
2019年 EGM0考古題

第6組

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題目翻譯

Problem 1

Day 1. Solutions

Problem 1 (Netherlands). Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

Problem 2

Problem 2 (Luxembourg). Let n be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is adjacent to exactly one cell covered by a domino. For each n , determine the largest number of dominoes that can be placed in this way.

(A *domino* is a tile of size 2×1 or 1×2 . Dominoes are placed on the board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. Two cells are said to be *adjacent* if they are different and share a common side.)

Problem 3

Problem 3 (Poland). Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I . Let X be the second point of intersection of ω and the circumcircle of ABC . Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC .

Problem 4

Day 2. Solutions

Problem 4 (Poland). Let ABC be a triangle with incentre I . The circle through B tangent to AI at I meets side AB again at P . The circle through C tangent to AI at I meets side AC again at Q . Prove that PQ is tangent to the incircle of ABC .

Problem 5

Problem 5 (Netherlands).

Let $n \geq 2$ be an integer, and let a_1, a_2, \dots, a_n be positive integers. Show that there exist positive integers b_1, b_2, \dots, b_n satisfying the following three conditions:

1. $a_i \leq b_i$ for $i = 1, 2, \dots, n$;
2. the remainders of b_1, b_2, \dots, b_n on division by n are pairwise different; and
3. $b_1 + \dots + b_n \leq n \left(\frac{n-1}{2} + \left\lfloor \frac{a_1 + \dots + a_n}{n} \right\rfloor \right)$.

(Here, $\lfloor x \rfloor$ denotes the integer part of real number x , that is, the largest integer that does not exceed x .)

Problem 6

Problem 6 (United Kingdom).

On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered *marked* if it is either

- (i) one of the 4038 endpoints of a chord; or
- (ii) an intersection point of at least two chords.

Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with k marked points has $k - 1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.

Alina finds that the $N + 1$ yellow labels take each value $0, 1, \dots, N$ exactly once. Show that at least one blue label is a multiple of 3.

(A *chord* is a line segment joining two different points on a circle.)

講解_Problem 1

Problem 1

Find all triples (a,b,c) of real numbers such that $ab+bc+ca=1$ and

$$\mathbf{a^2 b+c=b^2 c+a=c^2 a+b.}$$

solution

(i) **a, b, c** 其中一個為0

$$\mathbf{ab+bc+ca=1 \Rightarrow ab=1-bc-ca}$$

$$\Rightarrow \mathbf{a^2 b+c=b^2 c+a=a-abc-a^2 c+c}$$

$$\Rightarrow \mathbf{c(a^2+b^2+ab-1)=0}$$

$$\text{如果}\mathbf{c=0} \Rightarrow \mathbf{ab=1, a^2 b=b} \Rightarrow \mathbf{a=b=\pm 1}$$

同理，如果**a=0**，得 **b=c=±1**。如果**b=0**，得 **a=c=±1**

因此，如果 **a, b, c** 其中一個=0，另外兩個分別為**+1**和**-1**，就可以滿足條件。

(ii) **a, b, c** 均不為0

$$\mathbf{a^2+b^2+ab=1-(1)}$$

$$\mathbf{b^2+c^2+bc=1-(2)}$$

$$\mathbf{c^2+a^2+ca=1-(3)}$$

$$\mathbf{(1)+(2)+(3) \Rightarrow 2(a^2+b^2+c^2)+(ab+bc+ca)=3}$$

$$\Rightarrow \mathbf{(a^2+b^2+c^2)=1}$$

$$\Rightarrow \mathbf{(a+b)^2+(b+c)^2+(c+a)^2=0}$$

$$\Rightarrow \mathbf{a=b=c}$$
 代入 $\mathbf{a^2+b^2+ab=1}$

$$\Rightarrow \mathbf{a=b=c=\pm\sqrt{1/3}}$$

因此，如果 **a, b, c** 均不=0, **(a, b, c)= $\pm(\sqrt{1/3}, \sqrt{1/3}, \sqrt{1/3})$** ，就可以滿足條件。

相似的 Problem1

相似的 Problem 1

**Find all triples (a,b,c) of real numbers such that $ab+bc+ca=4$
and**

$$a^2 b+4c=b^2 c+4a=c^2 a+4b.$$

solution

(i) **a,b,c** 其中一個為0

$$\mathbf{ab+bc+ca=4 \Rightarrow ab=4-bc-ca}$$

$$\Rightarrow \mathbf{a^2b+4c=b^2c+4a=c^2a+4b}$$

$$\text{如果}\mathbf{c=0} \Rightarrow \mathbf{ab=1, a^2b=b \Rightarrow a=b=\pm 2}$$

同理，如果**a=0**，得 **b=c=±2**。如果**b=0**，得 **a=c=±2**

因此，如果 **a,b,c** 其中一個=0，另外兩個分別為**+2**和**-2**，就可以滿足條件。

(ii) **a,b,c** 均不為0

$$a^2+b^2+ab=4 \text{ --(1)}$$

$$b^2+c^2+bc=4 \text{ --(2)}$$

$$c^2+a^2+ca=4 \text{ --(3)}$$

$$(1)+(2)+(3) \Rightarrow 2(a^2+b^2+c^2)+(ab+bc+ca)=12$$

$$\Rightarrow a^2+b^2+c^2=4$$

$$\Rightarrow a^2+ab+b^2=4$$

$$\Rightarrow c^2=ab$$

$$\text{同理} \Rightarrow b^2=ca \text{ \& } a^2=bc$$

$$\Rightarrow a=b=c \text{ 代入 } a^2+b^2+ab=4$$

$$\Rightarrow a=b=c=\pm 2/\sqrt{3}$$

因此, 如果 **a,b,c** 均不=0, **(a,b,c)= $\pm(2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$**