

2-3

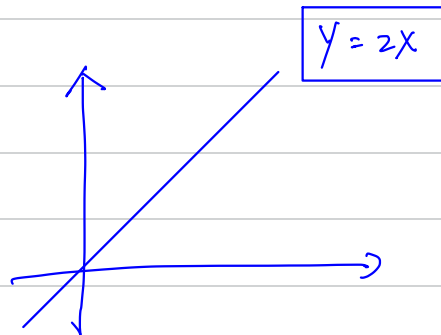
map  $f: \underset{\substack{\downarrow \text{domain}}}{X} \rightarrow \underset{\substack{\downarrow \text{codomain}}}{Y}$   
 $f(X) \rightarrow \text{range}$

ex:

$f: \{1, 2, 3, \dots, 10\} \rightarrow \overset{\textcircled{1}}{\mathbb{N}} \leftarrow \text{自然数}$   
 $x \mapsto f(x) = 2x$

$\textcircled{1} \{2, 4, 6, \dots, 20\}$   
 $\textcircled{2} \{1, 2, 3, \dots, 20\}$

$f(x) = 2x$



eg.  $f(x) = \frac{x-3}{x-2} \quad x \neq 2$

定义域 domain:  $\{1, 2, \dots, 10\}$

22 定义域  $\omega \text{ domain} = \mathbb{N}$

值域 range =  $\{2, 4, 6, 8, \dots, 20\} = \{f(x) \mid x \in X\}$

well-defined

$\textcircled{1}$   $f$  在  $X$  上可以送出去

$\textcircled{2}$   $f(X) \subseteq Y$

Def.

map

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation

if  $\left\{ \begin{array}{l} \textcircled{1} T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}), \quad \forall \vec{v}, \vec{u} \in \mathbb{R}^n \\ \textcircled{2} T(r\vec{v}) = rT(\vec{v}), \quad \forall r \in \mathbb{R}, \vec{v} \in \mathbb{R}^n \end{array} \right.$

Preservation of vector addition

$\textcircled{2} T(r\vec{v}) = rT(\vec{v}), \quad \forall r \in \mathbb{R}, \vec{v} \in \mathbb{R}^n$

Preservation of scalar multiplication

or P.S.  $T(\vec{0}) = T(0 \cdot \vec{0}) = 0 T(\vec{0}) = \vec{0}$

if  $T(r\vec{u} + s\vec{v}) = rT(\vec{u}) + sT(\vec{v}), \quad \forall r, s \in \mathbb{R}, \vec{u}, \vec{v} \in \mathbb{R}^n$

Preservation of linear combination

Prop.

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  : linear trans  $\{T(\vec{u}) \mid \vec{u} \in W\}$   
if  $W$  : subspace of  $\mathbb{R}^n$  then  $T(W)$  : subspace of  $\mathbb{R}^m$

p.f.

(i)  $\therefore$  range in codomain  $\therefore T(W)$  in  $\mathbb{R}^m$

(ii)  $\forall \vec{p}, \vec{q} \in T(W)$ ,  $r \in \mathbb{R}$ , claim:  $\begin{cases} \textcircled{1} \vec{p} + \vec{q} \in T(W) \\ \textcircled{2} r\vec{p} \in T(W) \end{cases}$

$\because \vec{p}, \vec{q} \in T(W) \therefore \exists \vec{u}, \vec{v} \in W$  s.t.  $T(\vec{u}) = \vec{p}$ ,  $T(\vec{v}) = \vec{q}$

$\leftarrow T$ : linear trans.

$\vec{p} + \vec{q} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}) \in T(W)$   $\left( \because \vec{u}, \vec{v} \in W, W: \text{subspace of } \mathbb{R}^n \right)$   
 $\therefore \vec{u} + \vec{v} \in W$

(iii)  $r\vec{p} = rT(\vec{u}) = T(r\vec{u}) \in T(W)$   $\star$   
 $\uparrow$   
 $T$ : linear trans.

Recall

$W$  = subspace of  $\mathbb{R}^m$

if  $\textcircled{1} W$  = subset of  $\mathbb{R}^m$

$\textcircled{2} \vec{u} + \vec{v} \in W, \forall \vec{u}, \vec{v} \in W$

$\textcircled{3} r\vec{v} \in W, \forall r \in \mathbb{R}$

$\star \vec{u}, \vec{v}$  不一定唯一

maybe  $T(\vec{\alpha}) = \vec{p}, T(\vec{\beta}) = \vec{p}$

ex:

given  $A_{m \times n}$ , define  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$   
 $\vec{x} \mapsto T(\vec{x}) = A\vec{x}$

