

## 應數一線性代數 2024 秋, 第一次期中考 **解答**

學號: \_\_\_\_\_, 姓名: \_\_\_\_\_

本次考試共有 9 頁 (包含封面), 有 12 題。如有缺頁或漏題, 請立刻告知監考人員。

### 考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可翻閱課本或筆記。**
- 計算題請寫出計算過程, 閱卷人員會視情況給予部份分數。沒有計算過程, 就算回答正確答案也不會得到滿分。答卷請清楚乾淨, 儘可能標記或是框出最終答案。
- 書寫空間不夠時, 可利用試卷背面, 但須標記清楚。

高師大校訓：**誠敬宏遠**

**誠**: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。  
**宏**: 開拓視界, 恢宏心胸。 **遠**: 任重致遠, 不畏艱難。

請簽名保證以下答題都是由你自己作答的, 並沒有得到任何的外部幫助。

簽名: \_\_\_\_\_

1. (5 points) Given  $\vec{u} = [-1, 1, 2]$ ,  $\vec{v} = [4, 2, -1]$  and  $\vec{w} = [5, 7, 4]$ .

Is  $\vec{w} \in sp(\vec{u}, \vec{v})$ ? (Yes / No) .

If so, write  $\vec{w}$  in the linear combination of  $\vec{v}$  and  $\vec{u}$ :  $\vec{w} = 3\vec{u} + 2\vec{v}$  .

2. (5 points) Given two vectors  $\vec{v} = [x, 2, -1, 1]$  and  $\vec{u} = [1, 6, -2, y]$ . Find all  $x, y \in \mathbb{R}$  so that

(a)  $\vec{v}, \vec{u}$  are parallel. None .

(b)  $\vec{v}, \vec{u}$  are perpendicular.  $x \in \mathbb{R}, y = -x - 14$  .

**Solution :**

(a) Since the second and third components of  $\vec{v}$  are [2, -1] and the second and third components of  $\vec{u}$  is [6, -2], we know that  $\vec{v}, \vec{u}$  are never parallel.

(b) It is fact that  $\vec{v}, \vec{u}$  perpendicular if  $\vec{v} \cdot \vec{u} = 0$ . Since  $\vec{v} \cdot \vec{u} = x + 12 + 2 + y$ ,  $\vec{v}$  and  $\vec{u}$  are perpendicular if  $x + 14 + y = 0$ .

3. (10 points) (a) Find the inverse of the matrix  $A$ , if it exists, and (b) express the inverse matrix as a product of elementary matrices.  $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Answer: (a)  $A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$  (b)  $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$  (答案不唯一)

4. (10 points) Describe all possible values for the unknowns  $x_i$  so that the matrix equation is valid.

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$$

**Solution :**

1-4, problem 35.  $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

5. (10 points) Find all values of  $r$  for which  $A$  and  $B$  commute.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & r \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Answer:  $r = \underline{\textcolor{red}{1}}$

6. (10 points) Let  $a$ ,  $b$  and  $c$  be scalar such that  $abc \neq 0$ . Prove that the plane  $ax + by + cz = 0$  is a subspace of  $\mathbb{R}^3$ .

**Solution :**

1-6 #12

7. (10 points) Assume the matrix  $A$  is row reduces to  $H$ , please answer the following questions.

$$A = \begin{bmatrix} 2 & 4 & 5 & 5 & 8 & 7 \\ -2 & -4 & -3 & 3 & 8 & 0 \\ 2 & 4 & 7 & 6 & 10 & -1 \\ 1 & 2 & 4 & 7 & 13 & 2 \end{bmatrix}, H = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) the **rank** of matrix  $A$ , is 4.

(b) a basis for the **row space** of  $A$  is  $[1, 2, 0, 0, -1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 2, 0], [0, 0, 0, 0, 0, 1]$

(c) a basis for the **column space** of  $A$  is  $\begin{bmatrix} 2 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ .

(d) a basis for the **nullspace** of  $A$  is  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ .

