Homework 2 Solution

2-1

題號： 3, 9

2-1 #3
Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.

a. [0, 1] b. [1, 3.2] c. [3.2, 4]

Answer:

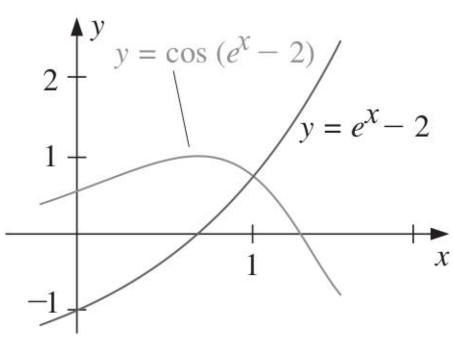
The Bisection method gives:

(a) $p_7 = 0.5859$ (b) $p_8 = 3.002$ (c) $p_7 = 3.419$

2-1 #9
a. Sketch the graphs of $y = e^x - 2$ and $y = \cos(e^x - 2)$.

b. Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos(e^x - 2)$.

Answer:

(a) 

(b) $p_17 = 1.00762177$

2-2

題號： 2, 19, 20

2-2 #1(不勾)

Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

(a) $g_1(x) = (3 + x - 2x^2)^{1/4}$,

(b) $g_2(x) = (\frac{x+3-x^4}{2})^{1/2}$,

(c) $g_3(x) = (\frac{x+3}{x^2+2})^{1/2}$,

(d) $g_4(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1}$,

2-2 #2

(a) Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

(b) Which function do you think gives the best approximation to the solution?

Answer:

(a) (a) $p_4 = 1.10782$; (b) $p_4 = 0.987506$; (c) $p_4 = 1.12364$; (d) $p_4 = 1.12412$;

(b) Part (d) gives the best answer since $|p_4 - p_3|$ is the smallest for (d).

2-2 #19

Let $g \in C^1[a, b]$ and p be in (a, b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Answer:

Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

2-2 #20

Let A be a given positive constant and $g(x) = 2x - Ax^2$.

(a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

(b) Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Answer:

(a) If fixed-point iteration converges to the limit p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2$$

(b) Any subinterval $[c, d]$ of $\left(\frac{1}{2A}, \frac{3}{2A}\right)$, containing $\frac{1}{A}$ suffices. Since

$$g(x) = 2x - Ax^2,\ g'(x) = 2 - 2Ax,$$

so $g(x)$ is continuous, and $g'(x)$ exists. Further, $g'(x) = 0$ only if $x = \frac{1}{A}$. Since

$$g(\frac{1}{A}) = \frac{1}{A},\ g(\frac{1}{2A}) = g(\frac{3}{2A}) = \frac{3}{4A},\ \text{ and we have } \frac{3}{4A} \leq g(x) \leq \frac{1}{A}$$

For x in $\left(\frac{1}{2A}, \frac{3}{2A}\right)$, we have

$$\left|x - \frac{1}{A}\right| < \frac{1}{2A},\ \text{ so } |g'(x)| = 2A\left|x - \frac{1}{A}\right| < 2A\left(\frac{1}{2A}\right) = 1$$