應數一線性代數 2020 秋, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 ?? 頁 (包含封面)，有 ?? 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

Answer: <u>-30</u>

$$\begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{vmatrix} = 1 \times \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 3 \end{vmatrix} \text{(expand on the } 3^{rd} \text{column.)}$$

$$= 1 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \times \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} \text{(by 4-3 problem 11)}$$

$$= 1 \times (-2-3) \times (6-0) = -30$$

2. (10 points) Suppose that A is a 5×5 matrix with determinant 7.

(a) Find $\det(3A) =$ $3^5 \times 7 = 1701$

(b) Find $\det(A^{-1}) =$ $1/7$

(c) Find $\det(2A^{-1}) =$ $2^5 \times 1/7 = 32/7$

(d) Find $\det((2A)^{-1}) =$ $1/(2^5 \times 7) = 1/224$

3. (5 points) Suppose that A is a 3×3 matrix with row vectors $\vec{a}, \vec{b}$, and $\vec{c}$, and that $det(A) = 3$. Find the determinant of the matrix having $\vec{a}, \vec{b}, 2\vec{a} + 3\vec{b} + 2\vec{c}$ as its row vectors

Determinant = ___6___

$$\begin{vmatrix} \vec{a} \\ \vec{b} \\ 2\vec{a}+3\vec{b}+2\vec{c} \end{vmatrix} = \begin{vmatrix} \vec{a} \\ \vec{b} \\ 3\vec{b}+2\vec{c} \end{vmatrix} (R_3 = R_3 - 2 \times R_1)$$

$$= \begin{vmatrix} \vec{a} \\ \vec{b} \\ 2\vec{c} \end{vmatrix} (R_3 = R_3 - 3 \times R_2)$$

$$= 2 \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix}$$

$$= 2 \times 3 = 6$$

4. (10 points)

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

The inverse of A = $\begin{bmatrix} 0.5 & -0.5 & 0 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$, and the adjoint matrix of A = $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 6 & -2 \\ 2 & -4 & 2 \end{bmatrix}$

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

$$adj(A) = [a'_{i,j}]^T, \text{where } a'_{i,j} = (-1)^{i+j} \det(A_{i,j})$$

5. (5 points) Let $\vec{a} = \vec{i} - 3\vec{k}$, $\vec{b} = -\vec{i} + 4\vec{j}$, $\vec{c} = \vec{i} + \vec{j} + \vec{k}$.

Find $\vec{a} \cdot (\vec{b} \times \vec{c}) =$ ____19____

$$\vec{i} = [1,0,0], \vec{j} = [0,1,0], \vec{k} = [0,0,1],$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix} = [4, 1, -5]$$

$$[1, 0, -3] \cdot [4, 1, -5] = 19$$

6. (10 points) Find out whether points (1, 2, 1), (3, 3, 4), (2, 2, 2) and (4, 3, 5) lie in a plane in $\mathbb{R}^3$

Answer: ____Yes____

Name A(1, 2, 1), B(3, 3, 4), C(2, 2, 2) and D(4, 3, 5),
then $\vec{AB} = [2, 1, 4], \vec{AC} = [1, 0, 1], \vec{AD} = [3, 1, 4]$

$$\begin{vmatrix} \vec{AB} \\ \vec{AC} \\ \vec{AD} \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0$$

7. (10 points) Using **Cramer's rule** to find the component y of the solution vector for the given linear system.

$$\begin{cases} 2x - 3y = 1 \\ -4x + 6y = -2 \end{cases}$$

$y =$ 無限多組解 or $\dfrac{2x-1}{3}$

$$|A| = \begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} = 0, |B_2| = \begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} = 0,$$

8. (10 points) Circle True or False. Read each statement in original Greek before answering.

   (a) True $\boxed{\textbf{False}}$ There's an unique coordinate vector associated with each vector $\vec{v} \in V$ relatve to a basis for V

   (b) $\boxed{\textbf{True}}$ False A linear transformation $T : V \to V'$ carries the zero vector of V into the zero vector of V'.

