葉均承
Jean Yeh

Homework 9-2

Section 9-2 How the "cross product" works in $\mathbb{C}^3$

Check the the idea for cross product in Ch4.

The Cross Product

Equation (2) defines a **second-order determinant**, associated with a 2×2 matrix. Another application of these second-order determinants appears when we find a vector in $\mathbb{R}^3$ that is perpendicular to each of two given independent vectors $\mathbf{b} = [b_1, b_2, b_3]$ and $\mathbf{c} = [c_1, c_2, c_3]$. Recall that the unit coordinate vectors in $\mathbb{R}^3$ are $\mathbf{i} = [1, 0, 0]$, $\mathbf{j} = [0, 1, 0]$, and $\mathbf{k} = [0, 0, 1]$. We leave as an exercise the verification that

$$\mathbf{p} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k} \tag{3}$$

is a vector perpendicular to both $\mathbf{b}$ and $\mathbf{c}$. (See Exercise 5.) This can be seen by computing $\mathbf{p} \cdot \mathbf{b} = \mathbf{p} \cdot \mathbf{c} = 0$. The vector $\mathbf{p}$ in formula (3) is known as the **cross product** of $\mathbf{b}$ and $\mathbf{c}$, and is denoted $\mathbf{p} = \mathbf{b} \times \mathbf{c}$.

There is a very easy way to remember formula (3) for the cross product $\mathbf{b} \times \mathbf{c}$. Form the 3×3 *symbolic matrix*

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Formula (3) can be obtained from this matrix in a simple way. Multiply the vector $\mathbf{i}$ by the determinant of the 2×2 matrix obtained by crossing out the row and column containing $\mathbf{i}$, as in

$$\begin{bmatrix} \cancel{\mathbf{i}} & \cancel{\mathbf{j}} & \cancel{\mathbf{k}} \\ \cancel{b_1} & b_2 & b_3 \\ \cancel{c_1} & c_2 & c_3 \end{bmatrix}.$$

Similarly, multiply $(-\mathbf{j})$ by the determinant of the matrix obtained by crossing out the row and column in which $\mathbf{j}$ appears. Finally, multiply $\mathbf{k}$ by the determinant of the matrix obtained by crossing out the row and column containing $\mathbf{k}$, and add these multiples of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to obtain formula (3).

Similarly, in the complex space, let $\vec{p} = \vec{b} \times \vec{c}$ and then have $\vec{b} \cdot \vec{p} = 0 = \vec{c} \cdot \vec{p}$. Try to build $\vec{p}$ in the same way, and note that the cross product will turn the first vector into its conjugation. $\vec{b} \cdot \vec{p} = \overline{\vec{b}^T} \vec{p}$.

If $\vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{vmatrix}$, then $\vec{b} \cdot \vec{p} = \begin{vmatrix} \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{vmatrix} = 0 = \begin{vmatrix} \overline{c_1} & \overline{c_2} & \overline{c_3} \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{vmatrix} = \vec{c} \cdot \vec{p}$

Hence, that is the "cross product" that ww are looking for.