The command **rref(A)** *in MATLAB will reduce the matrix A to reduced row-echelon form. Use this command to solve the following exercises.*

M6. Exercise 6

M7. Exercise 24

M8. Exercise 63

M9. Exercise 64

(MATLAB contains a demo command **rrefmovie(A)** designed to show the step-by-step reduction of A, but with our copy and a moderately fast computer, the demo goes so fast that it is hard to catch it with the Pause key in order to view it. If you are handy with a word processor, you might copy the file *rrefmovi.m* as *rrefmovp.m*, and then edit *rrefmovp.m* to supply *,pause* at the end of each of the four lines that start with *A(* . Entering **rrefmovp(A)** from MATLAB will then run the altered demo, which will pause after each change of a matrix. Strike any key to continue after a pause. You may notice that there seems to be unnecessary row swapping to create pivots. Look at the paragraph on partial pivoting in Section 10.3 to understand the reason for this.)

1.5 INVERSES OF SQUARE MATRICES

Matrix Equations and Inverses

A system of n equations in n unknowns $x_1, x_2, \ldots, x_n$ can be expressed in matrix form as

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where A is the $n \times n$ coefficient matrix, $\mathbf{x}$ is the $n \times 1$ column vector with ith entry x_i, and $\mathbf{b}$ is an $n \times 1$ column vector with constant entries. The analogous equation using scalars is

$$ax = b \tag{2}$$

for scalars a and b. If $a \neq 0$, we usually think of solving Eq. (2) for x by dividing by a, but we can just as well think of solving it by multiplying by $1/a$. Breaking the solution down into small steps, we have

$$\begin{aligned}
\left(\frac{1}{a}\right)ax &= \left(\frac{1}{a}\right)b && \textbf{Multiplication by } \mathbf{1/a}\\
\left[\left(\frac{1}{a}\right)a\right]x &= \left(\frac{1}{a}\right)b && \textbf{Associativity of multiplication}\\
1x &= \left(\frac{1}{a}\right)b && \textbf{Property of } \mathbf{1/a}\\
x &= \left(\frac{1}{a}\right)b. && \textbf{Property of 1}
\end{aligned}$$

Let us see whether we can solve Eq. (1) similarly if A is a nonzero matrix. Matrix multiplication is associative, and the $n \times n$ identity matrix I plays the same role for multiplication of $n \times n$ matrices that the number 1 plays for multiplication of numbers. The crucial step is to find an $n \times n$ matrix C such that $CA = I$, so that C plays for matrices the role that $1/a$ does for numbers. If such a matrix C exists, we can obtain from Eq. (1)

$$\begin{aligned} C(A\mathbf{x}) &= C\mathbf{b} && \textbf{Multiplication by } C \\ (CA)\mathbf{x} &= C\mathbf{b} && \textbf{Associativity of multiplication} \\ I\mathbf{x} &= C\mathbf{b} && \textbf{Property of } C \\ \mathbf{x} &= C\mathbf{b}, && \textbf{Property of } I \end{aligned}$$

which shows that our column vector $\mathbf{x}$ of unknowns must be the column vector $C\mathbf{b}$. Now an interesting problem arises. When we substitute $\mathbf{x} = C\mathbf{b}$ back into our equation $A\mathbf{x} = \mathbf{b}$ to verify that we do indeed have a solution, we obtain $A\mathbf{x} = A(C\mathbf{b}) = (AC)\mathbf{b}$. But how do we know that $AC = I$ from our assumption that $CA = I$? Matrix multiplication is not a commutative operation. This problem does not arise with our scalar equation $ax = b$, because multiplication of real numbers is commutative. It is indeed true that for square matrices, if $CA = I$ then $AC = I$, and we will work toward a proof of this. For example, the equation

$$\begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$$

illustrates that

$$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$$

satisfy $CA = I = AC$.

Unfortunately, it is not true that, for each nonzero $n \times n$ matrix A, we can find an $n \times n$ matrix C such that $CA = AC = I$. For example, if the first column of A has only zero entries, then the first column of CA also has only zero entries for any matrix C, so $CA \neq I$ for any matrix C. However, for many important $n \times n$ matrices A, there does exist an $n \times n$ matrix C such that $CA = AC = I$. Let us show that when such a matrix exists, it is unique.

