are subspaces are those that contain the origin.

6. Let $P = \{[x, y, z] \mid x, y, z \in \mathbb{R}, x = 2y + z\}$ which is a nonempty subset of $\mathbb{R}^3$. Let $\mathbf{v} = [2a + b, a, b]$ and $\mathbf{w} = [2c + d, c, d]$ be in P. Then

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= [2a + b + 2c + d, a + c, b + d] \\ &= [2(a + c) + (b + d), a + c, b + d]\end{aligned}$$

which has the form $[2y + z, y, z]$ and is in P. Also

$$r[2a + b, a, b] = [2ra + rb, ra, rb]$$

which is in P. Thus P is a subspace of $\mathbb{R}^3$.

7. Let $P = \{[x, y, z] \mid x, y, z \in \mathbb{R}, z = 1, y = 2x\}$. Clearly $[0, 0, 0]$ is not in P. Thus P is not a subspace of $\mathbb{R}^3$.

8. Let $P = \{[2x, x + y, y] \mid x, y \in \mathbb{R}\}$ which is a nonempty subset of $\mathbb{R}^3$. Let $\mathbf{v} = [2a, a + b, a]$ and $\mathbf{w} = [2c, c + d, d]$ be in P. Then

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= [2a + 2c, a + b + c + d, b + d] \\ &= [2(a + c), (a + c) + (b + d), b + d]\end{aligned}$$

which has the form $[2x, x + y, y]$ and is in P. Also

$$r[2a, a + b, b] = [2ra, ra + rb, rb]$$

which is in P. Thus P is a subspace of $\mathbb{R}^3$.

9. Since $[2x_1, 3x_2, 4x_3, 5x_4]$ can be any element of $\mathbb{R}^4$, we see that the given set is $\mathbb{R}^4$, which is of course a subspace of $\mathbb{R}^4$.

10. Let $H = \{[x_1, 0, x_3, \cdots, x_n] \mid x_i \in \mathbb{R}\}$ which is a nonempty subset of $\mathbb{R}^n$. Now

$$\begin{aligned}&[a_1, 0, a_3, \cdots, a_n] + [b_1, 0, b_3, \cdots, b_n] \\ &\qquad = [a_1 + b_1, 0, a_3 + b_3, \cdots, a_n + b_n]\end{aligned}$$

which is in H, and

$$r[a_1, 0, a_3, \cdots, a_n] = [ra_1, 0, ra_3, \cdots, ra_n]$$

which is in H, so H is a subspace of $\mathbb{R}^n$.

11. $W = \{[x, mx] \mid x \in \mathbb{R}\}$ is nonempty since $[0, 0] \in W$. Let $[x_1, mx_1]$ and $[x_2, mx_2]$ be in W. Then $[x_1, mx_1] + [x_2, mx_2] = [x_1 + x_2, mx_1 + mx_2] = [(x_1 + x_2), m(x_1 + x_2)] \in W$. Also $r[x_1, mx_1] = [rx_1, rmx_1] = [rx_1, m(rx_1)] \in W$ for any scalar r. Thus W is nonempty and closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^2$.

12. The plane $ax + by + cz = 0$ may be viewed as the solution set of a homogeneous linear system consisting of a single equation, and Theorem 1.13 shows this solution set is a subspace of $\mathbb{R}^3$.

13. a) Every subspace of $\mathbb{R}^2$ is either $\{\mathbf{0}\}$, all vectors along a line through the origin, or all of $\mathbb{R}^2$.

    b) Every subspace of $\mathbb{R}^3$ is either $\{\mathbf{0}\}$, all vectors along a line through the origin, all vectors in a plane through the origin, or all of $\mathbb{R}^3$.

14. Let W be a subspace of $\mathbb{R}^n$. Then W is nonempty, so we can choose $\mathbf{w} \in W$. Then $0\mathbf{w} \in W$ by the closure property for scalar multiplication. Since $0\mathbf{w} = \mathbf{0}$, this shows that $\mathbf{0} \in W$.

15. No, $\{\mathbf{0}\}$ is not a basis for the subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$, because $2\cdot\mathbf{0} = 3\cdot\mathbf{0} = \mathbf{0}$, so $\mathbf{0}$ is not a *unique* linear combination of $\mathbf{0}$.

16. $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\{[1, 1]\}$ is a basis for the solution space.

17. $\begin{bmatrix} 3 & 1 & 1 \\ 6 & 2 & 2 \\ -9 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -r/3 - s/3 \\ r \\ s \end{bmatrix} = \frac{r}{3}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{s}{3}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$,
    so $\{[-1, 3, 0], [-1, 0, 3]\}$ is a basis for the solution space.

18. $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & 4/5 \\ 0 & 0 & 1 & -4/5 \end{bmatrix}$,

$$\mathbf{x} = \begin{bmatrix} -3r/5 \\ -4r/5 \\ 4r/5 \\ r \end{bmatrix} = \frac{r}{5}\begin{bmatrix} -3 \\ -4 \\ 4 \\ 5 \end{bmatrix},$$ so {[-3, -4, 4, 5]} is a basis for the solution space.

