CHAPTER 2

Section 2.1

1. Two nonzero vectors in $\mathbb{R}^2$ are dependent if and only if they are parallel.

3. Two nonzero vectors in $\mathbb{R}^3$ are dependent if and only if they are parallel.

5. Three vectors in $\mathbb{R}^3$ are dependent if and only if they all lie in one plane through the origin.

7. $\{[-3, 1], [6, 4]\}$

9. $\{[2, 1], [1, 4]\}$

11. $\{[1, 2, 1, 2], [2, 1, 0, -1]\}$

13. $\{[1, 3, 5, 7], [2, 0, 4, 2]\}$

15. $\{[2, 3, 1], [5, 2, 1]\}$

17. Independent

19. Dependent

21. Independent

23. Dependent

25. Dependent

27. $\{[2, 1, 1, 1], [1, 0, 1, 1], [1, 0, 0, 0], [0, 0, 1, 0]\}$

31. Suppose that $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 = \mathbf{0}$, so that $r_1(2\mathbf{v}_1 + 3\mathbf{v}_2) + r_2(\mathbf{v}_2 - 2\mathbf{v}_3) + r_3(-\mathbf{v}_1 - 3\mathbf{v}_3) = \mathbf{0}$. Then $(2r_1 - r_3)\mathbf{v}_1 + (3r_1 + r_2)\mathbf{v}_2 + (-2r_2 - 3r_3)\mathbf{v}_3 = \mathbf{0}$. We try setting these scalar coefficients to zero and solve the linear system

$$\begin{aligned} 2r_1 \quad\quad &- r_3 = 0,\\ 3r_1 + r_2 \quad &\quad = 0,\\ -2r_2 &- 3r_3 = 0. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0\\ 3 & 1 & 0 & 0\\ 0 & -2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2} & 0\\ 0 & -2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2} & 0\\ 0 & 0 & 0 & 0 \end{array}\right].$$

This system has the nontrivial solution $r_3 = 2$, $r_2 = -3$, and $r_1 = 1$. Thus, $(2\mathbf{v}_1 + 3\mathbf{v}_2) - 3(\mathbf{v}_2 - 2\mathbf{v}_3) + 2(-\mathbf{v}_1 - 3\mathbf{v}_3) = \mathbf{0}$ for all choices of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$, so $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are dependent. (Notice that, if the system had only the trivial solution, then any choice of independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ would show this exercise to be false.)

33. No such scalars s exist.

35. Let $A = \begin{bmatrix} 1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}$. Then $A\mathbf{v} = A\mathbf{w} = \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$. For the second part of the problem, let $A = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}$. Then $A\mathbf{v} = \mathbf{v}$ and $A\mathbf{w} = \mathbf{w}$, so $A\mathbf{v}$ and $A\mathbf{w}$ are still independent.

39. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

41. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_6\}$

M1. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

M3. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_6\}$

Section 2.2

1. a. 2
   b. $\{[2, 0, -3, 1], [3, 4, 2, 2]\}$ by inspection
   c. The set consisting of $\begin{bmatrix} 2\\ 3 \end{bmatrix}, \begin{bmatrix} 0\\ 4 \end{bmatrix}$ or of $\begin{bmatrix} 1\\ 0 \end{bmatrix}, \begin{bmatrix} 0\\ 1 \end{bmatrix}$
   d. The set consisting of $\begin{bmatrix} 12\\ -13\\ 8\\ 0 \end{bmatrix}, \begin{bmatrix} -4\\ -1\\ 0\\ 8 \end{bmatrix}$

3. a. 3
   b. $\{[1, 0, -1, 0], [0, 1, 1, 0], [0, 0, 0, 1]\}$
   c. The set consisting of $\begin{bmatrix} 0\\ 1\\ 4\\ 1 \end{bmatrix}, \begin{bmatrix} 6\\ 2\\ 1\\ 3 \end{bmatrix}, \begin{bmatrix} 3\\ 1\\ 4\\ 0 \end{bmatrix}$
   d. The set consisting of the vector $\begin{bmatrix} 1\\ -1\\ 1\\ 0 \end{bmatrix}$