THEOREM 1.9 Uniqueness of an Inverse

Let A be an $n \times n$ matrix. If C and D are matrices such that $AC = DA = I$, then $C = D$. In particular, if $AC = CA = I$, then C is the *unique* matrix with this property.

PROOF Let C and D be matrices such that $AC = DA = I$. Because matrix multiplication is associative, we have

$$D(AC) = (DA)C.$$

But, because $AC = I$ and $DA = I$, we find that

$$D(AC) = DI = D \quad \text{and} \quad (DA)C = IC = C.$$

Therefore, $C = D$.

Now suppose that $AC = CA = I$, and let us show that C is the unique matrix with this property. To this end, suppose also that $AD = DA = I$. Then we have $AC = I = DA$, so $D = C$, as we just showed. ▲

From the title of the preceding theorem, we anticipate the following definition.

DEFINITION 1.15 Invertible Matrix

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that $CA = AC = I$, the $n \times n$ identity matrix. The matrix C is the inverse of A and is denoted by A^{-1}. If A is not invertible, it is **singular**.

Although A^{-1} plays the same role arithmetically as $a^{-1} = 1/a$ (as we showed at the start of this section), we will never write A^{-1} as $1/A$. The powers of an invertible $n \times n$ matrix A are now defined for all integers. That is, for $m > 0$, A^m is the product of m factors A, and A^{-m} is the product of m factors A^{-1}. We consider A^0 to be the $n \times n$ identity matrix I.

Inverses of Elementary Matrices

In Section 1.4, we saw that each elementary row operation can be undone by another (possibly the same) elementary row operation. Let us see how this fact

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HISTORICAL NOTE THE NOTION OF THE INVERSE OF A MATRIX first appears in an 1855 note of Arthur Cayley (1821–1895) and is made more explicit in an 1858 paper entitled "A Memoir on the Theory of Matrices." In that work, Cayley outlines the basic properties of matrices, noting that most of these derive from work with sets of linear equations. In particular, the inverse comes from the idea of solving a system

$$\begin{aligned} X &= a\,x + b\,y + c\,z \\ Y &= a'x + b'y + c'z \\ Z &= a''x + b''y + c''z \end{aligned}$$

for x, y, z in terms of X, Y, Z. Cayley gives an explicit construction for the inverse in terms of the determinants of the original matrix and of the minors.

In 1842, Arthur Cayley graduated from Trinity College, Cambridge, but could not find a suitable teaching post. So, like Sylvester, he studied law and was called to the bar in 1849. During his 14 years as a lawyer, he wrote about 300 mathematical papers; finally, in 1863 he became a professor at Cambridge, where he remained until his death. It was during his stint as a lawyer that he met Sylvester; their discussions over the next 40 years were extremely fruitful for the progress of algebra. Over his lifetime, Cayley produced about 1000 papers in pure mathematics, theoretical dynamics, and mathematical astronomy.

translates to the invertibility of the elementary matrices, and a description of their inverses.

Let E_1 be an elementary row-interchange matrix, obtained from the identity matrix I by the interchanging of rows i and k. Recall that E_1A effects the interchange of rows i and k of A for any matrix A such that E_1A is defined. In particular, taking $A = E_1$, we see that E_1E_1 interchanges rows i and k of E_1, and hence changes E_1 back to I. Thus,

$$E_1E_1 = I.$$

Consequently, E_1 is an invertible matrix and is its own inverse.

Now let E_2 be an elementary row-scaling matrix, obtained from the identity matrix by the multiplication of row i by a nonzero scalar r. Let E_2' be the matrix obtained from the identity matrix by the multiplication of row i by $1/r$. It is clear that

$$E_2'E_2 = E_2E_2' = I,$$

so E_2 is invertible, with inverse E_2'.

Finally, let E_3 be an elementary row-addition matrix, obtained from I by the addition of r times row i to row k. If E_3' is obtained from I by the addition of $-r$ times row i to row k, then

$$E_3'E_3 = E_3E_3' = I.$$

We have established the following fact:

> Every elementary matrix is invertible.

EXAMPLE 1 Find the inverses of the elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

SOLUTION Because E_1 is obtained from

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by the interchanging of the first and second rows, we see that $E_1^{-1} = E_1$.

The matrix E_2 is obtained from I by the multiplication of the first row by 3, so we must multiply the first row of I by $\frac{1}{3}$ to form

$$E_2^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, E_3 is obtained from I by the addition of 4 times row 3 to row 1. To form E_3^{-1} from I, we add -4 times row 3 to row 1, so

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

■

Inverses of Products

The next theorem is fundamental in work with inverses.

