§6-3 Derangements (錯位排列)

Q: **At a party, 10 gentlemen check their hats.**
In how many ways can their hats be returned so that no gentleman gets the hat with which he arrived?
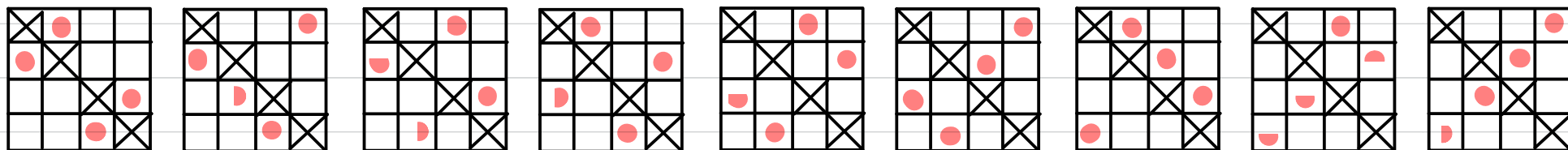
在一個派對上，有10位先生寄存了他們的帽子。有多少種歸還他們帽子的方式，讓每位先生都拿到不是自己寄存的那頂帽子？

<u>Def</u> : a **derangement** of $\{1,2,3,\cdots,n\}$ is a permutation $\pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n$ s.t. $\forall i$, $\pi_i \neq i$

✳ if $\pi_i = i$, we call it as a **fixed point**

∴ a **derangement** is a permutation without fixed point.

① $n = 2$, $\cancel{12}$, 21

② $n = 3$, $\cancel{123}$, $\cancel{132}$, $21\cancel{3}$, 231 , 312 , $3\cancel{21}$

⑨ $n = 4$, 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

Thm 6.3.1 : For $n \geq 1$, $D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!} \right)$

p.f.

using the inclusion-exclusion principle.

Let S = set of all permutation of $\{1, 2, 3, \cdots, n\} = S_n$

A_i = set of all permutation in S_n, with i is a fixed point.

$\quad = \{ \pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n \in S_n \mid \pi_i = i \}$

$\therefore D_n = | \bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \cdots \cap \bar{A_n} |$

$\quad = |S| - \sum_i^n |A_i| + \sum_{i,j}^n |A_i \cap A_j| - \sum_{i,j,k}^n |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$

$\quad = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n} 0!$

$\quad = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!} \right)$

$\triangle$

$A_1 : \underline{1}\_\_\_\cdots\_ \quad , \quad A_2 : \_\underline{2}\_\_\cdots\_ \quad \therefore (n-1)!$

$A_1 \cap A_2 : \underline{1}\ \underline{2}\_\_\cdots\_ \quad , \quad A_1 \cap A_3 : \underline{1}\_\underline{3}\_\cdots\_ \quad \therefore (n-2)!$

Recall : $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \therefore e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$

$\therefore D_n \approx n! \, e^{-1} \quad \text{or} \quad \lim_{n \to \infty} \frac{n!}{D_n} = e$

$\quad\quad \| \quad$

$\quad n \left((n-1)! \, e^{-1} \right) \approx n \, D_{n-1}$

Thm (eg. (6.8)) : $D_n = n D_{n-1} + (-1)^n$

p.f.
$$n D_{n-1} + (-1)^n = n \cdot (n-1)! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right) + (-1)^n \frac{n!}{n!}$$

$$= n! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right) = D_n$$

Thm (eg. (6.6)) : $D_n = (n-1)(D_{n-1} + D_{n-2})$

p.f. ①
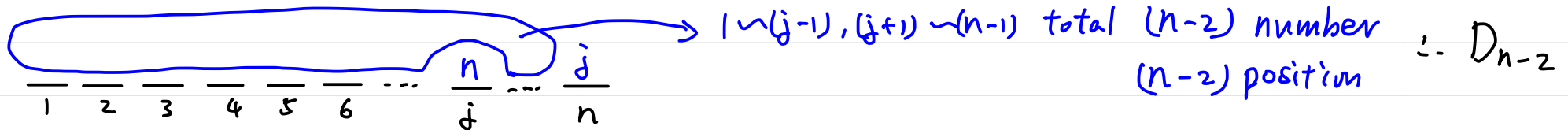$$D_n = n D_{n-1} + (-1)^n = (n-1)D_{n-1} + (-1)^n + D_{n-1} = (n-1)D_{n-1} + (-1)^n + (n-1)D_{n-2} + (-1)^{n-1}$$

