4. The eigenvectors of A corresponding to λ are the nontrivial solutions of the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, as illustrated in Examples 5 and 6.
5. Let k be a positive integer. If λ is an eigenvalue of A having $\mathbf{v}$ as eigenvector, then λ^k is an eigenvalue of A^k with $\mathbf{v}$ as eigenvector. If A is invertible, this statement is also true for $k = -1$.
6. Let λ be an eigenvalue of A. The set E_λ in n-space consisting of the zero vector and all eigenvectors corresponding to λ is a subspace of n-space.
7. Let T be a linear transformation of a vector space V into itself. A nonzero $\mathbf{v}$ in V is an eigenvector of T if $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ, which is called the eigenvalue of T corresponding to $\mathbf{v}$.
8. If T is a linear transformation of the vector space $\mathbb{R}^n$ into itself, the eigenvalues of T are the eigenvalues of the standard matrix representation of T.

EXERCISES

1. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

List the vectors that are eigenvectors of A_1 and the ones that are eigenvectors of A_2. Give the eigenvalue in each case.

In Exercises 2–16, find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors of the given matrix.

2. $\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$

3. $\begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$

4. $\begin{bmatrix} -7 & -5 \\ 16 & 17 \end{bmatrix}$

5. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}$

9. $\begin{bmatrix} 8 & 0 & 0 \\ 7 & -1 & -2 \\ -7 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -5 \\ 8 & 0 & 9 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}$

12. $\begin{bmatrix} -4 & 0 & 0 \\ -7 & 2 & -1 \\ 7 & 0 & 3 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 0 & 0 \\ 8 & 4 & 8 \\ 0 & 0 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 0 & 0 & 1 \\ -2 & -2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 0 & 1 \\ 6 & 4 & -3 \\ 2 & 0 & 3 \end{bmatrix}$

In Exercises 17–22, find the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{v}_i$ of the linear transformation T.

17. T defined on $\mathbb{R}^2$ by $T([x, y]) = [2x - 3y, -3x + 2y]$

18. T defined on $\mathbb{R}^2$ by $T([x, y]) = [x - y, -x + y]$

19. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1 + x_3, x_2, x_1 + x_3]$

T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1, 4x_2 + 7x_3, 2x_2 - x_3]$

T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1, -5x_1 + 3x_2 - 5x_3, -3x_1 - 2x_2]$

T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [3x_1 - x_2 + x_3, -2x_1 + 2x_2 - x_3, 2x_1 + x_2 + 4x_3]$

Mark each of the following True or False.

___ a. Every square matrix has real eigenvalues.

___ b. Every $n \times n$ matrix has n distinct (possibly complex) eigenvalues.

___ c. Every $n \times n$ matrix has n not necessarily distinct and possibly complex eigenvalues.

___ d. There can be only one eigenvalue associated with an eigenvector of a linear transformation.

___ e. There can be only one eigenvector associated with an eigenvalue of a linear transformation.

___ f. If $\mathbf{v}$ is an eigenvector of a matrix A, then $\mathbf{v}$ is an eigenvector of $A + cI$ for all scalars c.

___ g. If λ is an eigenvalue of a matrix A, then λ is an eigenvalue of $A + cI$ for all scalars c.

___ h. If $\mathbf{v}$ is an eigenvector of an invertible matrix A, then $c\mathbf{v}$ is an eigenvector of A^{-1} for all nonzero scalars c.

___ i. Every vector in a vector space V is an eigenvector of the identity transformation of V into V.

___ j. Every nonzero vector in a vector space V is an eigenvector of the identity transformation of V into V.

24. Let $T: V \to V$ be a linear transformation of a vector space V into itself, and let λ be a scalar. Prove that $\{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda\mathbf{v}\}$ is a subspace of V.

25. Prove that if A is a square matrix, then AA^T and A^TA have the same eigenvalues.

26. Following the idea in Example 8, show that the functions e^{ax}, e^{-ax}, $\sin ax$, and $\cos ax$ are eigenvectors for the linear transformation $T: D_\infty \to D_\infty$ defined by $T(f) = f^{(4)}$, the fourth derivative of f. Indicate the eigenvalue in each case.

27. Prove property 1 of Theorem 5.1.

28. Prove property 2 of Theorem 5.1.

29. Prove property 3 of Theorem 5.1.

30. Prove that a square matrix is invertible if and only if no eigenvalue is zero.

31. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

used to generate the Fibonacci sequence (2).

32. Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Compare the eigenvectors and eigenvalues of A with those of $A + rI$ for a scalar r.