19. $$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -6 & 1 & 0 \\ 3 & -5 & 2 & 1 \\ 5 & -4 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 1 & 0 \\ 0 & 13 & -1 & 1 \\ 0 & 13 & -1 & 1 \\ 0 & 26 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7/13 & 6/13 \\ 0 & 1 & -1/13 & 1/13 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} -7r/13 - 6s/13 \\ r/13 - s/13 \\ r \\ s \end{bmatrix} = \frac{r}{13}\begin{bmatrix} -7 \\ 1 \\ 13 \\ 0 \end{bmatrix} + \frac{s}{13}\begin{bmatrix} -6 \\ -1 \\ 0 \\ 13 \end{bmatrix}$$ so

{[-7, 1, 13, 0], [-6, -1, 0, 13]} is a basis for the solution space.

20. $$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & -7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -7 & 2 \\ 0 & 4 & 20 & -4 \\ 0 & 2 & 10 & -2 \\ 0 & 3 & 15 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} 2r - s \\ -5r + s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$ so

{[2, -5, 1, 0], [-1, 1, 0, 1]} is a basis for the solution space.

21. $$\begin{bmatrix} 1 & -1 & 6 & 1 & -1 \\ 3 & 2 & -3 & 2 & 5 \\ 4 & 2 & -1 & 3 & -1 \\ 3 & -2 & 14 & 1 & -8 \\ 2 & -1 & 8 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 6 & 1 & -1 \\ 0 & 5 & -21 & -1 & 8 \\ 0 & 6 & -25 & -1 & 3 \\ 0 & 1 & -4 & -2 & -5 \\ 0 & 1 & -4 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & -6 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & -1 & -1 & 33 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & 33 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 60 \\ 0 & 1 & 0 & 4 & -137 \\ 0 & 0 & 1 & 1 & -33 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 60 \\ 0 & 1 & 0 & 0 & -137 \\ 0 & 0 & 1 & 0 & -33 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} -60r \\ 137r \\ 33r \\ 0 \\ r \end{bmatrix} = r\begin{bmatrix} -60 \\ 137 \\ 33 \\ 0 \\ 1 \end{bmatrix},$$

so {[-60, 137, 33, 0, 1]} is a basis for the solution space.

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22. $\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$. The matrix is invertible so the set is a basis for $\mathbb{R}^2$ by Theorem 1.16.

23. Let $\mathbf{v} = [-1, 3, 1]$ and $\mathbf{w} = [2, 1, 4]$. Using Theorem 1.15, we need only test whether $r\mathbf{v} + s\mathbf{w} = \mathbf{0}$ implies $r = s = 0$. We solve this homogeneous vector equation by the reduction

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 6 & 0 \end{array}\right].$$ We see immediately that

$r = s = 0$ is the only solution, so by Theorem 1.15, $\{\mathbf{v}, \mathbf{w}\}$ is a basis for the space $\mathrm{sp}(\mathbf{v}, \mathbf{w})$.

24. $\begin{bmatrix} -1 & 1 & 1 \\ 3 & 5 & 13 \\ 4 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 8 & 16 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The matrix is not invertible so by Theorem 1.16, the given set is not a basis for $\mathbb{R}^3$.

25. $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 0 & -1 \\ -3 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -0 & -1 \\ 0 & 4 & 4 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$. The matrix is invertible so the set is a basis for $\mathbb{R}^3$ by Theorem 1.16.

26. $\begin{bmatrix} 2 & 2 & 3 & 5 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 2 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The matrix is invertible so the set is a basis for $\mathbb{R}^4$ by Theorem 1.16.

27. Let $\mathbf{v} = [2, -6, 1]$ and $\mathbf{w} = [1, -3, 4]$. Using Theorem 1.15, we need only test whether $r\mathbf{v} + s\mathbf{w} = 0$ implies $r = s = 0$. We solve this homogeneous vector equation by the reduction

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \\ 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 21 & 0 \\ 0 & -7 & 0 \end{array}\right].$$ We see immediately that

$r = s = 0$ is the only solution, so by Theorem 1.15, $\{\mathbf{v}, \mathbf{w}\}$ is a basis for the space $\mathrm{sp}(\mathbf{v}, \mathbf{w})$.

28. Let $\mathbf{v}$ be the first column vector and $\mathbf{w}$ the second column vector of the matrix. We observe that $3\mathbf{v} + \mathbf{w} = \mathbf{0}$, so by

Theorem 1.15, the column vectors are not a basis for the column space of the matrix.

29. Testing whether these four vectors in $\mathbb{R}^3$ form a basis for the space they span using Theorem 1.15 amounts to solving the homogeneous linear system having the transpose of the given matrix as coefficient matrix. This linear system has three equations in four unknowns, and has a nontrivial solution by Corollary 2 of Theorem 1.17. Theorem 1.15 then shows that the four vectors are not a basis for the row space.

30. Testing as indicated in Theorem 1.15, we perform the matrix reduction

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -3 \\ 1 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & -3 \\ 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -9 \\ 0 & 0 & 8 \end{bmatrix}.$$ We see at once that a homogeneous linear system having the first matrix as coefficient matrix has only the trivial solution. Theorem 1.15 shows that the column vectors are a basis for the column space of this matrix.

