

1-1/

How do we know if a math formula  
is true?

How do we know if an algorithm  
(like Euclid's GCD) "works"?

↖ ↘

correct effective

Does an algorithm exist?

What is an algorithm?

Does a program exist? ← problems

What is a program? ← models

I-2] A set is "a bunch of stuff"

$\emptyset$  - nothin' in it  
 $\forall x, x \notin \emptyset$

$\{ \text{pen}, \text{phone} \}$      $\{ \text{phone}, \text{pen} \}$

$\{\checkmark, \square\}$

$\nexists \text{ pen} \in \{ \text{pen}, \text{phone} \}$

$\forall x, x \in \{y\}$  iff  $x = y$

union -  $\cup$

$\forall x, x \in A \cup B$  iff  $x \in A$  or  $x \in B$

$\{ \text{pen}, \text{phone} \} = \{ \text{pen} \} \cup \{ \text{phone} \}$

[=3] "The set of all true math formulas"

A set IS its membership

" $1+1=2$ "  $\in TS \uparrow ?$

"Is there a god?"

"Will Buffy be remade?"

All sets "constructed" via  $\emptyset$ ,  $\{\cdot\}$ ,  $\cup$  are finite.

$$x \in \{\emptyset\} \cup \{\{\cdot\}\}$$

The Universe ( $U$ )

$A \subseteq B$  iff  $\forall x, x \in A \rightarrow x \in B$

↳ Our universe is made of strings  
and strings are sequences of characters  
and chars are elements of an alphabet  
an alphabet is a finite set



$$\Sigma = \{0, 1\}$$

↑  
chars

$$\{0, 1, \cup, \$, +\}$$

↑  
chars

"0100001" = a string = s

length = 7

$$s(0) = 0$$

$$s(1) = 1 \quad s(2) = 0$$

$\cup = \Sigma^*$  ← special notation

$$A^* = \{\epsilon\} \cup A \circ A^*$$

epsilon = " " = the string w/ no characters

$x \in A \circ B$  iff  $x(0) \in A$  and  
 $x(1..) \in B$

$$\{0, 1\} \circ \{0, 1\} = \{00, 01, 10, 11\}$$

$$\{1\} \circ \{0\} = \{10\}$$

LS / #1. Decide a data type to represent alphabets and characters.

Alphabet = List < Character >

Character = Object / void\*  
we need equality

#2. Decide a data type for strings

interface String { }

class MtString implements String { .. }

class OneString impl String { }

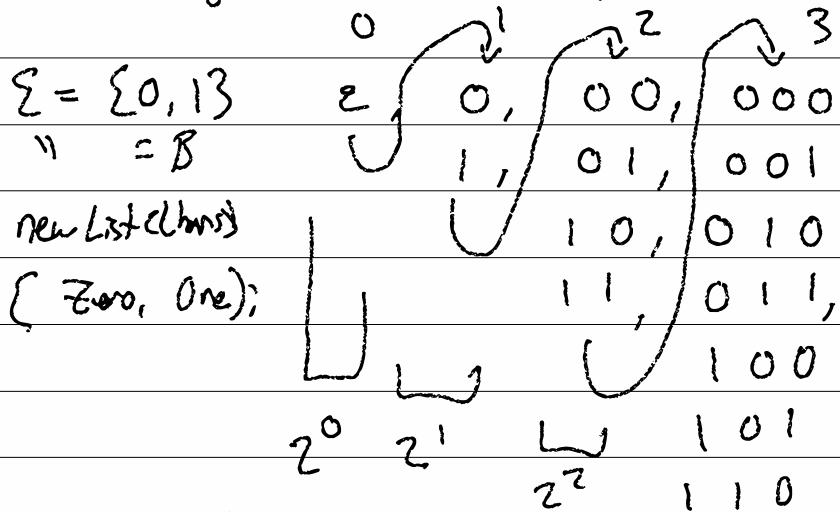
OneString ( char c, String s ) { ... }

Zero = new BasicChar('0'); One = new BC('1');

