EXERCISES

In Exercises 1–4, verify that the given matrix is orthogonal, and find its inverse.

1. $(1/\sqrt{2})\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{bmatrix}$

3. $\frac{1}{7}\begin{bmatrix} 2 & -3 & 6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

If A and D are square matrices, D is diagonal, and AD is orthogonal, then $(AD)^{-1} = (AD)^T$ and $D^{-1}A^{-1} = D^TA^T$ so that $A^{-1} = DD^TA^T = D^2A^T$. In Exercises 5–8, find the inverse of each matrix A by first finding a diagonal matrix D so that AD has column vectors of length 1, and then applying the formula $A^{-1} = D^2A^T$.

5. $\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 0 & 8 \\ -4 & 0 & 6 \\ 0 & 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 4 & -3 & 6 \\ 6 & 6 & 2 \\ -12 & 2 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -1 & 3 & 1 \\ -2 & 1 & 3 & 1 \\ 2 & 1 & -3 & 1 \\ 2 & 1 & 3 & -1 \end{bmatrix}$

9. Supply a third column vector so that the matrix

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & \\ 1/\sqrt{3} & 0 & \\ 1/\sqrt{3} & -1/\sqrt{2} & \end{bmatrix} \text{ is orthogonal.}$$

10. Repeat Exercise 9 for the matrix

$$\begin{bmatrix} \frac{2}{7} & 3/\sqrt{13} & \\ \frac{3}{7} & -2/\sqrt{13} & \\ \frac{6}{7} & 0 & \end{bmatrix}.$$

11. Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ be an ordered orthonormal basis for $\mathbb{R}^3$, and let $\mathbf{b}$ be a unit vector with coordinate vector $\left[\frac{1}{2}, \frac{1}{3}, c\right]$ relative to this basis. Find all possible values for c.

12. Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ be an ordered orthonormal basis for $\mathbb{R}^4$, and let $[2, 1, 4, -3]$ be the coordinate vector of a vector $\mathbf{b}$ in $\mathbb{R}^4$ relative to this basis. Find $\|\mathbf{b}\|$.

In Exercises 13–18, find a matrix C such that $D = C^{-1}AC$ is an orthogonal diagonalization of the given symmetric matrix A.

13. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -2 & 2 & 1 \\ 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

19. Mark each of the following True or False.

___ a. A square matrix is orthogonal if its column vectors are orthogonal.

___ b. Every orthogonal matrix has nullspace $\{\mathbf{0}\}$.

___ c. If A^T is orthogonal, then A is orthogonal.

___ d. If A is an $n \times n$ symmetric orthogonal matrix, then $A^2 = I$.

___ e. If A is an $n \times n$ symmetric matrix such that $A^2 = I$, then A is orthogonal.

___ f. If A and B are orthogonal $n \times n$ matrices, then AB is orthogonal.

___ g. Every orthogonal linear transformation carries every unit vector into a unit vector.

___ h. Every linear transformation that carries each unit vector into a unit vector is orthogonal.

___ i. Every map of the plane into itself that is an *isometry* (that is, preserves distance between points) is given by an orthogonal linear transformation.

___ j. Every map of the plane into itself that is an *isometry* and that leaves the origin fixed is given by an orthogonal linear transformation.

20. Let A be an orthogonal $n \times n$ matrix. Show that $\|A\mathbf{x}\| = \|A^{-1}\mathbf{x}\|$ for any vector $\mathbf{x}$ in $\mathbb{R}^n$.

21. Let A be an orthogonal matrix. Show that A^2 is an orthogonal matrix, too.

22. Show that, if A is an orthogonal matrix, then $\det(A) = \pm 1$.

23. Find a 2×2 matrix with determinant 1 that is not an orthogonal matrix.

24. Let $D = C^{-1}AC$ be a diagonal matrix, where C is an orthogonal matrix. Show that A is symmetric.

25. Let A be an $n \times n$ matrix such that $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Show that A is an orthogonal matrix.

26. Let A be an $n \times n$ matrix such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all vectors $\mathbf{x}$ in $\mathbb{R}^n$. Show that A is an orthogonal matrix. [HINT: Show that $\mathbf{x} \cdot \mathbf{y} = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$, and then use Exercise 25.]

27. Show that the real eigenvalues of an orthogonal matrix must be equal to 1 or -1. [HINT: Think in terms of linear transformations.]

28. Describe all real diagonal orthogonal matrices.

29. **a.** Show that a row-interchange elementary matrix is orthogonal.
b. Let A be a matrix obtained by permuting (that is, changing the order of) the rows of the $n \times n$ identity matrix. Show that A is an orthogonal matrix.

30. Let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be an orthonormal basis of column vectors for $\mathbb{R}^n$, and let C be an orthogonal $n \times n$ matrix. Show that

$$\{C\mathbf{a}_1, C\mathbf{a}_2, \ldots, C\mathbf{a}_n\}$$

is also an orthonormal basis for $\mathbb{R}^n$.

31. Let A and C be orthogonal $n \times n$ matrices. Show that $C^{-1}AC$ is orthogonal.

In Exercises 32–37, determine whether the given linear transformation is orthogonal.

32. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [y, x]$

33. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T([x, y, z]) = [x, y, 0]$

34. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [2x, y]$

35. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [x, -y]$

36. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [x/2 - \sqrt{3}y/2, -\sqrt{3}x/2 + y/2]$

37. $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T([x, y, z]) = [x/3 + 2y/3 + 2z/3, -2x/3 - y/3 + 2z/3, -2x/3 + 2y/3 - z/3]$

38. Find a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ that preserves the angle between vectors but is not an orthogonal transformation.

39. Show that every 2×2 orthogonal matrix is of one of two forms: either

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

for some angle θ.

40. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that T is orthogonal if and only if T maps unit vectors to unit vectors. [HINT: Use Exercise 26.]

41. (*Real Householder matrix*) Let $\mathbf{v}$ be a nonzero column vector in $\mathbb{R}^n$. Show that $C_{\mathbf{v}} = I - \frac{2}{\mathbf{v} \cdot \mathbf{v}}(\mathbf{v}\mathbf{v}^T)$ is an orthogonal matrix. (These Householder matrices can be used to perform certain important stable reductions of matrices.)