

應數一線性代數 2024 秋, 期末考 解答

學號: \_\_\_\_\_, 姓名: \_\_\_\_\_

本次考試共有 7 頁 (包含封面), 有 13 題。如有缺頁或漏題, 請立刻告知監考人員。

**考試須知:**

- 請在第一頁及最後一頁填上姓名學號。
- 不可翻閱課本或筆記。
- 1-7 題為填空題。
- 8-13 題為計算證明題。請寫出計算過程, 閱卷人員會視情況給予部份分數。沒有計算過程, 就算回答正確答案也不會得到滿分。答卷請清楚乾淨, 儘可能標記或是框出最終答案。
- 書寫空間不夠時, 可利用試卷背面, 但須標記清楚。

高師大校訓: 誠敬宏遠

誠: 一生動念都是誠實端正的。    敬: 就是對知識的認真尊重。  
宏: 開拓視界, 恢宏心胸。        遠: 任重致遠, 不畏艱難。

請簽名保證以下答題都是由你自己作答的, 並沒有得到任何的外部幫助。

簽名: \_\_\_\_\_

1. (10 points) Linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  satisfy  $T([1, 3]) = [2, 2, a]$ , and  $T([2, 1]) = [3, b, 6]$ . If  $T$  is NOT one-to-one, then  $a + b =$  (1) 7

**Solution :**

Since  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is not one-to-one and the dimension of the domain is 2, the vectors  $T([1, 3])$  and  $T([2, 1])$  must be linearly dependent. This implies that the vectors  $[2, 2, a]^T$  and  $[3, b, 6]^T$  are proportional:

$$\frac{2}{3} = \frac{2}{b} = \frac{a}{6}$$

From  $\frac{2}{3} = \frac{2}{b}$ , we get  $b = 3$ . From  $\frac{2}{3} = \frac{a}{6}$ , we get  $a = 4$ . Thus,  $a + b = 4 + 3 = 7$ .

2. (10 points) Given  $B$  and the inverse matrix of  $B$  are below, then  $a =$  (2) 0.8

$$B = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & 5 & -1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} * & * & * \\ a & * & * \\ * & * & * \end{bmatrix}$$

**Solution :**

Using the formula for the inverse matrix  $B^{-1} = \frac{1}{\det(B)} \text{adj}(B)$ , the element  $a$  at position  $(2, 1)$  of  $B^{-1}$  is:

$$a = \frac{1}{\det(B)} (-1)^{1+2} \det(B_{12})$$

$$\det(B) = 5, \quad \det(B_{12}) = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4 \implies a = \frac{1}{5} (-1) (-4) = 0.8$$

3. (10 points) Suppose that  $T$  is a linear transformation with standard matrix representation  $A$ , and that  $A$  is a  $9 \times 15$  matrix such that the nullspace of  $A$  has dimension 5.

(a) The dimension of the range of  $T$  is (3) 10. (b) The dimension of the kernel of  $T$  is (4) 5.

**Solution :**

By the Rank-Nullity Theorem:  $\text{rank}(T) + \text{nullity}(T) = \dim(\text{Domain})$ . Given  $\dim(\text{Domain}) = 15$  and  $\text{nullity}(A) = 5$ : (a) The dimension of the range of  $T$  is  $\text{rank}(A) = 15 - 5 = 10$ . (b) The dimension of the kernel of  $T$  is  $\text{nullity}(A) = 5$ .

4. (10 points) Find the area of the parallelogram(平行四邊形) in  $\mathbb{R}^3$  determined by the vectors  $[2, 1, 3]$  and  $[4, -3, 1]$ . The area is (5)  $10\sqrt{3}$ .

**Solution :**

The area of the parallelogram determined by vectors  $\mathbf{u}$  and  $\mathbf{v}$  is the magnitude of their cross product  $\|\mathbf{u} \times \mathbf{v}\|$ .