010 = new OneS(Zero, new OneS(One, new  
OneS(Zero, new MtS())));

(EP)

I-6/ Every alphabet has a lexicographical ordering of the strings in  $\Sigma^*$



$|A|^i$  where  $i$  is layer  $2^i$   
looking at  $\underbrace{111}_{2^3}$

lexi :  $\Sigma \times \mathbb{N} \rightarrow \Sigma^*$

lexi  $\emptyset 0 = \emptyset$

lexi  $\emptyset 1 = 0$

lexi  $\emptyset 2 = 1$

lexi  $\emptyset 6 = 101$

2-1 "1+1"  $\rightarrow$  "2"

"1+1 = 2"  $\in$  Truth

"1+1 = 3"  $\notin$  Truth

$\emptyset \quad \Sigma^3 \quad A \cup B$

Alphabet  $\Sigma$  Universe  $\Sigma^*$

{0, 1}

{ε, 0110, 000001,



3

$P(A) \quad 2^A$

$x \in P(A)$  iff  $x \subseteq A$  ( $x \subseteq A$ , iff  
 $\forall y \in x, y \in A$ )

$A = \{0, 1, 2, 3\}$

$\emptyset \in P(A) \quad \emptyset \subseteq A$

0110 {1, 2}

{0}  $\in$   $\emptyset \subseteq \{2, 3\} \in \{0, 1, 2, 3\} \quad \underline{\textcircled{1}} \quad 123$

$P(\Sigma^*) \quad \Sigma^* = \{ \epsilon, 0, 1, 00, 111111 \}$

$\emptyset \in P(\Sigma^*)$

...

{ε}  $\in P(\Sigma^*)$

0011, ...

all even length strings  $\in P(\Sigma^*) = \{ \epsilon, 00, 11, 01, \dots \}$

GIFS  $\in P(\Sigma^*)$

{GIFS of me}  $\in P(\Sigma^*)$

JPGs w/ a cat in them  $\in P(\Sigma^*)$

2-2  $\text{ALL} = \mathcal{P}(\Sigma^*)$

$\text{FIN} =$  the set of  
finite sets

- ALL
- True math
  - G-TMs
  - Even strings

$\overline{\text{FIN}}$

$\emptyset \in \text{FIN}$

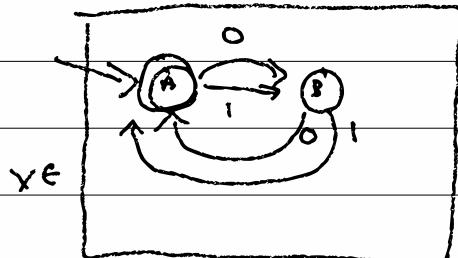
$\forall x \in \Sigma^*, \{x\} \in \text{FIN}$

$A \in \text{FIN} \wedge B \in \text{FIN}$

$\Rightarrow A \cup B \in \text{FIN}$

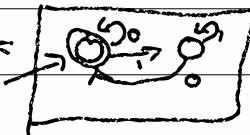
All even strings =

DFA - a deterministic finite automata



$\Sigma$  or  $\Sigma^*$

even numbers =



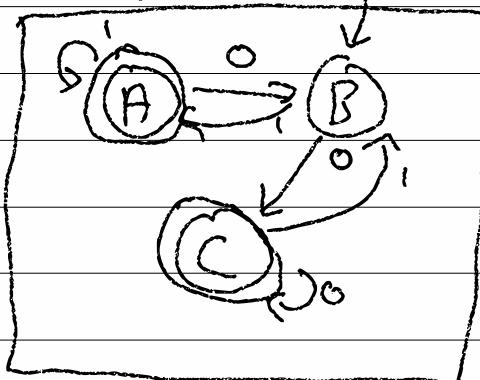
○ - states  $\Sigma A, B \}$

→ ○ - start state  $A$

○ - accepting states  $\Sigma A \}$

$\overset{x}{\rightarrow} \circ$  - transition

$x$  - labels are  $\Sigma$



$\Sigma A, B, C \}$

$\Sigma A, C \}$   $\Sigma = \Sigma_0, 1 \}$

A	0	1
B	C	A
C	C	B

2-3)  $x \in \text{DFA} (\underbrace{\text{states}, \text{alphabet}, \text{start}, \text{accepting}}_{\text{states } Q, \Sigma, q_0 \in Q, F \subseteq Q}, \delta: Q \times \Sigma \rightarrow Q - \text{transitions})$

