Def.

$A\vec{x}=\vec{b}$: linear system

1. the system is consistent if it has one or more solutions
2. the system is inconsistent if it has no solution

Thm.

$A\vec{x}=\vec{b}$: linear system , $[A|\vec{b}] \sim [H|\vec{c}]$, where H: r-e form

1. $A\vec{x}=\vec{b}$ is <u>inconsistent</u>
   iff $[H|\vec{c}]$ has a row with all 0 in the left part but non-zero in the right part
2. $A\vec{x}=\vec{b}$ is <u>consistent</u> and every column of H has a pivot
   $\Rightarrow$ <u>unique solution</u>.
3. $A\vec{x}=\vec{b}$ is <u>consistent</u> and some column of H has no pivot
   $\Rightarrow$ <u>infinitely many solution</u>, with as many free variables as the number of pivot-free column in H.

Def.

elementary matrix can be obtained by apply one elementary row operation to an identity matrix.

ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 + 4R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

elementary matrix

Thm.

$A_{m\times n}$: matrix , $E_{m\times m}$: elementary matrix

EA = apply the same elementary row operation from E to A.

ex :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \; R_3 \to \tfrac{1}{2}R_3 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \; R_2 \to R_2 + 4R_3$$

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \frac{a_{31}}{2} & \frac{a_{32}}{2} & \frac{a_{33}}{2} \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+4a_{31} & a_{22}+4a_{32} & a_{23}+4a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

ex: $A \overset{\substack{E_1 \\ R_1\leftrightarrow R_2}}{\sim} A_1 \overset{\substack{E_2 \\ R_2\to R_2+3R_3}}{\sim} A_2 \overset{E_3}{\sim} A_3 \sim \cdots A_n$

$$A_1 = E_1 A$$

$$A_2 = E_2 A_1 = E_2 E_1 A$$

$$A_3 = E_3 A_2 = E_3 E_2 E_1 A$$

ex:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 3 & -1 \\ 4 & 5 & -2 \end{bmatrix} \underbrace{\quad}_{} \xrightarrow{R_{(1)}:\ R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 4 & 5 & -2 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$R_{(1)} \leftrightarrow E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_{(2)}:\ R_3 \to R_3 - 2R_1} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$R_{(2)} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_{(3)}:\ R_3 \to R_3 + R_2} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} = H$$

$R_3 \to R_3 - R_2$

$$R_{(3)} \leftrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{cases} H = R_{(3)}\left(R_{(2)}\left(\left(R_{(1)}(A)\right)\right)\right) \\ H = E_3 E_2 E_1 A \end{cases}$$

$$A = R_{(1)}^{-1}\left(R_{(2)}^{-1}\left(R_{(3)}^{-1}((H))\right)\right) \quad \text{or} \quad A = E_1^{-1} E_2^{-1} E_3^{-1} H$$

$\boxed{1-5}$

- $a, b \in \mathbb{R}$,

$ax = b \Rightarrow x = b/a$

$3x = 5 \Rightarrow x = 5/3$

$ax = b$

$[\frac{1}{a} a] x = [\frac{1}{a}] ax = [\frac{1}{a}] b = b/a$

$x = \underline{\underline{\quad\quad\quad}}$

$\therefore x = \frac{1}{a} b$

- $A_{n\times n}$, $\vec{b}_{n\times 1}$,

$A\vec{x} = \vec{b}$

$C(A\vec{x}) = C\vec{b}$

$\vec{x} = I\vec{x}$

$\Rightarrow \vec{x} = C\vec{b}$

Q: How to find C s.t. $CA = I$

Q: if $CA = I \Rightarrow AC \overset{?}{=} I$

Thm

$A_{n\times n}$: matrix

If $\exists\, C_{n\times n},\ D_{n\times n}$ s.t. $AC = I$, $DA = I$.

then $C = D$

pf.

$$DAC = D(AC) = D\cdot I = D$$

$$DAC = (DA)C = I\cdot C = C$$

$\therefore C = D$

Def.

- $A_{n\times n}$: invertible if $\exists\, C_{n\times n}$ s.t. $CA = AC = I$. Denote $\underset{\text{inverse of } A}{C}$ by A^{-1}
- $A_{n\times n}$: singular if A is NOT invertible

Thm.

every elementary matrix is invertible

① $R_i \leftrightarrow R_j \rightsquigarrow R_j \leftrightarrow R_i$

② $R_i \to R_i + rR_j \rightsquigarrow R_i \to R_i - rR_j$

③ $R_i \to rR_i \rightsquigarrow R_i \to \frac{1}{r} R_i$

p.f.