THEOREM 1.10 Inverses of Products

Let A and B be invertible $n \times n$ matrices. Then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

PROOF By assumption, there exist matrices A^{-1} and B^{-1} such that $AA^{-1} = A^{-1}A = I$ and $BB^{-1} = B^{-1}B = I$. Making use of the associative law for matrix multiplication, we find that

$$(AB)(B^{-1}A^{-1}) = [A(BB^{-1})]A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

A similar computation shows that $(B^{-1}A^{-1})(AB) = I$. Therefore, the inverse of AB is $B^{-1}A^{-1}$; that is, $(AB)^{-1} = B^{-1}A^{-1}$. ▲

It is instructive to apply Theorem 1.10 to a product $E_t \cdots E_3E_2E_1$ of elementary matrices. In the expression

$$(E_t \ldots E_3E_2E_1)A,$$

the product $E_t \cdots E_3E_2E_1$ performs a sequence of elementary row operations on A. First, E_1 acts on A; then, E_2 acts on E_1A: and so on. To undo this sequence, we must first undo the last elementary row operation, performed by E_t. This is accomplished by using E_t^{-1}. Continuing, we should perform the sequence of operations given by

$$E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_t^{-1}$$

in order to effect $(E_t \cdots E_3E_2E_1)^{-1}$.

A Commutativity Property

We are now in position to show that if $CA = I$, then $AC = I$. First we prove a lemma (a result preliminary to the main result).

LEMMA 1.1 Condition for $A\mathbf{x} = \mathbf{b}$ to Be Solvable for All $\mathbf{b}$

> Let A be an $n \times n$ matrix. The linear system $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of column vector $\mathbf{b} \in \mathbb{R}^n$ if and only if A is row equivalent to the $n \times n$ identity matrix I.

PROOF Let $\mathbf{b}$ be any column vector in $\mathbb{R}^n$ and let the augmented matrix $[A \mid \mathbf{b}]$ be row-reduced to $[H \mid \mathbf{c}]$ where H is in reduced row-echelon form. If H is the identity matrix I, then the linear system $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = \mathbf{c}$.

For the converse, suppose that reduction of A to reduced row-echelon form yields a matrix H that is not the identity matrix. Then the bottom row of H must have every entry equal to 0. Now there exist elementary matrices $E_1, E_2, \ldots, E_t$ such that $(E_t \cdots E_2E_1)A = H$. Recall that every elementary matrix is invertible, and that a product of elementary matrices is invertible. Let $\mathbf{b} = (E_t \cdots E_2E_1)^{-1}\mathbf{e}_n$, where $\mathbf{e}_n$ is the column vector with 1 in its nth component and zeros elsewhere. Reduction of the augmented matrix $[A \mid \mathbf{b}]$ can be accomplished by multiplying both A and $\mathbf{b}$ on the left by $E_t \cdots E_2E_1$, so the reduction will yield $[H \mid \mathbf{e}_n]$, which represents a system of equations with no solution because the bottom row has entries 0 to the left of the partition and 1 to the right of the partition. This shows that if H is not the identity matrix, then $A\mathbf{x} = \mathbf{b}$ does not have a solution for some $\mathbf{b} \in \mathbb{R}^n$. ▲

THEOREM 1.11 A Commutativity Property

> Let A and C be $n \times n$ matrices. Then $CA = I$ if and only if $AC = I$.

PROOF To prove that $CA = I$ if and only if $AC = I$, it suffices to prove that if $AC = I$, then $CA = I$, because the converse statement is obtained by reversing the roles of A and C.

Suppose now that we do have $AC = I$. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every column vector $\mathbf{b}$ in $\mathbb{R}^n$; we need only notice that $\mathbf{x} = C\mathbf{b}$ is a solution because $A(C\mathbf{b}) = (AC)\mathbf{b} = I\mathbf{b} = \mathbf{b}$. By Lemma 1.1, we know that A is row equivalent to the $n \times n$ identity matrix I, so there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_t$ such that $(E_t \cdots E_2E_1)A = I$. By Theorem 1.9, the two equations

$$(E_t \cdots E_2E_1)A = I \quad \text{and} \quad AC = I$$

imply that $E_t \cdots E_2E_1 = C$, so we have $CA = I$ also. ▲

Computation of Inverses

Let $A = [a_{ij}]$ be an $n \times n$ matrix. To find A^{-1}, if it exists, we must find an $n \times n$ matrix $X = [x_{ij}]$ such that $AX = I$—that is, such that