DFA configuration =  $Q \times \Sigma^*$   
 $\stackrel{\uparrow}{[q]} w^{\uparrow}$

config update function : config  $\times \text{DFA} \rightarrow \text{config}$   
 $[q]w \rightarrow [q']w'$

$[q_i]x \rightarrow [q_j]y \text{ iff } \delta(q_i, x) = q_j$   
 $x \in \text{DFA} \text{ iff } [q_0]x \Rightarrow \Rightarrow \Rightarrow \Rightarrow [q_f] \in$   
 and  $q_f \in F$

0110  $\in \text{EvenLen}$  ;iff  $[A]0110 \rightarrow [B]110 \rightarrow [A]10$   
 $\rightarrow [B]0 \rightarrow [A] \in AF\{A\}$  ✓

class DFA  $\Sigma$

..  $Q, \Sigma, F, q_0, \delta \dots$

public bool accepts (String x) {

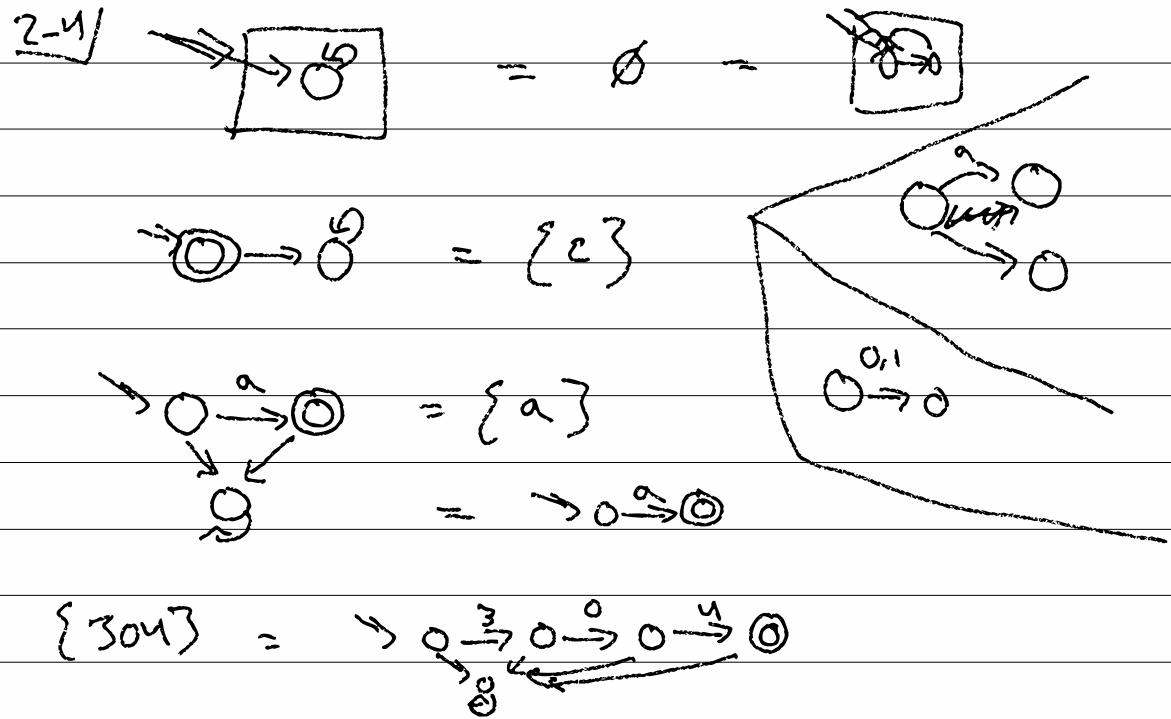
State  $q_i = q_0;$

while ( $(x, \cancel{\neq} \text{empty}) \Sigma$

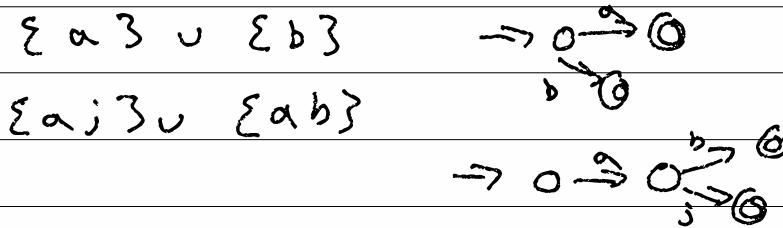
$q_i = \delta(q_i, x, \text{first}());$

$x = x, \text{rest}();$  } }

return  $F, \text{in}(q_i);$  } }



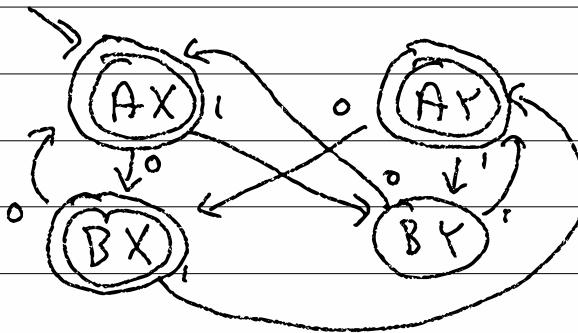
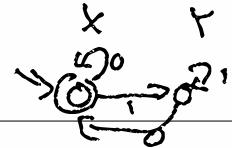
