

應數一線性代數 2022 春, 期末 考SOLUTION

考試時間：2022/06/23, 09:10 - 12:00,

收卷截止時間：12:10

考卷繳交位置：Google Classroom

考試須知:

- 需要開鏡頭麥克風。鏡頭需要看得到你的身邊，你在作答的紙面，還有你在使用的電子資源的畫面（例如電腦螢幕或平板螢幕）。我不需要直接閱讀螢幕內容，我只要看看畫面的形狀色塊，確定你在看什麼就好。
- 請將紙面答案卷掃成一份 pdf 檔，畫面請清晰並且轉正。第一頁左上寫明姓名學號，每一題前面註明題號，頁面請按照題號順序編排不要跳號。
- 注意事先準備充足的紙張。考試途中不能向外求助更多的計算紙。

1. (10 points) Given the coordinate vector $\vec{v}_B = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$. Please find the \vec{v} and \vec{v}'_B when the ordered basis B and

B' for P_2 are

$$B = (x^2 - x, 2x + 1, -x - 5), B' = (1, (1 + x), (1 + x)^2)$$

Answer: $\vec{v} = \underline{2x^2 - 6x + 7}$, $\vec{v}'_B = \underline{\begin{bmatrix} 15 \\ -10 \\ 2 \end{bmatrix}}$

From 7-1

Method 1

$$\vec{v} = 2(x^2 - x) + (-3)(2x + 1) + (-2)(-x - 5) = 2x^2 - 6x + 7 = 2(x + 1)^2 - 10(x + 1) + 15$$

$$\vec{v}_{B'} = \begin{bmatrix} 15 \\ -10 \\ 2 \end{bmatrix}$$

Method 2

$$\vec{v} = M_B \vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix} \Rightarrow 2x^2 - 6x + 7$$

$$\vec{v}_{B'} = C_{BB'} \vec{v}_B = M_{B'}^{-1} M_B \vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 2 \end{bmatrix}$$

2. (10 points) Express $\frac{z}{w}$ in the form $a + bi$, where $a, b \in \mathbb{R}$, if

$$z = 6 - i, \quad w = 2 - 3i$$

Answer: $\frac{z}{w} = \underline{\frac{15+16i}{13}}$

From 9-1

Method 1

$$\frac{1}{w} = \frac{\bar{w}}{|w|^2} = \frac{2 + 3i}{4 + 9}$$

$$\frac{z}{w} = (6 - i) \times \frac{(2 + 3i)}{4 + 9} = \frac{15 + 16i}{13}$$

Method 2

$$\frac{z}{w} = \frac{6 - i}{2 - 3i} = \frac{(6 - i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{(6 - i)(2 + 3i)}{4 + 9} = \frac{15 + 16i}{13}$$

3. (10 points) Find the five fifth roots of -32 . (need not simplify)

From 9-1

$$-32 = 2^5[\cos(\pi) + i \sin(\pi)]$$

$$2 \left(\cos \left(\frac{\pi + 2k\pi}{5} \right) + i \sin \left(\frac{\pi + 2k\pi}{5} \right) \right), \text{ for } i = 0, 1, 2, 3, 4$$

4. (10 points) Let A is an 3×3 complex matrix with $\det(A) = 2 + 5i$. Please the value for $\det(iA)$ and $\det(A^*)$.

Answer: $\det(iA) = \underline{(i)^3 \times (2 + 5i) = 5 - 2i}$, $\det(A^*) = \underline{\overline{2 + 5i} = 2 - 5i}$

From 9-2, using the technique from Sec. 4-2.

A is an $n \times n$, then $\det(aA) = a^n \det(A)$ and $\det(A) = \det(A^T)$.

Moreover, the definition of the conjugate transpose is also requested. If $A = [a_{ij}]$, $A^* = \overline{A}^T = [\overline{a_{ji}}] = [\overline{a_{ji}}]$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for linear transformation $T : P_2 \rightarrow P_2$ defined by $T(p(x)) = p(x-1) + 2p(x)$, $B = (x^2, x, 1)$, $B' = (x^2 - 1, x - 3, 2)$.

$$C_{B,B'} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}, C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}, R_{B',B'} = \frac{1}{2} \begin{bmatrix} 6 & 0 & 0 \\ -4 & 6 & 0 \\ -5 & -1 & 6 \end{bmatrix} \text{ and } R_{B,B} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}.$$

Is $C = C_{B,B'}$ or $C_{B',B}$? $C_{B',B}$.

