matrix A : diagonalizable

$$C^{-1}AC = D \Rightarrow A = CDC^{-1}$$

① $A^k = CD^kC^{-1}$

② T: linear trans. A: s.m.r. of T

$$T: (\mathbb{R}^n, E) \longrightarrow (\mathbb{R}^n, E)$$
$$\vec{v} \longmapsto T(\vec{v}) = A\vec{v}$$

$\exists$ basis $\mathcal{B}$ s.t. $T: (\mathbb{R}^n, \mathcal{B}) \longrightarrow (\mathbb{R}^n, \mathcal{B})$

$R_{\mathcal{B}}$: s.m.r. of T and $R_{\mathcal{B}}$: diag.

A: $n \times n$ $\Rightarrow$ $p(A)$: degree n poly

△ A : diagonalizable iff $\forall$ eigenvalue λ of A

$\underbrace{\text{alg. multi. of } \lambda}_{\text{重根}} = \underbrace{\text{gem. multi. of } \lambda}_{\dim(E_\lambda) = \text{nullity}(A - \lambda I)}$

ex:

$$J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad p(J) = (\lambda - 5)^3$$

$$J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{nullity}(J - 5I) = 2$$

$$\text{null}(J - 5I) = \text{sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \text{sp}(\vec{e}_1, \vec{e}_3)$$

check:

$$(J - 5I)\vec{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{e}_1$$

$(J - 5I)\vec{e}_1 = \vec{0}$

$$(J - 5I) : \quad \vec{e}_2 \to \vec{e}_1 \to \vec{0}$$

$$\vec{e}_3 \to \vec{0}$$

$$(J - 5I)^2 = 0$$

$$\triangle \ (J - 5I)\vec{e}_2 = \vec{e}_1 \Rightarrow J\vec{e}_2 = 5\vec{e}_2 + \vec{e}_1$$

$$\begin{cases} J\vec{e}_1 = 5\vec{e}_1 \\ J\vec{e}_3 = 5\vec{e}_3 \\ J\vec{e}_2 = 5\vec{e}_2 + \vec{e}_1 \end{cases}$$

ex:

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \Rightarrow \text{eigenvalues: } 1, 1, 1, 1, 1$$

$$(J - \lambda I) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{nullity } 1$$

$$(J - \lambda I)\vec{e}_1 = \vec{0} \quad , \quad (J - \lambda I)\vec{e}_5 = \vec{e}_4$$

$$(J - \lambda I) : \vec{e}_5 \longrightarrow \vec{e}_4 \longrightarrow \vec{e}_3 \longrightarrow \vec{e}_2 \longrightarrow \vec{e}_1 \longrightarrow \vec{0}$$

$$\begin{cases} J\vec{e}_1 = \lambda \vec{e}_1 \\ J\vec{e}_2 = \lambda \vec{e}_2 + \vec{e}_1 \\ J\vec{e}_3 = \lambda \vec{e}_3 + \vec{e}_2 \\ J\vec{e}_4 = \lambda \vec{e}_4 + \vec{e}_3 \\ J\vec{e}_5 = \lambda \vec{e}_5 + \vec{e}_4 \end{cases}$$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

Def

J : $m \times m$ jordan block

if 1. all diagonal entries are equal.

2. right above diagonal entries are 1.

3. rest are 0

Thm.

J : $m \times m$ jordan block , diagonal : λ

1. $(J-\lambda I)\vec{e}_i = \vec{e}_{i-1}$, $i = 2, 3 \ldots m$

$(J-\lambda I)\vec{e}_1 = \vec{0}$

2. $(J-\lambda I)^m = 0$, but $(J-\lambda I)^i \neq 0$, $i < m$

3. $J\vec{e}_i = \lambda\vec{e}_i + \vec{e}_{i-1}$, $i = 2, 3, \ldots, m$

$J\vec{e}_1 = \lambda\vec{e}_1$

p.f.

