SUMMARY

1. A subset W of a vector space V is a *subspace* of V if and only if it is nonempty and satisfies the two closure properties:

   $\mathbf{v} + \mathbf{w}$ is contained in W for all vectors $\mathbf{v}$ and $\mathbf{w}$ in W, and
   $r\mathbf{v}$ is contained in W for all vectors $\mathbf{v}$ in W and all scalars r.
2. Let X be a subset of a vector space V. The set $\mathrm{sp}(X)$ of all linear combinations of vectors in X is a subspace of V called the *span* of X, or the *subspace of V generated by the vectors in X*. It is the smallest subspace of V containing all the vectors in X.
3. A vector space V is *finitely generated* if $V = \mathrm{sp}(X)$ for some set $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ containing only a finite number of vectors in V.
4. A set X of vectors in a vector space V is *linearly dependent* if there exists a dependence relation

   $$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0}, \qquad \text{at least one } r_j \neq 0,$$

   where each $\mathbf{v}_i \in X$ and each $r_i \in \mathbb{R}$. The set X is *linearly independent* if no such dependence relation exists, and so a linear combination of vectors in X is the zero vector only if all scalar coefficients are zero.
5. A set B of vectors in a vector space V is a *basis* for V if B spans V and is independent.
6. A subset B of nonzero vectors in a vector space V is a basis for V if and only if every nonzero vector in V can be expressed as a linear combination of vectors in B in a *unique* way.
7. If X is a finite set of vectors spanning a vector space V, then X can be reduced, if necessary, to a basis for V by deleting in turn any vector that can be expressed as a linear combination of those remaining.
8. If a vector space V has a finite basis, then all bases for V have the same number of vectors. The number of vectors in a basis for V is the *dimension* of V, denoted by $\dim(V)$.
9. The following are equivalent for n vectors in a vector space V where $\dim(V) = n$.
   a. The vectors are linearly independent.
   b. The vectors generate V.

EXERCISES

In Exercises 1–6, determine whether the indicated subset is a subspace of the given vector space.

1. The set of all polynomials of degree greater than 3 together with the zero polynomial in the vector space P of all polynomials with coefficients in $\mathbb{R}$
2. The set of all polynomials of degree 4 together with the zero polynomial in the vector space P of all polynomials in x

3. The set of all functions f such that $f(0) = 1$ in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$
4. The set of all functions f such that $f(1) = 0$ in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$
5. The set of all functions f in the vector space W of differentiable functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Illustration 6) such that $f'(2) = 0$
6. The set of all functions f in the vector space W of differentiable functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Illustration 6) such that f has derivatives of all orders
7. Let F be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$. Show that
   a. $\mathrm{sp}(\sin^2 x, \cos^2 x)$ contains all constant functions,
   b. $\mathrm{sp}(\sin^2 x, \cos^2 x)$ contains the function $\cos 2x$,
   c. $\mathrm{sp}(7, \sin^2 2x)$ contains the function $8 \cos 4x$.
8. Let P be the vector space of polynomials. Prove that $\mathrm{sp}(1, x) = \mathrm{sp}(1 + 2x, x)$. [HINT: Show that each of these subspaces is a subset of the other.]
9. Let V be a vector space, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in V. Follow the hint of Exercise 8 to prove that
   a. $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{sp}(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2)$,
   b. $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{sp}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$.
10. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be vectors in a vector space V. Give a necessary and sufficient condition, involving linear combinations, for

$$\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \mathrm{sp}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m).$$

In Exercises 11–13, determine whether the given set of vectors is dependent or independent.

11. $\{x^2 - 1, x^2 + 1, 4x, 2x - 3\}$ in P
12. $\{1, 4x + 3, 3x - 4, x^2 + 2, x - x^2\}$ in P
13. $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$ in F

In Exercises 14–19, use the technique discussed following Example 3 to determine whether the given set of functions in the vector space F is independent or dependent.

14. $\{\sin x, \cos x\}$
15. $\{1, x, x^2\}$
16. $\{\sin x, \sin 2x, \sin 3x\}$
17. $\{\sin x, \sin(-x)\}$
18. $\{e^{2x}, e^{3x}, e^{4x}\}$
19. $\{1, e^x + e^{-x}, e^x - e^{-x}\}$

In Exercises 20 and 21, determine whether or not the given set of vectors is a basis for the indicated vector space.

