應數一線性代數 2022 春, 期末考SOLUTION

考試時間：2022/06/23, 09:10 - 12:00,

收卷截止時間：12:10

考卷繳交位置：Google Classroom

考試須知:

- 需要開鏡頭麥克風。鏡頭需要看得到你的身邊，你在作答的紙面，還有你在使用的電子資源的畫面（例如電腦螢幕或平板螢幕）。我不需要直接閱讀螢幕內容，我只要看看畫面的形狀色塊，確定你在看什麼就好。
- 請將紙面答案卷掃成一份 pdf 檔，畫面請清晰並且轉正。第一頁左上寫明姓名學號，每一題前面註明題號，頁面請按照題號順序編排不要跳號。
- 注意事先準備充足的紙張。考試途中不能向外求助更多的計算紙。

1. (10 points) Given the coordinate vector $\vec{v}_B = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$. Please find the $\vec{v}$ and $\vec{v}'_B$ when the ordered basis B and B' for P_2 are

$$B = (x^2 - x,\ 2x+1,\ -x-5),\ B' = (1,\ (1+x),\ (1+x)^2)$$

Answer: $\vec{v} =$ $2x^2 - 6x + 7$, $\vec{v}'_B = \begin{bmatrix} 15 \\ -10 \\ 2 \end{bmatrix}$

From 7-1

$\boxed{Method\ 1}$

$$\vec{v} = 2(x^2 - x) + (-3)(2x+1) + (-2)(-x-5) = 2x^2 - 6x + 7 = 2(x+1)^2 - 10(x+1) + 15$$

$$\vec{v}_{B'} = \begin{bmatrix} 15 \\ -10 \\ 2 \end{bmatrix}$$

$\boxed{Method\ 2}$

$$\vec{v} = M_B \vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix} \Rightarrow 2x^2 - 6x + 7$$

$$\vec{v}_{B'} = C_{BB'} \vec{v}_B = M_{B'}^{-1} M_B \vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 2 \end{bmatrix}$$

2. (10 points) Express $\frac{z}{w}$ in the form $a + bi$, where $a, b \in \mathbb{R}$, if

$$z = 6 - i, \quad w = 2 - 3i$$

Answer: $\frac{z}{w} =$ $\frac{15+16i}{13}$

From 9-1

$\boxed{Method\ 1}$

$$\frac{1}{w} = \frac{\overline{w}}{|w|^2} = \frac{2+3i}{4+9}$$

$$\frac{z}{w} = (6-i) \times \frac{(2+3i)}{4+9} = \frac{15+16i}{13}$$

$\boxed{Method\ 2}$

$$\frac{z}{w} = \frac{6-i}{2-3i} = \frac{(6-i)(2+3i)}{(2-3i)(2+3i)} = \frac{(6-i)(2+3i)}{4+9} = \frac{15+16i}{13}$$

3. (10 points) Find the five fifth roots of -32. (need not simplify)

From 9-1

$-32 = 2^5[\cos(\pi) + i\sin(\pi)]$

$$2\left(\cos\left(\frac{\pi+2k\pi}{5}\right) + i\sin\left(\frac{\pi+2k\pi}{5}\right)\right), \text{ for } i = 0, 1, 2, 3, 4$$

4. (10 points) Let A is an 3×3 complex matrix with $\det(A) = 2 + 5i$. Please the value for $\det(iA)$ and $\det(A^*)$.

Answer: $\det(iA) = \underline{(i)^3 \times (2+5i) = 5 - 2i}$, $\det(A^*) = \underline{\overline{2+5i} = 2 - 5i}$

From 9-2, using the technique from Sec. 4-2.

A is an $n \times n$, then $\det(aA) = a^n \det(A)$ and $\det(A) = \det(A^T)$.

Moreover, the definition of the conjugate transpose is also requested. If $A = [a_{ij}]$, $A^* = \overline{A}^T = \overline{[a_{ji}]} = [\overline{a_{ji}}]$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for linear transformation $T : P_2 \to P_2$ defined by $T(p(x)) = p(x-1) + 2p(x)$, $B = (x^2, x, 1)$, $B' = (x^2-1, x-3, 2)$.