31. $$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 2 \\ 0 & -33 & -9 \\ 0 & -11 & -3 \\ 0 & -44 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/11 \\ 0 & 1 & 3/11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ The solution space is given by $\begin{bmatrix} (-1/11)t \\ (-3/11)t \\ t \end{bmatrix}$ for all scalars t. A basis is $\{[-1, -3, 11]\}$.

32. $$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -6 & -6 & -12 \\ 0 & -7 & -7 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ The solution space is given by $\begin{bmatrix} -2s - r \\ -s - 2r \\ s \\ r \end{bmatrix}$ for all scalars r and s. A basis is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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33. $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k) = \mathrm{sp}(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m)$ if and only if each $\mathbf{v}_i$ is a linear combination of the $\mathbf{w}_j$ and each $\mathbf{w}_j$ is a linear combination of the $\mathbf{v}_i$.

34. $$\mathbf{x} = \begin{bmatrix} 7 + 2t - r - 5s \\ t \\ r \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2t - r - 5s \\ t \\ r \\ s \end{bmatrix}$$

35. $$\left[\begin{array}{cccc|c} 2 & -1 & 3 & 0 & -3 \\ 4 & 2 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & -1 & 3 & 0 & -3 \\ 0 & 4 & -6 & -1 & 7 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 3/4 & -1/8 & -5/8 \\ 0 & 1 & -3/2 & -1/4 & 7/4 \end{array}\right]; \qquad \mathbf{x} = \begin{bmatrix} -5/8 \\ 7/4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3r/4 + s/8 \\ 3r/2 + s/4 \\ r \\ s \end{bmatrix}$$

36. $$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 4 \\ 2 & 1 & -3 & -1 & 6 \\ 1 & -7 & -6 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 4 \\ 0 & 5 & -5 & -3 & -2 \\ 0 & -5 & -7 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/5 & 16/5 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & -12 & -2 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/30 & 16/5 \\ 0 & 1 & 0 & -13/30 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]; \qquad \mathbf{x} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s/30 \\ 13s/30 \\ -s/6 \\ s \end{bmatrix}$$

37. $$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 5 \\ 1 & -1 & 2 & 1 & 0 \\ 4 & -1 & 7 & 2 & 5 \\ -1 & -2 & -1 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & -2 & 5 \\ 0 & 3 & -1 & -2 & 5 \\ 0 & -3 & 1 & 2 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 5/3 & 1/3 & 5/3 \\ 0 & 1 & -1/3 & -2/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]; \qquad \mathbf{x} = \begin{bmatrix} 5/3 \\ 5/3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -(5r + s)/3 \\ (r + 2s)/3 \\ r \\ s \end{bmatrix}$$

38. F T T T F F T T F T

39. In this case a solution is not uniquely determined. The system is viewed as underdetermined since it is insufficient to determine a unique solution.

40. With more equations than unknowns, we naively expect that too many conditions are demanded of the unknowns to permit any solution. In practice, this is usually the case (see Section 6.5). If the system does have a solution, some of the equations could be deleted to produce a square system with the same solution.

41. $x - y = 1$
$2x - 2y = 2$
$3x - 3y = 3$

42. By Theorem 1.18 (part 1), if a homogeneous system has a nonzero solution $\mathbf{h}$, then $r\mathbf{h}$ is also a solution for each $r \in \mathbb{R}$. Thus the solution set is infinite in this case.

43. Assume $A\mathbf{x} = \mathbf{b}$ is consistent and has more than one solution. Theorem 1.18 (part 2) shows that $A\mathbf{x} = \mathbf{0}$ has more than one solution also. Exercise 42 then shows that $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. Thus $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions by Theorem 1.18 (part 2).

44. a) No. For example,

$$\begin{aligned} x_1 \phantom{{}-x_2} &= 0 \\ x_1 - x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

is a system $A\mathbf{x} = \mathbf{0}$ with only the trivial solution,

but $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent.

b) Yes. By Theorem 1.18, all solutions of $A\mathbf{x} = \mathbf{b}$ are of the form $\mathbf{x} = \mathbf{p} + \mathbf{h}$ for any fixed solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ and any solution $\mathbf{h}$ of $A\mathbf{x} = \mathbf{0}$. Thus if $A\mathbf{x} = \mathbf{b}$ is consistent, then $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have the same number of solutions.

