EXAMPLE 2 Illustrate the rank equation for the matrix A in Example 1.

SOLUTION The matrix A in Example 1 has $n = 5$ columns, and we saw that $\text{rank}(A) = 3$ and $\text{nullity}(A) = 2$. Thus the rank equation is $3 + 2 = 5$. ▪

Our work has given us still another criterion for the invertibility of a square matrix.

THEOREM 2.6 An Invertibility Criterion

An $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

SUMMARY

1. Let A be an $m \times n$ matrix. The dimension of the row space of A is equal to the dimension of the column space of A, and is called the *rank* of A, denoted by $\text{rank}(A)$. The rank of A is equal to the number of pivots in a row-echelon form H of A. The ***nullity*** of A, denoted by $\text{nullity}(A)$, is the dimension of the nullspace of A—that is, of the solution set of $A\mathbf{x} = \mathbf{0}$.
2. Bases for the row space, the column space, and the nullspace of a matrix A can be found as described in a box in the text.
3. (*Rank Equation*) For an $m \times n$ matrix A, we have

$$\text{rank}(A) + \text{nullity}(A) = n.$$

EXERCISES

For the matrices in Exercises 1–6, find (a) the rank of the matrix, (b) a basis for the row space, (c) a basis for the column space, and (d) a basis for the nullspace.

1. $\begin{bmatrix} 2 & 0 & -3 & 1 \\ 3 & 4 & 2 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 1 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$

In Exercises 7–10, determine whether the given matrix is invertible, by finding its rank.

7. $\begin{bmatrix} 0 & -9 & -9 & 2 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 3 & 1 \\ 4 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 0 & -1 & 2 \\ 4 & 2 & 1 & 8 \\ 1 & 4 & 0 & 1 \\ 2 & 6 & -3 & 1 \end{bmatrix}$

11. Mark each of the following True or False.

___ a. The number of independent row vectors in a matrix is the same as the number of independent column vectors.

___ b. If H is a row-echelon form of a matrix A, then the nonzero column vectors in H form a basis for the column space of A.

___ c. If H is a row-echelon form of a matrix A, then the nonzero row vectors in H are a basis for the row space of A.

___ d. If an $n \times n$ matrix A is invertible, then $\text{rank}(A) = n$.

___ e. For every matrix A, we have $\text{rank}(A) > 0$.

___ f. For all positive integers m and n, the rank of an $m \times n$ matrix might be any number from 0 to the maximum of m and n.

___ g. For all positive integers m and n, the rank of an $m \times n$ matrix might be any number from 0 to the minimum of m and n.

___ h. For all positive integers m and n, the nullity of an $m \times n$ matrix might be any number from 0 to n.

___ i. For all positive integers m and n, the nullity of an $m \times n$ matrix might be any number from 0 to m.

___ j. For all positive integers m and n, with $m \geq n$, the nullity of an $m \times n$ matrix might be any number from 0 to n.

12. Prove that, if A is a square matrix, the nullity of A is the same as the nullity of A^T.

13. Let A be an $m \times n$ matrix, and let $\mathbf{b}$ be an $n \times 1$ vector. Prove that the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution for $\mathbf{x}$ if and only if $\text{rank}(A) = \text{rank}(A \mid \mathbf{b})$, where rank $(A \mid \mathbf{b})$ represents the rank of the associated augmented matrix $[A \mid \mathbf{b}]$ of the system.

In Exercises 14–16, let A and C be matrices such that the product AC is defined.

14. Prove that the column space of AC is contained in the column space of A.

15. Is it true that the column space of AC is contained in the column space of C? Explain.

16. State the analogue of Exercise 14 concerning the row spaces of A and C.

17. Give an example of a 3×3 matrix A such that $\text{rank}(A) = 2$ and $\text{rank}(A^3) = 0$.

In Exercises 18–20, let A and C be matrices such that the product AC is defined.

18. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

19. Give an example where $\text{rank}(AC) < \text{rank}(A)$.

20. Is it true that $\text{rank}(AC) \leq \text{rank}(C)$? Explain.

It can be shown that $\text{rank}(A^TA) = \text{rank}(A)$ (see Theorem 6.10). Use this result in Exercises 21–23.

21. Let A be an $m \times n$ matrix. Prove that $\text{rank}(A(A^T)) = \text{rank}(A)$.

22. If $\mathbf{a}$ is an $n \times 1$ vector and $\mathbf{b}$ is a $1 \times m$ vector, prove that $\mathbf{ab}$ is an $n \times m$ matrix of rank at most one.

23. Let A be an $m \times n$ matrix. Prove that the column space and row space of $(A^T)A$ are the same.

24. Suppose that you are using computer software, such as LINTEK or MATLAB, that will compute and print the reduced row-echelon form of a matrix but does not indicate any row interchanges it may have made. How can you determine what rows of the *original* matrix form a basis for the row space?

In Exercises 25 and 26, use LINTEK or MATLAB to request a row reduction of the matrix, without seeing intermediate steps. Load data files as usual if they are available. (a) Give the rank of the matrix, and (b) use the software as suggested in Exercise 24 to find the lowest numbered rows, in consecutive order, of the given matrix that form a basis for its row space.

25. $A = \begin{bmatrix} 2 & -3 & 0 & 1 & 4 \\ 1 & 4 & -6 & 3 & -2 \\ 0 & 11 & -12 & 5 & -8 \\ 4 & -1 & 5 & 3 & 7 \end{bmatrix}$

26. $B = \begin{bmatrix} -1 & 1 & 3 & -6 & 8 & -2 \\ -3 & 5 & 3 & 1 & 4 & 8 \\ 1 & -3 & 3 & -13 & 12 & -12 \\ 0 & 2 & -6 & 19 & -20 & 14 \\ 5 & 13 & -21 & 3 & 11 & 6 \end{bmatrix}$