SUMMARY

1. A set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ *in* $\mathbb{R}^n$ is *linearly dependent* if there exists a *dependence relation*

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k = \mathbf{0}, \qquad \text{with at least one } r_j \neq 0.$$

   The set is *linearly independent* if no such dependence relation exists, so that a linear combination of the $\mathbf{w}_i$ is the zero vector only if all of the scalar coefficients are zero.
2. A set B of vectors in a subspace W of $\mathbb{R}^n$ is a *basis* for W if and only if the set is independent and the vectors span W. Equivalently, each vector in W can be written *uniquely* as a linear combination of the vectors in B.
3. If $W = \mathrm{sp}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$, then the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ can be cut down, if necessary, to a basis for W by reducing the matrix A having $\mathbf{w}_j$ as the jth column vector to row-echelon form H, and retaining $\mathbf{w}_j$ if and only if the jth column of H contains a pivot.
4. Every subspace W of $\mathbb{R}^n$ has a basis, and every independent set of vectors in W can be enlarged (if necessary) to a basis for W.
5. Let W be a subspace of $\mathbb{R}^n$. All bases of W contain the same number of vectors. The *dimension* of W, denoted by $\dim(W)$, is the number of vectors in any basis for W.
6. Let W be a subspace of $\mathbb{R}^n$ and let $\dim(W) = k$. A subset S of W containing exactly k vectors is a basis for W if either
   a. S is an independent set, or
   b. S spans W.

   That is, it is not necessary to check both conditions in Theorem 2.1 for a basis if S has the right number of elements for a basis.

EXERCISES

1. Give a geometric criterion for a set of two distinct nonzero vectors in $\mathbb{R}^2$ to be dependent.
2. Argue geometrically that any set of three distinct vectors in $\mathbb{R}^2$ is dependent.
3. Give a geometric criterion for a set of two distinct nonzero vectors in $\mathbb{R}^3$ to be dependent.
4. Give a geometric description of the subspace of $\mathbb{R}^3$ generated by an independent set of two vectors.
5. Give a geometric criterion for a set of three distinct nonzero vectors in $\mathbb{R}^3$ to be dependent.
6. Argue geometrically that every set of four distinct vectors in $\mathbb{R}^3$ is dependent.

In Exercises 7–11, use the technique of Example 2, described in the box on page 129, to find a basis for the subspace spanned by the given vectors.

7. $\mathrm{sp}([-3, 1], [6, 4])$ in $\mathbb{R}^2$
8. $\mathrm{sp}([-3, 1], [9, -3])$ in $\mathbb{R}^2$
9. $\mathrm{sp}([2, 1], [-6, -3], [1, 4])$ in $\mathbb{R}^2$
10. $\mathrm{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4])$ in $\mathbb{R}^3$

11. sp([1, 2, 1, 2], [2, 1, 0, −1], [−1, 4, 3, 8], [0, 3, 2, 5]) in $\mathbb{R}^4$

12. Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix}.$$

13. Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix}.$$

14. Find a basis for the column space of the matrix A in Exercise 13.

15. Find a basis for the row space of the matrix A in Exercise 12.

In Exercises 16–25, use the technique illustrated in Example 3 to determine whether the given set of vectors is dependent or independent.

16. {[1, 3], [−2, −6]} in $\mathbb{R}^2$
17. {[1, 3], [2, −4]} in $\mathbb{R}^2$
18. {[−3, 1], [6, 4]} in $\mathbb{R}^2$
19. {[−3, 1], [9, −3]} in $\mathbb{R}^2$
20. {[2, 1], [−6, −3], [1, 4]} in $\mathbb{R}^2$
21. {[−1, 2, 1], [2, −4, 3]} in $\mathbb{R}^3$
22. {[1, −3, 2], [2, −5, 3], [4, 0, 1]} in $\mathbb{R}^3$
23. {[1, −4, 3], [3, −11, 2], [1, −3, −4]} in $\mathbb{R}^3$
24. {[1, 4, −1, 3], [−1, 5, 6, 2], [1, 13, 4, 7]} in $\mathbb{R}^4$
25. {[−2, 3, 1], [3, −1, 2], [1, 2, 3], [−1, 5, 4]} in $\mathbb{R}^3$

In Exercises 26 and 27, enlarge the given independent set to a basis for the entire space $\mathbb{R}^n$.

26. {[1, 2, 1]} in $\mathbb{R}^3$
27. {[2, 1, 1, 1], [1, 0, 1, 1]} in $\mathbb{R}^4$
28. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. Mark each of the following True or False.

___ a. A subset of $\mathbb{R}^n$ containing two nonzero distinct parallel vectors is dependent.

___ b. If a set of nonzero vectors in $\mathbb{R}^n$ is dependent, then any two vectors in the set are parallel.

___ c. Every subset of three vectors in $\mathbb{R}^2$ is dependent.

___ d. Every subset of two vectors in $\mathbb{R}^2$ is independent.

___ e. If a subset of two vectors in $\mathbb{R}^2$ spans $\mathbb{R}^2$, then the subset is independent.

___ f. Every subset of $\mathbb{R}^n$ containing the zero vector is dependent.

___ g. If S is independent, then each vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of vectors in S.

___ h. If S is independent and spans $\mathbb{R}^n$, then each vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of vectors in S.

___ i. If each vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of vectors in S, then S is an independent set.

___ j. The subset S is independent if and only if each vector in sp($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$) has a unique expression as a linear combination of vectors in S.

