J. Math. Anal. Appl. 460 (2018) 1–16

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Positive solutions for the Kirchhoff-type problem involving general critical growth – Part I: Existence theorem involving general critical growth

Huixing Zhang [a], Cong Gu [b,*], Chun-Ming Yang [c], Jean Yeh [b], Juan Jiang [a]

[a] *School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, China*
[b] *Department of Mathematics, Texas A&M University, College Station, TX, USA*
[c] *Department of Mathematics, National Taiwan University, Taipei, Taiwan, ROC*

A R T I C L E   I N F O

Article history:
Received 31 May 2017
Available online 14 September 2017
Submitted by G. Chen

Keywords:
Kirchhoff-type problem
Positive solutions
Critical nonlinearity

A B S T R A C T

In this paper, we consider the following Kirchhoff-type problem

$$\begin{cases} \left(a+\lambda\int\limits_{\mathbb{R}^3}|\nabla u|^2dx+\lambda b\int\limits_{\mathbb{R}^3}|u|^2dx\right)(-\Delta u+bu)=f(u), \text{ in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3),\ u>0, \text{ in } \mathbb{R}^3, \end{cases}$$

where $\lambda \geq 0$ is a parameter, a, b are positive constants and f reaches the critical growth. Without the Ambrosetti–Rabinowitz condition, we prove the existence of positive solutions for the Kirchhoff-type problem with a general critical nonlinearity. We also study the asymptotics of solutions as $\lambda \to 0$. Numerical solutions for related problems will be discussed in the second part.

1. Introduction

This is the first paper in a series. It deals with theory while the second part is concerned with the numerical aspects. In this paper Part I, we consider the existence of positive solutions for the following nonlinear Kirchhoff-type problem

$$\begin{cases} \left(a+\lambda\int\limits_{\mathbb{R}^3}|\nabla u|^2dx+\lambda b\int\limits_{\mathbb{R}^3}|u|^2dx\right)(-\Delta u+bu)=f(u), \text{ in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3),\ u>0, \text{ in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail address: gucong@math.tamu.edu (C. Gu).

http://dx.doi.org/10.1016/j.jmaa.2017.09.010

where $\lambda \geq 0$, a, b are positive constants and the general nonlinearity f has a critical growth. Problem (1.1) arises from an interesting physical background. In fact, if we replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$ and let $\lambda = 1$ and $b = 0$, we obtain the following Kirchhoff-type problem

$$\begin{cases} -\left(a + \int\limits_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

which is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.3}$$

Equation (1.3) was first proposed by Kirchhoff in [18] describing the classical D'Alembert's wave equations for transversal oscillations of elastic strings, particularly, taking into account the change in string length caused by vibration. Lions introduced in [22] a functional analysis approach and described the abstract framework to the following problem

$$\begin{cases} u_{tt} - \left(a + b \int\limits_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{1.4}$$

Besides this, problem (1.4) also models several biological systems [1] from a mathematical biological point of view, where u shows a process that depends on the average of itself (for example, population density). For more detailed physical and biological background of Kirchhoff-type problem, we refer the reader to the papers [4,26] and the references therein.

Recently, problem (1.2) has been studied in literatures by variational methods, cf., for example [7,13,14, 27–29,37,39]. These works show an increasing interest in studying the existence of least energy solutions, positive solutions, multiple solutions, sign-changing solutions and semiclassical states. Meanwhile, various solvability conditions on the general nonlinearity f near infinity and zero, for example, the asymptotic case [31] and super-linear case [27], have been considered. Particularly, in [1], Alves, Corrêa and Ma considered problem (1.2) and proved the existence of positive solutions by the Mountain Pass Theorem. In [28], using the Young index and critical groups, Perera and Zhang obtained nontrivial solutions for problem (1.2). With the aid of mini–max methods and invariant sets of decent flow, Zhang and Perera [39], Mao and Zhang [27] studied the existence of three solutions (a sign-changing solution, a positive solution and a negative solution). In [13], He and Zou proved the existence of infinitely many solutions by Fountain Theorems. For more results of (1.2), we refer the reader to [9,10,24].

In terms of the Kirchhoff-type problem in $\mathbb{R}^N$, there are also several existence results, see for example [2,12,15,16,20,21,19,23,25,32,33,35,36] and the references therein. In these works, the existence of positive solutions, mountain pass solutions and high energy solutions were obtained with f satisfying various conditions. In particular, we mention the following two existence results for (1.1) with $\mathbb{R}^3$ replaced by $\mathbb{R}^N$. In [19], Li et al. considered problem (1.1) under the following assumptions:

(f_1) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $|f(t)| \leq c\left(|t| + |t|^{p-1}\right)$ for all $t \in \mathbb{R}_+ = [0, \infty)$ and some $p \in (2, 2^*)$, where $2^* = 2N/(N-2)$, for $N \geq 3$;

(f$_2$) $\lim\limits_{t\to 0^+} \dfrac{f(t)}{t} = 0$;

(f$_3$) $\lim\limits_{t\to \infty} \dfrac{f(t)}{t} = \infty$.

They used a cut-off functional to get a bounded Palais–Smale (PS) sequence and led to the following theorem.

Theorem A. *Assume that $N \geq 3$, a, b are positive constants and $\lambda \geq 0$ is a parameter. If the conditions (f$_1$)–(f$_3$) hold, then there exists a $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$, problem (1.1) has at least one positive solution.*

However, it is not clear whether the result in Theorem A still holds for large $\lambda > 0$. In [25], Liu, Liao and Tang again considered problem (1.1), but gave the following weaker conditions than the ones in [19]:

(f$_4$) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$ and $\lim\limits_{s\to 0^+} \dfrac{f(s)}{s} = 0$;

(f$_5$) $\lim\limits_{s\to \infty} \dfrac{f(s)}{s^{2^*-1}} = 0$ with $2^* = \dfrac{2N}{N-2}$;

(f$_6$) There exists $\xi > 0$ such that $F(\xi) := \displaystyle\int_0^{\xi} f(t)\, dt > \frac{ab}{2}\xi^2$.

They proved the existence of a positive solution for problem (1.1) using cut-off and monotonicity tricks. This result improved the results in [19]. They obtained another result that problem (1.1) has nonzero solution with large $\lambda > 0$ under some conditions.

However, the authors in both [19] and [25] considered problem (1.1) with the general nonlinearity f involving only subcritical growth. To our knowledge, no study has been conducted on problem (1.1) involving general critical growth. In this paper, we prove the existence of positive solutions to problem (1.1) with general critical nonlinearity. The main difficulties are as follows. On the one hand, because of the appearance of the terms $\int_{\mathbb{R}^3} |\nabla u|^2\, dx$ and $\int_{\mathbb{R}^3} |u|^2\, dx$, problem (1.1) is a non-local problem, which implies that equation (1.1) is not a pointwise identity. This phenomenon causes some mathematical difficulties which make the study of (1.1) interesting. On the other hand, the main difficulty comes from the general critical nonlinearity f. In [19] and [25], the authors used cut-off functionals to obtain the boundedness of (PS) sequences. This approach or trick is not suitable for problem (1.1) involving critical growth. Indeed, it is not easy to obtain bounded (PS) sequences due to the lack of the Ambrosetti–Rabinowitz condition. To overcome this difficulty, we adopt some ideas in [5] and [6]. First, we apply a local deformation argument as in [5] to obtain a bounded (PS) sequence. Second, we make a crucial modification on the min–max value as in [6]. In fact, we define the other min–max value C_λ and prove all paths to be uniformly bounded with respect to λ. The detailed arguments can be found in Section 2.

