§ 6-3 Derangements (錯位排列)

Q: **At a party, 10 gentlemen check their hats.**
In how many ways can their hats be returned so that no gentleman gets the hat with which he arrived?

在一個派對上，有10位先生寄存了他們的帽子。有多少種歸還他們帽子的方式，讓每位先生都拿到不是自己寄存的那頂帽子？

Def: a derangement of $\{1,2,3,\dots,n\}$ is a permutation $\pi = \pi_1\pi_2\pi_3\cdots\pi_n$ s.t. $\forall i$, $\pi_i \neq i$

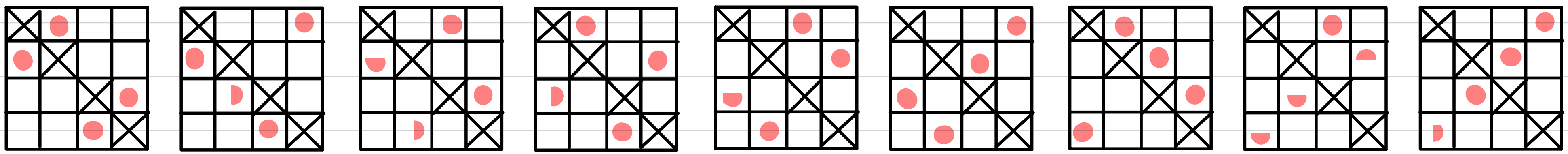
★ if $\pi_i = i$, we call it as a fixed point

$\therefore$ a derangement is a permutation without fixed point.

$n=2$, ~~12~~, 21 (1)

$n=3$, ~~123~~, ~~132~~, ~~213~~, 231, 312, ~~321~~ (2)

$n=4$, 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321 (9)



Thm 6.3.1 : For $n \geq 1$, $D_n = n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right)$

p.f.

using the inclusion-exclusion principle.

Let S = set of all permutation of $\{1, 2, 3, \ldots, n\} = S_n$

A_i = set of all permutation in S_n, with i is a fixed point.

$= \{\pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n \in S_n \mid \pi_i = i\}$

$$\therefore D_n = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$$
$$= |S| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$$
$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n} 0!$$
$$= n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

$\triangle$

A_1 : 1 _ _ ⋯ _ , A_2 : _ 2 _ _ ⋯ _ $\therefore (n-1)!$

$A_1 \cap A_2$: 1 2 _ _ ⋯ _ , $A_1 \cap A_3$: 1 _ 3 _ ⋯ _ $\therefore (n-2)!$

Recall : $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ $\therefore e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$

$\therefore D_n \simeq n!\, e^{-1}$ or $\lim_{n \to \infty} \frac{n!}{D_n} = e$

$\quad\quad \| $

$n\left((n-1)!\, e^{-1}\right) \simeq n D_{n-1}$

Thm (eq. (6.8)) : $D_n = n D_{n-1} + (-1)^n$

p.f.

$$n D_{n-1} + (-1)^n = n \cdot (n-1)! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n-1} \frac{1}{(n-1)!}\right) + (-1)^n \frac{n!}{n!}$$

$$= n! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right) = D_n$$

Thm (eq. (6.6)) : $D_n = (n-1)(D_{n-1} + D_{n-2})$

p.f. ①

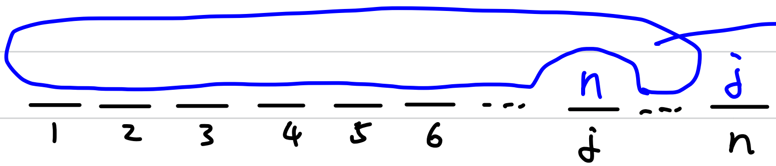
$$D_n = n D_{n-1} + (-1)^n = (n-1) D_{n-1} + (-1)^n + D_{n-1} = (n-1) D_{n-1} + (-1)^n + (n-1) D_{n-2} + (-1)^{n-1}$$

$$= (n-1)(D_{n-1} + D_{n-2})$$

p.f. ②

$\pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n$, what is 「π_n」? Δ $\pi_n \neq n$

i) $\pi_n = j$ & $\pi_j = n$ with $j = 1 \sim (n-1)$



$1 \sim (j-1), (j+1) \sim (n-1)$ total $(n-2)$ number

$(n-2)$ position $\therefore D_{n-2}$

there's $(n-1)$ difference 「j」 $\therefore (n-1) D_{n-2}$

(ii) $\pi_n = j$ & $\pi_j = l \neq n$ with $j = 1 \sim (n-1)$

$$\frac{\pi_1}{1}\ \frac{\pi_2}{2}\ \frac{\pi_3}{3}\ \frac{\pi_4}{4} \cdots \frac{\pi_{j-1}}{j-1}\ \frac{l}{j}\ \frac{\pi_{j+1}}{j+1} \cdots \frac{j}{n} \xrightarrow{\psi} \frac{\pi_1}{1}\ \frac{\pi_2}{2}\ \frac{\pi_3}{3}\ \frac{\pi_4}{4} \cdots \frac{\pi_{j-1}}{j-1}\ \overset{\times}{\boxed{\frac{j}{j}}}\ \frac{\pi_{j+1}}{j+1} \cdots \frac{l}{n}$$