20. $\{x, x^2 + 1, (x - 1)^2\}$ for P_2
21. $\{x, (x + 1)^2, (x - 1)^2\}$ for P_2

In Exercises 22–24, find a basis for the given subspace of the vector space.

22. $\mathrm{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$ in P
23. $\mathrm{sp}(1, 4x + 3, 3x - 4, x^2 + 2, x - x^2)$ in P
24. $\mathrm{sp}(1, \sin^2 x, \cos 2x, \cos^2 x)$ in F
25. Mark each of the following True or False.
   ___ a. The set consisting of the zero vector is a subspace for every vector space.
   ___ b. Every vector space has at least two distinct subspaces.
   ___ c. Every vector space with a nonzero vector has at least two distinct subspaces.
   ___ d. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subset of a vector space V, then $\mathbf{v}_i$ is in $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ for $i = 1, 2, \ldots, n$.
   ___ e. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subset of a vector space V, then the sum $\mathbf{v}_i + \mathbf{v}_j$ is in $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ for all choices of i and j from 1 to n.
   ___ f. If $\mathbf{u} + \mathbf{v}$ lies in a subspace W of a vector space V, then both $\mathbf{u}$ and $\mathbf{v}$ lie in W.
   ___ g. Two subspaces of a vector space V may have empty intersection.
   ___ h. If S is independent, each vector in V can be expressed uniquely as a linear combination of vectors in S.
   ___ i. If S is independent and generates V, each vector in V can be expressed uniquely as a linear combination of vectors in S.
   ___ j. If each vector in V can be expressed uniquely as a linear combination of vectors in S, then S is an independent set.

26. Let V be a vector space. Mark each of the following True or False.
___ a. Every independent set of vectors in V is a basis for the subspace the vectors span.
___ b. If $\{v_1, v_2, \ldots, v_n\}$ generates V, then each $v \in V$ is a linear combination of the vectors in this set.
___ c. If $\{v_1, v_2, \ldots, v_n\}$ generates V, then each $v \in V$ is a unique linear combination of the vectors in this set.
___ d. If $\{v_1, v_2, \ldots, v_n\}$ generates V and is independent, then each $v \in V$ is a unique linear combination of the vectors in this set.
___ e. If $\{v_1, v_2, \ldots, v_n\}$ generates V, then this set of vectors is independent.
___ f. If each vector in V is a unique linear combination of the vectors in the set $\{v_1, v_2, \ldots, v_n\}$, then this set is independent.
___ g. If each vector in V is a unique linear combination of the vectors in the set $\{v_1, v_2, \ldots, v_n\}$, then this set is a basis for V.
___ h. All vector spaces having a basis are finitely generated.
___ i. Every independent subset of a finitely generated vector space V is a part of some basis for V.
___ j. Any two bases in a finite-dimensional vector space V have the same number of elements.

27. Let W_1 and W_2 be subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is also a subspace of V.

28. Let $W_1 = \mathrm{sp}((1, 2, 3), (2, 1, 1))$ and $W_2 = \mathrm{sp}((1, 0, 1), (3, 0, -1))$ in $\mathbb{R}^3$. Find a set of generating vectors for $W_1 \cap W_2$.

29. Let V be a vector space with basis $\{v_1, v_2, v_3\}$. Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also a basis for V.

30. Let V be a vector space with basis $\{v_1, v_2, \ldots, v_n\}$, and let $W = \mathrm{sp}(v_3, v_4, \ldots, v_n)$. If $w = r_1v_1 + r_2v_2$ is in W, show that $w = 0$.

31. Let $\{v_1, v_2, v_3\}$ be a basis for a vector space V. Prove that the vectors $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, $w_3 = v_1 - v_3$ do not generate V.

32. Let $\{v_1, v_2, v_3\}$ be a basis for a vector space V. Prove that, if w is not in $\mathrm{sp}(v_1, v_2)$, then $\{v_1, v_2, w\}$ is also a basis for V.

33. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V, and let $w = t_1v_1 + t_2v_2 + \cdots + t_kv_k$, with $t_k \neq 0$. Prove that

$$\{v_1, v_2, \ldots, v_{k-1}, w, v_{k+1}, \ldots, v_n\}$$

is a basis for V.