Throughout the paper, we make the following assumptions:

(H$_1$) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$ and $\lim\limits_{s\to 0^+} \dfrac{f(s)}{s} = 0$;

(H$_2$) $\limsup\limits_{s\to \infty} \dfrac{f(s)}{s^5} \leq 1$;

(H$_3$) there exist $k \in (2, 6)$ and $\mu > \mu_k$ such that $f(s) \geq \mu s^{k-1}$ for all $s \geq 0$, where

$$\mu_k = \left[\frac{3(k-2)}{2kS^{\frac{3}{2}}}\right]^{\frac{k-2}{2}} a^{\frac{6-k}{4}} C_k{}^{\frac{k}{2}}. \tag{1.5}$$

Here, S and C_k in the definition of μ_k are the best constants of Sobolev embeddings $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ (cf. the definition of $D^{1,2}(\mathbb{R}^3)$ in Section 2) and $H^1(\mathbb{R}^3) \hookrightarrow L^k(\mathbb{R}^3)$ respectively, namely,

$$S \left(\int_{\mathbb{R}^3} |u|^6 \, dx \right)^{\frac{1}{3}} \leq \int_{\mathbb{R}^3} |\nabla u|^2 \, dx, \text{ for all } u \in D^{1,2}(\mathbb{R}^3)$$

and

$$C_k \left(\int_{\mathbb{R}^3} |u|^k dx \right)^{\frac{2}{k}} \leq \int_{\mathbb{R}^3} \left(|\nabla u|^2 + b|u|^2 \right) \, dx, \text{ for all } u \in H^1(\mathbb{R}^3).$$

Our main results are as follows.

Theorem 1.1. *Assume that f satisfies the conditions (H_1)–(H_3). Then there exists a positive constant λ^* such that, for every $\lambda \in (0, \lambda^*)$, problem (1.1) has at least one nontrivial positive solution.*

When $\lambda = 0$ in (1.1), the equation reduces to

$$-a\Delta u + abu = f(u), \text{ in } \mathbb{R}^3. \tag{1.6}$$

Equation (1.6) is viewed as the limiting problem of (1.1). Indeed, problem (1.6) plays an important role in studying problem (1.1). As is usually expected, if the limit problem (1.6) is well-behaved and undergoes a small perturbation, the perturbed problem (1.1) possesses a solution in the neighborhood of that of the limit problem. The result in this direction can be stated as the following.

Theorem 1.2. *For every $\lambda > 0$ small enough, there exists a positive solution $u_\lambda \in H^1(\mathbb{R}^3)$ for problem (1.1) such that, u_λ converges to u in $H^1(\mathbb{R}^3)$ as $\lambda \to 0$ along a subsequence, where u is a ground state solution for the limiting problem (1.6).*

Remark 1.3. In [3], if the general nonlinearity f satisfied conditions (H_1), (H_2) and

($H_3{}'$) there exist $k \in (2, 6)$ and $\mu > 0$ such that $f(s) \geq \mu s^{k-1}$ for all $s \geq 0$;
(H_4) $sf(s) - 2F(s) \geq 0$ for all $s \geq 0$, where $F(s) = \int_0^s f(\tau) \, d\tau$,

authors of [3] have proved the existence of a ground state solution for problem (1.6). In this paper, we notice that (H_4) can be removed at the cost of introducing a lower bound for μ, i.e., replacing ($H_3{}'$) with (H_3). In order to demonstrate that our results can be applied to nonlinearities that were not covered in [3], consider the following example of critical nonlinearity.

Example 1.4.

$$f(s) = \begin{cases} s^5 + \alpha s^2 |\ln s| + \beta s^3, & s > 0, \\ 0, & s = 0, \end{cases}$$

where $\alpha > 3 + \frac{9}{4}\beta$. The above nonlinearity satisfies (H_1)–(H_3) given sufficiently large β. However, direct computation shows that f doesn't satisfy the inequality in (H_4) on $s \in (1 - \delta, 1)$ for some $\delta > 0$.

The remainder of this paper is organized as follows. In Section 2, we give some notations and preliminary results and construct the min–max level. In Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2.

2. Some preliminary results and min–max levels

In this section, we first introduce the following notations. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space equipped with the inner product and norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+buv)\,dx,\quad \|u\|=(u,u)^{\frac{1}{2}}.$$

For any $1\le q<+\infty$, the usual norm of the Lebesgue space $L^q(\mathbb{R}^3)$ is denoted as $\|\cdot\|_q$. Let $D^{1,2}(\mathbb{R}^3)=\{u\in L^6(\mathbb{R}^3):\nabla u\in L^2(\mathbb{R}^3)\}$ be the Sobolev space with the norm $\|u\|^2_{D^{1,2}}=\int_{\mathbb{R}^3}|\nabla u|^2\,dx$ and $H^1_r(\mathbb{R}^3)$ be the subspace of $H^1(\mathbb{R}^3)$ that consists of radially symmetric functions. Let c_i denote various positive constants.

Since we study the positive solutions to problem (1.1), we may assume that $f(s)=0$ for all $s\le 0$. The energy functional for problem (1.1) is defined by

$$\Phi_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{\lambda}{4}\|u\|^4-\int_{\mathbb{R}^3}F(u)dx,$$

where $F(t)=\int_0^t f(s)\,ds$.

It is standard to prove that $\Phi_\lambda\in C^1(H^1(\mathbb{R}^3),\mathbb{R})$ and has the following variational derivative

$$\langle\Phi'_\lambda(u),v\rangle=a(u,v)+\lambda\|u\|^2(u,v)-\int_{\mathbb{R}^3}f(u)v\,dx,\quad \forall u,v\in H^1(\mathbb{R}^3).$$

Clearly, the critical points of Φ_λ are the weak solutions for problem (1.1). Since problem (1.1) is autonomous, we look for critical points of Φ_λ on $H^1_r(\mathbb{R}^3)$, which is a natural constraint (cf. Theorem 1.28 in [34]).

Next, we study the existence of ground state solutions to problem (1.6).

Proposition 2.1. *Assume that f satisfies (H_1)–(H_3). Then problem (1.6) has a ground state solution $u\in H^1_r(\mathbb{R}^3)$.*

The following Pohozăev identity is helpful.

Lemma 2.2 *(Pohozăev Identity). If u is a nonzero solution of the equation*

$$-a\Delta u+abu=f(u),\ \text{in } \mathbb{R}^3, \tag{2.1}$$

then the following Pohozăev identity

$$a\left(\int_{\mathbb{R}^3}|\nabla u|^2dx+3b\int_{\mathbb{R}^3}u^2dx\right)=6\int_{\mathbb{R}^3}F(u)\,dx \tag{2.2}$$

holds.