5. a. 3
   b. $\{[6, 0, 0, 5], [0, 3, 0, 1], [0, 0, 3, 1]\}$
   c. The set consisting of $\begin{bmatrix} 0\\ 2\\ 0 \end{bmatrix}, \begin{bmatrix} 1\\ 1\\ 2 \end{bmatrix}, \begin{bmatrix} 2\\ 0\\ 1 \end{bmatrix}$ or the set of column vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

. The set consisting of $\begin{bmatrix} -5 \\ -2 \\ -2 \\ 6 \end{bmatrix}$

7. The rank is 3; therefore, the matrix is not invertible.

). The rank is 3; therefore, the matrix is invertible.

l. T F T T F F T F F T

5. No. If A is $m \times n$ and C is $n \times s$, then AC is $m \times s$. The column space of AC lies in $\mathbb{R}^m$ whereas the column space of C lies in $\mathbb{R}^n$.

7. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

9. Let $A = I$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then rank $(AC) = 1 < 2 =$ rank (A).

5. **a.** 3 **b.** Rows 1, 2, and 4

tion 2.3

1. Yes. We have $T([x_1, x_2, x_3]) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Example 3 then shows that T is a linear transformation.

3. No. $T(0[1, 2, 3]) = T([0, 0, 0]) = [1, 1, 1, 1]$, whereas $0T([1, 2, 3]) = 0[1, 1, 1, 1] = [0, 0, 0, 0]$.

5. $[24, -34]$

7. $[2, 7, -15]$

9. $[4, 2, 4]$

1. $[802, -477, 398, 57]$

13. $\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

9. The matrix associated with $T' \circ T$ is $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$; $(T' \circ T)([x_1, x_2]) = [2x_1, 3x_1 + x_2]$.

:1. Yes; $T^{-1}([x_1, x_2]) = \left[\dfrac{3x_1 + x_2}{4}, \dfrac{x_1 - x_2}{4}\right]$.

23. Yes; $T^{-1}([x_1, x_2, x_3]) = [x_3, x_2 - x_3, x_1 - x_2]$.

25. Yes; $T^{-1}([x_1, x_2, x_3]) =$
$$\left[x_3, \frac{-x_1 + 3x_2 - 2x_3}{4}, \frac{x_1 + x_2 - 2x_3}{4}\right].$$

27. Yes; $(T' \circ T)^{-1}([x_1, x_2]) = \left[\dfrac{x_1}{2}, \dfrac{-3x_1 + 2x_2}{2}\right]$.

29. T F T T T F T T F T

33. **a.** Because
$$\begin{aligned} T_k(\mathbf{u} + \mathbf{v}) &= [u_1 + v_1, u_2 + v_2, \ldots, \\ &\quad u_k + v_k, 0, \ldots, 0] \\ &= [u_1, u_2, \ldots, u_k, 0, \ldots, 0] + \\ &\quad [v_1, v_2, \ldots, v_k, 0, \ldots, 0] \\ &= T_k(\mathbf{u}) + T_k(\mathbf{v}) \end{aligned}$$
and
$$\begin{aligned} T_k(r\mathbf{u}) &= [ru_1, ru_2, \ldots, ru_k, 0, \ldots, 0] \\ &= r[u_1, u_2, \ldots, u_k, 0, \ldots, 0] \\ &= rT_k(\mathbf{u}), \end{aligned}$$
we know that T_k is a linear transformation. We see that $T_k[V] \subseteq W_k$, which means that it is a subspace of W_k, by the box on page 143.

b. Because $T_k[V]$ is a subspace of $T_{k+1}[V]$, we have
$$d_1 \le d_2 \le d_3 \le \cdots \le d_n \le n.$$
If the jth column of H has a pivot but the $(j + 1)$st column has no pivot, then $d_{j+1} = d_j$. However, if the jth and $(j + 1)$st columns both have pivots, then $d_{j+1} = d_j + 1$. See parts c and d for illustrations.

