elementary matrices corresponding to the sequence of row operations used to reduce A to I.

3. To find A^{-1}, if it exists, form the augmented matrix $[A \mid I]$ and apply the Gauss–Jordan method to reduce this matrix to $[I \mid C]$. If this can be done, then $A^{-1} = C$. Otherwise, A is not invertible.
4. The inverse of a product of invertible matrices is the product of the inverses in the reverse order.

EXERCISES

In Exercises 1–8, (a) find the inverse of the square matrix, if it exists, and (b) express each invertible matrix as a product of elementary matrices.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$

4. $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ -3 & 1 & -7 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 12 \end{bmatrix}$

In Exercises 9 and 10, find the inverse of the matrix, if it exists.

9. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 11 and 12, determine whether the span of the column vectors of the given matrix is $\mathbb{R}^4$.

11. $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & -1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix}$

13. a. Show that the matrix
$$A = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$$
is invertible, and find its inverse.
b. Use the result in (a) to find the solution of the system of equations
$$2x_1 - 3x_2 = 4, \qquad 5x_1 - 7x_2 = -3.$$

14. Using the inverse of the matrix in Exercise 7, find the solution of the system of equations
$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 5 \\ 3x_1 + 2x_2 + 5x_3 &= 3 \\ -x_2 + x_3 &= 8. \end{aligned}$$

15. Find three linear equations that express x, y, z in terms of r, s, t, if
$$\begin{aligned} 2x + y + 4z &= r \\ 3x + 2y + 5z &= s \\ -y + z &= t. \end{aligned}$$
[HINT: See Exercise 14.]

16. Let

$$A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}.$$

If possible, find a matrix C such that

$$AC = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}.$$

17. Let

$$A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}.$$

If possible, find a matrix C such that

$$ACA = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}.$$

18. Let

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

If possible, find a matrix B such that $AB = 2A$.

19. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}.$$

If possible, find a matrix B such that $AB = A^2 + 2A$.

20. Find all numbers r such that

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

is invertible.

21. Find all numbers r such that

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

is invertible.

22. Let A and B be two $m \times n$ matrices. Show that A and B are row equivalent if and only if there exists an invertible $m \times m$ matrix C such that $CA = B$.

23. Mark each of the following True or False. The statements involve matrices A, B, and C, which are assumed to be of appropriate size.

___ a. If $AC = BC$ and C is invertible, then $A = B$.
___ b. If $AB = O$ and B is invertible, then $A = O$.
___ c. If $AB = C$ and two of the matrices are invertible, then so is the third.
___ d. If $AB = C$ and two of the matrices are singular, then so is the third.
___ e. If A^2 is invertible, then A^3 is invertible.
___ f. If A^3 is invertible, then A^2 is invertible.
___ g. Every elementary matrix is invertible.
___ h. Every invertible matrix is an elementary matrix.
___ i. If A and B are invertible matrices, then so is $A + B$, and $(A + B)^{-1} = A^{-1} + B^{-1}$.
___ j. If A and B are invertible, then so is AB, and $(AB)^{-1} = A^{-1}B^{-1}$.

24. Show that, if A is an invertible $n \times n$ matrix, then A^T is invertible. Describe $(A^T)^{-1}$ in terms of A^{-1}.

25. a. If A is invertible, is $A + A^T$ always invertible?

b. If A is invertible, is $A + A$ always invertible?

26. Let A be a matrix such that A^2 is invertible. Prove that A is invertible.

27. Let A and B be $n \times n$ matrices with A invertible.

a. Show that $AX = B$ has the unique solution $X = A^{-1}B$.

b. Show that $X = A^{-1}B$ can be found by the following row reduction:

$$[A \mid B] \sim [I \mid X].$$

That is, if the matrix A is reduced to the identity matrix I, then the matrix B will be reduced to $A^{-1}B$.

28. Note that

$$\frac{1}{a} + \frac{1}{b} = \frac{(a+b)}{(ab)}$$

for nonzero scalars $a, b \in \mathbb{R}$. Find an analogous equality for invertible $n \times n$ matrices A and B.

29. An $n \times n$ matrix A is **nilpotent** if $A^r = O$ (the $n \times n$ zero matrix) for some positive integer r.

a. Give an example of a nonzero nilpotent 2×2 matrix.

b. Show that, if A is an invertible $n \times n$ matrix, then A is not nilpotent.

30. A square matrix A is said to be **idempotent** if $A^2 = A$.

a. Give an example of an idempotent matrix other than O and I.

b. Show that, if a matrix A is both idempotent and invertible, then $A = I$.

31. Show that

$$\begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent. (See Exercise 29.)

32. A square matrix is **upper triangular** if all entries below the main diagonal are zero. **Lower triangular** is defined symmetrically. Give an example of a nilpotent 4×4 matrix that is not upper or lower triangular. (See Exercises 29 and 31.)

