

應數一線性代數 2024 秋, 第一次期中考 解答

學號: _____, 姓名: _____

本次考試共有 9 頁 (包含封面), 有 12 題。如有缺頁或漏題, 請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程, 閱卷人員會視情況給予部份分數。沒有計算過程, 就算回答正確答案也不會得到滿分。答卷請清楚乾淨, 儘可能標記或是框出最終答案。
- 書寫空間不夠時, 可利用試卷背面, 但須標記清楚。

高師大校訓: 誠敬宏遠

誠: 一生動念都是誠實端正的。 敬: 就是對知識的認真尊重。
宏: 開拓視界, 恢宏心胸。 遠: 任重致遠, 不畏艱難。

請簽名保證以下答題都是由你自己作答的, 並沒有得到任何的外部幫助。

簽名: _____

1. (5 points) Given $\vec{u} = [-1, 1, 2]$, $\vec{v} = [4, 2, -1]$ and $\vec{w} = [5, 7, 4]$.

Is $\vec{w} \in \text{sp}(\vec{u}, \vec{v})$? (Yes / No) .

If so, write \vec{w} in the linear combination of \vec{v} and \vec{u} : $\vec{w} = 3\vec{u} + 2\vec{v}$.

2. (5 points) Given two vectors $\vec{v} = [x, 2, -1, 1]$ and $\vec{u} = [1, 6, -2, y]$. Find all $x, y \in \mathbb{R}$ so that

(a) \vec{v}, \vec{u} are parallel. None .

(b) \vec{v}, \vec{u} are perpendicular. $x \in \mathbb{R}, y = -x - 14$.

Solution :

(a) Since the second and third components of \vec{v} are $[2, -1]$ and the second and third components of \vec{u} is $[6, -2]$, we know that \vec{v}, \vec{u} are never parallel.

(b) It is fact that \vec{v}, \vec{u} perpendicular if $\vec{v} \cdot \vec{u} = 0$. Since $\vec{v} \cdot \vec{u} = x + 12 + 2 + y$, \vec{v} and \vec{u} are perpendicular if $x + 14 + y = 0$.

3. (10 points) (a) Find the inverse of the matrix A , if it exists, and (b) express the inverse matrix as a product of elementary matrices. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Answer: (a) $A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ (答案不唯一)

4. (10 points) Describe all possible values for the unknowns x_i so that the matrix equation is valid.

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$$

Solution :

1-4, problem 35. $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

5. (10 points) Find all values of r for which A and B commute.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & r \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Answer: $r = \underline{\textcolor{red}{1}}$

6. (10 points) Let a , b and c be scalar such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace of \mathbb{R}^3 .

Solution :

1-6 #12

7. (10 points) Assume the matrix A is row reduces to H , please answer the following questions.

$$A = \begin{bmatrix} 2 & 4 & 5 & 5 & 8 & 7 \\ -2 & -4 & -3 & 3 & 8 & 0 \\ 2 & 4 & 7 & 6 & 10 & -1 \\ 1 & 2 & 4 & 7 & 13 & 2 \end{bmatrix}, H = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) the **rank** of matrix A , is 4 .

(b) a basis for the **row space** of A is $[1, 2, 0, 0, -1, 0]$, $[0, 0, 1, 0, 0, 0]$, $[0, 0, 0, 1, 2, 0]$, $[0, 0, 0, 0, 0, 1]$.

(c) a basis for the **column space** of A is $\begin{bmatrix} 2 \\ -2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ -3 \\ 7 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 3 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 0 \\ -1 \\ 2 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

8. (10 points) Use the previous question (前一題), let $\vec{a}_1 = [2, -2, 2, 1]$, $\vec{a}_2 = [4, -4, 4, 2]$, $\vec{a}_3 = [5, -3, 7, 4]$, $\vec{a}_4 = [5, 3, 6, 7]$, $\vec{a}_5 = [8, 8, 10, 13]$, $\vec{a}_6 = [7, 0, -1, 2]$

(a) Is $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ a linearly independent set? (Yes /) .

(b) Is $\{\vec{a}_1, \vec{a}_2\}$ a linearly independent set?? (Yes /) .

(c) Is $\{\vec{a}_4, \vec{a}_5\}$ a linearly independent set?? (/ No) .

(d) Is $\{\vec{a}_1, \vec{a}_5, \vec{a}_6\}$ a linearly independent set? (/ No) .

(e) Is $\{\vec{a}_2, \vec{a}_4, \vec{a}_5\}$ a linearly independent set? (Yes /) .

p.s. 記得每小題要分開給理由 !!

9. (15 points) Suppose the complete solution to the equation

$$A\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \quad \text{is} \quad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) The dimension of the row space of $A = \underline{\mathbf{1}}$

Solution :

Since \vec{x} and $\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ are both 3×1 matrices, we know that A is a 3×3 matrix.

\vec{x} has two free variables, thus the nullity of A is 2. Therefore, the rank of A is 1.

(b) What is the matrix A ? Answer: $A = \underline{\begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \\ 2.5 & 5 & 0 \end{bmatrix}}$.

Solution :

Method 1:

$$\vec{x} = \begin{bmatrix} 2-2r \\ r \\ s \end{bmatrix} \Rightarrow \left[A \mid \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[A \mid \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right] = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 4 \\ 1 & 2 & 0 & 2 \\ 0.5 & 5 & 0 & 5 \end{array} \right]$$

Method 2:

1. $r = s = 0$:

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

2. $r = 1, s = 0$:

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

3. $r = 1, s = 1$:

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{12} + a_{13} \\ a_{22} + a_{23} \\ a_{32} + a_{33} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Find all possible \vec{b} so that $A\vec{x} = \vec{b}$ can be solved. Answer: $\vec{b} = \underline{r \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}, \text{ for } r \in \mathbb{R}}$.

Solution :

$$\vec{b} \in \text{col}(A)$$

10. (10 points) Prove that if A^3 is invertible, then A^2 is invertible.

Solution :

1-5, problem 23f.

11. (10 points) Let W_1 and W_2 be two subspace of \mathbb{R}^n . Prove that their intersection $W_1 \cap W_2$ is also a subspace.

Solution :

1-6, problem 47

Clearly $W_1 \cap W_2$ is nonempty; it contains 0.

Let $\vec{v}, \vec{w} \in (W_1 \cap W_2)$. Then $\vec{v}, \vec{w} \in W_1$ and $\vec{v}, \vec{w} \in W_2$, so $\vec{v} + \vec{w} \in W_1$ and $\vec{v} + \vec{w} \in W_2$ since W_1 and W_2 are subspaces.

Thus $\vec{v} + \vec{w} \in (W_1 \cap W_2)$. Similarly, $r\vec{v} \in W_1$ and $r\vec{v} \in W_2$. Since W_1 and W_2 are subspaces. Thus $r\vec{v} \in (W_1 \cap W_2)$. Thus W_1 and W_2 are subspaces. Thus $W_1 \cap W_2$ is a subspace of \mathbb{R}^n .

