5. The volume of the box determined by nonzero vectors **a**, **b**, and **c** in $\mathbb{R}^3$ is the absolute value of the determinant of the matrix having row vectors **a**, **b**, and **c**. This determinant is also equal to $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

EXERCISES

In Exercises 1–4, find the indicated determinant.

1. $\begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix}$

2. $\begin{vmatrix} -1 & 0 \\ 0 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 0 & -3 \\ 5 & 0 \end{vmatrix}$

4. $\begin{vmatrix} 21 & -4 \\ 10 & 7 \end{vmatrix}$

5. Show that the vector $\mathbf{p} = \mathbf{b} \times \mathbf{c}$ given in Eq. (3) is perpendicular to both **b** and **c**.

In Exercises 6–9, find the indicated determinant.

6. $\begin{vmatrix} 1 & 4 & -2 \\ 5 & 13 & 0 \\ 2 & -1 & 3 \end{vmatrix}$

7. $\begin{vmatrix} 2 & -5 & 3 \\ 1 & 3 & 4 \\ -2 & 3 & 7 \end{vmatrix}$

8. $\begin{vmatrix} 1 & -2 & 7 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}$

9. $\begin{vmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \\ 2 & 1 & -4 \end{vmatrix}$

10. Show by direct computation that:

 a. $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0;$

 b. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0.$

11. Show by direct computation that

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}.$$

12. Show by direct computation that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

In Exercises 13–18, find $\mathbf{a} \times \mathbf{b}$.

13. $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$

14. $\mathbf{a} = -5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

15. $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

16. $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

17. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

18. $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 6\mathbf{j} + \mathbf{k}$

19. Mark each of the following True or False.

___ a. The determinant of a 2×2 matrix is a vector.

___ b. If two rows of a 3×3 matrix are interchanged, the sign of the determinant is changed.

___ c. The determinant of a 3×3 matrix is zero if two rows of the matrix are parallel vectors in $\mathbb{R}^3$.

___ d. In order for the determinant of a 3×3 matrix to be zero, two rows of the matrix must be parallel vectors in $\mathbb{R}^3$.

___ e. The determinant of a 3×3 matrix is zero if the points in $\mathbb{R}^3$ given by the rows of the matrix lie in a plane.

___ f. The determinant of a 3×3 matrix is zero if the points in $\mathbb{R}^3$ given by the rows of the matrix lie in a plane through the origin.

___ g. The parallelogram in $\mathbb{R}^2$ determined by nonzero vectors **a** and **b** is a square if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

___ h. The box in $\mathbb{R}^3$ determined by vectors **a**, **b**, and **c** is a cube if and only if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c}$.

___ i. If the angle between vectors **a** and **b** in $\mathbb{R}^3$ is $\pi/4$, then $\|\mathbf{a} \times \mathbf{b}\| = |\mathbf{a} \cdot \mathbf{b}|$.

___ j. For any vector **a** in $\mathbb{R}^3$, we have $\|\mathbf{a} \times \mathbf{a}\| = \|\mathbf{a}\|^2$.

In Exercises 20–24, find the area of the parallelogram with vertex at the origin and with the given vectors as edges.

20. $-\mathbf{i} + 4\mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j}$

21. $-5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} + 7\mathbf{j}$

22. $\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

23. $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

24. $\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

In Exercises 25–32, find the area of the given geometric configuration.

25. The triangle with vertices $(-1, 2)$, $(3, -1)$, and $(4, 3)$

26. The triangle with vertices $(3, -4)$, $(1, 1)$ and $(5, 7)$

27. The triangle with vertices $(2, 1, -3)$, $(3, 0, 4)$, and $(1, 0, 5)$

28. The triangle with vertices $(3, 1, -2)$, $(1, 4, 5)$, and $(2, 1, -4)$

29. The triangle in the plane $\mathbb{R}^2$ bounded by the lines $y = x$, $y = -3x + 8$, and $3y + 5x = 0$

30. The parallelogram with vertices $(1, 3)$, $(-2, 6)$, $(1, 11)$, and $(4, 8)$

31. The parallelogram with vertices $(1, 0, 1)$, $(3, 1, 4)$, $(0, 2, 9)$, and $(-2, 1, 6)$

32. The parallelogram in the plane $\mathbb{R}^2$ bounded by the lines $x - 2y = 3$, $x - 2y = 10$, $2x + 3y = -1$, and $2x + 3y = -8$

In Exercises 33–36, find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ *and* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

33. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{k}$

34. $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

35. $\mathbf{a} = \mathbf{i} - 3\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 4\mathbf{j}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

36. $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$

In Exercises 37–40, find the volume of the box having the given vectors as adjacent edges.

37. $-\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $4\mathbf{i} + 2\mathbf{k}$

38. $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}$

39. $-2\mathbf{i} + \mathbf{j}$, $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{i} - 2\mathbf{k}$

40. $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$, $5\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$

In Exercises 41–44, find the volume of the tetrahedron having the given vertices. (Consider how the volume of a tetrahedron having three vectors from one point as edges is related to the volume of the box having the same three vectors as adjacent edges.)

