

不可使用手機、計算器，禁止作弊!

1. Let A is a 5×5 matrix.

$$A = \begin{bmatrix} 73 & 30 & 2 & 3 & 7 \\ -6 & 92 & 4 & -2 & 6 \\ -2 & 4 & 76 & 10 & 2 \\ -1 & 2 & -2 & 85 & 1 \\ 6 & 4 & -4 & 2 & 74 \end{bmatrix}$$

(a) Find a Jordan canonical form and a Jordan basis for the given matrix.

(b) Find the $\det(A^{80}) = \underline{80^{400}}$.

Notice that

$$\text{rref}(A - 80I)$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{rref}((A - 80I)^2)$$

$$= \begin{bmatrix} 1 & -2 & \frac{-2}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{rref}((A - 80I)^3)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution :

(a)

$$\text{null}(A - 80I) = \text{sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right), \text{null}((A - 80I)^2) = \text{sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right), \text{null}((A - 80I)^3) = \mathbb{R}^5$$

$$(A - 80I) : \begin{matrix} \vec{b}_3 \rightarrow \vec{b}_2 \rightarrow \vec{b}_1 \rightarrow \vec{0} \\ \vec{b}_5 \rightarrow \vec{b}_4 \rightarrow \vec{0} \end{matrix}$$

$$\text{Pick } \vec{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ then } \vec{b}_2 = (A - 80I)\vec{b}_3 = \begin{bmatrix} -7 \\ -6 \\ -2 \\ -1 \\ 6 \end{bmatrix}, \text{ and then then } \vec{b}_1 = (A - 80I)\vec{b}_2 = \begin{bmatrix} -96 \\ 0 \\ 0 \\ 0 \\ -96 \end{bmatrix}$$

$$\text{Pick } \vec{b}_5 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \text{ then } \vec{b}_4 = (A - 80I)\vec{b}_5 = \begin{bmatrix} -8 \\ 0 \\ -16 \\ -8 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} | & | & | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} -96 & -7 & 1 & -8 & 2 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & -2 & 0 & -16 & 3 \\ 0 & -1 & 0 & -8 & 0 \\ -96 & 6 & 0 & 0 & 0 \end{bmatrix}, J = \begin{bmatrix} 80 & 1 & 0 & 0 & 0 \\ 0 & 80 & 1 & 0 & 0 \\ 0 & 0 & 80 & 0 & 0 \\ 0 & 0 & 0 & 80 & 1 \\ 0 & 0 & 0 & 0 & 80 \end{bmatrix}$$

We have

$$C^{-1}AC = J$$

(b)

$$\det(A) = \det(CJC^{-1}) = \det(C) \det(J) \det(C)^{-1} = \det(J) = 80^5$$

$$\det(A^{80}) = \det(A)^{80} = (80^5)^{80} = 80^{400}$$