$C_{B,B'} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, $C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$, $R_{B',B'} = \frac{1}{2}\begin{bmatrix} 6 & 0 & 0 \\ -4 & 6 & 0 \\ -5 & -1 & 6 \end{bmatrix}$ and $R_{B,B} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}$.

Is C=$C_{B,B'}$ or $C_{B',B}$? $\underline{\quad C_{B'B} \quad}$.

From 7-2

$$T(x^2) = (x-1)^2 + 2x^2 = 3x^2 - 2x + 1,\ T(x) = (x-1) + 2x = 3x - 1,\ T(1) = 1 + 2 \times 1 = 3$$

Thus

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = R_E\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We have

$$R_{B,B} = R_B = R_E = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}$$

By $C_{B',B} = M_B^{-1}M_{B'} = M_E^{-1}M_{B'} = I^{-1}M_{B'} = M_{B'}$,

$$C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$C_{B,B'} = C_{B',B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Since

$$R_{B'} = R_{B',B'} = C_{B,B'}R_BC_{B',B}$$

$$R_{B'} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & 0 & 0 \\ -4 & 6 & 0 \\ -5 & -1 & 6 \end{bmatrix}$$

6. (10 points) Use the process in Schur's Lemma to find an unitary matrix U such that $U^{-1}AU = R$ is an upper triangular.

$$A = \begin{bmatrix} 5 & 1 & -20 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

From 9-3

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A.

$$A = \begin{bmatrix} 3 & 0 & 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5i & 0 & 0 \\ 0 & 0 & 0 & 0 & 5i & 0 \\ 0 & 0 & 0 & 0 & 0 & 5i \end{bmatrix}$$

From 9-4

$$J = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5i & 0 & 0 \\ 0 & 0 & 0 & 0 & 5i & 0 \\ 0 & 0 & 0 & 0 & 0 & 5i \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3-5i \\ 0 \\ 0 \end{bmatrix}$$

$(A-3I)\vec{v}_1 = \vec{0},\ (A-3I)A\vec{v}_2 = \vec{v}_1,\ (A-3I)\vec{v}_3 = \vec{0},\ (A-5iI)\vec{v}_4 = \vec{0},\ (A-5iI)\vec{v}_5 = \vec{0},\ (A-5iI)\vec{v}_6 = \vec{0}$

8. (10 points) Prove or disprove the following:

$$det(C_{BB'}) = 1 \text{ if and only if } B = B'$$

From 7-1 #23 (h)

False!!

If $B = \{[1,0],[0,6]\}$, $B' = \{[2,0],[0,3]\}$, then

$$C_{BB'} = M_{B'}^{-1} M_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

$$det(C_{BB'}) = 1$$

9. (10 points) Answer the following question.

   1. Find the eigenvalues of the given Matrix J.
   2. Give the rank and nullity of $(J-\lambda)^k$ for each eigenvalue λ of J and for every positive integer k.
   3. Draw schemata of the strings of vectors in the standard basis arising from the Jordan blocks in J.
   4. For each standard basis vector $\vec{e}_k$, express $J\vec{e}_k$ as a linear combination of vectors in the standard basis.

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} = J = \begin{bmatrix} \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} & & & & & \text{\huge 0} \\ & \boxed{\begin{matrix} 9i & 1 \\ 0 & 9i \end{matrix}} & & & & \\ & & \boxed{\begin{matrix} 9i & 1 & 0 \\ 0 & 9i & 1 \\ 0 & 0 & 9i \end{matrix}} & & & \\ & & & \boxed{4} & & \\ \text{\huge 0} & & & & \boxed{4} & \end{bmatrix}$$

