will be $x^2 + y^2$. Entering **mesh(Z)** *will then create the mesh graph over the region $-3 \le x \le 3$, $-2 \le y \le 2$.*

Enter now

$$Z = X .* X + Y .* Y; \tag{9}$$

$$\text{mesh(Z)} \tag{10}$$

to see this graph.

M10. Using the up arrow, modify Eq. (8) to make both the x-increment and the y-increment 0.2. After pressing the Enter key, use the up arrow to get Eq. (9) and press the Enter key to form the larger matrix Z for these new grid points. Then create the mesh graph using Eq. (10).

M11. Modify Eq. (9) and create the mesh graph for $z = x^2 - y^2$.

M12. Change Eq. (8) so that the region will be $-3 \le x \le 3$ and $-3 \le y \le 3$ still with 0.2 for increments. Form the mesh graph for $z = 9 - x^2 - y^2$.

M13. Mesh graphs for cylinders are especially nice. Draw the mesh graphs for
a. $z = x^2$
b. $z = y^2$.

M14. Change the mesh domain to $-4\pi \le x \le 4\pi$, $-3 \le y \le 3$ with x-increment 0.2 and y-increment 6. Recall that π can be entered as **pi.** Draw the mesh graphs for
a. the cylinder $z = \sin(x)$,
b. the cylinder $z = x\sin(x)$, remembering to use **.***, and
c. the function $z = y\sin(x)$, which is not a cylinder but is pretty.

1.4 SOLVING SYSTEMS OF LINEAR EQUATIONS

As we have indicated, solving a system of linear equations is a fundamental problem of linear algebra. Many of the computational exercises in this text involve solving such linear systems. This section presents an algorithm for finding all solutions of any linear system.

The solution set of any system of equations is the intersection of the solution sets of the individual equations. That is, any solution of a system must be a solution of each equation in the system; and conversely, any solution of every equation in the system is considered to be a solution of the system. Bearing this in mind, we start with an intuitive discussion of the geometry of linear systems; a more detailed study of this geometry appears in Section 2.5.

The Geometry of Linear Systems

Frequently, students are under the impression that a linear system containing the same number of equations as unknowns always has a unique solution, whereas a system having more equations than unknowns never has a solution. The geometric interpretation of the problem shows that these statements are not true.

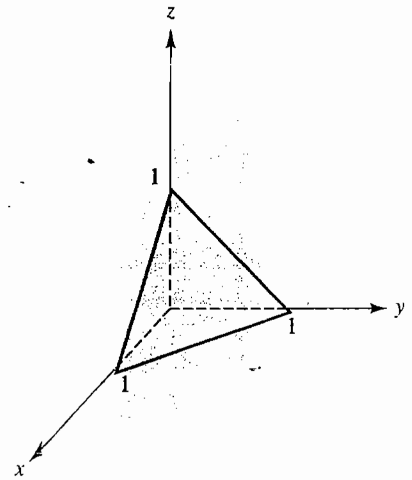

FIGURE 1.31
The plane $x + y + z = 1$.

We know that a single linear equation in two unknowns has a line in the plane as its solution set. Similarly, a single linear equation in three unknowns has a plane in space as its solution set. The solution set of $x + y + z = 1$ is the plane sketched in Figure 1.31. This geometric analysis can be extended to an equation that has more than three variables, but it is difficult for us to represent the solution set of such an equation graphically.

Two lines in the plane usually intersect at a single point; here the word *usually* means that, if the lines are selected in some random way, the chance

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HISTORICAL NOTE SYSTEMS OF LINEAR EQUATIONS are found in ancient Babylonian and Chinese texts dating back well over 2000 years. The problems are generally stated in real-life terms, but it is clear that they are artificial and designed simply to train students in mathematical procedures. As an example of a Babylonian problem, consider the following, which has been slightly modified from the original found on a clay tablet from about 300 B.C.: There are two fields whose total area is 1800 square yards. One produces grain at a rate of $\frac{2}{3}$ bushel per square yard, the other at a rate of $\frac{1}{2}$ bushel per square yard. The total yield of the two fields is 1100 bushels. What is the size of each field? This problem leads to the system

$$\begin{aligned} x + y &= 1800 \\ \frac{2}{3}x + \frac{1}{2}y &= 1100. \end{aligned}$$

