EXERCISES

In Exercises 1–8, find the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{v}_i$ of the given matrix A, and also find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

1. $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$

4. $A = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$

5. $A = \begin{bmatrix} -3 & 10 & -6 \\ 0 & 7 & -6 \\ 0 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$

7. $A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} -4 & 6 & -12 \\ 3 & -1 & 6 \\ 3 & -3 & 8 \end{bmatrix}$

In Exercises 9–12, determine whether the given matrix is diagonalizable.

9. $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 4 & 2 & -7 \\ 0 & 5 & -3 & 6 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 2 & 5 & 1 \\ 2 & 0 & 2 & 6 \\ 5 & 2 & 7 & -1 \\ 1 & 6 & -1 & 3 \end{bmatrix}$

13. Mark each of the following True or False.
___ a. Every $n \times n$ matrix is diagonalizable.
___ b. If an $n \times n$ matrix has n distinct real eigenvalues, it is diagonalizable.
___ c. Every $n \times n$ real symmetric matrix is real diagonalizable.
___ d. An $n \times n$ matrix is diagonalizable if and only if it has n distinct eigenvalues.
___ e. An $n \times n$ matrix is diagonalizable if and only if the algebraic multiplicity of each of its eigenvalues equals the geometric multiplicity.
___ f. Every invertible matrix is diagonalizable.
___ g. Every triangular matrix is diagonalizable.
___ h. If A and B are similar square matrices and A is diagonalizable, then B is also diagonalizable.
___ i. If an $n \times n$ matrix A is diagonalizable, there is a unique diagonal matrix D that is similar to A.
___ j. If A and B are similar square matrices, then $\det(A) = \det(B)$.

14. Give two different diagonal matrices that are similar to the matrix $\begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix}$.

15. Prove that, if a matrix is diagonalizable, so is its transpose.

16. Let P, Q, and R be $n \times n$ matrices. Recall that P is similar to Q if there exists an invertible $n \times n$ matrix C such that $C^{-1}PC = Q$. This exercise shows that similarity is an *equivalence relation*.
    a. (*Reflexive.*) Show that P is similar to itself.
    b. (*Symmetric.*) Show that, if P is similar to Q, then Q is similar to P.
    c. (*Transitive.*) Show that, if P is similar to Q and Q is similar to R, then P is similar to R.

17. Prove that, for every square matrix A all of whose eigenvalues are real, the product of its eigenvalues is $\det(A)$.

18. Prove that similar square matrices have the same eigenvalues with the same algebraic multiplicities.

19. Let A be an $n \times n$ matrix.
    a. Prove that if A is similar to rA where r is a real scalar other than 1 or -1, then all eigenvalues of A are zero. [HINT: See the preceding exercise.]
    b. What can you say about A if it is diagonalizable and similar to rA for some r where $|r| \neq 1$?
    c. Find a nonzero 2×2 matrix A which is similar to rA for every $r \neq 0$. (See part a.)
    d. Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is similar to $-A$. (Observe that the eigenvalues of A are not all zero.)

20. Let A be a real **tridiagonal matrix**—that is, an $n \times n$ matrix for $n > 2$ all of whose entries are zero except possibly those of the

form $a_{i,i-1}$, a_{ii}, or $a_{i-1,i}$. Show that if $a_{i,i-1}$ and $a_{i-1,i}$ are both positive, both negative, or both zero for $i = 2, 3, \ldots, n$, then A has real eigenvalues. [HINT: Show that a diagonal matrix D can be found such that DAD^{-1} is real symmetric.]

21. Find a formula for the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reflects vectors in the line $y = mx$. [HINT: Proceed as in Example 2.]

22. Let A and C be $n \times n$ matrices, and let C be invertible. Prove that, if $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ, then $C^{-1}\mathbf{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalue λ. Then prove that all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\mathbf{v}$, where $\mathbf{v}$ is an eigenvector of A.

23. Explain how we can deduce from Exercise 22 that, if A and B are similar square matrices, each eigenvalue of A has the same geometric multiplicity for A that it has for B. (See Exercise 18 for the corresponding statement on algebraic multiplicities.)

24. Prove that, if $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct real eigenvalues of an $n \times n$ real matrix A and if B_i is a basis for the eigenspace E_{λ_i}, then the union of the bases B_i is an independent set of vectors in $\mathbb{R}^n$. [HINT: Make use of Theorem 5.3.]

25. Let $T: V \rightarrow V$ be a linear transformation of a vector space V into itself. Prove that, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of T corresponding to distinct nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is independent.

26. Show that the set $\{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}\}$, where the λ_i are distinct, is independent in the vector space W of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and having derivatives of all orders.

27. Using Exercise 26, show that the infinite set $\{e^{kx} \mid k \in \mathbb{R}\}$ is an independent set in the vector space W described in Exercise 26. [HINT: How many vectors are involved in any dependence relation?]

28. In Section 5.1, we stated that if we allow complex numbers, then an $n \times n$ matrix with entries chosen at random has eigenvectors that form a basis for n-space with probability 1. In light of our work in this section, give the best justification you can for this statement.

