

應數一線性代數 2023 春, 期末考 解答

學號: _____, 姓名: _____

本次考試共有 10 頁 (包含封面), 有 11 題。如有缺頁或漏題, 請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號, 並在每一頁的最上方屬名, 避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程, 閱卷人員會視情況給予部份分數。
沒有計算過程, 就算回答正確答案也不會得到滿分。
答卷請清楚乾淨, 儘可能標記或是框出最終答案。

高師大校訓: 誠敬宏遠

誠, 一生動念都是誠實端正的。 敬, 就是對知識的認真尊重。
宏, 開拓視界, 恢宏心胸。 遠, 任重致遠, 不畏艱難。

請尊重自己也尊重其他同學, 考試時請勿東張西望交頭接耳。

1. (10 points) Express $\frac{z}{w}$ in the form $a + bi$, where $a, b \in \mathbb{R}$, if

$$z = -1 + i, \quad w = 3 + 4i$$

Answer: $\frac{z}{w} = \underline{\frac{1+7i}{25}}$.

Solution :

From 9-1.

2. (10 points) Find the five fifth roots of $\sin(30^\circ) + i \cos(30^\circ)$. (need not simplify)

Solution :

From 9-1.

$$\sin(30^\circ) + i \cos(30^\circ) = \cos(60^\circ) + i \sin(60^\circ) = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right).$$

$$\left(\cos\left(\frac{\frac{\pi}{3} + 2k\pi}{5}\right) + i \sin\left(\frac{\frac{\pi}{3} + 2k\pi}{5}\right) \right), \text{ for } i = 0, 1, 2, 3, 4$$

3. (10 points) Let A is an 3×3 complex matrix with $\det(A) = 2 + 3i$. Please the value for $\det(iA)$ and $\det(A^*)$.

Answer: $\det(iA) = \underline{i^3(2 + 3i) = 3 - 2i}$, $\det(A^*) = \underline{2 - 3i}$, $\det(A^2) = \underline{(2 + 3i)^2 = -5 + 12i}$.

Solution :

From 9-2, using the technique from Sec. 4-2.

A is an $n \times n$, then $\det(aA) = a^n \det(A)$ and $\det(A) = \det(A^T)$.

Moreover, the definition of the conjugate transpose is also requested. If $A = [a_{ij}]$, $A^* = \overline{A}^T = [\overline{a_{ji}}] = [\overline{a_{ji}}]$

4. (10 points) Given the coordinate vector $\vec{v}_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Please find the \vec{v} and \vec{v}'_B when the ordered basis B and B' for P_2 are

$$B = (x^2 - x, 2x + 1, -x - 5), B' = (1, (2 + x), (2 + x)^2)$$

Answer: $\vec{v} = \underline{2x^2 - 7x - 16}$, $\vec{v}'_B = \underline{\begin{bmatrix} 6 \\ -15 \\ 2 \end{bmatrix}}$

Solution :

From 7-1

$$\vec{v} = M_B \vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -16 \end{bmatrix} \Rightarrow 2x^2 - 7x - 16$$

$$\vec{v}_{B'} = C_{BB'} \vec{v}_B = M_{B'}^{-1} M_B \vec{v}_B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -15 \\ 2 \end{bmatrix}$$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for linear transformation $T : P_2 \rightarrow P_2$ defined by $T(p(x)) = \frac{d}{dx}p(x-1)$, $B = (x^2, x, 1)$, $B' = (x^2 - 1, x - 3, 2)$.

$$C_{B,B'} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \quad C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}, \quad R_{B',B'} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix} \quad \text{and}$$

$$R_{B,B} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}.$$

Is $C = C_{B,B'}$ or $C_{B',B}$? $C_{B',B}$.

Solution :

From 7-2

$$T(x^2) = \frac{d}{dx}(x-1)^2 = 2x-2, \quad T(x) = \frac{d}{dx}(x-1) = 1, \quad T(1) = \frac{d}{dx}1 = 0$$

Thus

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = R_E \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We have

$$R_{B,B} = R_B = R_E = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

By $C_{B',B} = M_B^{-1}M_{B'} = M_E^{-1}M_{B'} = I^{-1}M_{B'} = M_{B'}$,

$$C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$C_{B,B'} = C_{B',B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Since

$$R_{B'} = R_{B',B'} = C_{B,B'}R_B C_{B',B}$$

$$R_{B'} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix}$$

6. (10 points) Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$. Also find where

$$A = \begin{bmatrix} 2 & 0 & 1-i \\ 0 & -3 & 0 \\ 1+i & 0 & 1 \end{bmatrix}$$

Answer: $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $U = \begin{bmatrix} 0 & \frac{-1+i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Solution :

From 9-3.

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 0 & -1+i & 1-i \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \text{ such that } D = C^{-1}AC.$$

Since A is Hermitian matrix, it is unitarily diagonalizable.

We also notice that $\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1+i \\ 0 \\ 2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1+i \\ 0 \\ 2 \end{bmatrix} \right\rangle = 0$

$$U = \begin{bmatrix} 0 & \frac{-1+i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer: Jordan canonical form = $J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$

Jordan basis = $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solution :

From 9-4. Quiz 16.

8. (10 points) Find a polynomial in A that gives the zero matrix.

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Answer: $(A - 4I)^2(A - 9iI)^3 = O$ or $(A - 4I)^4(A - 9iI)^5 = O$.

9. (10 points) Prove that every 2×2 real matrix that is unitarily diagonalizable has one of the following forms: $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ -b & d \end{bmatrix}$, for $a, b, d \in \mathbb{R}$.

Solution :

Section 9-3 problem 17.

10. (30 points) Prove or disprove the following statement:

- (a) every unitarily diagonalizable matrix is Hermitian.

Solution :

9-3 problem 19(d): False!

From 9-3, Theorem 9.7.

A square matrix A is unitarily diagonalizable if and only if it is a normal matrix.

Therefore, just build a normal matrix which is not a Hermitian matrix as the counterexample.

- (b) If U is unitary, then $(\bar{U})^{-1} = U^T$.

Solution :

9-2 problem 33(e): True!

- (c) every unitary matrix is normal.

Solution :

9-2 problem 43(b): True!

(d) If $A^* = -A$, then A is normal.

Solution :

9-2 problem 43(c): True!

(e) $\det(C_{BB'}) = 1$ if and only if $B = B'$.

Solution :

From 7-1 problem 23 (h) : False!!

(f) If $C_{B,B'}$ is an orthogonal matrix and B is an orthonormal basis, then B' is an orthonormal basis.

Solution :

7-1 problem 23(e): True!