A typical Chinese problem, taken from the Han dynasty text *Nine Chapters of the Mathematical Art* (about 200 B.C.), reads as follows: There are three classes of corn, of which three bundles of the first class, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class? The system of equations here is

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26. \end{aligned}$$

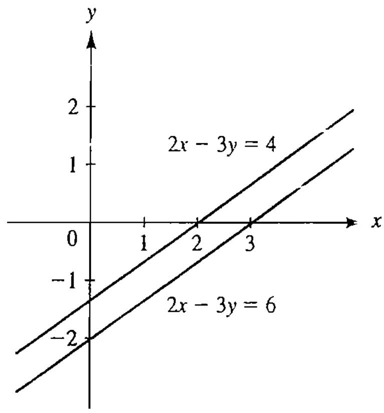

FIGURE 1.32
$2x - 3y = 4$ is parallel to
$2x - 3y = 6$.

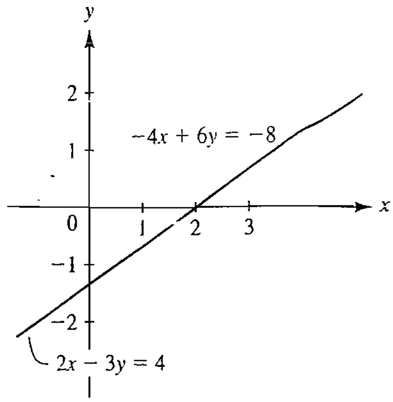

FIGURE 1.33
$2x - 3y = 4$ and $-4x + 6y = -8$
are the same line.

that they either are parallel (have empty intersection) or coincide (have an infinite number of points in their intersection) is very small. Thus we see that a system of two randomly selected equations in two unknowns can be expected to have a unique solution. However, it is *possible* for the system to have no solutions or an infinite number of solutions. For example, the equations

$$2x - 3y = 4$$
$$2x - 3y = 6$$

correspond to distinct parallel lines, as shown in Figure 1.32, and the system consisting of these equations has no solutions. Moreover, the equations

$$2x - 3y = 4$$
$$-4x + 6y = -8$$

correspond to the same line, as shown in Figure 1.33. All points on this line are solutions of this system of two equations. And because it is possible to have any number of lines in the plane—say, fifty lines—pass through a single point, it is possible for a system of fifty equations in only two unknowns to have a unique solution.

Similar illustrations can be made in space, where a linear equation has as its solution set a plane. Three randomly chosen planes can be expected to have a unique point in common. Two of them can be expected to intersect in a line (see Figure 1.34), which in turn can be expected to meet the third plane at a single point. However, it is possible for three planes to have no point in common, giving rise to a linear system with no solutions. It is also possible for all three planes to contain a common line, in which case the corresponding linear system will have an infinite number of solutions.

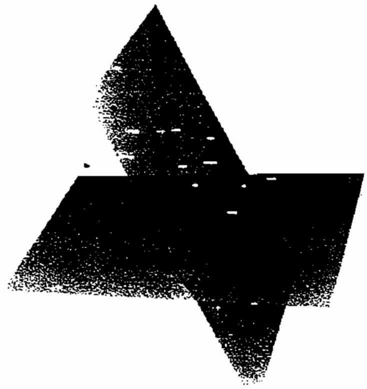

FIGURE 1.34
Two planes intersecting in a line.

Elementary Row Operations

We now describe operations that can be used to modify the equations of a linear system to obtain a system having the same solutions, but whose solutions are obvious. The most general type of linear system can have m equations in n unknowns. Such a system can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

System (1) is completely determined by its $m \times n$ **coefficient matrix** $A = [a_{ij}]$ and by the column vector $\mathbf{b}$ with ith component b_i. The system can be written as the single matrix equation

$$A\mathbf{x} = \mathbf{b}, \tag{2}$$

where $\mathbf{x}$ is the column vector with ith component x_i. Any column vector $\mathbf{s}$ such that $A\mathbf{s} = \mathbf{b}$ is a **solution** of system (1).

The **augmented matrix** or **partitioned matrix**

$$\left[\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{3}$$

is a shorthand summary of system (1). The coefficient matrix has been *augmented* by the column vector of constants. We denote matrix (3) by $[A \mid \mathbf{b}]$.

