應數一線性代數 **2020** 春, 期末考**SOLUTION**

學號: ______________________________, 姓名: ______________________________

本次考試共有 10 頁 (包含封面)，有 14 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁以及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________________

1. (10 points) Find the projection matrix P for the plane $W: 3x+2y+z=0$ in $\mathbb{R}^3$ and the find the projection $\vec{b}_w$ of $\vec{b}=[4,2,-1]$ on it.

   Answer: $\vec{b}_w = \underline{\frac{1}{14}\begin{bmatrix} 11 \\ -2 \\ -29 \end{bmatrix}}$, $P = \underline{\frac{1}{14}\begin{bmatrix} 5 & -6 & -3 \\ -6 & 10 & -2 \\ -3 & -2 & 13 \end{bmatrix}}$.

2. (10 points) Find the lease squares straight line fit to the four points (0,1) (1,3) (2,4) (3,4) and use it to approximate the fifth points (4, a).

   Answer: the line equation =____ $1.5+x$ ____, a=____ 5.5 ____.

3. (5 points) Find the coordinate vector of $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in M_2 relative to $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$

Answer: [3, 5, 1, 1]

4. (10 points) Find the five fifth roots of $-243i$. (need not simplify)

$$3\left[\cos\left(\frac{3\pi}{10}+\frac{2k\pi}{5}\right)+i\sin\left(\frac{3\pi}{10}+\frac{2k\pi}{5}\right)\right],\ k=0,1,2,3,4$$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for the given linear transformation T.
$T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as reflection of $\mathbb{R}^3$ through the plane $x+y-z=0$; $B = E$, $B' = ([1,0,1],[1,-1,0],[1,1,-1])$.

$C_{BB'} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $C_{BB'} = \begin{bmatrix} 1 & 11 & \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $R_{B'B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $R_{BB} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

Is C=$C_{BB'}$ or $C_{B'B}$? $C_{B'B}$.

6. (5 points) Express $(\sqrt{3}+i)^6$ in the form $a+bi$ for a, b are real numbers.

   Answer: a=$\underline{\quad -64 \quad}$, b=$\underline{\quad 0 \quad}$.

7. (10 points) Using the Gram-Schmidt process to transform the basis $\{[1, i, 1-i], [1+i, 1-i, 1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

   Answer: the found orthogonal basis for $\mathbb{C}^3$ is $\underline{\{[1, i, 1-i], [3+3i, 5-5i, 2], [-12i, 4, 8+8i]\}}$

8. (10 points) Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$. Also find where

$$A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, U = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

9. (10 points) Find a Jordan canonical form and a Jordan basis for the given matrix.

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$J = \begin{bmatrix} \boxed{i} & & \text{\Large 0} & \\ & \boxed{i} & & \\ \text{\Large 0} & & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & & & \boxed{2} \end{bmatrix}, \text{ basis: } \left\{ \vec{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2-i \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{b}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\begin{aligned} J\vec{b}_1 &= i\vec{b}_1, \\ J\vec{b}_2 &= i\vec{b}_2, \\ J\vec{b}_3 &= 2\vec{b}_3, \\ J\vec{b}_4 &= 2\vec{b}_4 + \vec{b}_3, \\ J\vec{b}_5 &= 2\vec{b}_5 \end{aligned}$$

10. (10 points) Find a polynomial in A that gives the zero matrix.

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$(A - iI)^2(A - 2I)^1$$

11. (5 points) Prove or disprove the following: All 2×2 matrix with determinant 1 is an orthogonal matrix.

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, such that $\det(A) = 1$, but A is NOT orthogonal!

12. (10 points) Find all the possible 2×2 real matrix that is unitarily diagonalizable.

Answer:

Every 2×2 real matrix A can written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is unitarily diagonalizable, A is normal, i.e. $A^*A = AA^*$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^*$$

$$\begin{bmatrix} a\overline{a} + c\overline{c} & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & b\overline{b} + d\overline{d} \end{bmatrix} = \begin{bmatrix} a\overline{a} + b\overline{b} & a\overline{c} + b\overline{d} \\ \overline{a}c + \overline{b}d & c\overline{c} + d\overline{d} \end{bmatrix}$$

Hence: (notice that a, b, c, d are real.)

(1). $a\overline{a} + c\overline{c} = a\overline{a} + b\overline{b} \Longrightarrow a^2 + c^2 = a^2 + b^2$

(2). $\overline{a}b + \overline{c}d = a\overline{c} + b\overline{d} \Longrightarrow ab + cd = ac + bd$

(3). $a\overline{b} + c\overline{d} = \overline{a}c + \overline{b}d \Longrightarrow ab + cd = ac + bd$

(4). $b\overline{b} + d\overline{d} = c\overline{c} + d\overline{d} \Longrightarrow b^2 + d^2 = c^2 + d^2$

by (1) and (4), we have $b = c$ or $b = -c$. And (2)(3) holds for for both cases.

13. (5 points) Prove that for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $(\vec{u}^*\vec{v})^* = \overline{\vec{u}^*\vec{v}} = \vec{v}^*\vec{u} = \vec{u}^T\overline{\vec{v}}$

Answer:

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$\begin{aligned}
(\vec{u}^*\vec{v})^* &= (\begin{bmatrix} \overline{u_1} & \overline{u_2} & \dots & \overline{u_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix})* = (\sum_{i=1}^{n} \overline{u_i} v_i)^* \\
&= \overline{\sum_{i=1}^{n} \overline{u_i} v_i} (= \overline{\vec{u}^*\vec{v}}) \\
&= \sum_{i=1}^{n} u_i \overline{v_i} \\
&= \begin{bmatrix} \overline{v_1} & \overline{v_2} & \dots & \overline{v_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} (= \vec{v}^*\vec{u}) \\
&= \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \\ \vdots \\ \overline{v_n} \end{bmatrix} (= \vec{u}^T\overline{\vec{v}})
\end{aligned}$$

14. (10 points) Prove the following:
    (a) Show that every Hermitian matrix is normal.
    (b) Show that every unitary matrix is normal.
    (c) Show that, if $A^* = -A$, then A is normal.

Answer:

(a) Let H are Hermitian matrices, i.e. $H^* = H$. $HH^* = HH = H^*H$.

(b) Let U are unitary matrices, i.e. $U^*U = I$, i.e. $U^{-1} = U^*$. $UU^* = I + U^*U$.

(c) If $A^* = -A$, $A * A = (-A)A = -AA = A(-A) = AA^*$.

應數一線性代數期末考SOLUTION, 學號: ______________, 姓名: ______________, 以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Points:	10	10	5	10	10	5	10	10	10	10	5	10	5	10	120
Score:															