2024-10-31

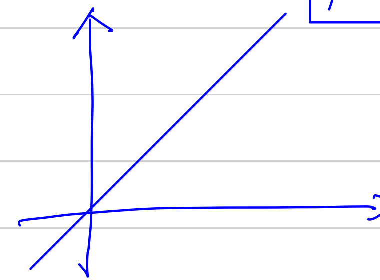
## 2-3

$$\text{map}\quad f: \underset{\text{domain}}{X} \longrightarrow \underset{\text{codomain}}{Y}$$

$$Y \supseteq f(X) \to \text{range}$$

$f(x) = 2x$

$\boxed{y = 2x}$



ex:

① $\{2, 4, 6, \ldots, 20\}$

② $\{1, 2, 3, \ldots, 20\}$

$$f: \{1, 2, 3, \ldots, 10\} \longrightarrow \mathbb{N} \ ③$$

← 自然數

$$x \longmapsto f(x) = 2x$$

e.g. $f(x) = \dfrac{x-3}{x-2}$ $\quad x \neq 2$

定義域 domain: $\{1, 2, \ldots, 10\}$

對應域 codomain $= \mathbb{N}$

值域 range $= \{2, 4, 6, 8, \ldots, 20\} = \{ f(x) \mid x \in X \}$

$\boxed{\text{Well-defined}}$

① $f$ 在 $X$ 上可以送出去

② $f(X) \subseteq Y$

**Def.**

$T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**

map

if

① $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, $\forall \vec{v}, \vec{u} \in \mathbb{R}^n$

Preservation of vector addition

② $T(r\vec{v}) = rT(\vec{v})$, $\forall r \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$

Preservation of scalar multiplication

P.S. $T(\vec{0}) = T(0 \cdot \vec{0}) = 0\,T(\vec{0}) = \vec{0}$

or

if $T(r\vec{u}+s\vec{v}) = rT(\vec{u}) + sT(\vec{v})$, $\forall r, s \in \mathbb{R}$, $\vec{u}, \vec{v} \in \mathbb{R}^n$

Preservation of linear combination

**Prop.**

$T: \mathbb{R}^n \to \mathbb{R}^m$ : linear trans

if $W$ : subspace of $\mathbb{R}^n$ then $\underset{\{T(\vec{u}) \mid \vec{u} \in W\}}{T(W)}$ : subspace of $\mathbb{R}^m$

Recall

$W$ : subspace of $\mathbb{R}^m$

if ① $W$ : subset of $\mathbb{R}^m$

② $\vec{u}+\vec{v} \in W$ , $\forall \vec{u}, \vec{v} \in W$

③ $r\vec{v} \in W$ , $\forall r \in \mathbb{R}$

p.f.

(i) $\because$ range in condomain $\therefore$ $T(W)$ in $\mathbb{R}^m$

(ii) $\forall \vec{p}, \vec{q} \in T(W)$ , $r \in \mathbb{R}$ , Claim: $\begin{cases} ① \ \vec{p}+\vec{q} \in T(W) \\ ② \ r\vec{p} \in T(W) \end{cases}$

☆ $\vec{u}, \vec{v}$ 不一定唯一

maybe $T(\vec{\alpha}) = \vec{p}$ , $T(\vec{\beta}) = \vec{p}$

$\because \vec{p}, \vec{q} \in T(W)$ $\therefore \exists \vec{u}, \vec{v} \in W$ s.t. $T(\vec{u}) = \vec{p}$ , $T(\vec{v}) = \vec{q}$

$\vec{p}+\vec{q} = \overbrace{T(\vec{u}) + T(\vec{v}) = T(\underbrace{\vec{u}+\vec{v}}_{\in W})}^{T:\ \text{linear trans.}} \in T(W)$ $\left( \begin{array}{l} \because \vec{u}, \vec{v} \in W, \ W: \text{subspace of } \mathbb{R}^n \\ \therefore \vec{u}+\vec{v} \in W \end{array} \right)$

(iii) $r\vec{p} = \underbrace{r T(\vec{u}) = T(\overbrace{r\vec{u}}^{\in W})}_{T:\ \text{linear trans.}} \in T(W)$ ※

ex:

given $A_{m\times n}$, define $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \longmapsto T(\vec{x}) = A\vec{x}$

