4. The least-squares solution $\bar{\mathbf{x}}$ of $A\mathbf{x} \approx \mathbf{b}$ and of $A\mathbf{x} = \mathbf{b}$ is given by the formula

$$\bar{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}.$$

Geometrically, $A\bar{\mathbf{x}}$ is the projection of $\mathbf{b}$ on the column space of A.

5. An alternative to using the formula for $\bar{\mathbf{x}}$ in summary item 4 is to convert the overdetermined system $A\mathbf{x} = \mathbf{b}$ to the consistent system $(A^TA)\mathbf{x} = A^T\mathbf{b}$ by multiplying both sides of $A\mathbf{x} = \mathbf{b}$ by A^T, and then to find its unique solution, which is the least-squares solution $\bar{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$.

EXERCISES

1. Let the length b_i of a spring with an attached weight a_i be determined by measurements, as shown in Table 6.8.
   a. Find the least-squares linear fit, in accordance with Hooke's law.
   b. Use the answer to part (a) to estimate the length of the spring if a weight of 5 ounces is attached.

TABLE 6.8

a_i = Weight in ounces	1	2	4	6
b_i = Length in inches	3	4.1	5.9	8.2

2. A company had profits (in units of \$10,000) of 0.5 in 1989, 1 in 1991, and 2 in 1994. Let time t be measured in years, with $t = 0$ in 1989.
   a. Find the least-squares linear fit of the data.
   b. Using the answer to part (a), estimate the profit in 1995.
3. Repeat Exercise 2, but find an exponential fit of the data, working with logarithms as explained in Example 2.
4. A publishing company specializing in college texts starts with a field sales force of ten people, and it has profits of \$100,000. On increasing this sales force to 20, it has profits of \$300,000; and increasing its sales force to 30 produces profits of \$400,000.
   a. Find the least-squares linear fit for these data. [HINT: Express the numbers of salespeople in multiples of 10 and the profit in multiples of \$100,000.]
   b. Use the answer to part (a) to estimate the profit if the sales force is reduced to 25.
   c. Does the profit obtained using the answer to part (a) for a sales force of 0 people seem in any way plausible?

In Exercises 5–7, find the least-squares fit to the given data by a linear function $f(x) = r_0 + r_1x$. Graph the linear function and the data points.

5. (0, 1), (2, 6), (3, 11), (4, 12)
6. (1, 1), (2, 4), (3, 6), (4, 9)
7. (0, 0), (1, 1), (2, 3), (3, 8)
8. Find the least-squares fit to the data in Exercise 6 by a parabola (a quadratic polynomial function).
9. Repeat Exercise 8, but use the data in Exercise 7 instead.

In Exercises 10–13, use the technique illustrated in Examples 6 and 7 to solve the least-squares problem.

10. Find the least-squares linear fit to the data points (−4, −2), (−2, 0), (0, 1), (2, 4), (4, 5).
11. Find the least-squares linear fit to the data points (0, 1), (1, 4), (2, 6), (3, 8), (4, 9).
12. The gallons of maple syrup made from the sugar bush of a Vermont farmer over the past five years were:

    80 gallons five years ago,
    70 gallons four years ago,
    75 gallons three years ago,
    65 gallons two years ago,
    60 gallons last year.

Use a least-squares linear fit of these data to project the number of gallons that will be produced this year. (Does this problem make practical sense? Why?)

13. The minutes required for a rat to find its way out of a maze on each repeated attempt were 8, 8, 6, 5, and 6 on its first five tries. Use a least-squares linear fit of these data to project the time the rat will take on its sixth try.

14. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)$ be data points. If $\sum_{i=1}^m a_i = 0$, show that the line that best fits the data in the least-squares sense is given by $r_0 + r_1x$, where

$$r_0 = \left(\sum_{i=1}^m b_i\right)\bigg/ m$$

and

$$r_1 = \left(\sum_{i=1}^m a_ib_i\right)\bigg/\left(\sum_{i=1}^m a_i^2\right).$$

15. Repeat Exercise 14, but do not assume that $\sum_{i=1}^m a_i = 0$. Show that the least-squares linear fit of the data is given by $y = r_0 + r_1(x - c)$, where $c = (\sum_{i=1}^m a_i)/m$ and r_0 and r_1 have the values given in Exercise 14.

In Exercises 16–20, find the least-squares solution of the given overdetermined system $A\mathbf{x} = \mathbf{b}$ by converting it to a consistent system and then solving, as illustrated in Example 5.

