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Determinantal representations of enumerative polynomials

Shi-Mei Ma [a], Hong Bian [b,*], Jun-Ying Liu [a], Jean Yeh [c], Yeong-Nan Yeh [d]

[a] School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255000, PR China
[b] School of Mathematical Sciences, Xinjiang Normal University, Urumqi Xinjiang 830017, PR China
[c] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
[d] College of Mathematics and Physics, Wenzhou University, Wenzhou 325035, PR China

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ABSTRACT

Based on a determinantal formula for the higher derivative of a quotient of two functions, we first present the determinantal expressions of Eulerian polynomials and André polynomials. In particular, we discover that the Euler number (number of alternating permutations) can be expressed as a lower Hessenberg determinant. We then investigate the determinantal representations of the up-down run polynomials and the types A and B alternating run polynomials. As applications, we deduce several new recurrence relations. And then, we provide two determinantal representations for the alternating run polynomials of dual Stirling permutations. In particular, we discover a close connection between the alternating run polynomials of dual Stirling permutations and the type B Eulerian polynomials.

1. Introduction

Let d_n be the *derangement number*, which counts permutations on $[n]=\{1,2,\ldots,n\}$ such that leaves no elements in their original places. The first few derangement numbers are $0, 1, 2, 9, 44, \ldots$. The generating function of the derangement numbers (see [4] for instance) is given by

$$\sum_{n=0}^{\infty} d_n \frac{z^n}{n!} = \frac{\mathrm{e}^{-z}}{1-z}. \tag{1}$$

In [24, Eq. (11)], it was given that for $n \geqslant 1$, one has

$$d_{n+2} = \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 3 & 3 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & n & n & -1 \\ 0 & 0 & 0 & \cdots & 0 & n+1 & n+1 \end{vmatrix}_{n\times n}.$$

* Corresponding author.
E-mail addresses: shimeimapapers@163.com (S.-M. Ma), bh1218@163.com (H. Bian), jyliu6@163.com (J.-Y. Liu), chunchenyeh@nknu.edu.tw (J. Yeh), mayeh@math.sinica.edu.tw (Y.-N. Yeh).

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Since then, various determinantal representations of d_n have been pursued by several authors [13]. The following result is stated in Bourbaki's book [3, p. 40] which was reformulated in Qi [34, Section 2.2], Qi-Chapman [36], and Wei-Qi [40, Lemma 2.1], see [12, Lemma 6] for a proof of it.

Lemma 1. *Let $u = u(x)$ and $v = v(x)$ be two real functions which are n times differentiable on an interval $I \subset \mathbb{R}$. If one puts $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{u}{v}\right) = (-1)^n \frac{w_n}{v^{n+1}}$ at every point where $v(x) \neq 0$, then*

$$w_n = \begin{vmatrix} u & v & 0 & 0 & \cdots & 0 & 0 \\ u' & v' & \binom{1}{1}v & 0 & \cdots & 0 & 0 \\ u'' & v'' & \binom{2}{1}v' & \binom{2}{2}v & \cdots & 0 & 0 \\ u^{(3)} & v^{(3)} & \binom{3}{1}v'' & \binom{3}{2}v' & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1}v^{(n-2)} & \binom{n-1}{2}v^{(n-3)} & \cdots & \binom{n-1}{n-2}v' & \binom{n-1}{n-1}v \\ u^{(n)} & v^{(n)} & \binom{n}{1}v^{(n-1)} & \binom{n}{2}v^{(n-2)} & \cdots & \binom{n}{n-2}v'' & \binom{n}{n-1}v' \end{vmatrix}_{(n+1)\times(n+1)}.$$

Based on a three-term recurrence relation satisfied by derangement polynomials $d_n(q)$ (major polynomial over the set of derangements on $[n]$), Munarini [33] showed that $d_n(q)$ can be expressed as the determinant of either an $n \times n$ tridiagonal matrix or an $n \times n$ lower Hessenberg matrix. Qi–Wang–Guo [37] made good use of Lemma 1 and found that d_n can be expressed as a tridiagonal determinant:

$$d_n = -\begin{vmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n-3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -(n-2) & n-2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -(n-1) & n-1 \end{vmatrix} = -\left|e_{ij}\right|_{(n+1)\times(n+1)},$$

where

$$e_{ij} = \begin{cases} 1, & \text{if } i-j=-1; \\ i-2, & \text{if } i-j=0; \\ 2-i, & \text{if } i-j=1; \\ 0, & \text{if } i-j\neq 0,\ \pm 1. \end{cases}$$

Moreover, Qi et al. [34,35,37] deduced determinantal representations for Eulerian polynomials, Bernoulli polynomials as well as the higher order derivatives of $\tan x$.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n] := \{1, 2, 3, \ldots, n\}$. For any $\pi \in \mathfrak{S}_n$, written as the word $\pi(1)\pi(2)\cdots\pi(n)$, the entry $\pi(i)$ is called

- a *descent* if $i \in [n-1]$ and $\pi(i) > \pi(i+1)$;
- an *excedance* if $i \in [n-1]$ and $\pi(i) > i$;
- a *drop* if $i \in \{2, 3, \ldots, n\}$ and $\pi(i) < i$;
- a *fixed point* if $i \in [n]$ and $\pi(i) = i$.

Let $\operatorname{des}(\pi)$ (resp. $\operatorname{exc}(\pi)$, $\operatorname{drop}(\pi)$, $\operatorname{fix}(\pi)$) be the number of descents (resp. excedances, drops, fixed points) of π. Then $d_n = \#\{\pi \in \mathfrak{S}_n : \operatorname{fix}(\pi) = 0\}$. It is well known that descents, excedances and drops are equidistributed over $\mathfrak{S}_n$, and their common enumerative polynomial is the *Eulerian polynomial* (see [38, A008292]):

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{des}(\pi)+1} = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{exc}(\pi)+1} = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{drop}(\pi)+1}.$$

The exponential generating function of $A_n(x)$ is given as follows:

$$A(x;z) = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!} = \frac{1-x}{1-xe^{z(1-x)}}. \tag{2}$$

Let $\{f_n(x)\}_{n\geq 0}$ be a sequences of polynomials, and let $f(x;z) = \sum_{n\geqslant 0} f_n(x)\frac{z^n}{n!}$. If $f(x;z)$ can be written as a quotient

$$f(x;z) = \frac{u(x;z)}{v(x;z)},$$

then using Lemma 1, one can derive that $f_n(x)$ can be expressed as a determinant of order $n+1$, since $f_n(x) = \lim_{z\to 0}\frac{\partial^n}{\partial z^n}f(x;z)$. It should be noted that both $u(x;z)$ and $v(x;z)$ have simple expressions in the most of the aforementioned studies. For example, let $u(x;z) = 1-x$ and $v(x;z) = 1-xe^{z(1-x)}$. Then for $k \geqslant 1$, $\frac{\partial^k}{\partial z^k}u(x;z) = 0$ and $\frac{\partial^k}{\partial z^k}v(x;z) = -x(1-x)^k e^{z(1-x)}$. Applying Lemma 1, and letting $z \to 0$, Chow [11] found that $A_n(x)$ can be expressed as the following lower Hessenberg determinant:

$$A_n(x) = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ 0 & x(1-x) & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x(1-x)^{n-2} & \binom{n-1}{1}x(1-x)^{n-3} & \cdots & \binom{n-1}{n-2}x & -1 \\ 0 & x(1-x)^{n-1} & \binom{n}{1}x(1-x)^{n-2} & \cdots & \binom{n}{n-2}x(1-x) & \binom{n}{n-1}x \end{vmatrix}_{(n+1)\times(n+1)}. \tag{3}$$

This paper is motivated by the following problem.

Problem 2. Assume that $f(x;z) = \sum_{n\geqslant 0} f_n(x)\frac{z^n}{n!} = \frac{u(x;z)}{v(x;z)}$. If both $u(x;z)$ and $v(x;z)$ have complicated expressions, or we cannot deduce explicit formulas for $u(x;z)$ and $v(x;z)$, how to deduce a determinantal formula of $f_n(x)$?

For example, the type A alternating run polynomials can be defined by

$$R_n(x) = \left(\frac{1+x}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right) \tag{4}$$

for $n \geqslant 2$ (see David–Barton [15, 157-162] and [28]), where $w = \sqrt{\frac{1-x}{1+x}}$. By the method of characteristics, Carlitz [8] proved that

$$R(x;z) = \sum_{n=0}^{\infty}(1-x^2)^{-n/2}\frac{z^n}{n!}\sum_{k=0}^{n}R(n+1,k)x^{n-k} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2. \tag{5}$$

The original purpose of this paper is to derive determinantal representation of $R_n(x)$ by using Lemma 1. Unlike Eqs. (1) or (2), the exponential generating function (5) seems complicated. In the next section, we collect the definitions, notations and preliminary results that will be used in the rest of this work. In particular, we discover that the Euler number E_{n+1} (the number of alternating permutations in $\mathfrak{S}_n$) can be expressed as a lower Hessenberg determinant of order n. In Section 3, we investigate the up-down run polynomials and the types A and B alternating run polynomials. For example, in Theorem 13, we find that the type A alternating run $R_{n+1}(x)$ can be expressed as a lower Hessenberg determinant of order n:

$$\begin{vmatrix} 2x & -1 & 0 & 0 & \cdots & 0 & 0 \\ 2xR_2(x) & f_1(x) & -\binom{1}{1} & 0 & \cdots & 0 & 0 \\ 2xR_3(x) & f_2(x) & \binom{2}{1}f_1(x) & -\binom{2}{2} & \cdots & 0 & 0 \\ 2xR_4(x) & f_3(x) & \binom{3}{1}f_2(x) & \binom{3}{2}f_1(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2xR_{n-1}(x) & f_{n-2}(x) & \binom{n-2}{1}f_{n-3}(x) & \binom{n-2}{2}f_{n-4}(x) & \cdots & \binom{n-2}{n-3}f_1(x) & -\binom{n-2}{n-2} \\ 2xR_n(x) & f_{n-1}(x) & \binom{n-1}{1}f_{n-2}(x) & \binom{n-1}{2}f_{n-3}(x) & \cdots & \binom{n-1}{n-3}f_2(x) & \binom{n-1}{n-2}f_1(x) \end{vmatrix},$$

where $f_n(x) = (-1)^{n+1}(1-x^2)^{\lfloor n/2\rfloor}$. When $n=3$, we see that

$$R_4(x) = \begin{vmatrix} 2x & -1 & 0 \\ 4x^2 & 1 & -1 \\ 4x^2+8x^3 & -1+x^2 & 2 \end{vmatrix} = 2x+12x^2+10x^3.$$

In Section 4, we discuss the alternating run polynomials of dual Stirling permutations.