We shall see how to determine all solutions of system (1) by manipulating augmented matrix (3) using *elementary row operations*. The elementary row operations correspond to the following familiar operations with equations of system (1):

R1 Interchange two equations in system (1).

R2 Multiply an equation in system (1) by a nonzero constant.

R3 Replace an equation in system (1) with the sum of itself and a multiple of a different equation of the system.

It is clear that operations R1 and R2 do not change the solution sets of the equations they affect. Therefore, they do not change the intersection of the solution sets of the equations; that is, the solution set of the system is unchanged. The fact that R2 does not change the solution set and the familiar algebraic principle, "Equals added to equals yield equals," show that any solution of both the ith and jth equations is also a solution of a new jth equation obtained by adding s times the ith equation to the jth equation. Thus operation R3 yields a system having all the solutions of the original one. Because the original system can be recovered from the new one by multiplying

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HISTORICAL NOTE A MATRIX-REDUCTION METHOD of solving a system of linear equations occurs in the ancient Chinese work, *Nine Chapters of the Mathematical Art*. The author presents the following solution to the system

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26. \end{aligned}$$

The diagram of the coefficients is to be set up on a "counting board":

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{matrix}$$

The author then instructs the reader to multiply the middle column by 3 and subsequently to subtract the right column "as many times as possible"; the same is to be done to the left column. The new diagrams are then

$$\begin{matrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{matrix}$$

The next instruction is to multiply the left column by 5 and then to subtract the middle column as many times as possible. This gives

$$\begin{matrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{matrix}$$

The system has thus been reduced to the system $3x + 2y + z = 39$, $5y + z = 24$, $36z = 99$, from which the complete solution is easily found.

the ith equation by $-s$ and adding it to the new jth equation (an R3 operation), we see that the original system has all the solutions of the new one. Hence R3, too, does not alter the solution set of system (1).

These procedures applied to system (1) correspond to **elementary row operations** applied to augmented matrix (3). We list these in a box together with a suggestive notation for each.

Elementary Row Operations	Notations
(*Row interchange*) Interchange the ith and jth row vectors in a matrix.	$R_i \leftrightarrow R_j$
(*Row scaling*) Multiply the ith row vector in a matrix by a nonzero scalar s.	$R_i \to sR_i$
(*Row addition*) Add to the ith row vector of a matrix s times the jth row vector.	$R_i \to R_i + sR_j$

If a matrix B can be obtained from a matrix A by means of a sequence of elementary row operations, then A is **row equivalent** to B. Each elementary row operation can be undone by another of the same type. A row-addition operation $R_i \to R_i + sR_j$ can be undone by $R_i \to R_i - sR_j$. Row scaling, $R_i \to sR_i$ for $s \neq 0$, can be undone using $R_i \to (1/s)R_i$, while a row-interchange operation undoes itself. Thus, if B is row equivalent to A, then A is row equivalent to B; we can simply speak of *row-equivalent matrices* A and B, which we denote by $A \sim B$. (See Exercise 55 in this regard.) We have just seen that the operations on a linear system $A\mathbf{x} = \mathbf{b}$ corresponding to these elementary row operations on the augmented matrix $[A \mid \mathbf{b}]$ do not change the solution set of the system. This gives us at once the following theorem, which is the foundation for the algorithm we will present for solving linear systems.

THEOREM 1.6 Invariance of Solution Sets Under Row Equivalence

If $[A \mid \mathbf{b}]$ and $[H \mid \mathbf{c}]$ are row-equivalent augmented matrices, then the linear systems $A\mathbf{x} = \mathbf{b}$ and $H\mathbf{x} = \mathbf{c}$ have the same solution sets.

Row-Echelon Form

We will solve a linear system $A\mathbf{x} = \mathbf{b}$ by row-reducing the augmented matrix $[A \mid \mathbf{b}]$ to an augmented matrix $[H \mid \mathbf{c}]$, where H is a matrix in *row-echelon form* (which we now define).

DEFINITION 1.12 Row-Echelon Form, Pivot

A matrix is in **row-echelon form** if it satisfies two conditions:

1. All rows containing only zeros appear below rows with nonzero entries.
2. The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row.