1. 乘開

3. 把 1. 展開

2. (i) $\because (J-\lambda I)^i \vec{e}_{i+1} = (J-\lambda I)^{i-1}\left[(J-\lambda I)\vec{e}_{i+1}\right]$

$$
\begin{aligned}
&= (J-\lambda I)^{i-1}\vec{e}_i \\
&= (J-\lambda I)^{i-2}\vec{e}_{i-1} \\
&\;\;\vdots \\
&= (J-\lambda I)\vec{e}_2 \\
&= \vec{e}_1 \neq \vec{0}
\end{aligned}
$$

$\therefore (J-\lambda I)^i \neq 0$

(ii) $\forall \vec{v} \in \mathbb{C}^m$, $\exists r_1, r_2, \dots, r_m$ s.t. $\vec{v} = r_1\vec{e}_1 + \dots + r_m\vec{e}_m$

$$(J-\lambda I)^m \vec{v} = (J-\lambda I)^m (r_1\vec{e}_1 + \dots + r_m\vec{e}_m)$$

$$= r_1 (J-\lambda I)^m \vec{e}_1 + \dots + r_m (J-\lambda I)^m \vec{e}_m$$

$$= r_1 \vec{0} + \dots + r_m \vec{0} = \vec{0}$$

$$\therefore (J-\lambda I)^m = 0$$

Def.

J : $n\times n$ jordan canonical form

if
$$J = \begin{bmatrix} J_1 & & & & & 0 \\ & J_2 & & & & \\ & & J_3 & & & \\ & & & J_4 & & \\ & & & & \ddots & \\ 0 & & & & & J_k \end{bmatrix}$$

J_i : jordan block

ex:

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$ $2 \neq 7$ X

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\lambda = 0$ ✓

$J_2 = [0]$, $\lambda = 0$

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \checkmark \qquad , \qquad J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \checkmark$$

$$J = \begin{bmatrix} \underline{2} & 1 & 0 \\ 0 & \underline{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad 2 \neq 3 \quad \times$$

ex:

$$J = \begin{bmatrix} -i & 1 & 0 & & & & & \\ 0 & -i & 1 & & & & 0 & \\ 0 & 0 & -i & & & & & \\ & & & -i & 1 & & & \\ & & & 0 & -i & & & \\ & 0 & & & & 2 & & \\ & & & & & & 5 & 1 \\ & & & & & & 0 & 5 \end{bmatrix} \quad \checkmark \qquad \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \vec{e}_1 \\ \leftarrow \vec{e}_2 \\ \leftarrow \vec{e}_3 \\ \leftarrow \vec{e}_4 \\ \leftarrow \vec{e}_5 \\ \leftarrow \vec{e}_6 \\ \leftarrow \vec{e}_7 \\ \leftarrow \vec{e}_8 \end{array}$$

$$\therefore (J + iI) : \vec{e}_3 \to \vec{e}_2 \to \vec{e}_1 \to \vec{0}$$
$$\vec{e}_5 \to \vec{e}_4 \to \vec{0}$$

$$(J - 2I) : \vec{e}_6 \to \vec{0}$$

$$(J - 5I) : \vec{e}_8 \to \vec{e}_7 \to \vec{0}$$

△ check

$$(J + iI) : \vec{e}_3 \to \vec{e}_2 \to \vec{e}_1 \to \vec{0}$$
$$\vec{e}_5 \to \vec{e}_4 \to \vec{0}$$

$\text{null}(J+iI) = \text{sp}(\vec{e}_1, \vec{e}_4)$, nullity 2

$\text{null}((J+iI)^2) = \text{sp}(\vec{e}_2, \vec{e}_1, \vec{e}_5, \vec{e}_4)$, nullity 4

$\text{null}((J+iI)^3) = \text{sp}(\vec{e}_3, \vec{e}_2, \vec{e}_1, \vec{e}_5, \vec{e}_4)$, nullity 5

$\text{null}((J+iI)^k)$ 〃 $k \geq 3$

△ $\text{null}((J-2I)^k) = \text{sp}(\vec{e}_6)$, nullity 1

△ $\text{null}((J-5I)) = \text{sp}(\vec{e}_7)$, nullity 1

$\text{null}((J-5I)^k) = \text{sp}(\vec{e}_7, \vec{e}_8)$, $k \geq 2$, nullity 2

ex:

J : 9×9 jordan. can. form.

