$\mathbb{R}$: real.

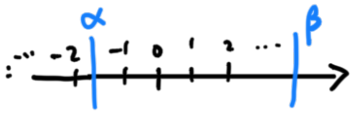
$\mathbb{C}$: complex $\rightarrow$ $a+bi$, where $a, b \in \mathbb{R}$

9.1 (數 + − × ÷) vector inner/dot product

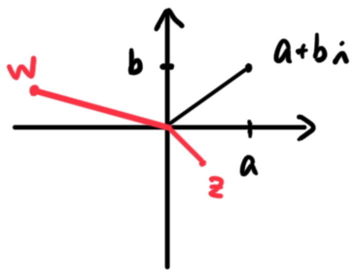
9.2 renew linear algebra

$A\vec{x} = \vec{b}$ $A \in \mathbb{C}^{n \times n}$, $\vec{b} \in \mathbb{C}^n$, 求 $\vec{x}$

$[A \mid \vec{b}] \sim [I \mid \vec{x}]$ ← 係數可為 $\mathbb{C}$

$\mathbb{R}$: 

∵ β 在 α 右 ∴ $\beta > \alpha$

$\mathbb{C}$ 

$z \not> \not< w$

沒有大小

∵ w 距 (0.0) 更遠

∴ $|w| > |z|$

norm , modulus , magnitude

$\mathbb{R}^1$: 1-dim , basis {1} , $\mathbb{R}^n$: n-dim ← basis has n elements
scalar with $\mathbb{R}$

$\mathbb{C}^1$: 2-dim (real vector space) , basis {1, i} , scalar with $\mathbb{R}$

1-dim (complex vector space) , basis {1} , scalar with $\mathbb{C}$

$\mathbb{C}^n$: 2n-dim real vector space

n-dim complex vector space

---

ex: $S = \{[1, 2i, 1+i], [1, 3i, i], [0, 1+i, -1]\}$

Is it linear indep for $\mathbb{C}^3$?

sol.

$$\begin{bmatrix} 1 & 2i & 1+i \\ 1 & 3i & i \\ 0 & 1+i & -1 \end{bmatrix} \underset{c}{\sim} \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \quad \therefore \text{ indep.}$$

$\therefore$ $W = sp(S)$: 3-dim complex V.S.

---

sol. linear system , inverse , $\vec{v}_B$, indep , basis . . .

$\mathbb{R}^n$, inner product

$\vec{x} = [x_1, x_2, \ldots, x_n]^T$, $\vec{y} = [y_1, y_2 \ldots y_n]^T$

② $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$

① $\boxed{\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}}$ $= x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$ 有交換性

$\vec{x}^T \vec{y} = [x_1 \; x_2 \; \ldots \; x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$\mathbb{C}^1$, inner product.

$\vec{x} = [\, z \,]_{1\times 1}$, $z = a+bi$, $|z| = \sqrt{a^2+b^2}$ $\quad \therefore |z|^2 = a^2+b^2$

$\vec{x} \cdot \vec{x} \underset{\text{hope!}}{=} a^2+b^2 = \underbrace{(a-bi)}_{\bar{z}}\underbrace{(a+bi)}_{z} = \bar{z} z$

the conjugate of z

$z = a+bi$, $\bar{z} = a-bi$ for $a, b \in \mathbb{R}$

$\vec{x} \cdot \vec{x} = [\bar{z}][z]$

Def inner product. in $\mathbb{C}^n$, $\vec{u}, \vec{v} \in \mathbb{C}^n$

$\vec{u} = [u_1, u_2, \ldots, u_n]^T$. $\vec{v} = [v_1, v_2, \ldots, v_n]^T$

$\langle \vec{u}, \vec{v} \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \ldots + \bar{u}_n v_n$ ← 無交換性

$$\boxed{\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= [\bar{u}_1, \ldots, \bar{u}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \bar{\vec{u}}^T \vec{v} \\ &= \vec{u}^* \vec{v} \end{aligned}}$$

Check

$\vec{v} \in \mathbb{C}^n$, $\vec{v} = [v_1, v_2 \ldots, v_n]$

$\langle \vec{v}, \vec{v} \rangle = \bar{v}_1 v_1 + \bar{v}_2 v_2 + \ldots + \bar{v}_n v_n = |v_1|^2 + |v_2|^2 + \ldots + |v_n|^2$

Thm

$\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$, $z \in \mathbb{C}$

1. $\langle \vec{u}, \vec{u} \rangle \geq 0$, $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$

2. $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ in $\mathbb{R}^n$ $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

3. $\langle (\vec{u} + \vec{v}), w \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

4. $\langle \vec{w}, (\vec{u} + \vec{v}) \rangle = \langle \vec{w}, \vec{u} \rangle + \langle \vec{w} + \vec{v} \rangle$

5. $\langle z\vec{u}, \vec{v} \rangle = \bar{z} \langle \vec{u}, \vec{v} \rangle$ in $\mathbb{R}^n$ $(r\vec{x}) \cdot \vec{y} = \vec{x} \cdot (r\vec{y}) = r(\vec{x} \cdot \vec{y})$

6. $\langle \vec{u}, z\vec{v} \rangle = z \langle \vec{u}, \vec{v} \rangle$

p.f.

