EXERCISES

In Exercises 1–6, find the indicated projection.

1. The projection of [2, 1] on sp([3, 4]) in $\mathbb{R}^2$
2. The projection of [3, 4] on sp([2, 1]) in $\mathbb{R}^2$
3. The projection of [1, 2, 1] on each of the unit coordinate vectors in $\mathbb{R}^3$
4. The projection of [1, 2, 1] on the line with parametric equations $x = 3t$, $y = t$, $z = 2t$ in $\mathbb{R}^3$
5. The projection of [−1, 2, 0, 1] on sp([2, −3, 1, 2]) in $\mathbb{R}^4$
6. The projection of [2, −1, 3, −5] on the line sp([1, 0, −1, 2]) in $\mathbb{R}^4$

In Exercises 7–12, find the orthogonal complement of the given subspace.

7. The subspace sp([1, 2, −1]) in $\mathbb{R}^3$
8. The line sp([2, −1, 0, −3]) in $\mathbb{R}^4$
9. The subspace sp([1, 3, 0], [2, 1, 4]) in $\mathbb{R}^3$
10. The plane $2x + y + 3z = 0$ in $\mathbb{R}^3$
11. The subspace sp([2, 1, 3, 4], [1, 0, −2, 1]) in $\mathbb{R}^4$
12. The subspace (hyperplane) $ax_1 + bx_2 + cx_3 + dx_4 = 0$ in $\mathbb{R}^4$ [Hint: See Illustration 3.]
13. Find a nonzero vector in $\mathbb{R}^3$ perpendicular to [1, 1, 2] and [2, 3, 1] by
    a. the methods of this section,
    b. computing a determinant.
14. Find a nonzero vector in $\mathbb{R}^4$ perpendicular to [1, 0, −1, 1], [0, 0, −1, 1], and [2, −1, 2, 0] by
    a. the methods of this section,
    b. computing a determinant.

In Exercises 15–22, find the indicated projection.

15. The projection of [1, 2, 1] on the subspace sp([3, 1, 2], [1, 0, 1]) in $\mathbb{R}^3$
16. The projection of [1, 2, 1] on the plane $x + y + z = 0$ in $\mathbb{R}^3$
17. The projection of [1, 0, 0] on the subspace sp([2, 1, 1], [1, 0, 2]) in $\mathbb{R}^3$
18. The projection of [−1, 0, 1] on the plane $x + y = 0$ in $\mathbb{R}^3$
19. The projection of [0, 0, 1] on the plane $2x - y - z = 0$ in $\mathbb{R}^3$
20. The projection in $\mathbb{R}^4$ of [−2, 1, 3, −5] on
    a. the subspace sp($\mathbf{e}_3$)
    b. the subspace sp($\mathbf{e}_1$, $\mathbf{e}_4$)
    c. the subspace sp($\mathbf{e}_1$, $\mathbf{e}_3$, $\mathbf{e}_4$)
    d. $\mathbb{R}^4$
21. The projection of [1, 0, −1, 1] on the subspace sp([1, 0, 0, 0], [0, 1, 1, 0], [0, 0, 1, 1]) in $\mathbb{R}^4$
22. The projection of [0, 1, −1, 0] on the subspace (hyperplane) $x_1 - x_2 + x_3 + x_4 = 0$ in $\mathbb{R}^4$ [Hint: See Example 5.]
23. Assume that **a**, **b**, and **c** are vectors in $\mathbb{R}^n$ and that W is a subspace of $\mathbb{R}^n$. Mark each of the following True or False.
    ___ a. The projection of **b** on sp(**a**) is a scalar multiple of **b**.
    ___ b. The projection of **b** on sp(**a**) is a scalar multiple of **a**.
    ___ c. The set of all vectors in $\mathbb{R}^n$ orthogonal to every vector in W is a subspace of $\mathbb{R}^n$.
    ___ d. The vector $\mathbf{w} \in W$ that minimizes $\|\mathbf{c} - \mathbf{w}\|$ is $\mathbf{c}_W$.
    ___ e. If the projection of **b** on W is **b** itself, then **b** is orthogonal to every vector in W.
    ___ f. If the projection of **b** on W is **b** itself, then **b** is in W.
    ___ g. The vector **b** is orthogonal to every vector in W if and only if $\mathbf{b}_W = \mathbf{0}$.
    ___ h. The intersection of W and $W^\perp$ is empty.
    ___ i. If **b** and **c** have the same projection on W, then $\mathbf{b} = \mathbf{c}$.
    ___ j. If **b** and **c** have the same projection on every subspace of $\mathbb{R}^n$, then $\mathbf{b} = \mathbf{c}$.
24. Let **a** and **b** be nonzero vectors in $\mathbb{R}^n$, and let θ be the angle between **a** and **b**. The scalar $\|\mathbf{b}\| \cos \theta$ is called the **scalar component of b along a**. Interpret this scalar graphically (see Figures 6.1 and 6.2), and give a formula for it in terms of the dot product.
25. Let W be a subspace of $\mathbb{R}^n$ and let **b** be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector **p** in W such that $\mathbf{b} - \mathbf{p}$ is

perpendicular to every vector in W. [HINT: Suppose that $\mathbf{p}_1$ and $\mathbf{p}_2$ are two such vectors, and show that $\mathbf{p}_1 - \mathbf{p}_2$ is in $W^\perp$.]