$$= (n-1)(D_{n-1} + D_{n-2})$$

p.f. ②

$\pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n$, what is $\lceil \pi_{n} \rfloor$? $\triangle \; \pi_n \neq n$

i) $\pi_n = j$ & $\pi_j = n$ with $j = 1 \sim (n-1)$

$1 \sim (j-1), (j+1) \sim (n-1)$ total $(n-2)$ number
$(n-2)$ position $\therefore D_{n-2}$

there's $(n-1)$ difference $\lceil j \rfloor$ $\therefore (n-1)D_{n-2}$

(ii) $\pi_n = j$ & $\pi_j = l \neq n$ with $j = 1 \sim (n-1)$

$$\frac{\pi_1}{1} \frac{\pi_2}{2} \frac{\pi_3}{3} \frac{\pi_4}{4} \cdots \frac{\pi_{j-1}}{j-1} \frac{l}{j} \frac{\pi_{j+1}}{j+1} \cdots \frac{j}{n} \xrightarrow{\psi} \frac{\pi_1}{1} \frac{\pi_2}{2} \frac{\pi_3}{3} \frac{\pi_4}{4} \cdots \frac{\pi_{j-1}}{j-1} \boxed{\frac{j}{j}} \frac{\pi_{j+1}}{j+1} \cdots \frac{l}{n}$$

X

$1 \sim (j-1), (j+1) \sim n$ total $(n-1)$ number $\therefore D_{n-1}$
$(n-1)$ position

there's $(n-1)$ difference $\lceil j \rfloor$ $\therefore (n-1) D_{n-1}$

$\triangle$ show $D_n = n D_{n-1} + (-1)^n$ by $D_n = (n-1)(D_{n-1} + D_{n-2})$

p.f.

$\overbrace{-n D_{n-1}}$

$D_n = (n-1)(D_{n-1} + D_{n-2}) - n D_{n-1} \Rightarrow D_n - n D_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2})$

$D_n - n D_{n-1} = (-1)(D_{n-1} - (n-1)D_{n-2}) = (-1)^2(D_{n-2} - (n-2)D_{n-3}) = (-1)^3(D_{n-3} - (n-3)D_{n-4})$
$= \cdots = (-1)^{n-2}(D_2 - 2D_1) = (-1)^{n-2} \times (1-0) = (-1)^{n-2} = (-1)^n$

☆ $D_n = (n-1)(D_{n-1} + D_{n-2})$

$D_2 = \{21\}$

$D_3 = \{312, 231\}$

$D_4 = \{2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321\}$

i) $\pi_n = j, \quad \pi_j = n$

$2143 \rightarrow \begin{array}{cc} 2 & 1 \\ \hline 1 & 2 \end{array}$

$3412 \rightarrow \begin{array}{cc} 3 & 1 \\ \hline 1 & 3 \end{array}$

$4321 \rightarrow \begin{array}{cc} 3 & 2 \\ \hline 2 & 3 \end{array}$

$\uparrow$

$3 \times D_2$

(ii) $\pi_n = j, \quad \pi_j = l \neq n$

$2341 \xrightarrow{\;4\;} 1342 \rightarrow \begin{array}{ccc} 3 & 4 & 2 \\ \hline 2 & 3 & 4 \end{array} \rightsquigarrow 231$

$2413 \xrightarrow{\;4\;} 2431 \rightarrow \begin{array}{ccc} 2 & 4 & 1 \\ \hline 1 & 2 & 4 \end{array} \rightsquigarrow 231$

$3142 \xrightarrow{\;4\;} 3241 \rightarrow \begin{array}{ccc} 3 & 4 & 1 \\ \hline 1 & 3 & 4 \end{array} \rightsquigarrow 231$