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & \vdots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{3}$$

Matrix equation (3) corresponds to n^2 linear equations in the n^2 unknowns x_{ij}; there is one linear equation for each of the n^2 positions in an $n \times n$ matrix. For example, equating the entries in the row 2, column 1 position on each side of Eq. (3), we obtain the linear equation

$$a_{21}x_{11} + a_{22}x_{21} + \cdots + a_{2n}x_{n1} = 0.$$

Of these n^2 linear equations, n of them involve the n unknowns x_{i1} for $i = 1, 2, \ldots, n$; and these equations are given by the column-vector equation

$$A\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4}$$

which is a square system of equations. There are also n equations involving the n unknowns x_{i2}, for $i = 1, 2, \ldots, n$; and so on. In addition to solving system (4), we must solve the systems

$$A\begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, A\begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{5}$$

where each system has the same coefficient matrix A. Whenever we want to solve several systems $A\mathbf{x} = \mathbf{b}_i$ with the same coefficient matrix but different vectors $\mathbf{b}_i$, we solve them all at once, rather than one at a time. The main job in solving a linear system is reducing the coefficient matrix to row-echelon or reduced row-echelon form, and we don't want to repeat that work over and over. We simply reduce one augmented matrix, where we line up all the vectors $\mathbf{b}_i$ to the right of the partition. Thus, to solve all the linear systems in Eqs. (4) and (5), we form the augmented matrix

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right], \tag{6}$$

which we abbreviate by $[A \mid I]$. The matrix A is to the left of the partition, and the identity matrix I is to the right. We then perform a Gauss–Jordan reduction on this augmented matrix. By Theorem 1.9, we know that if A^{-1} exists, it is *unique*, so that every column in the reduced row-echelon form of A has a pivot. Thus, A^{-1} exists if and only if the augmented matrix (6) can be reduced to

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 1 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ & \vdots & & & & \vdots & & \\ 0 & 0 & \cdots & 1 & c_{n1} & c_{n2} & \cdots & c_{nn} \end{array}\right],$$

where the $n \times n$ identity matrix I is to the left of the partition. The $n \times n$ solution matrix $C = [c_{ij}]$ to the right of the partition then satisfies $AC = I$, so $A^{-1} = C$. This is an efficient way to compute A^{-1}. We summarize the computation in the following box, and we state the theory in Theorem 1.12.

Computation of A^{-1}

To find A^{-1}, if it exists, proceed as follows:

Step 1 Form the augmented matrix $[A \mid I]$.

Step 2 Apply the Gauss–Jordan method to attempt to reduce $[A \mid I]$ to $[I \mid C]$. If the reduction can be carried out, then $A^{-1} = C$. Otherwise, A^{-1} does not exist.

EXAMPLE 2 For the matrix $A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$, compute the inverse we exhibited at the start of this section, and use this inverse to solve the linear system

$$\begin{aligned} 2x + 9y &= -5 \\ x + 4y &= 7. \end{aligned}$$

SOLUTION Reducing the augmented matrix, we have

$$\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 9 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -4 & 9 \\ 0 & 1 & 1 & -2 \end{array}\right].$$

Therefore,

$$A^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}.$$

If A^{-1} exists, the solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. Consequently, the solution of our system

$$\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 83 \\ -19 \end{bmatrix}.$$ ∎

We emphasize that the computation of the solution of the linear system in Example 2, using the inverse of the coefficient matrix, was for illustration only. When faced with the problem of solving a square system, $A\mathbf{x} = \mathbf{b}$, one should never start by finding the inverse of the coefficient matrix. To do so would involve row reduction of $[A \mid I]$ and subsequent computation of $A^{-1}\mathbf{b}$, whereas the shorter reduction of $[A \mid \mathbf{b}]$ provides the desired solution at once. The inverse of a matrix is often useful in symbolic computations. For example, if A is an invertible matrix and we know that $AB = AC$, then we can deduce that $B = C$ by multiplying both sides of $AB = AC$ on the left by A^{-1}. If we have r systems of equations

$$A\mathbf{x}_1 = \mathbf{b}_1, \qquad A\mathbf{x}_2 = \mathbf{b}_2, \qquad \ldots, \qquad A\mathbf{x}_r = \mathbf{b}_r,$$

all with the same invertible $n \times n$ coefficient matrix A, it might seem to be more efficient (for large r) to solve all the systems by finding A^{-1} and computing the column vectors

