應數三數值分析 2020 秋, 第一次期中考解答

學號: ________________________, 姓名: ________________________

本次考試共有 7 頁 (包含封面)，有 9 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ________________________

1. (10 points) Let $x = 2/3$, $y = 4/7$. Find $x \oslash y = fl(fl(x) \div fl(y))$ by using 4-digit chopping arithmetic.

$x \oslash y = fl(fl(2/3) \div fl(4/7)) = fl(0.6666/0.5714) = 1.1666 = 0.1166 \times 10^1$

2. (10 points) Neville's method is used to approximate $f(0.5)$ as follows. Complete the table.

i	x_i	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$
0	0	$Q_{0,0} = 0$		
1	0.4	$Q_{1,0} = 2.8$	$Q_{1,1} = 3.5$	
2	0.7	$Q_{2,0} =?$	$Q_{2,1} =?$	$Q_{2,2} = \frac{27}{7}$

Check the quiz 3 problem 2. $Q_{2,0} = 6.4, Q_{2,1} = 4$

3. (10 points) Use the Newton's Forward Difference Formula to approximate $\sqrt{2}$ with the function $f(x) = 2^x$ and the values list on the table. Also, compute the absolute error and relative error in this approximation.

i	x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$
0	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	1	1	1	
2	1	2	2		
3	2	4			

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	-1	$\frac{1}{2}$	$\frac{1-1/2}{0-(-1)} = \frac{1}{2}$	$\frac{1-\frac{1}{2}}{1-(-1)} = \frac{1}{4}$	$\frac{\frac{1}{2}-\frac{1}{4}}{2-(-1)} = \frac{1}{12}$
1	0	1	$\frac{2-1}{1-0} = 1$	$\frac{2-1}{2-0} = \frac{1}{2}$	
2	1	2	$\frac{4-2}{2-1} = 2$		
3	2	4			

$$f[x_i, x_{i+1}, ..., x_{i+k}] = \frac{1}{k!h^k}\Delta^k f(x_i)$$

Since $\sqrt{2} = 2^{0.5}$, $x = 0.5$. $h = 1$, Also, $s = \frac{x-x_0}{h} = \frac{0.5-(-1)}{0-(-1)} = 1.5$

$$f(0.5) \approx P(0.5)$$

$$(a) = f[x_0] + \sum_{k=1}^{3} f[x_0, x_1, ..., x_k] \prod_{i=0}^{k-1}(x - x_i)$$

$$(b) = f(x_0) + \frac{s}{1}\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-1)}{3!}\Delta^3 f(x_0)$$

$$= \frac{45}{32} = 1.40625$$

absolute error is $|\sqrt{2} - \frac{45}{32}| = 0.007963$ and relative error is $\left|\frac{\sqrt{2} - \frac{45}{32}}{\sqrt{2}}\right| = 0.005631$

4. (10 points) Use four steps of the Bisection Method to find an approximate root of $\sin(x) = 0.8x$ start with $a = 1, b = 1.5$.

Let $f(x) = \sin x - 0.8x = 0$. Since $f(a_0) = f(1) = \sin(1) - 0.8(1) = 0.041471 > 0$ and $f(b_0) = f(1.5) = \sin(1.5) - 0.8(1.5) = -0.20251 < 0$, this means $f(1)f(1.5) < 0$, there exist a root of f in $[1, 1.5]$. So we can use the Bisection Method:

n	a_n	b_n	$p_n = \frac{a_n+b_n}{2}$	$f(p_n)$
0	1	$\frac{3}{2} = 1.5$	$\frac{5}{4} = 1.25$	$-0.051015 < 0$
1	1	$\frac{5}{4} = 1.25$	$\frac{9}{8} = 1.125$	$0.0022676 > 0$
2	$\frac{9}{8} = 1.125$	$\frac{5}{4} = 1.25$	$\frac{19}{16} = 1.1875$	$-0.022563 < 0$
3	$\frac{9}{8} = 1.125$	$\frac{19}{16} = 1.1875$	$\frac{37}{32} = 1.15625$	-0.0097207

So the root is approximately $\frac{37}{32} = 1.15625$

5. (10 points) Use the Newton's Method to find a solution within $\epsilon = 10^{-4}$ for the function $f(x) = x - 0.8 - 0.2\sin(x) = 0$ where $0 \le x \le \frac{\pi}{2}$, starting with $p_0 = 0$

Since $f(x) = x - 0.8 - 0.2\sin(x) = 0 \Rightarrow f'(x) = 1 - 0.2\cos(x) \Rightarrow f'(0) \neq 0$, thus we can use the Newton Method. Note

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

n	p_n	$f(p_n)$
1	1	$0.03170580303842063 > 10^{-4}$
2	0.9644529683254768	$0.00010550745317697285 > 10^{-4}$
3	0.9643338890103158	$1.165181867657239 \times 10^{-9} < 10^{-4}$

So the root is approximately 0.9643338890103158

6. (10 points) Let $P_3(x)$ be the interpolating polynomial for the data $(0,0),(1,y),(2,3)$ and $(3,5)$. The coefficient of x^3 in $P_3(x)$ is 6. Find y.

