$\boxed{2\text{-}3}$

$T: \mathbb{R}^n \to \mathbb{R}^m$: linear transformation

Def 1 if $\begin{cases} ① \ T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v}) & \forall\, \vec{u}, \vec{v} \in \mathbb{R}^n \\ ② \ rT(\vec{u}) = T(r\vec{u}) & \forall\, r \in \mathbb{R} \end{cases}$

Def 2 if $rT(\vec{u}) + sT(\vec{v}) = T(r\vec{u} + s\vec{v}) \quad \forall\, \vec{u}, \vec{v} \in \mathbb{R}^n, \ \forall\, r, s \in \mathbb{R}$

Thm

$T: \mathbb{R}^n \to \mathbb{R}^m$: linear transformation, Given $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$: basis for $\mathbb{R}^n$

$\Rightarrow \forall\, \vec{v} \in \mathbb{R}^n$, $\underline{T(\vec{v}) \text{ is uniquely determined by } T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)}$

↳ i.e. 給定 $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$ 的值後，$\underline{T(\vec{v}) \text{ 的值也確定}}$]

↳ T 就是唯一的一個 linear transformation

Def. $T: \mathbb{R}^n \to \mathbb{R}^m$: linear transformation, A is the s.m.r. of T

if $\forall\, \vec{v} \in \mathbb{R}^n$, $T(\vec{v}) = A\vec{v}$

Moreover,

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

☆ the standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$

$$\therefore \ \forall \vec{v} \in \mathbb{R}^n, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n$$

$$\therefore \ T(\vec{v}) = v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) + \dots + v_n T(\vec{e}_n)$$

$$= \underline{\begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\therefore \ T(\vec{v}) = A\vec{v}$$

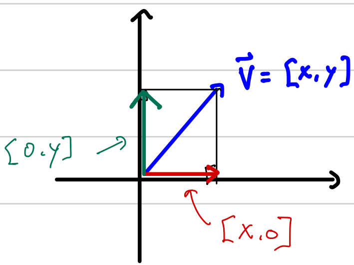
Def. kernel of T = nullspace of A

Def. $T: \mathbb{R}^n \to \mathbb{R}^n$: invertible linear transformation
if A is the s.m.r. of T and A is invertible

$\boxed{2\text{-}4}$

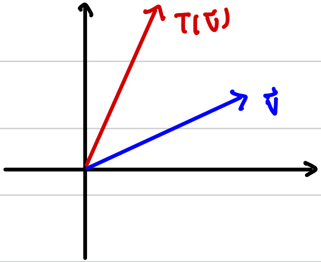
in $\mathbb{R}^2$, s.m.r. is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ i.e. projection on x-axis

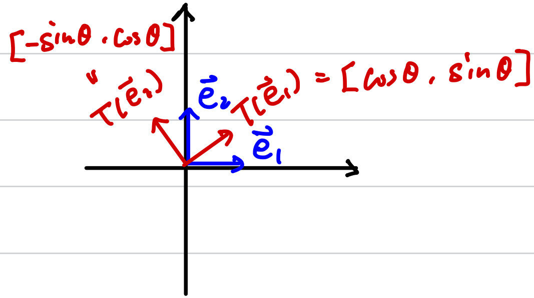
s.m.r. is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$ i.e. projection on y-axis



☆ Rotation counterclockwise through an angle θ, the s.m.r. of T is

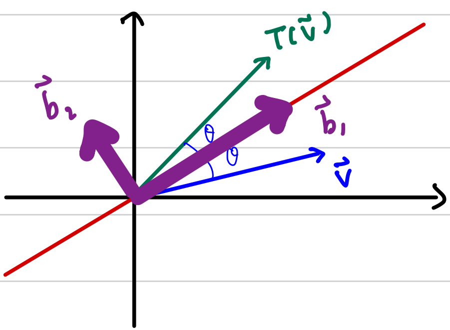
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$$





☆ Reflection in a line L

$$[v_1, v_2] = v_1\vec{e}_1 + v_2\vec{e}_2$$



$\because \{\vec{b}_1, \vec{b}_2\}$ is a basis for $\mathbb{R}^2$, $\vec{v} = r\vec{b}_1 + s\vec{b}_2$

$$\therefore T(\vec{v}) = r\,T(\vec{b}_1) + s\,T(\vec{b}_2)$$
$$= r\vec{b}_1 + s(-\vec{b}_2) = r\vec{b}_1 - s\vec{b}_2$$

$$T(\vec{v}) = v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2)$$

2-4. ex2

e.g. $L: y = 2x \Rightarrow \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \vec{b}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ s.m.r. $= \begin{bmatrix} | & | \\ T(\vec{e}_1), & T(\vec{e}_2) \\ | & | \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1/5 \\ 0 & 1 & -2/5 \end{array}\right] \Rightarrow \vec{e}_1 = \frac{1}{5}\vec{b}_1 - \frac{2}{5}\vec{b}_2$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right] \Rightarrow \vec{e}_1 = \frac{1}{5}\vec{b}_1 - \frac{2}{5}\vec{b}_2,\quad \vec{e}_2 = \frac{2}{5}\vec{b}_1 + \frac{1}{5}\vec{b}_2$$

$$T(\vec{e}_1) = \frac{1}{5}\vec{b}_1 + \frac{2}{5}\vec{b}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix},\quad T(\vec{e}_1) = \frac{2}{5}\vec{b}_1 - \frac{1}{5}\vec{b}_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$\begin{aligned} \therefore T(\vec{v}) &= v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) \\ &= v_1 T\left(\frac{1}{5}\vec{b}_1 - \frac{2}{5}\vec{b}_2\right) + v_2 T\left(\frac{2}{5}\vec{b}_1 + \frac{1}{5}\vec{b}_2\right) \\ &= v_1\left[\frac{1}{5}\vec{b}_1 + \frac{2}{5}\vec{b}_2\right] + v_2\left[\frac{2}{5}\vec{b}_1 - \frac{1}{5}\vec{b}_2\right] \\ &= \frac{1}{5}\left[(v_1 + 2v_2)\vec{b}_1 + (2v_1 - v_2)\vec{b}_2\right] \\ &= \frac{1}{5}\left[(v_1 + 2v_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (2v_1 - v_2)\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

✩ invertible linear transformation

Recall : every invertible matrix is a product of elementary matrices.

Recall : elementary row operation :

① $R_i \leftrightarrow R_j$ $\quad \begin{bmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & \ddots & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← reflection on $x=y$

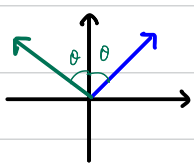
(2-1) $R_i \to (-1)R_i$ $\quad \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ ← reflection on x-axis / y-axis

(2-2) $R_i \to rR_i$, $r>0$ $\quad \begin{bmatrix} 1 & & & \\ & r & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ $\quad \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$ ← expansion

③ $R_i \to R_i + rR_j$ $\quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & r & \ddots & \\ & & & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$ ← shear

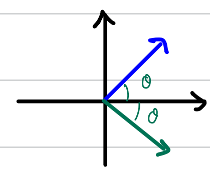
✩ (2-1) $R_i \to (-1)R_i$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



reflection on y-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



reflection on x-axis