

3-3

V : vector space, $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$: ordered basis for V

$\forall \vec{v} \in V$, $\exists r_1, r_2, \dots, r_n$: coefficient (scalar) s.t. $\vec{v} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \dots + r_n \vec{b}_n$

$$\vec{v}_B = [r_1, r_2, \dots, r_n]$$

Thm

$$1. (\vec{v} + \vec{u})_B = \vec{v}_B + \vec{u}_B \quad \forall \vec{u}, \vec{v} \in V$$

$$2. (r\vec{u})_B = r(\vec{u}_B) \quad \forall r \in \mathbb{R}$$

3-4

$T: V \rightarrow V'$: linear transformation, V, V' : vector space

Def 1 if $\left\{ \begin{array}{l} \textcircled{1} T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}) \\ \textcircled{2} rT(\vec{u}) = T(r\vec{u}) \end{array} \right. \quad \forall \vec{u}, \vec{v} \in V$

$$\forall r: \text{scalar}$$

Def 2 if $rT(\vec{u}) + sT(\vec{v}) = T(r\vec{u} + s\vec{v}) \quad \forall \vec{u}, \vec{v} \in V, \forall r, s: \text{scalar}$

★ kernel of $T = \ker(T) = \{ \vec{v} \in V \mid T(\vec{v}) = \underline{\vec{0}} \}$ → is the zero vector in V
✦ $A_3: \vec{0} + \vec{v} = \vec{v}$

Thm

$T(\underline{\vec{0}}_V) = \underline{\vec{0}}_{V'}$,
↑ is the zero vector in V
↖ is the zero vector in V'

Thm V, V' : vector space

$T: V \rightarrow V'$: linear transformation, Given $B = \{ \vec{b}_1, \vec{b}_2, \dots, \vec{b}_n \}$: basis for V

$\Rightarrow \forall \vec{v} \in V$, $T(\vec{v})$ is uniquely determined by $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$

↳ i.e. 给定 $T(\vec{b}_1), T(\vec{b}_2), \dots, T(\vec{b}_n)$ 的值後, $T(\vec{v})$ 的值也確定

↳ T 就是唯一的一個 linear transformation

Thm

A linear transformation $T \overset{V \rightarrow V'}{\text{is}} \underline{\text{one-to-one}}$ iff $\ker(T) = \{ \underline{\vec{0}}_V \}$

↳ i.e. $\forall \vec{u} \neq \vec{v} \in V \Leftrightarrow T(\vec{u}) \neq T(\vec{v})$

Def. $T: V \rightarrow V'$: invertible linear transformation

$$\exists \tilde{T}: V' \rightarrow V: \text{linear transformation s.t. } (\tilde{T} \circ T)(\vec{v}) = \vec{v} \quad , \quad \forall \vec{v} \in V$$
$$(T \circ \tilde{T})(\vec{u}') = \vec{u}' \quad , \quad \forall \vec{u}' \in V'$$

Denote T^{-1} ~~if A is the s.m.r. of T and A is invertible~~

Thm. $T: V \rightarrow V'$: invertible linear transformation

iff T is one-to-one and onto V'

\hookrightarrow if $T(\vec{v}_1) = T(\vec{v}_2)$ in $V' \Rightarrow \vec{v}_1 = \vec{v}_2$

$\forall \vec{v}' \in V' , \exists \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{v}'$

$$\text{Cor } T: V \rightarrow V' : \text{invertible linear transformation} \Rightarrow T^{-1}: V' \rightarrow V$$
$$\vec{v} \mapsto T(\vec{v}) \qquad T(\vec{v}) \mapsto \vec{v}$$