8. (10 points) Use the previous question (前一題), let  $\vec{a}_1 = [2, -2, 2, 1]$ ,  $\vec{a}_2 = [4, -4, 4, 2]$ ,  $\vec{a}_3 = [5, -3, 7, 4]$ ,  $\vec{a}_4 = [5, 3, 6, 7]$ ,  $\vec{a}_5 = [8, 8, 10, 13]$ ,  $\vec{a}_6 = [7, 0, -1, 2]$

- (a) Is  $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$  a linearly independent set? ( Yes / No ) .
- (b) Is  $\{\vec{a}_1, \vec{a}_2\}$  a linearly independent set?? ( Yes / No ) .
- (c) Is  $\{\vec{a}_4, \vec{a}_5\}$  a linearly independent set?? ( Yes / No ) .
- (d) Is  $\{\vec{a}_1, \vec{a}_5, \vec{a}_6\}$  a linearly independent set? ( Yes / No ) .
- (e) Is  $\{\vec{a}_2, \vec{a}_4, \vec{a}_5\}$  a linearly independent set? ( Yes / No ) .

p.s. 記得每小題要分開給理由 !!

9. (15 points) Suppose the complete solution to the equation

$$A\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \quad \text{is} \quad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) The dimension of the row space of  $A$  = 1

**Solution :**

Since  $\vec{x}$  and  $\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$  are both  $3 \times 1$  matrices, we know that  $A$  is a  $3 \times 3$  matrix.

$\vec{x}$  has two free variables, thus the nullity of  $A$  is 2. Therefore, the rank of  $A$  is 1.

(b) What is the matrix  $A$ ? Answer:  $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \\ 2.5 & 5 & 0 \end{bmatrix}$ .

### Solution :

### Method 1:

$$\vec{x} = \begin{bmatrix} 2 - 2r \\ r \\ s \end{bmatrix} \Rightarrow \left[ \begin{array}{c|cc} A & 4 \\ & 2 \\ & 5 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[ \begin{array}{c|cc} A & 4 \\ & 2 \\ & 5 \end{array} \right] = \left[ \begin{array}{ccc|c} 2 & 4 & 0 & 4 \\ 1 & 2 & 0 & 2 \\ 0.5 & 5 & 0 & 5 \end{array} \right]$$

### Method 2:

1.  $r = s = 0$ :

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

2.  $r = 1, s = 0$ :

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

3.  $r = 1, s = 1$ :

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} + a_{13} \\ a_{22} + a_{23} \\ a_{32} + a_{33} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Find all possible  $\vec{b}$  so that  $A\vec{x} = \vec{b}$  can be solved. Answer:  $\vec{b} = r \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ , for  $r \in \mathbb{R}$ .

**Solution :**

$$\vec{b} \in col(A)$$

10. (10 points) Prove that if  $A^3$  is invertible, then  $A^2$  is invertible.

**Solution :**

1-5, problem 23f.

11. (10 points) Let  $W_1$  and  $W_2$  be two subspaces of  $\mathbb{R}^n$ . Prove that their intersection  $W_1 \cap W_2$  is also a subspace.

**Solution :**

1-6, problem 47

Clearly  $W_1 \cap W_2$  is nonempty; it contains 0.

Let  $\vec{v}, \vec{w} \in (W_1 \cap W_2)$ . Then  $\vec{v}, \vec{w} \in W_1$  and  $\vec{v}, \vec{w} \in W_2$ , so  $\vec{v} + \vec{w} \in W_1$  and  $\vec{v} + \vec{w} \in W_2$  since  $W_1$  and  $W_2$  are subspaces.

Thus  $\vec{v} + \vec{w} \in (W_1 \cap W_2)$ . Similarly,  $r\vec{v} \in W_1$  and  $r\vec{v} \in W_2$ . Since  $W_1$  and  $W_2$  are subspaces. Thus  $r\vec{v} \in (W_1 \cap W_2)$ . Thus  $W_1$  and  $W_2$  are subspaces. Thus  $W_1 \cap W_2$  is a subspace of  $\mathbb{R}^n$ .

12. (10 points) Prove or disprove (證明或舉反例推翻) the following statement.

- (a) Let  $\vec{v}, \vec{w}$  be column vectors in  $\mathbb{R}^n$  and let  $A$  be an  $n \times n$  matrix. If  $A\vec{v}$  and  $A\vec{w}$  are linearly independent, then  $\vec{v}$  and  $\vec{w}$  are linearly independent

**Solution :**

It is true! 2-1, problem 36.

- (b) Let  $\vec{v}, \vec{w}$  be column vectors in  $\mathbb{R}^n$  and let  $A$  be an  $n \times n$  matrix. If  $\vec{v}$  and  $\vec{w}$  are linearly independent, then  $A\vec{v}$  and  $A\vec{w}$  are linearly independent

### Solution :

It is false! Compare with 2-1, problem 34, the hypothesis missing the condition that  $A$  is invertible.

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