$\overset{m\times n}{A}\ \overset{n\times 1}{\vec{x}}$

$(A\vec{x}) \to m\times 1$

check $T$ is an linear trans

① $\underset{T(\vec{x})+T(\vec{y})}{\underset{\parallel}{A\vec{x}+A\vec{y}}} = \underset{T(\vec{x}+\vec{y})}{\underset{\parallel}{A(\vec{x}+\vec{y})}}$

② $\underset{T(r\vec{x})}{\underset{\parallel}{A(r\vec{x})}} = \underset{rT(\vec{x})}{\underset{\parallel}{rA\vec{x}}}$

ex:

$T: \mathbb{R} \to \mathbb{R}$ : NOT linear trans.

$x \mapsto \sin(x)$

check: $\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) \overset{?}{\neq} \sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)$

$x=\frac{\pi}{4}$, $y=\frac{\pi}{4}$

**Thm**

$T: \mathbb{R}^n \to \mathbb{R}^m$ : linear trans , $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ : basis for $\mathbb{R}^n$

$\forall \vec{p} \in T(\mathbb{R}^n)$ can be expressed by $\mathcal{B}' = \{T(\vec{b}_1), \dots, T(\vec{b}_n)\}$ $\leftarrow$ $sp(T(\vec{b}_1), \dots, T(\vec{b}_n)) = T(\mathbb{R}^n)$

$\mathcal{B}' = T(\mathcal{B})$, $\quad T(\mathbb{R}^n) = sp(T(\mathcal{B}))$

p.f.

$\because \vec{p} \in T(\mathbb{R}^n) \quad \therefore \exists \vec{v}$ s.t. $\vec{p} = T(\vec{v})$

$\because \mathcal{B}$: basis for $\mathbb{R}^n$

$\therefore \exists! \, r_1, \dots, r_n \in \mathbb{R}$ s.t. $r_1\vec{b}_1 + \dots + r_n\vec{b}_n = \vec{v}$

$$
\begin{aligned}
\therefore \vec{p} &= T(\vec{v}) \\
&= T(r_1\vec{b}_1 + \dots + r_n\vec{b}_n) = T(r_1\vec{b}_1 + r_2\vec{b}_2 + \dots + r_{n-1}\vec{b}_{n-1} + r_n\vec{b}_n) = T(r_1\vec{b}_1 + \dots + r_{n-1}\vec{b}_{n-1}) + T(r_n\vec{b}_n) \\
&\;\vdots \\
&= T(r_1\vec{b}_1 + r_2\vec{b}_2 + r_3\vec{b}_3) + T(r_4\vec{b}_4) + \dots + T(r_n\vec{b}_n) \\
&= T(r_1\vec{b}_1 + r_2\vec{b}_2) + T(r_3\vec{b}_3) + \dots + T(r_n\vec{b}_n) \\
&= T(r_1\vec{b}_1) + T(r_2\vec{b}_2) + T(r_3\vec{b}_3) + \dots + T(r_n\vec{b}_n) \\
&= r_1 T(\vec{b}_1) + \dots + r_n T(\vec{b}_n)
\end{aligned}
$$

e.g. $\mathbb{R}^2$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ e.g. $\mathbb{R}^3$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

**Recall**

**Standard basis** $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$, $\vec{e}_i$ : 1 in the $i^{th}$ position, other are 0

**Def**

$T: \mathbb{R}^n \to \mathbb{R}^m$ : linear trans Let $A$ be the **standard matrix representation** (s.m.r) of $T$

if $A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$ ← $m \times n$

**Thm.**

$T: \mathbb{R}^n \to \mathbb{R}^m$ : linear trans Let $A$ be the standard matrix representation (s.m.r) of $T$

$\Rightarrow \forall \vec{x} \in \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$

p.f.

$\forall \vec{x} \in \mathbb{R}^n \Rightarrow \exists!\ r_1, r_2, \dots, r_n \in \mathbb{R}$ s.t. $\vec{x} = r_1\vec{e}_1 + r_2\vec{e}_2 + \dots + r_n\vec{e}_n = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$

$$T(\vec{x}) = r_1 T(\vec{e}_1) + r_2 T(\vec{e}_2) + \dots + r_n T(\vec{e}_n) = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = A\vec{x}$$

e.g. $\mathbb{R}^2$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $[1, 0]$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $[0, 1]$

ex:

$T: \mathbb{R}^2 \to \mathbb{R}^3$, $T([x, y]) = [2x+y, 3x, 4x-y]$, find the s.m.r. of $T$.

