Hermitian matrices provide the proper generalization of real symmetric matrices to enable us to prove in the next section that every Hermitian matrix is diagonalizable. The special case of this theorem for real matrices thus finally provides us with a proof of the fundamental theorem that every real symmetric matrix is diagonalizable. The fundamental theorem for real symmetric matrices is tough to prove if we stay within the real number system, but it is a corollary of a fairly easy theorem for complex matrices. This illustrates how we can obtain true insight into theorems in real analysis and linear algebra by studying analogous concepts that use complex numbers.

SUMMARY

1. $\mathbb{C}^n$ is an n-dimensional complex vector space.
2. If $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ are vectors in $\mathbb{C}^n$, then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{u_1}v_1 + \overline{u_2}v_2 + \cdots + \overline{u_n}v_n$ is the Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$ and satisfies the properties in Theorem 9.2. In general, $\langle \mathbf{u}, \mathbf{v} \rangle \neq \langle \mathbf{v}, \mathbf{u} \rangle$.
3. For $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, the vector $\mathbf{u} - \dfrac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$ is perpendicular to $\mathbf{v}$.
4. The conjugate transpose of an $m \times n$ matrix A is the $n \times m$ matrix $A^* = (\overline{A})^T$. The conjugate transpose operation satisfies the properties in Theorem 9.3.
5. A square matrix U is unitary if $U^*U = I$. A real unitary matrix is an orthogonal matrix.
6. A square matrix H is Hermitian if $H = H^*$. A real Hermitian matrix is a symmetric matrix.

EXERCISES

1. Explain how $\mathbb{C}^n$ can be viewed as a $2n$-dimensional real vector space and as an n-dimensional complex vector space. Give a basis in each case.
2. Is it appropriate to view $\mathbb{R}^n$ as a subspace of $\mathbb{C}^n$? Explain.
3. Find AB and BA if
$$A = \begin{bmatrix} 1 & i & i \\ 1+i & 1 & -i \\ i & 1+i & 1-i \end{bmatrix} \quad \text{and}$$
$$B = \begin{bmatrix} -1 & 1+i & i \\ 2+i & i & 1-i \\ i & 1 & i \end{bmatrix}.$$
4. Find A^2 and A^4 if $A = \begin{bmatrix} 1 & 1+i \\ -1+i & i \end{bmatrix}$.
5. Find A^{-1} if $A = \begin{bmatrix} 1 & i \\ 1+i & 2+i \end{bmatrix}$.
6. Find A^{-1} if $A = \begin{bmatrix} i & 1+i \\ 1 & i \end{bmatrix}$.
7. Find A^{-1} if $A = \begin{bmatrix} 1 & i & 1-i \\ 1 & 1 & 1+i \\ 0 & -1+i & 1 \end{bmatrix}$.
8. Find A^{-1} if $A = \begin{bmatrix} i & 1-i & 1+i \\ 0 & 1 & i \\ 1-i & -i & 1-i \end{bmatrix}$.
9. Solve the linear system $A\mathbf{z} = \begin{bmatrix} i \\ 1+i \\ i \end{bmatrix}$ if A is the matrix in Exercise 7.

10. Solve the linear system $A\mathbf{z} = \begin{bmatrix} -1+i \\ 2+i \\ 1 \end{bmatrix}$ if A is the matrix in Exercise 8.

11. Find the solution space of the homogeneous system

$$\begin{aligned} z_1 + \quad iz_2 + \quad (1-i)z_3 &= 0 \\ (1+2i)z_1 - \quad z_2 + \qquad\quad z_3 &= 0 \\ (1+i)z_1 + 2iz_2 + (3-2i)z_3 &= 0. \end{aligned}$$

12. Find the rank of the matrix
$$\begin{bmatrix} 1 & 1+i & i & 1-i \\ 1+i & 2+i & 1-i & 1 \\ 1+i & 1+4i & 1+3i & 2-i \end{bmatrix}.$$

13. Find the rank of the matrix
$$\begin{bmatrix} 2-i & i & 1-i & 1+3i \\ i & 1+i & -1+2i & i \\ 1-i & 1+2i & 1-3i & 2+3i \end{bmatrix}.$$

14. Find each of the indicated inner products.

a. $\langle [2+i, 2, i], [i, 1+i, 2-i] \rangle$ in $\mathbb{C}^3$

b. $\langle [1-i, 1+i], [1+i, 1-i] \rangle$ in $\mathbb{C}^2$

c. $\langle [1+i, 1, 1-i, 1], [i, 1+i, i, 1-i] \rangle$ in $\mathbb{C}^4$

d. $\langle [2-i, 3+i, 1+i], [1+i, 2-i, 1+i] \rangle$ in $\mathbb{C}^3$

15. Compute $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle \mathbf{v}, \mathbf{u} \rangle$ for each of the following.

