5-4

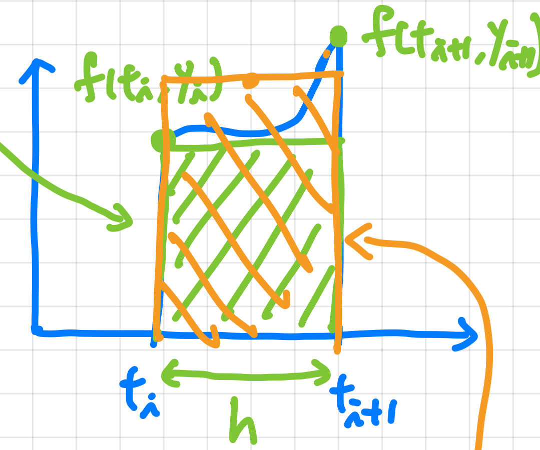
Sol: $y'(t) = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$

Euler's Method

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \underline{h f(t_i, w_i)} \qquad w_{i+1} \approx y(t_{i+1})$$

Modified Euler's Method



$$y'(t) = f(t,y)$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)\,dt$$

$$\approx y_i + \underline{h \cdot \left[f(t_i, y_i) + f(t_{i+1}, y_{i+1}) \right] / 2}$$

$$= y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_i + h f(t_i, y_i)) \right]$$

$\therefore$ ① $w_0 = \alpha$

② $w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + \underbrace{h f(t_i, w_i)}_{K_1}) \right]$

or

$$\boxed{\begin{aligned} &① \; w_0 = \alpha \\ &② \; \text{(i)} \; k_1 = h f(t_i, w_i) \\ &\quad \text{(ii)} \; k_2 = h f(t_{i+1}, w_i + k_1) \\ &\quad \text{(iii)} \; w_{i+1} = w_i + \frac{k_1 + k_2}{2} \end{aligned}}$$

Taylor in one variables

$$f(t) = f(t_0) + (t-t_0) f'(t_0) + \frac{(t-t_0)^2}{2!} f''(t_0) + \dots + \frac{(t-t_0)^n}{n!} f^{(n)}(t_0)$$

remainder $\boxed{+ \frac{(t-t_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)}$

Taylor in two variables

$$f(t,y) = f(t_0, y_0) + \left[(t-t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y-y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right]$$

$$+ \left[\frac{(t-t_0)^2}{2!} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + \frac{(t-t_0)(y-y_0)}{2!} \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right.$$

$$\left. + \frac{(t-t_0)(y-y_0)}{2!} \frac{\partial^2 f}{\partial y \partial t}(t_0, y_0) + \frac{(y-y_0)^2}{2!} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right]$$

$P_n(t,y) =$

nth Taylor poly. in two variable

$$+ \dots$$

$$+ \left[\frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (t-t_0)^{n-j} (y-y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right]$$

$R_n(t,y) =$

$$\boxed{+ \left[\frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t-t_0)^{n+1-j} (y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu) \right]}$$

Euler's Method

$$w_0 = \alpha$$

P_2

$$w_{i+1} = w_i + h\, T_2(t_i, w_i)$$

$$T_2(t,y) \approx a_1 f(t+\alpha_1,\ y+\beta_1)$$

$\triangle$ $a_1 f(t+\alpha_1,\ y+\beta_1)$

$$= a_1 \left\{ f(t,y) + \left[(t+\alpha_1 - t)\frac{\partial f}{\partial t}(t,y) + (y+\beta_1 - y)\frac{\partial f}{\partial y}(t,y) \right] + R_1(t+\alpha_1,\ y+\beta_1) \right\}$$

$$= a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+\alpha_1, y+\beta_1)$$

$$P_2 = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y)$$

$\triangle$ $T_2(t,y) = f(t,y) + \frac{h}{2}\frac{d}{dt} f(t,y)$

$$= f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot y'(t)$$

$$= f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot f(t,y)$$

Compare T_2, P_2

$$\Rightarrow a_1 = 1,\quad \alpha_1 = \frac{h}{2},\quad \beta_1 = \frac{h}{2} f(t,y)$$

$$\therefore\ T_2(t,y) = a_1 f(t+\alpha_1, y+\beta_1) - R_1(t+\alpha_1, y+\beta_1)$$

$$= f\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right) - R_1\left(t+\frac{h}{2}, y+\frac{h}{2}f(t,y)\right)$$

Midpoint Method (Runge-kutta Method of order 2)

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

truncation error $O(h^2)$