$T(\vec{e}_1) = [2, 3, 4]$, $T(\vec{e}_2) = [1, 0, -1]$

$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}$ ← s.m.r. of $T$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 3x \\ 4x-y \end{bmatrix}$

ex: $T: \mathbb{R}^4 \to \mathbb{R}^4$ : linear trans. $T([x_1, x_2, x_3, x_4]) = [x_1+2x_3, -x_1+2x_2-x_4, x_2-x_3+x_4, x_1+x_4]$

$T(\vec{e}_1) = T([1, 0, 0, 0]) = [1, -1, 0, 1]$

$T(\vec{e}_2) = T([0, 1, 0, 0]) = [0, 2, 1, 0]$

$T(\vec{e}_3) = T([0, 0, 1, 0]) = [2, 0, -1, 0]$

$T(\vec{e}_4) = T([0, 0, 0, 1]) = [0, -1, 1, 1]$

$\therefore A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

ex:

given $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ : linear trans. $T(\overset{\vec{u}}{[-1,2]}) = [-2,1,0]$ , $T(\overset{\vec{v}}{[3,-5]}) = [5,-7,1]$

① find the s.m.r. of T. $\because \vec{u}, \vec{v}$ : linear indep. $\therefore \{\vec{u}, \vec{v}\}$ : basis for $\mathbb{R}^2$

find $T(\vec{e}_1)$ , $T(\vec{e}_2)$ ② 求 $T([3,6])$

$\boxed{\text{sol}}$

$$\text{1. } \begin{array}{ccc} \vec{u} & \vec{v} & \vec{e}_1 \\ \left[\begin{matrix} -1 \\ 2 \end{matrix}\right. & \begin{matrix} 3 \\ -5 \end{matrix} & \left|\begin{matrix} 1 \\ 0 \end{matrix}\right] \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \vec{e}_1 = 5\vec{u} + 2\vec{v}$$

$$\text{2. } \begin{array}{ccc} & & \vec{e}_2 \\ \left[\begin{matrix} -1 \\ 2 \end{matrix}\right. & \begin{matrix} 3 \\ -5 \end{matrix} & \left|\begin{matrix} 0 \\ 1 \end{matrix}\right] \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \vec{e}_2 = 3\vec{u} + \vec{v}$$

$$\Rightarrow \begin{array}{cccc} \vec{u} & \vec{v} & \vec{e}_1 & \vec{e}_2 \\ \left[\begin{matrix} -1 \\ 2 \end{matrix}\right. & \begin{matrix} 3 \\ -5 \end{matrix} & \left|\begin{matrix} 1 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right] \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right] \Rightarrow \begin{cases} \vec{e}_1 = 5\vec{u} + 2\vec{v} \\ \vec{e}_2 = 3\vec{u} + \vec{v} \end{cases}$$

$$T(\vec{e}_1) = T(5\vec{u} + 2\vec{v}) = 5T(\vec{u}) + 2T(\vec{v}) = 5[-2,1,0] + 2[5,-7,1] = [0,-9,2]$$

$$T(\vec{e}_2) = T(3\vec{u} + \vec{v}) = 3T(\vec{u}) + T(\vec{v}) = 3[-2,1,0] + [5,-7,1] = [-1,-4,1]$$

$$A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -9 & -4 \\ 2 & 1 \end{bmatrix} , \quad A\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ -51 \\ 12 \end{bmatrix} \quad \therefore T([3,6]) = [-6,-51,12]$$

```
octave:1> A=[0 -1;-9 -4;2 1]
A =

   0  -1
  -9  -4
   2   1

octave:2> x=[3;6]
x =

   3
   6
```

```
octave:3> A*x
ans =

   -6
  -51
   12
```

**Def**

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear trans. $A$: **s.m.r.** of $T$, $\mathbb{R}^n$: **domain**, $\mathbb{R}^m$: **codomain**

