EXERCISES

In Exercises 1–6, determine whether the given matrix is a Jordan canonical form.

1. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

5. $\begin{bmatrix} i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

In Exercises 7–10:

a) Find the eigenvalues of the given matrix J.

b) Give the rank and nullity of $(J - \lambda)^k$ for each eigenvalue λ of J and for every positive integer k.

c) Draw schemata of the strings of vectors in the standard basis arising from the Jordan blocks in J.

d) For each standard basis vector $\mathbf{e}_k$, express $J\mathbf{e}_k$ as a linear combination of vectors in the standard basis.

7. $\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

8. $\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$

9. $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

10. $\begin{bmatrix} i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 11–14, find a Jordan canonical form for A from the given data.

11. A is 5×5, $A - 3I$ has nullity 2, $(A - 3I)^2$ has nullity 3, $(A - 3I)^3$ has nullity 4, $(A - 3I)^k$ has nullity 5 for $k \geq 4$.

12. A is 7×7, $A + I$ has nullity 3, $(A + I)^k$ has nullity 5 for $k \geq 2$; $A + iI$ has nullity 1, $(A + iI)^j$ has nullity 2 for $j \geq 2$.

13. A is 8×8, $A - I$ has nullity 2, $(A - I)^2$ has nullity 4, $(A - I)^k$ has nullity 5 for $k \geq 3$; $(A + 2I)^j$ has nullity 3 for $j \geq 1$.

14. A is 8×8; $A + iI$ has rank 4, $(A + iI)^2$ has rank 2, $(A + iI)^3$ has rank 1, $(A + iI)^k = O$ for $k \geq 4$.

In Exercises 15–22, find a Jordan canonical form and a Jordan basis for the given matrix.

15. $\begin{bmatrix} -10 & 4 \\ -25 & 10 \end{bmatrix}$

16. $\begin{bmatrix} 5 & -4 \\ 9 & -7 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 0 & 1 \\ 2 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

20. $\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

23. Mark each of the following True or False.

___ a. Every Jordan block matrix has just one eigenvalue.
___ b. Every matrix having a unique eigenvalue is a Jordan block.
___ c. Every diagonal matrix is a Jordan canonical form.
___ d. Every square matrix is similar to a Jordan canonical form.
___ e. Every square matrix is similar to a unique Jordan canonical form.
___ f. Every 1×1 matrix is similar to a unique Jordan canonical form.
___ g. There is a Jordan basis for every square matrix A.
___ h. There is a unique Jordan basis for every square matrix A.
___ i. Every 3×3 diagonalizable matrix is similar to exactly six Jordan canonical forms.
___ j. Every 3×3 matrix is similar to exactly six Jordan canonical forms.

24. Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Compute A^2, A^3, and A^4.

25. Let A be an $n \times n$ upper-triangular matrix with all diagonal entries 0. Compute A^m for all positive integers $m \geq n$. (See Exercise 24.) Prove that your answer is correct.

26. Let $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. Compute $(A - 2I)^3(A - 3I)^2$.

27. Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. Compute $(A - 2I)^2(A - 3I)^2$. Compare with Exercise 26.

28. Let $A = \begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. Find a polynomial in A (that is, a sum of terms a_jA^j with a term a_0I) that gives the zero matrix. (See Exercises 24–27.)

29. Repeat Exercise 28 for the matrix $A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & i \end{bmatrix}$.

30. The Cayley–Hamilton theorem states that, if $p(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ is the characteristic polynomial of a matrix A, then $p(A) = a_nA^n + \cdots + a_1A + a_0I = O$, the zero matrix. Prove it. [Hint: Consider $(A - \lambda_iI)^{n_i}\mathbf{b}$, where $\mathbf{b}$ is a vector in a Jordan basis corresponding to λ_i.] In view of Exercises 24–29, explain why you expect $p(J)$ to be O, where J is a Jordan canonical form for A. Deduce that $p(A) = O$.

31. Let $T: \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. A subspace W of $\mathbb{C}^n$ is **invariant under** T if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. Let A be the standard matrix representation of T.
   a. Describe the one-dimensional invariant subspaces of T.
   b. Show that every eigenspace E_λ of T is invariant under T.
   c. Show that the vectors in any string in a Jordan basis for A generate an invariant subspace of T.
   d. Is it true that, if S is a subspace of a subspace W that is invariant under T, then S is also invariant under T? If not, give a counterexample.
   e. Is it true that every subspace of $\mathbb{R}^n$ invariant under T is contained in the nullspace of $(A - \lambda I)^n$, where λ is some eigenvalue of T? If not, give a counterexample.

2. In Section 5.3, we considered systems $\mathbf{x}' = A\mathbf{x}$ of differential equations, and we saw that, if $A = CJC^{-1}$, then the system takes the form $\mathbf{y}' = J\mathbf{y}$, where $\mathbf{x} = C\mathbf{y}$. (We used D in place of J in Section 5.3, because we were concerned only with diagonalization.) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the (not necessarily distinct) eigenvalues of an $n \times n$ matrix A, and let J be a Jordan canonical form for A.
   a. Show that the system $\mathbf{y}' = J\mathbf{y}$ is of the form

$$\begin{aligned} y_1' &= \lambda_1 y_1 + c_1 y_2, \\ y_2' &= \lambda_2 y_2 + c_2 y_3, \\ &\vdots \\ y_{n-1}' &= \lambda_{n-1} y_{n-1} + c_{n-1} y_n, \\ y_n' &= \lambda_n y_n, \end{aligned}$$

   where each c_i is either 0 or 1.
   b. How can the system in part a be solved? [HINT: Start with the last equation.]
   c. Given that, for

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 0 & -\frac{5}{4} & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

   we have $C^{-1}AC = J$, find the solution of the differential system $\mathbf{x}' = A\mathbf{x}$.

33. Let A be an $n \times n$ matrix with eigenvalue λ. Prove that the algebraic multiplicity of λ is at least as large as its geometric multiplicity.