c. If $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$, then H must have the form
$$H = \begin{bmatrix} p & X & X & X \\ 0 & 0 & p & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
where a p denotes a pivot and an X denotes a possibly nonzero entry.

d. If $d_1 = 1$, $d_2 = d_3 = d_4 = 2$, and $d_5 = d_6 = 3$, then H must have the form

$$H = \begin{bmatrix} p & \times & \times & \times & \times & \times \\ 0 & p & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & p & \times \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where a p denotes a pivot and an $\times$ denotes a possibly nonzero entry.

e. The number of pivots in H is the number of distinct dimension numbers in the list $d_1, d_2, d_3, \ldots, d_n$. Moreover, a pivot occurs in the $(j+1)$st column if and only if $d_{j+1} = d_j + 1$; the row and column positions of pivots are completely determined by the list $d_1, d_2, d_3, \ldots, d_n$. The pivot in the kth row occurs in the jth column if and only if d_j is the kth *distinct* positive integer in the list $d_1, d_2, d_3, \ldots, d_n$. Because the numbers d_i depend only on A (they are defined just in terms of the row space of A), so do the number and locations of the pivots; the number and locations are the same for all echelon forms of A.

f. Let H be a *reduced* echelon form of A. Part e shows that the number of pivots and their locations depend only on A, and consequently the number of zero rows depends only on A and is the same for all choices of H. Suppose that the pivot in the kth row of H is in column j. Consider now a nonzero row vector in $\mathbb{R}^n$ that has entries zero in all components corresponding to columns of H containing pivots except for the jth component, where the entry is 1; entries in components not corresponding to pivot column locations in H may be arbitrary. We claim that there is a *unique* such vector in the row space of A—namely, the kth row vector of H. Such a vector must be a linear combination of the nonzero rows of H, which span the row space of A. The fact that there are zeros above as well as below pivots in the *reduced* row-echelon form H shows that the only possible such linear combination of nonzero rows of H is 1 times the kth row plus 0 times the other rows. This gives a characterization of the kth nonzero row of H in terms of the row space of A and completes the demonstration that the reduced echelon form of A is unique.

35. [588, 160, 8]

37. Undefined because $T_2 \circ T_3$ is not invertible.

Section 2.4

1. Because $\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 3y \\ 2x - 6y \end{bmatrix}$, which has the form $\begin{bmatrix} t \\ 2t \end{bmatrix}$, we see that the range of T_A consists of vectors along the line $y = 2x$. Now $T([1, 2]) = [-5, -10] \neq [1, 2]$, and projection onto a line must leave fixed the points that lie on the line, so T_A is not a projection on the line.

3. a. $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ b. $\begin{bmatrix} \frac{1}{2} & \sqrt{3}/2 \\ -\sqrt{3}/2 & \frac{1}{2} \end{bmatrix}$

c. $\begin{bmatrix} -\sqrt{3}/2 & \frac{1}{2} \\ -\frac{1}{2} & -\sqrt{3}/2 \end{bmatrix}$

7. $\begin{bmatrix} \dfrac{1 - m^2}{1 + m^2} & \dfrac{2m}{1 + m^2} \\ \dfrac{2m}{1 + m^2} & \dfrac{m^2 - 1}{1 + m^2} \end{bmatrix}$

11. In column-vector notation, we have $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, which represents a reflection in the line $y = x$ followed by a reflection in the y-axis.