45. a) Clearly $\mathbf{v}_1,\ 2\mathbf{v}_1 + \mathbf{v}_2 \in \mathrm{sp}(\mathbf{v}_1,\ \mathbf{v}_2)$ and therefore

$$\mathrm{sp}(\mathbf{v}_1,\ 2\mathbf{v}_1 + \mathbf{v}_2) \subseteq \mathrm{sp}(\mathbf{v}_1,\ \mathbf{v}_2)\ .$$

Also, $\mathbf{v}_1 = 1\mathbf{v}_1 + 0(2\mathbf{v}_1 + \mathbf{v}_2)$ and $\mathbf{v}_2 = (-2)\mathbf{v}_1 + 1(2\mathbf{v}_1 + \mathbf{v}_2)$ showing that $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{sp}(\mathbf{v}_1,\ 2\mathbf{v}_1 + \mathbf{v}_2)$ and therefore

$$\mathrm{sp}(\mathbf{v}_1,\ \mathbf{v}_2) \subseteq \mathrm{sp}(\mathbf{v}_1,\ 2\mathbf{v}_1 + \mathbf{v}_2).$$

Thus $\mathrm{sp}(\mathbf{v}_1,\ \mathbf{v}_2) = \mathrm{sp}(\mathbf{v}_1,\ 2\mathbf{v}_1 + \mathbf{v}_2)$.

b) Clearly $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \in \mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2)$ and therefore

$$\mathrm{sp}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \subseteq \mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2).$$

Also, $\mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) + \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2)$ and $\mathbf{v}_2 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) - \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2)$ so $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{sp}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$ and therefore

$$\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2) \subseteq \mathrm{sp}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2).$$

Thus $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{sp}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$.

46. a) In view of Theorem 1.15 and Exercise 45a, it remains to show that the zero vector is a *unique* linear combination of $\mathbf{v}_1$ and $2\mathbf{v}_1 + \mathbf{v}_2$ Suppose $c_1\mathbf{v}_1 + c_2(2\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0}$. Then $(c_1 + 2c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ so that $c_1 + 2c_2 = 0$ and $c_2 = 0$ because $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2)$. Thus $c_1 = c_2 = 0$, which is what we need to show.

b) In view of Theorem 1.15 and Exercise 45b, it remains to show that the zero vector is a *unique* linear combination of $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$. Suppose $c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$. Then $(c_1 + c_2)\mathbf{v} + (c_1 - c_2)\mathbf{v} = \mathbf{0}$ so that $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ because $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2)$. Adding the preceding equations we see that $2c_1 = 0$ so $c_1 = 0$. Thus $c_2 = 0$ also.

c) Following the criterion of Theorem 1.15, suppose that $c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 - \mathbf{v}_2) + c_3(2\mathbf{v}_1 - 3\mathbf{v}_2) = \mathbf{0}$. Then $(c_1 + c_2 + 2c_3)\mathbf{v}_1 + (c_1 - c_2 - 3c_3)\mathbf{v}_2 = \mathbf{0}$ so that

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 3, \\ c_1 - c_2 - 3c_3 &= 0 \end{aligned}$$

because $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2)$. This is a homogeneous system with fewer equations than unknowns and thus has nontrivial solutions. In view of Theorem 1.15, the given vectors cannot form a basis for any subspace of $\mathbb{R}^n$.

47. Clearly $W_1 \cap W_2$ is nonempty; it contains $\mathbf{0}$. Let $\mathbf{v},\mathbf{w} \in (W_1 \cap W_2)$. Then $\mathbf{v},\mathbf{w} \in W_1$ and $\mathbf{v},\mathbf{w} \in W_2$ so $\mathbf{v} + \mathbf{w} \in W_1$ and $\mathbf{v} + \mathbf{w} \in W_2$ since W_1 and W_2 are subspaces. Thus $\mathbf{v} + \mathbf{w} \in (W_1 \cap W_2)$. Similarly, $r\mathbf{v} \in W_1$ and $r\mathbf{v} \in W_2$ since W_1 and W_2 are subspaces, so $r\mathbf{v} \in (W_1 \cap W_2)$. Thus $W_1 \cap W_2$ is a subspace of R^n.

48. The matrix having $\mathbf{a}_i^T$ as ith column vector does not have as its reduced echelon form the 4×4 identity matrix, so by Theorem 1.17, the vectors do not form a basis for the subspace that they span.

49. The matrix having $\mathbf{b}_i^T$ as ith column vector has as its reduced echelon form the 4×4 identity matrix followed by a row of zeros, so by Theorem 1.17, the vectors form a basis for the subspace that they span.

50. The matrix having $\mathbf{v}_i^T$ as ith column vector has as its reduced echelon form the 4×4 identity matrix, so by Theorem 1.17, the vectors form a basis for the subspace that they span.

51. The matrix having $\mathbf{w}_i^T$ as ith column vector does not have as its reduced echelon form the 4×4 identity matrix, so by Theorem 1.17, the vectors do not form a basis for the subspace that they span.

M1-M4. See Exercises 48, 49, 50, and 51 respectively.

M5. 1

M6-M7. The probability is still 1. Our row reduction always yielded the 8×8 identity matrix, showing that the column vectors in each 8×8 "random" matrix spanned $\mathbb{R}^8$.