29. The MATLAB command

A = (2*r)*rand(n) − r*ones(n); eig(A)

should exhibit the eigenvalues of an $n \times n$ matrix A with random entries between $-r$ and r. Illustrate your argument in the preceding exercise for seven matrices with values of r from 1 to 50 and values of n from 2 to 8. (Recall that we used **rand(n)** in the MATLAB exercises of Section 1.6.)

In Exercises 30–41, use the routines MATCOMP and ALLROOTS in LINTEK or use MATLAB to classify the given matrix as real diagonalizable, complex (but not real) diagonalizable, or not diagonalizable. (Load the matrices from a matrix file if it is accessible.)

30. $\begin{bmatrix} 18 & 25 & -25 \\ 1 & 6 & -1 \\ 18 & 34 & -25 \end{bmatrix}$

31. $\begin{bmatrix} 8.3 & 8.0 & -6.0 \\ -2.0 & 0.3 & 3.0 \\ 0.0 & 0.0 & 4.3 \end{bmatrix}$

32. $\begin{bmatrix} 24.55 & 46.60 & 46.60 \\ -4.66 & -8.07 & -9.32 \\ -9.32 & -18.64 & -17.39 \end{bmatrix}$

33. $\begin{bmatrix} 0.8 & -1.6 & 1.8 \\ -0.6 & -3.8 & 3.4 \\ -20.6 & -1.2 & 9.6 \end{bmatrix}$

34. $\begin{bmatrix} 7 & -20 & -5 & 5 \\ 5 & -13 & -5 & 0 \\ -5 & 10 & 7 & 5 \\ 5 & -10 & -5 & -3 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 5 & -9 & 10 \\ 4 & 9 & 8 & -3 \\ 8 & 2 & 0 & 12 \\ 7 & -6 & 3 & 2 \end{bmatrix}$

36. $\begin{bmatrix} -22.7 & -26.9 & -6.3 & -46.5 \\ -59.7 & -40.9 & 20.9 & -99.5 \\ 15.9 & 9.6 & -8.4 & 26.5 \\ 43.8 & 36.5 & -7.3 & 78.2 \end{bmatrix}$

37. $\begin{bmatrix} 66.2 & 58.0 & -11.6 & 116.0 \\ 120.6 & 89.6 & -42.6 & 201.0 \\ -21.0 & -15.0 & 7.6 & -35.0 \\ -99.6 & -79.0 & 28.6 & -169.4 \end{bmatrix}$

38. $\begin{bmatrix} -253 & -232 & -96 & 1088 & 280 \\ 213 & 204 & 93 & -879 & -225 \\ -90 & -90 & -47 & 360 & 90 \\ -38 & -36 & -18 & 162 & 40 \\ 62 & 64 & 42 & -251 & -57 \end{bmatrix}$

39. $\begin{bmatrix} 154 & -24 & -36 & -1608 & -336 \\ -126 & 16 & 18 & 1314 & 270 \\ 54 & 0 & 4 & -540 & -108 \\ 24 & 0 & 0 & -236 & -48 \\ -42 & -12 & -18 & 366 & 70 \end{bmatrix}$

40. $\begin{bmatrix} -2513 & 596 & -414 & -2583 & 1937 \\ 127 & -32 & 33 & 132 & -81 \\ -421 & 94 & -83 & -434 & 306 \\ 2610 & -615 & 443 & 2684 & -1994 \\ 90 & -19 & 29 & 94 & -50 \end{bmatrix}$

41. $\begin{bmatrix} 2 & -4 & 6 & 2 & 1 \\ 6 & 3 & -8 & 1 & 2 \\ -4 & 0 & 5 & 1 & 2 \\ 4 & 3 & 5 & -11 & 3 \\ 6 & 7 & 9 & 2 & 3 \end{bmatrix}$

5.3 TWO APPLICATIONS

In this section, A will always denote a matrix with real-number entries.

In Section 5.1, we were motivated to introduce eigenvalues by our desire to compute $A^k\mathbf{x}$. Recall that we regard $\mathbf{x}$ as an initial information vector of some process, and A as a matrix that transforms, by left multiplication, an information vector at any stage of the process into the information vector at the next stage. In our first application, we examine this computation of $A^k\mathbf{x}$ and the significance of the eigenvalues of A. As illustration, we determine the behavior of the terms F_n of the Fibonacci sequence for large values of n. Our second application shows how diagonalization of matrices can be used to solve some linear systems of differential equations.

Application: Computing $A^k\mathbf{x}$

Let A be a real-diagonalizable $n \times n$ matrix, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n not necessarily distinct eigenvalues of A. That is, each eigenvalue of A is repeated in this list in accord with its algebraic multiplicity. Let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be an ordered basis for $\mathbb{R}^n$, where $\mathbf{v}_j$ is an eigenvector for λ_j. We have seen that, if C is the matrix having $\mathbf{v}_j$ as jth column vector, then

$$C^{-1}AC = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \Large 0 \\ & & \ddots & \\ \Large 0 & & & \lambda_n \end{bmatrix}.$$

For any vector $\mathbf{x}$ in $\mathbb{R}^n$, let $\mathbf{d} = [d_1, d_2, \ldots, d_n]$ be its coordinate vector relative to the basis B. Thus,

$$\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.$$