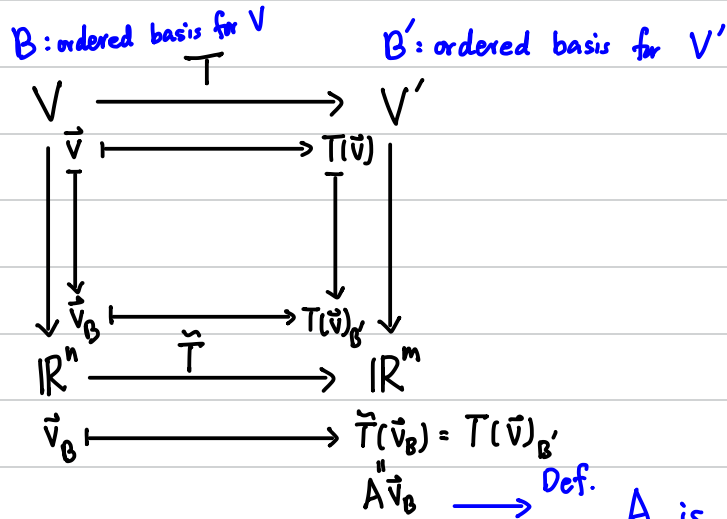
Def. $T: V \rightarrow V'$: isomorphism

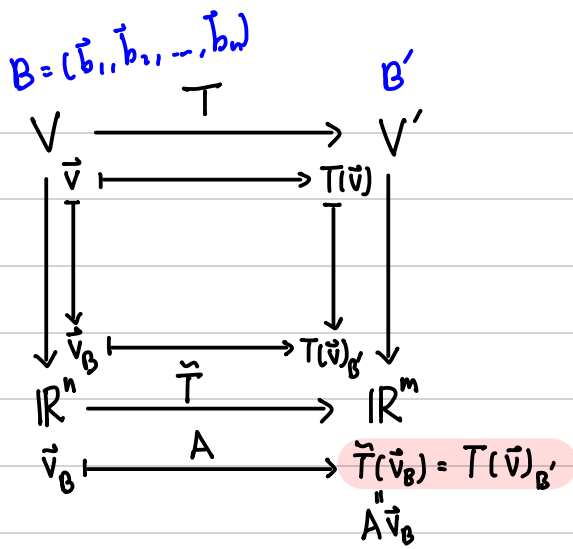
if T is invertible linear transformation (one-to-one and onto V')

Thm

$T: V \rightarrow \mathbb{R}^n$, where $\dim(V) = n$, $B = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$: ordered basis for V
 $\vec{v} \mapsto \vec{v}_B$

$\Rightarrow T$ is an invertible linear transformation (isomorphism)





$$A = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \Leftrightarrow \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$\star \tilde{T}(\vec{v}_B) = T(\tilde{v})_{B'}$$

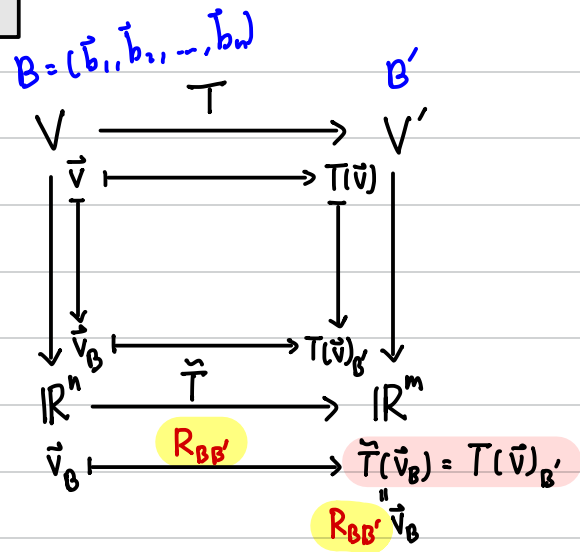
$$\therefore A = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \dots & T(\vec{b}_n)_{B'} \\ | & | & & | \end{bmatrix}$$

Def. $T: V \rightarrow V'$: linear transformation

\exists A is the matrix representation of T relative to B, B'

s.t. $\forall \vec{v} \in V, T(\vec{v})_{B'} = A \vec{v}_B$

7-2



$$R_{B,B'} = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \end{bmatrix}$$

$$\vec{v}_B = \vec{e}_j \Leftrightarrow \vec{v} = \vec{b}_j$$

$$\therefore \tilde{T}(\vec{e}_j) = \tilde{T}((\vec{b}_j)_B) = T(\vec{b}_j)_{B'}$$

$$\star \tilde{T}(\vec{v}_B) = T(\vec{v})_{B'}$$

$$R_{B,B'} = \begin{bmatrix} \tilde{T}(\vec{e}_1) & \tilde{T}(\vec{e}_2) & \dots & \tilde{T}(\vec{e}_n) \end{bmatrix} = \begin{bmatrix} T(\vec{b}_1)_{B'} & T(\vec{b}_2)_{B'} & \dots & T(\vec{b}_n)_{B'} \end{bmatrix}$$