2. Preliminaries

2.1. Lower hessenberg determinants and context-free grammars

The $n \times n$ *lower Hessenberg matrix* H_n is defined as follows:

$$H_n = \begin{pmatrix} h_{11} & h_{12} & 0 & \cdots & 0 & 0 \\ h_{21} & h_{22} & h_{23} & \cdots & 0 & 0 \\ h_{31} & h_{32} & h_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \cdots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & h_{n,3} & \cdots & h_{n,n-1} & h_{n,n} \end{pmatrix},$$

where $h_{ij} = 0$ if $j > i + 1$. Hessenberg matrices are one of the important matrices in numerical analysis [6,17,23]. Setting $H_0 = 1$, Cahill et al. [6] gave a recursion for the determinant of the matrix H_n as follows:

$$\det H_{n+1} = h_{n+1,n+1} \det H_n + \sum_{r=1}^{n} \left((-1)^{n+1-r} h_{n+1,r} \prod_{j=r}^{n} h_{j,j+1} \det H_{r-1} \right). \tag{6}$$

Following Chen [9], a *context-free grammar* G over an alphabet V is defined as a set of substitution rules replacing a letter in V by a formal function over V. The formal derivative D_G with respect to G satisfies the derivation rules:

$$D_G(u+v) = D_G(u) + D_G(v),\ \ D_G(uv) = D_G(u)v + uD_G(v).$$

So the *Leibniz rule* holds:

$$D_G^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u) D_G^{n-k}(v). \tag{7}$$

Recently, context-free grammars have been used extensively in the study of permutations, perfect matchings and increasing trees, see [10,21,29–31] for instances.

In this paper, we always let D_G be the formal derivative associated with the grammar G. As an illustration, we now recall a classical result in this topic.

Proposition 3 ([16]). *If $G = \{a \to ab, b \to ab\}$, then $D_G^n(a) = a^{n+1} A_n\left(\frac{b}{a}\right)$. Thus a grammatical description of the Eulerian polynomial is given as follows:*

$$D_G^n(a)|_{a=1,b=x} = A_n(x).$$

We now give the grammatical version of Lemma 1.

Lemma 4. *Let $G = \{a \to f_1(a,b,\ldots),\ b \to f_2(a,b,\ldots),\ldots\}$ be a given grammar. For formal functions $u(a,b,\ldots)$ and $v(a,b,\ldots)$, we have $D_G^n\left(\frac{u}{v}\right) = (-1)^n \frac{w_n}{v^{n+1}}$, where w_n is the following lower Hessenberg determinant of order $n+1$:*

$$w_n = \begin{vmatrix} u & v & 0 & \cdots & 0 & 0 \\ D_G(u) & D_G(v) & \binom{1}{1}v & \cdots & 0 & 0 \\ D_G^2(u) & D_G^2(v) & \binom{2}{1}D_G(v) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ D_G^{n-1}(u) & D_G^{n-1}(v) & \binom{n-1}{1}D_G^{n-2}(v) & \cdots & \binom{n-1}{n-2}D_G(v) & \binom{n-1}{n-1}v \\ D_G^n(u) & D_G^n(v) & \binom{n}{1}D_G^{n-1}(v) & \cdots & \binom{n}{n-2}D_G^2(v) & \binom{n}{n-1}D_G(v) \end{vmatrix}.$$

Proof. Set $w = \frac{u}{v}$. Then $u = wv$. By (7), we see that

$$\begin{pmatrix} u \\ D_G(u) \\ D_G^2(u) \\ \vdots \\ D_G^{n-1}(u) \\ D_G^n(u) \end{pmatrix} = \begin{pmatrix} v & 0 & 0 & \cdots & 0 & 0 \\ D_G(v) & \binom{1}{1}v & 0 & \cdots & 0 & 0 \\ D_G^2(v) & \binom{2}{1}D_G(v) & \binom{2}{2}v & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ D_G^{n-1}(v) & \binom{n-1}{1}D_G^{n-2}(v) & \binom{n-1}{2}D_G^{n-3}(v) & \cdots & \binom{n-1}{n-1}v & 0 \\ D_G^n(v) & \binom{n}{1}D_G^{n-1}(v) & \binom{n}{2}D_G^{n-2}(v) & \cdots & \binom{n}{n-1}D_G(v) & \binom{n}{n}v \end{pmatrix} \begin{pmatrix} w \\ D_G(w) \\ D_G^2(w) \\ \vdots \\ D_G^{n-1}(w) \\ D_G^n(w) \end{pmatrix}.$$

Note that the coefficient matrix is triangular with determinant v^{n+1}, by Cramer's rule, we get

$$D_G^n(w)=\frac{1}{v^{n+1}}\begin{vmatrix} v & 0 & 0 & \cdots & 0 & u\\ D_G(v) & \binom{1}{1}v & 0 & \cdots & 0 & D_G(u)\\ D_G^2(v) & \binom{2}{1}D_G(v) & \binom{2}{2}v & \cdots & 0 & D_G^2(u)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ D_G^{n-1}(v) & \binom{n-1}{1}D_G^{n-2}(v) & \binom{n-1}{2}D_G^{n-3}(v) & \cdots & \binom{n-1}{n-1}v & D_G^{n-1}(u)\\ D_G^n(v) & \binom{n}{1}D_G^{n-1}(v) & \binom{n}{2}D_G^{n-2}(v) & \cdots & \binom{n}{n-1}D_G(v) & D_G^n(u)\end{vmatrix}.$$

Interchanging of adjacent columns such that the last column becomes the first column, we must introduce a minus sign upon each swapping of a pair of adjacent columns, so we get the sign $(-1)^n$, as desired. This completes the proof. □

From a view of methods, Lemmas 1 and 4 are essentially the same. However, from a view of conceptions, Lemma 4 is more general than Lemma 1. In the next subsection, we give some illustrations of Lemma 4.

2.2. Eulerian polynomials and André polynomials

Consider Proposition 3. Let $G=\{a\to ab, b\to ab\}$. Note that $D_G^n(a)=D_G^n(\frac{u}{v})$, where $u(a,b)=1$ and $v(a,b)=1/a$. For $k\geqslant 1$, we have

$$D_G^k(u)=0,\ D_G^k(v)=-\frac{b}{a}(a-b)^{k-1}.$$

When $a=1$ and $b=x$ in the equations above, combining Lemma 4 and Proposition 3, we see that $A_n(x)=(-1)^n w_n$, where w_n is the following lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0\\ 0 & -x & \binom{1}{1} & \cdots & 0 & 0\\ 0 & -x(1-x) & -\binom{2}{1}x & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & -x(1-x)^{n-2} & -\binom{n-1}{1}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}x & \binom{n-1}{n-1}\\ 0 & -x(1-x)^{n-1} & -\binom{n}{1}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}x(1-x) & -\binom{n}{n-1}x\end{vmatrix}.$$

Except for the first column, multiplying each of the other columns of w_n by (-1), we get (3). If we set $u(a,b)=b$ and $v(a,b)=1/a$, then for $n\geqslant 1$, we have

$$D_G^{n+1}(a)=D_G^n(ab)=D_G^n\left(\frac{u}{v}\right),\ D_G^n(u)=D_G^n(b)=a^{n+1}A_n\left(\frac{b}{a}\right),\ D_G^n(v)=-\frac{b}{a}(a-b)^{n-1}.$$

Using Lemma 4 and Proposition 3, when $a=1$ and $b=x$ in the equations above, we find that $A_{n+1}(x)$ is expressible as $(-1)^n w_n$, where w_n is the lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} x & 1 & 0 & \cdots & 0 & 0\\ A_1(x) & -x & 1 & \cdots & 0 & 0\\ A_2(x) & -x(1-x) & -2x & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & \vdots\\ A_{n-1}(x) & -x(1-x)^{n-2} & -\binom{n-1}{1}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}x & 1\\ A_n(x) & -x(1-x)^{n-1} & -\binom{n}{1}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}x(1-x) & -\binom{n}{n-1}x\end{vmatrix}.$$

Except for the first column of w_n, multiplying each of the other columns by (-1), we get the following result.

Theorem 5. *For any $n \geqslant 0$, the Eulerian polynomial $A_{n+1}(x)$ can be expressed as the lower Hessenberg determinant of order $n+1$:*

$$A_{n+1}(x)=\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ A_1(x) & x & -1 & \cdots & 0 & 0 \\ A_2(x) & x(1-x) & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ A_{n-1}(x) & x(1-x)^{n-2} & \binom{n-1}{1}x(1-x)^{n-3} & \cdots & \binom{n-1}{n-2}x & -1 \\ A_n(x) & x(1-x)^{n-1} & \binom{n}{1}x(1-x)^{n-2} & \cdots & \binom{n}{n-2}x(1-x) & \binom{n}{n-1}x \end{vmatrix}.$$

Example 6. When $n=3$, one has

$$A_4(x)=\begin{vmatrix} x & -1 & 0 & 0 \\ x & x & -1 & 0 \\ x+x^2 & x(1-x) & 2x & -1 \\ x+4x^2+x^3 & x(1-x)^2 & 3x(1-x) & 3x \end{vmatrix}=x+11x^2+11x^3+x^4.$$

In the case of Theorem 5, it follows from (6) that $\det H_{n+1}=A_{n+1}(x)$, $h_{n+1,1}=A_n(x)$, $h_{n+1,n+1}=\binom{n}{n-1}x=nx$, $\prod_{j=r}^{n} h_{j,j+1}=(-1)^{n-r+1}$ for $1\leqslant r\leqslant n$, and

$$h_{n+1,r}=\binom{n}{r-2}x(1-x)^{n-r+1},$$

where $2\leqslant r\leqslant n$. So we get the following result.

