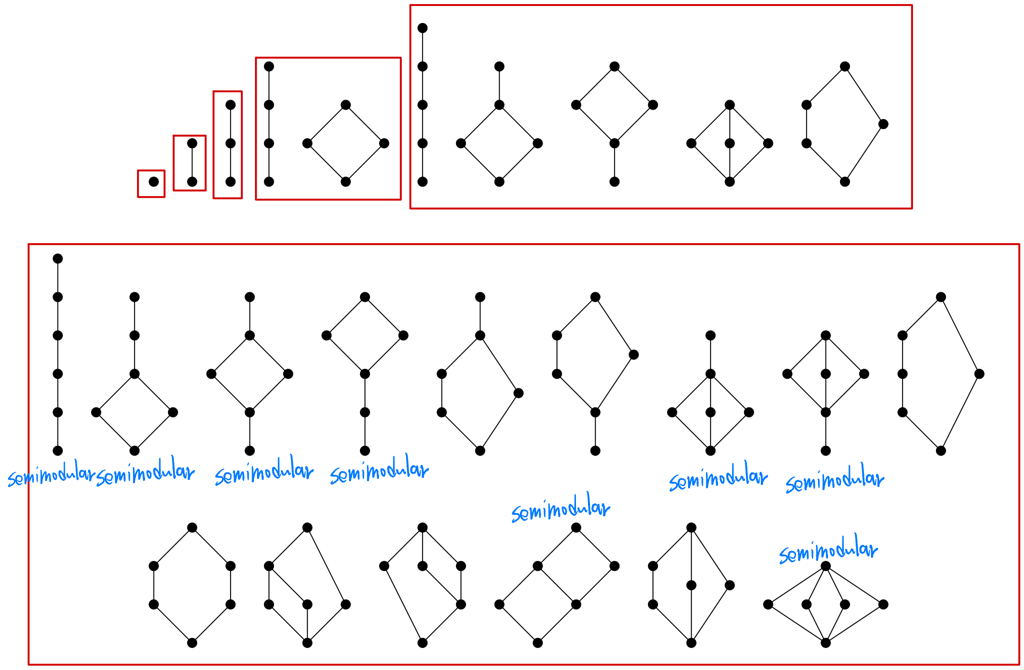


Figure 3.5: The lattices with at most six elements

3.3 Lattices

→ complete / bounded / isomorphic / distributive / complemented.

柘諺

簡短的調查 U(s,t).

We now turn to a brief survey of an important class of posets known as *lattices*. If s and t belong to a poset P, then an *upper bound* of s and t is an element $u \in P$ satisfying $u \geq s$ and $u \geq t$. A *least upper bound* (or *join* or *supremum*) of s and t is an upper bound u of s and t such that every upper bound v of s and t satisfies $v \geq u$. If a least upper bound of s and t exists, then it is clearly unique and is denoted $s \vee t$ (read "s join t" or "s sup t"). Dually one can define the *greatest lower bound* (or *meet* or *infimum*) $s \wedge t$ (read "s meet t" or "s inf t"), when it exists. A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound. One can also define a lattice axiomatically in terms of the operations $\vee$ and $\wedge$, but for combinatorial purposes this is not necessary. The reader should check, however, that in a lattice L:

合流 聚會

☆ Note: $\vee$, $\wedge$ are not partial ordered.

☆ prop.

a. the operations $\vee$ and $\wedge$ are associative, commutative, and idempotent (i.e., $t \wedge t = t \vee t = t$);

吸收律.

b. $s \wedge (s \vee t) = s = s \vee (s \wedge t)$ (absorption laws);

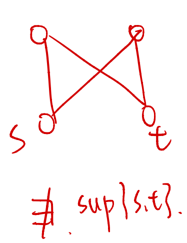
☆ $\hat{0} = \wedge s$ (sep), $\hat{1} = \vee t$ (tep).

c. $s \wedge t = s \Leftrightarrow s \vee t = t \Leftrightarrow s \leq t$.