Proof. The proof is similar to that of Lemma 2.6 in [19]. We omit the details. □

In order to prove Proposition 2.1, we introduce the following notations

$$\mathcal{K}=\left\{u\in H^1_r(\mathbb{R}^3)\backslash\{0\}:\int_{\mathbb{R}^3}G(u)dx=1\right\},$$

$$\mathcal{P} = \left\{ u \in H_r^1(\mathbb{R}^3)\backslash\{0\} : 6 \int_{\mathbb{R}^3} G(u)dx = a \int_{\mathbb{R}^3} |\nabla u|^2 dx \right\},$$

where $G(t) = F(t) - \frac{ab}{2}t^2$ and $\mathcal{P}$ is viewed as the Pohozăev manifold. The Pohozăev identity is used to simplify the energy functional. The Pohozăev manifold enables us to obtain weaker conditions than those of a Nehari manifold. By (H_3), it follows that there is a constant $\xi > 0$ such that $G(\xi) > 0$. Then, we easily obtain that $\mathcal{K} \neq \emptyset$ and $\mathcal{P} \neq \emptyset$. Define

$$N = \frac{a}{2} \inf_{u \in \mathcal{K}} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx, \quad p = \inf_{u \in \mathcal{P}} J(u),$$

and the min–max value

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} J(\gamma(t)),$$

where $\Gamma = \left\{\gamma \in C\left([0,1], H_r^1(\mathbb{R}^3)\right) : \gamma(0) = 0, J(\gamma(1)) < 0\right\}$ and

$$J(u) = \frac{a}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + b|u|^2\right) dx - \int_{\mathbb{R}^3} F(u)dx.$$

Lemma 2.3. *Assume that (H_1)–(H_3) hold. Then* $0 < N < \frac{\sqrt[3]{6}}{2} aS$ *and* $p < \frac{1}{3}(aS)^{\frac{3}{2}}$.

Proof. By (H_1)–(H_2), there exists $c_1 > 0$ such that

$$f(s) \le abs + c_1 s^5, \text{ for all } s \ge 0. \tag{2.3}$$

We can indeed prove $N > 0$. Assume the contrary that $N = 0$, then there exists $\{u_n\} \subset \mathcal{K}$ such that $\|\nabla u_n\|_2^2 \to 0$ as $n \to \infty$. By Sobolev's embedding theorem $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we have $\|u_n\|^6 \to 0$ as $n \to \infty$. Therefore, (2.3) implies that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} G(u_n) \, dx \le \limsup_{n \to \infty} \frac{c_1}{6} \int_{\mathbb{R}^3} |u_n|^6 \, dx = 0.$$

This is a contradiction with $\int_{\mathbb{R}^3} G(u_n) \, dx = 1$. Next, we claim that $p \le c$. This proof is similar to that of Lemma 4.1 in [17], so we omit the details. Furthermore, we use a similar idea in [11] to prove that $p = \frac{\sqrt{12}}{9} N^{\frac{3}{2}}$. Define an operator $\Psi : \mathcal{K} \to \mathcal{P}$ by $(\Psi(u))(x) = u(x/t_u)$, where $t_u = \sqrt{\frac{a}{6}}\|\nabla u\|_2$. It is easy to show that Ψ is a bijection. For any $u \in \mathcal{K}$, we get

$$\begin{aligned} J(\Psi(u)) &= \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u(x/t_u)|^2 \, dx - \int_{\mathbb{R}^3} G(u(x/t_u)) \, dx \\ &= \frac{a}{2} t_u \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - t_u^3 \int_{\mathbb{R}^3} G(u) \, dx \\ &= \frac{a}{2} t_u \cdot \frac{6}{a} t_u^2 - t_u^3 \\ &= \frac{\sqrt{6}}{18} a^{\frac{3}{2}} \|\nabla u\|_2^3. \end{aligned}$$

Thus,

$$\inf_{u\in\mathcal{P}} J(u) = \inf_{u\in\mathcal{K}} J(\Psi(u)) = \frac{\sqrt{6}}{18} a^{\frac{3}{2}} \inf_{u\in\mathcal{K}} \|\nabla u\|_2^3.$$

Note that $\inf_{u\in\mathcal{K}} \|\nabla u\|_2^2 = \frac{2N}{a}$, so $p = \frac{2\sqrt{3}}{9} N^{\frac{3}{2}}$. Finally, choosing a $\phi \in H_r^1(\mathbb{R}^3)$ with $\phi \geq 0$, $\|\phi\|_k^2 = C_k^{-1}$ and $\|\phi\| = 1$, then we have

$$\begin{aligned} c \leq \max_{t\geq 0} J(t\phi) &= \max_{t\geq 0}\left(\frac{at^2}{2}\|\phi\| - \int_{\mathbb{R}^3} F(t\phi)\,dx\right) \\ &\leq \max_{t\geq 0}\left(\frac{at^2}{2} - \mu\frac{t^k}{k}\|\phi\|_k^k\right) \\ &\leq \frac{k-2}{2k} a^{\frac{k}{k-2}} \mu^{-\frac{2}{k-2}} C_k^{\frac{k}{k-2}}. \end{aligned}$$

Together with $\mu > \mu_k$ in (1.5), we get $p < \frac{1}{3}(aS)^{\frac{3}{2}}$ and $N < \frac{\sqrt[3]{6}}{2} aS$. □

In order to prove that the limiting problem (1.6) has a ground state solution, we give the following Brezis–Lieb Lemma.

Lemma 2.4. *Let $h \in C(\mathbb{R}^3 \times \mathbb{R})$ and assume that*

$$\lim_{t\to 0} \frac{h(x,t)}{t} = 0 \text{ and } \lim_{|t|\to\infty} \frac{|h(x,t)|}{|t|^5} < \infty, \text{ uniformly in } x \in \mathbb{R}^3.$$

If $u_n \to u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$, then

$$\int_{\mathbb{R}^3} H(x,u_n)\,dx = \int_{\mathbb{R}^3} (H(x,u_n-u_0) + H(x,u_0))\,dx + o(1),$$

where $H(x,t) = \int_0^t h(x,s)\,ds$.