   (c) True $\boxed{\textbf{False}}$ The parallelogram (平行四邊形) in $\mathbb{R}^2$ determined by non-zero vectors $\vec{a}, \vec{b}$ is a square (正方形) if and only if $\vec{a} \cdot \vec{b} = 0$

   (d) True $\boxed{\textbf{False}}$ The product of a square matrix and its adjoint is the identity matrix.

   (e) $\boxed{\textbf{True}}$ False There is no square matrix A such that $\det(A^T A) = -1$.

(a) It needs to be an **ordered basis** to be TRUE.

(c) $\vec{a} = [1, 0], \vec{b} = [0, 2] \Rightarrow \vec{a} \cdot \vec{b} = 0$, but it's not a square.

(d) By Theorem 4.6: $A * adj(A) = \det(A)I$

9. (10 points) Let V and V'' be vector spaces with ordered bases $B = ([1,3,-2],[4,1,2],[-1,1,0])$ and $B' = ([1,0,1,0],[2,1,1,-1],[0,1,1,-1],[2,0,3,1])$, respectively, and let $T : V \longrightarrow V'$ be the linear transformation having the given matrix A as matrix representation relative to B, B'. Find $T([0,3,-6])$.

$$A = \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

(a) If $\vec{v} = [0,3,-6]$, then $\vec{v}_B =$ ____[2, -1, -2]____.

(b) $T([0,3,-6]) =$ ____[-14, -1, -12, -2]____.

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 3 & 1 & 1 & 3 \\ -2 & 2 & 0 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -14 \\ -1 \\ -12 \\ -2 \end{bmatrix}$$

10. (10 points) Let $T : P_3 \longrightarrow P_2$ be defined by $T(p(x)) = D(p(x+1))$, and let $B = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1)$.

(a) Find the matrix A as matrix representation of T relative to B, B'. $A = \underline{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}}$.

(b) Use A to compute $T(4x^3 - 5x^2 + 3x - 2) = \underline{12x^2 + 14x + 5}$.

(a) $T(x^3) = 3x^2 + 6x + 3$, $T(x^2) = 2x + 2$, $T(x^1) = 1$, $T(x^0) = 0$

(b)

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \\ 5 \end{bmatrix}$$

11. (10 points) Let $S = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x)\}$ is a set of functions in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

(a) Prove that S is an independent set in F.

(b) Find a basis for the subspace of F generated by the functions $\{f_1, f_2, f_3, f_4\}$, where

$$f_1(x) = 1 - 2\sin(x) + 4\cos(x) - \sin(2x) - 3\cos(2x), \quad f_2(x) = 1 - 2\sin(x),$$

$$f_3(x) = 4\cos(x) - 5\sin(2x) + 3\cos(2x), \quad f_4(x) = 1 + 2\sin(2x)$$

(a) Assume there exists $a, b, c, d, e \in \mathbb{R}$ such that $a + b\sin(x) + c\cos(x) + d\sin(2x) + e\cos(2x) = 0$.

$$\begin{array}{cccccccccccccc}
\boxed{x=0}, & a & + & b\sin(0) & + & c\cos(0) & + & d\sin(2\cdot 0) & + & e\cos(2\cdot 0) & = & 0 & \\
\Rightarrow & a & + & 0 & + & c & + & 0 & + & e & = & 0 & —(1) \\
\boxed{x=\pi}, & a & + & 0 & - & c & + & 0 & + & e & = & 0 & —(2) \\
\boxed{x=\frac{\pi}{2}}, & a & + & b & + & c & + & 0 & - & e & = & 0 & —(3) \\
\boxed{x=\frac{3\pi}{2}}, & a & - & b & + & c & + & 0 & - & e & = & 0 & —(4)
\end{array}$$

by (1), (2), we have $c = 0$, $\Rightarrow a + e = 0$ —(5)

by (3), (4), we have $b = 0$, $\Rightarrow a - e = 0$ —(6)

by (5), (6), we have $a = e = 0$.

Since $a = b = c = e = 0$, we have $d\sin(2x) = 0$ for all x, $\Rightarrow d = 0$.

Hence, we have $a = b = c = d = e = 0$. Therefore, its independent.

(b)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ -1 & 0 & -5 & 2 \\ -3 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the basis is $\{f_1, f_2, f_3, f_4\}$

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Run LaTeX again to produce the table