Section 1.7

SECTION 1.7 - Application to Population Distribution (Optional)

1. No, since the sum of the entries in the first column is not equal to 1.
2. Yes. The matrix is regular since its square has no zero entries.
3. No, since one of the entries is negative.
4. No, since the matrix is not square.
5. Yes. The matrix is regular since its third power has no zero entries.
6. Yes. The matrix is not regular since every power has the same third column containing some zero entries.
7. Yes. The matrix is not regular since every power has zeros in the first, second, and fourth entries of rows 3 and 5.
8. Yes. The matrix is regular since its fifth power has no zero entries.
9. We compute the entry in the 3rd row and 2nd column of T^2,

   obtaining $[.3 \quad .1 \quad .5]\begin{bmatrix} .7 \\ .2 \\ .1 \end{bmatrix} = .21 + .02 + .05 = 0.28.$

10. We compute the entry in the 2nd row and 3rd column of T^2,

    obtaining $[.4 \quad .2 \quad .1]\begin{bmatrix} .4 \\ .1 \\ .5 \end{bmatrix} = .16 + .02 + .05 = 0.23.$

11. We compute the entry in the third row of $T^2\mathbf{p}$.

    Computation shows that the third row vector of T^2 is

    $[.28 \quad .28 \quad .38]$, so the desired entry of $T^2\mathbf{p}$ is

    $$[.28 \quad .28 \quad .38]\begin{bmatrix} .3 \\ .2 \\ .5 \end{bmatrix} = .084 + .056 + .190 = .330.$$

12. We compute $T^2\mathbf{p} = \begin{bmatrix} .49 & .39 & .39 \\ .23 & .33 & .23 \\ .28 & .28 & .38 \end{bmatrix}\begin{bmatrix} .3 \\ .2 \\ .5 \end{bmatrix} = \begin{bmatrix} .42 \\ .25 \\ .33 \end{bmatrix}.$

13. $\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}^2$ again has the same form $\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$, so a matrix of this form is not regular.

14. Squaring repeatedly leads to the symbolic forms

$$\begin{bmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \times & \times & \times \\ \times & \times & 0 \\ \times & 0 & \times \end{bmatrix} \longrightarrow \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix};$$ the matrix is regular.

15. Squaring leads to the symbolic form

$$\begin{bmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$ so the matrix is regular.

16. A matrix of this form is not regular; its square has the same form. Alternatively, we can simply note that it is impossible to go from the second state to any other state.

17. A matrix of this form is not regular. We need only note that from the first or second state, you are back in the same state after any even number of time periods, and in the other of these two states after any odd number of time periods. Thus it impossible to achieve the third or fourth state from the first or second state. Alternatively, raising the matric symbolically to powers shows we never get rid of the 2×2 block of zeros in the lower left corner.

18. A matrix of this form is not regular. Just note that it is impossible to get from the first, second, or third state to the fourth state in any number of time periods. Alternatively, squaring repeatedly will never get rid of the first three zeros in row 4.

19. $T - I = \begin{bmatrix} -1/3 & 1 \\ 1/3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3r \\ r \end{bmatrix}$; $3r + r = 1$, $r = \frac{1}{4}$;

$\mathbf{s} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$.

20. $T - I = \begin{bmatrix} -1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 2r \\ r \end{bmatrix}$; $2r + r = 1$, $r = \frac{1}{3}$;

$\mathbf{s} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

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21. $T - I = \begin{bmatrix} -1 & 1/3 & 1/3 \\ 0 & -1/3 & 1/3 \\ 1 & 0 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & -1/3 & 1/3 \\ 0 & 1/3 & -1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{x} = \begin{bmatrix} 2r/3 \\ r \\ r \end{bmatrix}$; $r + r + \frac{2}{3}r = 1$, $\frac{8}{3}r = 1$, $r = \frac{3}{8}$; $\mathbf{s} = \begin{bmatrix} 1/4 \\ 3/8 \\ 3/8 \end{bmatrix}$.

22. $T - I = \begin{bmatrix} 0 & 1/4 & 1/2 \\ 0 & -3/4 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3/2 \\ 0 & 0 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$;

$r = 1$; $\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

23. $T - I = \begin{bmatrix} -3/4 & 1/2 & 1/3 \\ 1/4 & -1 & 1/3 \\ 1/2 & 1/2 & -2/3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3/2 & 0 \\ 1/4 & -1 & 1/3 \\ 1/2 & 1/2 & -2/3 \end{bmatrix}$ added (-row 2) to row 1

$\sim \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & -5/8 & 1/3 \\ 0 & 5/4 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4/5 \\ 0 & 1 & -8/15 \\ 0 & 0 & 0 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 4r/5 \\ 8r/15 \\ r \end{bmatrix}$;

$\frac{4}{5}r + \frac{8}{15}r + r = 1$, $\frac{35}{15}r = 1$, $r = \frac{15}{35}$; $\mathbf{s} = \begin{bmatrix} 12/35 \\ 8/35 \\ 15/35 \end{bmatrix}$

24. $T - I = \begin{bmatrix} -4/5 & 1/5 & 1/3 \\ 2/5 & -4/5 & 1/3 \\ 2/5 & 3/5 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 2/5 & 3/5 & -2/3 \\ 0 & -7/5 & 1 \\ 0 & 7/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 & -5/3 \\ 0 & 1 & -5/7 \\ 0 & 0 & 0 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & -25/42 \\ 0 & 1 & -5/7 \\ 0 & 0 & 0 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 25r/42 \\ 5r/7 \\ r \end{bmatrix}$; $\frac{25}{42}r + \frac{5}{7}r + r = 1$, $\frac{97}{42}r = 1$,

$r = \frac{42}{97}$; $\mathbf{s} = \begin{bmatrix} 25/97 \\ 30/97 \\ 42/97 \end{bmatrix}$

25. T F F T T F T F F T

26. By Theorem 1.19 and Exercise 20, $A^{100} \approx \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$.

27. By Theorem 1.19 and Exercise 23, $A^{100} \approx \begin{bmatrix} 12/35 & 12/35 & 12/35 \\ 8/35 & 8/35 & 8/35 \\ 15/35 & 15/35 & 15/35 \end{bmatrix}$.