Def. $T: V \rightarrow V'$: linear transformation

$\exists R_{B,B'}$ is the matrix representation of T relative to B, B'

s.t. $\forall \vec{v} \in V, T(\vec{v})_{B'} = R_{B,B'} \vec{v}_B$

3-5 Inner Product Space

Recall (V, \oplus, \otimes) is a vector space
if $A_0 \sim A_4$, $S_0 \sim S_4$ holds

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V &\longrightarrow \mathbb{R} \\ \uparrow \\ \vec{u}, \vec{v} &\longmapsto \langle \vec{u}, \vec{v} \rangle \end{aligned}$$

Def. $(V, \oplus, \otimes, \langle \cdot, \cdot \rangle)$ is an inner product space
if (V, \oplus, \otimes) is a vector space and $D_1 \sim D_4$ holds

$$\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall r, s \in \mathbb{R}$$

$$A_0 : \vec{v} \oplus \vec{u} \in V$$

$$S_0 : r \otimes \vec{v} \in V$$

$$A_1 : (\vec{u} \oplus \vec{v}) \otimes \vec{w} = \vec{u} \otimes \vec{w} \oplus (\vec{v} \otimes \vec{w})$$

$$S_1 : r \otimes (\vec{u} \oplus \vec{v}) = r \otimes \vec{u} \oplus r \otimes \vec{v}$$

$$A_2 : \vec{u} \oplus \vec{w} = \vec{w} \oplus \vec{u}$$

$$S_2 : (r+s) \otimes \vec{v} = (r \otimes \vec{v}) \oplus (s \otimes \vec{v})$$

$$A_3 : \vec{0} \oplus \vec{v} = \vec{v}$$

$$S_3 : r \otimes (s \otimes \vec{v}) = (rs) \otimes \vec{v}$$

$$A_4 : \vec{v} \oplus (-\vec{v}) = (-\vec{v}) \otimes \vec{v} = \vec{0}$$

$$S_4 : 1 \otimes \vec{v} = \vec{v}$$

$$\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall r, s : \text{scalar}$$

$$D_1 : \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$D_2 : \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \text{+ in } \mathbb{R}$$

$$D_3 : r \langle \vec{u}, \vec{v} \rangle = \langle r \otimes \vec{u}, \vec{v} \rangle = \langle \vec{u}, r \otimes \vec{v} \rangle$$

$$D_4 : \langle \vec{u}, \vec{u} \rangle \geq 0 \quad \text{and} \quad \langle \vec{u}, \vec{u} \rangle = 0 \quad \text{iff} \quad \vec{u} = \vec{0}_V$$

Def. $(V, \oplus, \otimes, \langle, \rangle)$ is an inner product space

$\forall \vec{v} \in V$, the magnitude or the norm of \vec{v} is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

Prop. $\forall \vec{u}, \vec{v} \in V$, the angle between \vec{u} and \vec{v} is $\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \times \|\vec{v}\|}$

Def. $\forall \vec{u}, \vec{v} \in V$, \vec{u}, \vec{v} are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

Thm (Schwarz Inequality) : $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \times \|\vec{v}\|$

Thm (Triangle Inequality) : $\|\vec{u} \oplus \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Recall (P, \oplus, \otimes) is a vector space, where P is the set of all polynomials with real coefficient.
 \oplus, \otimes are normal operator for polynomials

ex: $(P^{[0,1]}, \oplus, \otimes, \langle \cdot, \cdot \rangle)$ is an inner product space
, where $P^{[0,1]}$ is the set of all polynomials with real coefficient and domain $0 \leq x \leq 1$
 \oplus, \otimes are normal operator for polynomials
 $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)dx, \quad \forall f(x), g(x) \in P^{[0,1]}$

ex: $(P^{[a,b]}, \oplus, \otimes, \langle \cdot, \cdot \rangle_w)$ is an inner product space
, where $P^{[a,b]}$ is the set of all polynomials with real coefficient and domain $a \leq x \leq b$
 \oplus, \otimes are normal operator for polynomials
 $\langle f(x), g(x) \rangle_w = \int_a^b f(x)g(x)w(x)dx, \quad \forall f(x), g(x) \in P^{[a,b]}$