For such a matrix, the first nonzero entry in a row is the **pivot** for that row.

EXAMPLE 1 Determine which of the matrices

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in row-echelon form.

SOLUTION Matrix A is not in row-echelon form, because the second row (consisting of all zero entries) is not below the third row (which has a nonzero entry).

Matrix B is not in row-echelon form, because the first nonzero entry in the second row does not appear in a column to the right of the first nonzero entry in the first row.

Matrix C is in row-echelon form, because both conditions of Definition 1.12 are satisfied. The pivots are -1 and 3.

Matrix D satisfies both conditions as well, and is in row-echelon form. The pivots are the entries 1. ■

Solutions of $H\mathbf{x} = \mathbf{c}$

We illustrate by examples that, if a linear system $H\mathbf{x} = \mathbf{c}$ has coefficient matrix H in row-echelon form, it is easy to determine all solutions of the system. We color the pivots in H in these examples.

EXAMPLE 2 Find all solutions of $H\mathbf{x} = \mathbf{c}$, where

$$[H \mid \mathbf{c}] = \left[\begin{array}{ccc|c} -5 & -1 & 3 & 3 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 2 & -4 \end{array}\right].$$

SOLUTION The equations corresponding to this augmented matrix are

$$\begin{aligned} -5x_1 - x_2 + 3x_3 &= 3 \\ 3x_2 + 5x_3 &= 8 \\ 2x_3 &= -4. \end{aligned}$$

From the last equation, we obtain $x_3 = -2$. Substituting into the second equation, we have

$$3x_2 + 5(-2) = 8, \qquad 3x_2 = 18, \qquad x_2 = 6.$$

Finally, we substitute these values for x_2 and x_3 into the top equation, obtaining

$$-5x_1 - 6 + 3(-2) = 3, \qquad -5x_1 = 15, \qquad x_1 = -3.$$

Thus the only solution is

$$\mathbf{x} = \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix},$$

or equivalently, $x_1 = -3$, $x_2 = 6$, $x_3 = -2$. ▪

The procedure for finding the solution of $H\mathbf{x} = \mathbf{c}$ illustrated in Example 2 is called **back substitution**, because the values of the variables are found in backward order, starting with the variable with the largest subscript.

By multiplying each nonzero row in $[H \mid \mathbf{b}]$ by the reciprocal of its pivot, we can assume that each pivot in H is 1. We assume that this is the case in the next two examples.

EXAMPLE 3 Use back substitution to find all solutions of $H\mathbf{x} = \mathbf{c}$, where

$$[H \mid \mathbf{c}] = \left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right].$$

SOLUTION The equation corresponding to the last row of this augmented matrix is

$$0x_1 + 0x_2 + 0x_3 = -1.$$

This equation has no solutions, because the left side is 0 for any values of the variables and the right side is -1. ▪

DEFINITION 1.13 Consistent Linear System

A linear system having no solutions is **inconsistent**. If it has one or more solutions, the linear system is said to be **consistent**.

Now we illustrate a many-solutions case.

EXAMPLE 4 Use back substitution to find all solutions of $H\mathbf{x} = \mathbf{c}$, where

$$[H \mid \mathbf{c}] = \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

SOLUTION The linear system corresponding to this augmented matrix is

$$\begin{aligned} x_1 - 3x_2 \quad + 5x_4 \quad &= 4 \\ x_3 + 2x_4 \quad &= -7 \\ x_5 &= 1. \end{aligned}$$

We solve each equation for the variable corresponding to the colored pivot in the matrix. Thus we obtain

$$\begin{aligned} x_1 &= 3x_2 - 5x_4 + 4 \\ x_3 &= -2x_4 - 7 \\ x_5 &= 1. \end{aligned} \tag{4}$$

Notice that x_2 and x_4 correspond to columns of H containing no pivot. We can assign any value r we please to x_2 and any value s to x_4, and we can then use system (4) to determine corresponding values for x_1, x_3, and x_5. Thus the system has an infinite number of solutions. We describe all solutions by the vector equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 5s + 4 \\ r \\ -2s - 7 \\ s \\ 1 \end{bmatrix} \quad \text{for any scalars } r \text{ and } s. \tag{5}$$