1. $(J-3iI)^k$,
   - $k=1$, rank 7 $\Rightarrow$ nullity $\underline{2}$
   - $k=2$, rank 5 $\Rightarrow$ 4
   - $k \geq 3$, rank $=4$ $\Rightarrow$ 5

   $\left.\begin{matrix} 2 \\ 4 \end{matrix}\right\} 2 \qquad \left.\begin{matrix} 4 \\ 5 \end{matrix}\right\} 1$

2. $(J+I)^k$,
   - $k=1$, rank 6 $\Rightarrow$ 3
   - $k \geq 2$, rank 5 $\Rightarrow$ 4

sol.

$$(J-3iI): \quad \begin{array}{ccccccc} \vec{e}_3 & \rightarrow & \vec{e}_2 & \rightarrow & \vec{e}_1 & \rightarrow & \vec{0} \\ & & \vec{e}_5 & \rightarrow & \vec{e}_4 & \rightarrow & \vec{0} \\ \uparrow & & \uparrow & & \uparrow & & \\ 1 & & 2 & & 2 & & \end{array}$$

$$(J+I): \quad \vec{e}_7 \to \vec{e}_6 \to \vec{0}$$
$$\vec{e}_8 \to \vec{0}$$
$$\vec{e}_9 \to \vec{0}$$

$$\therefore \; J = \begin{bmatrix} \boxed{\begin{matrix} 3i & 1 & 0 \\ 0 & 3i & 1 \\ 0 & 0 & 3i \end{matrix}} & & & & 0 \\ & \boxed{\begin{matrix} 3i & 1 \\ 0 & 3i \end{matrix}} & & & \\ & & \boxed{\begin{matrix} -i & 1 \\ 0 & -i \end{matrix}} & & \\ & & & \boxed{-i} & \\ 0 & & & & \boxed{-i} \end{bmatrix}$$

or

$$J = \begin{bmatrix} \boxed{-i} & & & & 0 \\ & \boxed{\begin{matrix} -i & 1 \\ 0 & -i \end{matrix}} & & & \\ & & \boxed{\begin{matrix} 3i & 1 \\ 0 & 3i \end{matrix}} & & \\ & & & \boxed{-i} & \\ 0 & & & & \boxed{\begin{matrix} 3i & 1 & 0 \\ 0 & 3i & 1 \\ 0 & 0 & 3i \end{matrix}} \end{bmatrix}$$

T: linear trans. $\qquad$ A: s.m.r. of T

$$T: (\mathbb{R}^n, E) \longrightarrow (\mathbb{R}^n, E)$$
$$\vec{v} \longmapsto T(\vec{v}) = A\vec{v}$$

$\exists$ basis $\mathcal{B}$ s.t. $T: (\mathbb{R}^n, \mathcal{B}) \longrightarrow (\mathbb{R}^n, \mathcal{B})$

R_B: s.m.r. of T and R_B: Jordan. can. form.

$$\begin{tikzcd}
\mathbb{R}_B \arrow[r] \arrow[rrr, bend left, "T"] & \mathbb{R}_E \arrow[r, "T"] & \mathbb{R}_E \arrow[r] & \mathbb{R}_B
\end{tikzcd}$$

$$C_{EB} = M_B$$

$$\vec{v} \xmapsto{R_E = A} T(\vec{v}) = A\vec{v}$$

$$\vec{v}_B \xmapsto{R_B = C_{BE} R_E C_{EB}} T(\vec{v})_B$$

Def

A: $n \times n$, $\quad \mathcal{B} = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n) \subset \mathbb{C}^n$: $\underbrace{\text{Jordan Basis}}_{\text{for } A}$