5. $\langle z\vec{u}, \vec{v} \rangle = \overline{(z u_1)} v_1 + \overline{(z u_2)} v_2 + \dots + \overline{(z u_n)} v_n$

$= \bar{z} \bar{u}_1 v_1 + \bar{z} \bar{u}_2 v_2 + \dots + \bar{z} \bar{u}_n v_n$

$= \bar{z} (\bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n)$

$= \bar{z} \langle \vec{u}, \vec{v} \rangle$

6. $\langle \vec{u}, z\vec{v} \rangle = \bar{u}_1 (z v_1) + \bar{u}_2 (z v_2) + \dots + \bar{u}_n (z v_n)$

$= z(\bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n)$

$= z \langle \vec{u}, \vec{v} \rangle$

Def orthogonal

$\vec{u}, \vec{v} \in \mathbb{C}^n$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

---

in $\mathbb{R}^n$, Gram-Schmidt process

$S: \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$: linear indep.

(I) $\rightsquigarrow \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$: orthogonal. by $\vec{a}_i \rightsquigarrow \vec{v}_i = \vec{a}_i - (\vec{a}_i \text{ proj. onto } sp(\vec{a}_1, \dots \vec{a}_{i-1}))$

(II) $\rightsquigarrow \{\vec{q}_1, \vec{q}_2, \dots \vec{q}_n\}$: orthonormal by $\vec{v}_i \rightsquigarrow \vec{q}_i = \vec{v}_i / |\vec{v}_i|$

[I] $\vec{a}_i \rightsquigarrow \vec{v}_i$ in $\mathbb{R}^n$

① $\vec{v}_1 = \vec{a}_1$

② $\vec{v}_2 = \vec{a}_2 - \dfrac{\vec{a}_2 \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1}\vec{a}_1$ (hope $\vec{v}_1 \cdot \vec{v}_2 = 0$)

③ $\vec{v}_3 = \vec{a}_3 - \dfrac{\vec{a}_3 \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1}\vec{a}_1 - \dfrac{\vec{a}_3 \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2}\vec{a}_2$

⋮

in $\mathbb{C}^n$

① $\vec{v}_1 = \vec{a}_1$

② $\vec{v}_2$ s.t. $\langle \vec{v}_2, \vec{v}_1 \rangle = 0$ $\boxed{\text{check}}$

(i) $\vec{w}_2 = \vec{a}_2 - \dfrac{\langle \vec{a}_2, \vec{a}_1 \rangle}{\langle \vec{a}_1, \vec{a}_1 \rangle}\vec{a}_1$ X

(ii) $\boxed{\vec{w}_2' = \vec{a}_2 - \dfrac{\langle \vec{a}_1, \vec{a}_2 \rangle}{\langle \vec{a}_1, \vec{a}_1 \rangle}\vec{a}_1}$ ✓

③ ...

$\boxed{\text{Check}}$

$$\vec{w} = \vec{u} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v}$$

$$\begin{aligned}
\langle \vec{w}, \vec{v} \rangle &= \left\langle \vec{u} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v}, \vec{v} \right\rangle \\
&= \langle \vec{u}, \vec{v} \rangle - \left\langle \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v}, \vec{v} \right\rangle \\
&= \langle \vec{u}, \vec{v} \rangle - \overline{\left(\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle}\right)}\langle \vec{v}, \vec{v} \rangle \\
&= \langle \vec{u}, \vec{v} \rangle - \frac{\overline{\langle \vec{v}, \vec{u} \rangle}}{\overline{\langle \vec{v}, \vec{v} \rangle}}\langle \vec{v}, \vec{v} \rangle \\
&= \langle \vec{u}, \vec{v} \rangle - \overline{\langle \vec{v}, \vec{u} \rangle} \\
&= 0
\end{aligned}$$

$\because \langle \vec{v}, \vec{v} \rangle \in \mathbb{R}$

$\therefore \langle \vec{v}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{v} \rangle}$

[II] $\vec{v}_i \rightsquigarrow \vec{q}_i$ in $\mathbb{R}^n$

$$\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

in $\mathbb{C}^n$

$$\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|} = \frac{\vec{v}_i}{\sqrt{\langle \vec{v}_i, \vec{v}_i \rangle}}$$

ex: $\{ \underset{\vec{a}_1}{[1, i, i]}, \underset{\vec{a}_2}{[1, 0, -i]}, \underset{a_3}{[1, 0, 1]} \}$