Proof. The proof is similar to Lemma 2.5 in [38]. We omit the details. □

Proof of Proposition 2.1. For any $u \in H^1(\mathbb{R}^3)$, let

$$T(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx \quad \text{and} \quad V(u) = \int_{\mathbb{R}^3} G(u)\,dx.$$

We know that $N = \inf\{T(u) : V(u) = 1, u \in H_r^1(\mathbb{R}^3)\}$. Suppose that there exists $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ such that $a\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx \to 2N$ as $n \to \infty$, with $\int_{\mathbb{R}^3} G(u_n)\,dx = 1$. Together with ($H_1$)–($H_2$), we easily obtain that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. There is $u_0 \in H_r^1(\mathbb{R}^3)$ such that along a subsequence, $u_n \to u_0$ weakly in $H_r^1(\mathbb{R}^3)$. By Lemma 2.4, we have

$$T(u_n) = T(v_n) + T(u_0) + o(1)$$

and

$$V(u_n) = V(v_n) + V(u_0) + o(1),$$

where $v_n = u_n - u_0$. In the following, we prove that $u_0 \in \mathcal{K}$ and $u_n \to u_0$ strongly in $H_r^1(\mathbb{R}^3)$. We claim that

$$T(u) \geq N(V(u))^{\frac{1}{3}}$$

for any $u \in H_r^1(\mathbb{R}^3)$ and $V(u) > 0$. Indeed, for any $u \in H_r^1(\mathbb{R}^3)$ with $V(u) > 0$, there exists a $t \in \mathbb{R}$, such that $V(u(x/t)) = 1$. By direct computation, $V(u(x/t)) = t^3 V(u(x))$ and $T(u(x/t)) = tT(u(x))$. Thus, we have $V(u) = 1/t^3$. Together with $T(u(x/t)) = tT(u(x)) \geq N$, we get $T(u) \geq N(V(u))^{\frac{1}{3}}$. To prove the strong convergence of u_n to u_0 in $H_r^1(\mathbb{R}^N)$, it suffices to prove that $V(u_0) = 1$. Suppose $V(u_0) > 1$, then $T(u_0) \geq N(V(u_0))^{\frac{1}{3}} > N$, which contradicts $T(u_0) \leq N$. On the other hand, if $V(u_0) < 0$, then $V(v_n) > 1 - \frac{V(u_0)}{3} > 1$ for n large enough. We have

$$T(v_n) \geq N(V(v_n))^{\frac{1}{3}} > N\left(1 - \frac{V(u_0)}{3}\right)^{\frac{1}{3}}$$

which contradicts $T(v_n) \leq N + o(1)$, for n large enough. If $V(u_0) \in [0, 1)$, then $V(v_n) > 0$ for n large enough. We have

$$\begin{aligned}
N &= \lim_{n\to\infty} (T(u_0) + T(v_n)) \\
&\geq \lim_{n\to\infty} N\left[(V(u_0))^{\frac{1}{3}} + (V(v_n))^{\frac{1}{3}}\right] \\
&= N\left[(V(u_0))^{\frac{1}{3}} + (1 - V(u_0))^{\frac{1}{3}}\right] \\
&\geq N(V(u_0) + 1 - V(u_0)) = N.
\end{aligned}$$

If $V(u_0) \in (0, 1)$, this is a contradiction. So we deduce that $V(u_0) = 0$. Thus, $\lim_{n\to\infty} V(v_n) = 1$. By $V(u_n) = 1$, we get $u_0 = 0$ and $T(u_0) = 0$. From (H_1)–(H_2), we have

$$\limsup_{n\to\infty} \|u_n - u_0\|_6^2 > \sqrt[3]{6}.$$

Furthermore,

$$\begin{aligned}
N &= \frac{a}{2} \limsup_{n\to\infty} \|\nabla(u_n - u_0)\|_2^2 \\
&\geq \frac{aS}{2} \limsup_{n\to\infty} \|u_n - u_0\|_6^2 \\
&\geq \frac{\sqrt[3]{6}}{2} aS,
\end{aligned}$$

which is in contradiction with $N < \frac{\sqrt[3]{6}}{2} aS$ in Lemma 2.3. To sum up, we have $V(u_0) = 1$, namely, $u_0 \in \mathcal{K}$ and $V(v_n) \to 0$ as $n \to \infty$. It is easy to obtain that $\|\nabla v_n\|_2^2 = \|\nabla(u_n - u_0)\|_2^2 \to 0$ as $n \to \infty$. From $\int_{\mathbb{R}^3} G(u_n - u_0)\, dx \to 0$ as $n \to \infty$, we know that $u_n \to u_0$ strongly in $H_r^1(\mathbb{R}^3)$. Similar argument as the one in [11] shows that $U_0 = u_0(\cdot/t_u) \in \mathcal{P}$ is a ground state solution to problem (1.6). □

Define A_r as the set of radial ground state solutions to the problem (1.6). From Proposition 2.1, we have $U_0 \in A_r$, in other words, $A_r \neq \emptyset$. Furthermore, the following Lemma holds.

Lemma 2.5. *One has*

(i) A_r *is compact in* $H_r^1(\mathbb{R}^3)$.
(ii) $c = J(U_0)$, *that is, the mountain pass value agrees with the least energy level.*

Proof. The proof of (i) is similar to that in [6]. We omit the details. In the following, we give the proof of (ii). First, we know $p = J(U_0)$ and $p \leq c$. Second, we prove that $c \leq J(U_0)$. Since U_0 is a ground state solution to problem (1.6), similar to that in [17], there is a path $\gamma \in \Gamma$ that satisfies $\gamma(0) = 0$, $J(\gamma(1)) < 0$ and $\max_{t\in[0,1]} J(\gamma(t)) = J(U_0)$. This shows that $c \leq J(U_0)$. The proof is complete. $\square$

Let $W \in A_r$ be arbitrary but fixed and set $W_t(x) = W(x/t)$. By Lemma 2.2, we have

$$J(W_t) = \left(\frac{t}{2} - \frac{t^3}{6}\right) a \int\limits_{\mathbb{R}^3} |\nabla W|^2 \, dx. \tag{2.4}$$

It is easy to see that there exists a $t_1 > 1$ such that $J(W_t) < -2$ for $t \geq t_1$. We denote $D_\lambda = \max_{t\in[0,t_1]} \Phi_\lambda(W_t)$. Noting that the corresponding energy functional to the problem (1.1) is

$$\Phi_\lambda(u) = J(u) + \frac{\lambda}{4}\|u\|^4,$$

we obtain that $D_\lambda \to c$ as $\lambda \to 0$.

Lemma 2.6. *There exist* $\lambda^* > 0$ *and* $C^* > 0$, *such that for any* $0 < \lambda < \lambda^*$, *the following hold:*

$$\Phi_\lambda(W_{t_1}) < -2, \quad \|W_t\| \leq C^*, \ \textit{for all}\ t \in [0, t_1], \quad \textit{and} \quad \|W\| \leq C^*, \ \textit{for all}\ W \in A_r.$$

Proof. By Lemma 2.5, there exists a $C_0 > 0$ such that $\|W\| \leq C_0$, for any $W \in A_r$. For $W \in A_r$ and $t \in (0, t_1]$, we have

$$\|W_t\|^2 = t\|\nabla W\|_2^2 + t^3 b\|W\|_2^2 \leq \left(t + t^3 b\right) C_0^2.$$

Taking $C^* = C_0\sqrt{t_1 + t_1^3 b}$, we have

$$\|W_t\| \leq C^* \text{ and } \|W\| \leq C^*.$$

Since

$$\Phi_\lambda(W_{t_1}) = J(W_{t_1}) + \frac{\lambda}{4}\|W_{t_1}\|^4 \leq J(W_{t_1}) + \frac{\lambda}{4}(C^*)^4,$$

there exists $\lambda^* > 0$ small enough such that $\Phi_\lambda(W_{t_1}) < -2$ for any $\lambda \in (0, \lambda^*)$. $\square$

By Lemma 2.6, for any $\lambda \in (0, \lambda^*)$, we define a min–max value

$$C_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,t_1]} \Phi_\lambda(\gamma(t)),$$

where

$$\Gamma_\lambda = \left\{\gamma \in C\left([0, t_1], H_r^1(\mathbb{R}^3)\right) : \gamma(0) = 0, \gamma(t_1) = W_{t_1}, \|\gamma(t)\| \leq C^* + 2, t \in [0, t_1]\right\}.$$

It is obvious that $\Gamma_\lambda \neq \emptyset$ and $C_\lambda \leq D_\lambda$ for all $\lambda \in (0, \lambda^*)$. Furthermore, we can easily prove that $C_\lambda \to c$ as $\lambda \to 0$.