$A \cup B \leftarrow \text{DFA}$  (if  $A \in \text{DFA}$  and  $B \in \text{DFA}$ )



2-5 Even Len



Is Even



ε	✓
00	✓
11	✓
0	✓
110	✓

n

$(x, y) \in A \times B$

; if  $x \in A \wedge y \in B$

$$A = (Q_A, \Sigma, g_{0A}, \delta_A, F_A)$$

$$B = (Q_B, \Sigma, g_{0B}, \delta_B, F_B)$$

$$X = A \cup B$$

$$Q_X = Q_A \times Q_B \quad \delta_X = ((g_{0A}, g_{0B}), c) =$$

$$g_{0X} = (g_{0A}, g_{0B}) \quad ( \delta_A(g_{0A}, c),$$

$$F_X = F_A \times F_B - n \quad \delta_B(g_{0B}, c) )$$

$$F_A \times Q_B \cup Q_A \times F_B - V$$

$x \in A \cap B$  ; if  $x \in A \wedge x \in B$

2-6)  $x + A^c$  iff  $x \notin A \quad (x \in u)$

Even Len      odd Len  
 $\rightarrow \textcircled{0} \rightarrow \textcircled{0}$        $\Rightarrow \rightarrow \textcircled{0} \rightarrow \textcircled{0}$

$$F = \{A\}$$

complement

$$F' = Q - F$$

or  $F^c$  (wrt Q)

Algorithm for  $X \subseteq Y$  if  $X, Y$  are DFAs

### 3-11 DFA $\Rightarrow$ example or false

DFA:

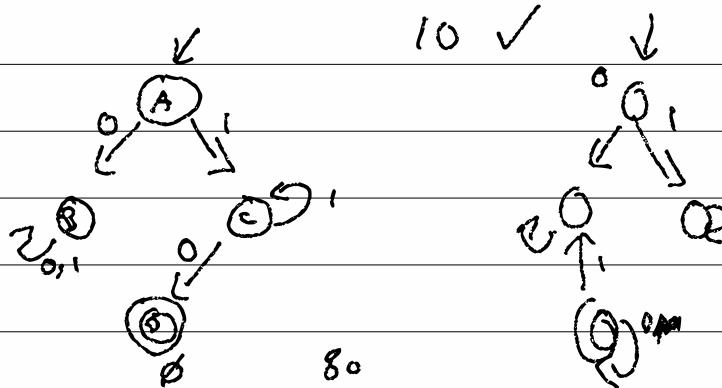
$Q$ : Sstate  $\rightarrow$  Bool

$\Sigma$ : list of characters

$s_0$ : state

$S$ : (state  $\times$  char)  $\rightarrow$  Sstate

$F$ : State  $\rightarrow$  Bool



$\Sigma = \{A, B, C, D\}$

$\Sigma = \{A\}$

$[A]$

$A \Rightarrow \Sigma$

$\Sigma = \{B, C, D\}$

$\Sigma = \{A\}$

$[B, C]$

$B \Rightarrow A, 0$   
 $C \Rightarrow A, 1$

$\Sigma = \{C, D\}$

$\Sigma = \{A, B\}$

$[C]$

'Yes, it is possible.'

$\Sigma = \{D\}$

$\Sigma = \{A, B, C\}$

$[D]$

$D \Rightarrow C, 0$

$\Sigma = \{E\}$

$\Sigma = \{A, B, C\}$

$[E]$

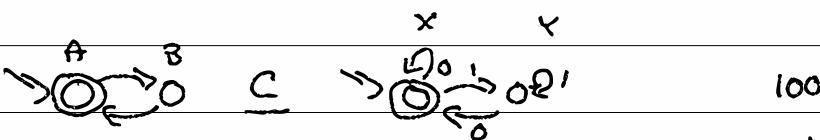
or, if not, No!

3-2/ subset

$A \subseteq B \iff \forall x \in A. x \in B \rightarrow x \in B$

$$\{\alpha, \beta\} \subseteq \{\alpha, \beta, \gamma\} \quad U = \{\alpha, \beta, \gamma\}$$

finite means naive works!



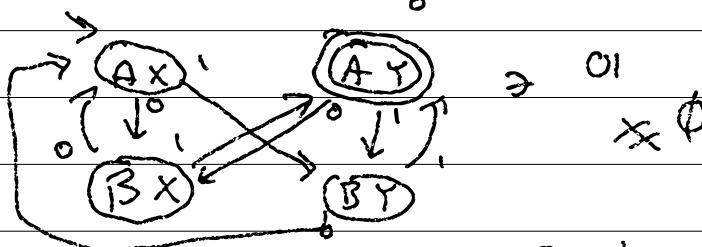
$$\boxed{\bar{A}} \subseteq \boxed{\bar{B}} \cap \boxed{\bar{A}} = \boxed{\bar{A}}$$

$$\boxed{B} \cap \boxed{\emptyset} = \boxed{\text{shaded}} \emptyset$$

Diagram illustrating set intersection and empty set:

- Set  $\bar{A}$  is shown intersecting with set  $\bar{B}$ , resulting in set  $\bar{A}$ .
- Set  $B$  is shown intersecting with the empty set, resulting in the empty set.

$$\mathbb{B} \text{ EvenNum} = \rightarrow \xrightarrow{x^0} \xrightarrow{x^1} \xrightarrow{y^0} \xrightarrow{y^1} \text{01}$$



model checking

soundness: model  $\subseteq$  theory  
completeness: theory  $\subseteq$  model  
model = theory

$$3-3) \quad 0, 1, 2, -1, 5 \quad \mathbb{Z}, \mathbb{P}, \mathbb{N}$$

$$\{\mathbb{P}\} + \{\mathbb{N}\} = \{\mathbb{P}, \mathbb{Z}, \mathbb{N}\}$$

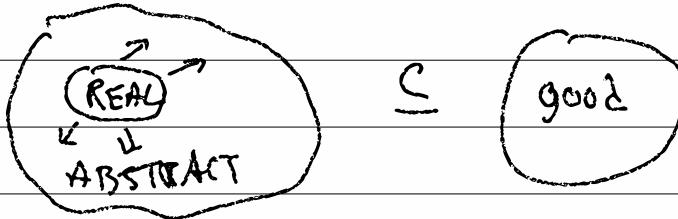
if  $x > 0$  then

$$A \quad y = 5 \Rightarrow \{\mathbb{P}\}$$

o.w.