34. Let W and U be subspaces of a vector space V, and let $W \cap U = \{0\}$. Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for W, and let $\{u_1, u_2, \ldots, u_m\}$ be a basis for U. Prove that, if each vector v in V is expressible in the form $v = w + u$ for $w \in W$ and $u \in U$, then

$$\{w_1, w_2, \ldots, w_k, u_1, u_2, \ldots, u_m\}$$

is a basis for V.

35. Illustrate Exercise 34 with nontrivial subspaces W and U of $\mathbb{R}^5$.

36. Prove that, if W is a subspace of an n-dimensional vector space V and $\dim(W) = n$, then $W = V$.

37. Let $v_1, v_2, \ldots, v_n$ be a list of nonzero vectors in a vector space V such that no vector in this list is a linear combination of its predecessors. Show that the vectors in the list form an independent set.

38. Exercise 37 indicates that a finite generating set for a vector space can be reduced to a basis by deleting, from left to right in a list of the vectors, each vector that is a linear combination of its predecessors. Use this technique to find a basis for the subspace

$$\mathrm{sp}(x^2 + 1, x^2 + x - 1, 3x - 6, x^3 + x^2 + 1, x^3)$$

of the polynomial space P.

39. We once watched a speaker in a lecture derive the equation $f(x) \sin x + g(x) \cos x = 0$, and then say, "Now everyone knows that $\sin x$ and $\cos x$ are independent functions, so $f(x) = 0$ and $g(x) = 0$." Was the statement correct or incorrect? Give a proof or a counterexample.

40. A homogeneous linear nth-order differential equation has the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

Show that the set of all solutions of this equation that lie in the space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ is a subspace of F.

41. Referring to Exercise 40, suppose that the differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = g(x)$$

does have a solution $y = p(x)$ in the space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. By analogy with Theorem 1.18 on p. 97, describe the structure of the set of solutions of this equation that lie in F.

42. Solve the differential equation $y' = 2x$ and describe your solution in terms of your answer to Exercise 41, or in terms of the answer in the back of the text.

It is a theorem of differential equations that if the functions $f_i(x)$ of the differential equation in Exercise 40 are all constant, then all the solutions of the equation lie in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and form a subspace of F of dimension n. Thus every solution can be written as a linear combination of n independent functions in F that form a basis for the solution space.

In Exercises 43–45, use your knowledge of calculus and the solution of Exercise 41 to describe the solution set of the given differential equation. You should be able to work these problems without having had a course in differential equations, using the hints.

43. a. $y'' + y = 0$ [HINT: You need to find two independent functions such that when you differentiate twice, you get the negative of the function you started with.]
b. $y'' + y = x$ [HINT: Find one solution by experimentation.]

44. a. $y'' - 4y = 0$ [HINT: What two independent functions, when differentiated twice, give 4 times the original function?]
b. $y'' - 4y = x$ [HINT: Find one solution by experimentation.]

45. a. $y^{(3)} - 9y' = 0$ [HINT: Try to find values of m such that $y = e^{mx}$ is a solution.]
b. $y^{(3)} - 9y' = x^2 + 2x$ [HINT: Find one solution by experimentation.]

46. Let S be any set and let F be the set of all functions mapping S into $\mathbb{R}$. Let W be the subset of F consisting of all functions $f \in F$ such that $f(s) = 0$ for all but a finite number of elements s in S.
a. Show that W is a subspace of F.
b. What condition must be satisfied to have $W = F$?

47. Referring to Exercise 46, describe a basis B for the subspace W of F. Explain why B is not a basis for F unless $F = W$.

3.3 COORDINATIZATION OF VECTORS

Much of the work in this text is phrased in terms of the Euclidean vector spaces $\mathbb{R}^n$ for $n = 1, 2, 3, \ldots$. In this section we show that, for finite-dimensional vector spaces, no loss of generality results from restricting ourselves to the spaces $\mathbb{R}^n$. Specifically, we will see that if a vector space V has dimension n, then V can be coordinatized so that it will look just like $\mathbb{R}^n$. We can then work with these coordinates by utilizing the matrix techniques we have developed for the space $\mathbb{R}^n$. Throughout this section, we consider V to be a finite-dimensional vector space.