$\forall j = 1 \sim n$, $\quad A\vec{b}_j = \lambda \vec{b}_j$ or $\lambda \vec{b}_j + \vec{b}_{j-1}$ for some λ

Note: $\underbrace{\text{if}}_{\exists k \text{ s.t.}}$ $A\vec{b}_k = \alpha \vec{b}_k + \vec{b}_{k-1} \Rightarrow A\vec{b}_{k-1} = \alpha \vec{b}_{k-1}$ or $\alpha \vec{b}_{k-1} + \vec{b}_{k-2}$

same

Thm.

$A : n \times n$

$\Rightarrow \exists\, C$: invertible , J: jordan. can. form

s.t. $J = C^{-1} A C$

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ex:

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$p(A) = |A - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & 5 & 0 & 0 & 1 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 & 0 \\ 0 & 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda) \begin{vmatrix} 2-\lambda & 5 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix} = \cdots = (2-\lambda)^2(-1-\lambda)^3$$

$$\Rightarrow \lambda = \underbrace{2, 2}_{\text{alg. multi} = 2}, \underbrace{-1, -1, -1}_{\text{alg. multi } 3}$$

$\triangle$ $\boxed{\lambda=2}$

$$(A-2I)=\begin{bmatrix}0&5&0&0&1\\0&0&0&0&0\\0&0&-3&0&0\\0&0&0&-3&0\\0&0&0&0&-3\end{bmatrix} \Rightarrow \text{nullity } 1$$

$$\text{null}(A-2I)=\text{sp}\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right)$$

$$(A-2I)^2=\begin{bmatrix}0&0&0&0&-3\\0&0&0&0&0\\0&0&9&0&0\\0&0&0&9&0\\0&0&0&0&9\end{bmatrix} \Rightarrow \text{nullity } \underline{2}$$

$$\text{null}((A-2I)^2)=\text{sp}\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right)$$

$$(A-2I):\ \vec{b}_2\longrightarrow\vec{b}_1\longrightarrow\vec{0} \quad\Rightarrow\quad J_1=\begin{bmatrix}2&1\\0&2\end{bmatrix}$$

$\triangle$ $\vec{b}_1\in\text{null}(A-2I)$, $\vec{b}_2\notin\text{null}(A-2I)$

$\vec{b}_1,\vec{b}_2\in\text{null}((A-2I)^2)$

$$\therefore\ \vec{b}_2=\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix},$$

$$\vec{b}_1=(A-2I)\vec{b}_2=\begin{bmatrix}0&5&0&0&1\\0&0&0&0&0\\0&0&-3&0&0\\0&0&0&-3&0\\0&0&0&0&-3\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix}$$

$$\boxed{\lambda = -1}$$

$$(A+I) = \begin{bmatrix} 3 & 5 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{nullity } 3$$

$$\text{null}(A+I) = \text{sp}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$(A+I)$: $\vec{b}_3 \to \vec{0} \Rightarrow J_2 = [-1]$

$\vec{b}_4 \to \vec{0} \Rightarrow J_3 = [-1]$

$\vec{b}_5 \to \vec{0} \Rightarrow J_4 = [-1]$

$$\therefore \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{b}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \sim J = \begin{bmatrix} J_1 & & & & \\ & J_2 & & 0 & \\ & & J_3 & & \\ & 0 & & J_4 & \\ & & & & J_5 \end{bmatrix} \text{ or } \begin{bmatrix} J_5 & & & & \\ & J_1 & & 0 & \\ & & J_4 & & \\ & 0 & & J_2 & \\ & & & & J_3 \end{bmatrix}$$