數學二離散數學 2023 秋, 第一次期中考 解答

學號: ________________________________, 姓名: ________________________________

本次考試共有 9 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find integers a and b such that

$$m^2 = a\binom{m}{2} + b\binom{m}{1}$$

for all m. Then use it to find the sum the series $1^2 + 2^2 + ... + n^2$.

Answer: $a =$ **2** , $b =$ **1** , $1^2 + 2^2 + ... + n^2 =$ $2\binom{n+1}{3} + \binom{n+1}{2} = \frac{n(n+1)(2n+1)}{6}$.

2. (10 points) Show that if n + 1 integers are chosen from the set 1, 2, . . . , 4n then there are always two which differ by at most 3.

3. (10 points) A group of mn people are to be arranged into m teams each with n players.

   (a) Determine the number of ways if each team has a different name.

   (b) Determine the number of ways if the teams don't have names.

   Answer: (a) $\underline{\dfrac{(mn)!}{(n!)^m}}$, (b) $\underline{\dfrac{(mn)!}{m!(n!)^m}}$.

4. (10 points) A bakery sells five different kinds of donuts, includes original, glazed, chocolated, mango and strawberry. If the bakery has at least two dozens of each kind, how many different options for twenty of donuts are there? What if a box is to contain at least five of chocolated donuts and two strawberry donuts?

   一家麵包店銷售 5 種不同的甜甜圈，口味是原味、糖霜、巧克力、芒果跟草莓。如果該店每種糕點至少有兩打（24 個），那麼 (a) 可能配製成多少盒（20 個）不同種類的甜甜圈？(b) 如果盒子裡至少要有 5 個巧克力口味和 2 個草莓口味的，又是多少種？

   Answer: (a) $\underline{\dbinom{24}{4}}$, (b) $\underline{\dbinom{17}{4}}$.

5. (10 points) A student has 35 days to prepare for an examination. From past experience she knows that she will require no more than 53 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 16 hours.

   一個學生有 35 天用來準備考試。根據以往的經驗，她知道她的唸書時間不超過 53 小時，她也希望她每天至少要唸一個小時的書。證明：無論她怎麼安排她的讀書計畫（每天的唸書時間都是一個整數小時），都存在連續的某幾天，在這幾天裡，她恰好唸了 16 個小時的書。

   Solution：

   Similar to textbook page 71, section 3.1 application 4.

6. (10 points) Determine the mobile integers in

   $$\overrightarrow{4}\ \overleftarrow{8}\ \overrightarrow{3}\ \overleftarrow{1}\ \overrightarrow{6}\ \overleftarrow{7}\ \overleftarrow{2}\ \overrightarrow{5}$$

   and then use the algorithm for generating the next six permutations.

   Answer: the mobile integers are **8, 3, 7** .

   (1) $\overleftarrow{8}\ \overrightarrow{4}\ \overrightarrow{3}\ \overleftarrow{1}\ \overrightarrow{6}\ \overleftarrow{7}\ \overleftarrow{2}\ \overrightarrow{5}$. (2) $\overrightarrow{8}\ \overrightarrow{4}\ \overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{7}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{5}$.

   (3) $\overrightarrow{4}\ \overrightarrow{8}\ \overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{7}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{5}$. (4) $\overrightarrow{4}\ \overrightarrow{3}\ \overrightarrow{8}\ \overleftarrow{1}\ \overleftarrow{7}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{5}$.

   (5) $\overrightarrow{4}\ \overrightarrow{3}\ \overleftarrow{1}\ \overrightarrow{8}\ \overleftarrow{7}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{5}$. (6) $\overrightarrow{4}\ \overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{7}\ \overrightarrow{8}\ \overrightarrow{6}\ \overleftarrow{2}\ \overrightarrow{5}$.

7. (10 points) Construct a permutation whose inversion sequences are 6, 2, 4, 4, 1, 1, 2, 0, 0.

   Answer: **852693147** .

8. (10 points) Show that an m-by-n chessboard has a perfect cover by dominoes if and only if at least one of m and n is even.

   Solution：

   Check ch1 problem #1.

9. (10 points) Determine the number of integral solutions of the equation

$$x + y + z + w = 20$$

that satisfy

$$3 \leq x \leq 8, \ \ -6 \leq y \leq 1, \ 5 \leq z \leq 9, \ 5 \leq w \leq 9.$$

Answer: $\binom{16}{13} - [\binom{10}{7} + \binom{8}{5} + \binom{11}{8} + \binom{11}{8}] + [\binom{5}{2} + \binom{5}{2} + \binom{3}{0} + \binom{3}{0} + \binom{6}{3}] = 96$.

10. (10 points) Determine the number of ways to place six non-attacking rooks on the following 6-by-6 board, with forbidden positions as shown.

x	x				
x	x	x			
		x			
				x	
				x	x

Answer: $6! - 9 \times 5! + 27 \times 4! - 32 \times 3! + 14 \times 2! - 2 \times 1! = 122$.

Solution：

11. (10 points) A collection of subsets of $1, 2, ..., n$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

$1, 2, ..., n$ 的子集集合具有這樣的屬性，即每對子集至少有一個共同元素。證明集合中至多有 2^{n-1} 個子集。

Solution：

Ch 3, problem 27.

12. (10 points) Use combinatorial reasoning (組合解釋) to prove the identity (in the form given)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

(Hint: Let S be a set with three distinguished elements a, b and c and count certain k-subsets of S.)

Solution：

Ch 5, problem 11.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	120
Score:													