From 9-4

1. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_8 = 2$, $\lambda_5 = \lambda_6 = \lambda_7 = \lambda_9 = i$
2. $(J-2I)$ has rank 6 and nullity 3,
   $(J-2I)^k$ has rank 4 and nullity 5, for $k \geq 2$,
   $(J-iI)$ has rank 7 and nullity 2,
   $(J-iI)^2$ has rank 6 and nullity 3,
   $(J-iI)^k$ has rank 5 and nullity 4 for $k \geq 3$,
3. The strings are: $(J-2I): \begin{cases} \vec{e}_2 \to \vec{e}_1 \to 0 \\ \vec{e}_4 \to \vec{e}_3 \to 0 \\ \qquad\quad \vec{e}_8 \to 0 \end{cases}$, $(J-iI): \begin{cases} \vec{e}_7 \to \vec{e}_6 \to \vec{e}_5 \to 0 \\ \qquad\qquad\qquad \vec{e}_9 \to 0 \end{cases}$
4. $\begin{cases} J\vec{e}_1 = 2\vec{e}_1, \\ J\vec{e}_2 = 2\vec{e}_2 + \vec{e}_1, \end{cases}$, $\begin{cases} J\vec{e}_3 = 2\vec{e}_3, \\ J\vec{e}_4 = 2\vec{e}_4 + \vec{e}_3, \end{cases}$, $\begin{cases} J\vec{e}_5 = i\vec{e}_5 \\ J\vec{e}_6 = i\vec{e}_6 + \vec{e}_5, \\ J\vec{e}_7 = i\vec{e}_7 + \vec{e}_6, \end{cases}$, $\{J\vec{e}_8 = 2\vec{e}_8$, $\{J\vec{e}_9 = i\vec{e}_9$

10. (10 points) Prove or disprove whether every unitarily diagonalizable matrix is Hermitian.

From 9-3, Theorem 9.7.

A square matrix A is unitarily diagonalizable if and only if it is a normal matrix.

Therefore, just build a normal matrix which is not a Hermitian matrix as the counterexample.

11. (10 points) Find all $a \in \mathbb{C}$, $b \in \mathbb{R}$ such that the following matrix is unitary diagonalizable.

$$\begin{bmatrix} a & -2i \\ bi & 1-i \end{bmatrix}$$

From 9-3 Let

$$A = \begin{bmatrix} a & -2i \\ bi & 1-i \end{bmatrix}$$

A is unitary diagonalizable if and only if A is normal.

$$AA^* = \begin{bmatrix} a & -2i \\ bi & 1-i \end{bmatrix} \begin{bmatrix} \overline{a} & -bi \\ 2i & \overline{1-i} \end{bmatrix} = A^*A = \begin{bmatrix} \overline{a} & -bi \\ 2i & \overline{1-i} \end{bmatrix} \begin{bmatrix} a & -2i \\ bi & 1-i \end{bmatrix}$$

$$\begin{bmatrix} a\overline{a}+4 & -abi+2-2i \\ \overline{a}bi+2+2i & b^2+2 \end{bmatrix} = \begin{bmatrix} a\overline{a}+b^2 & -2\overline{a}i-bi-b \\ 2ai+bi-b & 6 \end{bmatrix}$$

A is unitary diagonalizable if and only if

$$\begin{cases} b^2 = 4 \\ 2ai+bi-b = \overline{a}bi+2+2i \end{cases}$$

From $b^2 = 4$ and $b \in \mathbb{R}$, we can have $b = \pm 2$.

Let $a = x + yi$, $x, y \in \mathbb{R}$, we can rewrite the second equation as

$$(x-yi)bi + 2 + 2i = 2(x+yi)i + bi - b$$

$$(b+2)(y+1) = i(b-2)(x+1)$$

Therefore, A is unitary diagonalizable if and only if the following 2 conditions:

$$\begin{cases} 1. \quad b = 2,\ y = -1 \qquad \text{i.e. } b = 2,\ a = x - i,\ x \in \mathbb{R} \\ 2. \quad b = -2,\ x = -1 \quad \text{i.e. } b = -2,\ a = -1 + yi,\ y \in \mathbb{R} \end{cases}$$

Run LaTeX again to produce the table