- **range** of $T$: $\text{range}(T) = T(\mathbb{R}^n) = \{\underbrace{T(\vec{x})}_{=A\vec{x}} \mid \vec{x} \in \mathbb{R}^n\} = \text{col}(A) \Rightarrow \text{range}(T) \subseteq \mathbb{R}^m$
- **kernel** of $T$: $\ker(T) = \{\vec{x} \in \mathbb{R}^n \mid \underbrace{T(\vec{x}) = \vec{0}}_{=A\vec{x}=\vec{0}}\} = \text{null}(A) \Rightarrow \ker(T) \subseteq \mathbb{R}^n$
- **rank**$(T) = \dim(\text{range}(T)) = \text{rank}(A)$

**Thm**

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear trans. $A$: s.m.r. of $T$

1. $\text{range}(T) = \text{col}(A)$
2. $\ker(T) = \text{null}(A)$
3. $$\begin{aligned} n &= \dim(\text{col}(A)) + \dim(\text{null}(A)) \\ &= \dim(\text{range}(T)) + \dim(\ker(T)) \\ &= \text{rank}(T) + \dim(\ker(T)) \end{aligned}$$

**Thm** (Composition Thm)

$T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ , $T_2 : \mathbb{R}^m \longrightarrow \mathbb{R}^p$, $T_1, T_2$ : linear trans.

$\overset{m\times n}{A_1}$ : s.m.r. of $T_1$ , $\overset{p\times m}{A_2}$ : s.m.r. of $T_2$

$\Rightarrow \begin{cases} \text{① } T_2 \circ T_1 : \text{linear trans} \\ \text{② } T_3 = T_2 \circ T_1 \text{ , } A_3 : \text{s.m.r. of } T_3 \Rightarrow \overset{p\times n}{A_3} = \overset{p\times m}{A_2}\,\overset{m\times n}{A_1} \end{cases}$

p.f.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_1"] \arrow[rr, bend left, "T_2 \circ T_1 = T_3"] & \mathbb{R}^m \arrow[r, "T_2"] & \mathbb{R}^p \end{tikzcd}$$

$\vec{x} \longmapsto A_1\vec{x}$

$\vec{y} \longmapsto A_2\vec{y}$

$\vec{x} \longmapsto A_1\vec{x} \longmapsto A_2A_1\vec{x}$

$\vec{x} \longmapsto A_2A_1\vec{x}$

$\vec{x} \longmapsto (T_2\circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$

$\quad\quad\quad\quad \overset{\shortparallel}{T_3(\vec{x})}$

$\Rightarrow$

$$\mathbb{R}^n \xrightarrow{\ T_2\circ T_1 = T_3\ } \mathbb{R}^p$$

$\vec{x} \longmapsto \underset{p\times n}{A_3}\vec{x} = \underset{p\times m}{A_2}\,\underset{m\times n}{A_1}\vec{x}$

① $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$, $\forall r \in \mathbb{R}$

- $T_2 \circ T_1(\vec{u}+\vec{v}) = T_2(T_1(\vec{u}+\vec{v})) = T_2(T_1(\vec{u}) + T_1(\vec{v}))$
  $= T_2(T_1(\vec{u})) + T_2(T_1(\vec{v})) = T_2 \circ T_1(\vec{u}) + T_2 \circ T_1(\vec{v})$

- $T_2 \circ T_1(r\vec{u}) = T_2(T_1(r\vec{u})) = T_2(r T_1(\vec{u}))$
  $= r T_2(T_1(\vec{u})) = r\, T_2 \circ T_1(\vec{u})$

$\therefore T_2 \circ T_1$: linear trans

②

$\forall \vec{u} \in \mathbb{R}^n$, $A_3\vec{u} = T_2 \circ T_1(\vec{u}) = T_2(T_1(\vec{u})) = T_2(A_1\vec{u}) = A_2A_1\vec{u} \overset{\text{claim}}{\Rightarrow} A_3 = A_2A_1$

p.f. of claim:

(a) pick $\vec{u} = \vec{e}_1 \Rightarrow$ $A_3\vec{e}_1 = 1^{st}$ column of $A_3$