We call x_2 and x_4 **free variables**, and we refer to Eq. (5) as the **general solution** of the system. We obtain **particular solutions** by setting r and s equal to specific values. For example, we obtain

$$\begin{bmatrix} 2 \\ 1 \\ -9 \\ 1 \\ 1 \end{bmatrix} \text{ for } r = s = 1 \quad \text{and} \quad \begin{bmatrix} 25 \\ 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} \text{ for } r = 2,\ s = -3.$$ ■

Gauss Reduction of $A\mathbf{x} = \mathbf{b}$ to $H\mathbf{x} = \mathbf{c}$

We now show how to reduce an augmented matrix $[A \mid \mathbf{b}]$ to $[H \mid \mathbf{c}]$, where H is in row-echelon form, using a sequence of elementary row operations. Examples 2 through 4 illustrated how to use back substitution afterward to find solutions of the system $H\mathbf{x} = \mathbf{c}$, which are the same as the solutions of $A\mathbf{x} = \mathbf{b}$, by Theorem 1.6. This procedure for solving $A\mathbf{x} = \mathbf{b}$ is known as **Gauss**

reduction with back substitution. In the box below, we give an outline for reducing a matrix A to row-echelon form.

Reducing a Matrix A to Row-Echelon Form H

1. If the first column of A contains only zero entries, cross it off mentally. Continue in this fashion until the left column of the remaining matrix has a nonzero entry or until the columns are exhausted.
2. Use row interchange, if necessary, to obtain a nonzero entry (pivot) p in the top row of the first column of the remaining matrix. For each row below that has a nonzero entry r in the first column, add $-r/p$ times the top row to that row to create a zero in the first column. In this fashion, create zeros below p in the entire first column of the remaining matrix.
3. Mentally cross off this first column and the first row of the matrix, to obtain a smaller matrix. (See the shaded portion of the third matrix in the solution of Example 5.) Go back to step 1, and repeat the process with this smaller matrix until either no rows or no columns remain.

EXAMPLE 5 Reduce the matrix

$$\begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

to row-echelon form, making all pivots 1.

SOLUTION We follow the boxed outline and color the pivots of 1. Remember that the symbol $\sim$ denotes row-equivalent matrices.

$$\begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Multiply the first row by $\frac{1}{2}$, to produce a pivot of 1 in the next matrix.

$$R_1 \to \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Add -2 times row 1 to row 2, and then add -4 times row 1 to row 3, to obtain the next matrix.

$$R_2 \to R_2 - 2R_1; \quad R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Cross off the first shaded column of zeros (mentally), to obtain the next shaded matrix.

$$\sim\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Add row 2 to rows 3 and 4, to obtain the final matrix.

$$R_3 \to R_3 + 1R_2; \quad R_4 \to R_4 + 1R_2$$

$$\sim\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This last matrix is row-echelon form, with both pivots equal to 1. ■

To solve a linear system $A\mathbf{x} = \mathbf{b}$, we form the augmented matrix $[A \mid \mathbf{b}]$ and row-reduce it to $[H \mid \mathbf{c}]$, where H is in row-echelon form. We can follow the steps outlined in the box preceding Example 5 for row-reducing A to H. Of course, we always perform the elementary row operations on the full augmented matrix, including the entries in the column to the right of the partition.

EXAMPLE 6 Solve the linear system

$$\begin{aligned} x_2 - 3x_3 &= -5 \\ 2x_1 + 3x_2 - x_3 &= 7 \\ 4x_1 + 5x_2 - 2x_3 &= 10. \end{aligned}$$

SOLUTION We reduce the corresponding augmented matrix, using elementary row operations. Pivots are colored.

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 0 & -4 \end{array}\right] \qquad R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 0 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right]. \qquad R_3 \to R_3 + 1R_2$$

HISTORICAL NOTE THE GAUSS SOLUTION METHOD is so named because Gauss described it in a paper detailing the computations he made to determine the orbit of the asteroid Pallas. The parameters of the orbit had to be determined by observations of the asteroid over a 6-year period from 1803 to 1809. These led to six linear equations in six unknowns with quite complicated coefficients. Gauss showed how to solve these equations by systematically replacing them with a new system in which only the first equation had all six unknowns, the second equation included five unknowns, the third equation only four, and so on, until the sixth equation had but one. This last equation could, of course, be easily solved; the remaining unknowns were then found by back substitution.