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 4 & -3 & 1 \end{vmatrix} = \mathbf{i}(1 + 9) - \mathbf{j}(2 - 12) + \mathbf{k}(-6 - 4) = [10, 10, -10]$$

$$\text{Area} = \sqrt{10^2 + 10^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3}$$

5. (10 points) Let  $P_3$  be the vector space of polynomials with degree at most 3 with real coefficients. The coordinate vector of  $7x^3 + 3x^2 - 2x + 3$  relative to the ordered basis  $(x^2 + x, x^3, x^3 + x, 2x^2 + 1)$  is **(6)**  $[-3, 6, 1, 3]^T$ .

**Solution :**

Let the coordinate vector be  $[c_1, c_2, c_3, c_4]^T$ . We solve:

$$c_1(x^2 + x) + c_2(x^3) + c_3(x^3 + x) + c_4(2x^2 + 1) = 7x^3 + 3x^2 - 2x + 3$$

$$(c_2 + c_3)x^3 + (c_1 + 2c_4)x^2 + (c_1 + c_3)x + c_4 = 7x^3 + 3x^2 - 2x + 3$$

The coordinate vector is  $[-3, 6, 1, 3]^T$ .

6. (10 points) Suppose that  $C$  is a  $6 \times 6$  matrix with determinant 4. The  $\det(7C^{-1})$  is **(7)**  $\frac{7^6}{4}$ .

**Solution :**

Using the property  $\det(kA) = k^n \det(A)$  for an  $n \times n$  matrix, where  $n = 6$ :

$$\det(7C^{-1}) = 7^6 \cdot \det(C^{-1}) = 7^6 \cdot \frac{1}{\det(C)} = \frac{7^6}{4}$$

7. (10 points)

$$D = \begin{bmatrix} 2 & 4 & -2 & 0 & 5 & -1 & 9 \\ 1 & 2 & -1 & 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 3 & 1 & 2 \\ 0 & 5 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 4 & 0 & 3 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \text{The determinant of } D \text{ is } \textbf{(8) } -60.$$

**Solution :**

We compute the determinant of  $D$  by performing cofactor expansion on columns/rows with the most zeros.

- Expand along the 4th column:  $\det(D) = 3 \cdot (-1)^{2+4} \cdot \det(D_{24})$ .
- Expand the remaining  $6 \times 6$  along the 5th row (only element is 2 at position (5,2)):

$$\det(D) = 3 \cdot [(-1)^{5+2} \cdot 2 \cdot \det \begin{pmatrix} 2 & -2 & 5 & -1 & 9 \\ 1 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}]$$

- Continuing expansion along the 3rd row for 1, then the 4th row for 5:

$$\det(D) = -6 \cdot [1 \cdot 5 \cdot \begin{vmatrix} 2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}]$$

- Evaluating the  $3 \times 3$  determinant: 2.

Final result:  $\det(D) = -30 \times 2 = -60$ .

8. (10 points) Let  $P_3$  be the vector space of polynomials with degree at most 3 with real coefficients.  $T : P_3 \rightarrow P_3$  be defined by  $T(p(x)) = 2p(x) - 3\frac{d}{dx}p(x)$

(a) Prove that  $T$  is a linear transformation.

(b) Let the ordered basis for  $P_3$  is  $B = (1, x + 1, x^2, x^3 - 1)$ . Find the matrix representation  $A$  of  $T$  relative to the ordered bases  $B$ .

Answer: (b)  $A = \begin{bmatrix} 2 & -3 & 6 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**Solution :**

Similar with 3-4 example 9.

**(a) Proof that  $T$  is a linear transformation:** To show that  $T$  is a linear transformation, we must verify the properties of linear combination. Let  $p(x), q(x) \in P_3$  and  $r, s \in \mathbb{R}$ .

$$\begin{aligned} T(rp(x) + sq(x)) &= 2(rp(x) + sq(x)) - 3\frac{d}{dx}(rp(x) + sq(x)) \\ &= 2rp(x) + 2sq(x) - 3\left(r\frac{d}{dx}p(x) + s\frac{d}{dx}q(x)\right) \\ &= r\left(2p(x) - 3\frac{d}{dx}p(x)\right) + s\left(2q(x) - 3\frac{d}{dx}q(x)\right) \\ &= rT(p(x)) + sT(q(x)) \end{aligned}$$

Since the property holds,  $T$  is a linear transformation.

