9-3

## Schur's Lemma

$A$: $n \times n$ (complex) matrix.

$\Rightarrow \exists\, U$: unitary, $R$: upper triangular

s.t. $R = U^{-1} A U$

pf.

用歸納 on $n$,

① $n=1$ ✓

$A = [a]$, $U = [1]$ $\Rightarrow [1]^{-1}[a][1] = [a] \Leftarrow$ upper trian.

② $n = k-1$ ✓

③ $n = k$

(i) $\exists\, \lambda_1 \in \mathbb{C}$, $\vec{v}_1 \neq \vec{0}$ s.t. $A\vec{v}_1 = \lambda_1 \vec{v}_1$ $\nearrow$ $\{\vec{v}_1, \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

(ii) Use G-S process, extend $\{\vec{v}_1\}$ to $\{\vec{v}_1, \ldots, \vec{v}_n\}$

where $\{\vec{v}_1, \ldots, \vec{v}_n\}$: orthornormal basis for $\mathbb{C}$

(iii) let $U_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$, then $U$: unitary

$$U_1^* A U_1 = U_1^* A \begin{bmatrix} \vec v_1 & \vec v_2 & \cdots & \vec v_n \end{bmatrix} = U_1^* \begin{bmatrix} A\vec v_1 & A\vec v_2 & \cdots & A\vec v_n \end{bmatrix}$$

$$= U_1^* \begin{bmatrix} \lambda_1 \vec v_1 & A\vec v_2 & \cdots & A\vec v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 U_1^* \vec v_1 & | & | & | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & | & | & | & | \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

$$\because U_1^* \vec v_1 = \begin{bmatrix} \vec v_1^* \\ \vec v_2^* \\ \vdots \\ \vec v_n^* \end{bmatrix} \begin{bmatrix} \vec v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\therefore U_1^* A U_1 = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & \tilde A & \\ \vdots & & & \\ 0 & & & \end{array}\right], \quad \text{note: } \tilde A : (n-1)\times(n-1)$$

$\therefore$ by 數歸, $\exists\, C$ : unitary, $\tilde R = \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$, s.t. $C^* \tilde A C = \tilde R$

let $U_2 = \left[\begin{array}{c|c} 1 & \vec 0 \\ \hline \vec 0 & C \end{array}\right]$, $U = U_1 U_2$

then $U^* A U = U_2^* U_1^* A U_1 U_2 = U_2^* \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & \tilde A & \\ \vdots & & & \\ 0 & & & \end{array}\right] U_2$

$$= \left[\begin{array}{c|c} 1 & \vec 0 \\ \hline \vec 0 & C^* \end{array}\right] \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & \tilde A & \\ \vdots & & & \\ 0 & & & \end{array}\right] \left[\begin{array}{c|c} 1 & \vec 0 \\ \hline \vec 0 & C \end{array}\right] = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & C^*\tilde A & \\ \vdots & & & \\ 0 & & & \end{array}\right] \left[\begin{array}{c|c} 1 & \vec 0 \\ \hline \vec 0 & C \end{array}\right]$$

$$= \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & C^*\tilde A C & \\ \vdots & & & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & \tilde R & \\ \vdots & & & \\ 0 & & & \end{array}\right] = R : \text{upper triang.}$$

※Q.E.D.

# Recall

## Schur's Lemma

$A : n\times n \Rightarrow \exists\, U$: unitary s.t. $U^{-1}AU = \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$

**Thm**

$A$: hermitian $\Rightarrow \exists\, U$: unitary s.t. $U^{-1}AU = \begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix}$

Moreover, all eigenvalue of $A$ are real.

**Cor**

$A$: real symmetric $\Rightarrow \exists\, C$: orthogonal s.t. $C^{-1}AC = \begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix}$

**Thm**

$A$: hermitian has $\begin{cases} A\vec{v}_1 = \lambda_1 \vec{v}_1 \\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \end{cases}$, $\vec{v}_1, \vec{v}_2 \neq 0$

if $\lambda_1 \neq \lambda_2 \Rightarrow \vec{v}_1 \perp \vec{v}_2$

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**ex:** $A = \begin{bmatrix} -1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix}$, find unitary diag

$A^* = A \quad \therefore A$: hermitian

**sol.**

$P(A) = |A - \lambda I| = (1-\lambda)(\lambda^2 - 4) \Rightarrow \lambda = 1, +2, -2$

$\lambda_1 = 1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1+i \\ -i \end{bmatrix}$, $\lambda_2 = 2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1+i \\ 1-i \\ 2 \end{bmatrix}$, $\lambda_3 = -2 \Rightarrow \vec{v}_3 = \begin{bmatrix} -3-3i \\ 1-i \\ 2 \end{bmatrix}$

$\because \begin{cases} A: \text{hermitian} \\ \lambda_1, \lambda_2, \lambda_3 \text{ are distinct} \end{cases} \quad \therefore \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$: orthogonal

$\therefore$ let $\vec{q}_i = \vec{v}_i / \|\vec{v}_i\|$, $i = 1, 2, 3$, then $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$: orthonormal let $C = \begin{bmatrix} \vec{q}_{b_1} & \vec{q}_{b_2} & \vec{q}_{b_3} \end{bmatrix}$ , $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$