3. Proofs of the main results

Define

$$\Phi_\lambda^\alpha = \{u \in H_r^1(\mathbb{R}^3) : \Phi_\lambda(u) \le \alpha\}$$

and

$$A^d = \{u \in H_r^1(\mathbb{R}^3) : \inf_{v \in A_r} \|u - v\| \le d\}$$

for $\alpha, d > 0$. It is clear that $A^d \neq \emptyset$ for any $d > 0$ since $A_r \subset A^d$. In the following, we choose some positive constant d small enough and find a solution $u_\lambda \in A^d$ of problem (1.1) with $\lambda > 0$ small enough. In order to get a proper (PS) sequence for Φ_λ, we have the following Lemma.

Lemma 3.1. *Suppose that $\{u_{\lambda_i}\} \subset A^d$ with $\lim_{i\to\infty} \Phi_{\lambda_i}(u_{\lambda_i}) \le c$ and $\lim_{i\to\infty} \Phi'_{\lambda_i}(u_{\lambda_i}) = 0$, where $\lambda_i > 0$ and $\lambda_i \to 0$ as $i \to \infty$. Then there exists $u_0 \in A_r$ such that $u_{\lambda_i} \to u_0$ in $H_r^1(\mathbb{R}^3)$ for $d > 0$ small enough.*

Proof. By (H_1)–(H_2), we find a $c_2 > 0$ such that

$$F(s) \le \frac{a}{4}s^2 + \frac{c_2}{6}s^6. \tag{3.1}$$

We choose a constant d such that

$$0 < d < \min\left\{1, \frac{1}{3}\left(\frac{3aS^3}{2c_2}\right)^{\frac{1}{4}}, \sqrt{\frac{3c}{a}}\right\}. \tag{3.2}$$

For convenience, we replace λ_i by λ. As $\{u_\lambda\} \subset A^d$, there exist $W_\lambda \in A_r$ and $V_\lambda \in H^1(\mathbb{R}^3)$ such that $u_\lambda = W_\lambda + V_\lambda$ with $\|V_\lambda\| \le d$. By Lemma 2.5, we can obtain that there exist $W_0 \in A_r$ and $V_0 \in H_r^1(\mathbb{R}^3)$, such that $W_\lambda \to W_0$ strongly in $H^1(\mathbb{R}^3)$, $V_\lambda \to V_0$ weakly in $H^1(\mathbb{R}^3)$ with $\|V_0\| \le d$ and $V_\lambda \to V_0$ a.e. in $\mathbb{R}^3$. Set $u_0 = W_0 + V_0$, then $u_0 \in A^d$ and $u_\lambda \to u_0$ weakly in $H_r^1(\mathbb{R}^3)$. Together with $\lim_{i\to\infty} \Phi'_\lambda(u_\lambda) = 0$, we get $J'(u_0) = 0$.

We claim that $u_0 \not\equiv 0$. Indeed, if $u_0 \equiv 0$, then $\|W_0\| = \|V_0\| \le d$. By (3.2), we obtain $\|\nabla W_0\|_2 < \sqrt{\frac{3c}{a}}$. On the other hand, by (2.4) and Lemma 2.5, we get $\|\nabla W_0\|_2 = \sqrt{\frac{3c}{a}}$, which is a contradiction. Therefore $u_0 \not\equiv 0$ and $J(u_0) \ge c$. Moreover, by Lemma 2.4, we have

$$\Phi_\lambda(u_\lambda) = J(u_\lambda - u_0) + J(u_0) + o(1).$$

Together with $\lim_{\lambda\to 0} \Phi_\lambda(u_\lambda) \le c$, we have $J(u_\lambda - u_0) \le o(1)$. Also, by (3.1),

$$\begin{aligned} J(u_\lambda - u_0) &= \frac{a}{2}\|u_\lambda - u_0\|^2 - \int_{\mathbb{R}^3} F(u_\lambda - u_0)\,dx \\ &\ge \frac{a}{2}\|u_\lambda - u_0\|^2 - \frac{a}{4}\|u_\lambda - u_0\|^2 - \frac{c_2}{6}\|u_\lambda - u_0\|_6^6. \end{aligned}$$

Then by the Sobolev's embedding theorem, we have

$$\begin{aligned} \frac{a}{4}\|u_\lambda - u_0\|^2 &\le \frac{c_2}{6}S^{-3}\|\nabla(u_\lambda - u_0)\|_2^6 + o(1) \\ &\le \frac{c_2}{6}S^{-3}\|u_\lambda - u_0\|^6 + o(1). \end{aligned}$$

If $\liminf_{\lambda\to 0}\|u_\lambda - u_0\| > 0$, then $\liminf_{\lambda\to 0}\|u_\lambda - u_0\| \geq \left(3aS^3/(2c_2)\right)^{\frac{1}{4}}$. However, since $u_\lambda = W_\lambda + V_\lambda$, $u_0 = W_0 + V_0$ and $W_\lambda \to W_0$ strongly in $H_r^1(\mathbb{R}^3)$, $\|V_\lambda\|, \|V_0\| \leq d$, we have

$$\begin{aligned}\limsup_{\lambda\to 0}\|u_\lambda - u_0\| &\leq \limsup_{\lambda\to 0}(\|W_\lambda - W_0\| + \|V_\lambda\| + \|V_0\|)\\ &\leq 2d.\end{aligned}$$

This is a contradiction. Therefore, $\|u_\lambda - u_0\| \to 0$ in $H_r^1(\mathbb{R}^3)$. $\square$

Lemma 3.1 implies that for some $d > 0$ satisfying (3.2), $\beta > 0$ and $\lambda^* > 0$ such that for $u \in \Phi_\lambda^{D_\lambda} \cap (A^d \backslash A^{\frac{d}{2}})$ and $\lambda \in (0, \lambda^*)$, we have $\|\Phi'_\lambda(u)\| \geq \beta$.

