2-1 $f(x)=0$ bisec.

2-2 $\rightsquigarrow g(x)=x$

$\uparrow$

choice

fixed-point Thm

conv. (2.4 conv linear / qu)

2-3 newton method

$g(x)=x$

$\quad \| $

$x-\phi(x)f(x)$

where $\phi(x)=1/f'(x)$ if $f'(x)\neq 0$

$\Rightarrow \boxed{g(x)=x-\psi^{-1}(x)f(x) \quad \psi(x)=f'(x)}$

P. 80, after Thm 2.8 Thm 2.9

10-1 $\vec{F}(\vec{x})=\vec{0} \rightsquigarrow \vec{G}(\vec{x})=\vec{x}$

fixed-point Thm

10-2 newton method

$\vec{G}(\vec{x})=\vec{x}-A^{-1}(x)\vec{F}(\vec{x})$

Thm 2.9

let $p=g(p)$

求 $\psi(x)$

if ① $g'(p)=0$ and

② g'': conti with $|g''(x)|<M$ on $\underset{\Downarrow}{I} \ni p$

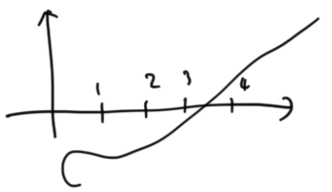
$\Rightarrow \delta>0$, $\forall p_0\in[p-\delta,p+\delta]$

$p_n=g(p_{n-1})$ $\underbrace{\text{conv.}}_{\text{at least quadratically}}$ to p

Moreover, n: large

$$|p_{n+1}-p|<\frac{M}{2}|p_n-p|^2$$

猜 $p=3.5$



Thm 10.7

let $\vec{p}=\vec{G}(\vec{p})$. $\delta>0$

if ① (i) $\dfrac{\partial g_i(\vec{x})}{\partial x_k}$: contin on $N_\delta=\{\vec{x}\mid \|\vec{x}-\vec{p}\|<\delta\}$

(ii) $\dfrac{\partial g_i}{\partial x_k}(\vec{p})=0$ 求 $A(x)$

② $\dfrac{\partial^2 g_i}{\partial x_j\partial x_k}(\vec{x})$: contin, with $\overline{\left|\dfrac{\partial^2 g_i}{\partial x_j\partial x_k}(\vec{x})\right|\le M}^{\forall \vec{x}\in N_\delta}$

$\Rightarrow \exists \hat{\delta}<\delta$, $\forall \vec{p}_0$ s.t. $\|\vec{p}_0-\vec{p}\|<\hat{\delta}$

$\vec{p}_n=\vec{G}(\vec{p}_{n-1})$ $\underbrace{\text{conv.}}_{\text{at least quadratically}}$ to $\vec{p}$

Moreover, n: large.

$$\|\vec{p}^{\,n}-\vec{p}\|_\infty \le \frac{n^2M}{2}\|\vec{p}_{n-1}-\vec{p}\|_\infty^2$$

$$\vec{G}(\vec{x}) = \vec{x} - A^{-1}(x)\,\vec{F}(\vec{x})$$

$$A^{-1}(\vec{x}) = [b_{ij}(\vec{x})]$$

$$\vec{G}(\vec{x}) = \begin{bmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \\ \vdots \\ g_n(\vec{x}) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} b_{11}(\vec{x}) & b_{12}(\vec{x}) & \cdots & b_{1n}(\vec{x}) \\ b_{21}(\vec{x}) & & & \\ \vdots & & & \\ b_{n1}(\vec{x}) & \cdots & & b_{nn}(\vec{x}) \end{bmatrix} \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix}$$

$$\frac{\partial g_i}{\partial x_k}(\vec{x}) = \begin{cases} 1 - \sum_{j=1}^{n} \left(b_{ij}(\vec{x}) \frac{\partial f_j(\vec{x})}{\partial x_k} + \frac{\partial b_{ij}(\vec{x})}{\partial x_k} f_j(\vec{x}) \right) & \text{if } i = k \\ 0 - \sum_{j=1}^{n} \left(b_{ij}(\vec{x}) \frac{\partial f_j(\vec{x})}{\partial x_k} + \frac{\partial b_{ij}(\vec{x})}{\partial x_k} f_j(\vec{x}) \right) & \text{if } i \neq k \end{cases}$$