**(b) Matrix representation  $A$  relative to the basis  $B = (1, x + 1, x^2, x^3 - 1)$ :** To find the matrix  $[T]_B$ , we apply  $T$  to each basis vector and express the result as a linear combination of the basis  $B$ .

1.  $T(1) = 2(1) - 3(0) = 2 = \underline{2} = \mathbf{2}(1) + 0(x + 1) + 0(x^2) + 0(x^3 - 1)$
2.  $T(x + 1) = 2(x + 1) - 3(1) = \underline{2x - 1} = 2(x + 1) - 3 = -\mathbf{3}(1) + \mathbf{2}(x + 1) + 0(x^2) + 0(x^3 - 1)$
3.  $T(x^2) = 2(x^2) - 3(2x) = \underline{2x^2 - 6x} = \mathbf{6}(1) + -\mathbf{6}(x + 1) + \mathbf{2}(x^2) + 0(x^3 - 1)$
4.  $T(x^3 - 1) = 2(x^3 - 1) - 3(3x^2) = \underline{2x^3 - 9x^2 - 2} = \mathbf{0}(1) + \mathbf{0}(x + 1) + -\mathbf{9}(x^2) + \mathbf{2}(x^3 - 1)$

Placing the coordinates of these images into the columns of matrix  $A$ :

$$A = [T]_B = \begin{bmatrix} 2 & -3 & 6 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

9. (10 points) Let  $F$  is the vector space of all functions mapping  $\mathbb{R}$  into  $\mathbb{R}$  and  $S = \{\sin(-x), 1, \sin(x), \sin(2x)\}$ .

Is  $S$  linear independent in  $F$ ? ( Yes / No ) . If not, find a basis of  $\text{sp}(S)$  {1, sin(x), sin(2x)} .

**Solution :**

注意一下，這邊是計算證明題，需要有解釋，不然沒分。然後第二部分答案不唯一。

**Is  $S$  linearly independent in  $F$ ? No.**

The set  $S = \{\sin(-x), 1, \sin(x), \sin(2x)\}$ . Recall the trigonometric identity for odd functions:  $\sin(-x) = -\sin(x)$ . This means we can write a non-trivial linear combination that equals the zero function:

$$1 \cdot \sin(-x) + 0 \cdot 1 + 1 \cdot \sin(x) + 0 \cdot \sin(2x) = -\sin(x) + \sin(x) = 0$$

Since there exists a set of coefficients (not all zero) such that the linear combination is zero, the set  $S$  is **linearly dependent**.

**Find a basis of  $\text{span}(S)$ :**

To find a basis, we remove the redundant vector(s) that can be expressed as a linear combination of others.

1. We observed that  $\sin(-x) = -\sin(x)$ , so  $\sin(-x)$  is in the span of  $\{\sin(x)\}$ .
2. The remaining set is  $\{1, \sin(x), \sin(2x)\}$ .
3. We check if these are linearly independent. Consider  $c_1(1) + c_2 \sin(x) + c_3 \sin(2x) = 0$  for all  $x \in \mathbb{R}$ .
  - Let  $x = 0$ :  $c_1(1) + 0 + 0 = 0 \implies c_1 = 0$ .
  - Let  $x = \pi/2$ :  $c_2 \sin(\pi/2) + c_3 \sin(\pi) = 0 \implies c_2(1) + 0 = 0 \implies c_2 = 0$ .
  - Let  $x = \pi/4$ :  $c_3 \sin(\pi/2) = 0 \implies c_3(1) = 0 \implies c_3 = 0$ .

Since  $c_1 = c_2 = c_3 = 0$  is the only solution, the set  $\{1, \sin(x), \sin(2x)\}$  is linearly independent and spans the same space as  $S$ .

