and

$$A^TA = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 14 \end{bmatrix}.$$

Thus,

$$\sqrt{\det(A^TA)} = \sqrt{70 - 25} = \sqrt{45} = 3\sqrt{5}.$$

A region G of $\mathbb{R}^2$ having area V is mapped by T into a plane region of area $3\sqrt{5} \cdot V$ in $\mathbb{R}^3$. Thus the disk $x^2 + y^2 \leq 4$ of area 4π is mapped into a plane region in $\mathbb{R}^3$ of area

$$(3\sqrt{5})(4\pi) = 12\pi\sqrt{5}.$$ ▪

SUMMARY

1. An n-box in $\mathbb{R}^m$, where $m \geq n$, is determined by n independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and consists of all vectors $\mathbf{x}$ in $\mathbb{R}^m$ such that
$$\mathbf{x} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n,$$
where $0 \leq t_i \leq 1$ for $i = 1, 2, \ldots, n$.
2. A 1-box in $\mathbb{R}^m$ is a line segment, and its "volume" is its length.
3. A 2-box in $\mathbb{R}^m$ is a parallelogram determined by two independent vectors, and the "volume" of the 2-box is the area of the parallelogram.
4. A 3-box in $\mathbb{R}^m$ is a skewed box (parallelepiped) in the usual sense, and its volume is the usual volume.
5. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be independent vectors in $\mathbb{R}^m$ for $m \geq n$, and let A be the $m \times n$ matrix with jth column vector $\mathbf{a}_j$. The volume of the n-box in $\mathbb{R}^m$ determined by the n vectors is $\sqrt{\det(A^TA)}$.
6. For the case of an n-box in the space $\mathbb{R}^n$ of the same dimension, the formula for its volume given in summary item 5 reduces to $|\det(A)|$.
7. If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation of rank n with standard matrix representation A, then T maps a region in its domain of volume V into a region of volume $|\det(A)|V$.
8. If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation of rank n with standard matrix representation A, then T maps a region in its domain of volume V into a region of $\mathbb{R}^m$ of volume $\sqrt{\det(A^TA)} \cdot V$.

EXERCISES

1. Find the area of the parallelogram in $\mathbb{R}^3$ determined by the vectors $[0, 1, 4]$ and $[-1, 3, -2]$.
2. Find the area of the parallelogram in $\mathbb{R}^5$ determined by the vectors $[1, 0, 1, 2, -1]$ and $[0, 1, -1, 1, 3]$.

3. Find the volume of the 3-box in $\mathbb{R}^4$ determined by the vectors $[-1, 2, 0, 1]$, $[0, 1, 3, 0]$, and $[0, 0, 2, -1]$.
4. Find the volume of the 4-box in $\mathbb{R}^5$ determined by the vectors $[1, 1, 1, 0, 1]$, $[0, 1, 1, 0, 0]$, $[3, 0, 1, 0, 0]$, and $[1, -1, 0, 0, 1]$.

In Exercises 5–10, find the volume of the n-box determined by the given vectors in $\mathbb{R}^n$.

5. $[-1, 4]$, $[2, 3]$ in $\mathbb{R}^2$
6. $[-5, 3]$, $[1, 7]$ in $\mathbb{R}^2$
7. $[1, 3, -5]$, $[2, 4, -1]$, $[3, 1, 2]$ in $\mathbb{R}^3$
8. $[-1, 4, 7]$, $[3, -2, -1]$, $[4, 0, 2]$ in $\mathbb{R}^3$
9. $[1, 0, 0, 1]$, $[2, -1, 3, 0]$, $[0, 1, 3, 4]$, $[-1, 1, -2, 1]$ in $\mathbb{R}^4$
10. $[1, -1, 0, 1]$, $[2, -1, 3, 1]$, $[-1, 4, 2, -1]$, $[0, 1, 0, 2]$ in $\mathbb{R}^4$
11. Find the area of the triangle in $\mathbb{R}^3$ with vertices $(-1, 2, 3)$, $(0, 1, 4)$, and $(2, 1, 5)$. [HINT: Think of vectors emanating from $(-1, 2, 3)$. The triangle may be viewed as half a parallelogram.]
12. Find the volume of the tetrahedron in $\mathbb{R}^3$ with vertices $(1, 0, 3)$, $(-1, 2, 4)$, $(3, -1, 2)$, and $(2, 0, -1)$. [HINT: Think of vectors emanating from $(1, 0, 3)$.]
13. Find the volume of the tetrahedron in $\mathbb{R}^4$ with vertices $(1, 0, 0, 1)$, $(-1, 2, 0, 1)$, $(3, 0, 1, 1)$, and $(-1, 4, 0, 1)$. [HINT: See the hint for Exercise 12.]
14. Give a geometric interpretation of the fact that an $n \times n$ matrix with two equal rows has determinant zero.
15. Using the results of this section, give a criterion that four points in $\mathbb{R}^n$ lie in a plane.
16. Determine whether the points $(1, 0, 1, 0)$, $(-1, 1, 0, 1)$, $(0, 1, -1, 1)$, and $(1, -1, 4, -1)$ lie in a plane in $\mathbb{R}^4$. (See Exercise 15.)
17. Determine whether the points $(2, 0, 1, 3)$, $(3, 1, 0, 1)$, $(-1, 2, 0, 4)$, and $(3, 1, 2, 4)$ lie in a plane in $\mathbb{R}^4$. (See Exercise 15.)