$m \times n$   $n \times 1$

$$(A \vec{x}) \rightarrow m \times 1$$

check  $T$  is a linear trans

$$\begin{array}{ll} \textcircled{1} & A\vec{x} + A\vec{y} = A(\vec{x} + \vec{y}) \\ & \text{"} T(\vec{x}) + T(\vec{y}) \quad \text{"} T(\vec{x} + \vec{y}) \end{array} \quad \begin{array}{ll} \textcircled{2} & A(r\vec{x}) = r A\vec{x} \\ & \text{"} T(r\vec{x}) \quad \text{"} r T(\vec{x}) \end{array}$$

ex:

$T: \mathbb{R} \rightarrow \mathbb{R}$  : NOT linear trans.  
 $x \mapsto \sin(x)$

$$\text{check: } \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) \neq \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)$$

$$x = \frac{\pi}{4}, y = \frac{\pi}{4}$$

Thm

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  : linear trans,  $B = \{\vec{b}_1, \dots, \vec{b}_n\}$  : basis for  $\mathbb{R}^n$

$\forall \vec{p} \in T(\mathbb{R}^n)$  can be expressed by  $B' = \{T(\vec{b}_1), \dots, T(\vec{b}_n)\} \leftarrow \text{sp}(T(\vec{b}_1), \dots, T(\vec{b}_n)) = T(\mathbb{R}^n)$

p.f.

$$\because \vec{p} \in T(\mathbb{R}^n) \therefore \exists \vec{v} \text{ s.t. } \vec{p} = T(\vec{v})$$

$\therefore B$ : basis for  $\mathbb{R}^n$

$$\therefore \exists! r_1, \dots, r_n \in \mathbb{R} \text{ s.t. } r_1 \vec{b}_1 + \dots + r_n \vec{b}_n = \vec{v}$$

$$\begin{aligned} \therefore \vec{p} &= T(\vec{v}) \\ &= T(r_1 \vec{b}_1 + \dots + r_n \vec{b}_n) = T(\overbrace{r_1 \vec{b}_1 + r_2 \vec{b}_2 + \dots + r_{n-1} \vec{b}_{n-1}} + \overbrace{r_n \vec{b}_n}) = T(r_1 \vec{b}_1 + \dots + r_{n-1} \vec{b}_{n-1}) \\ &\quad + T(r_n \vec{b}_n) \\ &= T(\underbrace{r_1 \vec{b}_1 + r_2 \vec{b}_2}_{\vec{v}_2}) + T(\underbrace{r_3 \vec{b}_3}_{\vec{v}_3}) + \dots + T(r_n \vec{b}_n) \\ &= T(\underbrace{r_1 \vec{b}_1}_{\vec{v}_1}) + T(\underbrace{r_2 \vec{b}_2}_{\vec{v}_2}) + T(r_3 \vec{b}_3) + \dots + T(r_n \vec{b}_n) \\ &= T(r_1 \vec{b}_1) + T(r_2 \vec{b}_2) + T(r_3 \vec{b}_3) + \dots + T(r_n \vec{b}_n) \\ &= r_1 T(\vec{b}_1) + \dots + r_n T(\vec{b}_n) \end{aligned}$$

e.g.  $\mathbb{R}^2$ ,  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

e.g.  $\mathbb{R}^3$ ,  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Recall

Standard basis  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  for  $\mathbb{R}^n$ ,  $\vec{e}_i$ : 1 in the  $i^{\text{th}}$  position, other are 0

Def

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ : linear trans (s.m.r.)  
Let  $A$  be the standard matrix representation of  $T$

if  $A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \leftarrow m \times n$

Thm.

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ : linear trans (s.m.r.)  
Let  $A$  be the standard matrix representation of  $T$

$\Rightarrow \forall \vec{x} \in \mathbb{R}^n, T(\vec{x}) = A\vec{x}$

p.f.

$\forall \vec{x} \in \mathbb{R}^n \Rightarrow \exists! r_1, r_2, \dots, r_n \in \mathbb{R} \text{ s.t. } \vec{x} = r_1 \vec{e}_1 + r_2 \vec{e}_2 + \dots + r_n \vec{e}_n = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$

$T(\vec{x}) = r_1 T(\vec{e}_1) + r_2 T(\vec{e}_2) + \dots + r_n T(\vec{e}_n) = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = A\vec{x}$

e.g.  $\mathbb{R}^2$ ,  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  or  $[1, 0]$ ,  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  or  $[0, 1]$

ex:

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $T([x, y]) = [2x+y, 3x, 4x-y]$ , find the s.m.r. of  $T$ .