Lemma 3.2. *There exist $\alpha_1 > 0$ and $\lambda > 0$ small enough such that $\Phi_\lambda(\gamma(s)) \geq C_\lambda - \alpha_1$ implies $\gamma(s) \in A^{\frac{d}{2}}$, where $\gamma(s) = U(\cdot/s)$, $s \in [0, t_1]$ and $U \in A_r$.*

Proof. By the Pohozǎev equality, we have

$$\Phi_\lambda(\gamma(s)) = \left(\frac{s}{2} - \frac{s^3}{6}\right) a \int_{\mathbb{R}^3} |\nabla U|^2 dx + \frac{\lambda}{4}\|U(\cdot/s)\|^4.$$

Noting that $\|U(\cdot/s)\|$ is bounded for $s \in (0, t_1]$, we have

$$\Phi_\lambda(\gamma(s)) = \left(\frac{s}{2} - \frac{s^3}{6}\right) a \int_{\mathbb{R}^3} |\nabla U|^2 dx + O(\lambda).$$

We easily obtain

$$\max_{s\in[0,t_1]} \left(\frac{s}{2} - \frac{s^3}{6}\right) a \int_{\mathbb{R}^3} |\nabla U|^2 dx = c$$

and that the maximum value is achieved only at $s = 1$. Then there exists a $\alpha_2 > 0$ small enough such that whenever $|s - 1| \leq \alpha_2$, we have $\gamma(s) = U(\cdot/s) \in A^{\frac{d}{2}}$. Together with $C_\lambda \to c$ as $\lambda \to 0$, there exists a $\alpha_1 > 0$ such that if $\lambda > 0$ small enough and $\Phi_\lambda(\gamma(s)) \geq C_\lambda - \alpha_1$, then $|s - 1| \leq \alpha_2$ and $\gamma(s) \in A^{\frac{d}{2}}$. $\square$

Lemma 3.3. *For any $d > 0$ and $\lambda > 0$ small enough, there exists $\{u_n\} \subset \Phi_\lambda^{D_\lambda} \cap A^d$ such that $\Phi'_\lambda(u_n) \to 0$ as $n \to \infty$.*

Proof. Assume by contradiction that for some $\lambda > 0$, there is $\beta(\lambda) > 0$ such that $|\Phi'_\lambda(u)| \geq \beta(\lambda)$ for all $u \in \Phi_\lambda^{D_\lambda} \cap A^d$. Similar arguments in [34] show that there exists a pseudo-gradient vector field Ψ_λ in $H_r^1(\mathbb{R}^3)$ on a neighborhood Y_λ of $\Phi_\lambda^{D_\lambda} \cap A^d$ such that

$$\|\Psi_\lambda(u)\| \leq 2\min\{1, |\Phi'_\lambda(u)|\}$$

and

$$\langle \Phi'_\lambda(u), \Psi_\lambda(u)\rangle \geq \min\{1, |\Phi'_\lambda(u)|\}\,|\Phi'_\lambda(u)|.$$

Let δ_λ be a Lipschitz continuous function on $H_r^1(\mathbb{R}^3)$ such that $\delta_\lambda \in [0, 1]$ and

$$\delta_\lambda(u) = \begin{cases} 1, & u \in \Phi_\lambda^{D_\lambda} \cap A^d, \\ 0, & u \in H_r^1(\mathbb{R}^3)\backslash Y_\lambda, \end{cases}$$

and let ξ_λ be a Lipschitz continuous function on $\mathbb{R}$ such that $\xi_\lambda \in [0,1]$ and

$$\xi_\lambda(t) = \begin{cases} 1, & |t - C_\lambda| \le \frac{\alpha_1}{2}, \\ 0, & |t - C_\lambda| \ge \alpha_1, \end{cases}$$

where α_1 is given in Lemma 3.2. If we set

$$E_\lambda(u) = \begin{cases} -\delta_\lambda(u)\xi_\lambda(\Phi_\lambda(u))\Psi_\lambda(u), & u \in Y_\lambda, \\ 0, & u \in H_r^1(\mathbb{R}^3)\backslash Y_\lambda, \end{cases}$$

then the following initial value problem

$$\begin{cases} \frac{d}{dt} Z_\lambda(u,t) = E_\lambda(Z_\lambda(u,t)), \\ Z_\lambda(u,0) = u, \end{cases}$$

admits a unique global solution $Z_\lambda : H_r^1(\mathbb{R}^3) \times \mathbb{R}_+ \to H_r^1(\mathbb{R}^3)$ satisfying

(i) $Z_\lambda(u,t) = u$, if $t = 0$ or $u \notin Y_\lambda$ or $|\Phi_\lambda(u) - C_\lambda| \ge \alpha_1$;
(ii) $\left\| \dfrac{d}{dt} Z_\lambda(u,t) \right\| \le 2$, for $(u,t) \in H_r^1(\mathbb{R}^3) \times \mathbb{R}_+$;
(iii) $\dfrac{d}{dt}\Phi_\lambda(Z_\lambda(u,t)) \le 0$.

We adopt similar ideas in [6,8] and obtain that for any $s \in (0, t_1]$, there is a $t_s > 0$ such that

$$Z_\lambda(\gamma(s), t_s) \in \Phi_\lambda^{C_\lambda - \frac{\alpha_1}{2}}, \text{ where } \gamma(s) = W(\cdot/s),\ s \in (0, t_1].$$

Let $\gamma_0(s) = Z_\lambda(\gamma(s), t_*(s))$, where

$$t_*(s) = \inf\left\{ t \ge 0 : Z_\lambda(\gamma(s), t) \in \Phi_\lambda^{C_\lambda - \frac{\alpha_1}{2}} \right\}.$$

Then we can prove that $\gamma_0(s)$ is continuous in $[0, t_1]$ and $\|\gamma_0(s)\| \le C^* + 2$. Therefore, we have $\gamma_0 \in \Gamma_\lambda$ with $\max_{t\in[0,t_1]} \Phi_\lambda(\gamma_0(t)) \le C_\lambda - \frac{\alpha_1}{2}$. This is in contradiction with $C_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{s\in[0,t_1]} \Phi_\lambda(\gamma(s))$. The proof is complete. □

Finally, we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. In what follows, we fix $d > 0$ small, in particular, $d < \frac{1}{3}\left(aS^3\right)^{1/4}$. By Lemma 3.3, there exist a $\lambda^* > 0$ with $\lambda \in (0, \lambda^*)$ and $\{u_n\} \subset \Phi_\lambda^{D_\lambda} \cap A^d$ such that $\Phi_\lambda(u_n) \le D_\lambda$ and $\Phi'_\lambda(u_n) \to 0$ as $n \to \infty$. Noting that $\{u_n\} \subset A^d$, thanks to Lemma 2.5, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$ and we may assume that $\lim_{n\to\infty} \|u_n\|^2 = \kappa$, where $\kappa \le \left(d + \sup_{u\in A_r} \|u\|\right)^2$. Assume $u_n \to u_\lambda$ weakly in $H_r^1(\mathbb{R}^3)$. Then by [34, Corollary 1.26], up to a subsequence, $u_n \to u_\lambda$ strongly in $L^p(\mathbb{R}^3)$, $p \in (2,6)$ and a.e. in $\mathbb{R}^3$. Since $u_n \in A^d$, there exist $U_n \in A_r$ and $w_n \in H_r^1(\mathbb{R}^3)$ such that $u_n = U_n + w_n$ and $\|w_n\| \le d$. Due to the compactness of A_r, we may assume that for some $U \in A_r$, $U_n \to U$ strongly in $H_r^1(\mathbb{R}^3)$. Let $v_n = u_n - u_\lambda$, we have $\|v_n\| \le 3d$ for n sufficiently large.

