Quiz 14

1. 請框出答案. 2. 禁止作弊!

1. First check if Euler's method can be applied. If so, find the formula for error bound when using Euler's method to approximate the solutions for the following initial-value problems.

$$y' = e^{t-y},\ 0 \le t \le 1,\ y(0) = 1,\ \text{with } h = 0.5$$

Check the Well-Posed

We know that $y'(t) = f(t, y) = e^{t-y} \ge 0$, $y(0) = 1$. Obviously, $y(t)$ is non-decreasing when $0 \le t \le 1$. Hence, $y(t) \ge 1$ when $0 \le t \le 1$. Therefore, we can restrict D as $D = \{(t, y) | 0 \le t \le 1, y \ge 1\}$

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = \left|\frac{\partial}{\partial y} e^{t-y}\right| = \left|-e^{t-y}\right| = e^{t-y} \le e^{t-1} \le e^{1-1} = e^0 = 1 = L.$$

Hence by theorem 5.3 we know $f(t, y)$ satisfies a Lipschitz condition on D in the variable y with Lipschitz constant $L = 1$.

Find the error bound

$$\begin{aligned}\left|\frac{d^2}{dt^2} y(t)\right| &= \left|\frac{d}{dt} f(t, y)\right| = \left|\frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t)\right| = \left|\frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot f(t, y)\right| \\ &= \left|\frac{\partial}{\partial t} e^{t-y} + \left[\frac{\partial}{\partial y} e^{t-y}\right] \cdot e^{t-y}\right| = \left|e^{t-y} + [-e^{t-y}] \cdot e^{t-y}\right| \le \frac{1}{4} (\text{explained below.})\end{aligned}$$

Let $z = e^{t-y}$, $|e^{t-y} + [-e^{t-y}] \cdot e^{t-y}| = |z - z^2| = |-(z - \frac{1}{2})^2 + \frac{1}{4}| \le \frac{1}{4}$

Therefore $M = \frac{1}{4}$.

By Theorem 5.9, $h = 0.5, L = 1, M = \frac{1}{4}, a = 0$ the error bound is

$$|y(t_i) - w_i| \le \frac{hM}{2L}[e^{L(t_i - a)} - 1] = \frac{1}{16}[e^{t_i} - 1]$$

t_i	error bound
0.5	0.04054507941875801
1.0	0.10739261427869032