16. $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & 5 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

20. $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -5 & -1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 5 \\ 0 \end{bmatrix}$

21. Mark each of the following True or False.

___ a. There is a unique polynomial function of degree k with graph passing through any given k points in $\mathbb{R}^2$.

___ b. There is a unique polynomial function of degree k with graph passing through any k points in $\mathbb{R}^2$ having distinct first coordinates.

___ c. There is a unique polynomial function of degree at most $k - 1$ with graph passing through any k points in $\mathbb{R}^2$ having distinct first coordinates.

___ d. The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is unique.

___ e. The least-squares solution of $A\mathbf{x} = \mathbf{b}$ can be an actual solution only if A is a square matrix.

___ f. The least-squares solution vector of $A\mathbf{x} = \mathbf{b}$ is the projection of $\mathbf{b}$ on the column space of A.

___ g. The least-squares solution vector of $A\mathbf{x} = \mathbf{b}$ is the vector $\overline{\mathbf{x}}$ such that $A\overline{\mathbf{x}}$ is the projection of $\mathbf{b}$ on the column space of A.

___ h. The least-squares solution vector of $A\mathbf{x} = \mathbf{b}$ is the vector $\mathbf{x} = \overline{\mathbf{x}}$ that minimizes the magnitude of $A\mathbf{x} - \mathbf{b}$.

___ i. A linear system has a least-squares solution only if the number of equations is greater than the number of unknowns.

___ j. Every linear system has a least-squares solution.

In Exercises 22–26, use LINTEK or MATLAB to find the least-squares solution of the linear system. If matrix files are available, the matrix A and the vector $\mathbf{b}$ for Exercise 22 are denoted in the file by A22 and b22, and similarly for Exercises 23–26. Recall that in MATLAB, the command **rref([A'*A A'*b])** *will row-reduce the augmented matrix $[A^TA \mid A^T\mathbf{b}]$.*

22. Exercise 16
23. Exercise 17
24. Exercise 18
25. Exercise 19
26. Exercise 20

The routine YOUFIT in LINTEK can be used to illustrate graphically the fitting of data points by linear, quadratic, or exponential functions. In Exercises 27–31, use YOUFIT to try visually to fit the given data with the indicated type of graph. When this is done, enter the zero data suggested to see the computer's fit. Run twice more with the same data points but without trying to fit the data visually, and determine whether the data are best fitted by a linear, quadratic, or (logarithmically fitted) exponential function, by comparing the least-squares sums for the three cases.

27. Fit (1, 2), (4, 6), (7, 10), (10, 14), (14, 19) by a linear function.
28. Fit (2, 2), (6, 10), (10, 12), (16, 2) by a quadratic function.
29. Fit (1, 1), (10, 8), (14, 12), (16, 20) by an exponential function. Try to achieve a lower squares sum than the computer obtains with its least-squares fit that uses logarithms of y-values.
30. Repeat Exercise 29 with data (1, 9), (5, 1), (6, .5), (9, .01).
31. Fit (2, 9), (4, 6), (7, 1), (8, .1) by a linear function.

The routine QRFACTOR in LINTEK has an option to use a QR-factorization of A to find the least-squares solution of a linear system $A\mathbf{x} = \mathbf{b}$, executing back substitution on $R\mathbf{x} = Q^T\mathbf{b}$ as in Eq. (12).

Recall that in MATLAB, if A is $n \times k$, then Q is $n \times n$ and R is $n \times k$. Cutting Q and R down to the text sizes $n \times k$ and $k \times k$, respectively, we can use the command lines

```
[Q R] = qr(A); [n, k] = size(A);
rref([R(1:k,1:k) Q(:,1:k)'*b])
```

to compute the solution of $R\mathbf{x} = Q^T\mathbf{b}$.

Use LINTEK or MATLAB in this fashion for Exercises 32–37. You must supply the matrix A and the vector $\mathbf{b}$.

32. Find the least-squares linear fit for the data points (−3, 10), (−2, 8), (−1, 7), (0, 6), (1, 4), (2, 5), (3, 6).
33. Find the least-squares quadratic fit for the data points in Exercise 32.
34. Find the least-squares cubic fit for the data points in Exercise 32.
35. Find the least-squares quartic fit for the data points in Exercise 32.
36. Find the quadratic polynomial function whose graph passes through the points (1, 4), (2, 15), (3, 32).
37. Find the cubic polynomial function whose graph passes through the points (−1, 13), (0, −5), (2, 7), (3, 25).