Corollary 7. *For $n\geqslant 1$, we have $A_1(x)=x$ and*

$$\begin{aligned} A_{n+1}(x)&=(1+nx)A_n(x)+x\sum_{r=2}^{n}\binom{n}{r-2}(1-x)^{n-r+1}A_{r-1}(x) \\ &=A_n(x)+x\sum_{r=2}^{n+1}\binom{n}{r-2}(1-x)^{n-r+1}A_{r-1}(x). \end{aligned}$$

As another illustration of Lemma 4, we shall consider the famous André polynomials. An increasing tree on $[n]$ is a rooted tree with vertex set $\{0,1,2,\ldots,n\}$ in which the labels of the vertices are increasing along any path from the root 0. The degree of a vertex is meant to be the number of its children. A *0-1-2 increasing tree* is an increasing tree in which the degree of any vertex is at most two. Given a 0-1-2 increasing tree T, let $\ell(T)$ denote the number of leaves of T, and $u(T)$ denote the number of vertices of T with degree 1. The *André polynomials* are defined by

$$E_n(x,y)=\sum_{T\in\mathcal{E}_n}x^{\ell(T)}y^{u(T)},$$

where $\mathcal{E}_n$ is the set of 0-1-2 increasing trees on $[n-1]$. Here are the first few polynomials:

$$E_1(x,y)=x,\;\; E_2(x,y)=xy,\;\; E_3(x,y)=xy^2+x^2,\;\; E_4(x,y)=xy^3+4x^2y.$$

Using a grammatical labeling of 0-1-2 increasing trees, Dumont [16] discovered that

$$D_G^n(y)=E_n(x,y), \tag{8}$$

where $G=\{x\rightarrow xy,\ y\rightarrow x\}$. Let $E(x,y;z)=\sum_{n=0}^{\infty}E_n(x,y)\frac{z^n}{n!}$. By solving a differential equation, Foata–Schützenberger [18] found that

$$E(x,y;z)=\frac{x\sqrt{2x-y^2}+y(2x-y^2)\sin(z\sqrt{2x-y^2})-(x-y^2)\sqrt{2x-y^2}\cos(z\sqrt{2x-y^2})}{(x-y^2)\sin(z\sqrt{2x-y^2})+y\sqrt{2x-y^2}\cos(z\sqrt{2x-y^2})}. \tag{9}$$

We say that $\pi\in\mathfrak{S}_n$ is *alternating*, if $\pi(1)>\pi(2)<\pi(3)>\cdots\pi(n)$, i.e., $\pi(2i-1)>\pi(2i)$ and $\pi(2i)<\pi(2i+1)$ for all i. The classical *Euler number* E_n counts alternating permutations in $\mathfrak{S}_n$, see [38, A000111]. In particular, $E_3=\#\{213,312\}=2$. Dumont [16] noted that

$$E(1,1;z)=\tan z+\sec z=\sum_{n=0}^{\infty}E_n\frac{z^n}{n!}. \tag{10}$$

Thus $E_n = \#\mathcal{E}_n = E_n(1,1)$. Following [26, Theorem 11], a close connection between André polynomials and the type A alternating run polynomials is given as follows:

$$R_n(x) = 2(1+x)^{n-1}E_n\left(\frac{x}{1+x}, 1\right) \text{ for } n \geqslant 2.$$

We now give a dual formula of (9).

Theorem 8. *For any $n \geqslant 1$, we define*

$$e_n(x,y) = \begin{cases} (-1)^{k-1}y(2x-y^2)^{k-1}, & \text{if } n = 2k-1; \\ (-1)^{k-1}(x-y^2)(2x-y^2)^{k-1}, & \text{if } n = 2k. \end{cases}$$

In particular, $e_1(x,y) = y$, $e_2(x,y) = x - y^2$ and $e_3(x,y) = -y(2x-y^2)$. The André polynomial $E_{n+1}(x,y)$ can be expressed as the Hessenberg determinant of order n:

$$\begin{vmatrix} xe_1(x,y) & -\binom{1}{1} & 0 & \cdots & 0 & 0 \\ xe_2(x,y) & \binom{2}{1}e_1(x,y) & -\binom{2}{2} & \cdots & 0 & 0 \\ xe_3(x,y) & \binom{3}{1}e_2(x,y) & \binom{3}{2}e_1(x,y) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ xe_{n-1}(x,y) & \binom{n-1}{1}e_{n-2}(x,y) & \binom{n-1}{2}e_{n-3}(x,y) & \cdots & \binom{n-1}{n-2}e_1(x,y) & -\binom{n-1}{n-1} \\ xe_n(x,y) & \binom{n}{1}e_{n-1}(x,y) & \binom{n}{2}e_{n-2}(x,y) & \cdots & \binom{n}{n-2}e_2(x,y) & \binom{n}{n-1}e_1(x,y) \end{vmatrix}.$$

Proof. Let $G = \{x \to xy,\ y \to x\}$. It follows from (8) that

$$E_{n+1}(x,y) = D_G^{n+1}(y) = D_G^n(x) = D_G^n\begin{pmatrix} 1 \\ \frac{1}{x} \end{pmatrix}.$$

Put $u = 1$ and $v = \frac{1}{x}$. Then $D_G^k(u) = 0$ for $k \geqslant 1$. Note that

$$D_G(v) = D_G\left(\frac{1}{x}\right) = -\frac{y}{x},\ D_G^2(v) = D_G\left(-\frac{y}{x}\right) = \frac{y^2-x}{x},$$

$$D_G^3(v) = D_G^3\left(\frac{y^2-x}{x}\right) = \frac{y}{x}(2x-y^2).$$

Since $D_G(2x-y^2) = 0$ and $D_G\left(\frac{y}{x}\right) = \frac{x-y^2}{x}$, we find that

$$D_G^4(v) = \frac{x-y^2}{x}(2x-y^2),\ D_G^5(v) = -\frac{y}{x}(2x-y^2)^2,$$

$$D_G^6(v) = -\frac{x-y^2}{x}(2x-y^2)^2,\ D_G^7(v) = \frac{y}{x}(2x-y^2)^3.$$

By induction, it is easy to verify that for $k \geqslant 1$, we have

$$D_G^{2k-1}(v) = (-1)^k\frac{y}{x}(2x-y^2)^{k-1},\ D_G^{2k}(v) = (-1)^k\frac{x-y^2}{x}(2x-y^2)^{k-1}.$$

We define

$$\widetilde{e}_n(x,y) = \begin{cases} (-1)^k\frac{y}{x}(2x-y^2)^{k-1}, & \text{if } n = 2k-1; \\ (-1)^k\frac{x-y^2}{x}(2x-y^2)^{k-1}, & \text{if } n = 2k. \end{cases}$$

Using Lemma 4, we obtain that $E_{n+1}(x,y) = (-1)^n x^{n+1} w_n$, where w_n is the lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} 1 & \frac{1}{x} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{y}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0 \\ 0 & \widetilde{e}_2(x,y) & -\binom{2}{1}\frac{y}{x} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \widetilde{e}_{n-1}(x,y) & \binom{n-1}{1}\widetilde{e}_{n-2}(x,y) & \cdots & -\binom{n-1}{n-2}\frac{y}{x} & \binom{n-1}{n-1}\frac{1}{x} \\ 0 & \widetilde{e}_n(x,y) & \binom{n}{1}\widetilde{e}_{n-1}(x,y) & \cdots & \binom{n}{n-2}\widetilde{e}_2(x,y) & -\binom{n}{n-1}\frac{y}{x} \end{vmatrix}.$$

Expanding w_n along the first column, and then multiplying the first column of the resulting determinant by $-x^2$ and multiplying each of the other columns by $-x$, we get

$$\begin{vmatrix} xy & -\binom{1}{1} & \cdots & 0 & 0 \\ -x^2\widetilde{e}_2(x,y) & \binom{2}{1}y & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x^2\widetilde{e}_{n-1}(x,y) & -x\binom{n-1}{1}\widetilde{e}_{n-2}(x,y) & \cdots & \binom{n-1}{n-2}y & -\binom{n-1}{n-1} \\ -x^2\widetilde{e}_n(x,y) & -x\binom{n}{1}\widetilde{e}_{n-1}(x,y) & \cdots & -x\binom{n}{n-2}\widetilde{e}_2(x,y) & \binom{n}{n-1}y \end{vmatrix},$$

which yields the desired result. □

When $x = y = 1$, combining Theorem 8 and (6), we arrive at the following result.

Corollary 9. *For any $n \geqslant 1$, let $a_{2n-1} = (-1)^{n-1}$ and $a_{2n} = 0$. The Euler number E_{n+1} can be expressed as the Hessenberg determinant of order n:*

$$\begin{vmatrix} a_1 & -\binom{1}{1} & 0 & \cdots & 0 & 0 \\ a_2 & \binom{2}{1}a_1 & -\binom{2}{2} & \cdots & 0 & 0 \\ a_3 & \binom{3}{1}a_2 & \binom{3}{2}a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & \binom{n-1}{1}a_{n-2} & \binom{n-1}{2}a_{n-3} & \cdots & \binom{n-1}{n-2}a_1 & -\binom{n-1}{n-1} \\ a_n & \binom{n}{1}a_{n-1} & \binom{n}{2}a_{n-2} & \cdots & \binom{n}{n-2}a_2 & \binom{n}{n-1}a_1 \end{vmatrix}.$$

Moreover, for $n \geqslant 1$, the Euler numbers E_n satisfy the recurrence relation

$$E_{n+1} = \sum_{r=1}^{n} \binom{n}{r-1} a_{n-r+1} E_r,$$

with the initial values $E_1 = E_2 = 1$ and $E_3 = 2$.

2.3. Alternating run and up-down run polynomials

For each $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$, a *run* of π is a maximal segment (a subsequence consisting of consecutive elements) that is either increasing or decreasing. For example, the alternating runs of $\pi = 82315467$ are $82, 23, 31, 15, 54, 467$. Let $\mathrm{run}(\pi)$ denote the number of alternating runs of π. The *type A alternating run polynomials* are defined by

$$R_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{run}(\pi)} = \sum_{k=0}^{n-1} R(n,k)x^k.$$

André [1] found that

$$R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2), \tag{11}$$

where $R(1,0) = 1$ and $R(1,k) = 0$ for $k \geqslant 1$.