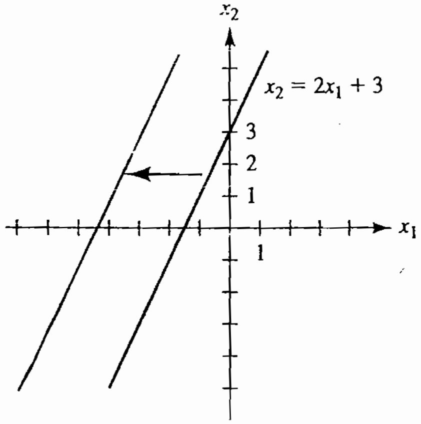
13. In column-vector notation, we have $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, which represents a reflection in the x-axis followed by a reflection in the y-axis.

15. In column-vector notation, we have
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$
which represents a horizontal shear followed by a vertical expansion followed by a vertical shear.

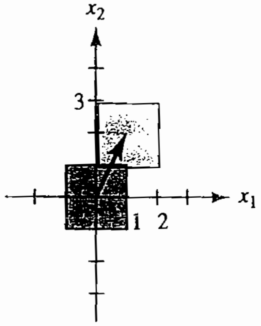
9. $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ and $\theta = \cos^{-1} \dfrac{\mathbf{u} \cdot \mathbf{v}}{\sqrt{\mathbf{u} \cdot \mathbf{u}}\sqrt{\mathbf{v} \cdot \mathbf{v}}}$.

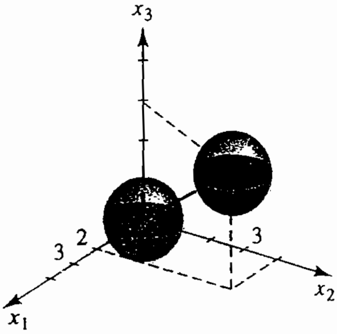
tion 2.5

1.

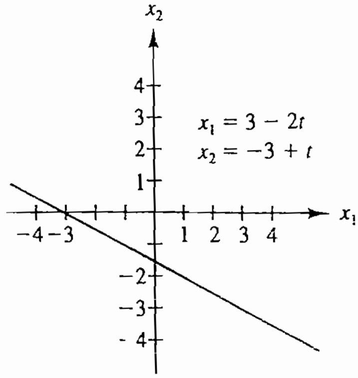


3.





7. $x_1 = 3 - 2t$
$x_2 = -3 + t$



9. $x_1 = \dfrac{d_1c}{d_1^2 + d_2^2} + d_2t,\ x_2 = \dfrac{d_2c}{d_1^2 + d_2^2} - d_1t$

11. a. $x_1 = -2 + t,\ x_2 = 4 - t$;
b. $x_1 = 3 - 3t,\ x_2 = -1 - 2t,\ x_3 = 6 - 7t$;
c. $x_1 = 2 - 3t,\ x_2 = 5t,\ x_3 = 4 - 12t$

13. The lines contain exactly the same set $\{(5 - 3t, -1 + t) \mid t \in \mathbb{R}\}$ of points.

15. a. $\left(\frac{1}{2}, \frac{3}{2}\right)$ b. $\left(\frac{3}{2}, -2, \frac{5}{2}\right)$ c. $\left(-2, \frac{9}{2}, \frac{17}{2}\right)$

17. $\left(-\frac{3}{2}, -\frac{1}{2}, \frac{15}{4}\right)$

19. $\left(\frac{3}{2}, \frac{3}{2}, 1, \frac{7}{2}, -\frac{1}{2}\right)$

21. (2, 3, 4)

23. $x_1 + x_2 + x_3 = 1$

25. $2x_1 + x_2 - x_3 = 2$

27. $x_1 + 4x_2 + x_3 = 10$

29. $x_2 + x_3 = 3,\ -3x_1 - 7x_2 + 8x_3 + 3x_4 = 0$
(There are infinitely many other correct linear systems.)

31. $9x_1 - x_2 + 2x_3 - 6x_4 + 3x_5 = 6$

33. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$

35. The 0-flat $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

37. The 1-flat $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

39. The 1-flat $\mathbf{x} = \begin{bmatrix} -8 \\ -23 \\ -7 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -5 \\ 1 \\ 2 \end{bmatrix}$