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Enumerations for Parking Functions by leading term and maximum value

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Abstract. Given a parking function $\alpha = (a_1, \dots, a_n)$, define the maximum value of α, denoted by $M(\alpha)$, as $M(\alpha) = \max\{a_i \mid 1 \leq i \leq n\}$. A parking function $(a_1, \cdots, a_n)$ is said to be k-leading if $a_1 = k$. In this paper, we give the enumerations of the parking functions according to their maximum value and leading term. Let $p^l_{n;s}$ be the number of parking functions $\alpha = (a_1, \dots, a_n)$ of length n such that $M(\alpha) = s$ and $a_1 = l$. We derive some formulas and recurrence relations for the sequences $p^l_{n;s}$, and study the generating functions and the asymptotic behaviors of the sequence.

Keywords: Asymptotic behavior; Leading term; Parking function.

1. Introduction

Following [7], we define the parking function as follows: n parking spaces are arranged in a line, numbered 1 to n from left to right; n cars, arriving successively, have initial parking preferences, a_i for i, chosen independently and at random; we call $(a_1, \cdots, a_n)$ a *preference function* of length n; if the space a_i is occupied, car i moves to the first unoccupied space to the right; if all the cars can be parked, then we say the preference function is a *parking function* of length n.

Let $\mathcal{P}_n$ denote the set of parking functions of length n and $p_n = |\mathcal{P}_n|$. Riordan [7] derived that $p_n = (n+1)^{n-1}$, which coincides with the number of labeled trees on $n+1$ vertices by Cayley's formula. Several bijections between the two sets are known (e.g., see [3, 7, 8, 11]). A parking function $\alpha = (a_1, \ldots, a_n)$ is said to be *ordered* if $a_i \leq a_{i+1}$ for each i. Riordan concluded that the number of ordered parking functions is $\frac{1}{n+1}\binom{2n}{n}$, which is also equals the number of Dyck path of semilength n. In [5], Huang, Ma and Yeh studied ordered k-flaw preference sets. Parking functions have been found in connection to many other combinatorial structures such as acyclic mappings, polytopes, non-crossing partitions, non-nesting partitions, hyperplane arrangements,etc. Refer to [2, 3, 4, 6, 9, 10, 11] for more information.

A parking function $(a_1, \cdots, a_n)$ is said to be l-leading if $a_1 = l$. Let p_n^l denote the number of l-leading parking functions of length n. The sequence p_n^l is a refinement of parking functions of length n. Foata and Riordan [3] derived a generating function for p_n^l algebraically. Recently, Sen-peng Eu, Tung-shan Fu and Chun-Ju Lai [1] gave a combinatorial approach to the enumeration of parking functions by their leading terms.

Given a parking function $\alpha = (a_1, \ldots, a_n)$, define the maximum value of α, denoted by $M(\alpha)$, as $M(\alpha) = \max\{a_i \mid 1 \leq i \leq n\}$. For any $1 \leq s \leq n$, define $\mathcal{P}_{n;s}$ as a set of all the parking functions α of length n such that $M(\alpha) = s$ and let $p_{n;s}$ be the number of the parking functions in the set $\mathcal{P}_{n;s}$.

In this paper, motivated by the works of Foata and Riordan in [3] as well as Sen-Peng Eu et al. in [1], we consider a refinement of the conception of parking functions of length n according to their leading term and maximum value. Let $\mathcal{P}_{n;s}^l$ be the set of parking functions $\alpha = (a_1, \ldots, a_n)$ such that $a_1 = l$ and $M(\alpha) = s$, and let $p_{n;s}^l$ be the number of parking functions in the set $\mathcal{P}_{n;s}^l$. We study the recurrence relation and the generating function. Since there is no explicit formula for the sequence $p_{n;s}$, we are interested in the asymptotic behavior of the sequence $p_{n;s}$. Let $\lambda_{l,k} = \lim\limits_{n\to\infty} \frac{p_{n;n-k}^l}{(n+1)^{n-2}}$ for any $k \geq 0$ and define $\lambda(x,y) = \sum\limits_{l\geq 1}\sum\limits_{k\geq 0} \lambda_{l,k}x^k y^l$. Also we derive the generating function for $\lambda_{l,k}$.

Since there are some difficulties in computing the sequences $p_{n;s}^l$ directly, we define $\mathcal{P}_{n;\leq s}^l$ as a set of parking functions $\alpha = (a_1, \ldots, a_n)$ of length n such that $a_1 = l$ and $M(\alpha) \leq s$, and let $p_{n;\leq s}^l = |\mathcal{P}_{n;\leq s}^l|$. Obviously, $p_{n;s+1}^l = p_{n;\leq s+1}^l - p_{n;\leq s}^l$. As intermediate steps, we obtain some results related to the sequences $p_{n;\leq s}^l$.

This paper is organized as follows. In Section 2, we will give enumeration of parking functions according to their leading terms and maximum values. In Section 3, we will study the generating function for $p_{n;n-k}^l$. In Section 4, we study the asymptotic behavior for $p_{n;s}^l$. In the Appendix, we list the values of

$p^l_{n;\le s}$ for $n \le 7$.

2. Enumerating parking functions by their leading term and maximum value

Throughout the paper, we let $[n] := \{1, 2, \ldots, n\}$ and $[m, n] := \{m, m+1, \ldots, n\}$. In this section, we investigate the problem of enumerations of parking functions in the set $\mathcal{P}^l_{n;s}$. Let p^l_n be the number of parking functions $\alpha = (a_1, \ldots, a_n)$ of length n such that $a_1 = l$. Sen-peng Eu et al.[1] have deduced the following lemma.

Lemma 2.1.[1] *The sequence p^l_n satisfies the recurrence relation*

$$p^l_n - p^{l+1}_n = \binom{n-1}{l-1} l^{l-2}(n-l+1)^{n-l-1} \tag{1}$$

for $1 \le l \le n-1$, with the initial condition $p^1_n = 2(n+1)^{n-2}$.

Let $\mathcal{P}_{n;\le s}$ be the set of parking functions $\alpha = (a_1, \ldots, a_n)$ of length n such that $M(\alpha) \le s$ and $p_{n;\le s} = |\mathcal{P}_{n;\le s}|$. In the following lemma, we give a recurrence relation for $p_{n;\le n-k}$. Recall that $\mathcal{P}^l_{n;\le s}$ is the set of the parking functions $\alpha = (a_1, \ldots, a_n)$ of length n such that $M(\alpha) \le s$ and $\mathcal{P}^l_{n;\le s} = \{\alpha \in \mathcal{P}_{n;\le s} \mid a_1 = l\}$, where $1 \le l \le s \le n$.

Lemma 2.2.

(i) *Let k be an integer with $k \ge 1$. For any $n \ge k$, the sequence $p_{n;\le n-k}$ satisfies the following recurrence relation:*

$$p_{n;\le n-k} = p_{n;\le n-k+1} - \sum_{i=1}^{k} \binom{n}{i} p_{n-i;\le n-k}.$$

(ii) *For any $1 \le l \le s \le n$,*

$$p^l_{n;\le s} = s^{n-1} - \sum_{i=0}^{l-2} \binom{n-1}{i} p_i (s-i-1)^{n-i-1} - \sum_{i=l}^{s-2} \binom{n-1}{i-1} p^l_i (s-i-1)^{n-i}.$$

Proof.