a. $\mathbf{u} = [1, i], \mathbf{v} = [1+i, 1-i]$

b. $\mathbf{u} = [1+i, 2-i, i], \mathbf{v} = [1, 1-i, 1+i]$

16. Verify property 1 of Theorem 9.2.

17. Verify property 2 of Theorem 9.2.

18. Verify properties 3 and 4 of Theorem 9.2.

19. Verify property 5 of Theorem 9.2.

20. Find the magnitude of each of the following vectors.

a. $[1, i, i]$

b. $[1+i, 1-i, 1+i]$

c. $[1+i, 2+i, 3+i]$

d. $[i, 1+i, 1-i, i]$

e. $[1+i, 1-i, i, 1-i]$

21. Determine whether the given pairs of vectors are parallel, perpendicular, or neither.

a. $[1, i], [i, 1]$

b. $[1+i, i, 1-i], [1-i, 1, -1-i]$

c. $[1+i, 2-i], [3i, 3+i]$

d. $[1, i, 1-i], [1-i, 1+i, 2]$

e. $[1+i, 1-i, 1], [i, 1-i, -3-i]$

22. Find a unit vector parallel to $[1+i, 1-i, i]$.

23. Find a vector of length 2 parallel to $[i, 1-i, 1+i, 1-i]$.

24. Find a unit vector perpendicular to $[2-i, 1+i]$.

25. Find a vector perpendicular to both $[1, i, 1-i]$ and $[1+i, 1-i, 1]$.

In Exercises 26–29, transform the given basis of $\mathbb{C}^n$ into an orthogonal basis, using the Gram–Schmidt process.

26. $\{[1+i, 1-i], [1, 1]\}$ in $\mathbb{C}^2$

27. $\{[2+i, 1+i], [1+i, i]\}$ in $\mathbb{C}^2$

28. $\{[1-i, 1+i, 1+i], [i, 1, -1-i], [1, i, -i]\}$ in $\mathbb{C}^3$

29. $\{[1, i, i], [1+i, 1, i], [i, 1+i, 0]\}$ in $\mathbb{C}^3$

30. Find the conjugate transpose of each of the following matrices.

a. $\begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 1+i & 2 \end{bmatrix}$

b. $\begin{bmatrix} i & 1+i \\ 2+i & 1-i \end{bmatrix}$

c. $\begin{bmatrix} 1+i & 1+i \\ 2-i & 1-i \\ 1 & 1-2i \end{bmatrix}$

d. $\begin{bmatrix} 1+2i & i & 1-i \\ i & 1-i & 1+i \\ 1+i & 2-i & 1+3i \end{bmatrix}$

31. Label each of the following matrices as Hermitian, unitary, both, or neither.

a. $\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

b. $\dfrac{1}{2}\begin{bmatrix} 1 & i & 1-i \\ -i & -2 & 1 \\ 1+i & 1 & 1 \end{bmatrix}$

c. $\dfrac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2}i & \sqrt{2}i \\ \sqrt{3}i & 0 & \sqrt{3} \\ -i & -2 & 1 \end{bmatrix}$

d. $\dfrac{1}{4}\begin{bmatrix} 1 & i & 1-i \\ i & 3 & 1 \\ 1-i & 1 & 2 \end{bmatrix}$

32. Prove the properties of the conjugate transpose operation given in Theorem 9.3. [HINT: From Section 1.3, we know that analogous properties of the *transpose* operation hold for real matrices and real scalars and can be derived using just field properties of $\mathbb{R}$, so they are also true for matrices with complex entries. Thus we can focus on the effect of the *conjugation*. From Theorem 9.1, we know that $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, and $\bar{\bar{z}} = z$, for $z, w \in \mathbb{C}$. Use these properties of conjugation to complete the proof of Theorem 9.3.]

33. Mark each of the following True or False. Assume that all matrices and scalars are complex.

___ a. The definition of a determinant, properties of determinants (the transpose property, the row-interchange property, and so on), and techniques for computing them are developed using only field properties of $\mathbb{R}$ in Chapter 4, and thus they remain equally valid for square complex matrices.

___ b. Cramer's rule is valid for square linear systems with complex coefficients.

___ c. If A is any square matrix and $\det(A) \neq 0$, then $\det(iA) \neq \det(A)$.

___ d. If U is unitary, then $U^{-1} = U^T$.

___ e. If U is unitary, then $(\overline{U})^{-1} = U^T$.

___ f. The Euclidean inner product in $\mathbb{C}^n$ is not commutative.

___ g. For $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, we have $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle$ is a real number.