$1 \sim (j-1), (j+1) \sim n$ total $(n-1)$ number
$(n-1)$ position $\therefore D_{n-1}$

there's $(n-1)$ difference 「j」 $\therefore (n-1) D_{n-1}$

△ show $D_n = n D_{n-1} + (-1)^n$ by $D_n = (n-1)(D_{n-1} + D_{n-2})$

p.f.

$$D_n \overset{-nD_{n-1}}{=} (n-1)(D_{n-1} + D_{n-2}) - nD_{n-1} \Rightarrow D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$$

$$D_n - nD_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2}) = (-1)^2(D_{n-2} - (n-2)D_{n-3}) = (-1)^3(D_{n-3} - (n-3)D_{n-4})$$
$$= \cdots = (-1)^{n-2}(D_2 - 2D_1) = (-1)^{n-2} \times (1-0) = (-1)^{n-2} = (-1)^n$$

$\star \quad D_n = (n-1)(D_{n-1} + D_{n-2})$

$D_2 = \{21\}$

$D_3 = \{312,\ 231\}$

$D_4 = \{2143,\ 2341,\ 2413,\ 3142,\ 3412,\ 3421,\ 4123,\ 4312,\ 4321\}$

i) $\pi_n = j.\ \pi_j = n$

$\underline{2}\,1\,\underline{4}\,\underline{3} \rightarrow \frac{2}{1}\,\frac{1}{2}$

$3\,\underline{4}\,1\,\underline{2} \rightarrow \frac{3}{1}\,\frac{1}{3}$

$\underline{4}\,3\,2\,\underline{1} \rightarrow \frac{3}{2}\,\frac{2}{3}$

$\uparrow$

$3 \times D_2$

(ii) $\pi_n = j,\ \pi_j = l \neq n$

$\underline{2}\,3\,4\,\underline{1} \xrightarrow{4} \underline{1}\,3\,4\,2 \rightarrow \frac{3}{2}\,\frac{4}{3}\,\frac{2}{4} \hookrightarrow 231$

$2\,4\,\underline{1}\,\underline{3} \xrightarrow{4} 2\,4\,\underline{3}\,1 \rightarrow \frac{2}{1}\,\frac{4}{2}\,\frac{1}{4} \hookrightarrow 231$

$3\,\underline{1}\,4\,\underline{2} \xrightarrow{4} 3\,\underline{2}\,4\,1 \rightarrow \frac{3}{1}\,\frac{4}{3}\,\frac{1}{4} \hookrightarrow 231$

$\underline{3}\,4\,2\,\underline{1} \xrightarrow{4} \underline{1}\,4\,2\,3 \rightarrow \frac{4}{2}\,\frac{2}{3}\,\frac{3}{4} \hookrightarrow 312$

$4\,1\,\underline{2}\,\underline{3} \xrightarrow{4} 4\,1\,\underline{3}\,2 \rightarrow \frac{4}{1}\,\frac{1}{2}\,\frac{2}{4} \hookrightarrow 312$

$4\,\underline{3}\,1\,\underline{2} \xrightarrow{4} 4\,\underline{2}\,1\,3 \rightarrow \frac{4}{1}\,\frac{1}{3}\,\frac{3}{4} \hookrightarrow 312$

$\uparrow$

$3 \times D_1$

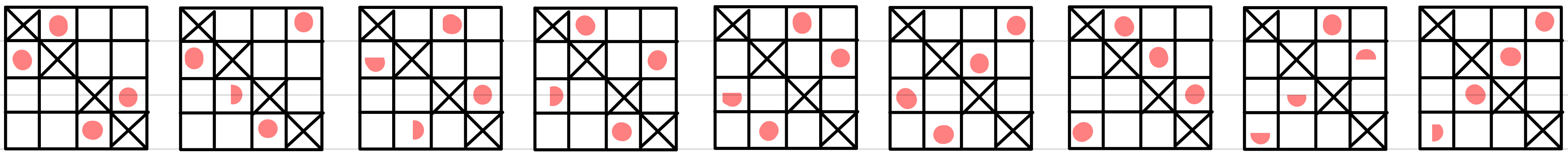
§ 6-4 Permutation with Forbidden (禁止) Position

Def: rook: (西洋棋:城堡, 象棋:車) attack horizontally (橫) and vertically (直)