x_i	0	1	2	3
$f(x_i)$	0	y	3	5

$$
\begin{aligned}
P_3(x) &= \sum_{k=0}^{3} f(x_k) \prod_{i=0}^{3} \frac{(x-x_i)}{(x_k-x_i)} \\
&= 0\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + y\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} + 3\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} + 5\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} \\
&= 0 + y\frac{(x-0)(x-2)(x-3)}{2} + 3\frac{(x-0)(x-1)(x-3)}{-2} + 5\frac{(x-0)(x-1)(x-2)}{6}
\end{aligned}
$$

The coefficient of x^3 in $P_3(x)$ is $\frac{y}{2} - \frac{3}{2} + \frac{5}{6} = 6$, hence $y = \frac{40}{3}$

7. (10 points) Let $f(x) = e^{2x}$. Find the Hermite polynomial that agrees with the function and its derivative at the points $x_0 = 0, x_1 = 0.5$. Then use your function to approximate $f(0.43)$

solution 1

z_i	$f(z_i)$	$f[z_i, z_{i+1}]$	$f[z_i, z_{i+1}, z_{i+2}]$	$f[z_i, z_{i+1}, z_{i+2}, z_{i+3}]$
$z_0 = 0$	1	$2e^0 = 2$	$\frac{2(e-1)-2}{0.5-0} = 2.87312$	$\frac{4-(4e-8)}{0.5-0} = 2.25376$
$z_1 = 0$	1	$\frac{e-1}{0.5-0} = 3.43656$	$\frac{2e-2(e-1)}{0.5-0} = 4$	
$z_2 = 0.5$	$e = 2.71828$	$2e^{2\times 0.5} = 5.43656$		
$z_3 = 0.5$	$e = 2.71828$			

$$
\begin{aligned}
H_{2n+1}(x) &= f[z_0] + \sum_{k=1}^{2n+1} f[z_0, ..., z_k](x-z_0)(x-z_1)...(x-z_{k-1}) \\
&= 1 + 2e^0(x-0) + 2.87312(x-0)^2 + 2.25376(x-0)^2(x-0.5)
\end{aligned}
$$

$$H_{2n+1}(0.43) = 2.362069472$$

8. (15 points) The iterative method to solve $f(x) = 0$, given by the fixed-point method $g(x) = x$, where

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} - \frac{f''(p_{n-1})}{2f'(p_{n-1})}\left[\frac{f(p_{n-1})}{f'(p_{n-1})}\right]^2 \text{ for } n = 1, 2, 3, ...,$$

has $g'(p) = g''(p) = 0$. This will generally yield cubic ($\alpha = 3$) convergence.

Expanding $g(x)$ in Taylor polynomial for $x \in [p-\delta, p+\delta]$ gives

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(\xi)}{3!}(x-p)^3$$

where ξ lies between x and p. The problem gave $g'(p) = g''(p) = 0$ imply that

$$g(x) = p + \frac{g'''(\xi)}{6}(x-p)^3$$

In particular, when $x = p_n$

$$p_{n+1} = g(p_n) = p + \frac{g'''(\xi_n)}{6}(p_n - p)^3$$

with ξ_n lies between p_n and p. Thus

$$p_{n+1} - p = \frac{g'''(\xi_n)}{6}(p_n - p)^3$$

Since

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)}\left[\frac{f(x)}{f'(x)}\right]^2 \text{ and } f(x) = e^x - x - 1$$

, we have $|g'(x)| \le k < 1$ on $[p-\delta, p+\delta]$ and g maps $[p-\delta, p+\delta]$ into itself, it follows from the Fixed-Point Theorem that $\{p_n\}_{n=0}^{\infty}$ converges to p. But ξ_n is between p and p_n for each n, so $\{\xi_n\}_{n=0}^{\infty}$ also converges to p, and we have

$$\lim_{n\to\infty} g'''(\xi_n) = g'''(p)$$

Thus

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lim_{n\to\infty}\frac{|g'''(\xi_n)||p_n-p|^{3-\alpha}}{6} = \frac{|g'''(p)|\times 0}{6} = 0, \text{ for } \alpha = 1, 2$$

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^3} = \lim_{n\to\infty}\frac{|g'''(\xi_n)|}{6} = \frac{|g'''(p)|}{6}$$

Hence, if $g'''(p) \neq 0$, fixed-point iteration exhibits cubic convergence with asymptotic error constant $|g'''(p)|$.

9. (15 points) Show that $g(x) = 2^{-x}$ has an unique fixed point in the interval $[\frac{1}{3}, 1]$. Also estimate the number of iterations required to achieve 10^{-4} accuracy.
Hint: first show the existence and then the uniqueness.

Since $|g'(x)| = \dfrac{\ln 2}{2^{-x}}$, g is continuous and g' exists on $[\frac{1}{3}, 1]$. Further, $g'(x) \neq 0$ on $[1/3, 1]$, so that $1 \geq g(\frac{1}{3}) = 0.7937 \geq g(x) \geq g(1) = 1/2 = 0.5 > \frac{1}{3}$ and $|g'(x)| \leq k = 0.5502$, for $\frac{1}{3} \leq x \leq 1$. Theorem 2.3 implies that a unique fixed point p exists in $[\frac{1}{3}, 1]$.

With $k = 0.5502$ and $p_0 = 1$, we have $p_1 = \frac{1}{2}$. Corollary 2.5 implies that

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| = \frac{1}{2 \times (1 - 0.5502)}(0.5502)^n$$

For the bound to be less than 10^{-4}, we need $n \geq 16$.
Note: this number is depend on p_0.

應數三數值分析, 第一次期中考, 學號: ____________, 姓名: ____________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	15	15	100
Score:										