in $\mathbb{R}$ v.s. ① A: $n\times n$ $\mathbb{R}$ matrix, hope $\exists C$: orthogonal, s.t. $C^T A C = D$

② C: orthogonal, 保長

Thm

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{v} \longmapsto T(\vec{v}) = A\vec{v} \quad \Rightarrow \|\vec{v}\| = \|T(\vec{v})\| \quad \text{if } A: \text{orthogonal}$$

in $\mathbb{C}$ v.s. ① A: $n\times n$ $\mathbb{C}$ matrix, hope $\exists U$: unitary, s.t. $U^* A U = D$

② U: unitary, 保長

Thm

$$T: \mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$\vec{z} \longmapsto T(\vec{z}) = U\vec{z} \quad \Rightarrow \|\vec{z}\| = \|T(\vec{z})\| \quad \text{if } U: \text{unitary}$$

Thm

$$T: \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad \text{if } U: \text{unitary, s.m.v. of } T$$
$$\vec{z} \longmapsto T(\vec{z}) = U\vec{z} \quad \Rightarrow \|\vec{z}\| = \|T(\vec{z})\|$$

p.f.

$$\|\vec{z}\|^2 = \langle \vec{z}, \vec{z} \rangle = \vec{z}^* \vec{z}$$

$$\|U\vec{z}\|^2 = (U\vec{z})^*(U\vec{z}) = \vec{z}^* \underbrace{U^* U}_{= I} \vec{z} = \vec{z}^* \vec{z} = \|\vec{z}\|^2 \qquad \therefore \|\vec{z}\| = \|U\vec{z}\|$$

Def.

A: symmetric if $A^T = A$

△ A: real symmetric $\Rightarrow \overset{\text{orthogonal}}{C}{}^T A C = D$

Def

H: hermitian if $H^* = H$

△ H: hermitian $\Rightarrow U^* H \overset{\text{unitary}}{U} = D$

Cross Product

△ in $\mathbb{R}^3$, $\vec{a}, \vec{b}$, 找 $\vec{c}$ s.t. $\vec{a} \perp \vec{c}$, $\vec{b} \perp \vec{c}$

pick $\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

check $\vec{a} \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$, $\vec{b} \cdot \vec{c} = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

△ in $\mathbb{C}^3$, $\vec{a}, \vec{b}$, 找 $\vec{c}$ s.t. $\langle \vec{a}, \vec{c} \rangle = 0 = \langle \vec{b}, \vec{c} \rangle$

pick $\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{vmatrix}$

check $\langle \vec{a}, \vec{c} \rangle = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{vmatrix} = 0$

in $\mathbb{R}$ v.s.

A : $n\times n$ real , $A\vec{v}=\lambda\vec{v}$, $\vec{v}\neq 0$, λ : eigenvalue , $\vec{v}$: eigenvector corr. to λ

①求 λ : sol $p(\lambda)=|A-\lambda I|=0$

② 求 $\text{null}(A-\lambda_i I)$ 得 E_{λ_i}

in $\mathbb{C}$ v.s.

A : $n\times n$, $A\vec{v}=\lambda\vec{v}$, $\vec{v}\neq 0$, $\lambda \in \mathbb{C}$: eigenvalue , $\vec{v} \in \mathbb{C}^n$: eigenvector corr. to λ

有 n 个

①求 λ : sol $p(\lambda)=|A-\lambda I|=0$

② 求 $\text{null}(A-\lambda_i I)$ 得 E_{λ_i}

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ex:

$$A=\begin{bmatrix}1&0&i\\0&2&0\\-i&0&1\end{bmatrix}$$

$$p(\lambda)=\begin{vmatrix}1-\lambda&0&i\\0&2-\lambda&0\\-i&0&1-\lambda\end{vmatrix}=(2-\lambda)(1-\lambda)^2-(2-\lambda)\,i\,(-i)=-\lambda(2-\lambda)^2$$

$\therefore \lambda=0,2,2$

$\boxed{\lambda=0}$

$$[A-0I]=\begin{bmatrix}1&0&i\\0&2&0\\-i&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&i\\0&1&0\\0&0&0\end{bmatrix}\rightsquigarrow E_0=sp\left(\begin{bmatrix}-i\\0\\1\end{bmatrix}\right)$$

eigenvector corr. to 0

← dim 1

$\boxed{\lambda=2}$

$$[A-2I]=\begin{bmatrix}-1&0&i\\0&0&0\\-i&0&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-i\\0&0&0\\0&0&0\end{bmatrix}\rightsquigarrow E_2=sp\left(\begin{bmatrix}i\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

← dim 2

$$\therefore C=\begin{bmatrix}-i&i&0\\0&0&1\\1&1&0\end{bmatrix},\quad D=\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}\Rightarrow D=C^{-1}AC$$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

發現 $\vec{v}_1\perp\vec{v}_2$, $\vec{v}_1\perp\vec{v}_3$, $\vec{v}_2\perp\vec{v}_3$, $\|\vec{v}_1\|=\|\vec{v}_2\|=\sqrt{2}$, $\|\vec{v}_3\|=1$, $\vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$, $\vec{u}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}$, $\vec{u}_3=\frac{\vec{v}_3}{\|\vec{v}_3\|}$

let $U=\begin{bmatrix}\frac{-i}{\sqrt{2}}&\frac{i}{\sqrt{2}}&0\\0&0&1\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\end{bmatrix}$: unitary s.t. $D=U^*AU$

$\boxed{\lambda=2}$

$$[A-2I]=\begin{bmatrix}-1&0&i\\0&0&0\\-i&0&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-i\\0&0&0\\0&0&0\end{bmatrix}$$

← 求 nullspace

$$\begin{array}{ccc} & r & s \\ & \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $y = r$, $z = s$