$3421 \xrightarrow{\;4\;} 1423 \rightarrow \begin{array}{ccc} 4 & 2 & 3 \\ \hline 2 & 3 & 4 \end{array} \rightsquigarrow 312$

$4123 \xrightarrow{\;4\;} 4132 \rightarrow \begin{array}{ccc} 4 & 1 & 2 \\ \hline 1 & 2 & 4 \end{array} \rightsquigarrow 312$

$4312 \xrightarrow{\;4\;} 4213 \rightarrow \begin{array}{ccc} 4 & 1 & 3 \\ \hline 1 & 3 & 4 \end{array} \rightsquigarrow 312$

$\uparrow$

$3 \times D_1$

§ 6-4 Permutation with Forbidden Position (禁止)

Def: rook: (西洋棋：城堡 , 象棋：車) attack horizontally (橫) and vertically (直)

△ place n nonattacking rooks on a $n \times n$ chessboard with forbidden position

or permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with sets $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n$ s.t. $\forall i$, $\pi_i \notin \underline{X}_i$

D_4: 2143 , 2341 , 2413 , 3142 , 3412 , 3421 , 4123 , 4312 , 4321
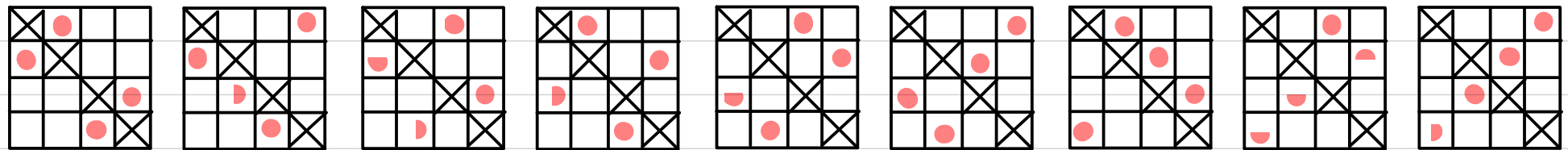
place n nonattacking rooks on a $n \times n$ chessboard, avoiding the diagonal

or permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with sets $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n$ s.t. $\forall i$, $\pi_i \notin \underline{X}_i = \{i\}$

ex:

$S = \{1,2,3,4,5\}$

$\overline{X}_1 = \{1,3,4,5\}$, $\overline{X}_2 = \{1,2,3\}$, $\overline{X}_3 = \{5\}$, $\overline{X}_4 = \phi$, $\overline{X}_5 = \{1,2,3\}$

Question

$S = \{1,2,3,4,\ldots, n\}$ $\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4, \ldots, \overline{X}_n$: forbidden position.

sol:

$A_i = \{ \pi = \pi_1 \pi_2 \pi_3 \cdots \pi_n \in S_n \mid \pi_i \in \overline{X}_i \}$ or place rook in $\overline{X}_i$ in column i

$\therefore |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \cdots \cap \overline{A}_n|$

$= |S| - \sum_i |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$

$\triangle$ $|A_i| = |\overline{X}_i| (n-1)!$, $|A_i \cap A_j| = r_2 (n-2)!$, $\ldots$, $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k (n-k)!$

r_k: place k-nonattacking rooks in the ⌜Assign position⌟

$= n! - r_1 (n-1)! + r_2 (n-2)! - r_3 (n-3)! + \cdots + (-1)^n r_n \cdot 0!$

ex:
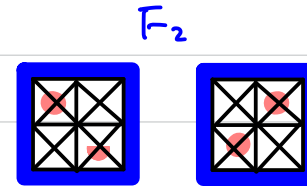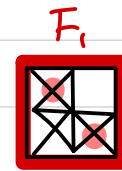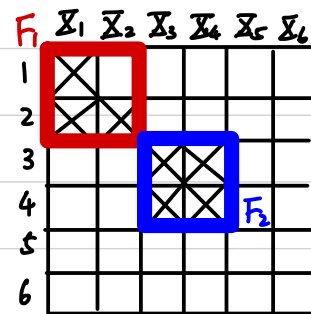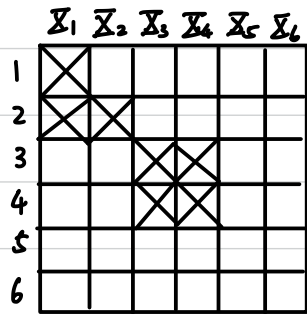
$r_1 = 3 + 4 = 7$

