

Section 9-2

課本第 35 題

Prove that an $n \times n$ matrix U unitary if and only if the rows of U form an orthonormal basis for \mathbb{C}^n .

解答：

\Rightarrow If U is unitary, by the definition, its column vectors are orthogonal unit vectors $(\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n)$. Assume there's $r_1, r_2, \dots, r_n \in \mathbb{C}$, such that $r_1\vec{c}_1 + \dots + r_n\vec{c}_n = \vec{0}$.

$$\because \langle \vec{c}_1, r_1\vec{c}_1 + \dots + r_n\vec{c}_n \rangle = \langle \vec{c}_1, \vec{0} \rangle = 0$$

$$\therefore r_1 \langle \vec{c}_1, \vec{c}_1 \rangle + \dots + r_n \langle \vec{c}_1, \vec{c}_n \rangle = r_1 \langle \vec{c}_1, \vec{c}_1 \rangle = r_1 = 0$$

Similarly, $r_2 = \dots = r_n = 0$. Therefore, $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ are linearly independent, that is, the column vectors of U are an orthonormal basis.

Since $U^*U = I = UU^* = \overline{U}U^T = (U^T)^*U^T$, U^T is also a unitary matrix. That is, the column vectors of U^T is also an orthonormal basis. Therefore, the row vectors of U is an orthonormal basis.

\Leftarrow Let the row vectors of U is $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

$$\overline{U}U^T = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} \vec{v}_1^T & \vec{v}_2^T & \dots & \vec{v}_n^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} = I$$

$I^T = (\overline{U}U^T)^T = UU^*$. Hence U is unitary.

課本第 37 題

Prove that the product of two commuting $n \times n$ Hermitian matrices is also a Hermitian matrix. What can you say about the sum of two Hermitian matrices?

解答：

Let H_1, H_2 are Hermitian matrices, i.e. $H_1^* = H_1, H_2^* = H_2$. Since H_1, H_2 are commuting, i.e. $H_1H_2 = H_2H_1$ $(H_1H_2)^* = H_2^*H_1^* = H_2H_1 = H_1H_2$. Hence H_1H_2 is a Hermitian matrix.

$(H_1 + H_2)^* = H_1^* + H_2^* = H_1 + H_2$. Hence $H_1 + H_2$ is a Hermitian matrix.

課本第 39 題

Let $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear transformation whose standard matrix representation is a unitary matrix U . Show that $\langle T(\vec{u}), T(\vec{v}) \rangle = \langle \vec{u}, \vec{v} \rangle$, for all $\vec{u}, \vec{v} \in \mathbb{C}^n$

解答：

I already proved in class.

課本第 40 題

Prove that for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $(\vec{u}^* \vec{v})^* = \overline{\vec{u}^* \vec{v}} = \vec{v}^* \vec{u} = \vec{u}^T \vec{v}$

解答：

$$\text{Let } \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{aligned} (\vec{u}^* \vec{v})^* &= (\overline{u_1} \quad \overline{u_2} \quad \dots \quad \overline{u_n}) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^* = \left(\sum_{i=1}^n \overline{u_i} v_i \right)^* \\ &= \overline{\sum_{i=1}^n \overline{u_i} v_i} (= \overline{\vec{u}^* \vec{v}}) \\ &= \sum_{i=1}^n u_i \overline{v_i} \\ &= [\overline{v_1} \quad \overline{v_2} \quad \dots \quad \overline{v_n}] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} (= \vec{v}^* \vec{u}) \\ &= [u_1 \quad u_2 \quad \dots \quad u_n] \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \\ \vdots \\ \overline{v_n} \end{bmatrix} (= \vec{u}^T \vec{v}) \end{aligned}$$

課本第 43 題

A square matrix A is normal if $A^* A = A A^*$

- (a) Show that every Hermitian matrix is normal.
- (b) Show that every unitary matrix is normal.
- (c) Show that, if $A^* = -A$, then A is normal.

解答：

(a) Let H are Hermitian matrices, i.e. $H^* = H$. $H H^* = H H = H^* H$.

(b) Let U are unitary matrices, i.e. $U^* U = I$, i.e. $U^{-1} = U^*$. $U U^* = I = U^* U$.

(c) If $A^* = -A$, $A^* A = (-A) A = -A A = A(-A) = A A^*$.

課本第 44 題

Let A be an $n \times n$ matrix. Referring to Exercise 43, prove that , if A is normal, then $\|A\vec{z}\| = \|A^*\vec{z}\|$ for all $\vec{z} \in \mathbb{C}^n$.

解答：

$$\|A\vec{z}\|^2 = (A\vec{z})^*(A\vec{z}) = \vec{z}^* A^* A \vec{z} = \vec{z}^* A A^* \vec{z} = (A^* \vec{z})^* (A^* \vec{z}) = \|A^* \vec{z}\|^2.$$

Since all norms are real and positive, $\|A\vec{z}\| = \|A^* \vec{z}\|$.