28. $T = \begin{array}{cc} \begin{array}{ccc} U & S & R \end{array} & \\ \begin{bmatrix} .4 & .3 & .1 \\ .5 & .5 & .2 \\ .1 & .2 & .7 \end{bmatrix} & \begin{array}{c} U \\ S \\ R \end{array} \end{array}$

29. The answer is the entry in the second row first column of T^2 which is $[.5 \quad .5 \quad .2]\begin{bmatrix} .4 \\ .5 \\ .1 \end{bmatrix} = .20 + .25 + .02 = .47$.

30. The answer is the entry in the third row third column of T^2 which is $[.1 \quad .2 \quad .7]\begin{bmatrix} .1 \\ .2 \\ .7 \end{bmatrix} = .01 + .04 + .49 = .54$.

31. $\begin{bmatrix} .4 & .3 & .1 \\ .5 & .5 & .2 \\ .1 & .2 & .7 \end{bmatrix}\begin{bmatrix} .4 \\ .5 \\ .1 \end{bmatrix} = \begin{bmatrix} .32 \\ .47 \\ .21 \end{bmatrix}$

32. We can compute T^2 and then $T^2\begin{bmatrix} .4 \\ .5 \\ .1 \end{bmatrix}$, or we can use the answer to Exercise 31 and compute $\begin{bmatrix} .4 & .3 & .1 \\ .5 & .5 & .2 \\ .1 & .2 & .7 \end{bmatrix}\begin{bmatrix} .32 \\ .47 \\ .21 \end{bmatrix} = \begin{bmatrix} .290 \\ .437 \\ .273 \end{bmatrix}$.

33. The Markov chain is regular because T has no zero entry.

$$T - I = \begin{bmatrix} -.6 & .3 & .1 \\ .5 & -.5 & .2 \\ .1 & .2 & -.3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 15 & -17 \\ 0 & -15 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -11/15 \\ 0 & 1 & -17/15 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{x} = \begin{bmatrix} 11r/15 \\ 17r/15 \\ r \end{bmatrix}; \quad \frac{11}{15}r + \frac{17}{15}r + r = 1, \ \frac{43}{15}r = 1, \ r = \frac{15}{43};$$

$$\mathbf{s} = \begin{bmatrix} 11/43 \\ 17/43 \\ 15/43 \end{bmatrix}$$

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34. Offspring of one dominant and one hybrid parent inherit G from the dominant parent. Half of them inherit g and half inherit G from the hybrid parent. This gives column 1 of the transition matrix. Column 3 is obtained similarly. For column 2, half the offspring of two hybrid parents inherit G from the mother, and half of those also inherit G from the father. Thus the proportion of dominant offspring is $(1/2)(1/2) = 1/4$. The other two numbers in column 2 are obtained similarly.

In Exercises 35 through 39, we use the fact that

$$T^2 = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}^2 = \begin{bmatrix} 3/8 & 1/4 & 1/8 \\ 1/2 & 1/2 & 1/2 \\ 1/8 & 1/4 & 3/8 \end{bmatrix}.$$

35. The entry in the first row third column of T^2, namely 1/8.

36. The sum of the entries in the first and third rows of the second column, namely $1/4 + 1/4 = 1/2$.

37. $T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$ 38. $T^2\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

39. The Markov chain is regular because T^2 has no zero entries.

$$T - I = \begin{bmatrix} -1/2 & 1/4 & 0 \\ 1/2 & -1/2 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & -1/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{x} = \begin{bmatrix} r \\ 2r \\ r \end{bmatrix}; \quad r + 2r + r = 1, \ r = \frac{1}{4}; \quad \mathbf{s} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

40. The transition matrix has at least one entry 1 on the main diagonal. It cannot be regular if it has more than one state, for it is impossible to get from such an absorbing state, corresponding to a diagonal entry 1, to any other state.

41. $\mathrm{T} = \begin{array}{c} \begin{array}{ccc} D & H & R \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ D \\ H \\ R \end{array}$. The recessive state is absorbing since there is the entry 1 in the 3rd row 3rd column position.

42. a) $\begin{bmatrix} 1 & .5 & .1 \\ 0 & .2 & .4 \\ 0 & .3 & .5 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & .2 \\ 0 & .5 & .8 \end{bmatrix}$

43. Let m and q be as suggested, and consider the suggested new chain with transition matrix $C = \begin{array}{cc} A & F \\ \end{array} \begin{bmatrix} 1 & q \\ 0 & 1 - q \end{bmatrix}$. Then

$C^r = \begin{bmatrix} 1 & 1 - (1 - q)^r \\ 0 & (1 - q)^r \end{bmatrix}$ so $C^r \approx \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ for large integers r.

