應數一線性代數 2024 春, 期中考 解答

學號: ________________________________, 姓名: ________________________________

本次考試共有 10 頁 (包含封面)，有 10 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，忘記填寫扣十分！
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。　**敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。　　　**遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Is A orthogonal diagonalizable? (Yes / $\boxed{No}$) .

why? **No! the algebraic multiplicity of 2 is 2, but the geometric multiplicity is 1**

Solution：

First, A is orthogonal diagonalizable = there exist D is diagonal matrix and C is an orthogonal matrix such that $D = C^{-1}AC$.

The eigenvalues and eigenvectors of A are $(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}), (3, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix})$.

Follow 課本 5-2 Theorem 5.4.

$\because$ the algebraic multiplicity of 2 is 2, but the geometric multiplicity is 1.

$\therefore$ A is NOT diagonalizable.

$\therefore$ A is NOT orthogonal diagonalizable.

2. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{bmatrix}$$

Is A orthogonal diagonalizable? (Yes / $\boxed{No}$) .

why? **Yes! A is symmetric!**

Solution：

Method 1: Follow 課本 6-3 Theorem 6.8

Method 2:

$$det(A - \lambda I) = -(\lambda^3 - 4\lambda^2 - 53\lambda + 124)$$

Let $f(x) = x^3 - 4x^2 - 53x + 124$, $f(0) = 124 > 0$, $f(6) = -122 < 0$.

By 高中堪根定理，$f(x)$ 有三相異實根分別在 $(-\infty, 0)$, $(0, 6)$, $(6, \infty)$ 的區間內。

定理 5.3 可知 A is diagonalizable.

定理 6.7 可知 symmtrix matrix 中，相異的 eigenvalue 所對應的 eigenvector 是 orthogonal，所以 A 是 orthogonal diagonalizable.

3. (10 points) Solve the system $\begin{cases} x_1' = 4x_1 - 2x_2 + x_3 \\ x_2' = -2x_1 + 3x_2 - 2x_3 \\ x_3' = x_1 - 2x_2 + 4x_3 \end{cases}$

Answer: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C \begin{bmatrix} k_1e^t \\ k_2e^{3t} \\ k_3e^{7t} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1e^t \\ k_2e^{3t} \\ k_3e^{7t} \end{bmatrix} = \begin{bmatrix} k_1e^t - k_2e^{3t} + k_3e^{7t} \\ 2k_1e^t \qquad - k_3e^{7t} \\ k_1e^t + k_2e^{3t} + k_3e^{7t} \end{bmatrix}$.

Solution：

Follow 課本 5-3 example 3

Follow 109-2 midterm problem 1.

$$C = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

4. (15 points) Use Gram-Schmidt process to find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by [1, 0, 1, 0], [1, 1, -1, 0], [1, 1, 0, 1] and then use it to find the QR-factorization of A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

Q= $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{42}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} \\ 0 & 0 & \frac{-6}{\sqrt{42}} \end{bmatrix}$, R= $\begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{7}}{\sqrt{6}} \end{bmatrix}$,

an orthonormal basis of W = $\frac{1}{\sqrt{2}}[1,0,1,0], \frac{1}{\sqrt{3}}[1,1,-1,0], \frac{1}{\sqrt{42}}[-1,2,1,6],$

Solution：

Follow 課本 6-2 example 5.

Follow 111-2 quiz 6.

Follow 109-2 midterm problem 6.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{42}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ 0 & 0 & \frac{6}{\sqrt{42}} \end{bmatrix}, \ R = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{7}}{\sqrt{6}} \end{bmatrix}$$

5. (10 points) Find the formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects in the line $3x - 7y = 0$.

Answer: $T([x, y]) =$ $\frac{1}{58}[40x + 42y, 42x + 40y]$.

Solution：

Follow 課本 5-2 example 2

Follow 109-2 midterm problem 2.

$$C = \begin{bmatrix} -3 & 7 \\ 7 & 3 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let A is the s.m.r. of T

$$A = CDC^{-1} = \frac{1}{-58}\begin{bmatrix} -40 & -42 \\ -42 & 40 \end{bmatrix}$$

6. (10 points) Find the projection of [2, 4, 1] on the plane $P:\ 2x - y - 2z = 0$

Answer: the projection = $\frac{1}{9}[22, 34, 5]$, and the $P^{\perp}$ = $sp(\begin{bmatrix} 2 & -1 & -2 \end{bmatrix})$.

Solution：

Let $\vec{n} = \begin{bmatrix} 2 & -1 & -2 \end{bmatrix}$, then $P^{\perp} = sp(\vec{n})$.

