姓名: SOLUTION　　　　　　　　　　　　　　　　　　　　葉均承　　應數一線性代數

學號: ________　　　　**Quiz 14**　　　　考試日期: 2025/06/04

不可使用手機、計算器，禁止作弊!

1. Let A is a 5×5 matrix.

$$A = \begin{bmatrix} 73 & 30 & 2 & 3 & 7 \\ -6 & 92 & 4 & -2 & 6 \\ -2 & 4 & 76 & 10 & 2 \\ -1 & 2 & -2 & 85 & 1 \\ 6 & 4 & -4 & 2 & 74 \end{bmatrix}$$

(a) Find a Jordan canonical form and a Jordan basis for the given matrix.

(b) Find the $\det(A^{50}) =$ $\underline{\ 80^{400}\ }$.

Notice that

$$rref(A-80I) = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ rref((A-80I)^2) = \begin{bmatrix} 1 & -2 & -2/3 & 1/3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$rref((A-80I)^3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution：

(a)

$$null(A-80I) = sp(\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\1\\0\end{bmatrix}),\ null((A-80I)^2) = sp(\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\3\\0\end{bmatrix}, \begin{bmatrix}2\\0\\3\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},),\ null((A-80I)^3) = \mathbb{R}^5$$

$$\begin{aligned}(A-80I):\quad & \vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 \to \vec{0} \\ & \qquad\ \ \vec{b}_5 \to \vec{b}_4 \to \vec{0}\end{aligned}$$

Pick $\vec{b}_3 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$, then $\vec{b}_2 = (A-80I)\vec{b}_3 = \begin{bmatrix}-7\\-6\\-2\\-1\\6\end{bmatrix}$, and then then $\vec{b}_1 = (A-80I)\vec{b}_2 = \begin{bmatrix}-96\\0\\0\\0\\-96\end{bmatrix}$

Pick $\vec{b}_5 = \begin{bmatrix}2\\0\\3\\0\\0\end{bmatrix}$, then $\vec{b}_4 = (A-80I)\vec{b}_5 = \begin{bmatrix}-8\\0\\-16\\-8\\0\end{bmatrix}$

$$C = \begin{bmatrix} | & | & | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} -96 & -7 & 1 & -8 & 2 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & -2 & 0 & -16 & 3 \\ 0 & -1 & 0 & -8 & 0 \\ -96 & 6 & 0 & 0 & 0 \end{bmatrix}, \; J = \begin{bmatrix} 80 & 1 & 0 & 0 & 0 \\ 0 & 80 & 1 & 0 & 0 \\ 0 & 0 & 80 & 0 & 0 \\ 0 & 0 & 0 & 80 & 1 \\ 0 & 0 & 0 & 0 & 80 \end{bmatrix}$$

We have

$$C^{-1}AC = J$$

(b)

$$\det(A) = \det(CJC^{-1}) = \det(C)\det(J)\det(C)^{-1} = \det(J) = 80^5$$

$$\det(A^{80}) = \det(A)^{80} = (80^5)^{80} = 80^{400}$$