Section 1.3 Matrices and Their Algebra

32. Method 1: The $(i,j)^{th}$ entry of $(AB)^T$ is the $(j,i)^{th}$ entry in AB, which is

$$\begin{aligned} & (j^{th} \text{ row of } A)\cdot(i^{th} \text{ column of } B) \\ = \ & (i^{th} \text{ column of } B)\cdot(j^{th} \text{ row of } A) \\ = \ & (i^{th} \text{ row of } B^T)\cdot(j^{th} \text{ column of } A^T) \end{aligned}$$

which is the $(i,j)^{th}$ entry of B^TA^T. Since $(AB)^T$ and B^TA^T have the same size, they are equal.

32. Method 2:

Let

$$A = [a_{ij}]_{m\times n}, \quad B = [b_{ij}]_{n\times s}, \quad AB = C = [c_{ij}]_{m\times s}.$$

Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

So

$$(AB)^T = C^T = [c'_{ij}], \quad \text{where } c'_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki}.$$

Now, consider

$$A^T = [a'_{ij}]_{n\times m}, \quad B^T = [b'_{ij}]_{s\times n}, \quad B^TA^T = D = [d_{ij}]_{s\times m}.$$

Here

$$a'_{ij} = a_{ji}, \quad b'_{ij} = b_{ji}.$$

Thus

$$d_{ij} = \sum_{k=1}^{n} b'_{ik}a'_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki} = c'_{ij}.$$

Therefore,

$$C^T = D \quad \Rightarrow \quad (AB)^T = B^TA^T.$$

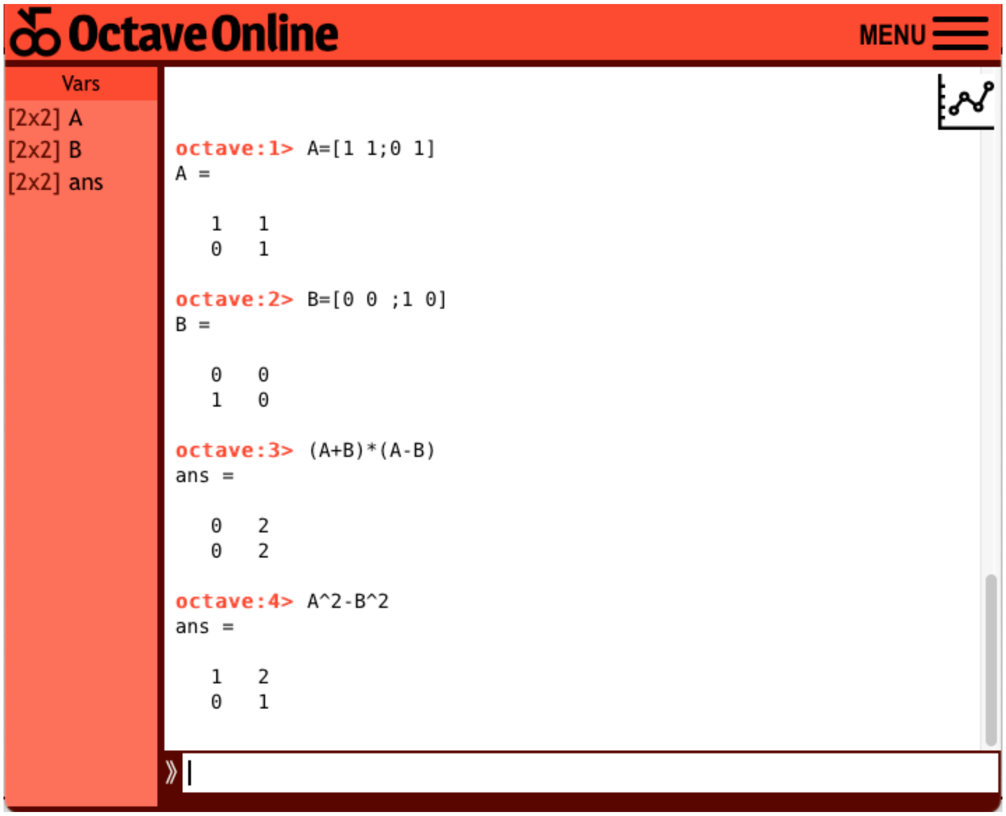
43.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$(A+B)(A-B) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \ A^2 - B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Since $(A+B)(A-B) = (A^2-B^2)+(BA-AB)$, we know that $(A+B)(A-B) = (A^2-B^2)$ only if $BA - AB = 0$. Therefore, the state holds only under the conditions that A, B are commutative.

p.s. You can using `https://octave-online.net` to check your example as below:



45. ** 課本答案錯了喔！

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ r & 0 & r \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} = \begin{bmatrix} 2 & 0 & r \\ 0 & 1 & 0 \\ 2 & 0 & r \end{bmatrix}$$

Therefore,$AB = BA$ if and only if $r = 2$.