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1, \qquad \mathbf{x}_2 = A^{-1}\mathbf{b}_2, \qquad \ldots, \qquad \mathbf{x}_r = A^{-1}\mathbf{b}_r.$$

Section 10.1 will show that using the Gauss method with back substitution on the augmented matrix $[A \mid \mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_r]$ remains more efficient. Thus, inversion of a coefficient matrix is not a good numerical way to solve a linear system. However, we will find inverses very useful for solving other kinds of problems.

THEOREM 1.12 Conditions for A^{-1} to Exist

The following conditions for an $n \times n$ matrix A are equivalent:

(i) A is invertible.

(ii) A is row equivalent to the identity matrix I.

(iii) The system $A\mathbf{x} = \mathbf{b}$ has a solution for each n-component column vector $\mathbf{b}$.

(iv) A can be expressed as a product of elementary matrices.

(v) The span of the column vectors of A is $\mathbb{R}^n$.

PROOF Step 2 in the box preceding Example 2 shows that parts (i) and (ii) of Theorem 1.12 are equivalent. For the equivalence of (ii) with (iii), Lemma 1.1 shows that $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$ if and only if (ii) is true. Thus, (ii) and (iii) are equivalent. The equivalence of (iii) and (v) follows from the box on page 63.

Turning to the equivalence of parts (ii) and (iv), we know that the matrix A is row equivalent to I if and only if there is a sequence of elementary matrices $E_1, E_2, \ldots, E_t$ such that $E_t \cdots E_2E_1A = I$; and this is the case if and only if A is expressible as a product $A = E_1^{-1}E_2^{-1} \cdots E_t^{-1}$ of elementary matrices. ▲

EXAMPLE 3 Using Example 2, express $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ as a product of elementary matrices.

SOLUTION The steps we performed in Example 2 can be applied in sequence to 2×2 identity matrices to generate elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \textbf{Interchange rows 1 and 2.}$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \textbf{Add } -2 \textbf{ times row 1 to row 2.}$$

$$E_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}. \quad \textbf{Add } -4 \textbf{ times row 2 to row 1.}$$

Thus we see that $E_3E_2E_1A = I$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

■

Example 3 illustrates the following boxed rule for expressing an invertible matrix A as a product of elementary matrices.

> **Expressing an Invertible Matrix A as a Product of Elementary Matrices**
>
> Write in left-to-right order the inverses of the elementary matrices corresponding to successive row operations that reduce A to I.

EXAMPLE 4 Determine whether the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix}$$

is invertible, and find its inverse if it is.

SOLUTION We have

$$\left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 14 & -8 & -1 \\ 0 & 1 & 0 & -17 & 10 & 1 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right].$$

Therefore, A is an invertible matrix, and

$$A^{-1} = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}. \quad \blacksquare$$

EXAMPLE 5 Express the matrix A of Example 4 as a product of elementary matrices.

SOLUTION In accordance with the box that follows Example 3, we write in left-to-right order the successive inverses of the elementary matrices corresponding to the row reduction of A in Example 4. We obtain

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \quad \blacksquare$$

EXAMPLE 6 Determine whether the span of the vectors $[1, -2, 1]$, $[3, -5, 4]$, and $[4, -3, 9]$ is all of $\mathbb{R}^3$.

SOLUTION Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & -5 & -3 \\ 1 & 4 & 9 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 3 & 4 \\ -2 & -5 & -3 \\ 1 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

We do not have a pivot in the row 3, column 3 position, so we are not able to reduce A to the identity matrix. By Theorem 1.12, the span of the given vectors is not all of $\mathbb{R}^3$. ■

SUMMARY

1. Let A be a square matrix. A square matrix C such that $CA = AC = I$ is the inverse of A and is denoted by $C = A^{-1}$. If such an inverse of A exists, then A is said to be *invertible*. The inverse of an invertible matrix A is unique. A square matrix that has no inverse is called *singular*.
2. The inverse of a square matrix A exists if and only if A can be reduced to the identity matrix I by means of elementary row operations or (equivalently) if and only if A is a product of elementary matrices. In this case, A is equal to the product, in left-to-right order, of the inverses of the successive