41. $(-3, 0, 1)$, $(4, 2, 1)$, $(0, 1, 7)$, $(1, 1, 1)$

42. $(0, 1, 1)$, $(8, 2, -7)$, $(3, 1, 6)$, $(-4, -2, 0)$

43. $(-1, 1, 2)$, $(3, 1, 4)$, $(-1, 6, 0)$, $(2, -1, 5)$

44. $(-1, 2, 4)$, $(2, -3, 0)$, $(-4, 2, -1)$, $(0, 3, -2)$

In Exercises 45–48, use a determinant to ascertain whether the given points lie on a line in $\mathbb{R}^2$. [HINT: *What is the area of a "parallelogram" with collinear vertices?*]

45. $(0, 0)$, $(3, 5)$, $(6, 9)$

46. $(0, 0)$, $(4, 2)$, $(-6, -3)$

47. $(1, 5)$, $(3, 7)$, $(-3, 1)$

48. $(2, 3)$, $(1, -4)$, $(6, 2)$

In Exercises 49–52, use a determinant to ascertain whether the given points lie in a plane in $\mathbb{R}^3$. [HINT: *What is the "volume" of a box with coplanar vertices?*]

49. $(0, 0, 0)$, $(1, 4, 3)$, $(2, 5, 8)$, $(-1, 2, -5)$

50. $(0, 0, 0)$, $(2, 1, 1)$, $(3, -2, 1)$, $(-1, 2, 3)$

51. $(1, -1, 3)$, $(4, 2, 3)$, $(3, 1, -2)$, $(5, 5, -5)$

52. $(1, 2, 1)$, $(3, 3, 4)$, $(2, 2, 2)$, $(4, 3, 5)$

Let $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *be any vectors in* $\mathbb{R}^3$. *In Exercises 53–56, simplify the given expression.*

53. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$

54. $(\mathbf{b} \times \mathbf{c}) - (\mathbf{c} \times \mathbf{b})$

55. $\|\mathbf{a} \times \mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2$

56. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$

57. Prove property (2) of Theorem 4.1.

58. Prove property (3) of Theorem 4.1.

59. Prove property (6) of Theorem 4.1.

60. Option 7 of the routine VECTGRPH in LINTEK provides drill on the determinant of a 2×2 matrix as the area of the parallelogram determined by its row vectors, with an associated plus or minus sign. Run this option until you can regularly achieve a score of 80% or better.

MATLAB has a function **det(A)** *which gives the determinant of a matrix A. In Exercises 61–63, use the routine MATCOMP in LINTEK or MATLAB to find the volume of the box having the given vectors in $\mathbb{R}^3$ as adjacent edges. (We have not supplied matrix files for these problems.)*

61. $-\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$, $4\mathbf{i} + 23\mathbf{j} - 13\mathbf{k}$, $12\mathbf{i} - 17\mathbf{j} - 31\mathbf{k}$

62. $4.1\mathbf{i} - 2.3\mathbf{k}$, $5.3\mathbf{j} - 2.1\mathbf{k}$, $6.1\mathbf{i} + 5.7\mathbf{j}$

63. $2.13\mathbf{i} + 4.71\mathbf{j} - 3.62\mathbf{k}$, $5\mathbf{i} - 3.2\mathbf{j} + 6.32\mathbf{k}$, $8.3\mathbf{i} - 0.45\mathbf{j} + 1.13\mathbf{k}$

MATLAB

M1. Enter the data vectors **x** = [1 5 7] and **y** = [−3 2 4] into MATLAB. Then enter a line **crossxy** = [ ], which will compute the cross product $\mathbf{x} \times \mathbf{y}$ of vectors [x(1) x(2) x(3)] and [y(1) y(2) y(3)]. [Hint: The first component in [ ] will be **x(2)*y(3)−y(2)*x(3)**.] Be sure you use no spaces except one between the vector components. Check that the value given for **crossxy** is the correct vector $6\mathbf{i} - 25\mathbf{j} + 17\mathbf{k}$ for the data vectors entered.

M2. Use the **norm** function in MATLAB to find the area of the parallelogram in $\mathbb{R}^3$ having the vectors **x** and **y** in the preceding exercise as adjacent edges.

M3. Enter the vectors $\mathbf{x} = 4.2\mathbf{i} - 3.7\mathbf{j} + 5.6\mathbf{k}$ and $\mathbf{y} = -7.3\mathbf{i} + 4.5\mathbf{j} + 11.4\mathbf{k}$.
- **a.** Using the up-arrow key to access your line defining **crossxy**, find $\mathbf{x} \times \mathbf{y}$.
- **b.** Find the area of the parallelogram in $\mathbb{R}^3$ having **x** and **y** as adjacent edges.

M4. Find the area of the triangle in $\mathbb{R}^3$ having vertices $(-1.2, 3.4, -6.7)$, $(2.3, -5.2, 9.4)$, and $(3.1, 8.3, -3.6)$. [Hint: Enter vectors **a**, **b**, and **c** from the origin to these points and set **x** and **y** equal to appropriate differences of them.]

NOTE: If you want to add a function **cross(x, y)** to your own personal MATLAB, do so following a procedure analogous to that described at the very end of Section 1.2 for adding the function **angl(x, y)**.

4.2 THE DETERMINANT OF A SQUARE MATRIX

The Definition

We defined a third-order determinant in terms of second-order determinants in Eq. (5) on page 245. A second-order determinant can be defined in terms of first-order determinants if we interpret the determinant of a 1×1 matrix to be its sole entry. We define an nth-order determinant in terms of determinants of order $n - 1$. In order to facilitate this, we introduce the **minor matrix** A_{ij} of an $n \times n$ matrix $A = [a_{ij}]$; it is the $(n - 1) \times (n - 1)$ matrix obtained by cross