Using column generating functions (see [7, Section 4]), Canfield–Wilf obtained that

$$R(n,k) = \frac{1}{2^{k-2}}k^n - \frac{1}{2^{k-4}}(k-1)^n + \psi_2(n,k)(k-2)^n + \cdots + \psi_{k-1}(n,k),\ \ n \geqslant 2,$$

in which each $\psi_i(n,k)$ is a polynomial in n whose degree in n is $\lfloor i/2 \rfloor$. By solving partial differential equations, Stanley [39] discovered that

$$R(n,k) = \sum_{i=0}^{k} \frac{1}{2^{i-1}}(-1)^{k-i}z_{k-i} \sum_{\substack{r+2m\leqslant i\\ r\equiv i \bmod 2}} (-2)^m \binom{i-m}{(i+r)/2}\binom{n}{m} r^n,$$

where $z_0 = 2$ and $z_n = 4$ for $n \geqslant 1$. Ma [25] found another explicit formula by expressing $R_n(x)$ in terms of the derivative polynomials of tangent function. The reader is referred to [28,42] for the recent work on alternating run polynomials.

An *alternating subsequence* of $\pi = \pi(1)\pi(2)\cdots\pi(n)$ is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k),$$

where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Denote by $\mathrm{as}(\pi)$ the length of the longest alternating subsequence of π. We also define an *up-down run* of π to be a run of π endowed with a 0 in the front. It should be noted that $\mathrm{as}(\pi)$ equals the number of up-down runs of π. For example,

$$\mathrm{as}(82315467) = \mathrm{run}(082315467) = 7.$$

The *up-down run polynomials* are defined by

$$S_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{as}(\pi)} = \sum_{k=1}^{n} S(n,k)x^k.$$

Let $S(x;z) = \sum_{n=0}^{\infty} S_n(x)\frac{z^n}{n!}$. Remarkably, Stanley [39, Theorem 2.3] obtained the following closed-form formula:

$$S(x;z) = (1-x)\frac{1+\rho+2xe^{\rho z}+(1-\rho)e^{2\rho z}}{1+\rho-x^2+(1-\rho-x^2)e^{2\rho z}}, \tag{12}$$

where $\rho = \sqrt{1-x^2}$. When $n \geqslant 2$, Bóna [2, Section 1.3.2] obtained a remarkable identity:

$$S_n(x) = \frac{1}{2}(1+x)R_n(x). \tag{13}$$

Ma [26, Eq. (14)] found that

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n} R(n+1,k)x^{n-k} = \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n} S(n,k)x^{n-k}\right)^2.$$

Let $\mathcal{B}_n$ be the hyperoctahedral group of rank n. Elements π of $\mathcal{B}_n$ are signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. In this paper, we always identify a signed permutation $\pi = \pi(1)\cdots\pi(n)$ with the word $\pi(0)\pi(1)\cdots\pi(n)$, where $\pi(0) = 0$. A *run* of a signed permutation π is defined as a maximal consecutive subword that is monotonic in the order: $\cdots < \overline{2} < \overline{1} < 0 < 1 < 2 < \cdots$, where we adopt the standard convention that $\overline{i}$ is being used for $-i$. The *up signed permutations* are defined as signed permutations with $\pi(1) > 0$. Let $T(n,k)$ denote the number of up signed permutations in $\mathcal{B}_n$ with k alternating runs. The *type B alternating run polynomials* over up signed permutations are defined by

$$T_n(x) = \sum_{k=1}^{n} T(n,k)x^k.$$

Let $T(x;z) = 1 + \sum_{n=1}^{\infty} T_n(x)z^n/n!$. By the method of characteristic, Chow–Ma [14, Theorem 12] found that

$$T(x,z) = \frac{1}{1+x} + \frac{x\sqrt{x-1}}{(1+x)\sqrt{x-\cos(2z\sqrt{x^2-1})-\sqrt{x^2-1}\sin(2z\sqrt{x^2-1})}}. \tag{14}$$

The polynomials $R_n(x), S_n(x)$ and $T_n(x)$ satisfy the following recursions

$$R_{n+2}(x) = x(nx+2)R_{n+1}(x) + x\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}R_{n+1}(x),$$
$$S_{n+1}(x) = x(nx+1)S_n(x) + x\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}S_n(x),$$
$$T_{n+1}(x) = (2nx^2+3x-1)T_n(x) + 2x\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}T_n(x),$$

with $R_0(x) = R_1(x) = S_0(x) = 1$ and $T_1(x) = x$, see [14,22,41,42]. The first few polynomials are given as follows (see [38, A059427, A186370] and [26,42]):

$$R_2(x) = 2x,\ R_3(x) = 2x+4x^2,\ R_4(x) = 2x+12x^2+10x^3;$$
$$S_1(x) = x,\ S_2(x) = x+x^2,\ S_3(x) = x+3x^2+2x^3,\ S_4(x) = x+7x^2+11x^3+5x^4;$$
$$T_2(x) = x+3x^2,\ T_3(x) = x+12x^2+11x^3,\ T_4(x) = x+39x^2+95x^3+57x^4.$$

The generating functions (5), (12) and (14) are all seem complicated. In the next section, we shall deduce the determinantal representations of $R_n(x), S_n(x)$ and $T_n(x)$.

3. Up-down run and alternating run polynomials

3.1. Up-down run and alternating run polynomials over the symmetric group

As given by (13), the polynomial $R_n(x)$ is closely related to $S_n(x)$, which can also be illustrated by the following grammatical descriptions.

Lemma 10 ([26]). *If $G = \{a \to ab, b \to bc, c \to b^2\}$, then we have*

$$D_G^n(a) = ac^nS_n\left(\frac{b}{c}\right), \; D_G^n(a^2) = a^2c^nR_{n+1}\left(\frac{b}{c}\right). \tag{15}$$

Theorem 11. *For any $n \geqslant 1$, let $s_n(x) = (-1)^{n+1}(1+x)(1-x^2)^{\lfloor (n-1)/2 \rfloor}$ and let $r_n(x) = (-1)^{n-1}(x-1)(1-x^2)^{\lfloor (n-1)/2 \rfloor}$. In particular, $s_1(x) = 1+x$ and $r_1(x) = x-1$. Then the polynomial $S_{n+1}(x)$ is expressible as the following lower Hessenberg determinant of order n:*

$$\begin{vmatrix} xs_1(x) & -1 & 0 & \cdots & 0 & 0 \\ xs_2(x) & \binom{2}{1}s_1(x) & -1 & \cdots & 0 & 0 \\ xs_3(x) & \binom{3}{1}s_2(x) & \binom{3}{2}s_1(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ xs_{n-1}(x) & \binom{n-1}{1}s_{n-2}(x) & \binom{n-1}{2}s_{n-3}(x) & \cdots & \binom{n-1}{n-2}s_1(x) & -1 \\ xs_n(x) & \binom{n}{1}s_{n-1}(x) & \binom{n}{2}s_{n-2}(x) & \cdots & \binom{n}{n-2}s_2(x) & \binom{n}{n-1}s_1(x) \end{vmatrix}.$$

While $R_{n+1}(x)$ is expressible as the following lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ x-1 & s_1(x) & -1 & 0 & \cdots & 0 & 0 \\ r_2(x) & s_2(x) & \binom{2}{1}s_1(x) & -1 & \cdots & 0 & 0 \\ r_3(x) & s_3(x) & \binom{3}{1}s_2(x) & \binom{3}{2}s_1(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n-1}(x) & s_{n-1}(x) & \binom{n-1}{1}s_{n-2}(x) & \binom{n-1}{2}s_{n-3}(x) & \cdots & \binom{n-1}{n-2}s_1(x) & -1 \\ r_n(x) & s_n(x) & \binom{n}{1}s_{n-1}(x) & \binom{n}{2}s_{n-2}(x) & \cdots & \binom{n}{n-2}s_2(x) & \binom{n}{n-1}s_1(x) \end{vmatrix}.$$

Proof (*A*). By Lemma 10, we see that $D_G^{n+1}(a) = D_G^n(ab) = ac^{n+1}S_{n+1}\left(\frac{b}{c}\right)$. Since $D_G^n(ab) = D_G^n\left(\frac{1}{\frac{1}{ab}}\right)$, let $u(a,b,c) = 1$ and $v(a,b,c) = \frac{1}{ab}$. Note that

$$\begin{aligned} &D_G(v) = -\frac{b+c}{ab}, \; D_G\left(\frac{b+c}{ab}\right) = -c\frac{b+c}{ab}, \\ &D_G\left(c\frac{b+c}{ab}\right) = -\frac{b+c}{ab}(c^2-b^2), \; D_G(c^2-b^2) = 0. \end{aligned} \tag{16}$$

For $k \geqslant 1$, $D_G^k(u) = 0$, and

$$D_G^{2k}(v) = c\frac{b+c}{ab}(c^2-b^2)^{k-1}, \; D_G^{2k+1}(v) = -\frac{b+c}{ab}(c^2-b^2)^k. \tag{17}$$

When $a = c = 1$ and $b = x$ in the equations above, we get

$$D_G^n(v)\,|_{a=c=1,b=x} = (-1)^n\frac{1+x}{x}(1-x^2)^{\lfloor (n-1)/2 \rfloor}.$$

Applying Lemmas 4, we arrive at

$$ac^{n+1}S_{n+1}\left(\frac{b}{c}\right)|_{a=c=1,b=x} = S_{n+1}(x) = (-1)^nx^{n+1}w_n^{(1)},$$

where the $(n+1)$ order determinant $w_n^{(1)}$ can be given as follows:

$$w_n^{(1)}=\begin{vmatrix} 1 & \frac{1}{x} & 0 & \cdots & 0 & 0\\ 0 & -\frac{1+x}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0\\ 0 & \frac{1+x}{x} & -\binom{2}{1}\frac{1+x}{x} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & (-1)^{n-1}\frac{1+x}{x}(1-x^2)^{\lfloor (n-2)/2\rfloor} & \cdots & \cdots & -\binom{n-1}{n-2}\frac{1+x}{x} & \binom{n-1}{n-1}\frac{1}{x}\\ 0 & (-1)^{n}\frac{1+x}{x}(1-x^2)^{\lfloor (n-1)/2\rfloor} & \cdots & \cdots & \binom{n}{n-2}\frac{1+x}{x} & -\binom{n}{n-1}\frac{1+x}{x} \end{vmatrix}$$

$$=\begin{vmatrix} -\frac{1+x}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0\\ \frac{1+x}{x} & -\binom{2}{1}\frac{1+x}{x} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ (-1)^{n-1}\frac{1+x}{x}(1-x^2)^{\lfloor (n-2)/2\rfloor} & \cdots & \cdots & -\binom{n-1}{n-2}\frac{1+x}{x} & \binom{n-1}{n-1}\frac{1}{x}\\ (-1)^{n}\frac{1+x}{x}(1-x^2)^{\lfloor (n-1)/2\rfloor} & \cdots & \cdots & \binom{n}{n-2}\frac{1+x}{x} & -\binom{n}{n-1}\frac{1+x}{x} \end{vmatrix}.$$

The last determinant is obtained by expanding $w_n^{(1)}$ along the first column. Multiplying each column of the last determinant by $-x$, and then multiplying the first column by x, we get the desired expression of $S_{n+1}(x)$.