consider $\hat{\ } \otimes \mathbb{N}^*$

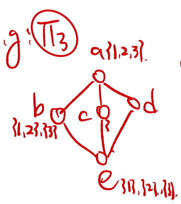
Clearly all finite lattices have a $\hat{0}$ and $\hat{1}$. If L and M are lattices, then so are L^*, $L \times M$, and $L \oplus M$. However, $L + M$ will never be a lattice unless one of L or M is empty, but $\widehat{L+M}$ (i.e., $L + M$ with an $\hat{0}$ and $\hat{1}$ adjoined) is always a lattice. Figure 3.5 shows the Hasse diagrams of all lattices with at most six elements.

e.g:



∄ sup{s,t}.

e.g: (T_3)



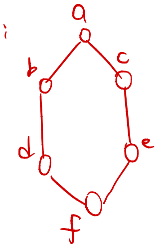
① $b \vee (c \wedge d) = b \vee e = b$

② $(b \vee c) \wedge (b \vee d) = a \wedge a = a$

$\Rightarrow$ ① ≠ ②

not distributive.

e.g:



① $(d \vee e) \wedge f = a \wedge f = f$

② $d \vee (e \wedge f) = d \vee f = d$

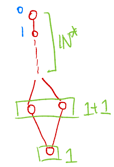
① ≠ ② → ∄ mixed associative

In checking whether a (finite) poset is a lattice, it is sometimes easy to see that meets, say, exist, but the existence of joins is not so clear. Thus the criterion of the next proposition can be useful. If every pair of elements of a poset P has a meet (respectively, join), then we say that P is a *meet-semilattice* (respectively, *join-semilattice*).

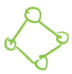
3.3.1 Proposition. *Let P be a finite meet-semilattice with $\hat{1}$. Then P is a lattice. (Of course, dually a finite join-semilattice with $\hat{0}$ is a lattice.)*

explain why sup always exists ?

Proof. If $s, t \in P$, then the set $S = \{u \in P : u \geq s \text{ and } u \geq t\}$ is finite (since P is finite) and nonempty (since $\hat{1} \in P$). Clearly by induction the meet of finitely many elements of a meet-semilattice exists. Hence we have $s \vee t = \bigwedge_{u \in S} u$. $\square$



結構.



$1 \oplus (1+1) \oplus \mathbb{N}^*$

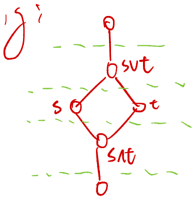
Proposition 3.3.1 fails for infinite lattices because an *arbitrary* subset of L need not have a meet or join. (See Exercise 3.26.) If in fact every subset of L does have a meet and join, then L is called a *complete lattice*. Clearly a complete lattice has a $\hat{0}$ and $\hat{1}$.

We now consider one of the types of lattices of most interest to combinatorics.

亮鈞.

3.3.2 Proposition. *Let L be a finite lattice. The following two condtions are equivalent.*

e.g.:



i. *L is graded, and the rank function ρ of L satisfies*

$\rho(s)+\rho(t) \leq \rho(s \wedge t)+\rho(s \vee t)$

$\Rightarrow \rho(t)-1 \leq \rho(s \wedge t) < \rho(t)$

$\Rightarrow \rho(s \wedge t) = \rho(t)-1.$

$$\rho(s)+\rho(t) \geq \rho(s \wedge t)+\rho(s \vee t)$$

$\Rightarrow \rho(t)+1 \geq \rho(s \vee t) > \rho(t)$

$\Rightarrow \rho(s \vee t) = \rho(t)+1.$

for all $s, t \in L$.

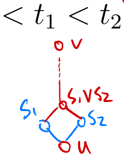
ii. *If s and t both cover $s \wedge t$, then $s \vee t$ covers both s and t.*

Proof. (i)⇒(ii) Suppose that s and t cover $s \wedge t$. Then $\rho(s) = \rho(t) = \rho(s \wedge t) + 1$ and $\rho(s \vee t) > \rho(s) = \rho(t)$. Hence by (i), $\rho(s \vee t) = \rho(s) + 1 = \rho(t) + 1$, so $s \vee t$ covers both s and t.

