4. The span of any k vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. If A is an $m \times n$ matrix, the *row space* of A is the span in $\mathbb{R}^n$ of the row vectors of A, the *column space* of A is the span in $\mathbb{R}^m$ of the column vectors, and the *nullspace* of A is the solution set of $A\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^n$.
5. A subset $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ of a subspace W of $\mathbb{R}^n$ is a *basis for* W if every vector in W can be expressed uniquely as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.
6. The set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis for $\mathrm{sp}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$ if and only if $0\mathbf{w}_1 + 0\mathbf{w}_2 + \cdots + 0\mathbf{w}_k$ is the unique linear combination of the $\mathbf{w}_i$ that is equal to the zero vector.
7. A consistent linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns has a unique solution if and only if the reduced row-echelon form of A appears as the $n \times n$ identity matrix followed by $m - n$ rows of zeros.
8. A consistent linear system having fewer equations than unknowns is underdetermined—that is, it has an infinite number of solutions.
9. A square linear system has a unique solution if and only if its coefficient matrix is row equivalent to the identity matrix.
10. The solutions of any consistent linear system $A\mathbf{x} = \mathbf{b}$ are precisely the vectors $\mathbf{p} + \mathbf{h}$, where $\mathbf{p}$ is any one particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{h}$ varies through the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

EXERCISES

In Exercises 1–10, determine whether the indicated subset is a subspace of the given Euclidean space $\mathbb{R}^n$.

1. $\{[r, -r] \mid r \in \mathbb{R}\}$ in $\mathbb{R}^2$
2. $\{[x, x + 1] \mid x \in \mathbb{R}\}$ in $\mathbb{R}^2$
3. $\{[n, m] \mid n$ and m are integers$\}$ in $\mathbb{R}^2$
4. $\{[x, y] \mid x,y \in \mathbb{R}$ and $x,y \geq 0\}$ (the first quadrant of $\mathbb{R}^2$)
5. $\{[x, y, z] \mid x,y,z \in \mathbb{R}$ and $z = 3x + 2\}$ in $\mathbb{R}^3$
6. $\{[x, y, z] \mid x,y,z \in \mathbb{R}$ and $x = 2y + z\}$ in $\mathbb{R}^3$
7. $\{[x, y, z] \mid x,y,z \in \mathbb{R}$ and $z = 1, y = 2x\}$ in $\mathbb{R}^3$
8. $\{[2x, x + y, y] \mid x,y \in \mathbb{R}\}$ in $\mathbb{R}^3$
9. $\{[2x_1, 3x_2, 4x_3, 5x_4] \mid x_i \in \mathbb{R}\}$ in $\mathbb{R}^4$
10. $\{[x_1, x_2, \ldots, x_n] \mid x_i \in \mathbb{R}, x_2 = 0\}$ in $\mathbb{R}^n$
11. Prove that the line $y = mx$ is a subspace of $\mathbb{R}^2$. [HINT: Write the line as $W = \{[x, mx] \mid x \in \mathbb{R}\}$.]
12. Let a, b, and c be scalars such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$.
13. a. Give a geometric description of all subspaces of $\mathbb{R}^2$.
    b. Repeat part (a) for $\mathbb{R}^3$.
14. Prove that every subspace of $\mathbb{R}^n$ contains the zero vector.
15. Is the zero vector a basis for the subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$? Why or why not?

In Exercises 16–21, find a basis for the solution set of the given homogeneous linear system.

16. $$\begin{aligned} x - y &= 0 \\ 2x - 2y &= 0 \end{aligned}$$
17. $$\begin{aligned} 3x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 2x_3 &= 0 \\ -9x_1 - 3x_2 - 3x_3 &= 0 \end{aligned}$$

18. $$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0 \\ x_2 + x_3 \phantom{{}-x_4} &= 0 \\ x_1 + 2x_2 - x_3 + 3x_4 &= 0 \end{aligned}$$

19. $$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 6x_2 + x_3 \phantom{{}+x_4} &= 0 \\ 3x_1 - 5x_2 + 2x_3 + x_4 &= 0 \\ 5x_1 - 4x_2 + 3x_3 + 2x_4 &= 0 \end{aligned}$$

20. $$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 0 \\ 3x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 + x_2 + 3x_3 \phantom{{}+x_4} &= 0 \\ x_1 - x_2 - 7x_3 + 2x_4 &= 0 \end{aligned}$$

21. $$\begin{aligned} x_1 - x_2 + 6x_3 + x_4 - x_5 &= 0 \\ 3x_1 + 2x_2 - 3x_3 + 2x_4 + 5x_5 &= 0 \\ 4x_1 + 2x_2 - x_3 + 3x_4 - x_5 &= 0 \\ 3x_1 - 2x_2 + 14x_3 + x_4 - 8x_5 &= 0 \\ 2x_1 - x_2 + 8x_3 + 2x_4 - 7x_5 &= 0 \end{aligned}$$

In Exercises 22–30, determine whether the set of vectors is a basis for the subspace of $\mathbb{R}^n$ that the vectors span.

