We replace $\mathbf{v}_2$ by $3\mathbf{v}_2$, obtaining $\mathbf{v}_2 = 3\sqrt{x} - 2$, and compute $\mathbf{v}_3$ as

$$\begin{aligned}\mathbf{v}_3 &= x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx}1 - \frac{\int_0^1 x(3\sqrt{x}-2)\,dx}{\int_0^1 (3\sqrt{x}-2)^2\,dx}(3\sqrt{x}-2)\\ &= x - \frac{1/2}{1} - \frac{6/5-1}{9/2-8+4}(3\sqrt{x}-2) = x - \frac{1}{2} - \frac{2}{5}(3\sqrt{x}-2)\\ &= x - \frac{6}{5}\sqrt{x} + \frac{3}{10}.\end{aligned}$$

Replacing $\mathbf{v}_3$ by $10\mathbf{v}_3$, we obtain the orthogonal basis

$$\{1, 3\sqrt{x} - 2, 10x - 12\sqrt{x} + 3\}.$$ ■

SUMMARY

1. A basis for a subspace W is orthogonal if the basis vectors are mutually perpendicular, and it is orthonormal if the vectors also have length 1.
2. Any orthogonal set of vectors in $\mathbb{R}^n$ is a basis for the subspace it generates.
3. Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis. The projection of a vector $\mathbf{b}$ in $\mathbb{R}^n$ on W is equal to the sum of the projections of $\mathbf{b}$ on each basis vector.
4. Every nonzero subspace W of $\mathbb{R}^n$ has an orthonormal basis. Any basis can be transformed into an orthogonal basis by means of the Gram–Schmidt process, in which each vector $\mathbf{a}_j$ of the given basis is replaced by the vector $\mathbf{v}_j$ obtained by subtracting from $\mathbf{a}_j$ its projection on the subspace generated by its predecessors.
5. Any orthogonal set of vectors in a subspace W of $\mathbb{R}^n$ can be expanded, if necessary, to an orthogonal basis for W.
6. Let A be an $n \times k$ matrix of rank k. Then A can be factored as QR, where Q is an $n \times k$ matrix with orthonormal column vectors and R is a $k \times k$ upper-triangular invertible matrix.

EXERCISES

In Exercises 1–4, verify that the generating set of the given subspace W is orthogonal, and find the projection of the given vector $\mathbf{b}$ *on* W.

1. $W = \mathrm{sp}([2, 3, 1], [-1, 1, -1])$; $\mathbf{b} = [2, 1, 4]$
2. $W = \mathrm{sp}([-1, 0, 1], [1, 1, 1])$; $\mathbf{b} = [1, 2, 3]$
3. $W = \mathrm{sp}([1, -1, -1, 1], [1, 1, 1, 1], [-1, 0, 0, 1])$; $\mathbf{b} = [2, 1, 3, 1]$
4. $W = \mathrm{sp}([1, -1, 1, 1], [-1, 1, 1, 1], [1, 1, -1, 1])$; $\mathbf{b} = [1, 4, 1, 2]$
5. Find an orthonormal basis for the plane $2x + 3y + z = 0$.

6. Find an orthonormal basis for the subspace

$$W = \{[x_1, x_2, x_3, x_4] \mid x_1 = x_2 + 2x_3, \; x_4 = -x_2 + x_3\}$$

of $\mathbb{R}^4$.

7. Find an orthonormal basis for the subspace sp([0, 1, 0], [1, 1, 1]) of $\mathbb{R}^3$.

8. Find an orthonormal basis for the subspace sp([1, 1, 0], [−1, 2, 1]) of $\mathbb{R}^3$.

9. Transform the basis {[1, 0, 1], [0, 1, 2], [2, 1, 0]} for $\mathbb{R}^3$ into an orthonormal basis, using the Gram–Schmidt process.

10. Repeat Exercise 9, using the basis {[1, 1, 1], [1, 0, 1], [0, 1, 1]} for $\mathbb{R}^3$.

11. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by [1, 0, 1, 0], [1, 1, 1, 0], and [1, −1, 0, 1].

12. Find an orthonormal basis for the subspace sp([1, −1, 1, 0, 0], [−1, 0, 0, 0, 1], [0, 0, 1, 0, 1], [1, 0, 0, 1, 1]) of $\mathbb{R}^5$.

13. Find the projection of [5, −3, 4] on the subspace in Exercise 7, using the orthonormal basis found there.

14. Repeat Exercise 13, but use the subspace in Exercise 8.

15. Find the projection of [2, 0, −1, 1] on the subspace in Exercise 11, using the orthonormal basis found there.

16. Find the projection of [−1, 0, 0, 1, −1] on the subspace in Exercise 12, using the orthonormal basis found there.

17. Find an orthonormal basis for $\mathbb{R}^4$ that contains an orthonormal basis for the subspace sp([1, 0, 1, 0], [0, 1, 1, 0]).

18. Find an orthogonal basis for the orthogonal complement of sp([1, −1, 3]) in $\mathbb{R}^3$.

19. Find an orthogonal basis for the nullspace of the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 5 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}.$$

20. Find an orthonormal basis for $\mathbb{R}^3$ that contains the vector $(1/\sqrt{3})[1, 1, 1]$.

21. Find an orthonormal basis for sp([2, 1, 1], [1, −1, 2]) that contains $(1/\sqrt{6})[2, 1, 1]$.

22. Find an orthogonal basis for sp([1, 2, 1, 2], [2, 1, 2, 0]) that contains [1, 2, 1, 2].

23. Find an orthogonal basis for sp([2, 1, −1, 1], [1, 1, 3, 0], [1, 1, 1, 1]) that contains [2, 1, −1, 1] and [1, 1, 3, 0].