$\because$ s, t 已經 cover $s \wedge t$, 不會有 $\rho(s \vee t) = \rho(s)$ 的情況發生

prove L is graded

(ii)⇒(i) Suppose that L is not graded, and let $[u, v]$ be an interval of L of minimal length that is not graded. Then there are elements s_1, s_2 of $[u, v]$ that cover u and such that all maximal chains of each interval $[s_i, v]$ have the same length ℓ_i, where $\ell_1 \neq \ell_2$. By (ii), there are saturated chains in $[s_i, v]$ of the form $s_i < s_1 \vee s_2 < t_1 < t_2 < \cdots < t_k = v$, contradicting $\ell_1 \neq \ell_2$. Hence L is graded.



need to be graded.

if "=", then $[u,v]$ is graded.

prove the inequality.

Now suppose that there is a pair $s, t \in L$ with

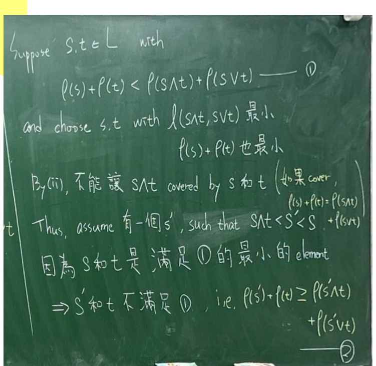
$$\rho(s) + \rho(t) < \rho(s \wedge t) + \rho(s \vee t), \tag{3.4}$$

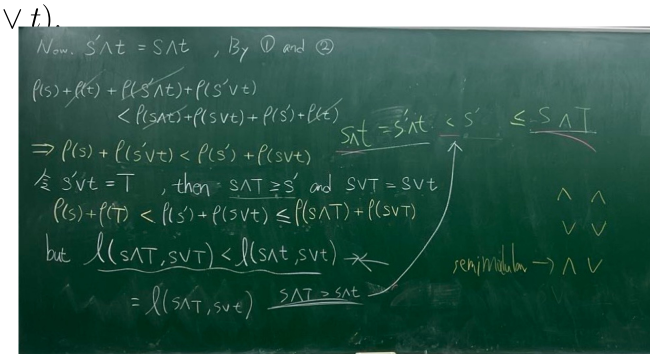
and choose such a pair with $\ell(s \wedge t, s \vee t)$ minimal, and then with $\rho(s) + \rho(t)$ minimal. By (ii), we cannot have both s and t covering $s \wedge t$. Thus assume that $s \wedge t < s' < s$, say. By the minimality of $\ell(s \wedge t, s \vee t)$ and $\rho(s) + \rho(t)$, we have → if yes, trivial.

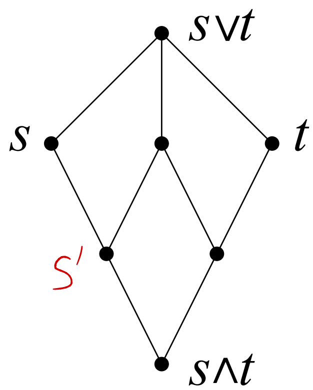
$$\rho(s') + \rho(t) \geq \rho(s' \wedge t) + \rho(s' \vee t). \tag{3.5}$$

Now $s' \wedge t = s \wedge t$, so equations (3.4) and (3.5) imply

$$\rho(s) + \rho(s' \vee t) < \rho(s') + \rho(s \vee t).$$







$\Rightarrow$

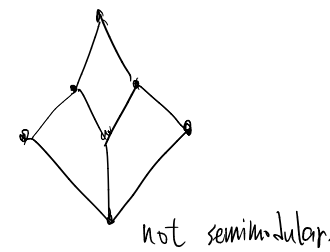


Figure 3.6: A semimodular but nonmodular lattice

Clearly $s \wedge (s' \vee t) \geq s'$ and $s \vee (s' \vee t) = s \vee t$. Hence setting $S = s$ and $T = s' \vee t$, we have found a pair $S, T \in L$ satisfying $\rho(S) + \rho(T) < \rho(S \wedge T) + \rho(S \vee T)$ and $\ell(S \wedge T, S \vee T) < \ell(s \wedge t, s \vee t)$, a contradiction. This completes the proof. □

assume it's minimal.

山白比良

半模组化.