E : elementary matrix if $\exists\, R$: elementary row operation

s.t. $E = R(I)$

$\exists\, R^{-1}$: elementary row operation s.t. $R^{-1} \circ R$ = identity

let $\tilde{E} = R^{-1}(I)$

$\therefore \tilde{E} \cdot E = R^{-1}(E) = R^{-1}(R(I)) = I$

$E\tilde{E} = R(\tilde{E}) = R(R^{-1}(I)) = I$

$\therefore \tilde{E} = E^{-1}$

ex: $E_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: R : $R_1 \to R_1 + 4R_3$ $\therefore$ R^{-1} : $R_1 \to R_1 - 4R_3$

$E_3^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thm

A, B : invertible $n \times n$ matrix

$\Rightarrow$ AB : invertible and $(AB)^{-1} = B^{-1}A^{-1}$

p.f.

A, B : invertible $\Rightarrow$ $\exists\, A^{-1}, B^{-1}$ s.t. $AA^{-1} = A^{-1}A = I$, $BB^{-1} = B^{-1}B = I$

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$

$\therefore$ $(AB)^{-1} = B^{-1}A^{-1}$

Thm

The following are equivalent:

1. $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}$
2. $A \sim I$
3. A : invertible

p.f. $(2) \Rightarrow (3)$

$A \sim I \Rightarrow R_{(k)}(\cdots R_{(2)}(R_{(1)}(A))\cdots) = I$

let $E_i = R_{(i)}(I)$ $\quad \therefore I = E_k \cdots E_2 E_1 A$

$(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) \cdot I = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) E_k \cdots E_2 E_1 A = A$

$\therefore A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} = (E_k \cdots E_2 E_1)^{-1}, \quad A^{-1} = E_k \cdots E_2 E_1$

☆ $[A \mid I] \overset{R_{(1)}}{\sim} [E_1 A \mid E_1] \overset{R_{(2)}}{\sim} [E_2 E_1 A \mid E_2 E_1] \sim \cdots$

$\cdots \overset{R_{(k)}}{\sim} [E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1] = [I \mid A^{-1}]$

$(3) \Rightarrow (1)$

A : invertible iff A^{-1} : exist

$\forall \vec{b}$ s.t. $A\vec{x}=\vec{b} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{b}$, where $A^{-1}(A\vec{x}) = \vec{x}$

$\therefore \vec{x} = A^{-1}\vec{b}$

(check: $A(A^{-1}\vec{b}) = \vec{b}$)

1. $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$
2. $A \sim I$

(1) $\Rightarrow$ (2)

reduced row-echelon form

$A\vec{x} = \vec{b}$ has a solution $\Rightarrow$ $\text{rref}([A|\vec{b}]) = [H|\vec{c}]$

then $H = \text{rref}(A)$

① $H = I$ ✓

② $H \neq I$ $\because$ each column: above, below pivots are 0's

$\therefore H \neq I \Rightarrow$ some row has no pivot

i.e. row: all 0's

$\therefore$ the bottom row of H is $\vec{0}^T$

$\because A \sim H$ $\therefore \exists E_1, E_2, \ldots, E_k$: elementary matrices s.t. $H = E_k \cdots E_2 E_1 A$

pick $\vec{b} = (E_k \cdots E_2 E_1)^{-1} \vec{e}_n$, where $\vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$

then $[A|\vec{b}] \sim [H|\vec{e}_n] = \left[\begin{array}{ccccccc|c} & & & * & & & & 0 \\ & & & & & & & \vdots \\ & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$ No Solution $\times$

Thm

A, C : $n \times n$ matrices

$\Rightarrow$ $CA = I$ iff $AC = I$

p.f.

Assume $AC = I$, $\forall \vec{b}$, sol: $A\vec{x} = \vec{b}$, then $\vec{x} = C\vec{b}$ is a solution

Check: $A\vec{x} = AC\vec{b} = I\vec{b} = \vec{b}$ ✓

$\therefore$ $\forall \vec{b}$, $A\vec{x} = \vec{b}$ has a solution $\therefore$ $A \sim I$

$\therefore$ $\exists$ $E_k \cdots E_2 E_1$: elementary matrices s.t. $I = E_k \cdots E_2 E_1 A$

let $D = E_k \cdots E_2 E_1$ $\therefore$ $\begin{cases} AC = I \\ DA = I \end{cases}$ $\therefore$ $C = D$

ex:

$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$, find A^{-1}

sol.

