Section 5-2 Diagonalization

17. Prove that, for every square matrix A all of whose eigenvalues are real, the product of its eigenvalues is $\det(A)$

Answer: If the characteristic polynomial of A is $p(\lambda) = |A - \lambda I|$, then $p(0) = |A| = \det(A)$. Also,

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

, so

$$p(0) = (-1)^{2n}\lambda_1\lambda_2\cdots\lambda_n = \lambda_1\lambda_2\cdots\lambda_n = \det(A).$$

18. Prove that similar square matrices have the same eigenvalues with the same algebraic multiplicities.

Answer: Let A and B are similar and $B = C^{-1}AC$. Then

$$\begin{aligned}\det(B - \lambda I) &= \det(C^{-1}AC - \lambda I) = \det(C^{-1}AC - C^{-1}(\lambda I)C) \\ &= \det(C^{-1}(A - \lambda I)C) = \det(C^{-1})\det(A - \lambda I)\det(C) \\ &= \det(A - \lambda I)\end{aligned}$$

Thus we know that A and B have the same characteristic polynomial. Therefore they have the same roots with the same multiplicities.

22. Let A and Cbe $n \times n$ matrices, and let C be invertible. Prove that, if $\vec{v}$ is an eigenvector of A with corresponding eigenvalues λ, then $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalues λ. Then prove that all eigenvectors of $C^{-1}AC$ are form $C^{-1}\vec{v}$, where $\vec{v}$ is an eigenvector of A.

Answer: Let $A\vec{v} = \lambda\vec{v}$. Then

$$(C^{-1}AC)(C^{-1}\vec{v}) = C^{-1}A(CC^{-1})\vec{v} = C^{-1}(A\vec{v}) = C^{-1}(\lambda\vec{v}) = \lambda(C^{-1}\vec{v})$$

Therefore, $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalues λ.

Given an eigenvector $\vec{u}$ of $C^{-1}AC$ with corresponding eigenvalue α so that $C^{-1}AC\vec{u} = \alpha\vec{u}$. Then

$$A(C\vec{u}) = (CC^{-1})AC\vec{u} = C(C^{-1}AC\vec{u}) = C\alpha\vec{u} = \alpha C\vec{u}$$

Hence we know $C\vec{u}$ is an eigenvector of A with corresponding eigenvalue α. Thus $\vec{u} = C^{-1}(C\vec{u})$ has the requested form.