A finite lattice satisfying either of the (equivalent) conditions of the previous proposition is called a *finite upper semimodular lattice*, or a just a *finite semimodular lattice*. The reader may check that of the 15 lattices with six elements, exactly eight are semimodular.

A finite lattice L whose dual L^* is semimodular is called *lower semimodular*. A finite lattice that is both upper and lower semimodular is called a *modular lattice*. By Proposition 3.3.2, a finite lattice L is modular if and only if it is graded, and its rank function ρ satisfies

$$\rho(s) + \rho(t) = \rho(s \wedge t) + \rho(s \vee t) \quad \text{for all } s, t \in L. \tag{3.6}$$

For instance, the lattice $B_n(q)$ of subspaces (ordered by inclusion) of an n-dimensional vector space over the field $\mathbb{F}_q$ is modular, since the rank of a subspace is just its dimension, and equation (3.6) is then familiar from linear algebra. Every semimodular lattice with at most six elements is modular. There is a unique seven-element non-modular, semimodular lattice, which is shown in Figure 3.6. This lattice is not modular since $s \vee t$ covers s and t, but s and t don't cover $s \wedge t$. It can be shown that a finite lattice L is modular if and only if for all $s, t, u \in L$ such that $s \leq u$, we have

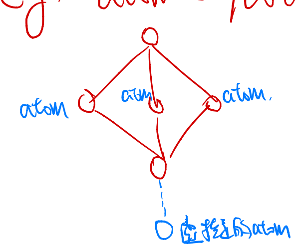
$$s \vee (t \wedge u) = (s \vee t) \wedge u. \tag{3.7}$$

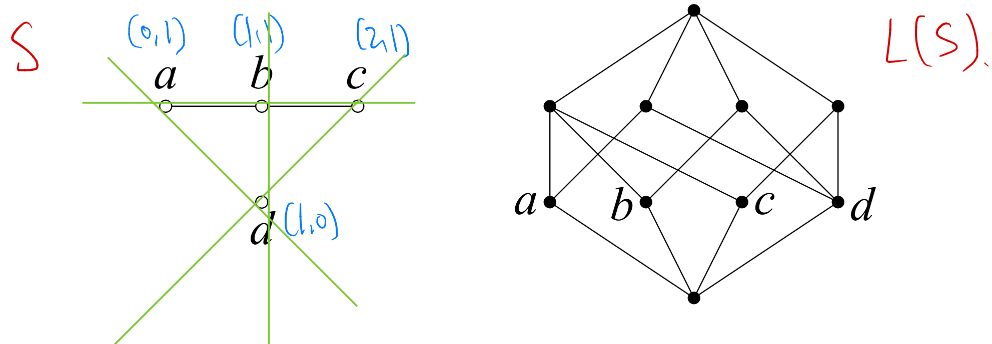
This allows the concept of modularity to be extended to nonfinite lattices, though we will only be concerned with the finite case. Equation (3.7) also shows immediately that a sublattice of a modular lattice is modular. (A subset M of a lattice L is a *sublattice* if it is closed under the operations of $\wedge$ and $\vee$ in L.)

補元的.

A lattice L with $\hat{0}$ and $\hat{1}$ is *complemented* if for all $s \in L$ there is a $t \in L$ such that $s \wedge t = \hat{0}$ and $s \vee t = \hat{1}$. If for all $s \in L$ the complement t is unique, then L is *uniquely complemented*. If every interval $[s, t]$ of L is itself complemented, then L is *relatively complemented*. An *atom* 原子. of a finite lattice L is an element covering $\hat{0}$, and L is said to be *atomic* (or a *point lattice*) if every element of L is a join of atoms. (We always regard $\hat{0}$ as the join of an empty set of

eg: atomic / coatomic. → B_2





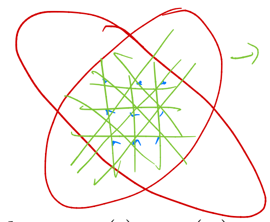
(a) (b)

Figure 3.7: A subset S of the affine plane and its corresponding geometric lattice $L(S)$

atoms.) Dually, a *coatom* is an element that $\hat{1}$ covers, and a *coatomic* lattice is defined in the obvious way. Another simple result of lattice theory, whose proof we omit, is the following.