(i) Given $1 \le k \le n$, for any $\alpha = (a_1, \cdots, a_n) \in \mathcal{P}_{n;\le n-k+1}$, let $\lambda(\alpha) = |\{j \in [n] \mid a_j = n-k+1\}|$ and $A_{k,i} = \{\alpha \in \mathcal{P}_{n;\le n-k+1} \mid \lambda(\alpha) = i\}$.

Obviously, $p_{n;\le n-k} = p_{n;\le n-k+1} - \sum\limits_{i=1}^{k} |A_{k,i}|$. It is easy to see that $|A_{k,i}| = \binom{n}{i} p_{n-i;\le n-k}$ for any $i \in [k]$. Hence, we have

$$p_{n;\le n-k} = p_{n;\le n-k+1} - \sum_{i=1}^{k} \binom{n}{i} p_{n-i;\le n-k}.$$

(*ii*) Let A be the set of the sequences $(l, a_2, \cdots, a_n)$ of length n with leading term l such that $a_i \le s$ for all i. Then $|A| = s^{n-1}$. Next, we count the number of the elements α in A which aren't parking functions. Since α isn't a parking function, we may suppose that the $(i+1)$-th parking space is the leftmost parking space which isn't occupied by cars. Obviously, $0 \le i \le s-2$ and $i \ne l-1$. Let $S = \{j \mid a_j \le i, a_j \in \alpha\}$ and let α_S be a subsequence of α determined by the subscripts in S. Then $|S| = i$ and $\alpha_S \in \mathcal{P}_i$. Let $T = [n] \setminus S$ and let α_T be a subsequence of α determined by the subscripts in T. Then $|T| = n - i$ and $i + 2 \le a_j \le s$ for any $j \in T$. Now, we discuss the following two cases.

Case I. $0 \le i \le l-2$

Obviously, $1 \in T$. There are $\binom{n-1}{i}$ ways to choose i numbers from $[2, n]$ for the elements in S. There are p_i and $(s-i-1)^{n-i-1}$ possibilities for α_S and α_T, respectively.

Case II. $l \le i \le s-2$

Note that $1 \in S$; hence, there are $\binom{n-1}{i-1}$ ways to choose $i-1$ numbers from $[2, n]$ for the other elements in S. Since the leading term of α_S is l, there are p_i^l and $(s-i-1)^{n-i}$ possibilities for α_S and α_T, respectively.

So, the number of the elements α in A which aren't parking functions is equal to

$$\sum_{i=0}^{l-2} \binom{n-1}{i} p_i (s-i-1)^{n-i-1} + \sum_{i=l}^{s-2} \binom{n-1}{i-1} p_i^l (s-i-1)^{n-i}.$$

Hence,

$$p_{n;\le s}^l = s^{n-1} - \sum_{i=0}^{l-2} \binom{n-1}{i} p_i (s-i-1)^{n-i-1} - \sum_{i=l}^{s-2} \binom{n-1}{i-1} p_i^l (s-i-1)^{n-i}.$$

■

Example 2.3. Take $n = 7$, $s = 6$ and $l = 3$. By the data in the Appendix, we find $p_{7;\le 6}^3 = 23667$, $p_0 = p_1 = 1$, $p_3^3 = 3$, $p_4^3 = 25$. It is easy to check that $p_{7;\le 6}^3 = 6^6 - \sum\limits_{i=0}^{1} \binom{6}{i} p_i (5-i)^{6-i} - \sum\limits_{i=3}^{4} \binom{6}{i-1} p_i^3 (5-i)^{7-i}$.

Lemma 2.4.

(i) $p^n_{n;\leq n} = p_{n-1}$ *for any* $n \geq 1$.

(ii) *For any* $1 \leq s \leq n-1$, *we have* $p^s_{n;\leq s} = p_{n-1;\leq s}$.

Proof. We first prove the identity (ii). For any $\alpha = (s, a_1, \cdots, a_{n-1}) \in \mathcal{P}^s_{n;\leq s}$, it is easy to see that $(a_1, \cdots, a_{n-1}) \in \mathcal{P}_{n-1;\leq s}$. Obviously, this is a bijection. Hence, $p^s_{n;\leq s} = p_{n-1;\leq s}$ for any $1 \leq s \leq n-1$.

Using a similar method, it is easy to derive the identity (i). ■

Let $\mathcal{B}_{i,k}$ denote the set of parking functions of length i with k parking spaces for $i \leq k$. Riordan [7] derived the explicit formula

$$b_{i,k} = |\mathcal{B}_{i,k}| = (k-i+1)(k+1)^{i-1}$$

for $i \leq k$. In the following lemma, we will derive two recurrence relations for $p^l_{n;\leq n-k}$.

Lemma 2.5. *Let* k *be an integer with* $k \geq 1$. *For any* $n \geq k+1$ *and* $1 \leq l \leq n-k$, *the sequence* $p^l_{n;\leq n-k}$ *satisfies the recurrence relations*

$$p^l_{n;\leq n-k} = p^l_{n;\leq n-k+1} - \sum_{i=1}^{k} \binom{n-1}{i} p^l_{n-i;\leq n-k} \tag{2}$$

and

$$p^l_{n;\leq n-k} = p^l_n - \sum_{i=1}^{k} \binom{n-1}{i} (k-i+1)(k+1)^{i-1} p^l_{n-i;\leq n-k}. \tag{3}$$

Proof. We first prove the identity (2). Given $1 \leq k \leq n$, for any $\alpha = (a_1, \cdots, a_n) \in \mathcal{P}^l_{n;\leq n-k+1}$, let $\lambda(\alpha) = |\{j \mid a_j = n-k+1\}|$ and $A_{k,i} = \{\alpha \in \mathcal{P}^l_{n;\leq n-k+1} \mid \lambda(\alpha) = i\}$. Obviously, $p^l_{n;\leq n-k} = p^l_{n;\leq n-k+1} - \sum\limits_{i=1}^{k} |A_{k,i}|$. It is easy to see that $|A_{k,i}| = \binom{n-1}{i} p^l_{n-i;\leq n-k}$ for any $i \in [k]$. Hence, we have

$$p^l_{n;\leq n-k} = p^l_{n;\leq n-k+1} - \sum_{i=1}^{k} \binom{n-1}{i} p^l_{n-i;\leq n-k}.$$

For deriving the identity (3), we count the number of the parking functions $\alpha = (a_1, \cdots, a_n) \in \mathcal{P}^l_n$ satisfying $a_i > n-k$ for some i. Let $S = \{j \mid a_j \leq n-k, a_j \in \alpha\}$ and let α_S be a subsequence of α determined by the subscripts in S. Let $T = [n] \backslash S$ and let α_T be a subsequence of α determined by the subscripts in T. Then $1 \le |T| \le k$ and $n-k+1 \le a_j \le n$ for any $j \in T$. Suppose $|T| = i$ and $\alpha_T = (b_1, \cdots, b_i)$. Then $(b_1 - n + k, \cdots, b_i - n + k) \in \mathcal{B}_{i,k}$.