___ k. The zero subspace of $\mathbb{R}^n$ has dimension 0.

___ l. Any two bases of a subspace W of $\mathbb{R}^n$ contain the same number of vectors.

___ m. Every independent subset of $\mathbb{R}^n$ is a subset of every basis for $\mathbb{R}^n$.

___ n. Every independent subset of $\mathbb{R}^n$ is a subset of some basis for $\mathbb{R}^n$.

29. Let $\mathbf{u}$ and $\mathbf{v}$ be two different vectors in $\mathbb{R}^n$. Prove that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

30. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be independent vectors in $\mathbb{R}^n$. Prove that $\mathbf{w}_1 = 3\mathbf{v}_1$, $\mathbf{w}_2 = 2\mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_3$ are also independent.

31. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be any vectors in $\mathbb{R}^n$. Prove that $\mathbf{w}_1 = 2\mathbf{v}_1 + 3\mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 - 2\mathbf{v}_3$, and $\mathbf{w}_3 = -\mathbf{v}_1 - 3\mathbf{v}_3$ are dependent.

32. Find all scalars s, if any exist, such that [1, 0, 1], [2, s, 3], [2, 3, 1] are independent.

33. Find all scalars s, if any exist, such that [1, 0, 1], [2, s, 3], [1, $-s$, 0] are independent.

34. Let $\mathbf{v}$ and $\mathbf{w}$ be independent column vectors in $\mathbb{R}^3$, and let A be an invertible 3×3 matrix. Prove that the vectors $A\mathbf{v}$ and $A\mathbf{w}$ are independent.

35. Give an example showing that the conclusion of the preceding exercise need not hold if A is nonzero but singular. Can you also find specific independent vectors $\mathbf{v}$ and $\mathbf{w}$ and a singular matrix A such that $A\mathbf{v}$ and $A\mathbf{w}$ are still independent?

36. Let $\mathbf{v}$ and $\mathbf{w}$ be column vectors in $\mathbb{R}^n$, and let A be an $n \times n$ matrix. Prove that, if $A\mathbf{v}$ and $A\mathbf{w}$ are independent, $\mathbf{v}$ and $\mathbf{w}$ are independent.

37. Generalizing Exercise 34, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be independent column vectors in $\mathbb{R}^n$, and let C be an invertible $n \times n$ matrix. Prove that the vectors $C\mathbf{v}_1, C\mathbf{v}_2, \ldots, C\mathbf{v}_k$ are independent.

38. Prove that if W is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$, then $W = \mathbb{R}^n$.

In Exercises 39–42, use LINTEK to find a basis for the space spanned by the given vectors in $\mathbb{R}^n$.

39. $\mathbf{v}_1 = [5, 4, 3]$, $\mathbf{v}_2 = [2, 1, 6]$, $\mathbf{v}_3 = [4, 5, -12]$, $\mathbf{v}_4 = [6, 1, 4]$, $\mathbf{v}_5 = [1, 1, 1]$

40. $\mathbf{a}_1 = [0, 1, 1, 2]$, $\mathbf{a}_2 = [-3, -2, 4, 5]$, $\mathbf{a}_3 = [1, 2, 0, 1]$, $\mathbf{a}_4 = [-1, 4, 6, 11]$, $\mathbf{a}_5 = [1, 1, 1, 3]$, $\mathbf{a}_6 = [3, 7, 3, 9]$

41. $\mathbf{u}_1 = [3, 1, 2, 4, 1]$,
$\mathbf{u}_2 = [3, -2, 6, 7, -3]$,
$\mathbf{u}_3 = [3, 4, -2, 1, 5]$,
$\mathbf{u}_4 = [1, 2, 3, 2, 1]$,
$\mathbf{u}_5 = [7, 1, 11, 13, -1]$,
$\mathbf{u}_6 = [2, -1, 2, 3, 1]$

42. $\mathbf{w}_1 = [2, -1, 3, 4, 1, 2]$,
$\mathbf{w}_2 = [-2, 5, 3, -2, 1, -4]$,
$\mathbf{w}_3 = [2, 4, 6, 5, 2, 1]$,
$\mathbf{w}_4 = [1, -1, 1, -1, 2, 2]$,
$\mathbf{w}_5 = [1, 8, 10, 2, 5, -1]$,
$\mathbf{w}_6 = [3, 0, 0, 2, 1, 5]$

MATLAB

Access MATLAB and work the indicated exercise. If the data files for the text are available, enter **fbc2s1** *for the vector data. Otherwise, enter the vector data by hand.*

M1. Exercise 39

M2. Exercise 40

M3. Exercise 41

M4. Exercise 42

2.2 THE RANK OF A MATRIX

In Section 1.6 we discussed three subspaces associated with an $m \times n$ matrix A: its *column space* in $\mathbb{R}^m$, its *row space* in $\mathbb{R}^n$, and its *nullspace* (solution space of $A\mathbf{x} = \mathbf{0}$) in $\mathbb{R}^n$. In this section we consider how the dimensions of these subspaces are related.

We can find the dimension of the column space of A by row-reducing A to row-echelon form H. This dimension is the number of columns of H having pivots.

Turning to the row space, note that interchange of rows does not change the row space, and neither does multiplication of a row by a nonzero scalar. If we multiply the ith row vector $\mathbf{v}_i$ by a scalar r and add it to the kth row vector $\mathbf{v}_k$, then the new kth row vector is $r\mathbf{v}_i + \mathbf{v}_k$, which is still in the row space of A