(B) Note that

$$D_G^n(a^2)=D_G^n\left(\frac{\frac{a}{b}}{\frac{1}{ab}}\right),\ D_G\left(\frac{a}{b}\right)=\frac{a}{b}(b-c),\ D_G\left(\frac{a}{b}(b-c)\right)=-\frac{ac}{b}(b-c),$$

$$D_G\left(\frac{ac}{b}(b-c)\right)=\frac{a}{b}(b-c)(b^2-c^2)=-\frac{a}{b}(b-c)(c^2-b^2).$$

Let $u(a,b,c)=\frac{a}{b}$ and $v(a,b,c)=\frac{1}{ab}$. Then $D_G^k(v)$ can be computed as in (17). Note that $D_G(b^2-c^2)=0$. For $k\geqslant 1$, we have

$$D_G^{2k-1}(u)=D_G^{2k-1}\left(\frac{a}{b}\right)=\frac{a}{b}(b-c)(c^2-b^2)^{k-1},D_G^{2k}(u)=D_G^{2k}\left(\frac{a}{b}\right)=-\frac{ac}{b}(b-c)(c^2-b^2)^{k-1}.$$

When $a=c=1$ and $b=x$ in the equations above, we get

$$D_G^n(u)\mid_{a=c=1,b=x}=(-1)^{n-1}\frac{x-1}{x}(1-x^2)^{\lfloor (n-1)/2\rfloor}.$$

Applying Lemmas 4 and 10, we arrive at

$$a^2c^nR_{n+1}\left(\frac{b}{c}\right)\mid_{a=c=1,b=x}=R_{n+1}(x)=(-1)^nx^{n+1}w_n^{(2)},$$

where $w_n^{(2)}$ is the following lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} \frac{1}{x} & \frac{1}{x} & 0 & \cdots & 0 & 0\\ \frac{x-1}{x} & -\frac{1+x}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0\\ -\frac{x-1}{x} & \frac{1+x}{x} & -\binom{2}{1}\frac{1+x}{x} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ D_G^{n-1}(u)\mid_{a=c=1,b=x} & D_G^{n-1}(v)\mid_{a=c=1,b=x} & \cdots & \cdots & -\binom{n-1}{n-2}\frac{1+x}{x} & \binom{n-1}{n-1}\frac{1}{x}\\ D_G^{n}(u)\mid_{a=c=1,b=x} & D_G^{n}(v)\mid_{a=c=1,b=x} & \cdots & \cdots & \binom{n}{n-2}\frac{1+x}{x} & -\binom{n}{n-1}\frac{1+x}{x} \end{vmatrix}.$$

Multiplying the first column of the above determinant by x, and then multiplying each of the other columns by $-x$, we get the desired expression of $R_{n+1}(x)$. □

When $n=4$, Theorem 11 says that

$$S_5(x)=\begin{vmatrix} x(1+x) & -1 & 0 & 0\\ -x(1+x) & \binom{2}{1}(1+x) & -1 & 0\\ x(1+x)(1-x^2) & -\binom{3}{1}(1+x) & \binom{3}{2}(1+x) & -1\\ -x(1+x)(1-x^2) & \binom{4}{1}(1+x)(1-x^2) & -\binom{4}{2}(1+x) & \binom{4}{3}(1+x) \end{vmatrix},$$

$$R_5(x) = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ x-1 & 1+x & -1 & 0 & 0 \\ -(x-1) & -(1+x) & \binom{2}{1}(1+x) & -1 & 0 \\ (x-1)(1-x^2) & (1+x)(1-x^2) & -\binom{3}{1}(1+x) & \binom{3}{2}(1+x) & -1 \\ -(x-1)(1-x^2) & -(1+x)(1-x^2) & \binom{4}{1}(1+x)(1-x^2) & -\binom{4}{2}(1+x) & \binom{4}{3}(1+x) \end{vmatrix}.$$

Combining (6) and Theorem 11, set

$$\det H_n = S_{n+1}(x),\ h_{nn} = n(1+x),\ \prod_{j=r}^{n-1} h_{j,j+1} = (-1)^{n-r},\ h_{n,r} = \binom{n}{r-1} s_{n-r+1}(x),$$

we obtain

$$S_{n+1}(x) = n(1+x)S_n(x) + \sum_{r=1}^{n-1} \binom{n}{r-1} s_{n-r+1}(x) S_r(x).$$

Similarly, combining (6) and Theorem 11, set

$$\det H_{n+1} = R_{n+1}(x),\ h_{n+1,n+1} = n(1+x),\ h_{n+1,1} = r_n(x),$$

$$\prod_{j=r}^{n} h_{j,j+1} = (-1)^{n-r+1} \text{ for } 1 \leqslant r \leqslant n,\ h_{n+1,r} = \binom{n}{r-2} s_{n-r+2}(x) \text{ for } 2 \leqslant r \leqslant n,$$

we arrive at

$$\begin{aligned} R_{n+1}(x) &= n(1+x)R_n(x) + r_n(x) + \sum_{r=2}^{n} \binom{n}{r-2} s_{n-r+2} R_{r-1}(x) \\ &= r_n(x) + \sum_{r=2}^{n+1} \binom{n}{r-2} s_{n-r+2} R_{r-1}(x). \end{aligned}$$

After simplifying, we get the following result.

Corollary 12. *For any $n \geqslant 1$, we have*

$$\begin{aligned} S_{n+1}(x) &= (1+x)\sum_{r=1}^{n} \binom{n}{r-1} (-1)^{n-r} (1-x^2)^{\lfloor (n-r)/2 \rfloor} S_r(x), \\ R_{n+1}(x) &= (-1)^{n-1}(x-1)(1-x^2)^{\lfloor (n-1)/2 \rfloor} + \\ &\quad (1+x)\sum_{r=2}^{n+1} \binom{n}{r-2} (-1)^{n-r+1} (1-x^2)^{\lfloor (n-r+1)/2 \rfloor} R_{r-1}(x). \end{aligned}$$

We can now conclude the following result.

Theorem 13. *Let $f_n(x) = (-1)^{n+1}(1-x^2)^{\lfloor n/2 \rfloor}$. The up-down run polynomial $S_{n+1}(x)$ can be expressed as a lower Hessenberg determinant of order $n+1$:*

$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ xS_1(x) & 1 & -\binom{1}{1} & \cdots & 0 & 0 \\ xS_2(x) & -(1-x^2) & \binom{2}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ xS_{n-1}(x) & f_{n-1}(x) & \binom{n-1}{1} f_{n-2}(x) & \cdots & \binom{n-1}{n-2} & -\binom{n-1}{n-1} \\ xS_n(x) & f_n(x) & \binom{n}{1} f_{n-1}(x) & \cdots & -\binom{n}{n-2}(1-x^2) & \binom{n}{n-1} \end{vmatrix}.$$

While the polynomial $R_{n+1}(x)$ can be expressed as a lower Hessenberg determinant of order n:

$$\begin{vmatrix} 2x & -1 & 0 & \cdots & 0 & 0 \\ 2xR_2(x) & 1 & -\binom{1}{1} & \cdots & 0 & 0 \\ 2xR_3(x) & -(1-x^2) & \binom{2}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2xR_{n-1}(x) & f_{n-2}(x) & \binom{n-2}{1}f_{n-3}(x) & \cdots & \binom{n-2}{n-3} & -\binom{n-2}{n-2} \\ 2xR_n(x) & f_{n-1}(x) & \binom{n-1}{1}f_{n-2}(x) & \cdots & -\binom{n-1}{n-3}(1-x^2) & \binom{n-1}{n-2} \end{vmatrix}.$$

Proof (A)**.** Consider the grammar $G=\{a\to ab, b\to bc, c\to b^2\}$. It is clear that

$$D_G^{n+1}(a)=D_G^n(ab)=D_G^n\left(\frac{a}{\frac{1}{b}}\right).$$

Let $u(a,b,c)=a$ and $v(a,b,c)=\frac{1}{b}$. From Lemma 10, we see that $D_G^n(u)|_{a=c=1,b=x}=S_n(x)$. Note that

$$D_G(v)=D_G\left(\frac{1}{b}\right)=-\frac{c}{b},\ D_G\left(\frac{c}{b}\right)=\frac{b^2-c^2}{b}=-\frac{c^2-b^2}{b}.$$

Then for $k\geqslant 1$, the higher derivatives of v are given as follows:

$$D_G^{2k-1}(v)=-\frac{c}{b}(c^2-b^2)^{k-1},\ D_G^{2k}(v)=\frac{1}{b}(c^2-b^2)^k.$$

When $a=c=1$ and $b=x$ in the equations above, we obtain

$$D_G^{2k-1}(v)|_{a=c=1,b=x}=-\frac{1}{x}(1-x^2)^{k-1},\ D_G^{2k}(v)|_{a=c=1,b=x}=\frac{1}{x}(1-x^2)^k. \tag{18}$$