Since $1 \in S$, there are $\binom{n-1}{i}$ ways to choose i numbers from $[2, n]$ for the elements in T. Note that $\alpha_S \in \mathcal{P}^l_{n-i;\le n-k}$. There are $p^l_{n-i;\le n-k}$ and $b_{i,k}$ possibilities for parking functions α_S and α_T, respectively. We have

$$\begin{aligned} p^l_{n;\le n-k} &= p^l_n - \sum_{i=1}^{k} \binom{n-1}{i} b_{i,k} p^l_{n-i;\le n-k} \\ &= p^l_n - \sum_{i=1}^{k} \binom{n-1}{i} (k-i+1)(k+1)^{i-1} p^l_{n-i;\le n-k}. \end{aligned}$$

■

Example 2.6. Take $n = 7$, $k = 1$ and $l = 3$. By the data in the Appendix, we find $p^3_{7;\le 6} = 23667$, $p^3_7 = 40953$, $p^3_{6;\le 6} = 2881$. It is easy to check that $p^3_{7;\le 6} = p^3_7 - \binom{6}{1} p^3_{6;\le 6}$.

Recall that $p^l_{n;s}$ is the number of parking functions $\alpha = (a_1, \ldots, a_n)$ such that $a_1 = l$ and $M(\alpha) = s$.

Theorem 2.7.

(i) $p^n_{n;n} = p_{n-1}$ *and* $p^{n-k}_{n;n-k} = p_{n-1;\le n-k}$ *for any* $k \ge 1$.

(ii) Let $k \ge 1$, *then*

$$p^l_{n;n-k} = p^l_{n;n-k+1} + \binom{n-1}{k+1} p^l_{n-k-1} - \sum_{i=1}^{k} \binom{n-1}{i} p^l_{n-i;n-k}$$

for any $n \ge k+2$ *and* $l \le n-k-1$.

Proof.

(*i*) Obviously, $p^{n-k}_{n;n-k} = p^{n-k}_{n;\le n-k}$. Hence, we easily obtain that $p^n_{n;n} = p_{n-1}$ and $p^{n-k}_{n;n-k} = p_{n-1;\le n-k}$ for any $k \ge 1$.

(*ii*) Note that $p^l_{n;n-k+1} = p^l_{n;\le n-k+1} - p^l_{n;\le n-k}$ for $l \le n-k$. By the identity

(2) of Lemma 2.5, we obtain

$$\begin{aligned}
p^l_{n;n-k} &= p^l_{n;\le n-k} - p^l_{n;\le n-k-1}\\
&= p^l_{n;\le n-k+1} - \sum_{i=1}^{k}\binom{n-1}{i}p^l_{n-i;\le n-k}\\
&\quad -p^l_{n;\le n-k} + \sum_{i=1}^{k+1}\binom{n-1}{i}p^l_{n-i;\le n-k-1}\\
&= p^l_{n;n-k+1} + \binom{n-1}{k+1}p^l_{n-k-1;\le n-k-1} - \sum_{i=1}^{k}\binom{n-1}{i}p^l_{n-i;n-k}\\
&= p^l_{n;n-k+1} + \binom{n-1}{k+1}p^l_{n-k-1} - \sum_{i=1}^{k}\binom{n-1}{i}p^l_{n-i;n-k}
\end{aligned}$$

for any $n \ge k+2$ and $l \le n-k-1$.

■

3. The generating function for the sequence $p^l_{n;n-k}$

In this section, we will study the generating function for the sequence $p^l_{n;n-k}$. First, define $P(x)$ as $P(x) = \sum_{n\ge 0}(n+1)^{n-1}\frac{x^n}{n!}$ which is the generating function of the parking functions of length n. Let $\psi(x) = xe^{-x}$. It is well known that $xP(x)$ is the inverse function of $\psi(x)$, i.e., $\psi(xP(x)) = x$. Hence, we have $\psi'(xP(x))[P(x) + xP'(x)] = 1$, which implies that $P(x)$ satisfies the differential equation $P'(x) = [P(x)]^2 + xP(x)P'(x)$.

Let $L(x) = \sum_{n\ge 1}\frac{p^1_n}{(n-1)!}x^n$. For a fixed $k \ge 0$, we define a generating function by letting $T_k(x) = \sum_{n\ge k+1}\frac{p^{n-k}_n}{(n-1)!}x^n$ and $T(x,y) = \sum_{k\ge 0}T_k(x)y^k$. We need the following lemma.

Lemma 3.1. *Let $T_0(x)$ and $L(x)$ be the generating functions for the sequences p^n_n and p^1_n, respectively. Then $L(x)$ and $T_0(x)$ satisfy the following differential equations:*

$$L(x) + xL'(x) = 2xP'(x)$$

and

$$T_0(x) + xT_0'(x) = 2xP(x) + x^2P'(x),$$

respectively. Furthermore, we have $L(x) = x[P(x)]^2$ and $T_0(x) = xP(x)$.

Proof. Lemma 2.1 tells us that $p^1_n = 2(n+1)^{n-2}$. By comparing coefficients, it is easy to prove that $L(x)$ satisfies the differential equation $L(x) + xL'(x) =$

$2xP'(x)$. Since $P'(x) = [P(x)]^2 + xP(x)P'(x)$ and $L(0) = 0$, solving this equation, we have $L(x) = x[P(x)]^2$.

Take $l = n - k$ in Lemma 2.1. Then

$$p_n^{n-k} - p_n^{n-k+1} = \binom{n-1}{k}(n-k)^{n-k-2}(k+1)^{k-1} \tag{4}$$

for $1 \le k \le n-1$. So, we have

$$p_n^1 - p_n^n = \sum_{k=1}^{n-1}\binom{n-1}{k}(n-k)^{n-k-2}(k+1)^{k-1}.$$

Therefore,

$$\sum_{n\ge1}\frac{p_n^1}{(n-1)!}x^n - \sum_{n\ge1}\frac{p_n^n}{(n-1)!}x^n = \sum_{n\ge1}\sum_{k=1}^{n-1}\frac{(n-k)^{n-k-2}}{(n-k-1)!}\frac{(k+1)^{k-1}}{k!}x^n.$$

We obtain

$$T_0(x) = L(x) + xP(x) - x[P(x)]^2.$$

Since $L(x) + xL'(x) = 2xP'(x)$ and $P'(x) = [P(x)]^2 + xP(x)P'(x)$, we have $T_0(x) + xT_0'(x) = 2xP(x) + x^2P'(x)$. Note that $T_0(0) = 0$. Solving this equation, we obtain $T_0(x) = xP(x)$. ∎

In the following lemma, we give bijection proofs for $T_0(x) = xP(x)$ and $L(x) = x[P(x)]^2$.

