4. The methods of Chapter 1 are far more efficient than those described in this section for actual computation of both the inverse of A and the solution of the system $A\mathbf{x} = \mathbf{b}$.
5. Let A be an $n \times n$ matrix, and let A' be its matrix of cofactors. The adjoint adj(A) is the matrix $(A')^T$ and satisfies $(\text{adj}(A))A = A(\text{adj}(A)) = (\det(A))I$, where I is the $n \times n$ identity matrix.
6. The inverse of an invertible matrix A is given by the explicit formula

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

EXERCISES

In Exercises 1–10, find the determinant of the given matrix.

1. $\begin{bmatrix} 2 & 3 & -1 \\ 5 & -7 & 1 \\ -3 & 2 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -3 & 2 \\ -1 & -1 & 1 \\ -5 & 5 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 2 & 4 & 0 \\ 2 & -3 & -1 & 2 \\ 3 & -4 & 3 & 7 \\ 1 & -1 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -5 & -1 & 7 \\ 0 & 3 & 1 & -6 \\ 2 & -5 & -1 & 8 \\ -8 & 8 & 2 & -9 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ -1 & 4 & 1 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & -3 \\ 0 & -2 & 1 & 0 & 0 \\ 5 & -3 & 2 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -4 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ -5 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 8 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 & 1 \\ -1 & 2 & 4 & 0 & 0 \\ 3 & 1 & -2 & 0 & 0 \\ 5 & 1 & 5 & 0 & 0 \end{bmatrix}$

11. The matrices in Exercises 8 and 9 have zero entries except for entries in an $r \times r$ submatrix R and a separate $s \times s$ submatrix S whose main diagonals lie on the main diagonal of the whole $n \times n$ matrix, and where $r + s = n$. Prove that, if A is such a matrix with submatrices R and S, then $\det(A) = \det(R) \cdot \det(S)$.
12. The matrix A in Exercise 10 has a structure similar to that discussed in Exercise 11, except that the square submatrices R and S lie along the other diagonal. State and prove a result similar to that in Exercise 11 for such a matrix.
13. State and prove a generalization of the result in Exercise 11, when the matrix A has zero entries except for entries in k submatrices positioned along the diagonal.

In Exercises 14–19, use the corollary to Theorem 4.6 to find A^{-1} if A is invertible.

14. $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

16. $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

17. $A = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

18. $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix}$

19. $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

20. Find the adjoint of the matrix $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix}$.

21. Find the adjoint of the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$.

22. Given that $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A^{-1}) = 3$, find the matrix A.

23. If A is a matrix with integer entries and if $\det(A) = \pm 1$, prove that A^{-1} also has the same properties.

In Exercises 24–31, solve the given system of linear equations by Cramer's rule wherever it is possible.

24. $\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 3 \end{aligned}$

25. $\begin{aligned} 2x_1 - 3x_2 &= 1 \\ -4x_1 + 6x_2 &= -2 \end{aligned}$

26. $\begin{aligned} 3x_1 + x_2 &= 5 \\ 2x_1 + x_2 &= 0 \end{aligned}$

27. $\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 2 \end{aligned}$

28. $\begin{aligned} 5x_1 - 2x_2 + x_3 &= 1 \\ x_2 + x_3 &= 0 \\ x_1 + 6x_2 - x_3 &= 4 \end{aligned}$

29. $\begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 + 5x_3 &= 1 \end{aligned}$

30. $\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$

31. $\begin{aligned} 3x_1 + 2x_2 - x_3 &= 1 \\ x_1 - 4x_2 + x_3 &= -2 \\ 5x_1 + 2x_2 &= 1 \end{aligned}$

In Exercises 32 and 33, find the component x_2 of the solution vector for the given linear system.

32. $\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= 1 \\ 2x_1 + x_2 + 2x_4 &= 0 \\ x_2 - 6x_3 - x_4 &= 5 \\ 3x_1 + x_2 + x_4 &= 1 \end{aligned}$

33. $\begin{aligned} 6x_1 + x_2 - x_3 &= 4 \\ x_1 - x_2 + 5x_4 &= -2 \\ -x_1 + 3x_2 + x_3 &= 2 \\ x_1 + x_2 - x_3 + 2x_4 &= 0 \end{aligned}$

34. Find the unique solution (assuming that it exists) of the system of equations represented by the partitioned matrix

$$\left[\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & 3b_1 \\ a_2 & b_2 & c_2 & d_2 & 3b_2 \\ a_3 & b_3 & c_3 & d_3 & 3b_3 \\ a_4 & b_4 & c_4 & d_4 & 3b_4 \end{array}\right].$$

35. Let A be a square matrix. Mark each of the following True or False.

___ a. The determinant of a square matrix is the product of the entries on its main diagonal.

___ b. The determinant of an upper-triangular square matrix is the product of the entries on its main diagonal.

___ c. The determinant of a lower-triangular square matrix is the product of the entries on its main diagonal.

___ d. A square matrix is nonsingular if and only if its determinant is positive.

___ e. The column vectors of an $n \times n$ matrix are independent if and only if the determinant of the matrix is nonzero.

