應數一線性代數 2020 春, 期末考

學號: sol., 姓名:

本次考試共有 10 頁 (包含封面)，有 14 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁以及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：誠敬弘遠

誠，一生動念都是誠實端正的。敬，就是對知識的認真尊重。宏，開拓視界，恢宏心胸。遠，任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名:

1. (10 points) Find the projection matrix P for the plane $W : 3x + 2y + z = 0$ in $\mathbb{R}^3$ and the find the projection $\vec{b}_w$ of $\vec{b} = [4, 2, -1]$ on it.

Answer: $\vec{b}_w = \frac{1}{14}\begin{bmatrix} 11 \\ -2 \\ -29 \end{bmatrix}$, $P = \frac{1}{14}\begin{bmatrix} 5 & -6 & -3 \\ -6 & 10 & -2 \\ -3 & -2 & 13 \end{bmatrix}$.

2. (10 points) Find the lease squares straight line fit to the four points (0,1) (1,3) (2,4) (3,4) and use it to approximate the fifth points (4, a).

Answer: the line equation = $1.5 + x$, a= 5.5.

3. (5 points) Find the coordinate vector of $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in M_2 relative to $\left(\overset{a}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \overset{+b}{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}, \overset{+c}{\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}}, \overset{+d}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \right)$

Answer: [3, 5, 1, 1]

$$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c & -c \\ 0 & 3c \end{bmatrix} + \begin{bmatrix} 0 & d \\ 0 & d \end{bmatrix}$$

$1 = 0 + 0 + c + 0$

$-2 =$

$3 =$

$4 =$

4. (10 points) Find the five fifth roots of $-243i$. (need not simplify)

5th

A: $3\left(\cos\left(\frac{3\pi}{10} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{3\pi}{10} + \frac{2k\pi}{5}\right)\right)$, $k = 0, 1, 2, 3, 4$

$-243i = r(\cos\theta + i\sin\theta)$

$\vec{v} = \sqrt[5]{r}\left(\cos\left(\frac{\theta}{5} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\theta}{5} + \frac{2\pi k}{5}\right)\right)$ $\quad k = 0 \sim 4$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for the given linear transformation T.
$T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as reflection of $\mathbb{R}^3$ through the plane $x+y-z=0$; $B = E$, $B' = ([1,0,1],[1,-1,0],[1,1,-1])$.

$C_{BB'}$= $\frac{1}{3}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $C_{BB'}$= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $R_{B'B'}$= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and R_{BB}= $\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

Is C=$C_{BB'}$ or $C_{BB'}$? $C_{B'B}$.

$\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \in x+y-z=0 \quad \Rightarrow \quad T(\vec{v}_1) = \vec{v}_1$

$T(\vec{v}_2) = \vec{v}_2$

normal of $x+y-z=0$; $\overset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$ $\quad T(\vec{v}_3) = -\vec{v}_3$

6. (5 points) Express $(\sqrt{3}+i)^6$ in the form $a+bi$ for a, b are real numbers.

   Answer: a=$\underline{\ -64\ }$, b=$\underline{\ 0\ }$.

$\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$(\sqrt{3}+i)^6 = r^6(\cos 6\theta + i\sin 6\theta)$

$\vec{v}_1$ $\vec{v}_2$ $[1,0,0], [0,1,0], [0,0,1]\}$

7. (10 points) Using the Gram-Schmidt process to transform the basis $\{[1, i, 1-i], [1+i, 1-i, 1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

   Answer: the found orthogonal basis for $\mathbb{C}^3$ is $\{[1, i, 1-i], \underbrace{[3+3i, 5-5i, 2]}_{\vec{v}_2 - \vec{v}_{2\vec{v}_1}}, [-12i, 4, 8+8i]\}$

G-S: $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$

8. (10 points) Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$. Also find where

$$A = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{g}_1 \quad \vec{g}_2 \quad \vec{g}_3$$

① $|A - \lambda I| = \lambda(\lambda-2)^2$

②

$\boxed{\lambda = 0}$

$\text{rref}(A - 0I) \to A\vec{v}_1 = 0\vec{v}_1$

$\boxed{\lambda = 2}$

$\text{rref}(A - 2I) \Rightarrow sp(\vec{v}_2, \vec{v}_3)$

③ G-S. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\to \{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$

☆ $\because 0 \neq 2$

$\therefore \vec{v}_1 \perp sp(\vec{v}_2, \vec{v}_3)$

$\vec{g}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|}$

G-S: $\{\vec{v}_2, \vec{v}_3\}$

9. (10 points) Find a Jordan canonical form and a Jordan basis for the given matrix.

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$J = \begin{bmatrix} \underbrace{\boxed{i}}_{J_1} & & & 0 \\ & \underbrace{\boxed{i}}_{J_2} & & \\ & & \underbrace{\boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}}}_{J_3} & \\ 0 & & & \underbrace{\boxed{2}}_{J_4} \end{bmatrix}$$

$$\text{basis}: \left\{ \underset{b_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 2-i \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}}, \underset{b_3}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{b_4}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}}, \underset{b_5}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$$

$(J-iI): \begin{cases} \vec{b}_1 \to \vec{0} \\ \vec{b}_2 \to \vec{0} \end{cases}$

$(J-2I): \vec{b}_4 \to \vec{b}_3 \to \vec{0}$

$\vec{b}_5 \to \vec{0}$

10. (10 points) Find a polynomial in A that gives the zero matrix.

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$(A-iI)^2(A-2I)^1$

11. (5 points) Prove or disprove the following: All 2×2 matrix with determinant 1 is an orthogonal matrix.

$\exists A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ s.t. $\det(A) = 1$, but A is NOT orthogonal.

12. (10 points) Find all the possible 2×2 real matrix that is unitarily diagonalizable.

13. (5 points) Prove that for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $(\vec{u}^* \vec{v})^* = \overline{\vec{u}^* \vec{v}} = \vec{v}^* \vec{u} = \vec{u}^T \overline{\vec{v}}$

14. (10 points) Prove the following:
   (a) Show that every Hermitian matrix is normal.
   (b) Show that every unitary matrix is normal.
   (c) Show that, if $A^* = -A$, then A is normal.

應數一線性代數期末考, 學號: ____________, 姓名: ____________, 以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	8
Points:	10	10	5	10	10	5	10	10
Score:								

Question:	9	10	11	12	13	14		Total
Points:	10	10	5	10	5	10		120
Score:								