Lemma 3.2. *For any $n \ge 1$, there is a bijection from the sets $\mathcal{P}_n^n$ (resp. $\mathcal{P}_n^1$) to $\mathcal{P}_{n-1}$ (resp. $\mathcal{B}_{n-1,n}$). Furthermore, we have $T_0(x) = xP(x)$ and $L(x) = x[P(x)]^2$.*

Proof. For any $\alpha = (n, a_1, \cdots, a_{n-1}) \in \mathcal{P}_n^n$, it is easy to check that $(a_1, \cdots, a_{n-1}) \in \mathcal{P}_{n-1}$. Obviously, this is a bijection. Hence, $p_n^n = p_{n-1}$ for any $n \ge 1$. Similarly, for any $\beta = (1, b_1, \cdots, b_{n-1}) \in \mathcal{P}_n^1$, we have $(b_1, \cdots, b_{n-1}) \in \mathcal{B}_{n-1,n}$. Clearly, this is a bijection. So, $p_n^1 = b_{n-1,n}$ for any $n \ge 1$.

By these two bijections, simple computations tell us that $T_0(x) = xP(x)$ and $L(x) = x[P(x)]^2$. ∎

Lemma 3.3. *Let $k \ge 1$ and let $T_k(x)$ be the generating function for the sequence p_n^{n-k}. Then $T_k(x)$ satisfies the recurrence relation*

$$T_k(x) = T_{k-1}(x) + \frac{(k+1)^{k-1}}{k!}x^{k+1}P(x) - \frac{2(k+1)^{k-2}}{(k-1)!}x^k.$$

Equivalently,

$$T_k(x) = P(x)\sum_{i=0}^{k}\frac{(i+1)^{i-1}}{i!}x^{i+1} - \sum_{i=1}^{k}\frac{2(i+1)^{i-2}}{(i-1)!}x^{i}.$$

Proof. From Identity (4), we have

$$\sum_{n\geq k+1}\frac{p_n^{n-k}}{(n-1)!}x^n - \sum_{n\geq k+1}\frac{p_n^{n-k+1}}{(n-1)!}x^n = \frac{(k+1)^{k-1}}{k!}x^k\sum_{n\geq k+1}\frac{(n-k)^{n-k-2}}{(n-k-1)!}x^{n-k}.$$

Hence,

$$T_k(x) = T_{k-1}(x) + \frac{(k+1)^{k-1}}{k!}x^{k+1}P(x) - \frac{2(k+1)^{k-2}}{(k-1)!}x^k.$$

■

Corollary 3.4. *Let $T(x,y) = \sum\limits_{k\geq 0} T_k(x)y^k$. Then $T(x,y)$ satisfies the equations*

$$\begin{aligned}(1-y)\left[T(x,y) + x\frac{\partial T(x,y)}{\partial y}\right] &= 2xP(x)P(xy) + x^2P(xy)P'(x) + x^2yP(x)P'(xy)\\ &\quad -2xyP'(xy)\end{aligned}$$

and

$$T(x,y) - xP(x) = yT(x,y) + xP(x)[P(xy)-1] - xy[P(xy)]^2.$$

The second identity implies that

$$T(x,y) = \frac{xP(xy)[P(x) - yP(xy)]}{1-y}.$$

Proof. By Lemma 3.3, we have

$$\sum_{k\geq 1}T_k(x)y^k = y\sum_{k\geq 0}T_k(x)y^k + xP(x)\sum_{k\geq 1}\frac{(k+1)^{k-1}}{k!}(xy)^k - \sum_{k\geq 1}\frac{2(k+1)^{k-2}}{(k-1)!}(xy)^k.$$

Since $T(x,y) = \sum\limits_{k\geq 0} T_k(x)y^k$, we have

$$T(x,y) - T_0(x) = yT(x,y) + xP(x)[P(xy)-1] - L(xy).$$

Furthermore, by Lemma 3.1, $T(x,y)$ satisfies the following equations:

$$\begin{aligned}&(1-y)\left[T(x,y) + x\frac{\partial T(x,y)}{\partial y}\right]\\ &= 2xP(x)P(xy) + x^2P(xy)P'(x) + x^2yP(x)P'(xy) - 2xyP'(xy)\end{aligned}$$

and

$$T(x,y) - xP(x) = yT(x,y) + xP(x)[P(xy) - 1] - xy[P(xy)]^2.$$

Hence,

$$T(x,y) = \frac{xP(xy)[P(x) - yP(xy)]}{1-y}.$$

■

Given $s \geq k \geq 0$, we define a generating function by letting

$$F_{k,s}(x) = \sum_{n \geq s+1} \frac{p_{n;\leq n-k}^{n-s}}{(n-1)!} x^n.$$

Let $F_k(x,y) = \sum_{s \geq k} F_{k,s}(x) y^s$ and $F(x,y,z) = \sum_{k \geq 0} F_k(x,y) z^k$.

Lemma 3.5. *Let $s \geq k \geq 0$ and $F_{k,s}(x)$ and $T_s(x)$ be the generating function for $p_{n;\leq n-k}^{n-s}$ and p_n^{n-s}, respectively. Then $F_{k,s}(x)$ satisfies the recurrence relation*

$$F_{k,s}(x) = T_s(x) - \sum_{i=1}^{k} \frac{(k-i+1)(k+1)^{i-1}}{i!} x^i F_{k-i,s-i}(x) \tag{5}$$

for any $1 \leq k \leq s$, with the initial condition $F_{0,s}(x) = T_s(x)$. Furthermore, we have

$$F_{k,s}(x) = \sum_{i=0}^{k} \frac{(-1)^i (k+1-i)^i}{i!} x^i T_{s-i}(x). \tag{6}$$

Proof. Taking $l = n - s$ in the identity (3) of Lemma 2.5, we have

$$p_{n;\leq n-k}^{n-s} = p_n^{n-s} - \sum_{i=1}^{k} \binom{n-1}{i} (k-i+1)(k+1)^{i-1} p_{n-i;\leq n-k}^{n-s}. \tag{7}$$

Note that $p_{n;\leq n}^{n-s} = p_n^{n-s}$. Hence,

$$F_{0,s}(x) = T_s(x) = \sum_{n \geq s+1} \frac{p_n^{n-s}}{(n-1)!} x^n.$$

By Identity (7), we have

$$\begin{aligned}
&\sum_{n \geq s+1} \frac{p_{n;\leq n-k}^{n-s}}{(n-1)!} x^n \\
&= \sum_{n \geq s+1} \frac{p_n^{n-s}}{(n-1)!} x^n - \sum_{n \geq s+1} \sum_{i=1}^{k} \frac{(k-i+1)(k+1)^{i-1}}{i!} \frac{p_{n-i;\leq n-k}^{n-s}}{(n-i-1)!} x^n.
\end{aligned}$$

Hence,

$$F_{k,s}(x) = T_s(x) - \sum_{i=1}^{k} \frac{(k-i+1)(k+1)^{i-1}}{i!} x^i F_{k-i,s-i}(x).$$