24. Let B be the ordered orthonormal basis

$$\left(\left[\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right], \left[\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right], \left[-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right]\right) \text{ for } \mathbb{R}^3.$$

a. Find the coordinate vectors $[c_1, c_2, c_3]$ for [1, 2, −4] and $[d_1, d_2, d_3]$ for [5, −3, 2], relative to the ordered basis B.

b. Compute [1, 2, −4] · [5 − 3, 2], and then compute $[c_1, c_2, c_3] \cdot [d_1, d_2, d_3]$. What do you notice?

25. Mark each of the following True or False.

___ a. All vectors in an orthogonal basis have length 1.

___ b. All vectors in an orthonormal basis have length 1.

___ c. Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis.

___ d. Every vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.

___ e. Every nonzero vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.

___ f. Every unit vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.

___ g. Every $n \times k$ matrix A has a factorization $A = QR$, where the column vectors of Q form an orthonormal set and R is an invertible $k \times k$ matrix.

___ h. Every $n \times k$ matrix A of rank k has a factorization $A = QR$, where the column vectors of Q form an orthonormal set and R is an invertible $k \times k$ matrix.

___ i. It is advantageous to work with an orthogonal basis for W when projecting a vector $\mathbf{b}$ in $\mathbb{R}^n$ on a subspace W of $\mathbb{R}^n$.

___ j. It is even more advantageous to work with an orthonormal basis for W when performing the projection in part (i).

In Exercises 26–28, use the text answers for the indicated earlier exercise to find a QR-factorization of the matrix having as column vectors the transposes of the row vectors given in that exercise.

26. Exercise 7 27. Exercise 9 28. Exercise 11

29. Let A be an $n \times k$ matrix. Prove that the column vectors of A are orthonormal if and only if $A^TA = I$.

30. Let A be an $n \times n$ matrix. Prove that A has orthonormal column vectors if and only if A is invertible with inverse $A^{-1} = A^T$.

31. Let A be an $n \times n$ matrix. Prove that the column vectors of A are orthonormal if and only if the row vectors of A are orthonormal. [HINT: Use Exercise 30 and the fact that A commutes with its inverse.]

Exercises 32–35 involve inner-product spaces.

32. Let V be an inner-product space of dimension n, and let B be an ordered orthonormal basis for V. Prove that, for any vectors $\mathbf{a}$ and $\mathbf{b}$ in V, the inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ is equal to the dot product of the coordinate vectors of $\mathbf{a}$ and $\mathbf{b}$ relative to B. (See Exercise 24 for an illustration.)

33. Find an orthonormal basis for sp(sin x, cos x) for $0 \le x \le \pi$ if the inner product is defined by $\langle f, g \rangle = \int_0^\pi f(x)g(x)\,dx$.

34. Find an orthonormal basis for sp$(1, x, x^2)$ for $-1 \le x \le 1$ if the inner product is defined by $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

35. Find an orthonormal basis for sp$(1, e^x)$ for $0 \le x \le 1$ if the inner product is defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$.

The routine QRFACTOR in LINTEK allows the user to enter k independent row vectors in $\mathbb{R}^n$ for n and k at most 10. The program can then be used to find an orthonormal set of vectors spanning the same subspace. It will also exhibit a QR-factorization of the $n \times k$ matrix A having the entered vectors as column vectors.

For an $n \times k$ matrix A of rank k, the command [Q,R] = qr(A) *in MATLAB produces an $n \times n$ matrix Q whose columns form an orthonormal basis for $\mathbb{R}^n$ and an $n \times k$ upper-triangular matrix R (that is, with $r_{ij} = 0$ for $i > j$) such that $A = QR$. The first k columns of Q comprise the $n \times k$ matrix Q described in Corollary 1 of Theorem 6.4, and R is the $k \times k$ matrix R described in Corollary 1 with $n - k$ rows of zeros supplied at the bottom to make it the same size as A.*

In Exercises 36–38, use MATLAB or LINTEK as just described to check the answers you gave for the indicated preceding exercise. (Note that the order you took for the vectors in the Gram–Schmidt process in those exercises must be the same as the order in which you supply them in the software to be able to check your answers.)

36. Exercises 7–12

37. Exercises 17, 20–23

38. Exercises 26–28

6.3 ORTHOGONAL MATRICES

Let A be the $n \times n$ matrix with column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Recall that these vectors form an orthonormal basis for $\mathbb{R}^n$ if and only if

$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 0 & \text{if } i \neq j, \quad \mathbf{a}_i \perp \mathbf{a}_j \\ 1 & \text{if } i = j. \quad \|\mathbf{a}_i\| = 1 \end{cases}$$

Because

$$A^TA = \begin{bmatrix} \text{———} & \mathbf{a}_1 & \text{———} \\ \text{———} & \mathbf{a}_2 & \text{———} \\ & \vdots & \\ \text{———} & \mathbf{a}_n & \text{———} \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}$$