___ f. A homogeneous square linear system has a nontrivial solution if and only if the determinant of its coefficient matrix is zero.

___ g. The product of a square matrix and its adjoint is the identity matrix.

___ h. The product of a square matrix and its adjoint is equal to some scalar times the identity matrix.

___ i. The transpose of the adjoint of A is the matrix of cofactors of A.

___ j. The formula $A^{-1} = (1/\det(A))\text{adj}(A)$ is of practical use in computing the inverse of a large nonsingular matrix.

36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.

37. Prove that a square matrix is invertible if and only if its adjoint is an invertible matrix.

38. Let A be an $n \times n$ matrix. Prove that $\det(\text{adj}(A)) = \det(A)^{n-1}$.

39. Let A be an invertible $n \times n$ matrix with $n > 1$. Using Exercises 37 and 38, prove that $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2}A$.

The routine YUREDUCE in LINTEK has a menu option D that will compute and display the product of the diagonal elements of a square matrix. The routine MATCOMP has a menu option D to compute a determinant. Use YUREDUCE or MATLAB to compute the determinant of the matrices in Exercises 40–42. Write down your results. If you used YUREDUCE, use MATCOMP to compute the determinants of the same matrices again and compare the answers.

40. $\begin{bmatrix} 11 & -9 & 28 \\ 32 & -24 & 21 \\ 10 & 13 & -19 \end{bmatrix}$

41. $\begin{bmatrix} 13 & -15 & 33 \\ -15 & 25 & 40 \\ 12 & -33 & 27 \end{bmatrix}$

42. $\begin{bmatrix} 7.6 & 2.8 & -3.9 & 19.3 & 25.0 \\ -33.2 & 11.4 & 13.2 & 22.4 & 18.3 \\ 21.4 & -32.1 & 45.7 & -8.9 & 12.5 \\ 17.4 & 11.0 & -6.8 & 20.3 & -35.1 \\ 22.7 & 11.9 & 33.2 & 2.5 & 7.8 \end{bmatrix}$

43. MATCOMP computes determinants in essentially the way described in this section. The matrix

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

has determinant 1, so every power of it should have determinant 1. Use MATCOMP with single-precision printing and with the default roundoff control ratio r. Start computing determinants of powers of A. Find the smallest positive integer m such that $\det(A^m) \neq 1$, according to MATCOMP. How bad is the error? What does MATCOMP give for $\det(A^{20})$? At what integer exponent does the break occur between the incorrect value 0 and incorrect values of large magnitude?

Repeat the above, taking zero for roundoff control ratio r. Try to explain why the results are different and what is happening in each case.

44. Using MATLAB, find the smallest positive integer m such that $\det(A^m) \neq 1$, according to MATLAB, for the matrix A in Exercise 43.

In Exercises 45–47, use MATCOMP in LINTEK or MATLAB and the corollary of Theorem 4.6 to find the matrix of cofactors of the given matrix.

45. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 0 \\ 3 & 1 & 4 \end{bmatrix}$

46. $\begin{bmatrix} -52 & 31 & 47 \\ 21 & -11 & 28 \\ 43 & -71 & 87 \end{bmatrix}$

47. $\begin{bmatrix} 6 & -3 & 2 & 14 \\ -3 & 7 & 8 & 1 \\ 4 & 9 & -5 & 3 \\ -8 & -40 & 47 & 29 \end{bmatrix}$

[Hint: Entries in the matrix of cofactors are integers. The cofactors of a matrix are continuous functions of its entries; that is, changing an entry by a very slight amount will change a cofactor only slightly. Change some entry just a bit to make the determinant nonzero.]

4.4 LINEAR TRANSFORMATIONS AND DETERMINANTS (OPTIONAL)

We continue our program of exhibiting the relationship between matrices and linear transformations. Associated with an $m \times n$ matrix A is the linear transformation T mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, where $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. If $n = m$, so