33. Let a square matrix A with real eigenvalues have a unique eigenvalue of greatest magnitude. Numerical computation of this eigenvalue by the *power method* (discussed in Section 8.4) can be difficult if there is another eigenvalue of almost equal magnitude, so the ratio of these magnitudes is close to 1.
    a. Suppose we know that a 4×4 matrix A has eigenvalues of approximately 20, 2, -3, and -19.5. Using Exercise 32, how might we modify A so that the aforementioned ratio is not so close to 1?
    b. Repeat part (a), given that the eigenvalues are known to be approximately 19.5, 2, -3, and -20.

34. Let A be an $n \times n$ real matrix. An eigenvector $\mathbf{w}$ in $\mathbb{R}^n$ and a corresponding eigenvalue α of A^T are also called a *left eigenvector and eigenvalue* of A. Explain the reason for this name.

35. (*Principle of biorthogonality*) Let A be an $n \times n$ real matrix. Let $\mathbf{v}$ in $\mathbb{R}^n$ be an eigenvector of A with corresponding eigenvalue λ, and let $\mathbf{w}$ in $\mathbb{R}^n$ be an eigenvector of A^T with corresponding eigenvalue α. Prove that if $\lambda \neq \alpha$, then $\mathbf{v}$ and $\mathbf{w}$ are perpendicular vectors. [HINT: Refer to Exercise 34, and compute $\mathbf{w}^TA\mathbf{v}$ in two ways, using associativity of matrix multiplication.]

36. a. Prove that the eigenvalues of an $n \times n$ real matrix A are the same as the eigenvalues of A^T.
    b. With reference to part (a), show by a counterexample that an eigenvector of A need not be an eigenvector of A^T.

37. The trace of an $n \times n$ matrix A is defined by

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Let the characteristic polynomial $p(\lambda)$ factor into linear factors, so that A has n (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that

$$\begin{aligned}\operatorname{tr}(A) &= (-1)^{n-1}(\text{Coefficient of } \lambda^{n-1} \text{ in } p(\lambda)) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_n.\end{aligned}$$

38. Let A be an $n \times n$ matrix, and let C be an invertible $n \times n$ matrix. Show that the eigenvalues of A and of $C^{-1}AC$ are the same. [HINT: Show that the characteristic polynomials of the two matrices are the same.]

39. *Cayley–Hamilton theorem: Every square matrix A satisfies its characteristic equation. That is, if the characteristic equation is* $p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0$, *then* $p_nA^n + p_{n-1}A^{n-1} + \cdots + p_1A + p_0I = O$, *the zero matrix.* Illustrate the Cayley–Hamilton theorem for the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

40. Let A be an invertible $n \times n$ matrix. Using the Cayley–Hamilton theorem stated in the preceding exercise, show that A^{-1} can be computed as a linear combination of the powers $A, A^2, \ldots, A^n$ of A. Compute the inverse of the matrix A in the preceding exercise in this fashion.

41. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of a linear transformation $T: V \rightarrow V$ with corresponding eigenvalues λ_1 and λ_2, respectively. Prove that, if $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent vectors.

42. The analogue of Exercise 41 for a list of r eigenvectors in V having distinct eigenvalues is also true; that is, the vectors are independent. See if you can prove it. [HINT: Suppose that the vectors are dependent; consider the first vector in the list that is a linear combination of its predecessors, and apply T.]

43. State the result for matrices corresponding to Exercise 42. Explain why successful completion of Exercise 42 gives a proof of this statement for matrices.

44. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It can be shown that, if $\mathbf{x}$ is any column vector in $\mathbb{R}^2$, then $A\mathbf{x}$ can be obtained geometrically from $\mathbf{x}$ by rotating $\mathbf{x}$ counterclockwise through an angle of 90°. For example, we find that $A\mathbf{e}_1 = \mathbf{e}_2$ and $A\mathbf{e}_2 = -\mathbf{e}_1$.

a. Argue geometrically that A has no real eigenvalues.
b. Find the complex eigenvalues and eigenvectors of A.
c. Argue geometrically that A^2 should have real eigenvalues _____. (Fill in the blank.)
d. Use part (b) and Theorem 5.1 to find the eigenvalues of A^2, and compare with the answer obtained for part (c). What are the real eigenvectors of A^2? the complex eigenvectors of A^2?
e. Find eigenvalues and eigenvectors for A^3.
f. Find eigenvalues and eigenvectors for A^4.