Step 1. We claim that for any $\delta > 1$, up to a subsequence, there holds

$$\int_{\mathbb{R}^3} f(u_n)u_n \, dx \le \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda \, dx + \delta \int_{\mathbb{R}^3} v_n^6 \, dx + o_n(1).$$

In fact, by (H_2), there exists $s_0 > 1$ such that $f(s) \le \delta s^5$ for all $s \ge s_0$. Take a function $\chi(s) \in C(\mathbb{R})$ such that $\chi(s) = 0$ if $s \le 1$, $\chi(s) = f(s)/s^5$ if $s \ge s_0$ and $\chi(s) \in [0, \delta]$ for any $s \in \mathbb{R}$. Let $g(s) = f(s) - \chi(s)s^5$, then $\lim_{s\to 0} g(s)/s \to 0$ and $\lim_{s\to\infty} g(s)/s^5 \to 0$. It follows from the compactness lemma of Strauss [30] that

$$\int_{\mathbb{R}^3} g(u_n)u_n \, dx = \int_{\mathbb{R}^3} g(u_\lambda)u_\lambda \, dx + o_n(1).$$

Meanwhile, due to the boundedness of $\chi(s)$, similar as to Brezis–Lieb Lemma [34, Lemma 1.32], we have

$$\int_{\mathbb{R}^3} \chi(u_n) \left(u_n^6 - u_\lambda^6 - v_n^6\right) \, dx = o_n(1).$$

By the Lebesgue dominated theorem, we get

$$\int_{\mathbb{R}^3} \chi(u_n)u_n^6 \, dx = \int_{\mathbb{R}^3} \chi(u_n)v_n^6 \, dx + \int_{\mathbb{R}^3} \chi(u_\lambda)u_\lambda^6 \, dx + o_n(1).$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}^3} f(u_n)u_n \, dx &= \int_{\mathbb{R}^3} g(u_n)u_n \, dx + \int_{\mathbb{R}^3} \chi(u_n)u_n^6 \, dx \\
&= \int_{\mathbb{R}^3} g(u_\lambda)u_\lambda \, dx + \int_{\mathbb{R}^3} \chi(u_n)v_n^6 \, dx + \int_{\mathbb{R}^3} \chi(u_\lambda)u_\lambda^6 \, dx + o_n(1) \\
&= \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda \, dx + \int_{\mathbb{R}^3} \chi(u_n)v_n^6 \, dx + o_n(1) \\
&\le \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda \, dx + \delta \int_{\mathbb{R}^3} v_n^6 \, dx + o_n(1).
\end{aligned}$$

Step 2. We show that $\|\nabla v_n\|_2^2 \to 0$ as $n \to \infty$. In fact, one can get that u_λ is a weak solution of

$$(a + \lambda\kappa)(-\Delta u + bu) = f(u), \quad u \in H^1(\mathbb{R}^3).$$

By Step 1 and $\langle \Phi'_\lambda(u_n), u_n \rangle \to 0$,

$$(a + \lambda\kappa)(\|v_n\|^2 + \|u_\lambda\|^2) \le \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda \, dx + \delta \int_{\mathbb{R}^3} v_n^6 \, dx + o_n(1).$$

Noting that

$$(a + \lambda\kappa)\|u_\lambda\|^2 = \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda \, dx,$$

we get $(a + \lambda\kappa)\|v_n\|^2 \leq \delta \int_{\mathbb{R}^3} v_n^6\,dx + o_n(1)$. If $\|\nabla v_n\|_2^2 \not\to 0$ as $n \to \infty$, then by Sobolev's embedding, we know

$$a\|\nabla v_n\|_2^2 \leq \delta \int\limits_{\mathbb{R}^3} v_n^6\,dx + o_n(1) \leq \delta S^{-3}\|\nabla v_n\|_2^6 + o_n(1),$$

which implies that

$$\liminf_{n\to\infty} \|\nabla v_n\|_2 \geq \left(\delta^{-1} a S^3\right)^{1/4}.$$

Due to the arbitrariness of $\delta > 1$,

$$\liminf_{n\to\infty} \|\nabla v_n\|_2 \geq \left(a S^3\right)^{1/4},$$

which is impossible since $d < \frac{1}{3}\left(aS^3\right)^{1/4}$. Thus, $\|\nabla v_n\|_2^2 \to 0$ as $n \to \infty$.

Step 3. We prove that $u_n \to u_\lambda$ strongly in $H^1(\mathbb{R}^3)$. If we have this claim in hand, we immediately get $\Phi'_\lambda(u_\lambda) = 0$ and $u_\lambda \in \Phi_\lambda^{D_\lambda} \cap A^d$. By Step 2, $u_n \to u_\lambda$ strongly in $D^{1,2}(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$. It follows that

$$(a + \lambda\kappa)(-\Delta u_\lambda + b u_\lambda) = f(u_\lambda), \;\; u_\lambda \in H^1(\mathbb{R}^3),$$

and $\int_{\mathbb{R}^3} \chi(u_n) v_n^6\,dx \to 0$ as $n \to \infty$. By Step 1, $f(u_n)u_n \to f(u_\lambda)u_\lambda$ strongly in $L^1(\mathbb{R}^3)$. Thus, by $\langle \Phi'_\lambda(u_n), u_n\rangle \to 0$,

$$\begin{aligned}(a + \lambda\kappa)\|u_n\|^2 &= \int\limits_{\mathbb{R}^3} f(u_n)u_n\,dx + o_n(1)\\ &= \int\limits_{\mathbb{R}^3} f(u_\lambda)u_\lambda\,dx + o_n(1) = (a + \lambda\kappa)\|u_\lambda\|^2 + o_n(1).\end{aligned}$$

So, $\|u_n\| \to \|u_\lambda\|$ as $n \to \infty$. Therefore, $u_n \to u_\lambda$ strongly in $H^1(\mathbb{R}^3)$ for sufficiently small d given and $u_\lambda \neq 0$. The proof is completed. $\square$

Proof of Theorem 1.2. Noting that

$$\Phi_\lambda(u_\lambda) = J(u_\lambda) + \frac{\lambda}{4}\|u_\lambda\|^4 \leq D_\lambda,$$

we have $J(u_\lambda) \leq D_\lambda$. Meanwhile, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$0 = \Phi'_\lambda(u_\lambda)\varphi = J'(u_\lambda)\varphi + \lambda\|u_\lambda\|^2 \int\limits_{\mathbb{R}^3} u_\lambda \varphi\,dx.$$

Combining with the fact that $u_\lambda \in A^d$, we know

$$J'(u_\lambda)\varphi = -\lambda\|u_\lambda\|^2 \int\limits_{\mathbb{R}^3} u_\lambda \varphi\,dx \to 0 \; as \; \lambda \to 0.$$

From the above, we have

$$u_\lambda \in \Phi_\lambda^{D_\lambda} \cap A^d, \quad J(u_\lambda) \leq D_\lambda \;\text{ and }\; J'(u_\lambda) \to 0 \; as \; \lambda \to 0.$$

Assuming that $u_\lambda \to u$ weakly in $H_r^1(\mathbb{R}^3)$, we have $J'(u) = 0$. Together with $D_\lambda \to c$ as $\lambda \to 0$ and $c \le \frac{1}{3}(aS)^{\frac{3}{2}}$, we can obtain that $u_\lambda \to u$ strongly in $H_r^1(\mathbb{R}^3)$ by similar arguments as in Lemma 3.1.