___ h. For a square matrix A, we have $\det(\overline{A}) = \overline{\det(A)}$.

___ i. For a square matrix A, we have $\det(A^*) = \det(A)$.

___ j. If U is a unitary matrix, then $\det(U^*) = \pm 1$.

34. Prove that, for vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{C}^n$, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{C}^n$ if and only if $\{\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \ldots, \overline{\mathbf{v}}_n\}$ is a basis for $\mathbb{C}^n$.

35. Prove that an $n \times n$ matrix U is unitary if and only if the rows of U form an orthonormal basis for $\mathbb{C}^n$.

36. Prove that, if A is a square matrix, then AA^* is a Hermitian matrix.

37. Prove that the product of two commuting $n \times n$ Hermitian matrices is also a Hermitian matrix. What can you say about the sum of two Hermitian matrices?

38. Prove that the product of two $n \times n$ unitary matrices is also a unitary matrix. What about the sum?

39. Let $T: \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation whose standard matrix representation is a unitary matrix U. Show that $\langle T(\mathbf{u}), T(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$. [HINT: Remember that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^*\mathbf{v}$.]

40. Prove that for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, we have $(\mathbf{u}^*\mathbf{v})^* = \overline{\mathbf{u}^*\mathbf{v}} = \mathbf{v}^*\mathbf{u} = \mathbf{u}^T\overline{\mathbf{v}}$.

41. Describe the unitary diagonal matrices.

42. Prove that, if U is unitary, then $\overline{U}$, U^T, and U^* are unitary matrices also.

43. A square matrix A is **normal** if $A^*A = AA^*$.

    a. Show that every Hermitian matrix is normal.

    b. Show that every unitary matrix is normal.

    c. Show that, if $A^* = -A$, then A is normal.

44. Let A be an $n \times n$ matrix. Referring to Exercise 43, prove that, if A is normal, then $\|A\mathbf{z}\| = \|A^*\mathbf{z}\|$ for all $\mathbf{z} \in \mathbb{C}^n$.

45. Prove the converse of the statement in Exercise 44.

MATLAB

MATLAB can work with complex numbers. When typing a complex number $a + bi$ as a component of a vector or as an entry of a matrix, be sure to type **a+b*i** with the * denoting multiplication and with *no spaces* before or after the + or the *. A space before the + would cause MATLAB to interpret a +**b*i** as two numbers, the real number a followed by another entry containing the complex number bi.

For a matrix A, MATLAB interprets A′ as the *conjugate* transpose of A—that is, as the transpose of A with every entry replaced by its conjugate. Here are three more MATLAB functions:

real(A) is the matrix of real parts of the entries in A,

imag(A) is the matrix of complex parts of the entries in A,

conj(A) is the matrix of conjugates of the entries in A.

Matrices for the exercises that follow are in a file. Some of the exercises ask you to check your answers to some of the more tedious pencil-and-paper computations in the exercises for this section.

M1. Check Exercise 3. (The file matrices are E3A and E3B.)

M2. Check Exercise 7.

M3. Check Exercise 8.

M4. Check Exercise 9.

M5. Check Exercise 10.

M6. Check Exercise 11.

M7. Check Exercise 12.

M8. Check Exercise 13.

M9. Check Exercise 28.

M10. Check Exercise 29.

M11. Consider the matrix

$$\begin{bmatrix} 2-3i & 3+7i & -5+2i & 7-3i & -10+4i \\ 8-i & 2+5i & 11-3i & 6+2i & 14-4i \\ 13+2i & 3-4i & 9+9i & 3-2i & 7+6i \\ 5+i & 8-4i & 12+8i & -3+2i & 1-5i \end{bmatrix}.$$

a. Find the norms of the four row vectors by multiplying appropriate matrices.

b. Find the norms of the five column vectors by multiplying appropriate matrices.

c. Find the inner product $\langle \mathbf{r}_1, \mathbf{r}_3 \rangle$ where $\mathbf{r}_1$ is the first row vector and $\mathbf{r}_3$ is the third row vector.

d. Find the inner product $\langle \mathbf{c}_2, \mathbf{c}_5 \rangle$ where $\mathbf{c}_2$ is the second column vector and $\mathbf{c}_5$ is the fifth column vector.

9.3 EIGENVALUES AND DIAGONALIZATION

Recall the fundamental theorem of real symmetric matrices that we stated without proof as Theorem 6.8:

> Every real symmetric matrix is diagonalizable by a real orthogonal matrix.