We assume that

$$F_{k',s}(x) = \sum_{i=0}^{k'} \frac{(-1)^i (k'+1-i)^i}{i!} x^i T_{s-i}(x)$$

for any $k' \leq k$. Then

$$\begin{aligned}
&F_{k+1,s}(x)\\
&= T_s(x) - \sum_{i=1}^{k+1} \frac{(k-i+2)(k+2)^{i-1}}{i!} x^i \sum_{j=0}^{k+1-i} \frac{(-1)^j (k+2-i-j)^j}{j!} x^j T_{s-i-j}(x)\\
&= T_s(x) - \sum_{i=1}^{k+1} \frac{(-1)^i (k+2-i)^i}{i!} x^i T_{s-i}(x) \sum_{j=1}^{i} \binom{i}{j} (k+2-j)(k+2)^{j-1} (-\frac{1}{k+2-m})^j\\
&= \sum_{i=0}^{k+1} \frac{(-1)^i (k+2-i)^i}{i!} x^i T_{s-i}(x)
\end{aligned}$$

∎

Corollary 3.6. *Let $k \geq 0$ and $F_k(x,y) = \sum_{s \geq k} F_{k,s}(x) y^s$. Then $F_k(x,y)$ satisfies the recurrence relation*

$$F_k(x,y) = T(x,y) - \sum_{i=0}^{k-1} T_i(x) y^i - \sum_{i=1}^{k} \frac{(k-i+1)(k+1)^{i-1}}{i!} (xy)^i F_{k-i}(x,y).$$

Equivalently,

$$\begin{aligned}
F_k(x,y) \ &= \ T(x,y) \sum_{i=0}^{k} \frac{(-1)^i (k+1-i)^i}{i!} (xy)^i\\
&\quad - \sum_{s=0}^{k-1} \sum_{i=0}^{k-1-s} \frac{(-1)^i (k+1-i)^i}{i!} (xy)^i T_s(x) y^s.
\end{aligned}$$

Proof. Clearly, $F_0(x,y) = T(x,y)$. Identity (5) in Lemma 3.5 implies that

$$\sum_{s \geq k} F_{k,s}(x) y^s = \sum_{s \geq k} T_s(x) y^s - \sum_{s \geq k} \sum_{i=1}^{k} \frac{(k-i+1)(k+1)^{i-1}}{i!} x^i F_{k-i,s-i}(x) y^s.$$

Hence,

$$F_k(x,y) = T(x,y) - \sum_{s=0}^{k-1} T_s(x)y^s - \sum_{i=1}^{k} \frac{(k-i+1)(k+1)^{i-1}}{i!}(xy)^i F_{k-i}(x,y),$$

Furthermore, by the identity (6) in Lemma 3.5, for any $k \geq 1$, we have

$$\begin{aligned}
&F_k(x,y)\\
&= \sum_{s\geq k}\sum_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!} x^i T_{s-i}(x)y^s\\
&= \sum_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!}(xy)^i \left[T(x,y) - \sum_{s=0}^{k-1-i} T_s(x)y^s\right]\\
&= T(x,y)\sum_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!}(xy)^i - \sum_{s=0}^{k-1}\sum_{i=0}^{k-1-s} \frac{(-1)^i(k+1-i)^i}{i!}(xy)^i T_s(x)y^s.
\end{aligned}$$

■

Corollary 3.7. *Let $F(x,y,z) = \sum_{k\geq 0} F_k(x,y)z^k$. Then*

$$F(x,y,z) = \frac{T(x,y) - zT(x,yz)}{e^{xyz} - z}.$$

Proof. By Corollary 3.6, we have

$$\begin{aligned}
F(x,y,z) &= F_0(x,y) + \sum_{k\geq 1} F_k(x,y)z^k\\
&= T(x,y)\sum_{k\geq 0}\sum_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!}(xy)^i z^k\\
&\quad - \sum_{k\geq 1}\sum_{s=0}^{k-1}\sum_{i=0}^{k-1-s} \frac{(-1)^i(k+1-i)^i}{i!}(xy)^i T_s(x)y^s z^k\\
&= \frac{T(x,y) - zT(x,yz)}{e^{xyz} - z}.
\end{aligned}$$

■

Now, for a fixed k, we define a generating function by letting

$$R_k(x) = \sum_{n\geq k} \frac{p_{n;\leq n-k}}{n!} x^n.$$

Lemma 3.8. *Suppose $k \geq 0$. Let $R_k(x)$ be the generating function for $p_{n;\leq n-k}$. Then $R_0(x) = P(x)$ and $R_k(x)$ satisfies the recurrence relation*

$$R_{k+1}(x) = R_k(x) - \sum_{i=1}^{k+1} \frac{x^i}{i!} R_{k+1-i}(x)$$

for any $k \geq 1$, with initial condition $R_1(x) = (1-x)P(x) - 1$. Furthermore, we have

$$R_k(x) = P(x) \sum_{i=0}^{k} \frac{(-1)^i (k+1-i)^i}{i!} x^i - \sum_{i=0}^{k-1} \frac{(-1)^i (k-i)^i}{i!} x^i.$$

Proof. By Lemma 2.2 (i), we have

$$\begin{aligned}
R_{k+1}(x) &= \sum_{n \geq k+1} \frac{p_{n;\leq n-k-1}}{n!} x^n \\
&= \sum_{n \geq k+1} \frac{p_{n;\leq n-k} - \sum_{i=1}^{k+1} \binom{n}{i} p_{n-i;\leq n-k-1}}{n!} x^n \\
&= \sum_{n \geq k+1} \frac{p_{n;\leq n-k}}{n!} x^n - \sum_{n \geq k+1} \sum_{i=1}^{k+1} \frac{p_{n-i;\leq n-k-1}}{i!(n-i)!} x^n \\
&= R_k(x) - \sum_{i=1}^{k+1} \frac{x^i}{i!} \sum_{n \geq k+1} \frac{p_{n-i;\leq n-k-1}}{(n-i)!} x^{n-i} \\
&= R_k(x) - \sum_{i=1}^{k+1} \frac{x^i}{i!} R_{k+1-i}(x).
\end{aligned}$$

This gives the first identity.

Note that $R_1(x) = (1-x)P(x) - 1$. Assume that the identity (9) holds for any $k' \leq k$, i.e.,

$$R_{k'}(x) = P(x) \sum_{i=0}^{k'} \frac{(-1)^i (k'+1-i)^i}{i!} x^i - \sum_{i=0}^{k'-1} \frac{(-1)^i (k'-i)^i}{i!} x^i. \tag{8}$$

Then

$$\begin{aligned}
&R_{k+1}(x) \\
&= R_k(x) - \sum_{i=1}^{k+1} \frac{x^i}{i!} R_{k+1-i}(x) \\
&= P(x) \sum_{i=0}^{k} \frac{(-1)^i (k+1-i)^i}{i!} x^i - \sum_{i=0}^{k-1} \frac{(-1)^i (k-i)^i}{i!} x^i
\end{aligned}$$

$$-\sum_{i=1}^{k+1}\frac{x^i}{i!}\Big(P(x)\sum_{j=0}^{k+1-i}\frac{(-1)^j(k+2-i-j)^j}{j!}x^j-\sum_{j=0}^{k-i}\frac{(-1)^j(k+1-i-j)^j}{j!}x^j\Big).$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{k+1}\frac{x^i}{i!}P(x)\sum_{j=0}^{k+1-i}\frac{(-1)^j(k+2-i-j)^j}{j!}x^j\\
&= P(x)\sum_{i=1}^{k+1}\sum_{j=0}^{k+1-i}\frac{(-1)^j(k+2-i-j)^j}{i!j!}x^{j+i}\\
&= P(x)\sum_{i=1}^{k+1}\sum_{m=i}^{k+1}\frac{(-1)^{m-i}(k+2-m)^{m-i}}{i!(m-i)!}x^m\\
&= P(x)\sum_{m=1}^{k+1}\sum_{i=1}^{m}\frac{(-1)^{m-i}(k+2-m)^{m-i}}{i!(m-i)!}x^m\\
&= P(x)\sum_{m=1}^{k+1}\frac{x^m}{m!}\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}(k+2-m)^{m-i}\\
&= P(x)\sum_{m=1}^{k+1}\frac{x^m}{m!}((m-k-1)^m-(m-k-2)^m).
\end{aligned}$$