We define $\widetilde{f}_n(x)=D_G^n(v)|_{a=c=1,b=x}$. Then

$$\widetilde{f}_n(x)=(-1)^n\frac{1}{x}(1-x^2)^{\lfloor n/2\rfloor}.$$

It follows from Lemma 4 that $S_{n+1}(x)=(-1)^nx^{n+1}w_n^{(1)}$, where $w_n^{(1)}$ is the following lower Hessenberg determinant of order $n+1$:

$$w_n^{(1)}=\begin{vmatrix} 1 & \frac{1}{x} & 0 & \cdots & 0 & 0 \\ S_1(x) & -\frac{1}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0 \\ S_2(x) & \frac{1}{x}(1-x^2) & -\binom{2}{1}\frac{1}{x} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{n-1}(x) & \widetilde{f}_{n-1}(x) & \binom{n-1}{1}\widetilde{f}_{n-2}(x) & \cdots & -\binom{n-1}{n-2}\frac{1}{x} & \binom{n-1}{n-1}\frac{1}{x} \\ S_n(x) & \widetilde{f}_n(x) & \binom{n}{1}\widetilde{f}_{n-1}(x) & \cdots & \binom{n}{n-2}\frac{1}{x}(1-x^2) & -\binom{n}{n-1}\frac{1}{x} \end{vmatrix}.$$

First multiplying each column by x, and then except for the first column, multiplying each of the other columns by -1, we see that $(-1)^nx^{n+1}w_n^{(1)}$ can be expressed as follows:

$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ xS_1(x) & 1 & -\binom{1}{1} & \cdots & 0 & 0 \\ xS_2(x) & -(1-x^2) & \binom{2}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ xS_{n-1}(x) & f_{n-1}(x) & \binom{n-1}{1}f_{n-2}(x) & \cdots & \binom{n-1}{n-2} & -\binom{n-1}{n-1} \\ xS_n(x) & f_n(x) & \binom{n}{1}f_{n-1}(x) & \cdots & -\binom{n}{n-2}(1-x^2) & \binom{n}{n-1} \end{vmatrix},$$

where $f_n(x)=(-x)\widetilde{f}_n(x)=(-1)^{n+1}(1-x^2)^{\lfloor n/2\rfloor}$.

(B) For $n\geqslant 1$, it is clear that

$$D_G^n(a^2)=D_G^{n-1}(2a^2b)=2D_G^{n-1}(a^2b)=2D_G^{n-1}\left(\frac{a^2}{\frac{1}{b}}\right).$$

Let $u(a,b,c)=a^2$ and $v(a,b,c)=\frac{1}{b}$. Then for $k\geqslant 1$, it follows from (15) and (18) that

$$D_G^k(u)|_{a=c=1,b=x}=R_{k+1}(x),\ D_G^k(v)|_{a=c=1,b=x}=(-1)^k\frac{1}{x}(1-x^2)^{\lfloor k/2\rfloor}.$$

Applying Lemma 4, we see that $R_{n+1}(x)=2(-1)^{n-1}x^n w_n^{(2)}$, where $w_n^{(2)}$ is a lower Hessenberg determinant of order n. Similarly, multiplying the first column by $2x$ and multiplying each of the other columns by $-x$, we immediately get the desired representation of $R_{n+}(x)$. $\square$

Combining (6) and Theorem 13, it is routine to verify the following result.

Corollary 14. *For $n\geqslant 1$, we have*

$$S_{n+1}(x)=(n+x)S_n(x)+(1-x^2)\sum_{r=2}^{n}\binom{n}{r-2}(-1)^{n-r+1}(1-x^2)^{\lfloor (n-r)/2\rfloor}S_{r-1}(x),$$

$$R_{n+1}(x)=2xR_n(x)+\sum_{r=2}^{n}\binom{n-1}{r-2}(-1)^{n-r}(1-x^2)^{\lfloor (n-r+1)/2\rfloor}R_r(x).$$

3.2. On the type B alternating run polynomials over up signed permutations

Let $T_n(x)=\sum_{k=1}^n T(n,k)x^k$. For $n\geqslant 2$ and $1\leqslant k\leqslant n$, Zhao [41, Theorem 4.2.1] showed that the numbers $T(n,k)$ satisfy the following recurrence relation

$$T(n,k)=(2k-1)T(n-1,k)+3T(n-1,k-1)+(2n-2k+2)T(n-1,k-2), \tag{19}$$

where $T(1,1)=1$ and $T(1,k)=0$ for $k>1$. We now present a grammatical description.

Lemma 15. *If $G=\{a\to a(b+c), b\to 2bc, c\to 2b^2\}$, then for $n\geqslant 1$, we have*

$$D_G^n(a)|_{a=b=x,c=1}=(1+x)T_n(x). \tag{20}$$

Proof. Note that $D_G(a)=a(b+c)$, $D_G^2(a)=a(b+c)(c+3b)$ and $D_G^3(a)=a(b+c)(c^2+12bc+11b^2)$. So the result holds for $n\leqslant 3$. We proceed by induction. Assume that

$$D_G^n(a)=a(b+c)\sum_{k=1}^{n}T(n,k)b^{k-1}c^{n-k}. \tag{21}$$

Then

$$\begin{aligned}D_G^{n+1}(a)&=D\left(a(b+c)\sum_{k=1}^{n}T(n,k)b^{k-1}c^{n-k}\right)\\&=a(b+c)(c+3b)\sum_{k}T(n,k)b^{k-1}c^{n-k}+\\&a(b+c)\sum_{k=1}^{n}T(n,k)\left[2(k-1)b^{k-1}c^{n-k+1}+2(n-k)b^{k+1}c^{n-k-1}\right].\end{aligned}$$

Exacting the coefficients of $a(b+c)b^{k-1}c^{n-k+1}$ in the last expression, we obtain

$$T(n,k)+3T(n,k-1)+2(k-1)T(n,k)+2(n-k+2)T(n,k-2)=T(n+1,k).$$

It follows from (19) that $D_G^{n+1}(a)=a(b+c)\sum_{k=1}^{n+1}T(n+1,k)b^{k-1}c^{n-k+1}$, as desired. $\square$

Theorem 16. *For any $n\geqslant 1$, let*

$$t_n(x)=\begin{cases}(1+x)(1-x^2)^{k-1}, & \text{if } n=2k-1;\\ -(1-x^2)^k, & \text{if } n=2k.\end{cases}$$

Then $T_n(x)$ is expressible as the following lower Hessenberg determinant of order n:

$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ -x(1-x) & \binom{2}{1}(1+x) & -1 & \cdots & 0 & 0 \\ \frac{x}{1+x}t_3(x) & \binom{3}{1}t_2(x) & \binom{3}{2}(1+x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{x}{1+x}t_{n-1}(x) & \binom{n-1}{1}t_{n-2}(x) & \binom{n-1}{2}t_{n-3}(x) & \cdots & \binom{n-1}{n-2}(1+x) & -1 \\ \frac{x}{1+x}t_n(x) & \binom{n}{1}t_{n-1}(x) & \binom{n}{2}t_{n-2}(x) & \cdots & -\binom{n}{n-2}(1-x^2) & \binom{n}{n-1}(1+x) \end{vmatrix}.$$

Proof. Consider $G = \{a \to a(b+c), b \to 2bc, c \to 2b^2\}$. By Lemma 15, we find that

$$D_G^n(a) = D_G^n\left(\frac{1}{\frac{1}{a}}\right).$$

Let $u(a,b,c) = 1$ and $v(a,b,c) = \frac{1}{a}$. Then $D_G^k(u) = 0$ for $k \geqslant 1$. Note that

$$D_G(v) = D_G\left(\frac{1}{a}\right) = -\frac{b+c}{a},\ D_G\left(\frac{b+c}{a}\right) = \frac{b^2-c^2}{a} = -\frac{c^2-b^2}{a}.$$

Since $D_G(c^2-b^2) = 0$, it follows that for $k \geqslant 1$, we have

$$D_G^{2k-1}(v) = -\frac{b+c}{a}(c^2-b^2)^{k-1},\ D_G^{2k}(v) = \frac{1}{a}(c^2-b^2)^k. \tag{22}$$

When $a = b = x$ and $c = 1$ in the equations above, we see that

$$D_G^{2k-1}(v)|_{a=b=x,c=1} = -\frac{1+x}{x}(1-x^2)^{k-1},\ D_G^{2k}(v)|_{a=b=x,c=1} = \frac{1}{x}(1-x^2)^k.$$

For $n \geqslant 1$, we define

$$\widetilde{t}_n(x) = \begin{cases} -\frac{1+x}{x}(1-x^2)^{k-1}, & \text{if } n = 2k-1; \\ \frac{1}{x}(1-x^2)^k, & \text{if } n = 2k. \end{cases}$$

Applying Lemma 4 and (21), we arrive at

$$a(b+c)\sum_{k=1}^{n} T(n,k)b^{k-1}c^{n-k}|_{a=b=x,c=1} = (1+x)T_n(x) = (-1)^n x^{n+1} w_n,$$

where the $(n+1)$ order lower Hessenberg determinant w_n is given by

$$\begin{vmatrix} 1 & \frac{1}{x} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1+x}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0 \\ 0 & \frac{1}{x}(1-x^2) & -\binom{2}{1}\frac{1+x}{x} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \widetilde{t}_{n-1}(x) & \binom{n-1}{1}\widetilde{t}_{n-2}(x) & \cdots & -\binom{n-1}{n-2}\frac{1+x}{x} & \binom{n-1}{n-1}\frac{1}{x} \\ 0 & \widetilde{t}_n(x) & \binom{n}{1}\widetilde{t}_{n-1}(x) & \cdots & \binom{n}{n-2}\frac{1}{x}(1-x^2) & -\binom{n}{n-1}\frac{1+x}{x} \end{vmatrix}.$$

Expanding along the first column of w_n, we see that

$$w_n = \begin{vmatrix} -\frac{1+x}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0 \\ \frac{1}{x}(1-x^2) & -\binom{2}{1}\frac{1+x}{x} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{t}_{n-1}(x) & \binom{n-1}{1}\widetilde{t}_{n-2}(x) & \cdots & -\binom{n-1}{n-2}\frac{1+x}{x} & \binom{n-1}{n-1}\frac{1}{x} \\ \widetilde{t}_n(x) & \binom{n}{1}\widetilde{t}_{n-1}(x) & \cdots & \binom{n}{n-2}\frac{1}{x}(1-x^2) & -\binom{n}{n-1}\frac{1+x}{x} \end{vmatrix}.$$

Multiplying each column of the above determinant by $-x$, we see that

$$(-1)^n x^n w_n = \begin{vmatrix} 1+x & -1 & \cdots & 0 & 0 \\ -(1-x^2) & \binom{2}{1}(1+x) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-1}(x) & \binom{n-1}{1}t_{n-2}(x) & \cdots & \binom{n-1}{n-2}(1+x) & -1 \\ t_n(x) & \binom{n}{1}t_{n-1}(x) & \cdots & -\binom{n}{n-2}(1-x^2) & \binom{n}{n-1}(1+x) \end{vmatrix},$$

where

$$t_n(x) = \begin{cases} (1+x)(1-x^2)^{k-1}, & \text{if } n=2k-1; \\ -(1-x^2)^k, & \text{if } n=2k. \end{cases}$$

Multiplying the first column by $\frac{x}{1+x}$ leads to the stated formula. □

When $n=4$, Theorem 16 says that

$$\frac{1}{x}T_4(x) = \begin{vmatrix} 1 & -1 & 0 & 0 \\ -(1-x) & \binom{2}{1}(1+x) & -1 & 0 \\ 1-x^2 & -\binom{3}{1}(1-x^2) & \binom{3}{2}(1+x) & -1 \\ -(1-x)(1-x^2) & \binom{4}{1}(1+x)(1-x^2) & -\binom{4}{2}(1-x^2) & \binom{4}{3}(1+x) \end{vmatrix}.$$

Combining (6) and Theorem 16, we get the following.