3.3.3 Proposition. *Let L be a finite semimodular lattice. The following two conditions are equivalent.*

↳ s,t cover s∧t ⇒ s∨t cover s&t.



- ***i.*** *L is relatively complemented.*
- ***ii.*** *L is atomic.*

L(S) = S choice in V(K): vector space

A finite semimodular lattice satisfying either of the two (equivalent) conditions (i) or (ii) above is called a *finite geometric lattice*. A basic example is the following. Take any finite set S of points in some affine space (respectively, vector space) V over a field K (or even over a division ring). Then the subsets of S of the form $S \cap W$, where W is an affine subspace (respectively, linear subspace) of V, ordered by inclusion, form a geometric lattice $L(S)$. For instance, taking $S \subset \mathbb{R}^2$ (regarded as an affine space) to be as in Figure 3.7(a), then the elements of $L(S)$ consist of $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, c, d\}$. For this example, $L(S)$ is in fact modular and is shown in Figure 3.7(b).

dim 0 dim 1 dim 2

密切相关 广播阵

NOTE. A geometric lattice is intimately related to the subject of *matroid theory*. A (finite) *matroid* may be defined as a pair $(S, \mathcal{I})$, where S is a finite set and $\mathcal{I}$ is a collection of subsets of S satisfying the two conditions:

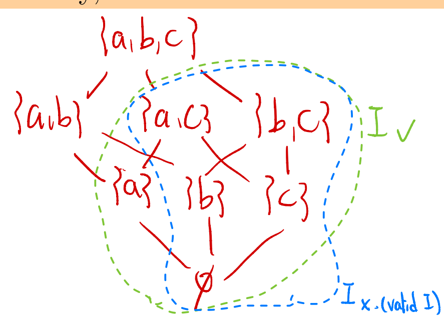
refinement 的感觉 ⇒ F = G ∪̇ U ⇒ G, U ∈ I.

- If $F \in \mathcal{I}$ and $G \subseteq F$, then $G \in \mathcal{I}$. In other words, $\mathcal{I}$ is an order ideal of the boolean algebra B_S of all subsets of S (defined in Section 3.4).
- For any $T \subseteq S$, let $\mathcal{I}_T$ be the restriction of $\mathcal{I}$ to T, i.e., $\mathcal{I}_T = \{F \in \mathcal{I} : F \subseteq T\}$. Then all maximal (under inclusion) elements of $\mathcal{I}_T$ have the same number of elements.

(basis) 抽象概念

(There are several equivalent definitions of a matroid.) The elements of $\mathcal{I}$ are called *independent* sets. They are an abstraction of linear independent sets of a vector space or affinely independent subsets of an affine space. Indeed, if S is a finite subset of a vector space (respectively, affine subset of an affine space) and $\mathcal{I}$ is the collection of linearly independent

e.g: S = B₃



e.g: S = ℝ₂ {(0,1),(0,2)} not linear independent

I = { {one elt}, {two elt} }.

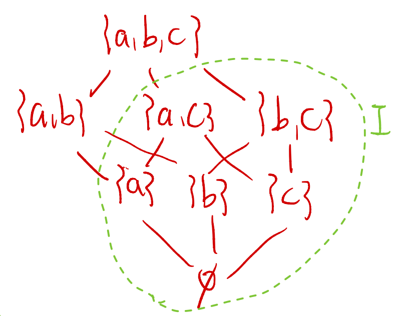
(respectively, affinely independent) subsets of S, then $(S,\mathcal{I})$ is a matroid. A matroid is *simple* if every two-element subset of $\mathcal{I}$ is independent. Every matroid can be "simplified" (converted to a simple matroid) by removing all elements of S not contained in any independent set and by identifying any two points that are not independent. It is not hard to see that matroids on a set S are in bijection with geometric lattices L whose set of atoms is S, where a set $T \subseteq S$ is independent if and only if its join in L has rank $\#T$.

