we see that

$$d_1 = 0, \quad d_2 = -1, \quad \text{and} \quad d_3 = 1.$$

Thus, Eq. (8) has the form

$$T^k(x + 2) = 2^k(-1)x^2 + 4^k(1)(x^2 + x + 2).$$

In particular,

$$T^4(x + 2) = -16x^2 + 256(x^2 + x + 2) = 240x^2 + 256x + 512. \qquad \blacksquare$$

SUMMARY

Let $T: V \to V$ be a linear transformation of a finite-dimensional vector space into itself.

1. If B and B' are ordered bases of V, then the matrix representations R_B and $R_{B'}$ of T relative to B and to B' are similar. That is, there is an invertible matrix C—namely, $C = C_{B',B}$—such that
$$R_{B'} = C^{-1}R_BC.$$
2. Conversely, two similar $n \times n$ matrices represent the same linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^n$ relative to two suitably chosen ordered bases.
3. Similar matrices have the same eigenvalues with the same algebraic and geometric multiplicities.
4. If A and R are similar matrices with $R = C^{-1}AC$, and if $\mathbf{v}$ is an eigenvector of A, then $C^{-1}\mathbf{v}$ is an eigenvector of R corresponding to the same eigenvalue.
5. The transformation T is diagonalizable if V has a basis B consisting of eigenvectors of T. In this case, the matrix representation R_B is a diagonal matrix and computation of $T^k(\mathbf{v})$ by $R_B{}^k(\mathbf{v}_B)$ becomes relatively easy.

EXERCISES

In Exercises 1–14, find the matrix representations R_B and $R_{B'}$ and an invertible matrix C such that $R_{B'} = C^{-1}R_BC$ for the linear transformation T of the given vector space with the indicated ordered bases B and B'.

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [x - y, x + 2y]$; $B = ([1, 1], [2, 1])$, $B' = E$
2. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [2x + 3y, x + 2y]$; $B = ([1, -1], [1, 1])$, $B' = ([2, 3], [1, 2])$
3. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T([x, y, z]) = [x + y, x + z, y - z]$; $B = ([1, 1, 1], [1, 1, 0], [1, 0, 0])$, $B' = E$
4. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T([x, y, z]) = [5x, 2y, 3z]$; B and B' as in Exercise 3
5. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T([x, y, z]) = [z, 0, x]$; $B = ([3, 1, 2], [1, 2, 1], [2, -1, 0])$, $B' = ([1, 2, 1], [2, 1, -1], [5, 4, 1])$
6. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as reflection of the plane through the line $5x = 3y$; $B = ([3, 5], [5, -3])$, $B' = E$

7. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as reflection of $\mathbb{R}^3$ through the plane $x + y + z = 0$; $B = ([1, 0, -1], [1, -1, 0], [1, 1, 1])$, $B' = E$
8. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as reflection of $\mathbb{R}^3$ through the plane $2x + 3y + z = 0$; $B = ([2, 3, 1], [0, 1, -3], [1, 0, -2])$, $B' = E$
9. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as projection on the plane $x_2 = 0$; $B = E$, $B' = ([1, 0, 1], [1, 0, -1], [0, 1, 0])$
10. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as projection on the plane $x + y + z = 0$; B and B' as in Exercise 7.
11. $T: P_2 \to P_2$ defined by $T(p(x)) = p(x + 1) + p(x)$; $B = (x^2, x, 1)$, $B' = (1, x, x^2)$
12. $T: P_2 \to P_2$ as in Exercise 11, but using $B = (x^2, x, 1)$ and $B' = (x^2 + 1, x + 1, 2)$
13. $T: P_3 \to P_3$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$; $B = (x^3, x^2, x, 1)$, $B' = (1, x + 1, x^2 + 1, x^3 + 1)$
14. $T: W \to W$, where $W = \text{sp}(e^x, xe^x)$ and T is the derivative transformation; $B = (e^x, xe^x)$, $B' = (2xe^x, 3e^x)$
15. Let $T: P_2 \to P_2$ be the linear transformation and B' the ordered basis of P_2 given in Example 3. Find the matrix representation $R_{B'}$ of T by computing the matrix with column vectors $T(\mathbf{b}_1')_{B'}$, $T(\mathbf{b}_2')_{B'}$, $T(\mathbf{b}_3')_{B'}$.
16. Repeat Exercise 15 for the transformation in Example 5 and the basis $B' = (x + 1, -1, x^2 + x)$ of P_2.

In Exercises 17–22, find the eigenvalues λ_i and the corresponding eigenspaces of the linear transformation T. Determine whether the linear transformation is diagonalizable.

17. T defined on $\mathbb{R}^2$ by $T([x, y]) = [2x - 3y, -3x + 2y]$
18. T defined on $\mathbb{R}^2$ by $T([x, y]) = [x - y, -x + y]$
19. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1 + x_3, x_2, x_1 + x_3]$
20. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1, 4x_2 + 7x_3, 2x_2 - x_3]$
21. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [5x_1, -5x_1 + 3x_2 - 5x_3, -3x_1 - 2x_2]$
22. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [3x_1 - x_2 + x_3, -2x_1 + 2x_2 - x_3, 2x_1 + x_2 + 4x_3]$
23. Mark each of the following True or False

___ a. Two similar $n \times n$ matrices represent the same linear transformation of $\mathbb{R}^n$ into itself relative to the standard basis.

___ b. Two different $n \times n$ matrices represent different linear transformations of $\mathbb{R}^n$ into itself relative to the standard basis.

___ c. Two similar $n \times n$ matrices represent the same linear transformation of $\mathbb{R}^n$ into itself relative to two suitably chosen bases for $\mathbb{R}^n$.

___ d. Similar matrices have the same eigenvalues and eigenvectors.

___ e. Similar matrices have the same eigenvalues with the same algebraic and geometric multiplicities.

___ f. If A and C are $n \times n$ matrices and C is invertible and $\mathbf{v}$ is an eigenvector of A, then $C^{-1}\mathbf{v}$ is an eigenvector of $C^{-1}AC$.

___ g. If A and C are $n \times n$ matrices and C is invertible and $\mathbf{v}$ is an eigenvector of A, then $C\mathbf{v}$ is an eigenvector of CAC^{-1}.

___ h. Any two $n \times n$ diagonal matrices are similar.

___ i. Any two $n \times n$ diagonalizable matrices having the same eigenvectors are similar.

___ j. Any two $n \times n$ diagonalizable matrices having the same eigenvalues of the same algebraic multiplicities are similar.

24. Prove statement 2 of Theorem 7.2.
25. Prove statement 3 of Theorem 7.2.
26. Let A and R be similar matrices. Prove in two ways that A^2 and R^2 are similar matrices: using a matrix argument, and using a linear transformation argument.
27. Give a determinant proof that similar matrices have the same eigenvalues.