Corollary 17. *For any $n \geqslant 1$, let*

$$t_n(x) = \begin{cases} (1+x)(1-x^2)^{k-1}, & \text{if } n=2k-1; \\ -(1-x^2)^k, & \text{if } n=2k. \end{cases}$$

Then we have

$$T_n(x) = \frac{x}{1+x}t_n(x) + \sum_{r=2}^{n}\binom{n}{r-1}t_{n-r+1}T_{r-1}(x).$$

We now give a deep connection between the up-down run polynomials $S_n(x)$ and the type B alternating run polynomials $T_n(x)$.

Theorem 18. *For any $n \geqslant 1$, let*

$$t_n(x) = \begin{cases} (1+x)(1-x^2)^{k-1}, & \text{if } n=2k-1; \\ -(1-x^2)^k, & \text{if } n=2k. \end{cases}$$

Then $T_{n+1}(x)$ is expressible as the following lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 2xS_1(x) & 1+x & -\binom{1}{1} & \cdots & 0 & 0 \\ 2^2xS_2(x) & -(1-x^2) & \binom{2}{1}(1+x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{n-1}xS_{n-1}(x) & t_{n-1}(x) & \binom{n-1}{1}t_{n-2}(x) & \cdots & \binom{n-1}{n-2}(1+x) & -\binom{n-1}{n-1} \\ 2^nxS_n(x) & t_n(x) & \binom{n}{1}t_{n-1}(x) & \cdots & -\binom{n}{n-2}(1-x^2) & \binom{n}{n-1}(1+x) \end{vmatrix}.$$

Proof. Recall Lemma 15. It follows from (21) that

$$D_G^{n+1}(a) = a(b+c)\sum_{k=1}^{n+1} T(n+1,k)b^{k-1}c^{n+1-k}. \tag{23}$$

On the other hand,

$$D_G^{n+1}(a) = D_G^n(a(b+c)) = D_G^n\left(\frac{b+c}{1/a}\right).$$

Let $u(a,b,c)=b+c$ and $v(a,b,c)=\frac{1}{a}$. For any $k\geqslant 1$, the expansion of $D_G^k(v)$ is given by (22). When $u=b+c$, then $D_G(u)=2b(b+c)$ and for $k\geqslant 2$, we observe that

$$D_G^k(u)=2^{k-1}(b+c)^2c^{k-1}R_k\left(\frac{b}{c}\right)=2^{k-1}(b+c)^2\sum_{i=1}^{k-1}R(k,i)b^ic^{k-1-i}, \tag{24}$$

where $R(k,i)$ is defined by (11), i.e., the number of permutations in $\mathfrak{S}_k$ with i alternating runs.

We now prove (24) by induction. Note that $D_G^2(u)=2(b+c)^22b$, $D_G^3(u)=4(b+c)^2(2bc+4b^2)$ and $D_G^4(u)=8(b+c)^2(2bc^2+12b^2c+10b^3)$. Thus (24) holds for $k\leqslant 4$. Assume it holds for $k=n$. Then

$$\begin{aligned}D_G^{n+1}(u)&=D_G\left(2^{n-1}(b+c)^2\sum_{k=1}^{n}R(n,k)b^kc^{n-1-k}\right)\\&=2^{n-1}D_G\left((b+c)^2\sum_{k=1}^{n}R(n,k)b^kc^{n-1-k}\right)\\&=2^{n-1}\left(4b(b+c)^2\sum_{k=1}^{n}R(n,k)b^kc^{n-1-k}\right)+\\&\quad 2^{n-1}(b+c)^2\sum_{k}R(n,k)\left(2kb^kc^{n-k}+2(n-1-k)b^{k+2}c^{n-2-k}\right)\end{aligned}$$

Exacting the coefficient of $2^n(b+c)^2b^kc^{n-k}$, we get

$$2R(n,k-1)+kR(n,k)+(n-k+1)R(n,k-2)=R(n+1,k),$$

as desired. Thus (24) holds for $k=n+1$. When $b=x$ and $c=1$, then $u|_{b=x,c=1}=1+x$, $D_G(u)|_{b=x,c=1}=2x(1+x)$ and (24) reduces to

$$D_G^n(u)|_{b=x,c=1}=2^{n-1}(1+x)^2R_n(x).$$

When $a=b=x$ and $c=1$, applying Lemma 4 and (23), we see that $(1+x)T_{n+1}(x)=(-1)^nx^{n+1}w_n$, where w_n is given by

$$\begin{vmatrix}1+x & \frac{1}{x} & 0 & \cdots & 0 & 0\\ 2x(1+x) & -\frac{1+x}{x} & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0\\ 2(1+x)^2R_2(x) & \frac{1}{x}(1-x^2) & -\binom{2}{1}\frac{1+x}{x} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 2^{n-2}(1+x)^2R_{n-1}(x) & \widetilde{t}_{n-1}(x) & \binom{n-1}{1}\widetilde{t}_{n-2}(x) & \cdots & -\binom{n-1}{n-2}\frac{1+x}{x} & \binom{n-1}{n-1}\frac{1}{x}\\ 2^{n-1}(1+x)^2R_n(x) & \widetilde{t}_n(x) & \binom{n}{1}\widetilde{t}_{n-1}(x) & \cdots & \binom{n}{n-2}\frac{1}{x}(1-x^2) & -\binom{n}{n-1}\frac{1+x}{x}\end{vmatrix},$$

where

$$\widetilde{t}_n(x)=\begin{cases}-\frac{1+x}{x}(1-x^2)^{k-1}, & \text{if } n=2k-1;\\ \frac{1}{x}(1-x^2)^k, & \text{if } n=2k.\end{cases}$$

Multiplying the first column of w_n by $\frac{x}{1+x}$, and then multiplying each of the other columns by $-x$, and applying (13), we get the desired result. This completes the proof. □

By (6) and Theorem 18, we find the following result. The proof is omitted for simplicity.

Corollary 19. *For $n\geqslant 1$, let*

$$t_n(x)=\begin{cases}(1+x)(1-x^2)^{k-1}, & \text{if } n=2k-1;\\ -(1-x^2)^k, & \text{if } n=2k.\end{cases}$$

Then the type B alternating run polynomials satisfy the following recursion:

$$\begin{aligned}T_{n+1}(x)&=2^nxS_n(x)+\sum_{r=2}^{n+1}\binom{n}{r-2}t_{n-r+2}T_{r-1}(x)\\&=2^{n-1}x(1+x)R_n(x)+\sum_{r=2}^{n+1}\binom{n}{r-2}t_{n-r+2}T_{r-1}(x),\end{aligned} \tag{25}$$

where $S_n(x)$ are the up-down polynomials and $R_n(x)$ are the type A alternating run polynomials.

4. On the alternating run polynomials over dual stirling permutations

For a signed permutation $\sigma \in \mathcal{B}_n$, let

$$\mathrm{des}_B(\sigma) = \#\{i \in \{0, 1, 2, \ldots, n-1\} : \ \sigma(i) > \sigma(i+1),\ \sigma(0) = 0\}$$

be the number of *descents* of σ. The *type B Eulerian polynomials* are defined by

$$B_n(x) = \sum_{\sigma \in \mathcal{B}_n} x^{\mathrm{des}_B(\sigma)}.$$

They satisfy the recursion

$$B_{n+1}(x) = (2nx + 1 + x)B_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_n(x),$$

with $B_0(x) = 1,\ B_1(x) = 1 + x$ and $B_2(x) = 1 + 6x + x^2$. It is now well known that the types A and B Eulerian polynomials share several similar properties, see [20] Let $\left\{ {n \atop k} \right\}$ be the Stirling number of the second kind, i.e., the number of ways to partition $[n]$ into k blocks. Brenti [5, Theorem 3.14] found an explicit formula:

$$B_n(x) = \sum_{k=0}^{n} k! \sum_{i=k}^{n} \binom{n}{i} 2^i \left\{ {i \atop k} \right\} (x-1)^{n-k}.$$

Let $r(x) = \sqrt{\frac{1+x}{1-x}}$. Following [28, p. 14], the *alternating run polynomials of dual Stirling permutations* can be defined by

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n r(x) = \frac{r(x)F_n(x)}{(1-x^2)^n}.$$

They satisfy the recursion

$$F_{n+1}(x) = (x + 2nx^2)F_n(x) + x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}F_n(x),$$

with $F_0(x) = 1$. The first few alternating run polynomials of dual Stirling permutations are

$$\begin{aligned}
&F_1(x) = x,\\
&F_2(x) = x + x^2 + x^3,\\
&F_3(x) = x + 3x^2 + 7x^3 + 3x^4 + x^5,\\
&F_4(x) = x + 7x^2 + 29x^3 + 31x^4 + 29x^5 + 7x^6 + x^7.
\end{aligned}$$