From the last augmented matrix, we could proceed to write the corresponding equations (as in Example 2) and to solve in succession for x_3, x_2, and x_1 by back substitution. However, it makes sense to keep using our shorthand, without writing out variables, and to do our back substitution in terms of augmented matrices. Starting with the final augmented matrix in the preceding set, we obtain

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right] \sim \left[\begin{array}{rrr|r} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{c} R_3 \to -\tfrac{1}{3}R_3 \\ \text{(This shows that } x_3 = 3.) \end{array}$$

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 2 & 3 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{c} R_1 \to R_1 + 1R_3; \quad R_2 \to R_2 + 3R_3 \\ \text{(This shows that } x_2 = 4.) \end{array}$$

$$\left[\begin{array}{rrr|r} 2 & 3 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]. \qquad \begin{array}{c} R_1 \to R_1 - 3R_2 \\ \text{(This shows that } x_1 = -1.) \end{array}$$

We have found the solution: $\mathbf{x} = [x_1, x_2, x_3] = [-1, 4, 3]$. ■

In Example 6, we had to write down as many matrices to execute the back substitution as we wrote to reduce the original augmented matrix to row-echelon form. We can avoid this by creating zeros *above* as well as below each pivot as we reduce the matrix to row-echelon form. This is known as the **Gauss–Jordan method.** We show in Chapter 10 that, for a large system, it takes about 50% more time for a computer to use the Gauss–Jordan method than to use the Gauss method with back substitution illustrated in Example 6; there are actually about 50% more arithmetic operations involved. Creating the zeros above the pivots requires less computation if we do it *after* the matrix is reduced to row-echelon form than if we do it as we go along. However, when one is working with pencil and paper, fixing up a whole column in a single step avoids writing so many matrices. Our next example illustrates the Gauss–Jordan procedure.

EXAMPLE 7 Determine whether the vector $\mathbf{b} = [1, -7, -4]$ is in the span of the vectors $\mathbf{v} = [2, 1, 1]$ and $\mathbf{w} = [1, 3, 2]$.

SOLUTION We know that $\mathbf{b}$ is in $\text{sp}(\mathbf{v}, \mathbf{w})$ if and only if $\mathbf{b} = x_1\mathbf{v} + x_2\mathbf{w}$ for some scalars x_1 and x_2. This vector equation is equivalent to the linear system

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -4 \end{bmatrix}.$$

Reducing the appropriate augmented matrix, we obtain

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 3 & -7 \\ 1 & 2 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 2 & 1 & 1 \\ 1 & 2 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & -5 & 15 \\ 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right].$$

$R_1 \leftrightarrow R_2$ (to avoid fractions); $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 1R_1$; $R_2 \to \frac{1}{5}R_2$, $R_1 \to R_1 - 3R_2$, $R_3 \to R_3 + 1R_2$

The left side of the final augmented matrix is in *reduced row-echelon form.* From the solution $x_1 = 2$ and $x_2 = -3$, we see that $\mathbf{b} = 2\mathbf{v} - 3\mathbf{w}$, which is indeed in sp $(\mathbf{v}, \mathbf{w})$. ■

The linear system $A\mathbf{x} = \mathbf{b}$ displayed in Eq. (1) can be written in the form

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}.$$

This equation expresses a typical column vector $\mathbf{b}$ in $\mathbb{R}^m$ as a linear combination of the column vectors of the matrix A if and only if scalars $x_1, x_2, \ldots, x_n$ can be found to satisfy that equation. Example 7 illustrates this. We phrase this result as follows:

> Let A be an $m \times n$ matrix. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ in $\mathbb{R}^m$ is in the span of the column vectors of A.