$$B \quad y = 0 \Rightarrow \{\mathbb{Z}\}$$

$\Rightarrow$  assume  $y = \{\mathbb{P}, \mathbb{Z}\}$



<u>3-w)</u>	Finite	=	$\emptyset$	$\Sigma^3$	$A \cup B$	EDFA
			$A^c$	$A \cap B$	$A \circ B$	

Infinite =  $A^*$

\*  $x \in \Sigma^* \wedge y \in \Sigma^*$  then  $xoy \in A \circ B$  iff  
 $x \in A \wedge y \in B$

$$\varepsilon \circ y = y \quad \text{if } a \in \Sigma, (a \circ x) \circ y = a \circ (xoy)$$

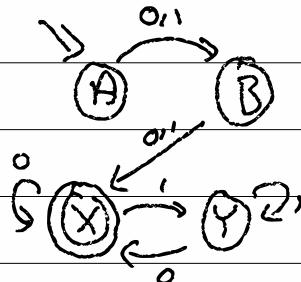
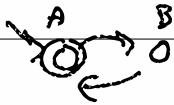
$$abcd = ab \circ cd$$

$$\{\text{jm}\} \circ \{\text{mj, nj}\} = \{\text{jim, jn}\}$$

$x \in A^*$  iff  $x = x_0 \circ x_1 \circ \dots \circ x_n$  for  $n \in N$   
and  $x_i \in A$

$$\{\text{jm}\}^* \ni \varepsilon, \text{ jm, jmjmjmjmjm}$$

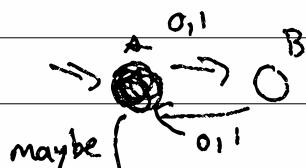
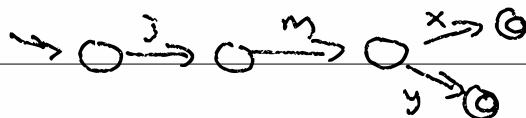
3-5/ Even Len  $\circ$  Even Num



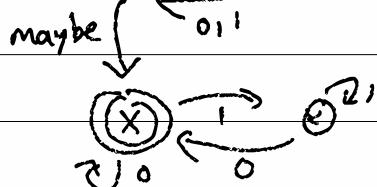
00110 ✓  
0011 X

~~00110011~~

$\{ \text{im } 3 \circ \text{Ex, y} \}$

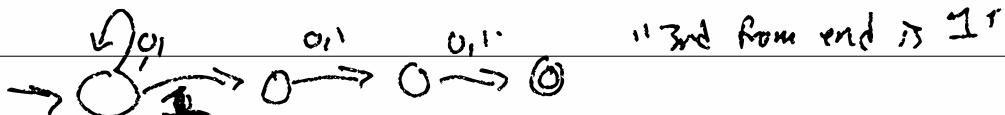


$\Sigma = \{0, 1\}$



$x \in \text{DFA}$  iff

There is some path  
from  $q_0$  to  $q_f \in F$   
labelled w/  $x$



"3rd from end is 1"

3.6) NFA = non-deterministic  
finite automata

old world: the next step was obvious

$$\delta: Q \times \Sigma \rightarrow Q$$

new world: crazy options

- do you even read achar?
- which path do you take?

$$\delta': Q \times \{\text{maybe}\} \cup \Sigma \rightarrow P(Q)$$

$$\delta'(A, r) = \{A, B\}$$

$$\delta'(A, \text{maybe}) = \{C\}$$

epsilon

$$\epsilon \in \Sigma$$

4-11  $A \circ B \in \text{DFA}$  iff  $A \in \text{DFA}$   
 $A^*$   $\wedge B \in \text{DFA}$