Stirling permutations were introduced by Gessel and Stanley [19] and have been extensively studied in recent years [30,31]. A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each i, $1 \le i \le n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. Let Φ be the injection which maps each first occurrence of entry j in σ to $2j$ and the second occurrence of j to $2j-1$, where $j \in [n]$. For example, $\Phi(221331) = 432651$. Let $\Phi(\mathcal{Q}_n) = \{\pi \mid \sigma \in \mathcal{Q}_n, \Phi(\sigma) = \pi\}$ be the set of *dual Stirling permutations* of order n. A main result in [32] says that

$$F_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\mathrm{run}\,(\pi)}.$$

Let $F(x; z) = \sum_{n=0}^{\infty} F_n(x)\frac{z^n}{n!}$. According to [32, Eq. (3.7)], we have

$$F(x; z) = \frac{e^{z(x-1)(x+1)} + x}{1 + x}\sqrt{\frac{1-x^2}{e^{2z(x-1)(x+1)} - x^2}}.$$

An occurrence of an *ascent-plateau* of $\sigma \in \mathcal{Q}_n$ is an index i such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{2, 3, \ldots, 2n-1\}$. Let $\mathrm{ap}\,(\sigma)$ be the number of ascent-plateaus of σ. The number of *flag ascent-plateaus* of σ is defined by

$$\mathrm{fap}\,(\sigma) = \begin{cases} 2\mathrm{ap}\,(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2;\\ 2\mathrm{ap}\,(\sigma), & \text{otherwise.}\end{cases}$$

It is easy to check that $\mathrm{fap}\,(\sigma) = \mathrm{altrun}\,(\Phi(\sigma))$ for any $\sigma \in \mathcal{Q}_n$. Hence another interpretation of $F_n(x)$ is given as follows:

$$F_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{fap}\,(\sigma)}.$$

The grammatical interpretations of $B_n(x)$ and $F_n(x)$ are given as follows.

Lemma 20 ([27,28,32]). *If $G=\{a\to abc, b\to bc^2, c\to b^2c\}$, then we have*

$$D_G^n(bc)=bc^{2n+1}B_n\left(\frac{b^2}{c^2}\right),\ D_G^n(a)=ac^{2n}F_n\left(\frac{b}{c}\right).$$

Theorem 21. *For any $n\geqslant 1$, the polynomial $F_{n+1}(x)$ can be expressed as the following lower Hessenberg determinant of order $n+1$:*

$$\begin{vmatrix} 1 & -1 & 0 & \cdots & 0 & 0\\ xB_1(x^2) & -F_1(-x) & -1 & \cdots & 0 & 0\\ xB_2(x^2) & -F_2(-x) & -\binom{2}{1}F_1(-x) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ xB_{n-1}(x^2) & -F_{n-1}(-x) & -\binom{n-1}{1}F_{n-2}(-x) & \cdots & -\binom{n-1}{n-2}F_1(-x) & -1\\ xB_n(x^2) & -F_n(-x) & -\binom{n}{1}F_{n-1}(-x) & \cdots & -\binom{n}{n-2}F_2(-x) & -\binom{n}{n-1}F_1(-x)\end{vmatrix}.$$

Proof. Consider the grammar $G=\{a\to abc, b\to bc^2, c\to b^2c\}$. Note that

$$ac^{2n+2}F_{n+1}\left(\frac{b}{c}\right)=D_G^{n+1}(a)=D_G^n(abc)=D_G^n\left(\frac{bc}{\frac{1}{a}}\right). \tag{26}$$

Let $u(a,b,c)=bc$ and $v(a,b,c)=\frac{1}{a}$. By Lemma 20, we have $D_G^n(u)=bc^{2n+1}B_n\left(\frac{b^2}{c^2}\right)$. Note that

$$D_G(v)=D_G\left(\frac{1}{a}\right)=\frac{c}{a}(-b),\ D_G^2(v)=\frac{c}{a}(-bc^2+b^2c-b^3),$$

$$D_G^3(v)=\frac{c}{a}(-bc^4+3b^2c^3-7b^3c^2+3b^4c-b^5).$$

By induction, it is routine to verify that

$$D_G^n(v)=\frac{c^{2n}}{a}F_n\left(-\frac{b}{c}\right),$$

and we omit the details for simplicity. When $a=c=1$ and $b=x$, applying (26) and Lemma 4, we arrive at $F_{n+1}(x)=(-1)^n w_n$, where w_n is given by

$$\begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0\\ xB_1(x^2) & F_1(-x) & \binom{1}{1} & \cdots & 0 & 0\\ xB_2(x^2) & F_2(-x) & \binom{2}{1}F_1(-x) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ xB_{n-1}(x^2) & F_{n-1}(-x) & \binom{n-1}{1}F_{n-2}(-x) & \cdots & \binom{n-1}{n-2}F_1(-x) & \binom{n-1}{n-1}\\ xB_n(x^2) & F_n(-x) & \binom{n}{1}F_{n-1}(-x) & \cdots & \binom{n}{n-2}F_2(-x) & \binom{n}{n-1}F_1(-x)\end{vmatrix}.$$

Except for the first column, multiplying each of the other columns by -1 gives the desired result. This completes the proof. □

By (6) and Theorem 21, we immediately find the following result.

Corollary 22. *We have*

$$F_{n+1}(x)=xB_n(x^2)+nxF_n(x)-\sum_{r=2}^{n}\binom{n}{r-2}F_{n-r+2}(-x)F_{r-1}(x).$$

A dual of Theorem 21 is given as follows.

Theorem 23. *Let*

$$f_n(x)=\begin{cases}(1+x^2)(x^2-1)^{2k-2}, & \text{if } n=2k-1;\\ -(x^2-1)^{2k}, & \text{if } n=2k.\end{cases}$$

Then $F_{n+1}(x)$ can be expressed as the following lower Hessenberg determinant of order $n+1$:

$$\begin{vmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ xF_1(x) & f_1(x) & -\binom{1}{1} & \cdots & 0 & 0 \\ xF_2(x) & f_2(x) & \binom{2}{1}f_1(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ xF_{n-1}(x) & f_{n-1}(x) & \binom{n-1}{1}f_{n-2}(x) & \cdots & \binom{n-1}{n-2}f_1(x) & -\binom{n-1}{n-1} \\ xF_n(x) & f_n(x) & \binom{n}{1}f_{n-1}(x) & \cdots & \binom{n}{n-2}f_2(x) & \binom{n}{n-1}f_1(x) \end{vmatrix}.$$

Proof. Consider the grammar $G=\{a\to abc, b\to bc^2, c\to b^2c\}$. Note that

$$ac^{2n+2}F_{n+1}\left(\frac{b}{c}\right)=D_G^{n+1}(a)=D_G^n(abc)=D_G^n\left(\frac{a}{\frac{1}{bc}}\right). \tag{27}$$

Let $u(a,b,c)=a$ and $v(a,b,c)=\frac{1}{bc}$. By Lemma 20, we have $D_G^n(u)=ac^{2n}F_n\left(\frac{b}{c}\right)$. Note that

$$D_G(v)=D_G\left(\frac{1}{bc}\right)=-\frac{1}{bc}(b^2+c^2),\ D_G\left(\frac{1}{bc}(b^2+c^2)\right)=-\frac{1}{bc}(b^2-c^2)^2.$$

Since $D_G(b^2-c^2)=0$, it follows that

$$D_G^{2k-1}\left(\frac{1}{bc}\right)=-\frac{b^2+c^2}{bc}(b^2-c^2)^{2k-2},\ D_G^{2k}\left(\frac{1}{bc}\right)=\frac{1}{bc}(b^2-c^2)^{2k}.$$

When $a=c=1$ and $b=x$ in the equations above, we see that

$$D_G^n(u)|_{a=c=1,b=x}=F_n(x),\ D_G^n(v)|_{a=c=1,b=x}=\widetilde{f}_n(x),$$

where

$$\widetilde{f}_n(x)=\begin{cases} -\frac{1+x^2}{x}(x^2-1)^{2k-2}, & \text{if } n=2k-1; \\ \frac{1}{x}(x^2-1)^{2k}, & \text{if } n=2k. \end{cases}$$

Applying (27) and Lemma 4, we arrive at $F_{n+1}(x)=(-1)^n x^{n+1} w_n$, where w_n is given by

$$\begin{vmatrix} 1 & \frac{1}{x} & 0 & \cdots & 0 & 0 \\ F_1(x) & \widetilde{f}_1(x) & \binom{1}{1}\frac{1}{x} & \cdots & 0 & 0 \\ F_2(x) & \widetilde{f}_2(x) & \binom{2}{1}\widetilde{f}_1(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-1}(x) & \widetilde{f}_{n-1}(x) & \binom{n-1}{1}\widetilde{f}_{n-2}(x) & \cdots & \binom{n-1}{n-2}\widetilde{f}_1(x) & \binom{n-1}{n-1}\frac{1}{x} \\ F_n(x) & \widetilde{f}_n(x) & \binom{n}{1}\widetilde{f}_{n-1}(x) & \cdots & \binom{n}{n-2}\widetilde{f}_2(x) & \binom{n}{n-1}\widetilde{f}_1(x) \end{vmatrix}.$$

Multiplying the first column by x, and then multiplying each of the other columns by $-x$, we get the desired result. This completes the proof. □

By (6) and Theorem 23, we arrive at the following result.

Corollary 24. *Let*

$$f_n(x)=\begin{cases} (1+x^2)(x^2-1)^{2k-2}, & \text{if } n=2k-1; \\ -(x^2-1)^{2k}, & \text{if } n=2k. \end{cases}$$

Then the alternating run polynomials of dual Stirling permutations satisfy the recursion

$$F_{n+1}(x)=xF_n(x)+\sum_{r=2}^{n+1}\binom{n}{r-2}f_{n-r+2}(x)F_{r-1}(x).$$

5. Concluding remarks

In this paper, we develop a general method to deduce determinantal representations of enumerative polynomials. When we get the recurrence relation of a polynomial and its corresponding grammar, the key point in solving this problem is deducing iterated expressions. Some problems may be difficult. Along the same lines, it would be interesting to deal with any other Eulerian-type polynomial that was collected by Hwang et al. [22].

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Data availability

No data was used for the research described in the article.

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