Section 7.2 Matrix Representations and Similarity

24. Prove statement (ii) of Theorem 7.2

Answer:
See *the prrof of Theorem 7.2* below.

25. Prove statement (iii) of Theorem 7.2

Answer:
See *the prrof of Theorem 7.2* below.

27. Give a determinant proof that similar matrices have the same eigenvalues.

Answer:
See the statement (i) of *the proof of Theorem 7.2* below.

Theorem 7.2 Eigenvalues and Eigenvectors of Similar Matrices

Let A and R be similar $n\times$ matrices, so that $R = C^{-1}AC$ for some inevitable $n\times n$ matrix C. Let the eigenvalues of A be the (not necessarily distinct) numbers $\lambda_1, \lambda_2, ..., \lambda_n$.

(i) The eigenvalues of R are also $\lambda_1, \lambda_2, ..., \lambda_n$.

(ii) The algebraic and geometric multiplicity of each λ_i as an eigenvalue of A remains the same as when it is viewed as an eigenvalue of R.

(iii) If $\vec{v}_i$ in $\mathbb{R}^n$ is an eigenvector of the matrix A corresponding to λ_i, then $C^{-1}\vec{v}_i$ is an eigenvector of the matrix R corresponding to λ_i

Proof of (i):
The characteristic equation for matrix R is $\det(R-\lambda I)$ and so

$$\begin{aligned}\det(R-\lambda I) &= \det(C^{-1}AC-\lambda I) = \det(C^{-1}AC-\lambda C^{-1}C)\\ &= \det(C^{-1}(A-\lambda)C) = \det(C^{-1})\det(A-\lambda)\det(C)\\ &\frac{1}{\det(C)}\det(A-\lambda)\det(C) = \det(A-\lambda)\end{aligned}$$

Therefore the characteristic equation of R and A are the same, and so R and A have the same eigenvalues.

Proof of (iii):
Suppose $A\vec{v} = \lambda\vec{v}$. Then as $A = CRC^{-1}$ we have

$$(CRC^{-1})\vec{v} = \lambda\vec{v} \Rightarrow RC^{-1}\vec{v} = C^{-1}\lambda\vec{v} \Rightarrow R(C^{-1}\vec{v}) = \lambda(C^{-1}\vec{v})$$

So, $C^{-1}\vec{v}$ is an eigenvector for R, with eigenvalue λ. Note that as C is invertible so is, of course, C^{-1}, and so C^{-1} has full rank. Having full rank means its columns are linearly independent, and so if $\vec{v} = \vec{0}$ we must have $C^{-1}\vec{v} = \vec{0}$.

Proof of (ii):
By the *Proof of (i)*, we know the characteristic equation of R and A are the same, hence each λ_i as an eigenvalue of A and R has the same algebraic multiplicity.

The geometric multiplicity of an eigenvalue λ of R is the dimension of the eigenspace

$$\begin{aligned} E_\lambda &= \{\vec{v} \mid \text{ the eigenvecotrs of } R \text{ corresponding to } \lambda\} \\ &= \{\vec{v} \mid R\vec{v} = \lambda\vec{v}\} \\ &= \{\vec{v} \mid (R - \lambda I)\vec{v} = \vec{0}\} \end{aligned}$$

Denote the eigenspace of matrix A associated with λ as E_λ^A. Now we need to prove that $\dim(E_\lambda^A) = \dim(E_\lambda^R)$

By the *Proof of (iii)*, we know that each $\vec{v} \in E_\lambda^A$, there exists $C^{-1}\vec{v} \in E_\lambda^R$. Obviously, for $\vec{x}, \vec{y} \in \mathbb{R}^n$, if $\vec{x} \neq \vec{y}$ then $C^{-1}\vec{x} \neq C^{-1}\vec{y}$.

$$\dim(E_\lambda^A) < \dim(E_\lambda^R).$$

Conversely, $\dim(E_\lambda^A) > \dim(E_\lambda^R)$. Finally, $\dim(E_\lambda^A) = \dim(E_\lambda^R)$

extra 1. Prove that if two matrices have the same repeated eigenvalues they may not be similar.

Answer:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Notice that both A, B has the repeated eigenvalue 0, but is only similar to itself.

extra 2. Prove that if two matrices have the same distinct eigenvalues they are similar.

Answer:
Suppose A and B have the same distinct eigenvalues. Then they are both diagonalizable with the same diagonal matrix D. So, both A and B are similar to D, and therefore A is similar to B.