△ place n nonattacking rooks on a $n \times n$ chessboard with forbidden position

or permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with sets $X_1, X_2, \cdots, X_n$ s.t. $\forall i$, $\pi_i \notin X_i$

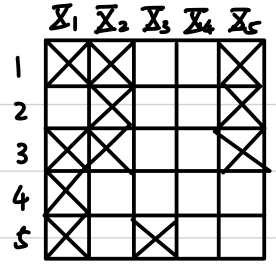
D_4: 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321



place n nonattacking rooks on a $n \times n$ chessboard, avoiding the diagonal

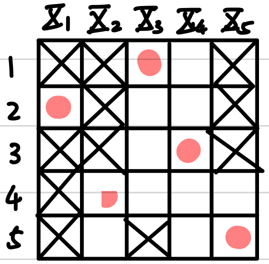
or permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with sets $X_1, X_2, \cdots, X_n$ s.t. $\forall i$, $\pi_i \notin X_i = \{i\}$

ex:

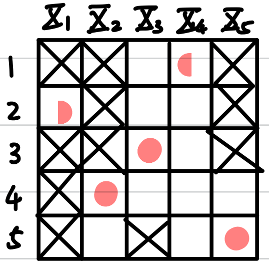


$S = \{1.2.3.4.5\}$

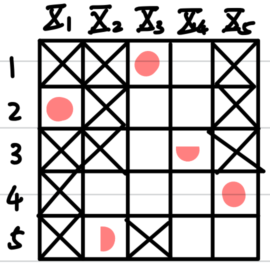
$X_1 = \{1.3.4.5\}$, $X_2 = \{1.2.3\}$, $X_3 = \{5\}$, $X_4 = \phi$, $X_5 = \{1.2.3\}$



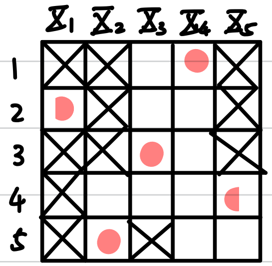
2 4 1 3 5



2 4 3 1 5



2 5 1 3 4



2 5 3 1 4

Question

$S = \{1.2.3.4.\dots, n\}$ $\quad X_1, X_2, X_3, X_4, \dots, X_n$: forbidden position.

sol:

$A_i = \{\pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n \in S_n \mid \pi_i \in X_i\}$ or place rook in X_i in column i

$\therefore |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$

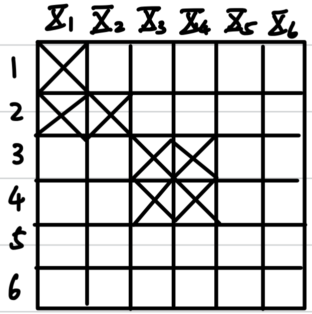
$= |S| - \sum_i |A_i| + \sum_{i.j} |A_i \cap A_j| - \sum_{i.j.k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$

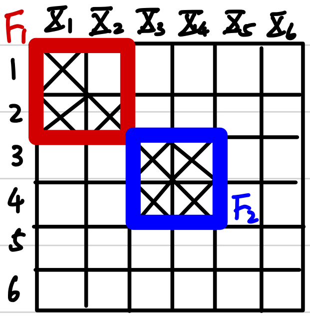
Δ $|A_i| = |X_i|(n-1)!$, $|A_i \cap A_j| = r_2 (n-2)!$, ..., $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k (n-k)!$

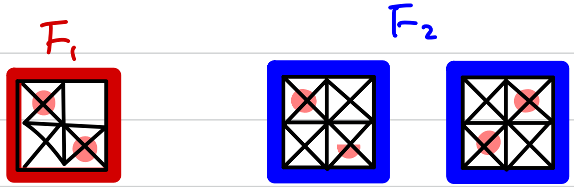
r_k: place k-nonattacking rooks in the 「Assign position」

$= n! - r_1 (n-1)! + r_2 (n-2)! - r_3 (n-3)! + \cdots + (-1)^n r_n \cdot 0!$

ex:







$r_1 = 3 + 4 = 7$

F_1 , F_2

$r_2 = 3 \times 4 + 1 + 2 = 15$

(F_1, F_2) , (F_1, F_1), (F_2, F_2)

$r_3 = 1 \times 4 + 3 \times 2 = 10$

(F_1, F_1, F_2) , (F_1, F_2, F_2)

$r_4 = 1 \times 2 = 2$

(F_1, F_1, F_2, F_2)

$r_5 = r_6 = 0$

$\therefore \left| \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n \right|$

$= n! - r_1 (n-1)! + r_2 (n-2)! - r_3 (n-3)! + \cdots + (-1)^n r_n \cdot 0!$

$= 6! - 7 \times 5! + 15 \times 4! - 10 \times 3! + 2 \times 2! + 0$

$= 184$