$T(\vec{e}_1) = [2, 3, 4]$ ,  $T(\vec{e}_2) = [1, 0, -1]$

$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}$ ,  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 3x \\ 4x-y \end{bmatrix}$

$\downarrow$     $\downarrow$   
 $\swarrow$  s.m.r. of  $T$

ex:  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$  : linear trans.  $T([x_1, x_2, x_3, x_4]) = [x_1 + 2x_3, -x_1 + 2x_2 - x_4, x_2 - x_3 + x_4, x_1 + x_4]$

$T(\vec{e}_1) = T([1, 0, 0, 0]) = [1, -1, 0, 1]$

$T(\vec{e}_2) = T([0, 1, 0, 0]) = [0, 2, 1, 0]$

$T(\vec{e}_3) = T([0, 0, 1, 0]) = [2, 0, -1, 0]$

$T(\vec{e}_4) = T([0, 0, 0, 1]) = [0, -1, 1, 1]$

$\therefore A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$

ex:

given  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ : linear trans.  $T(\vec{u}) = [-2, 1, 0]$ ,  $T(\vec{v}) = [5, -7, 1]$

① find the s.m.r. of  $T$ .  $\because \vec{u}, \vec{v}$ : linear indep.  $\therefore \{\vec{u}, \vec{v}\}$ : basis for  $\mathbb{R}^2$

$\uparrow$  find  $T(\vec{e}_1), T(\vec{e}_2)$  ② find  $T([3, 6])$

**sol**

$$\begin{aligned} \textcircled{1} \left[ \begin{array}{cc|c} \vec{u} & \vec{v} & \vec{e}_1 \\ -1 & 3 & 1 \\ 2 & -5 & 0 \end{array} \right] &\sim \left[ \begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array} \right] \Rightarrow \vec{e}_1 = 5\vec{u} + 2\vec{v} \\ \textcircled{2} \left[ \begin{array}{cc|c} & & \vec{e}_2 \\ -1 & 3 & 0 \\ 2 & -5 & 1 \end{array} \right] &\sim \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow \vec{e}_2 = 3\vec{u} + \vec{v} \end{aligned} \quad \Rightarrow \left[ \begin{array}{cc|cc} \vec{u} & \vec{v} & \vec{e}_1 & \vec{e}_2 \\ -1 & 3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \end{array} \right] \Rightarrow \begin{cases} \vec{e}_1 = 5\vec{u} + 2\vec{v} \\ \vec{e}_2 = 3\vec{u} + \vec{v} \end{cases}$$

$$T(\vec{e}_1) = T(5\vec{u} + 2\vec{v}) = 5T(\vec{u}) + 2T(\vec{v}) = 5[-2, 1, 0] + 2[5, -7, 1] = [0, -9, 2]$$

$$T(\vec{e}_2) = T(3\vec{u} + \vec{v}) = 3T(\vec{u}) + T(\vec{v}) = 3[-2, 1, 0] + [5, -7, 1] = [-1, -4, 1]$$

$$A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}, \quad A \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ -51 \\ 12 \end{bmatrix} \quad \therefore T([3, 6]) = [-6, -51, 12]$$



```
octave:1> A=[0 -1;-9 -4;2 1]
```

```
A =
```

```
0  -1  
-9 -4  
2   1
```

```
octave:2> x=[3;6]
```

```
x =
```

```
3  
6
```

```
octave:3> A*x
```

```
ans =
```

```
-6  
-51  
12
```

Def

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear trans .  $A$ : s.m.r. of  $T$ ,  $\mathbb{R}^n$ : domain,  $\mathbb{R}^m$ : codomain

- range of  $T$ :  $\text{range}(T) = T(\mathbb{R}^n) = \{ \underbrace{T(\vec{x})}_{A\vec{x}} \mid \vec{x} \in \mathbb{R}^n \} \Rightarrow \text{range}(T) \subseteq \mathbb{R}^m$   
"col(A)"
- kernel of  $T$ :  $\ker(T) = \{ \vec{x} \in \mathbb{R}^n \mid \underbrace{T(\vec{x})}_{A\vec{x}} = \vec{0} \} = \text{null}(A) \Rightarrow \ker(T) \subseteq \mathbb{R}^n$   
" $A\vec{x} = \vec{0}$ "
- $\text{rank}(T) = \dim(\text{range}(T))$   
"rank(A)"

Thm

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear trans .  $A$ : s.m.r. of  $T$

1.  $\text{range}(T) = \text{col}(A)$

2.  $\ker(T) = \text{null}(A)$

3.  $n = \dim(\text{col}(A)) + \dim(\text{null}(A))$   
 $= \dim(\text{range}(T)) + \dim(\ker(T))$   
 $= \text{rank}(T) + \dim(\ker(T))$

**Thm** (Composition Thm)

$T_1: \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $T_2: \mathbb{R}^m \rightarrow \mathbb{R}^p$ ,  $T_1, T_2$ : linear trans.

$A_1$ : s.m.r. of  $T_1$ ,  $A_2$ : s.m.r. of  $T_2$

$\Rightarrow \left\{ \begin{array}{l} \textcircled{1} T_2 \circ T_1 : \text{linear trans} \end{array} \right.$

$\textcircled{2} T_3 = T_2 \circ T_1$ ,  $A_3$ : s.m.r. of  $T_3 \Rightarrow A_3 = A_2 A_1$