Similarly, we have

$$\sum_{i=1}^{k+1}\frac{x^i}{i!}\sum_{j=0}^{k-i}\frac{(-1)^j(k+1-i-j)^j}{j!}x^j=\sum_{m=1}^{k}\frac{x^m}{m!}((m-k)^m-(m-k-1)^m)$$

So,

$$\begin{aligned}
R_{k+1}(x) &= P(x)\sum_{i=0}^{k}\frac{(-1)^i(k+1-i)^i}{i!}x^i-\sum_{i=0}^{k-1}\frac{(-1)^i(k-i)^i}{i!}x^i\\
&\quad -P(x)\sum_{i=1}^{k+1}\frac{x^i}{i!}((i-k-1)^i-(i-k-2)^i)\\
&\quad +\sum_{i=1}^{k}\frac{x^i}{i!}((i-k)^i-(i-k-1)^i)\\
&= P(x)\sum_{i=0}^{k+1}\frac{(-1)^i(k+2-i)^i}{i!}x^i-\sum_{i=0}^{k}\frac{(-1)^i(k+1-i)^i}{i!}x^i.
\end{aligned}$$

Hence, the second identity holds. ■

Now, given $s \geq k \geq 0$, we define a generating function by letting

$$D_{k,s}(x)=\sum_{n\geq s+1}\frac{p_{n;n-k}^{n-s}}{(n-1)!}x^n.$$

Lemma 3.9. *Let $s \geq k \geq 0$ and $D_{k,s}(x)$ be the generating function for $p_{n;n-k}^{n-s}$. Then, when $s=k$,*

$$D_{k,k}(x) = \begin{cases} xP(x) & if \;\; k=0; \\ x[P(x)-1] & if \;\; k=1; \\ P(x)\sum\limits_{i=0}^{k-1} \frac{(-1)^i(k-i)^i}{i!}x^{i+1} - \sum\limits_{i=0}^{k-2} \frac{(-1)^i(k-1-i)^i}{i!}x^{i+1} & if \;\; k \geq 2. \end{cases}$$

When $s \geq k+1$, $D_{k,s}(x)$ satisfies the recurrence relation

$$D_{k,s}(x) = D_{k-1,s}(x) + \frac{x^{k+1}}{(k+1)!}T_{s-k-1}(x) - \sum_{i=1}^{k} \frac{x^i}{i!} D_{k-i,s-i}(x),$$

with initial conditions $D_{0,0}(x) = xP(x)$ and $D_{0,s}(x) = T_{s-1}(x)$ for any $s \geq 1$. Furthermore, for any $k \geq 0$ and $s \geq k+1$, we have

$$D_{k,s}(x) = \sum_{i=1}^{k+1} (-1)^i \frac{x^i}{i!} T_{s-i}(x) \left[(k+1-i)^i - (k+2-i)^i\right].$$

Proof. Theorem 2.7 implies that $D_{0,0}(x) = xP(x)$ and

$$D_{k,k}(x) = x\left[R_{k-1}(x) - \frac{p_{k-1;\leq 0}}{(k-1)!}x^{k-1}\right]$$

for any $k \geq 1$, where $R_k(x) = \sum\limits_{n \geq k} p_{n;\leq n-k} \frac{x^n}{n!}$. Hence, $D_{1,1}(x) = x[P(x)-1]$ since $p_{0;\leq 0} = 1$. When $k \geq 2$, $D_{k,k}(x) = xR_{k-1}(x)$ since $p_{k-1;\leq 0} = 0$. By Lemma 3.8, we have

$$D_{k,k}(x) = P(x)\sum_{i=0}^{k-1} \frac{(-1)^i(k-i)^i}{i!}x^{i+1} - \sum_{i=0}^{k-2} \frac{(-1)^i(k-1-i)^i}{i!}x^{i+1}$$

for any $k \geq 2$.

Furthermore, by Theorem 2.7, for any $k \geq 1$ and $s \geq k+1$, we have

$$\begin{aligned} \sum_{n \geq s+1} \frac{p_{n;n-k}^{n-s}}{(n-1)!}x^n &= \sum_{n \geq s+1} \frac{p_{n;n-k+1}^{n-s}}{(n-1)!}x^n + \sum_{n \geq s+1} \frac{\binom{n-1}{k+1}p_{n-k-1}^{n-s}}{(n-1)!}x^n \\ &\quad - \sum_{n \geq s+1} \sum_{i=1}^{k} \frac{\binom{n-1}{i}p_{n-i;n-k}^{n-s}}{(n-1)!}x^n. \end{aligned}$$

Hence,

$$D_{k,s}(x) = D_{k-1,s}(x) + \frac{x^{k+1}}{(k+1)!}T_{s-k-1}(x) - \sum_{i=1}^{k} \frac{x^i}{i!} D_{k-i,s-i}(x).$$

For any $s \geq k \geq 1$, by Lemma 2.5, we have $p^{n-s}_{n;n-k+1} = \sum\limits_{i=1}^{k} \binom{n-1}{i} p^{n-s}_{n-i;\leq n-k}$. Hence, $D_{k-1,s}(x) = \sum\limits_{i=1}^{k} \frac{x^i}{i!} F_{k-i,s-i}(x)$. By Lemma 3.5, for any $k \geq 0$ and $s \geq k+1$, we obtain

$$\begin{aligned} D_{k,s}(x) &= \sum_{i=1}^{k+1} \frac{x^i}{i!} F_{k+1-i,s-i}(x) \\ &= \sum_{i=1}^{k+1} \frac{x^i}{i!} \sum_{j=0}^{k+1-i} \frac{(-1)^j (k+2-i-j)^j}{j!} x^j T_{s-i-j}(x) \\ &= \sum_{i=1}^{k+1} \frac{x^i}{i!} T_{s-i}(x) \sum_{j=1}^{i} \binom{i}{j} (i-k-2)^{i-j} \\ &= \sum_{i=1}^{k+1} (-1)^i \frac{x^i}{i!} T_{s-i}(x) \left[(k+1-i)^i - (k+2-i)^i \right]. \end{aligned}$$