$$\left[\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 9 & 1 & 0 \end{array}\right] \underset{\sim}{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\underset{\sim}{R_1 \to R_1 - 4R_2} \left[\begin{array}{cc|cc} 1 & 0 & -4 & 9 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$$

$$\boxed{\text{if } A^{-1} \text{ exist} \Rightarrow [A \mid I] \sim [I \mid A^{-1}]}$$

Thm

The following are equivalent:

1. $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}$
2. $A \sim I$
3. A : invertible
4. $\exists\, E_1, E_2, \dots, E_k$: elementary matrices s.t. $A = E_1 E_2 \cdots E_k$
5. let $\vec{a}_i$ = the i^{th} column vector of A, then $sp(\vec{a}_1, \dots, \vec{a}_n) = \mathbb{R}^n$

p.f.

(5) $\Leftrightarrow$ (1)

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$A\vec{x}=\vec{b} \Rightarrow \vec{b} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n}_{\text{linear comb. of } \{\vec{a}_i\}} \in sp(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$$

$\boxed{1-6}$

Def.

a linea system ($A\vec{x}=\vec{b}$) is homogeneous if $\vec{b}=\vec{0}$

$\triangle$ $A\vec{x}=\vec{b}$ has solution $\vec{x}_1 \neq \vec{x}_2$

$\Rightarrow A\vec{x}_1=\vec{b}=A\vec{x}_2 \Rightarrow A\vec{x}_1-A\vec{x}_2=A(\vec{x}_1-\vec{x}_2)=\vec{b}-\vec{b}=\vec{0}$

$\therefore A\vec{x}=\vec{0}$ has non-trivial solution $(\neq\vec{0})$

Thm

$A\vec{x}=\vec{0}$ has solutions $\vec{h}_1, \vec{h}_2$

$\Rightarrow \forall r, s \in \mathbb{R}$, $r\vec{h}_1 + s\vec{h}_2$ still a solution for $A\vec{x}=\vec{0}$

p.f.

$A\vec{h}_1 = \vec{0}$, $A\vec{h}_2 = \vec{0}$

$\forall r, s \in \mathbb{R}$, $A(r\vec{h}_1 + s\vec{h}_2) = A(r\vec{h}_1) + A(s\vec{h}_2) = rA\vec{h}_1 + sA\vec{h}_2$

$= r\vec{0} + s\vec{0} = \vec{0} + \vec{0} = \vec{0}$

Def.

W is a **subspace** of $\mathbb{R}^n$

if ① W : subset of $\mathbb{R}^n$

② (i) **closed under vector addition** : $\forall \vec{u}, \vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W$

(ii) **closed under scalar multiplication** : $\forall r \in \mathbb{R}, \forall \vec{u} \in W \Rightarrow r\vec{u} \in W$

6

Given A: matrix. Let $W = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$\Rightarrow W$: subspace of $\mathbb{R}^n$

ex: $W = \{[x, 2x] \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$

check : ② (i) $\forall a, b \in \mathbb{R}$, let $\vec{u} = [a, 2a]$, $\vec{v} = [b, 2b]$

$\vec{u} + \vec{v} = [a, 2a] + [b, 2b] = [a+b, 2a+2b] = [a+b, 2(a+b)] \in W$

② (ii) $\forall r$, $r\vec{u} = r[a, 2a] = [ra, 2(ra)] \in W$

$\therefore W$: subspace of $\mathbb{R}^2$

ex: $W = \{\vec{0}\} \subset \mathbb{R}^n$

Check : ② (i) $\vec{0} + \vec{0} = \vec{0} \in W$ $\therefore W$: subspace of $\mathbb{R}^n$

② (ii) $\forall r \in \mathbb{R}$, $r\vec{0} = \vec{0} \in W$

Thm

Given $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \mathbb{R}^n$, let $W = sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$

$\Rightarrow$ W : subspace of $\mathbb{R}^n$

p.f.

① W : subset of $\mathbb{R}^n$

② (i) $\vec{u}, \vec{v} \in W$

let $\vec{u} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \ldots + r_k \vec{w}_k$, where $r_1, \ldots, r_k, s_1, \ldots, s_k \in \mathbb{R}$

$\vec{v} = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \ldots + s_k \vec{w}_k$

$\vec{u} + \vec{v} = (r_1 + s_1)\vec{w}_1 + (r_2 + s_2)\vec{w}_2 + \ldots + (r_k + s_k)\vec{w}_k \in W$

(ii) $\forall \alpha \in \mathbb{R}$

$\alpha \vec{u} = \alpha (r_1 \vec{w}_1 + r_2 \vec{w}_2 + \ldots + r_k \vec{w}_k)$