$\boxed{\text{G-S}}$

(I) ① $\vec{v}_1 = \vec{a}_1 = [1, i, i]$

② $\vec{v}_2 = \vec{a}_2 - \dfrac{\langle \vec{v}_1, \vec{a}_2 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1$

$\because \langle \vec{v}_1, \vec{a}_2 \rangle = 1 \times 1 + (-i) \times 0 + (-i)(-i) = 0$

$\therefore \vec{v}_1 \perp \vec{a}_2$

$\vec{v}_2 = \vec{a}_2$

③ $\vec{v}_3 = \vec{a}_3 - \dfrac{\langle \vec{v}_1, \vec{a}_3 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \dfrac{\langle \vec{v}_2, \vec{a}_3 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2$

$= \vec{a}_3 - \dfrac{1 \times 1 + (-i) \times 0 + (-i) \times 1}{|1|^2 + |i|^2 + |i|^2} \vec{v}_1 - \dfrac{1 \times 1 + 0 \times 0 + i \times 1}{|1|^2 + |0|^2 + |-i|^2} \vec{v}_2$

$= \vec{a}_3 - \dfrac{1-i}{3} \vec{v}_1 - \dfrac{1+i}{2} \vec{v}_2$

$= \vec{a}_3 - \frac{1}{3} [1-i, 1+i, 1+i] - \frac{1}{2} [1+i, 0, 1-i]$

$= \frac{1}{6} [1-i, -2-2i, 1+i]$

$\therefore \{\vec{v}_i\} = \{ \underset{\vec{v}_1}{[1, i, i]}, \underset{\vec{v}_2}{[1, 0, -i]}, \underset{\vec{v}_3}{[1-i, -2-2i, 1+i]} \}$

(II) $\|\vec{v}_1\| = \sqrt{|1|^2 + |i|^2 + |i|^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\|\vec{v}_2\| = \sqrt{|1|^2 + |0|^2 + |-i|^2} = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$\|\vec{v}_3\| = \sqrt{|1-i|^2 + |-2-2i|^2 + |1+i|^2} = \sqrt{(1^2+1^2) + (2^2+2^2) + (1^2+1^2)}$

$= \sqrt{12}$

$\therefore \{\vec{q}_i\} = \{ \frac{1}{\sqrt{3}} [1, i, i], \frac{1}{\sqrt{2}} [1, 0, -i], \frac{1}{\sqrt{12}} [1-i, -2-2i, 1+i] \}$

Def. $A: m\times n$ $A \in \mathbb{C}^{m\times n}$, $A = [a_{ij}]$

1. conjugate of A : $\boxed{\overline{A}} = [\overline{a_{ij}}]_{m\times n}$

2. conjugate transpose of A ; $\boxed{A^*} = \left([\overline{a_{ij}}]^T\right)_{n\times m}$

$\therefore$ $\boxed{\langle \vec{u}, \vec{v} \rangle = \vec{u}^* \vec{v} \quad , \vec{u}, \vec{v} \in \mathbb{C}^n}$

Def.

A : orthogonal if $\boxed{A^T A = I}$

Note:

$A^T A = I \Rightarrow$ col. of A are orthornormal in $\mathbb{R}$ v.s.

Def.

A : unitary

if the col. of A are orthornormal in $\mathbb{C}$ v.s.

$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$, i.e. $\{\vec{v}_1, \dots, \vec{v}_n\}$: orthornormal

i.e. $\begin{cases} \forall\, i \neq j, & \langle \vec{v}_i, \vec{v}_j \rangle = 0 = \vec{v}_i^* \vec{v}_j = \begin{bmatrix} \overline{\quad\quad} \end{bmatrix} \begin{bmatrix} \; \end{bmatrix} \\ i = j, & \langle \vec{v}_i, \vec{v}_i \rangle = 1 \end{cases}$

i.e. $\overline{A}^T A = \boxed{I = A^* A}$

---

ex:

$A = \begin{bmatrix} 1 & i \\ 1+i & 1-i \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 1+i \\ i & 1-i \end{bmatrix}$, $A^* = \overline{A^T} = \begin{bmatrix} 1 & 1-i \\ -i & 1+i \end{bmatrix}$

$A^* A = \begin{bmatrix} 1 & 1-i \\ -i & 1+i \end{bmatrix} \begin{bmatrix} 1 & i \\ 1+i & 1-i \end{bmatrix} = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}$ NOT unitary ex: $A = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$A^* A = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore \text{unitary}$$