33. Give an example of two invertible 4×4 matrices whose sum is singular.

34. Give an example of two singular 3×3 matrices whose sum is invertible.

35. Consider the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and let $h = ad - bc$.

a. Show that, if $h \neq 0$, then

$$\begin{bmatrix} d/h & -b/h \\ -c/h & a/h \end{bmatrix}$$

is the inverse of A.

b. Show that A is invertible if and only if $h \neq 0$.

Exercises 36–38 develop elementary column operations.

36. For each type of elementary matrix E, explain how E can be obtained from the identity matrix by means of operations on columns.

37. Let A be a square matrix, and let E be an elementary matrix of the same size. Find the effect on A of multiplying A on the right by E. [HINT: Use Exercise 36.]

38. Let A be an invertible square matrix. Recall that $(BA)^{-1} = A^{-1}B^{-1}$, and use Exercise 37 to answer the following questions:

a. If two rows of A are interchanged, how does the inverse of the resulting matrix compare with A^{-1}?

b. Answer the question in part (a) if, instead, a row of A is multiplied by a nonzero scalar r.

c. Answer the question in part (a) if, instead, r times the ith row of A is added to the jth row.

In Exercises 39–42, use the routine YUREDUCE in LINTEK to find the inverse of the matrix, if it exists. If a printer is available, make a copy of the results. Otherwise, copy down the answers to three significant figures.

39. $\begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 1 \\ 0 & 3 & -4 \end{bmatrix}$

40. $\begin{bmatrix} -2 & 1 & 4 \\ 3 & 6 & 7 \\ 13 & 15 & -2 \end{bmatrix}$

41. $\begin{bmatrix} 2 & -1 & 3 & 4 \\ -5 & 2 & 0 & 11 \\ 12 & 13 & -6 & 8 \\ 18 & -10 & 3 & 0 \end{bmatrix}$

42. $\begin{bmatrix} 4 & -10 & 3 & 17 \\ 2 & 0 & -3 & 11 \\ 14 & 2 & 12 & -15 \\ 0 & -10 & 9 & -5 \end{bmatrix}$

In Exercises 43–48, follow the instructions for Exercises 39–42, but use the routine MATCOMP in LINTEK. Check to ensure that $AA^{-1} = I$ for each matrix A whose inverse is found.

43. The matrix in Exercise 9
44. The matrix in Exercise 10
45. The matrix in Exercise 41
46. The matrix in Exercise 40

47. $\begin{bmatrix} 4 & 1 & -3 & 2 & 6 \\ 0 & 1 & 5 & 2 & 1 \\ 3 & 8 & -11 & 4 & 6 \\ 2 & 1 & -8 & 7 & 2 \\ 1 & 3 & -1 & 4 & 8 \end{bmatrix}$

48. $\begin{bmatrix} 2 & -1 & 0 & 1 & 6 \\ 3 & -1 & 2 & 4 & 6 \\ 0 & 1 & 3 & 4 & 8 \\ -1 & 1 & 1 & 1 & 8 \\ 3 & 1 & 4 & -11 & 10 \end{bmatrix}$

MATLAB

Access MATLAB and, if the data files for our text are accessible, enter **fbc1s5**. *Otherwise, enter these four matrices by hand. [In MATLAB, ln(x) is denoted by* **log(x)**.]

$$A = \begin{bmatrix} -2 & 3 & 2/7 \\ \pi/2 & 1 & 3.2 \\ 5 & -6 & 1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 3\pi & \cos 2 & 21/8 \\ \sqrt{7} & \ln 4 & 2/3 \\ \sqrt{2} & \sin 4 & 8.3 \end{bmatrix},$$

$$C = \begin{bmatrix} -3.2 & 1.4 & 5.3 \\ 1.7 & -3.6 & 4.1 \\ 10.3 & 8.5 & -7.6 \end{bmatrix}$$

As you work the problems, write down the entry in the 2nd row, 3rd column position of the answer, with four-significant-figure accuracy, to hand in.

Enter **help inv**, *read the information, and then use the function* **inv** *to work problems M1 through M4.*

M1. Compute C^{-3}.

M2. Compute $A^3B^{-2}C$.

M3. Find the matrix X such that $XB = C$.

M4. Find the matrix X such that $B^2XC = A$.

Enter **help /** *and then* **help **, *read the information, and then use / and \\ rather than the function* **inv** *to work problems M5 through M8.*

M5. Compute $A^{-1}B^2C^{-1}B$.

M6. Compute $B^{-2}CA^{-3}B^3$.

M7. Find the matrix X such that $CX = B^{-2}$.

M8. Find the matrix X such that $AXC^3 = B^4$.