**應數一線性代數 2021 秋, 第一次期中考 解答**

學號: ________________________, 姓名: ________________________

本次考試共有 8 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

## 考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

**誠**: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
**宏**: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ________________________

1. (5 points) Find all numbers $r$ such that $\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is invertible.

   Answer: $r=$ $\mathbb{R} \setminus \{0\}$

   $det(\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{bmatrix}) = 4r + 2 + 12 - 2r - 8 - 6 = 2r.$

   Therefore, determine equal to 0 only if $r = 0$.

2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T([1,0,0]) = [2,3,0], T([0,1,0]) = [-2,0,1]$, and $T([1,2,3]) = [4,15,2]$. Find $T^{-1}([4,-3,2]) =$ ______

   $T([0,0,1]) = \frac{1}{3}(T([1,2,3]) - T([1,0,0]) - 2T([0,1,0])) = [2,4,0]$ Let $A$ is the standard matrix representation of $T$.

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

   The standard matrix representation of $T^{-1}$ is $A^{-1}$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & 1 \\ -1.5 & 1 & -3 \end{bmatrix}$$

$$T^{-1}([4,-3,2]) = (A^{-1}\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix})^T = [19,2,-15]$$

3. (5 points) Classify $\vec{v} = [4, 1, 2, 1, 6]$ and $\vec{u} = [8, 2, 4, 2, 3]$ are parallel, perpendicular, or neither.

   Answer: $\vec{v}$ and $\vec{u}$ are neither .

   Since $\vec{v} \cdot \vec{u} \neq 0$, it is NOT perpendicular.

4. (10 points) Find the homogeneous solution and general solution of the given linear system and express the solution set.

$$\begin{cases} x_1 \quad + \ x_3 + 5x_4 = -1 \\ \quad x_2 + 2x_3 + 6x_4 = 3 \\ x_1 - x_2 \quad + 2x_4 = 3 \end{cases}$$

   Answer: the homogeneous solution is ____________________.

   The general solution is ____________________.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & -1 \\ 0 & 1 & 2 & 6 & 3 \\ 1 & -1 & 0 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -8 \\ 0 & 1 & 0 & 0 & -11 \\ 0 & 0 & 1 & 3 & 7 \end{array}\right]$$

   NOT!!!

   Let $x_4 = r$, we get $x_3 = 0, x_2 + 2r = 8, x_1 + 3r = 13$. Thus $x_1 = 13 - 3r, x_2 = 8 - 2r, x_3 = 0, x_4 = r$. Then solution are

$$\left\{ \begin{bmatrix} 13 - 3r \\ 8 - 2r \\ 0 \\ r \end{bmatrix} \middle| \, r \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 13 \\ 8 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$$

5. (10 points) Assume the the matrix $A$ can be row reduces to $H$, please answer the following questions.

$$A = \begin{bmatrix} 5 & 1 & 0 & 3 & -3 \\ 1 & 0 & -1 & 1 & 8 \\ 0 & 3 & 1 & -6 & 1 \\ 1 & 1 & 0 & -1 & 7 \end{bmatrix}, H = \begin{bmatrix} 5 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & -4 & -6 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) the **rank** of matrix A, is 3.

(b) a basis for the **row space** of A is [5, 0, 0, 5, 0], [0, 2, 0, -4, -6],[0, 0, -1, 0, -2].

(c) a basis for the **column space** of A is $\begin{bmatrix} 5 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

Let $x_4 = r, x_5 = s$, we get $-x_3 - 2s = 0, x_2 - 4r - 6s = 0, 5x_1 + 5r = 0$. Thus $x_1 = -r, x_2 = 4r + 6s, x_3 = -2s, x_4 = r, x_5 = s$. Then homogeneous solution are

$$\left\{ \begin{bmatrix} -r \\ 4r+6s \\ -2s \\ r \\ s \end{bmatrix} \middle| \, r \in \mathbb{R} \right\} = \left\{ r \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 6 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$$

6. (5 points) Suppose that $T$ is a linear transformation with standard matrix representation $A$, and that $A$ is a $8 \times 11$ matrix such that the nullspace of $A$ has dimension 6. What is the dimension of the range of $T$?

   Since the nullity of A is equal to 6, the rank of A is equal to 5. Thus the dimension of the range of T is 5.

7. (10 points) Given set $S = \{[-2,3,1,0],[0,1,5,-2],[1,-1,2,-1]\}$ in $\mathbb{R}^4$.

   (a) Determine whether the set $S$ is linearly dependent or linearly independent. If it is linearly dependent, find a basis for $sp(S)$.

   Answer: $\{[-2,3,1,0],[0,1,5,-2],\}$ .

   (b) Enlarge the basis you found in part (a) to be a basis for $\mathbb{R}^4$.

