Homework 4 Solution

2-4

題號： 8, 13, extra

2-4 #8

a. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.

b. Show that the sequence $p_n = 10^{-n^k}$ does not converge to 0 quadratically, regardless of the size of the exponent $k > 1$.

Answer:

(a) Since

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1$$

The sequence is quadratically convergent.

(b) We have

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} = \lim_{n\to\infty} 10^{2n^k-(n+1)^k} = \lim_{n\to\infty} 10^{n^k(2-(\frac{n+1}{n})^k} = \infty$$

so the sequence $p_n = 10^{-n^k}$ does not converge quadratically.

2-4 Example 1

Let $f(x) = e^x - x - 1$. (a) Show that f has a zero of multiplicity 2 at $x = 0$. (b) Show that Newton’s method with $p_0 = 1$ converges to this zero but not quadratically.

2-4 #13

The iterative method to solve $f(x) = 0$, given by the fixed-point method $g(x) = x$, where

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} - \frac{f''(p_{n-1})}{2f'(p_{n-1})}\left[\frac{f(p_{n-1})}{f'(p_{n-1})}\right]^2 \text{ for } n = 1, 2, 3, ...,$$

has $g'(p) = g''(p) = 0$. This will generally yield cubic ($\alpha = 3$) convergence. Expand the analysis of Example 1 to compare quadratic and cubic convergence.

Answer:

Expanding $g(x)$ in Taylor polynomial for $x \in [p - \delta, p + \delta]$ gives

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(\xi)}{3!}(x-p)^3$$

where ξ lies between x and p. The problem gave $g'(p) = g''(p) = 0$ imply that

$$g(x) = p + \frac{g'''(\xi)}{6}(x-p)^3$$

In particular, when $x = p_n$

$$p_{n+1} = g(p_n) = p + \frac{g'''(\xi_n)}{6}(p_n - p)^3$$

with ξ_n lies between p_n and p. Thus

$$p_{n+1} - p = \frac{g'''(\xi_n)}{6}(p_n - p)^3$$

Since

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)}\left[\frac{f(x)}{f'(x)}\right]^2 \text{ and } f(x) = e^x - x - 1$$

, we have $|g'(x)| \le k < 1$ on $[p-\delta, p+\delta]$ and g maps $[p-\delta, p+\delta]$ into itself, it follows from the Fixed-Point Theorem that $\{p_n\}_{n=0}^{\infty}$ converges to p. But ξ_n is between p and p_n for each n, so $\{\xi_n\}_{n=0}^{\infty}$ also converges to p, and we have

$$\lim_{n\to\infty} g'''(\xi_n) = g'''(p)$$

Thus

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lim_{n\to\infty} \frac{|g'''(\xi_n)||p_n - p|^{3-\alpha}}{6} = \frac{|g'''(p)| \times 0}{6} = 0, \text{ for } \alpha = 1, 2$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} = \lim_{n\to\infty} \frac{|g'''(\xi_n)|}{6} = \frac{|g'''(p)|}{6}$$

Hence, if $g'''(p) \neq 0$, fixed-point iteration exhibits cubic convergence with asymptotic error constant $|g'''(p)|$.