$x - iz = 0 \Rightarrow x - is = 0 \Rightarrow x = is$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} is \\ r \\ s \end{bmatrix} = s\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \therefore E_2 = \text{null}(A - 2I) = \text{sp}\left(\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

---

in $\mathbb{R}$, $\mathbb{C}$

A : diagonalizable iff $\exists\, C$: invertible, D : diag. s.t. $D = C^{-1}AC$

☆ $C_{n\times n}$: invertible iff $\text{rank}(C) = n$

linear comb. coef. 在 $\mathbb{C}$

iff { column vector of C } : basis for $\mathbb{R}^n$, $\mathbb{C}^n$

△ $AC = CD$, if C : invertible

$\Rightarrow$ iff { column vector of C } : basis for $\mathbb{C}^n$

eigenvectors for A

$\because$ for eigenvalue λ, $\dim(E_\lambda) \le$ alg. multi. of λ

and $\Sigma \dim(E_\lambda) = n$

$\therefore$ for each eigenvalue λ, $\dim(E_\lambda) =$ geo. multi of λ = alg. multi of λ

△ $\boxed{\text{geo. multi of } \lambda = \text{alg. multi of } \lambda \Rightarrow A : \text{diagonalizable}}$

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ex: $A = \begin{bmatrix} i & c & 1 \\ 0 & i & 2i \\ 0 & 0 & 1 \end{bmatrix}$, A : diagonalizable, 求 c.

sol $|A - \lambda I| = \begin{vmatrix} i-\lambda & c & 1 \\ 0 & i-\lambda & 2i \\ 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - i)^2 \Rightarrow \lambda = 1, i, i$

alg multi. 2

$\boxed{\lambda = i}$

$$[A - iI] = \begin{bmatrix} 0 & c & 1 \\ 0 & 0 & 2i \\ 0 & 0 & 1-i \end{bmatrix} \sim \begin{bmatrix} 0 & c & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dim(E_i) = \begin{cases} 1 & \text{if } c \neq 0 \\ 2 & \text{if } c = 0 \end{cases}$$

$\therefore$ A : diag-able iff $c = 0$

Schur's Lemma

A : $n\times n$ (complex) $\Rightarrow \exists\, U$: unitary s.t. $U^* A U$: $\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$

Thm

A : hermitian $\Rightarrow \exists\, U$: unitary s.t. $U^* A U$: diag.

Moreover, all eigenvalues of A are real.

pf. of Thm

① by Schur's Lemma, $\exists\, U$: unitary s.t. $U^* A U = R$: $\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$

② $\underbrace{(U^* A U)^*}_{R} = U^* A^* (U^*)^* = U^* A^* U = U^* A U = R$

A : hermitian, i.e. $A^* = A$

$\therefore R^* = R$, $R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ 0 & & \ddots & r_{nn} \end{bmatrix} = R^* = \begin{bmatrix} \bar{r}_{11} & & & \\ \bar{r}_{12} & \bar{r}_{22} & & 0 \\ \vdots & & \ddots & \\ \bar{r}_{1n} & \cdots & & \bar{r}_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & & & 0 \\ & r_{22} & & \\ 0 & & \ddots & r_{nn} \end{bmatrix}$

$= 0$ $\qquad \forall\, r_{ii} \in \mathbb{R}$

$\therefore U^* A U = R$: real diag.

Cor.

A : real symmetric $\Rightarrow \exists\, C$: orthogonal s.t. $C^T A C = D$: diag.

pf.

$\because A$: real symmetric $\therefore A$: hermitian $\therefore$ all eigenvalue are real

$\therefore A - \lambda_i I$: real matrix $\sim \begin{bmatrix} \ddots & \\ & \end{bmatrix}$: real $\therefore E_{\lambda_i}$: has $\overbrace{\text{basis}}^{\text{orthornormal}}$ in $\mathbb{R}^n$

let $C = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$, $\{\vec{v}_i\}$: basis for each E_{λ_j}, $\underbrace{\text{then}}_{\text{by next thm}}$ C : orthogonal.

Thm.

H : hermitian has $\begin{cases} H\vec{v}_1 = \lambda_1 \vec{v}_1 \\ H\vec{v}_2 = \lambda_2 \vec{v}_2 \end{cases}$, $\vec{v}_1 \neq 0$, $\vec{v}_2 \neq 0$, $\lambda_1 \neq \lambda_2$

$\Rightarrow \vec{v}_1 \perp \vec{v}_2$

p.f.

$$\lambda_2 \langle \vec{v}_1 , \vec{v}_2 \rangle = \lambda_2 \vec{v}_1^* \vec{v}_2 = \vec{v}_1^* (\lambda_2 \vec{v}_2) = \vec{v}_1^* (H \vec{v}_2) = \vec{v}_1^* H \vec{v}_2$$

$\|$ ← H: hermitian

$$\lambda_1 \langle \vec{v}_1 , \vec{v}_2 \rangle = \bar{\lambda}_1 \vec{v}_1^* \vec{v}_2 = (\lambda_1 \vec{v}_1)^* \vec{v}_2 = (H \vec{v}_1)^* \vec{v}_2 = \vec{v}_1^* H^* \vec{v}_2$$

↑ all eigenvalues for hermitian are real

$$\therefore \lambda_2 \langle \vec{v}_1 , \vec{v}_2 \rangle = \lambda_1 \langle \vec{v}_1 , \vec{v}_2 \rangle \qquad \therefore \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \langle \vec{v}_1 , \vec{v}_2 \rangle = 0$$

$$\therefore \langle \vec{v}_1 , \vec{v}_2 \rangle = 0 \quad , \text{ i.e. } \vec{v}_1 \perp \vec{v}_2$$