☆ $\frac{\partial g_i}{\partial x_k}(\vec{p}) = 0$

☆ $\vec{G}(\vec{p}) = \vec{p}$, $\vec{F}(\vec{p}) = \vec{0}$

① $\boxed{i = k}$

$$\frac{\partial g_i}{\partial x_i}(\vec{p}) = 0 = 1 - \sum_{j=1}^{n} \left(b_{ij}(\vec{p}) \frac{\partial f_j(\vec{p})}{\partial x_i} + \boxed{\frac{\partial b_{ij}(\vec{p})}{\partial x_i} \underbrace{f_j(\vec{p})}_{=0}} \right)_{=0}$$

$$0 = 1 - \boxed{\sum_{j=1}^{n} b_{ij}(\vec{p}) \frac{\partial f_j(\vec{p})}{\partial x_i}}$$

$$\boxed{\sum_{j=1}^{n} b_{ij}(\vec{p}) \frac{\partial f_j(\vec{p})}{\partial x_i}} = 1$$

☆ $\vec{G}(\vec{p}) = \vec{p}$, $\vec{F}(\vec{p}) = \vec{0}$

② $\boxed{i \neq k}$

$$\frac{\partial g_i}{\partial x_k}(\vec{p}) = 0 = 0 - \sum_{j=1}^{n} \left(b_{ij}(\vec{p}) \frac{\partial f_j(\vec{p})}{\partial x_k} + \boxed{\frac{\partial b_{ij}(\vec{p})}{\partial x_k} \underbrace{f_j(\vec{p})}_{=0}} \right)_{=0}$$

$$0 = 0 - \boxed{\sum_{j=1}^{n} b_{ij}(\vec{p}) \frac{\partial f_j(\vec{p})}{\partial x_k}}$$

$$\boxed{\sum_{j=1}^{n} b_{ij}(\vec{p}) \frac{\partial f_j(\vec{p})}{\partial x_k}} = 0$$

①, ②

$\Downarrow$

$$\therefore \sum_{j=1}^{n} \overset{i,j}{\boxed{b_{ij}(\vec{p})}} \overset{j,k}{\boxed{\frac{\partial f_j(\vec{p})}{\partial x_k}}} = \begin{cases} 1 & \text{if } i=k \quad ① \\ 0 & \text{if } i \neq k \quad ② \end{cases}$$

$$\left[b_{ij}(\vec{p}) \right] \left[\frac{\partial f_j(\vec{p})}{\partial x_k} \right] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$(A(\vec{p}))^{-1} (J(\vec{p})) = I$$

$$\therefore (J(\vec{p})) = (A(\vec{p}))$$

$$\therefore A(\vec{p}) = J(\vec{p}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{p}) & \frac{\partial f_1}{\partial x_2}(\vec{p}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{p}) \\ \frac{\partial f_2}{\partial x_1}(\vec{p}) & \cdots & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{p}) \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1}(\vec{p}) & \cdots & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{p}) \end{bmatrix}$$

$\therefore$ let $\vec{G}(\vec{x}) = \vec{x} - (J(\vec{x}))^{-1} \vec{F}(\vec{x})$

Newton Method.

Ch 2 Newton Method

$$g(x) = x - \frac{1}{f'(x)} f(x)$$

pick x_0

$$\left.\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_2) \\ &\vdots \\ x_n &= g(x_{n-1}) \end{aligned}\right\} \rightarrow \text{fixed-point iteration}$$

△ error $= |x_n - x_{n-1}|$

error < tol ex: 10^{-6}

Ch 10 Newton Method.

$$\vec{G}(\vec{x}) = \vec{x} - (J(\vec{x}))^{-1} \vec{F}(\vec{x})$$

※ 每步的 $\vec{F}(\vec{x}_0)$ $J(\vec{x}_0)$ 不同

pick $\vec{x}_0$

$$\vec{x}_1 = \vec{G}(\vec{x}_0) \rightarrow \begin{cases} ① \text{ 求 } \vec{y}_0 \text{ s.t. } (J(\boxed{\vec{x}_0}))\, \vec{y}_0 = \vec{F}(\vec{x}_0) \\ ② \ \vec{x}_1 = \vec{x}_0 - \vec{y}_0 \end{cases}$$

$$\vec{x}_2 = \vec{G}(\vec{x}_1)$$

$$\vdots$$

$$\vec{x}_n = \vec{G}(\vec{x}_{n-1})$$

$\ell = 1, 2, \infty$ or else

△ error $= \| \vec{x}_n - \vec{x}_{n-1} \|_{\ell}$ ∞

error < tol

Example 1 The nonlinear system

$$\vec{F}(x_1,x_2,x_3)=\vec{F}(\vec{x}):\quad \begin{bmatrix} f_1(\vec{x})= 3x_1-\cos(x_2x_3)-\dfrac{1}{2} \\ f_2(\vec{x})\ \text{"}\ x_1^2-81(x_2+0.1)^2+\sin x_3+1.06 \\ f_3(\vec{x})= e^{-x_1x_2}+20x_3+\dfrac{10\pi-3}{3} \end{bmatrix} = \begin{bmatrix} 0, \\ 0, \\ 0 \end{bmatrix}$$

was shown in Example 2 of Section 10.1 to have the approximate solution $(0.5, 0, -0.52359877)^t$. Apply Newton's method to this problem with $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.