$(S,I) \cong L(S)$

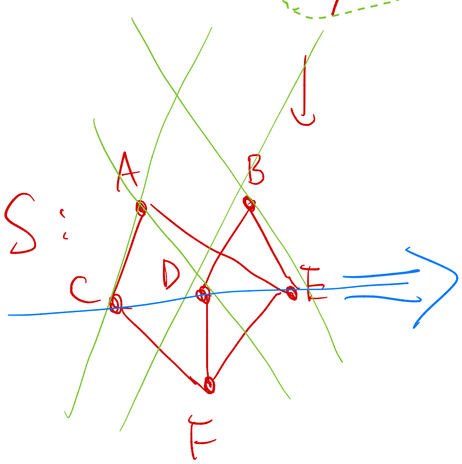
The reader may wish to verify the (partly redundant) entries of the following table concerning the posets of Example 3.1.1.

Poset P	**Properties that P possesses**	**Properties that P lacks (n large)**
$\boldsymbol{n}$	modular lattice	complemented, atomic, coatomic, geometric
B_n	modular lattice, relatively complemented, uniquely complemented, atomic, coatomic, geometric	
D_n	modular lattice	complemented, atomic coatomic, geometric (unless n is squarefree, in which case $D_n \cong B_k$)
Π_n	geometric lattice	modular
$B_n(q)$	modular lattice, complemented, atomic, coatomic, geometric	uniquely complemented

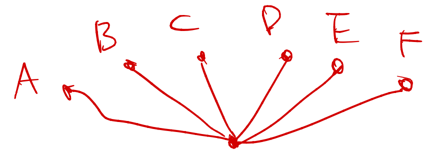
e.g.: $S = B_3$



S :

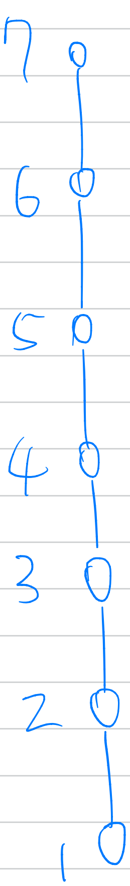


L(S):

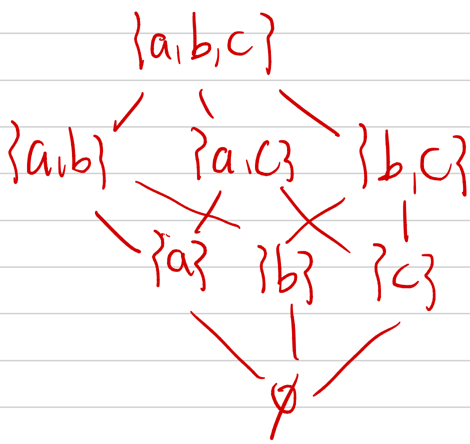


Poset P	Properties that P possesses	Properties that P lacks (n large)
$\boldsymbol{n}$	modular lattice	complemented, atomic, coatomic, geometric
B_n	modular lattice, relatively complemented, uniquely complemented, atomic, coatomic, geometric	
D_n	modular lattice	complemented, atomic coatomic, geometric (unless n is squarefree, in which case $D_n \cong B_k$)
Π_n	geometric lattice	modular
$B_n(q)$	modular lattice, complemented, atomic, coatomic, geometric	uniquely complemented

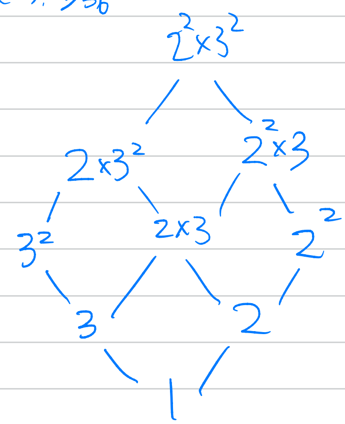
(a). $\boldsymbol{7}$



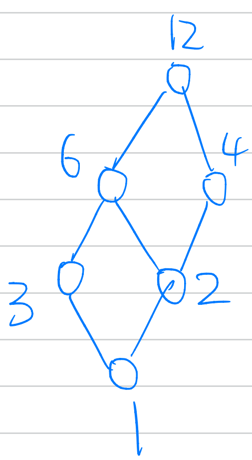
(b). B_3



(c). D_{36}

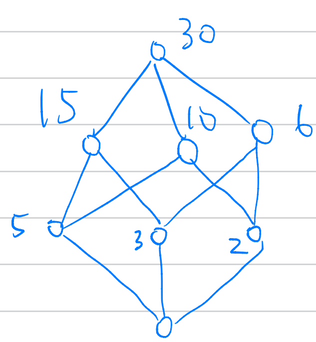


(c). D_{12}



用refinement 想.