■

Lemma 3.10. *Let $k \geq 0$ and $D_k(x,y) = \sum\limits_{s \geq k} D_{k,s}(x) y^s$. Then $D_k(x,y)$ satisfies the recurrence relation*

$$\begin{aligned} D_k(x,y) &= D_{k-1}(x,y) - D_{k-1,k-1}(x) y^{k-1} - D_{k-1,k}(x) y^k + D_{k,k}(x) y^k \\ &\quad + \frac{(xy)^{k+1}}{(k+1)!} T(x,y) - \sum_{i=1}^{k} \frac{(xy)^i}{i!} [D_{k-i}(x,y) - D_{k-i,k-i}(x) y^{k-i}] \end{aligned}$$

with initial condition $D_0(x) = xP(x) + T(x,y)$. Furthermore, we have

$$\begin{aligned} D_k(x) &= D_{k,k}(x) y^k + \sum_{i=1}^{k+1} (-1)^i \frac{x^i}{i!} \left[(k+1-i)^i - (k+2-i)^i \right] T(x,y) \\ &\quad - \sum_{s=0}^{k-1} \sum_{i=1}^{k-s} (-1)^i \frac{x^i}{i!} \left[(k+1-i)^i - (k+2-i)^i \right] T_s(x) y^s. \end{aligned}$$

Proof. By Lemma 3.9, it is easy to check that

$$D_0(x) = xP(x) + T(x,y).$$

1 Furthermore, for any $k \geq 1$, we have

$$\begin{aligned}\sum_{s\geq k+1} D_{k,s}(x)y^s &= \sum_{s\geq k+1} D_{k-1,s}(x)y^s + \sum_{s\geq k+1} \frac{x^{k+1}}{(k+1)!}T_{s-k-1}(x)y^s \\ &\quad - \sum_{s\geq k+1}\sum_{i=1}^{k} \frac{x^i}{i!}D_{k-i,s-i}(x)y^s.\end{aligned}$$

2 Hence,

$$\begin{aligned}D_k(x,y) &= D_{k-1}(x,y) - D_{k-1,k-1}(x)y^{k-1} - D_{k-1,k}(x)y^k + D_{k,k}(x)y^k \\ &\quad + \frac{(xy)^{k+1}}{(k+1)!}T(x,y) - \sum_{i=1}^{k} \frac{(xy)^i}{i!}[D_{k-i}(x,y) - D_{k-i,k-i}(x)y^{k-i}].\end{aligned}$$

3 On the other hand,

$$\begin{aligned}D_k(x,y) &= \sum_{s\geq k} D_{k,s}(x)y^s \\ &= D_{k,k}(x)y^k + \sum_{s\geq k+1} D_{k,s}(x)y^s \\ &= D_{k,k}(x)y^k + \sum_{s\geq k+1}\sum_{i=1}^{k+1}(-1)^i\frac{x^i}{i!}T_{s-i}(x)\left[(k+1-i)^i - (k+2-i)^i\right]y^s \\ &= D_{k,k}(x)y^k + \sum_{i=1}^{k+1}(-1)^i\frac{x^i}{i!}\left[(k+1-i)^i - (k+2-i)^i\right][T(x,y) - \sum_{s=0}^{k-i} T_s(x)y^s] \\ &= D_{k,k}(x)y^k + \sum_{i=1}^{k+1}(-1)^i\frac{x^i}{i!}\left[(k+1-i)^i - (k+2-i)^i\right]T(x,y) \\ &\quad - \sum_{s=0}^{k-1}\sum_{i=1}^{k-s}(-1)^i\frac{x^i}{i!}\left[(k+1-i)^i - (k+2-i)^i\right]T_s(x)y^s.\end{aligned}$$

4 ∎

5 **Theorem 3.11.** *Let* $D(x,y,z) = \sum_{k\geq 0} D_k(x,y)z^k$. *Then*

$$D(x,y,z) = \frac{xe^{xyz}(P(x) - yz)}{e^{xyz} - yz} + \frac{e^{xyz} - 1}{z}F(x,y,z).$$

Proof. For any $s \geq k \geq 1$, by Lemma 2.7, we have

$$p_{n;n-k+1}^{n-s} = \sum_{i=1}^{k}\binom{n-1}{i}p_{n-i;\leq n-k}^{n-s}.$$

Hence, $D_{k-1,s}(x) = \sum\limits_{i=1}^{k} \frac{x^i}{i!} F_{k-i,s-i}(x)$. So,

$$\begin{aligned} D_k(x,y) &= \sum_{s\geq k} D_{k,s}(x) y^s \\ &= D_{k,k}(x)y^k + \sum_{s\geq k+1} D_{k,s}(x)y^s \\ &= D_{k,k}(x)y^k + \sum_{s\geq k+1} \sum_{i=1}^{k+1} \frac{x^i}{i!} F_{k+1-i,s-i}(x) y^s \\ &= D_{k,k}(x)y^k + \sum_{i=1}^{k+1} \frac{(xy)^i}{i!} F_{k+1-i}(x,y). \end{aligned}$$

Therefore,

$$\begin{aligned} D(x,y,z) &= \sum_{k\geq 0} D_{k,k}(x)(yz)^k + \sum_{k\geq 0} \sum_{i=1}^{k+1} \frac{(xy)^i}{i!} F_{k+1-i}(x,y) z^k \\ &= \sum_{k\geq 0} D_{k,k}(x)(yz)^k + \sum_{k\geq 0} \sum_{i=0}^{k} \frac{(xy)^{i+1}}{(i+1)!} F_{k-i}(x,y) z^k \\ &= xP(x) - xyz + xyzR(x,yz) + \frac{e^{xyz}-1}{z} F(x,y,z) \\ &= \frac{xe^{xyz}(P(x)-yz)}{e^{xyz}-yz} + \frac{e^{xyz}-1}{z} F(x,y,z). \end{aligned}$$

■

4. The asymptotic behavior for the sequence $p^l_{n;n-k}$

In this section, we consider the asymptotic behavior for the sequences $p^l_{n;n-k}$. Given $l \geq 1$, let $\tau_l = \lim\limits_{n\to\infty} \frac{p^l_n}{(n+1)^{n-2}}$. Clearly, $\tau_1 = 2$. We define a generating function by letting $\tau(x) = \sum\limits_{l\geq 1} \tau_l x^l$.

Lemma 4.1. *The sequence τ_l satisfies the recurrence relation*

$$\tau_l - \tau_{l+1} = \frac{l^{l-2}}{(l-1)!} e^{-l}$$

for any $l \geq 1$, with the initial condition $\tau_1 = 2$. Let $\tau(x)$ be the generating function for τ_l. Then

$$\tau(x) = \frac{\frac{x^2}{e} P(\frac{x}{e}) - 2x}{x-1}.$$

Proof. For a fixed $l \geq 1$, we have $\lim\limits_{n\to\infty} \frac{\binom{n-1}{l-1}(n-l+1)^{n-l-1}}{(n+1)^{n-2}} = \frac{e^{-l}}{(l-1)!}$. So, by Lemma 2.1, we have $\tau_l - \tau_{l+1} = \frac{l^{l-2}}{(l-1)!}e^{-l}$. Hence, $\sum\limits_{l\geq 1} \tau_l x^l - \sum\limits_{l\geq 1} \tau_{l+1}x^l = \sum\limits_{l\geq 1} \frac{l^{l-2}}{(l-1)!}e^{-l}x^l$. $\tau(x)$ satisfies the equation $(x-1)\tau(x) + 2x = \frac{x^2}{e}P(\frac{x}{e})$. Equivalently, $\tau(x) = \frac{\frac{x^2}{e}P(\frac{x}{e})-2x}{x-1}$. ∎

For fixed l and k, let $\rho_{l,k} = \lim\limits_{n\to\infty} \frac{p^l_{n;\leq n-k}}{(n+1)^{n-2}}$ and define a generating function by letting $\rho_l(x) = \sum\limits_{k\geq 0} \rho_{l,k}x^k$.