Since q is the smallest entry in its row in T, we are more likely to reach an absorbing state in *mr time periods in the original chain than we are in r* time periods in the two-state with transition matrix C. Since the steady state distribution vector in the two-state chain is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the vector $T^m\mathbf{p}$ must approach the vector with one in the component corresponding to the absorbing state and zero elsewhere.

44. All nonzero entries are positive. Since more than half the entries in each row and each column are positive, the dot product of any row vector with any column vector has to involve the product of at least one positive component of the row with one positive component of the column vector. Thus the dot product is positive. Consequently, the square of the matrix has all positive entries, so the matrix is regular.

45. $\begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$ 46. $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ 47. $\begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$ 48. $\begin{bmatrix} 5/18 \\ 1/3 \\ 7/18 \end{bmatrix}$

49. $\begin{bmatrix} .4037 \\ .3975 \\ .1988 \end{bmatrix}$ 50. $\begin{bmatrix} .2949 \\ .1667 \\ .5385 \end{bmatrix}$ 51. $\begin{bmatrix} .2115 \\ .3462 \\ .4423 \end{bmatrix}$

52. $\begin{bmatrix} .2020 \\ .2626 \\ .2862 \\ .2492 \end{bmatrix}$ 53. $\begin{bmatrix} .2682 \\ .2235 \\ .1916 \\ .1676 \\ .1490 \end{bmatrix}$ 54. $\begin{bmatrix} .2325 \\ .0318 \\ .2452 \\ .2357 \\ .2548 \end{bmatrix}$

SECTION 1.8 - Binary Linear Codes (Optional)

1. 0001011 0000000 0111001 1111111 1111111 0100101 0000000 0100101 1111111 0111001
2. GONE HOME
3. $x_4 = x_1 + x_2$, $x_5 = x_1 + x_3$, $x_6 = x_2 + x_3$
4. 000000, 001011, 010101, 011110, 100110, 101101, 110011, 111000
5. Note that each x_i appears in at least two parity check equations and for each combination i,j of positions in a message word, some parity-check equation contains one of x_i, x_j but not the other. As explained after Eq.(2) in the text, this shows that the distance between code words is at least 3. Consequently, any two errors can be detected.
6. Since the minimum distance between code words is 3 as explained in the previous solution, any single error can be corrected.
7. a) 110 b) 001 c) 110 d) 001 or 100 or 111 e) 101
8. (See the first matrix in the next solution.)
9. We compute, using binary addition,

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

where the first matrix is the parity-check matrix and the second one has as *columns* the received words.

a) Since the first column of the product is the fourth column of the parity-check matrix, we change the fourth position of the received word 110111 to get the code word 110011 and decode as 110.

b) Since the second column of the product is the zero vector, the received word 001011 is a code word and we decode it as 001.

c) Since the third column of the product is the third column of the parity-check matrix, we change the third position of the received word 111011 to get the code word 110011 and decode as 110.

d) Since the fourth column of the product is not the zero vector and not a column of the parity-check matrix, there are at least two errors, and we ask for retransmission.

e) Since the fifth column of the product is the third column of the parity-check matrix, we change the third position of the received word 100101 to get the code word 101101 and decode as 101.

10. a) 7 b) 6 c) $\mathrm{wt}(u + v) = \mathrm{wt}(1010011001) = 5$
d) $\mathrm{wt}(u - v) = \mathrm{wt}(1010011001) = 5$

11. Since we add by components, this follows from the fact that using binary addition, $1 + 1 = 1 - 1 = 0$, $1 + 0 = 1 - 0 = 1$, $0 + 1 = 0 - 1 = 1$, and $0 + 0 = 0 - 0 = 0$.

12. If the transmitted word v and the received word w agree in component i, then the sum of the ith components is either $0 + 0 = 0$ or $1 + 1 = 0$. If they are different in their jth components, the sum of the jth components is $0 + 1 = 1$ or $1 + 0 = 1$. Thus $u + v$ has 1's in precisely the components where the error was made; hence the name *error pattern*.

13. From the solution to Exercises 11 and 12, we see that $u - v = u + v$ contains 1's in precisely the positions where the two words differ. The number of places where u and v differ is distance between them, and the number of 1's in $u - v$ is $\mathrm{wt}(u - v)$, so these numbers are equal.

14. a) By definition, $d(u, v)$ is the number of components where u and v differ, which is 0 if and only if u and v are the same word.

b) The number of components in which u and v differ is the same as the number of places where v and u differ.

c) If the ith characters of u and w disagree then the ith characters of u, v, and w can't all agree. Thus the contribution of the ith component to $d(u, v) + d(v, w)$ is greater than or equal to its contribution to $d(u, w)$. Summing over all components, we obtain the desired inequality.

d) From Exercise 13, we obtain $d(u + w, v + w) = \mathrm{wt}((u + w) - (v + w)) = \mathrm{wt}(u - v) = d(u, v)$.