Solution Define

$$\mathbf{F}(x_1,x_2,x_3) = (f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3), f_3(x_1,x_2,x_3))^t,$$

where

$$f_1(x_1,x_2,x_3) = 3x_1-\cos(x_2x_3)-\frac{1}{2},$$

$$f_2(x_1,x_2,x_3) = x_1^2-81(x_2+0.1)^2+\sin x_3+1.06,$$

and

$$f_3(x_1,x_2,x_3) = e^{-x_1x_2}+20x_3+\frac{10\pi-3}{3}.$$

The Jacobian matrix $J(\mathbf{x})$ for this system is

$$\vec{G}(\vec{x})=\vec{x}-(J(\vec{x}))^{-1}\vec{F}(\vec{x})$$

$$\begin{array}{cccc} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ J(x_1,x_2,x_3) = & \left[\begin{matrix} 3 \\ 2x_1 \\ -x_2e^{-x_1x_2} \end{matrix}\right. & \begin{matrix} x_3\sin x_2x_3 \\ -162(x_2+0.1) \\ -x_1e^{-x_1x_2} \end{matrix} & \left.\begin{matrix} x_2\sin x_2x_3 \\ \cos x_3 \\ 20 \end{matrix}\right]. \end{array}$$

Let $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$. Then $\mathbf{F}\left(\mathbf{x}^{(0)}\right) = (-0.199995, -2.269833417, 8.462025346)^t$ and

$$J\left(\mathbf{x}^{(0)}\right) = \begin{bmatrix} 3 & 9.999833334\times10^{-4} & 9.999833334\times10^{-4} \\ 0.2 & -32.4 & 0.9950041653 \\ -0.09900498337 & -0.09900498337 & 20 \end{bmatrix}.$$

求 $(J(\vec{x}_0))^{-1}$

Solving the linear system, $J\left(\mathbf{x}^{(0)}\right)\mathbf{y}^{(0)} = -\mathbf{F}\left(\mathbf{x}^{(0)}\right)$ gives

$$\mathbf{y}^{(0)} = \begin{bmatrix} 0.3998696728 \\ -0.08053315147 \\ -0.4215204718 \end{bmatrix} \quad\text{and}\quad \mathbf{x}^{(1)} = \mathbf{x}^{(0)}+\mathbf{y}^{(0)} = \begin{bmatrix} 0.4998696782 \\ 0.01946684853 \\ -0.5215204718 \end{bmatrix}.$$

Continuing for $k = 2, 3, \ldots,$ we have

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix},$$

where

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix} = -\left(J\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)\right)^{-1} \mathbf{F}\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right).$$

Thus, at the kth step, the linear system $J\left(\mathbf{x}^{(k-1)}\right)\mathbf{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$ must be solved, where

$$J\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} 3 & x_3^{(k-1)} \sin x_2^{(k-1)} x_3^{(k-1)} & x_2^{(k-1)} \sin x_2^{(k-1)} x_3^{(k-1)} \\ 2x_1^{(k-1)} & -162\left(x_2^{(k-1)} + 0.1\right) & \cos x_3^{(k-1)} \\ -x_2^{(k-1)} e^{-x_1^{(k-1)} x_2^{(k-1)}} & -x_1^{(k-1)} e^{-x_1^{(k-1)} x_2^{(k-1)}} & 20 \end{bmatrix},$$

$$\mathbf{y}^{(k-1)} = \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix},$$

and

$$\mathbf{F}\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} 3x_1^{(k-1)} - \cos x_2^{(k-1)} x_3^{(k-1)} - \frac{1}{2} \\ \left(x_1^{(k-1)}\right)^2 - 81\left(x_2^{(k-1)} + 0.1\right)^2 + \sin x_3^{(k-1)} + 1.06 \\ e^{-x_1^{(k-1)} x_2^{(k-1)}} + 20x_3^{(k-1)} + \frac{10\pi - 3}{3} \end{bmatrix}.$$

The results using this iterative procedure are shown in Table 10.3. ■

Table 10.3

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\Vert\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\Vert_\infty$
0	0.1000000000	0.1000000000	−0.1000000000	
1	0.4998696728	0.0194668485	−0.5215204718	0.4215204718
2	0.5000142403	0.0015885914	−0.5235569638	1.788×10^{-2}
3	0.5000000113	0.0000124448	−0.5235984500	1.576×10^{-3}
4	0.5000000000	8.516×10^{-10}	−0.5235987755	1.244×10^{-5}
5	0.5000000000	-1.375×10^{-11}	−0.5235987756	8.654×10^{-10}