**Basis of  $\text{span}(S)$ :**  $\{1, \sin(x), \sin(2x)\}$

10. (10 points) (a) Build a linear transformation that is one-to-one but not onto.

**Solution :**

Consider  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by:

$$T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

- **One-to-one:** The kernel is found by setting  $T(\mathbf{x}) = \mathbf{0}$ , which implies  $x_1 = 0$  and  $x_2 = 0$ . Since  $\ker(T) = \{\mathbf{0}\}$ ,  $T$  is one-to-one.
- **Not onto:** The range of  $T$  is the  $xy$ -plane in  $\mathbb{R}^3$ . Any vector with a non-zero  $z$ -component (e.g.,  $[0, 0, 1]^T$ ) has no preimage. Since  $\text{rank}(T) = 2 < \dim(\mathbb{R}^3)$ , it is not onto.

(b) Build a linear transformation that is onto but not one-to-one.

**Solution :**

Consider  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by:

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **Onto:** For any vector  $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ , we can choose  $\mathbf{x} = [y_1, y_2, 0]^T$  such that  $T(\mathbf{x}) = \mathbf{y}$ . Since the range is all of  $\mathbb{R}^2$ , it is onto.
- **Not one-to-one:** The kernel consists of all vectors where  $x_1 = 0$  and  $x_2 = 0$ , specifically  $\ker(T) = \{[0, 0, x_3]^T \mid x_3 \in \mathbb{R}\}$ . Since the kernel contains non-zero vectors,  $T$  is not one-to-one.

\* 這兩小題在回答時，記得也要告知 domain 跟 codomain。然後 domain, codomain 本身要是 vector space。

\*\* 另外 mapping 本身也要記得有 linear transformation 的性質（由 2-3 example 3 知道，用矩陣構造不需驗）。

11. (10 points) Determine the set  $S_1$  of all functions  $f$  such that  $f(0) = 0$  is a subspace in the vector space  $F$  of all functions mapping  $\mathbb{R}$  into  $\mathbb{R}$ .

Answer: Is  $S_1$  a subspace of  $F$ ? ( Yes / No )

**Solution :**

1. **Closed under Addition:** Let  $f(x), g(x) \in S_1$ . By definition,  $f(0) = 0$  and  $g(0) = 0$ . Consider the sum function  $(f \oplus g)$ :

$$(f \oplus g)(0) = f(0) + g(0) = 0 + 0 = 0 \implies (f \oplus g) \in S_1$$

2. **Closed under Scalar Multiplication:** Let  $f(x) \in S_1$  and  $r \in \mathbb{R}$ . By definition,  $f(0) = 0$ . Consider the scalar product function  $(r \otimes f)$ :

$$(r \otimes f)(0) = r \cdot f(0) = r \cdot 0 = 0 \implies (r \otimes f) \in S_1$$

Thus  $S_1$  is a subspace of  $F$ .

12. (10 points) Consider the set  $\mathbb{R}^2$ , with the addition defined by  $[x, y] \oplus [a, b] = [x + a + 2, y + b]$ , and with scalar multiplication defined by  $r \otimes [x, y] = [r(x + 2) - 2, ry]$ .

a. Is this set a vector space? ( Yes / No )

*Hint:* Show by verifying the closed under two operations, A1-A4 and S1-S4.

b. If the set is a vector space, then find the zero vector and the additive inverse (加法反元素) in this vector space. *Hint:* The zero vector may NOT be the vector  $[0, 0]$ .

**Answer:** the zero vector is [-2, 0], for any vectors  $[x, y]$ , the  $-[x, y]$  is [-x-4, -y]

**Solution :**

By the Theorem 3.1, if  $(V, \oplus, \otimes)$  is a vector space, then

1. **Find the Zero Vector  $\vec{0}$ :**

We require  $\vec{0} = 0 \otimes [x, y] \implies \vec{0} = 0 \otimes [x, y] = [0(x + 2) - 2, 0y] = [-2, 0]$ .