$= (\alpha r_1)\vec{w}_1 + (\alpha r_2)\vec{w}_2 + \ldots + (\alpha r_k)\vec{w}_k \in W$

$\therefore$ W : subspace of $\mathbb{R}^n$

Def. $A_{m\times n}$

1. The nullspace of matrix $A_{m\times n}$ is $\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\}$
2. The row space of A = sp(row vectors of A) $\subset \mathbb{R}^n$
3. The column space of A = sp(column vectors of A) $\subset \mathbb{R}^m$

Moreover, nullspace, row space are subspaces of $\mathbb{R}^n$, column space is subspace of $\mathbb{R}^m$.

ex:

$$A = \begin{array}{c} \begin{array}{ccc} & & \downarrow r \end{array} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \end{array} \qquad A\vec{x}=\vec{0} \Rightarrow \begin{cases} x_1 + 3r = 0 \\ x_2 - 1r = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3r \\ x_2 = r \\ x_3 = r \end{cases}$$

1. nullspace of A = $sp\left(\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right) \subset \mathbb{R}^3$
2. row space of A = $sp([1,0,3],[0,1,-1]) \subset \mathbb{R}^3$
3. column space of A = $sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) \subset \mathbb{R}^2$

2021-10-14

Thm.

$\vec{p}$ is particular solution of $A\vec{x} = \vec{b}$

then 「 $\vec{v}$ is a solution of $A\vec{x} = \vec{b}$ 」

iff 「 $\vec{v}$ has the form $\vec{p} + \vec{h}$, where $\vec{h}$ is a solution of $A\vec{x} = \vec{0}$ 」

pf.

($\Leftarrow$)

$$A(\vec{p}+\vec{h}) = A\vec{p} + A\vec{h} = \vec{b} + \vec{0} = \vec{b}$$

($\Rightarrow$)

$$\left.\begin{cases} A\vec{v} = \vec{b} \\ A\vec{p} = \vec{b} \end{cases}\right] \Rightarrow \underset{\vec{b}-\vec{b}=\vec{0}}{\underset{\parallel}{A\vec{v} - A\vec{p}}} = A(\vec{v}-\vec{p})$$

$$\therefore \vec{v} = \vec{p} + (\underbrace{\vec{v}-\vec{p}}_{\vec{h}})$$

Thm. $A_{n\times n}$

A : invertible

iff $A\vec{x}=\vec{0}$ has only one solution

iff if $A\vec{x}=\vec{b}$ has a solution, then the solution is unique.

iff $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^n$

pf.

if A : invertible, $A(A^{-1}\vec{b})=\vec{b}$

$\therefore \vec{x}=A^{-1}\vec{b}$ is a solution

if $\vec{v}$ is another solution, $A^{-1}(\ulcorner A\vec{v}=\vec{b}\lrcorner) \Rightarrow \vec{v}=A^{-1}\vec{b}=\vec{x}$

Thm

$A_{m\times n}$

(1) if $A\vec{x}=\vec{b}$ has a solution, then the solution is unique

iff (2) $H=\text{rref}(A)$, $H=\begin{bmatrix} I_{n\times n} \\ \hline O_{m-n,n} \end{bmatrix}$ ← $m\geq n$

pf.

(2) ⇒ (1)

$[A|\vec{b}] \sim [H|\vec{c}] = \left[\begin{array}{c|c} \frac{I}{O} & \vec{c} \end{array}\right]$ ⇒ the solution is unique if it exists.

(1) ⇒ (2)

every column of H has a pivot. #(pivot) = n

every row of H has at most one pivot ⇒ $m\geq n$

$A=[a_{ij}]$, say pivots are $a_{1\sigma(1)}, a_{2\sigma(2)}, \dots, a_{n\sigma(n)}$

$1\leq \sigma(1) < \sigma(2) < \sigma(3) < \cdots < \sigma(n) \leq n \Rightarrow \sigma(j)=j$

Cor.

$A_{m\times n}$, $m<n$ (m : 方程數， n : 變數个數)

$\Rightarrow$ (1) $A\vec{x}=\vec{0}$ has infinitely many solution.

(2) $A\vec{x}=\vec{b}$ has a solution $\Rightarrow$ the solution is NOT unique.

Recall

1. If $A\vec{x}=\vec{b}$ has a solution $\Rightarrow$ $\vec{b}\in col(A)$
2. the column space of A (col(A)) is a <u>subspace of $\mathbb{R}^n$</u>

   and the # of <u>free variable</u> is equal to the # of non-pivot column of A