$\qquad \qquad \qquad \qquad \; \shortparallel$

$\qquad \qquad \qquad \quad A_2A_1\vec{e}_1 = 1^{st}$ column of $A_2A_1$

(b) pick $\vec{u} = \vec{e}_i \Rightarrow$ $A_3\vec{e}_i = i^{th}$ column of $A_3$

$\qquad \qquad \qquad \qquad \; \shortparallel$

$\qquad \qquad \qquad \quad A_2A_1\vec{e}_i = i^{th}$ column of $A_2A_1$

(c) pick $\vec{u} = \vec{e}_1, \vec{e}_2, \vec{e}_3, \cdots, \vec{e}_n \Rightarrow$ all column of $A_3$ = all column of $A_2A_1$

**Def.**

1. identity transformation $I : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

   if $\forall \vec{x} \in \mathbb{R}^n$, $I(\vec{x}) = \vec{x}$

2. $T$ : linear trans

   if $\exists\ \tilde{T}$ : linear trans s.t. $\begin{cases} \tilde{T} \circ T(\vec{y}) = \vec{y} \\ T \circ \tilde{T}(\vec{x}) = \vec{x} \end{cases}$ $\quad (\tilde{T} \circ T = I)$ $\quad (T \circ \tilde{T} = I)$

   then call $\tilde{T}$ : inverse of $T$, Denote $T^{-1}$

   and $T$ : invertible

**Thm**

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ : linear trans, $A$ : s.m.r. $T$

then ① 「$T$: invertible」 iff 「$A$ : invertible」

② $T^{-1}$ : linear trans, $A^{-1}$ : s.m.r. of $T^{-1}$

pf.

① (i) Assume $A$: invertible $\Rightarrow$ $A^{-1}$: exist , let $S$: map s.t. $S(\vec{x}) = A^{-1}\vec{x}$ $\Rightarrow$ $S$: linear trans

$$S\circ T(\vec{x}) = S(T(\vec{x})) = S(A\vec{x}) = A^{-1}A\vec{x} = \vec{x}$$
$$T\circ S(\vec{y}) = T(S(\vec{y})) = T(A^{-1}\vec{y}) = AA^{-1}\vec{y} = \vec{y}$$
$\Rightarrow$ $T$: invertible & $S = T^{-1}$

(ii) Assume $T$: invertible , $T^{-1}$: exists , prove $T^{-1}$: linear trans.

$$\begin{cases} T\circ T^{-1}(\vec{x}+\vec{y}) = T(T^{-1}(\vec{x}+\vec{y})) = \vec{x}+\vec{y} \\ T(T^{-1}(\vec{x})+T^{-1}(\vec{y})) = T(T^{-1}(\vec{x}))+T(T^{-1}(\vec{y})) = \vec{x}+\vec{y} \end{cases}$$
$\Rightarrow$ $\therefore T(T^{-1}(\vec{x}+\vec{y})) = T(T^{-1}(\vec{x})+T^{-1}(\vec{y}))$

$$T^{-1}(\vec{x}+\vec{y}) = \overset{I}{\boxed{T^{-1}(T}}(T^{-1}(\vec{x}+\vec{y}))) = \overset{I}{\boxed{T^{-1}(T}}(T^{-1}(\vec{x})+T^{-1}(\vec{y}))) = T^{-1}(\vec{x})+T^{-1}(\vec{y})$$

$$\left.\begin{aligned} &\therefore T^{-1}(\vec{x}+\vec{y}) = T^{-1}(\vec{x})+T^{-1}(\vec{y}) \\ &T^{-1}(r\vec{x}) = rT^{-1}(\vec{x}) \quad (\text{同法}) \end{aligned}\right] \Rightarrow T^{-1}: \text{linear trans.}$$

let $B$: s.m.r. of $T^{-1}$

$AB\vec{x} = A\,T^{-1}(\vec{x}) = T(T^{-1}(\vec{x})) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$

$\Rightarrow AB = I$

$BA\vec{x} = B\,T(\vec{x}) = T^{-1}(T(\vec{x})) = \vec{x}, \quad \forall \vec{x} \in \mathbb{R}^n$

$\Rightarrow BA = I$

$\Rightarrow$ $\therefore$ $A$: invertible

$\therefore B = A^{-1}$