From 7-2

$$T(x^2) = (x-1)^2 + 2x^2 = 3x^2 - 2x + 1, T(x) = (x-1) + 2x = 3x - 1, T(1) = 1 + 2 \times 1 = 3$$

Thus

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = R_E \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We have

$$R_{B,B} = R_B = R_E = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

By $C_{B',B} = M_B^{-1}M_{B'} = M_E^{-1}M_{B'} = I^{-1}M_{B'} = M_{B'}$,

$$C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$C_{B,B'} = C_{B',B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Since

$$R_{B'} = R_{B',B'} = C_{B,B'}R_{B,B}C_{B',B} \\ R_{B'} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & 0 & 0 \\ -4 & 6 & 0 \\ -5 & -1 & 6 \end{bmatrix}$$

6. (10 points) Use the process in Schur's Lemma to find an unitary matrix U such that $U^{-1}AU = R$ is an upper triangular.

$$A = \begin{bmatrix} 5 & 1 & -20 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

From 9-3

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A .

$$A = \begin{bmatrix} 3 & 0 & 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5i & 0 & 0 \\ 0 & 0 & 0 & 0 & 5i & 0 \\ 0 & 0 & 0 & 0 & 0 & 5i \end{bmatrix}$$

From 9-4

$$J = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5i & 0 & 0 \\ 0 & 0 & 0 & 0 & 5i & 0 \\ 0 & 0 & 0 & 0 & 0 & 5i \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 - 5i \\ 0 \\ 0 \end{bmatrix}$$

$$(A - 3I)\vec{v}_1 = \vec{0}, (A - 3I)A\vec{v}_2 = \vec{v}_1, (A - 3I)\vec{v}_3 = \vec{0}, (A - 5iI)\vec{v}_4 = \vec{0}, (A - 5iI)\vec{v}_5 = \vec{0}, (A - 5iI)\vec{v}_6 = \vec{0}$$

8. (10 points) Prove or disprove the following:

$$\det(C_{BB'}) = 1 \text{ if and only if } B = B'$$

From 7-1 #23 (h)

False!!

If $B = \{[1, 0], [0, 6]\}$, $B' = \{[2, 0], [0, 3]\}$, then

$$C_{BB'} = M_{B'}^{-1} M_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(C_{BB'}) = 1$$

9. (10 points) Answer the following question.

1. Find the eigenvalues of the given Matrix J .
2. Give the rank and nullity of $(J - \lambda)^k$ for each eigenvalue λ of J and for every positive integer k .
3. Draw schemata of the strings of vectors in the standard basis arising from the Jordan blocks in J .
4. For each standard basis vector \vec{e}_k , express $J\vec{e}_k$ as a linear combination of vectors in the standard basis.

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} = J = \left[\begin{array}{ccccccc} \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} & & & & & & \\ & \boxed{\begin{matrix} 9i & 1 \\ 0 & 9i \end{matrix}} & & & & & \\ & & \boxed{\begin{matrix} 9i & 1 & 0 \\ 0 & 9i & 1 \\ 0 & 0 & 9i \end{matrix}} & & & & \\ & & & \boxed{4} & & & \\ & & & & \boxed{4} & & \\ & & & & & & \end{array} \right]$$

From 9-4

1. $\lambda_1 = \lambda_2 = \lambda_8 = \lambda_9 = 2$, $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 9i$
2. $(J - 4I)$ has rank 6 and nullity 3,
 $(J - 4I)^k$ has rank 5 and nullity 4, for $k \geq 2$,
 $(J - 9iI)$ has rank 7 and nullity 2,
 $(J - 9iI)^2$ has rank 5 and nullity 4,
 $(J - 9iI)^k$ has rank 4 and nullity 5 for $k \geq 3$,
3. The strings are: $(J - 4I) : \begin{cases} \vec{e}_2 \rightarrow \vec{e}_1 \rightarrow 0 \\ \vec{e}_8 \rightarrow 0 \\ \vec{e}_9 \rightarrow 0 \end{cases}$, $(J - 9iI) : \begin{cases} \vec{e}_4 \rightarrow \vec{e}_3 \rightarrow 0 \\ \vec{e}_7 \rightarrow \vec{e}_6 \rightarrow \vec{e}_5 \rightarrow 0 \end{cases}$
4. $\begin{cases} J\vec{e}_1 = 4\vec{e}_1, \\ J\vec{e}_2 = 4\vec{e}_2 + \vec{e}_1, \\ J\vec{e}_8 = 4\vec{e}_8, \\ J\vec{e}_9 = 4\vec{e}_9, \end{cases}, \begin{cases} J\vec{e}_3 = 9i\vec{e}_3 \\ J\vec{e}_4 = 9i\vec{e}_4 + \vec{e}_3, \\ J\vec{e}_5 = 9i\vec{e}_5 \\ J\vec{e}_6 = 9i\vec{e}_6 + \vec{e}_5, \\ J\vec{e}_7 = 9i\vec{e}_7 + \vec{e}_6, \end{cases}$

10. (10 points) Prove or disprove whether every unitarily diagonalizable matrix is Hermitian.

From 9-3, Theorem 9.7.

A square matrix A is unitarily diagonalizable if and only if it is a normal matrix.

Therefore, just build a normal matrix which is not a Hermitian matrix as the counterexample.