F_1, F_2

$r_2 = 3 \times 4 + 1 + 2 = 15$

$(F_1, F_2), (F_1, F_1), (F_2, F_2)$

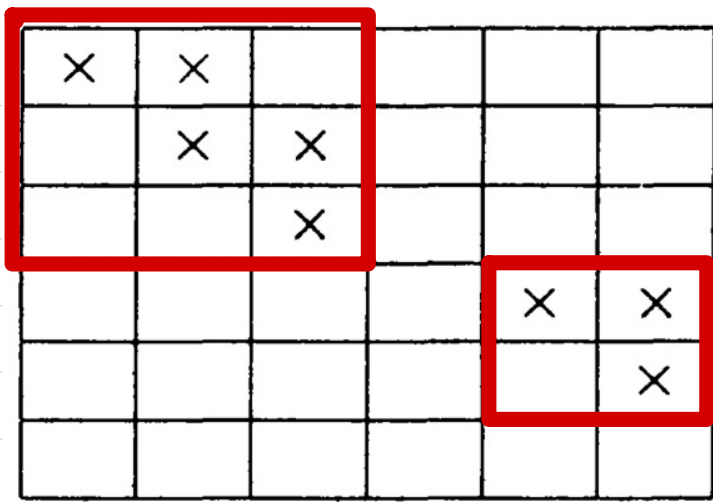$r_3 = 1 \times 4 + 3 \times 2 = 10$

$(F_1, F_1, F_2), (F_1, F_2, F_2)$

$r_4 = 1 \times 2 = 2$

(F_1, F_1, F_2, F_2)

$r_5 = r_6 = 0$

$\therefore |\bar{A_1} \wedge \bar{A_2} \wedge \bar{A_3} \wedge \cdots \wedge \bar{A_n}|$

$= n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + \cdots + (-1)^n r_n \cdot 0!$

$= 6! - 7 \times 5! + 15 \times 4! - 10 \times 3! + 2 \times 2! + 0$

$= 184$
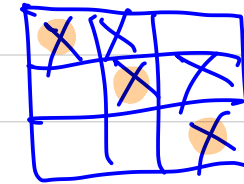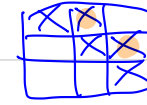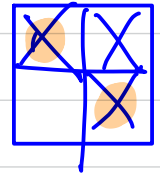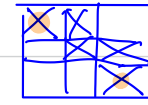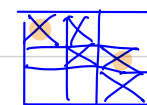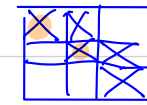
F_1 (red box around top-left 3×3 region)

F_2 (red box around 2×2 region)

$x_1 \quad x_2 \quad x_3 \quad 4 \quad 5 \quad 6$

$x_1 = \{1\}$
$x_2 = \{1,2\}$
$x_3 = \{2,3\}$
$x_4 = \phi$
$x_5 = \{4\}$
$x_6 = \{4,5\}$

$r_1 = 8$

$r_2 = 5 \times 3 + 6 + 1 = 22$
$\quad (F_1, F_2) \quad (F_1, F_1) \quad (F_2, F_2)$

$r_3 = 1 + 6 \times 3 + 5 \times 1 = 24$
$\quad (F_1 \times 3), \ (F_1 \times 2, F_2), \ (F_1, F_2 \times 2)$

$r_4 = 1 \times 3 + 6 \times 1 = 9$
$\quad (F_1 \times 3, F_2), \ (F_1 \times 2, F_2 \times 2),$

$r_5 = 1 \times 1 = 1$
$\quad (F_1 \times 3, F_2 \times 2)$

$r_6 = 0$

$\therefore \ |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$
$= n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! + \cdots + (-1)^n r_n \cdot 0!$

$= 6! - 8 \times 5! + 22 \times 4! - 24 \times 3! + 9 \times 2! + 1 \times 1! - 0$