22. $\{[-1, 1], [1, 2]\}$ in $\mathbb{R}^2$

23. $\{[-1, 3, 1], [2, 1, 4]\}$ in $\mathbb{R}^3$

24. $\{[-1, 3, 4], [1, 5, -1], [1, 13, 2]\}$ in $\mathbb{R}^3$

25. $\{[2, 1, -3], [4, 0, 2], [2, -1, 3]\}$ in $\mathbb{R}^3$

26. $\{[2, 1, 0, 2], [2, -3, 1, 0], [3, 2, 0, 0]\}$ in $\mathbb{R}^4$

27. The set of row vectors of the matrix

$$\begin{bmatrix} 2 & -6 & 1 \\ 1 & -3 & 4 \end{bmatrix}.$$

28. The set of column vectors of the matrix in Exercise 27.

29. The set of row vectors of the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -3 \\ 1 & -3 & 4 \end{bmatrix}.$$

30. The set of column vectors of the matrix in Exercise 29.

31. Find a basis for the nullspace of the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix}.$$

32. Find a basis for the nullspace of the matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix}.$$

33. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be vectors in a vector space V. Give a necessary and sufficient condition, involving linear combinations, for

$$\operatorname{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \operatorname{sp}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m).$$

In Exercises 34–37, solve the given linear system and express the solution set in a form that illustrates Theorem 1.18.

34. $x_1 - 2x_2 + x_3 + 5x_4 = 7$

35. $$\begin{aligned} 2x_1 - x_2 + 3x_3 \phantom{{}-x_4} &= -3 \\ 4x_1 + 2x_2 \phantom{{}+3x_3} - x_4 &= 1 \end{aligned}$$

36. $$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 4 \\ 2x_1 + x_2 - 3x_3 - x_4 &= 6 \\ x_1 - 7x_2 - 6x_3 + 2x_4 &= 6 \end{aligned}$$

37. $$\begin{aligned} 2x_1 + x_2 + 3x_3 \phantom{{}+x_4} &= 5 \\ x_1 - x_2 + 2x_3 + x_4 &= 0 \\ 4x_1 - x_2 + 7x_3 + 2x_4 &= 5 \\ -x_1 - 2x_2 - x_3 + x_4 &= -5 \end{aligned}$$

38. Mark each of the following True or False.

___ **a.** A linear system with fewer equations than unknowns has an infinite number of solutions.

___ **b.** A consistent linear system with fewer equations than unknowns has an infinite number of solutions.

___ **c.** If a square linear system $A\mathbf{x} = \mathbf{b}$ has a solution for *every* choice of column vector $\mathbf{b}$, then the solution is unique for each $\mathbf{b}$.

___ **d.** If a square system $A\mathbf{x} = \mathbf{0}$ has only the trivial soltuion, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.

___ **e.** If a linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.

___ **f.** The sum of two solution vectors of any linear system is also a solution vector of the system.

___ g. The sum of two solution vectors of any homogeneous linear system is also a solution vector of the system.

___ h. A scalar multiple of a solution vector of any homogeneous linear system is also a solution vector of the system.

___ i. Every line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

___ j. Every line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

39. We have defined a linear system to be *underdetermined* if it has an infinite number of solutions. Explain why this is a reasonable term to use for such a system.

40. A linear system is **overdetermined** if it has more equations than unknowns. Explain why this is a reasonable term to use for such a system.

41. Referring to Exercises 39 and 40, give an example of an overdetermined underdetermined linear system!

42. Use Theorem 1.13 to explain why a homogeneous system of linear equations has either a unique solution or an infinite number of solutions.

43. Use Theorem 1.18 to explain why no system of linear equations can have exactly two solutions.

44. Let A be an $m \times n$ matrix such that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

a. Does it follow that every system $A\mathbf{x} = \mathbf{b}$ is consistent?

b. Does it follow that every consistent system $A\mathbf{x} = \mathbf{b}$ has a unique solution?

45. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbb{R}^n$. Prove the following set equalities by showing that each of the spans is contained in the other.

a. $\operatorname{sp}(\mathbf{v}_1, \mathbf{v}_2) = \operatorname{sp}(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2)$

b. $\operatorname{sp}(\mathbf{v}_1, \mathbf{v}_2) = \operatorname{sp}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$

46. Referring to Exercise 45, prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\operatorname{sp}(\mathbf{v}_1, \mathbf{v}_2)$, then

a. $\{\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2\}$ is also a basis.

b. $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ is also a basis.

c. $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, 2\mathbf{v}_1 - 3\mathbf{v}_2\}$ is not a basis.

47. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace.

In Exercises 48–51, use LINTEK or MATLAB to determine whether the given vectors form a basis for the subspace of $\mathbb{R}^n$ that they span.

48. $\mathbf{a}_1 = [1, 1, -1, 0]$
$\mathbf{a}_2 = [5, 1, 1, 2]$
$\mathbf{a}_3 = [5, -3, 2, -1]$
$\mathbf{a}_4 = [9, 3, 0, 3]$

49. $\mathbf{b}_1 = [3, -4, 0, 0, 1]$
$\mathbf{b}_2 = [4, 0, 2, -6, 2]$
$\mathbf{b}_3 = [0, 1, 1, -3, 0]$
$\mathbf{b}_4 = [1, 4, -1, 3, 0]$

50. $\mathbf{v}_1 = [4, -1, 2, 1]$
$\mathbf{v}_2 = [10, -2, 5, 1]$
$\mathbf{v}_3 = [-9, 1, -6, -3]$
$\mathbf{v}_4 = [1, -1, 0, 0]$

51. $\mathbf{w}_1 = [1, 4, -8, 16]$
$\mathbf{w}_2 = [1, 1, -1, 1]$
$\mathbf{w}_3 = [1, 4, 8, 16]$
$\mathbf{w}_4 = [1, 1, 1, 1]$

MATLAB

Access MATLAB and enter **fbc1s6** *if our text data files are available; otherwise, enter the vectors in Exercises 48–51 by hand. Use MATLAB matrix commands to form the necessary matrix and reduce it in problems M1–M4.*

M1. Solve Exercise 48.

M2. Solve Exercise 49.

M3. Solve Exercise 50.

M4. Solve Exercise 51.