A matrix in row-echelon form with all pivots equal to 1 and with zeros above as well as below each pivot is said to be in **reduced row-echelon form.** Thus the Gauss–Jordan method consists of using elementary row operations on an augmented matrix $[A \mid \mathbf{b}]$ to bring the coefficient matrix A into reduced

HISTORICAL NOTE THE JORDAN HALF OF THE GAUSS–JORDAN METHOD is essentially a systematic technique of back substitution. In this form, it was first described by Wilhelm Jordan (1842–1899), a German professor of geodesy, in the third (1888) edition of his *Handbook of Geodesy.* Although Jordan's arrangement of his calculations is different from the one presented here, partly because he was always applying the method to the symmetric system of equations arising out of a least-squares application in geodesy (see Section 6.5), Jordan's method uses the same arithmetic and arrives at the same answers for the unknowns.

Wilhelm Jordan was prominent in his field in the late nineteenth century, being involved in several geodetic surveys in Germany and in the first major survey of the Libyan desert. He was the founding editor of the German geodesy journal and was widely praised as a teacher of his subject. His interest in finding a systematic method of solving large systems of linear equations stems from their frequent appearance in problems of triangulation.

row-echelon form. It can be shown that the reduced row-echelon form of a matrix A is unique. (See Section 2.3, Exercise 33.)

The examples we have given illustrate the three possibilities for solutions of a linear system—namely, no solutions (inconsistent system), a unique solution, or an infinite number of solutions. We state this formally in a theorem and prove it.

THEOREM 1.7 Solutions of $A\mathbf{x} = \mathbf{b}$

Let $A\mathbf{x} = \mathbf{b}$ be a linear system, and let $[A \mid \mathbf{b}] \sim [H \mid \mathbf{c}]$, where H is in row-echelon form.

1. The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if the augmented matrix $[H \mid \mathbf{c}]$ has a row with all entries 0 to the left of the partition and a nonzero entry to the right of the partition.
2. If $A\mathbf{x} = \mathbf{b}$ is consistent and every column of H contains a pivot, the system has a unique solution.
3. If $A\mathbf{x} = \mathbf{b}$ is consistent and some column of H has no pivot, the system has infinitely many solutions, with as many free variables as there are pivot-free columns in H.

PROOF If $[H \mid \mathbf{c}]$ has an ith row with all entries 0 to the left of the partition and a nonzero entry c_i to the right of the partition, the corresponding ith equation in the system $H\mathbf{x} = \mathbf{c}$ is $0x_1 + 0x_2 + \cdots + 0x_n = c_i$, which has no solutions; therefore, the system $A\mathbf{x} = \mathbf{b}$ has no solutions, by Theorem 1.6. The next paragraph shows that, if H contains no such row, we can find a solution to the system. Thus the system is inconsistent if and only if H contains such a row.

Assume now that $[H \mid \mathbf{c}]$ has no row with all entries 0 to the left of the partition and a nonzero entry to the right. If the ith row of $[H \mid \mathbf{c}]$ is a zero row vector, the corresponding equation $0x_1 + 0x_2 + \cdots + 0x_n = 0$ is satisfied for all values of the variables x_j, and thus it can be deleted from the system $H\mathbf{x} = \mathbf{c}$. Assume that this has been done wherever possible, so that $[H \mid \mathbf{c}]$ has no zero row vectors. For each j such that the jth column has no pivot, we can set x_j equal to any value we please (as in Example 4) and then, starting from the last remaining equation of the system and working back to the first, solve in succession for the variables corresponding to the columns containing the pivots. If some column j has no pivot, there are an infinite number of solutions, because x_j can be set equal to any value. On the other hand, if every column has a pivot (as in Examples 2, 6, and 7), the value of each x_j is uniquely determined. ▲

With reference to item (3) of Theorem 1.7, the number of free variables in the solution set of a system $A\mathbf{x} = \mathbf{b}$ depends only on the system, and not on the way in which the matrix A is reduced to row-echelon form. This follows from the uniqueness of the reduced row-echelon form. (See Exercise 33 in Section 2.3.)

Elementary Matrices

The elementary row operations we have performed can actually be carried out by means of matrix multiplication. Although it is not efficient to row-reduce a matrix by multiplying it by other matrices, representing row reduction as a product of matrices is a useful theoretical tool. For example, we use elementary matrices in Section 1.5 to show that, for square matrices A and C, if $AC = I$, then $CA = I$. We use them again in Section 4.2 to demonstrate the multiplicative property of determinants, and again in Section 10.2 to exhibit a factorization of some square matrices A into a product LU of a lower-triangular matrix L and an upper-triangular matrix U.