15. This follows immediately from the fact that $\mathbb{B}^n$ itself is closed under addition modulo 2.

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16. Let G be the generator matrix of the Hamming (7, 4) code C. In vector form, we have $C = \{\mathbf{x}G \mid \mathbf{x} \in \mathbf{B}^4\}$. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{B}^4$ then $\mathbf{u}G + \mathbf{v}G = (\mathbf{u} + \mathbf{v})G$ is in C. Since $\mathbf{0} \in C$, we know that C is nonempty.

17. Suppose that $d(u, v)$ is minimum in C. By Exercise 13, $d(u, v) = \text{wt}(u - v)$, showing that the minimum weight of nonzero code words is less than or equal to the minimum distance between two of them. On the other hand, if w is a nonzero code word of minimum weight, then $\text{wt}(w)$ is the distance from w to the zero code word, showing the opposite inequality, so we have the equality stated in the exercise.

18. When the distance between code words is $m + 1$, then no received word that differs from the transmitted word in m or fewer components can be a code word, so the error in transmission will be detected.

19. The triangle inequality in Exercise 14 shows that if the distance from received word v to code word u is at most m and from code word w is at most m, then $d(u, v) \leq 2m$. Thus if the distance between code words is at least $2m + 1$, a received word v with at most m incorrect components has a *unique* nearest neighbor code word. This number $2m + 1$ can't be improved since if $d(u, v) = 2m$, then a possible received word w at distance m from both u and v can be constructed by having its components agree with those of u and v where the components of u and v agree, and making m of the 2m components of w in which u and v differ opposite from the components of u and the other m components of w opposite from those of v.

20. By Exercise 17, the minimum nonzero weight of code words is equal to the minimum distance between them. By the answer to Exercise 18, if this distance is $m = 2t + 1$, then any $m - 1 = 2t$ or fewer errors can be detected. By the answer to Exercise 19, if the distance is $2m + 1 = 2t + 1$, then any $m = t$ or fewer errors can be corrected.

21. Let e_i be the word in $\mathbf{B}^n$ with 1 in the ith position and 0's elsewhere. Now e_i is not in C since the distance from e_i to $000\cdots0$ is 1, and $000\cdots0 \in C$. Also, $v + e_i \neq w + e_j$ for any two distinct words v and w in C since otherwise $v - w = e_j - e_i$ would be in C with $\text{wt}(e_j - e_i) = 2$. Let $e_i + C = \{e_i + v \mid v \in C\}$. Note that C and $e_i + C$ have the same

number of words. The disjoint sets C and $e_i + C$ for $i = 1, 2, \cdots, k$ contain all words whose distance from some word in C is at most 1. Thus $\mathbb{B}^n$ must be large enough to contain all of these $(n + 1)2^k$ elements. That is, $2^n \geq (n + 1)2^k$. Dividing by 2^k gives the desired result.

22. Since the minimum distance between words in C is at least 5, no word $w \in \mathbb{B}^n$ has distance ≤ 2 from two different words $u, v \in C$, for if $d(u, w) \leq 2$ and $d(v, w) \leq 2$, then $d(u, v) \leq 4$ by the triangle inequality in Exercise 14. Let e_i be the word in $\mathbb{B}^n$ with 1 in the ith component and 0's elsewhere, and let e_{ij} for $i \neq j$ be the word in $\mathbb{B}^n$ with 1's in components i and j and 0's elsewhere. By the first sentence, the sets $e_i + C = \{e_i + v \mid v \in C\}$ and $e_{ij} + C = \{e_{ij} + v \mid v \in C\}$ for $i, j = 1, 2, \cdots, n, i \neq j$, are distinct from each other, and are distinct from C. Thus n must be large enough to have $\mathbb{B}^n$ contains all these words. Now C has 2^k elements, and the number of e_i is n while the the number of e_{ij} is $n(n - 1)/2$. Thus we must have

$$2^n \geq 2^k + n\cdot 2^k + \frac{n(n - 1)}{2}\cdot 2^k.$$

Dividing by $2k$, we obtain $2^{n-k} \geq 1 + n + n(n - 1)/2$.

23. a) 3 b) 3 c) 4 d) 5 e) 6 f) 7

24. $x_6 = x_1 + x_2 + x_3$, $x_7 = x_2 + x_4 + x_5$, $x_8 = x_1 + x_2 + x_4$, $x_9 = x_1 + x_3 + x_5$ (Other answers are possible.)

25. $x_9 = x_1 + x_2 + x_3 + x_6 + x_7$, $x_{10} = x_5 + x_6 + x_7 + x_8$
$x_{11} = x_2 + x_3 + x_4 + x_6 + x_8$, $x_{12} = x_1 + x_3 + x_4 + x_5$
(Other answers are possible.)

26. Since the last $n - k$ columns of H form the $(n - k) \times (n - k)$ identity matrix, the reduced echelon form of H, obtained using modulo 2 arithmetic, will have $n - k$ pivots. Thus the system $H\mathbf{x} = \mathbf{0}$ gives rise to k free variables. Since the only possible scalars are 0 and 1, we obtain 2^k possible distinct linear combinations comprising the solution set of $H\mathbf{x} = \mathbf{0}$.

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Since C already has 2^k elements and is contained in this solution space, it must be the solution space.