In Exercises 18–21, let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T([x, y]) = [4x - 2y, 2x + 3y]$. Find the area of the image under T of each of the given regions in $\mathbb{R}^2$.

18. The square $0 \le x \le 1, 0 \le y \le 1$
19. The rectangle $-1 \le x \le 1, 1 \le y \le 2$
20. The parallelogram determined by $2\mathbf{e}_1 + 3\mathbf{e}_2$ and $4\mathbf{e}_1 - \mathbf{e}_2$
21. The disk $(x - 1)^2 + (y + 2)^2 \le 9$

In Exercises 22–25, let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T([x, y, z]) = [x - 2y, 3x + z, 4x + 3y]$. Find the volume of the image under T of each of the given regions in $\mathbb{R}^3$.

22. The cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$
23. The box $0 \le x \le 2, -1 \le y \le 3, 2 \le z \le 5$
24. The box determined by $2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3$, $4\mathbf{e}_1 - 2\mathbf{e}_3$, and $\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$
25. The ball $x^2 + (y - 3)^2 + (z + 2)^2 \le 16$

In Exercises 26–29, let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T([x, y]) = [y, x, x + y]$. Find the area of the image under T of each of the given regions in $\mathbb{R}^2$.

26. The square $0 \le x \le 1, 0 \le y \le 1$
27. The rectangle $2 \le x \le 3, -1 \le y \le 4$
28. The triangle with vertices $(0, 0)$, $(6, 0)$, $(0, 3)$
29. The disk $x^2 + y^2 \le 25$

In Exercises 30–32, let $T: \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $T([x, y]) = [x - y, x, -y, 2x + y]$. Find the area of the image under T of each of the given regions in $\mathbb{R}^2$.

30. The square $0 \le x \le 1, 0 \le y \le 1$
31. The square $-1 \le x \le 3, -1 \le y \le 3$
32. The disk $x^2 + y^2 \le 9$
33. a. If one attempts to define an n-box in $\mathbb{R}^m$ for $n > m$, what will its volume as an n-box be?
    b. Let A be an $m \times n$ matrix with $n > m$. Find $\det(A^TA)$.
34. We have seen that, for $n \times n$ matrices A and B, we have $\det(AB) = \det(A) \cdot \det(B)$, but the proof was not intuitive. Give an intuitive

geometric argument showing that at least $|\det(AB)| = |\det(A)| \cdot |\det(B)|$. [HINT: Use the fact that, if A is the standard matrix representation of $T: \mathbb{R}^n \to \mathbb{R}^n$ and B is the standard matrix representation of $T': \mathbb{R}^n \to \mathbb{R}^n$, then AB is the standard matrix representation $T \circ T'$.]

35. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of rank n with standard matrix representation A. Mark each of the following True or False.

___ a. The image under T of a box in $\mathbb{R}^n$ is again a box in $\mathbb{R}^n$.

___ b. The image under T of an n-box in $\mathbb{R}^n$ of volume V is a box in $\mathbb{R}^n$ of volume $\det(A) \cdot V$.

___ c. The image under T of an n-box in $\mathbb{R}^n$ of volume > 0 is a box in $\mathbb{R}^n$ of volume > 0.

___ d. If the image under T of an n-box B in $\mathbb{R}^n$ has volume 12, the box B has volume $|\det(A)| \cdot 12$.

___ e. If the image under T of an n-box B in $\mathbb{R}^n$ has volume 12, the box B has volume $12/|\det(A)|$.

___ f. If $n = 2$, the image under T of the unit disk $x^2 + y^2 \leq 1$ has area $|\det(A)|$.

___ g. The linear transformation T is an isomorphism.

___ h. The image under $T \circ T$ of an n-box in $\mathbb{R}^n$ of volume V is a box in $\mathbb{R}^n$ of volume $\det(A^2) \cdot V$.

___ i. The image under $T \circ T \circ T$ of an n-box in $\mathbb{R}^n$ of volume V is a box in $\mathbb{R}^n$ of volume $\det(A^3) \cdot V$.

___ j. The image under T of a nondegenerate 1-box is again nondegenerate.

36. Prove Eq. (1); that is, prove that the square of the length of the line segment determined by $\mathbf{a}_1$ in $\mathbb{R}^n$ is $\|\mathbf{a}_1\|^2 = \det([\mathbf{a}_1 \cdot \mathbf{a}_1])$.