If we interchange its second and third rows, the 3×3 identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{becomes} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If $A = [a_{ij}]$ is a 3×3 matrix, we can compute EA, and we find that

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

We have interchanged the second and third rows of A by multiplying A on the left by E.

DEFINITION 1.14 Elementary Matrix

Any matrix that can be obtained from an identity matrix by means of one elementary row operation is an **elementary matrix.**

We leave the proof of the following theorem as Exercises 52 through 54.

THEOREM 1.8 Use of Elementary Matrices

Let A be an $m \times n$ matrix, and let E be an $m \times m$ elementary matrix. Multiplication of A on the left by E effects the same elementary row operation on A that was performed on the identity matrix to obtain E.

Thus row reduction of a matrix to row-echelon form can be accomplished by successive multiplication on the left by elementary matrices. In other words, if A can be reduced to H through elementary row operations, there exist elementary matrices $E_1, E_2, \ldots, E_t$ such that

$$H = (E_t \cdots E_2E_1)A.$$

Again, this is by no means an efficient way to execute row reduction, but such an algebraic representation of H in terms of A is sometimes handy in proving theorems.

EXAMPLE 8 Let

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix}.$$

Find a matrix C such that CA is a matrix in row-echelon form that is row equivalent to A.

SOLUTION We row reduce A to row-echelon form H and write down, for each row operation, the elementary matrix obtained by performing the same operation on the 3×3 identity matrix.

Reduction of A	Row Operation	Elementary Matrix
$A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{bmatrix}$	$R_3 \to R_3 - 2R_1$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix}$	$R_3 \to R_3 + 1R_2$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} = H.$		

Thus, we must have $E_3(E_2(E_1A)) = H$; so the desired matrix C is

$$C = E_3E_2E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

To compute C, we do not actually have to multiply out $E_3E_2E_1$. We know that multiplication of E_1 on the left by E_2 simply adds -2 times row 1 of E_1 to its row 3, and subsequent multiplication on the left by E_3 adds row 2 to row 3 of the matrix E_2E_1. Thus we can find C by executing the same row-reduction steps on I that we executed to change A to H—namely,

$$\underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \sim \underset{E_1}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \sim \underset{E_2E_1}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \sim \underset{C = E_3E_2E_1}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}}.$$

■

We can execute analogous *elementary column* operations on a matrix by multiplying the matrix on the *right* by an elementary matrix. Column reduction of a matrix A is not important for us in this chapter, because it does not preserve the solution set of $A\mathbf{x} = \mathbf{b}$ when applied to the augmented matrix $[A \mid \mathbf{b}]$. However, we will have occasion to refer to column reduction when computing determinants. The effect of multiplication of a matrix A on the right by elementary matrices is explored in Exercises 36–38 in Section 1.5.

SUMMARY

1. A linear system has an associated augmented (or partitioned) matrix, having the coefficient matrix of the system on the left of the partition and the column vector of constants on the right of the partition.
2. The elementary row operations on a matrix are as follows:

   (*Row interchange*) Interchange of two rows,

   (*Row scaling*) Multiplication of a row by a nonzero scalar;

   (*Row addition*) Addition of a multiple of a row to a different row.

3. Matrices A and B are row equivalent (written $A \sim B$) if A can be transformed into B by a sequence of elementary row operations.
4. If $A\mathbf{x} = \mathbf{b}$ and $H\mathbf{x} = \mathbf{c}$ are systems such that the augmented matrices $[A \mid \mathbf{b}]$ and $[H \mid \mathbf{c}]$ are row equivalent, the systems $A\mathbf{x} = \mathbf{b}$ and $H\mathbf{x} = \mathbf{c}$ have the same solution set.
5. A matrix is in row-echelon form if:
   a. All rows containing only zero entries are grouped together at the bottom of the matrix.
   b. The first nonzero element (the pivot) in any row appears in a column to the right of the first nonzero element in any preceding row.
6. A matrix is in reduced row-echelon form if it is in row-echelon form and, in addition, each pivot is 1 and is the only nonzero element in its column. Every matrix is row equivalent to a unique matrix in reduced row-echelon form.
7. In the Gauss method with back substitution, we solve a linear system by reducing the augmented matrix so that the portion to the left of the