Method 1

$$\vec{b}_{P^{\perp}} = \frac{\vec{b}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{-2}{9}[2, -1, -2] \ \Rightarrow\ \vec{b}_P = \vec{b} - \vec{b}_{P^{\perp}} = \frac{1}{9}[22, 34, 5]$$

Method 2 (i)

Since P is dim 2 subspace in $\mathbb{R}^3$, we know $P = sp(\vec{a}_1, \vec{a}_2)$ fro any $\vec{a}_1, \vec{a}_2 \in P$ and $\vec{a}_1, \vec{a}_2$ are not paralleled. Pick $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then the projection is

$$\vec{b}_P = A(A^TA)^{-1}A^T\vec{b} = \frac{1}{9}[22, 34, 5]^T$$

Method 2 (ii)

Since $\vec{a}_1, \vec{a}_2, \vec{n}$ is a basis for $\mathbb{R}^3$, and

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 0 & -1 & 4 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17/9 \\ 0 & 1 & 0 & 5/9 \\ 0 & 1 & 1 & -2/9 \end{array}\right] \ \Rightarrow\ \vec{b} = \frac{17}{9}\vec{a}_1 + \frac{5}{9}\vec{a}_2 + \frac{-2}{9}\vec{n}$$

$$\vec{b}_P = \frac{17}{9}\vec{a}_1 + \frac{5}{9}\vec{a}_2 = \frac{1}{9}[22, 34, 5],\ \vec{b}_{P^{\perp}} = \frac{-2}{9}\vec{n} = \frac{-2}{9}[2, -1, -2]$$

Method 3

Pick $\vec{v}_1 = [1, 0, 1]$ from P, and let $\vec{v}_2 = \vec{v}_1 \times \vec{n} = [1, 4, -1]$. Find $\vec{q}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{\sqrt{2}}[1, 0, 1]$, $\vec{q}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{1}{\sqrt{18}}[1, 4, -1]$. Let $Q = \begin{bmatrix} \vec{q}_1^T & \vec{q}_2^T \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ 0 & \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} \end{bmatrix}$. Then the projection is

$$\vec{b}_P = QQ^T\vec{b} = \frac{1}{9}\begin{bmatrix} 22 \\ 34 \\ 5 \end{bmatrix}$$

7. (10 points) Show that orthogonal matrices preserve the dot product of vectors. (i.e. $(A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y})$.)

Solution：

Theorem 6.6 (1).

8. (10 points) Let A is an $n \times n$ invertible matrix and if λ is an eigenvalue of A with $\overrightarrow{v}$ as a corresponding eigenvector. Prove that (a) $\lambda \neq 0$ and (b) $1/\lambda$ is an eigenvalue of A^{-1} with $\overrightarrow{v}$ as a corresponding eigenvector.

Solution：

Section 5-1 # 28, 我上課有證過 and Quiz 1。

9. (15 points) Circle True or False and then prove (證明) or disprove (反駁) it. Read each statement in original Greek before answering. *** **只圈對錯，沒有論述一律不給分** ***

(a) True **False** Every $n \times k$ matrix A has a factorization $A = QR$, where the column vectors of Q form an orthonormal set and R is an invertible $k \times k$ matrix.

Solution：

Section 6-2, problem 25g.

(b) True **False** Every vector in a vector space V is an eigenvector of the identity transformation of V into V.

Solution：

Section 5-1, problem 23i

(c) **True** False Given W is a subspace of $\mathbb{R}^n$. If a vector $\vec{v}$ belongs to both W and $W^{\perp}$, then $\vec{v} = \vec{0}$.

Solution：

上課證過

10. (10 points) Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector $\vec{p}$ in W such that $\vec{b}-\vec{p}$ is perpendicular(垂直) to every vector in W.

Solution：

Assume there are two vectors $\vec{p}_1, \vec{p}_2 \in W$ such that $\vec{b}-\vec{p}_1$ and $\vec{b}-\vec{p}_2$ are both perpendicular to every vector in W . i.e. $\vec{b}-\vec{p}_1$ and $\vec{b}-\vec{p}_2$ are both in $W^{\perp}$.

For all vector $\vec{v} \in W$

$$0 = \vec{v}\cdot(\vec{b}-\vec{p}_1) = \vec{v}\cdot\vec{b} - \vec{v}\cdot\vec{p}_1 \therefore \vec{v}\cdot\vec{b} = \vec{v}\cdot\vec{p}_1$$

$$0 = \vec{v}\cdot(\vec{b}-\vec{p}_2) = \vec{v}\cdot\vec{b} - \vec{v}\cdot\vec{p}_2 \therefore \vec{v}\cdot\vec{b} = \vec{v}\cdot\vec{p}_2$$

$$\therefore \forall \vec{v} \in W, \quad \vec{v}\cdot(\vec{p}_1-\vec{p}_2) = 0$$

$$\therefore \vec{p}_1-\vec{p}_2 \in W^{\perp}$$

Note that W is a vector space and $\vec{p}_1, \vec{p}_2 \in W$, we will have $\vec{p}_1-\vec{p}_2 \in W^{\perp}$. Since $\vec{p}_1-\vec{p}_2$ in both W and $W^{\perp}$, we can easily checked that $\vec{p}_1-\vec{p}_2 = \vec{0}$.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	15	10	10	10	10	15	10	110
Score:											