45. Use MATCOMP in LINTEK, or MATLAB, and relation (4) to find the following terms of the Fibonacci sequence (2) as accurately as possible. (Use double-precision printing if possible.)

a. F_8 (Note that $F_8 = 21$; this part is to check procedure.)
b. F_{30}
c. F_{50}
d. F_{77}
e. F_{150}

46. The first two terms of a sequence are $a_0 = 0$ and $a_1 = 1$. Subsequent terms are generated using the relation

$$a_k = 2a_{k-1} + a_{k-2} \quad \text{for} \quad k \geq 2.$$

a. Write the terms of the sequence through a_8.
b. Find a matrix that can be used to generate the sequence, as the matrix A in Exercise 31 can be used to generate the Fibonacci sequence.
c. Use MATCOMP in LINTEK, or MATLAB, to find a_{30}.

47. Repeat Exercise 46 for a sequence where $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and $a_k = 2a_{k-1} - 3a_{k-2} + a_{k-3}$ for $k \geq 3$.

For a square matrix A, the MATLAB command **eig(A)** *produces the eigenvalues (both real and complex) of A. The command* **[V, D] = eig(A)** *produces a matrix V whose column vectors are (perhaps complex) eigenvectors of A and a diagonal matrix D whose entry d_{ii} is the eigenvalue for the eigenvector in the ith column of V. In Exercises 48–51, use either MATLAB or the routine MATCOMP in LINTEK to find the real eigenvalues and corresponding eigenvectors of the given matrix.*

48. $\begin{bmatrix} 7 & 10 & 6 \\ 2 & -1 & -6 \\ -2 & -5 & 0 \end{bmatrix}$

49. $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ -3 & 1 & 0 & 3 \end{bmatrix}$

50. $\begin{bmatrix} 0 & 0 & -2 & 2 \\ -3 & 4 & -3 & 3 \\ -4 & 0 & 2 & 4 \\ -2 & 0 & 2 & 4 \end{bmatrix}$

51. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ -6 & 16 & -6 & 6 \\ -16 & 0 & 20 & 16 \\ -16 & 0 & 16 & 20 \end{bmatrix}$

The routine ALLROOTS in LINTEK can be used to find both real and complex roots of a polynomial. The program uses Newton's method, which finds a solution by successive approximations of the polynomial function by a linear one. ALLROOTS is designed so that the user can watch the approximations approach a solution. Of course, a program such as MATLAB, which is designed for research, simply spits out the answers. In Exercises 52–55, either

a. *use the command* **eig(A)** *in MATLAB to find all eigenvalues of the matrix or*

b. *first use MATCOMP in LINTEK to find the characteristic equation of the given matrix. Copy down the equation, and then use ALLROOTS to find all eigenvalues of the matrix.*

52. $\begin{bmatrix} -1 & 4 & 6 \\ 2 & 7 & 9 \\ -3 & 11 & 13 \end{bmatrix}$

53. $\begin{bmatrix} 10 & -13 & 8 \\ 3 & -20 & 5 \\ -11 & 7 & -6 \end{bmatrix}$

54. $\begin{bmatrix} -7 & 11 & -7 & 10 \\ 5 & 8 & -13 & 3 \\ -15 & 8 & -9 & 2 \\ 3 & -4 & 20 & -6 \end{bmatrix}$

55. $\begin{bmatrix} 21 & -8 & 0 & 32 \\ -14 & 17 & -6 & 9 \\ 15 & 11 & -13 & 16 \\ -18 & 30 & 43 & 31 \end{bmatrix}$

56. Use the routine MATCOMP in LINTEK, or MATLAB, to illustrate the Cayley–Hamilton theorem for the matrix

$$\begin{bmatrix} -2 & 4 & 6 & -1 \\ 5 & -8 & 3 & 2 \\ 11 & -3 & 7 & 1 \\ 0 & -5 & 9 & 10 \end{bmatrix}.$$

(See Exercise 39.)

57. The eigenvalue option of the routine VECTGRPH in LINTEK is designed to illustrate graphically, for a linear transformation of $\mathbb{R}^2$ into itself having real eigenvalues, how repeatedly transforming a vector makes it swing in the direction of an eigenvector having eigenvalue of maximum magnitude. By finding graphically a vector whose transform is parallel to it, one can estimate from the graph an eigenvector and the corresponding eigenvalue. Read the directions for this option, and then work with it until you can reliably achieve a score of 85% or better.

MATLAB

The command **v** = **poly(A)** in MATLAB produces a vector **v** whose components are the coefficients of the characteristic polynomial $p(\lambda)$ of A, appearing in order of decreasing powers of λ. The command **roots(v)** then produces the solutions of $p(\lambda) = 0$.