   Answer: $\{[-2,3,1,0],[0,1,5,-2],[1,0,0,0],[0,1,0,0]\}$

$$\begin{bmatrix} -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 5 & 2 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -0.5 & 0 & 0 & 1 & 2.5 \\ 0 & 1 & 0.5 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -3 & -7 \end{bmatrix}$$

8. (10 points) Determine if the set $W = \{(x,y,z) \in \mathbb{R}^3 | x = 2y+z, y = 5z\}$ is a subspace of $\mathbb{R}^3$

If $y = 5z$, we have $x = 2y + z = 11z$. Let $w_1 = (11z_1, 5z_1, z_1)$ and $w_2 = (11z_2, 5z_2, z_2)$ be arbitrary vectors in $W$.
Then for any real number $c$, $cw_1 = (11cz_1, 5cz_1, cz_1) \in W$
$w_1 + w_2 = (11z_1 + 11z_2, 5z_1 + 5z_2, z_1 + z_2) = (11(z_1+z_2), 5(z_1+z_2), (z_1+z_2)) \in W$
So since $W$ is closed under scalar multiplication and vector addition, $W$ is a subspace of $\mathbb{R}^3$

9. (5 points) Let $F$ bet he set of all real-valued functions on a (nonempty) set $S$; that is , let $F$ be the set of all functions mapping $S$ into $\mathbb{R}$. For $f, g \in F$, let the sum $f \oplus g$ of two functions $f$ and $g$ in $F$, and for any scalar $r$, let scalar multiplication be defined below. Is this set a vector space?

$$(f \oplus g)(x) = f(x) + 3g(x) \quad \text{for all } x \in S$$
$$(rf)(x) = rf(x) \quad \text{for all } x \in S$$

Define $(f \oplus g) = f(x) + 3g(x)$, for all $x \in \mathbb{R}$.

A2: $(f \oplus g) = (g \oplus f)$ for all $f, g \in F$.

However, pick $f(x) = x, g(x) = 6x$, $(f \oplus g) = f(x) + 3g(x) \neq 3f(x) + g(x) = (g \oplus f)$.

Therefore, It is NOT a vector space.

10. (10 points) Let $\vec{v}_1$ and $\vec{v}_2$ be two vectors in $\mathbb{R}^n$. Prove that $sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2) = sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$

Clearly, $\vec{v}_1 = \frac{1}{2}[(\vec{v}_1 - \vec{v}_2) + (\vec{v}_1 + \vec{v}_2)]$ and $\vec{v}_1 + \vec{v}_2 = 0 \cdot (\vec{v}_1 - \vec{v}_2) + 1 \cdot (\vec{v}_1 + \vec{v}_2)$. Hence $\vec{v}_1, \vec{v}_1 + \vec{v}_2 \in sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$ and therefore $sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2) \subset sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$.

Also, $\vec{v}_1 + \vec{v}_2 = 0 \cdot \vec{v}_1 + 1 \cdot (\vec{v}_1 + \vec{v}_2)$ and $\vec{v}_1 - \vec{v}_2 = 1 \cdot \vec{v}_1 - 1 \cdot (\vec{v}_1 + \vec{v}_2)$. Hence $\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2 \in sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$ and therefore $sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2) \subset sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2)$.

Thus $sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2) = sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2)$

11. (10 points) Suppose that the vectors $\vec{v}, \vec{w}$, and $\vec{x}$ are mutually perpendicular (i.e. $\vec{v}$ and $\vec{w}$ are perpendicular, $\vec{v}$ and $\vec{x}$ are perpendicular, and $\vec{w}$ and $\vec{x}$ are perpendicular). Use dot products to find $\|\vec{v}+3\vec{w}+2\vec{x}\|$ in terms of the magnitudes (lengths) of $\vec{v}, \vec{w}$, and $\vec{x}$. Hint: Start by computing $\|\vec{v}+3\vec{w}+2\vec{x}\|^2$.

$\|\vec{v}+3\vec{w}+2\vec{x}\|^2 = (\vec{v}+3\vec{w}+2\vec{x}) \cdot (\vec{v}+3\vec{w}+2\vec{x})$
$= \vec{v}\cdot\vec{v} + \vec{v}\cdot 3\vec{w} + \vec{v}\cdot 2\vec{x} + 3\vec{w}\cdot\vec{v} + 3\vec{w}\cdot 3\vec{w} + 3\vec{w}\cdot 2\vec{x} + 2\vec{x}\cdot\vec{v} + 2\vec{x}\cdot 3\vec{w} + 2\vec{x}\cdot 2\vec{x} = \|\vec{v}\|^2 + 9\|\vec{w}\|^2 + 4\|\vec{x}\|^2$
Thus $\|\vec{v}+3\vec{w}+2\vec{x}\| = \sqrt{\|\vec{v}\|^2 + 9\|\vec{w}\|^2 + 4\|\vec{x}\|^2}$

12. (10 points) Let $A$ and $C$ be matrices such that the product $AC$ is defined. Whether the column space of $AC$ is contained in the column space of $A$ or $C$? Explain your answer.

Answer: the column space of $AC$ is contained in the column space of ___$A$___.

1. Let $A$ be $m \times n$ matrix. Every vector in the column space of $AC$ is of the form $\vec{v} = (AC)\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. For every $\vec{x}$, $(C\vec{x}) \in \mathbb{R}^n$. Then $\vec{v} = A(C\vec{x})$ which is the vector in the column space of $A$. Thus $colspace(AC) \subseteq colspace(A)$.
2. Let $A$ be $m \times n$ matrix, $C$ be $n \times s$ matrix. Thus $AC$ is $m \times s$ matrix.
   Since the column vectors of $AC$ are belong to $\mathbb{R}^m$ and the column vectors of $C$ are belong to $\mathbb{R}^n$, the column space of $AC$ can not be contained in the column space of $C$.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 5 | 10 | 5 | 10 | 10 | 5 | 10 | 10 | 5 | 10 | 10 | 10 | 100 |
| Score: | | | | | | | | | | | | | |