2. **Find the Additive Inverse  $-[x, y]$ :**

We require  $-[x, y] = (-1) \otimes [x, y] \implies -[x, y] = (-1) \otimes [x, y] = [-(x + 2) - 2, -y] = [-x - 4, -y]$ .

**Verification of Vector Space Axioms:** Let  $\vec{u} = [x, y]$ ,  $\vec{v} = [a, b]$ ,  $\vec{w} = [c, d] \in \mathbb{R}^2$  and  $r, s \in \mathbb{R}$ .

- **Closure under Addition:**  $\vec{u} \oplus \vec{v} = [x + a + 2, y + b]$  in  $\mathbb{R}^2$ .
- **Closure under Scalar Multiplication:**  $r \otimes \vec{u} = [r(x + 2) - 2, ry]$  in  $\mathbb{R}^2$ .
- **A1 (Associativity of  $\oplus$ ):**  
 $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = [x + a + 2, y + b] \oplus [c, d] = [(x + a + 2) + c + 2, y + b + d] = [x + a + c + 4, y + b + d]$ .  
 $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = [x, y] \oplus [a + c + 2, b + d] = [x + (a + c + 2) + 2, y + b + d] = [x + a + c + 4, y + b + d]$ . (Holds)
- **A2 (Commutativity of  $\oplus$ ):**  $\vec{u} \oplus \vec{v} = [x + a + 2, y + b] = [a + x + 2, b + y] = \vec{v} \oplus \vec{u}$ . (Holds)
- **A3 (Identity Element):** There exists  $\vec{0} \oplus \vec{u} = [-2, 0] \oplus [x, y] = [(-2) + x + 2, 0 + y] = [x, y] = \vec{u}$ . (Holds)
- **A4 (Inverse Element):** For each  $\vec{u}$ , there exists  $-\vec{u} = [-x - 4, -y]$  such that  $\vec{u} \oplus (-\vec{u}) = \vec{0}$ .  
 $\vec{u} \oplus (-\vec{u}) = [x, y] \oplus [-x - 4, -y] = [x + (-x - 4) + 2, y + (-y)] = [-2, 0] = \vec{0}$ . (Holds)
- **S1 (Scalar Distributivity over Vector Addition):**  
 $r \otimes (\vec{u} \oplus \vec{v}) = r \otimes [x + a + 2, y + b] = [r(x + a + 2 + 2) - 2, r(y + b)] = [r(x + a + 4) - 2, r(y + b)]$ .  
 $(r \otimes \vec{u}) \oplus (r \otimes \vec{v}) = [r(x + 2) - 2, ry] \oplus [r(a + 2) - 2, rb] = [(r(x + 2) - 2) + (r(a + 2) - 2) + 2, ry + rb] = [r(x + 2 + a + 2) - 2, r(y + b)]$ . (Holds)
- **S2 (Vector Distributivity over Scalar Addition):**  
 $(r + s) \otimes \vec{u} = [(r + s)(x + 2) - 2, (r + s)y] = [r(x + 2) + s(x + 2) - 2, ry + sy]$ .  
 $(r \otimes \vec{u}) \oplus (s \otimes \vec{u}) = [r(x + 2) - 2, ry] \oplus [s(x + 2) - 2, sy] = [(r(x + 2) - 2) + (s(x + 2) - 2) + 2, ry + sy] = [r(x + 2) + s(x + 2) - 2, ry + sy]$ . (Holds)
- **S3 (Associativity of Scalar Multiplication):**  
 $r \otimes (s \otimes \vec{u}) = r \otimes [s(x + 2) - 2, sy] = [r(s(x + 2) - 2 + 2) - 2, r(sy)] = [rs(x + 2) - 2, rsy]$ .  
 $(rs) \otimes \vec{u} = [rs(x + 2) - 2, rsy]$ . (Holds)
- **S4 (Scalar Identity):**  $1 \otimes \vec{u} = [1(x + 2) - 2, 1(y)] = [x, y]$ . (Holds)

