SUMMARY

1. The cofactor of an element a_{ij} in a square matrix A is $(-1)^{i+j}|A_{ij}|$, where A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.
2. The *determinant* of an $n \times n$ matrix may be defined inductively by expansion by minors on the first row. The determinant can be computed by expansion by minors on any row or on any column; it is the sum of the products of the entries in that row or column by the cofactors of the entries. For large matrices, such a computation is hopelessly long.
3. The elementary row operations have the following effect on the determinant of a square matrix A.
   a. If two different rows of A are interchanged, the sign of the determinant is changed.
   b. If a single row of A is multiplied by a scalar, the determinant is multiplied by the scalar.
   c. If a multiple of one row is added to a different row, the determinant is not changed.
4. We have $\det(A) = \det(A^T)$. As a consequence, the properties just listed for elementary row operations are also true for elementary column operations.
5. If two rows or two columns of a matrix are the same, the determinant of the matrix is zero.
6. The determinant of an upper-triangular matrix or of a lower-triangular matrix is the product of the diagonal entries.
7. An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.
8. If A and B are $n \times n$ matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

EXERCISES

In Exercises 1–10, find the determinant of the given matrix.

1. $\begin{bmatrix} 5 & 2 & 1 \\ 1 & -1 & 4 \\ 3 & 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 6 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -1 & 2 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 4 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 6 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 & 4 & 6 \\ 2 & 0 & -9 & 6 \\ 4 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 0 & -1 & 2 & 4 \\ 6 & 2 & 8 & 1 & -1 & 1 \\ 4 & 2 & 1 & 2 & 2 & -5 \\ 4 & 5 & 4 & 5 & 1 & 2 \\ 1 & 2 & 0 & -1 & 2 & 4 \\ 1 & 0 & 1 & 8 & 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 4 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

11. Find the cofactor of 5 for the matrix in Exercise 2.

12. Find the cofactor of 3 for the matrix in Exercise 4.

13. Find the cofactor of 7 for the matrix in Exercise 8.

14. Find the cofactor of -5 for the matrix in Exercise 9.

In Exercises 15–20, let A be a 3×3 matrix with $\det(A) = 2$.

15. Find $\det(A^2)$.

16. Find $\det(A^k)$.

17. Find $\det(3A)$.

18. Find $\det(A + A)$.

19. Find $\det(A^{-1})$.

20. Find $\det(A^T)$.

21. Mark each of the following True or False.

___ a. The determinant $\det(A)$ is defined for any matrix A.

___ b. The determinant $\det(A)$ is defined for each square matrix A.

___ c. The determinant of a square matrix is a scalar.

___ d. If a matrix A is multiplied by a scalar c, the determinant of the resulting matrix is $c \cdot \det(A)$.

___ e. If an $n \times n$ matrix A is multiplied by a scalar c, the determinant of the resulting matrix is $c^n \cdot \det(A)$.

___ f. For every square matrix A, we have $\det(AA^T) = \det(A^TA) = [\det(A)]^2$.

___ g. If two rows and also two columns of a square matrix A are interchanged, the determinant changes sign.

___ h. The determinant of an elementary matrix is nonzero.

___ i. If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

___ j. If $\det(A) = 2$ and $\det(B) = 3$, then $\det(AB) = 6$.

In Exercises 22–25, let A be a 3×3 matrix with row vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and with determinant equal to 3. Find the determinant of the matrix having the indicated row vectors.*

22. $\mathbf{a} + \mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{c}$

23. $\mathbf{a}, \mathbf{b}, 2\mathbf{a} + 3\mathbf{b}$

24. $\mathbf{a}, \mathbf{b}, 2\mathbf{a} + 3\mathbf{b} + 2\mathbf{c}$

25. $\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}$

In Exercises 26–29, find the values of λ for which the given matrix is singular.

26. $\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$

27. $\begin{bmatrix} -\lambda & 5 \\ 2 & 3-\lambda \end{bmatrix}$

28. $\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{bmatrix}$

29. $\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 3 \\ 0 & 4 & -\lambda \end{bmatrix}$

30. If A and B are $n \times n$ matrices and if A is singular, prove (without using Theorem 4.4) that AB is also singular. [HINT: Assume that AB is invertible, and derive a contradiction.]

31. Prove that if A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

32. If A and C are $n \times n$ matrices, with C invertible, prove that $\det(A) = \det(C^{-1}AC)$.

33. Without using the multiplicative property of determinants (Theorem 4.4), prove that $\det(AB) = \det(A) \cdot \det(B)$ for the case where B is a diagonal matrix.

34. Continuing Exercise 33, find two other types of matrices B for which it is easy to show that $\det(AB) = \det(A) \cdot \det(B)$.

35. Prove that, if three $n \times n$ matrices A, B, and C are identical except for the kth rows $\mathbf{a}_k$, $\mathbf{b}_k$, and $\mathbf{c}_k$, respectively, which are related by $\mathbf{a}_k = \mathbf{b}_k + \mathbf{c}_k$, then

$$\det(A) = \det(B) + \det(C).$$

36. Notice that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22}) = (-a_{12}a_{21})$$

is a sum of signed products, where each product contains precisely one factor from each row and one factor from each column of the corresponding matrix. Prove by induction that this is true for an $n \times n$ matrix $A = [a_{ij}]$.

37. (*Application to permutation theory*) Consider an arrangement of n objects, lined up in a column. A rearrangement of the order of the objects is called a *permutation* of the objects. Every such permutation can be achieved by successively swapping the positions of pairs of the objects. For example, the first swap might be to interchange the first object with whatever one you want to be first in the new arrangement, and then continuing this procedure with the second, the third, etc. However, there are many possible sequences of swaps that will achieve a given permutation. Use the theory of determinants to prove that it is impossible to achieve the same permutation using both an even number and an odd number of swaps. [HINT: It doesn't matter what the objects actually are—think of them as being the rows of an $n \times n$ matrix.]

38. This exercise is for the reader who is skeptical of our assertion that the solar system would be dead long before a present-day computer could find the determinant of a 50×50 matrix using just Definition 4.1 with expansion by minors.

    a. Recall that $n! = n(n-1)\cdots(3)(2)(1)$. Show by induction that expansion of an $n \times n$ matrix by minors requires at least $n!$ multiplications for $n > 1$.

    b. Run the routine EBYMTIME in LINTEK and find the time required to perform $n!$ multiplications for $n = 8, 12, 16, 20, 25, 30, 40, 50, 70$, and 100.

39. Use MATLAB or the routine MATCOMP in LINTEK to check Example 2 and Exercises 5–10. Load the appropriate file of matrices if it is accessible. The determinant of a matrix A is found in MATLAB using the command **det(A)**.

4.3 COMPUTATION OF DETERMINANTS AND CRAMER'S RULE

We have seen that computation of determinants of high order is an unreasonable task if it is done directly from Definition 4.1, relying entirely on repeated expansion by minors. In the special case where a square matrix is triangular, Example 4 in Section 4.2 shows that the determinant is simply the product of the diagonal entries. We know that a matrix can be reduced to row-echelon form by means of elementary row operations, and row-echelon form for a square matrix is always triangular. The discussion leading to Theorem 4.3 in the previous section actually shows how the determinant of a matrix can be computed by a row reduction to echelon form. We rephrase part of this discussion in a box as an algorithm that a computer might follow to find a determinant.

Computation of a Determinant

The determinant of an $n \times n$ matrix A can be computed as follows:

1. Reduce A to an echelon form, using only row addition and row interchanges.
2. If any of the matrices appearing in the reduction contains a row of zeros, then $\det(A) = 0$.