Lemma 4.2. *Let $l \geq 1$, then the sequence $\rho_{l,k}$ satisfies the recurrence relation*

$$\rho_{l,k} = \rho_{l,k-1} - \sum_{i=1}^{k} \frac{e^{-i}}{i!}\rho_{l,k-i}.$$

for any $k \geq 1$. Furthermore, we have $\rho_{l,k} = \rho_{l,0}\sum\limits_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!}e^{-i}$. Let $\rho_l(x)$ be the generating function for $\rho_{l,k}$. Then $\rho_l(x) = \rho_{l,0}(e^{\frac{x}{e}} - x)^{-1}$. Furthermore, let $\rho(x,y) = \sum\limits_{l\geq 1} \rho_l(x)y^l$. Then we have

$$\rho(x,y) = \frac{\frac{y^2}{e}P(\frac{y}{e}) - 2y}{(y-1)(e^{\frac{x}{e}} - x)}$$

Proof. For a fixed $i \geq 0$, it is easy to see that

$$\lim_{n\to\infty} \binom{n-1}{i} \frac{(n-i+1)^{n-i-2}}{(n+1)^{n-2}} = \frac{e^{-i}}{i!}.$$

Hence, by the identity (2) in Lemma 2.5, we have

$$\rho_{l,k} = \rho_{l,k-1} - \sum_{i=1}^{k} \frac{e^{-i}}{i!}\rho_{l,k-i}.$$

Now, assume that $\rho_{l,k'} = \rho_{l,0}\sum\limits_{i=0}^{k'} \frac{(-1)^i(k'+1-i)^i}{i!}e^{-i}$ for any $k' \leq k$. Then

$$\begin{aligned}
\rho_{l,k+1} &= \rho_{l,k} - \sum_{i=1}^{k+1} \frac{e^{-i}}{i!}\rho_{l,k+1-i} \\
&= \rho_{l,0}\sum_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!}e^{-i} - \sum_{i=1}^{k+1} \frac{e^{-i}}{i!}\rho_{l,0}\sum_{j=0}^{k+1-i} \frac{(-1)^j(k+2-i-j)^j}{j!}e^{-j} \\
&= \rho_{l,0}\sum_{i=0}^{k+1} \frac{(-1)^i(k+2-i)^i}{i!}e^{-i}
\end{aligned}$$

Hence, $\rho_l(x) = \sum\limits_{k\geq 0} \rho_{l,0} \sum\limits_{i=0}^{k} \frac{(-1)^i(k+1-i)^i}{i!} e^{-i}x^k = \rho_{l,0}(e^{\frac{x}{e}} - x)^{-1}$. Since $\rho_{l,0} = \tau_l$, by Lemma , we have

$$\begin{aligned}\rho(x,y) &= \sum_{l\geq 1} \tau_l(e^{\frac{x}{e}} - x)^{-1}y^l \\ &= \frac{\frac{y^2}{e}P(\frac{y}{e}) - 2y}{(y-1)(e^{\frac{x}{e}} - x)}\end{aligned}$$

∎

For a fixed k, let $\lambda_{l,k} = \lim\limits_{n\to\infty} \frac{p^l_{n;n-k}}{(n+1)^{n-2}}$ and define a generating function by letting $\lambda_l(x) = \sum\limits_{k\geq 0} \lambda_{l,k}x^k$.

Theorem 4.3. *Let $\lambda_l(x)$ be the generating function for $\lambda_{l,k}$. Then $\lambda_l(x) = \frac{e^{\frac{x}{e}}-1}{x}\rho_l(x)$. Let $\lambda(x,y) = \sum\limits_{l\geq 1} \lambda_l(x)y^l$. Then $\lambda(x,y) = \dfrac{e^{\frac{x}{e}} - 1}{x(e^{\frac{x}{e}} - x)} \dfrac{\frac{y^2}{e}P(\frac{y}{e}) - 2y}{y-1}$.*

Proof. Lemma 2.5 implies that $\lambda_{l,k} = \sum\limits_{i=1}^{k+1} \frac{e^{-i}}{i!}\rho_{l,k+1-i}$. So,

$$\lambda_l(x) = \sum_{k\geq 0}\sum_{i=1}^{k+1} \frac{e^{-i}}{i!}\rho_{l,k+1-i}x^k = \frac{e^{\frac{x}{e}} - 1}{x}\rho_l(x).$$

Hence,

$$\lambda(x,y) = \frac{e^{\frac{x}{e}} - 1}{x}\rho(x,y) = \frac{e^{\frac{x}{e}} - 1}{x(e^{\frac{x}{e}} - x)} \frac{\frac{y^2}{e}P(\frac{y}{e}) - 2y}{y-1}.$$

∎

5. Appendix

For $n \leq 7$ we give the number of parking functions in the set $\mathcal{P}^l_{n;\leq s}$ in Table 1, as obtained by a computer search.

Table.1. The values of $p^l_{n;\leq s}$ for $1 \leq n \leq 7$

	$l=1$	2	3	4	5	6	7	8	$p_{n;\leq s}$
$(n,s)=(1,1)$	1	0							1
$(2,1)$	1	0							1
$(2,2)$	2	1	0						3
$(3,1)$	1	0							1
$(3,2)$	4	3	0						7
$(3,3)$	8	5	3	0					16
$(4,1)$	1	0							1
$(4,2)$	8	7	0						15
$(4,3)$	26	19	16	0					61
$(4,4)$	50	34	25	16	0				125
$(5,1)$	1	0							1
$(5,2)$	16	15	0						31
$(5,3)$	80	65	61	0					206
$(5,4)$	232	171	143	125	0				671
$(5,5)$	432	307	243	189	125	0			1296
$(6,1)$	1	0							1
$(6,2)$	32	31	0						63
$(6,3)$	242	211	206	0					659
$(6,4)$	982	776	701	671	0				3130
$(6,5)$	2642	1971	1666	1456	1296	0			9031
$(6,6)$	4802	3506	2881	2401	1921	1296	0		16807
$(7,1)$	1	0							1
$(7,2)$	64	63	0						127
$(7,3)$	728	665	659	0					2052
$(7,4)$	4020	3361	3175	3130	0				13686
$(7,5)$	14392	11262	10026	9351	9031	0			54062
$(7,6)$	36724	27693	23667	20922	18682	16807	0		144495
$(7,7)$	65536	48729	40953	35328	30208	24583	16807	0	262144

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