**Def**

$A: n\times n$ , normal

if $AA^* = A^*A$

$\forall A: n\times n$

$\triangle$ $(AA^*)^* = (A^*)^* A^* = AA^*$ : hermitian

$\triangle$ $(A^*A)^* = A^*A$ : hermitian

**Thm**

$A: n\times n$ ,

$A$: unitary diagonalizable iff $A$: normal

**ex:** $A = \begin{bmatrix} i & a \\ 2 & i \end{bmatrix}$ , 求 $a$ s.t $A$: unitary diagonalizable

**sol.**

by Thm , $A$: normal

i.e. $AA^* = A^*A$

$$\begin{bmatrix} i & a \\ 2 & i \end{bmatrix} \begin{bmatrix} -i & 2 \\ \bar{a} & -i \end{bmatrix} = \begin{bmatrix} -i & 2 \\ \bar{a} & -i \end{bmatrix} \begin{bmatrix} i & a \\ 2 & i \end{bmatrix}$$

$$\begin{bmatrix} \underline{1+|a|^2} & \underline{2i - ai} \\ \underline{-2i + \bar{a}i} & \underline{4+1} \end{bmatrix} = \begin{bmatrix} \underline{1+4} & \underline{2i - ai} \\ \underline{-2i + \bar{a}i} & \underline{|a|^2+1} \end{bmatrix} \Rightarrow 1+|a|^2 = 5$$

$\therefore |a|^2 = 4$

$\therefore |a| = 2$

$\therefore \forall a$ s.t. $|a| = 2 \Rightarrow A$: unitary diagonalizable

**Thm** $A : n\times n$

$A$: unitary diagonalizable iff $A$: normal.

**p.f. of Thm**

($\Rightarrow$) 「$A$: unitary diagonalizable $\Rightarrow$ $A$: normal」

(i) claim : $D$: diag. $= \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \Rightarrow D$: normal

check: $DD^* = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}\begin{bmatrix} \bar{d}_1 & & & 0 \\ & \bar{d}_2 & & \\ & & \ddots & \\ 0 & & & \bar{d}_n \end{bmatrix} = \begin{bmatrix} |d_1|^2 & & & 0 \\ & |d_2|^2 & & \\ & & \ddots & \\ 0 & & & |d_n|^2 \end{bmatrix} = D^*D$

(ii) claim: $A$: normal , $\forall\, U$: unitary s.t. $B = U^*AU \Rightarrow B$: normal

※ $A, B$: unitarily equivalent

check : $BB^* = (U^*AU)(U^*AU)^* = U^*AUU^*A^*U = U^*AA^*U$

$B^*B = (U^*AU)^*(U^*AU) = U^*A^*UU^*A\ U = U^*A^*AU$

(iii) combin (i) , (ii) , we have

「$A$: unitary diagonalizable $\Rightarrow$ $A$: normal」

($\Leftarrow$) 「$A$: normal $\Rightarrow$ $A$: unitary diagonalizable」

(iv) $A$: normal ,

by schur's lemma , $\exists\, B : \left[\begin{smallmatrix} & \nabla \\ 0 & \end{smallmatrix}\right]$ , $U$: unitary s.t. $A = U^*BU$

use (ii) , we have $B$: normal

(v) claim: $B \cdot \begin{bmatrix} * \\ 0 & \end{bmatrix}$ and normal $\Rightarrow$ $B: \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$

歸納 on $n$.

① $n=1$, $B=[b]$ ✓

② $n=k-1$, ✓

③ $n=k$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & & \vdots \\ & 0 & \ddots & \\ & & & b_{nn} \end{bmatrix}, \text{ let } \underset{\displaystyle [c_{ij}]}{\overset{}{C}} = BB^* = B^*B$$

$$\begin{array}{c} BB^* \\ \| \\ C=[c_{ij}] \end{array} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & & \vdots \\ & 0 & \ddots & \\ & & & b_{nn} \end{bmatrix} \begin{bmatrix} \bar{b}_{11} & & & \\ \bar{b}_{12} & \bar{b}_{22} & & 0 \\ \vdots & & \ddots & \\ \bar{b}_{1n} & \cdots & & \bar{b}_{nn} \end{bmatrix} \Rightarrow c_{11} = |b_{11}|^2 + \underbrace{|b_{12}|^2 + \cdots + |b_{1n}|^2}_{= 0}$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$B^*B = \begin{bmatrix} \bar{b}_{11} & & & \\ \bar{b}_{12} & \bar{b}_{22} & & 0 \\ \vdots & & \ddots & \\ \bar{b}_{1n} & \cdots & & \bar{b}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & & \vdots \\ & 0 & \ddots & \\ & & & b_{nn} \end{bmatrix} \Rightarrow c_{11} = |b_{11}|^2$$

$|b_{12}|^2 + \cdots + |b_{1n}|^2 = 0 \quad \therefore |b_{12}|^2 = \cdots = |b_{1n}|^2 = 0 \quad \therefore b_{12} = \cdots = b_{1n} = 0$

$$\therefore B = \left[\begin{array}{c|ccc} b_{11} & 0 & \cdots & 0 \\ \hline & b_{22} & b_{23} \cdots & b_{2n} \\ \vec{0} & & \ddots & \vdots \\ & 0 & & b_{nn} \end{array}\right] = \left[\begin{array}{c|c} b_{11} & 0 \\ \hline 0 & B_1 \end{array}\right]$$

$\therefore B_1 = \begin{bmatrix} * \\ 0 & \end{bmatrix}$ and normal and $(n-1)\times(n-1)$ matrix $\therefore$ by 歸納, $B_1 = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$

$$\therefore B = \left[\begin{array}{c|c} b_{11} & 0 \\ \hline 0 & B_1 \end{array}\right] = \left[\begin{array}{c|c} b_{11} & 0 \\ \hline 0 & \begin{matrix} \ddots & 0 \\ 0 & \ddots \end{matrix} \end{array}\right] = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

(vi) combin (iv), (v)

「A: normal $\Rightarrow$ A: unitary diagonalizable」 ※Q.E.D.