(c). $D_{30=2\times3\times5} \cong B_3$



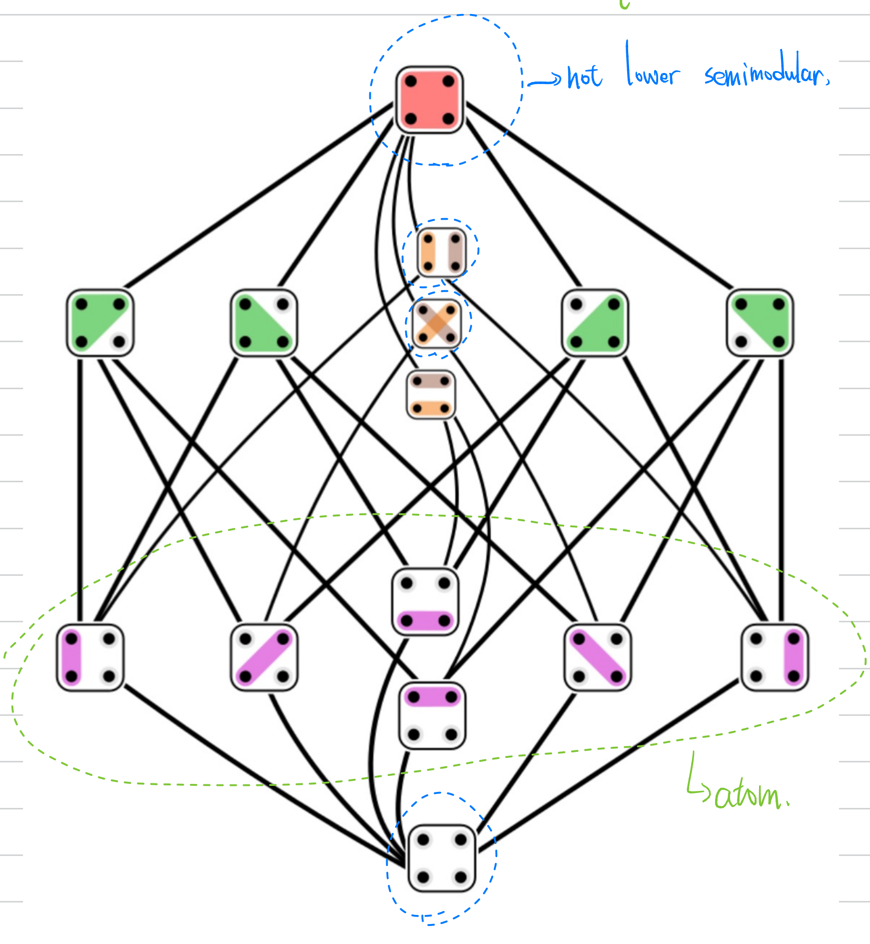
Note: $\boldsymbol{7} = \boldsymbol{7}^*$

Poset P	Properties that P possesses	Properties that P lacks (n large)
$\boldsymbol{n}$	modular lattice	complemented, atomic, coatomic, geometric
B_n	modular lattice, relatively complemented, uniquely complemented, atomic, coatomic, geometric	
D_n	modular lattice	complemented, atomic coatomic, geometric (unless n is squarefree, in which case $D_n \cong B_k$)
Π_n	have semimodular, atomic. geometric lattice	modular
$B_n(q)$	modular lattice, complemented, atomic, coatomic, geometric	uniquely complemented

(e). $\mathbb{F}_q^2 \to \{(1,0)\} \to \{(1,0),\underbrace{(0,1)}_{t}\}$

$\{(0,1)\} \to \{(0,1),\underbrace{(1,1)}_{t}\}$

(d) Π_4



Lattice of Partitions of a 4-Element Set – Tilman Piesk