Since $J(u_\lambda) \le D_\lambda$ and $\lim_{\lambda\to 0} D_\lambda = c$, we have $J(u) \le c$. On the other hand, by the choice of d in (3.2), we can prove that $u \not\equiv 0$, and then $J(u) \ge c$. Therefore $J(u) = c$. In other words, by Lemma 2.5, u is a ground state solution of the limit problem of (1.6). The proof of Theorem 1.2 is complete. $\square$

The numerical study of the Kirchhoff-type problem will be continued in Part II of the series.

Acknowledgments

This research was partially supported by the Fundamental Research Funds for the Central Universities 2015XKMS072 of China and by Qatar National Research Fund project #NPRP 9-166-1-031 and by National Center for Theoretical Sciences (NCTS) and Ministry of Science and Technology (MOST) grant 103-2115-M-002-005 of Taiwan.

References

[1] C. Alves, F. Corra, T.F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49 (1) (2005) 85–93.
[2] C. Alves, G. Figueiredo, Nonlinear perturbations of a periodic Kirchhoff equation in RN, Nonlinear Anal. 75 (5) (2012) 2750–2759.
[3] C. Alves, M. Souto, M. Montenegro, Existence of a ground state solution for a nonlinear scalar field equation with critical growth, Calc. Var. Partial Differential Equations 43 (2012).
[4] A. Azzollini, et al., The elliptic Kirchhoff equation in RN perturbed by a local nonlinearity, Differential Integral Equations 25 (5/6) (2012) 543–554.
[5] J. Byeon, L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Ration. Mech. Anal. 185 (2) (2007) 185–200.
[6] J. Byeon, J. Zhang, W. Zou, Singularly perturbed nonlinear Dirichlet problems involving critical growth, Calc. Var. Partial Differential Equations (2013) 1–21.
[7] C.-y. Chen, Y.-c. Kuo, T.-f. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations 250 (4) (2011) 1876–1908.
[8] Z. Chen, W. Zou, Standing waves for a coupled system of nonlinear Schrödinger equations, Ann. Mat. Pura Appl. 194 (1) (2015) 183–220.
[9] B. Cheng, New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems, J. Math. Anal. Appl. 394 (2) (2012) 488–495.
[10] B. Cheng, X. Wu, Existence results of positive solutions of Kirchhoff type problems, Nonlinear Anal. 71 (10) (2009) 4883–4892.
[11] S. Coleman, V. Glaser, A. Martin, Action minima among solutions to a class of Euclidean scalar field equations, Comm. Math. Phys. 58 (2) (1978) 211–221.
[12] G. Figueiredo, N. Ikoma, J. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, Arch. Ration. Mech. Anal. 213 (3) (2014).
[13] X. He, W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, Nonlinear Anal. 70 (3) (2009) 1407–1414.
[14] X.-m. He, W.-m. Zou, Multiplicity of solutions for a class of Kirchhoff type problems, Acta Math. Appl. Sin. Engl. Ser. 26 (3) (2010) 387–394.
[15] Y. Huang, Z. Liu, On a class of Kirchhoff type problems, Arch. Math. 102 (2) (2014) 127–139.
[16] N. Ikoma, Existence of ground state solutions to the nonlinear Kirchhoff type equations with potentials, Discrete Contin. Dyn. Syst. 35 (3) (2015) 943–966.
[17] L. Jeanjean, K. Tanaka, A remark on least energy solutions in R N, Proc. Amer. Math. Soc. (2003) 2399–2408.
[18] G. Kirchhoff, Vorlesungen über mathematische Physik: Mechanik, vol. 1, BG Teubner, 1876.
[19] Y. Li, F. Li, J. Shi, Existence of a positive solution to Kirchhoff type problems without compactness conditions, J. Differential Equations 253 (7) (2012) 2285–2294.
[20] G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in R3, J. Differential Equations 257 (2) (2014) 566–600.
[21] G. Li, H. Ye, Existence of positive solutions for nonlinear Kirchhoff type problems in R3 with critical Sobolev exponent, Math. Methods Appl. Sci. 37 (16) (2014) 2570–2584.
[22] J. Lions, On Some Questions in Boundary Value Problems of Mathematical Physics, North-Holl. Math. Stud., vol. 30, 1978, pp. 284–346.
[23] Z. Liu, S. Guo, Positive solutions for asymptotically linear Schrödinger–Kirchhoff-type equations, Math. Methods Appl. Sci. 37 (4) (2014) 571–580.
[24] Z. Liu, S. Guo, On ground states for the Kirchhoff-type problem with a general critical nonlinearity, J. Math. Anal. Appl. 426 (1) (2015) 267–287.

[25] J. Liu, J.-F. Liao, C.-L. Tang, Positive solution for the Kirchhoff-type equations involving general subcritical growth, Commun. Pure Appl. Anal. 15 (2) (2016) 445–455.

[26] T. Ma, J.M. Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, Appl. Math. Lett. 16 (2) (2003) 243–248.

[27] A. Mao, Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the PS condition, Nonlinear Anal. 70 (3) (2009) 1275–1287.

[28] K. Perera, Z. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, J. Differential Equations 221 (1) (2006) 246–255.

[29] W. Shuai, Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains, J. Differential Equations 259 (4) (2015) 1256–1274.

[30] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (2) (1977) 149–162.

[31] J.-J. Sun, C.-L. Tang, Existence and multiplicity of solutions for Kirchhoff type equations, Nonlinear Anal. 74 (4) (2011) 1212–1222.

[32] J. Sun, T.-f. Wu, Ground state solutions for an indefinite Kirchhoff type problem with steep potential well, J. Differential Equations 256 (4) (2014) 1771–1792.

[33] J. Wang, L. Tian, J. Xu, F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, J. Differential Equations 253 (7) (2012) 2314–2351.

[34] M. Willem, Minimax Theorems, vol. 24, Springer Science & Business Media, 1997.

[35] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in RN, Nonlinear Anal. Real World Appl. 12 (2) (2011) 1278–1287.

[36] Y. Wu, Y. Huang, Z. Liu, On a Kirchhoff type problem in RN, J. Math. Anal. Appl. 425 (1) (2015) 548–564.

[37] Y. Yang, J. Zhang, Nontrivial solutions of a class of nonlocal problems via local linking theory, Appl. Math. Lett. 23 (4) (2010) 377–380.

[38] J. Zhang, J.M. do Ó, M. Squassina, Schrödinger–Poisson systems with a general critical nonlinearity, Commun. Contemp. Math. (2015) 1650028.

[39] Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl. 317 (2) (2006) 456–463.