26. Let A be an $m \times n$ matrix.
    a. Prove that the set W of row vectors $\mathbf{x}$ in $\mathbb{R}^m$ such that $\mathbf{x}A = \mathbf{0}$ is a subspace of $\mathbb{R}^m$.
    b. Prove that the subspace W in part (a) and the column space of A are orthogonal complements.

27. Subspaces U and W of $\mathbb{R}^n$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{w} = 0$ for all $\mathbf{u}$ in U and all $\mathbf{w}$ in W. Let U and W be orthogonal subspaces of $\mathbb{R}^n$, and let $\dim(U) = n - \dim(W)$. Prove that each subspace is the orthogonal complement of the other.

28. Let W be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\mathbf{a} = \mathbf{a}_W + \mathbf{a}_{W^\perp}$, as in Theorem 6.1, prove that

$$\|\mathbf{a}\| = \sqrt{\|\mathbf{a}_W\|^2 + \|\mathbf{a}_{W^\perp}\|^2}.$$

[HINT: Use the formula $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$.]

29. (*Distance from a point to a subspace*) Let W be a subspace of $\mathbb{R}^n$. Figure 6.5 suggests that the distance from the tip of $\mathbf{a}$ in $\mathbb{R}^n$ to the subspace W is equal to the magnitude of the projection of the vector $\mathbf{a}$ on the orthogonal complement of W. Find the distance from the point $(1, 2, 3)$ in $\mathbb{R}^3$ to the subspace (plane) $\text{sp}([2, 2, 1], [1, 2, 1])$.

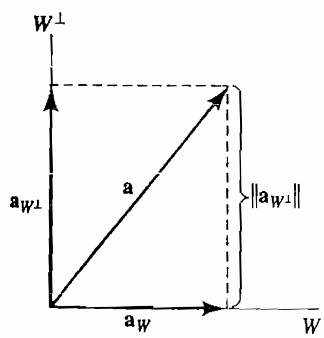

FIGURE 6.5
The distance from $\mathbf{a}$ to W is $\|\mathbf{a}_{W^\perp}\|$.

30. Find the distance from the point $(2, 1, 3, 1)$ in $\mathbb{R}^4$ to the plane $\text{sp}([1, 0, 1, 0], [1, -1, 1, 1])$. [HINT: See Exercise 29.]

In Exercises 31–36, use the idea in Exercise 29 to find the distance from the tip of $\mathbf{a}$ *to the given one-dimensional subspace (line).* [NOTE: *To calculate* $\|\mathbf{a}_{W^\perp}\|$, *first calculate* $\|\mathbf{a}_W\|$ *and then use Exercise 28.*]

31. $\mathbf{a} = [1, 2, 1]$,
    $W = \text{sp}([2, 1, 0])$ in $\mathbb{R}^3$

32. $\mathbf{a} = [2, -1, 3]$,
    $W = \text{sp}([1, 2, 4])$ in $\mathbb{R}^3$

33. $\mathbf{a} = [1, 2, -1, 0]$,
    $W = \text{sp}([3, 1, 4, -1])$ in $\mathbb{R}^4$

34. $\mathbf{a} = [2, 1, 1, 2]$,
    $W = \text{sp}([1, 2, 1, 3])$ in $\mathbb{R}^4$

35. $\mathbf{a} = [1, 2, 3, 4, 5]$,
    $W = \text{sp}([1, 1, 1, 1, 1])$ in $\mathbb{R}^5$

36. $\mathbf{a} = [1, 0, 1, 0, 1, 0, 1]$,
    $W = \text{sp}([1, 2, 3, 4, 3, 2, 1])$ in $\mathbb{R}^7$

Exercises 37–39 involve inner-product spaces discussed in optional Section 3.5.

37. Referring to Example 6, find the projection of $f(x) = 1$ on $\text{sp}(x)$ in P_2.

38. Referring to Example 6, find the projection of $f(x) = x$ on $\text{sp}(1 + x)$.

39. Let S and T be nonempty subsets of an inner-product space V with the property that every vector in S is orthogonal to every vector in T. Prove that the span of S and the span of T are orthogonal subspaces of V.

40. Work with Topic 3 of the routine VECTGRPH in LINTEK until you are able to get a score of at least 80% most of the time.