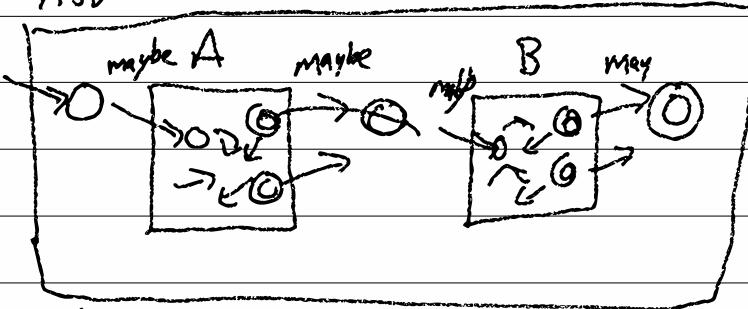
NFA ( $N - \underline{\text{non}} \text{ D-deterministic}$ )

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$S: Q \times (\Sigma \cup \{\text{maybe}\}) \rightarrow P(Q)$$

$$S: Q \times \Sigma \rightarrow Q$$

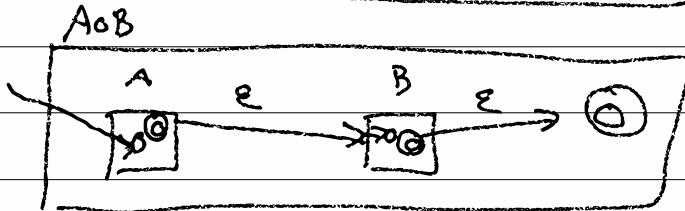
$A \circ B$



maybe written  
as "ε"

what does (NFA)

this mean?



$\text{NFA} \leftrightarrow \text{DFA}$

## 4-2] what do NFAs mean?

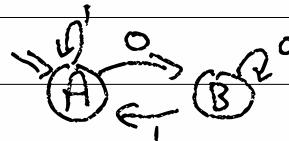
A DFA represents a set and  
a set "is" a membership function

$$U \rightarrow \{0,1\}$$

$$\subseteq \Sigma^* \rightarrow \{\text{Y}, N\}$$

$$\text{config} = \Sigma^* \times Q$$

$$\Sigma^* \rightarrow Q^*$$



$$0110 \rightarrow \underline{ABAAB} \rightarrow \text{a trace}$$

$$\Sigma^* \rightarrow (\underline{Q}, \delta)^*$$

$$0110 \rightarrow \underbrace{(0, B)(1, A)(1, A)(0, B)}_{\text{a trace}} = \Sigma^* \cup \Sigma \epsilon^3$$

$$0A1A1A0B \rightarrow N$$

$$\text{valid? } : \delta(\boxed{\Sigma}, Q)^* \rightarrow \{\text{Y}, N\}$$

$$\text{valid } g; \epsilon = Y$$

$$\text{valid } g; (c, g_j) : \text{more} = \text{if } \delta(g_j, c) = \boxed{g_j}$$

$$\text{Nvalid? } : Q \times (\boxed{\Sigma} \times Q)^* \rightarrow B$$

$$\text{valid } g; \text{ more}$$

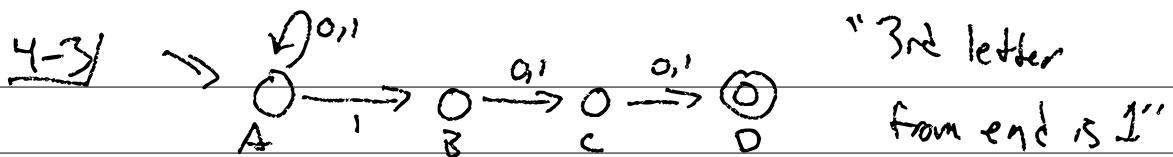
$$\text{Nvalid } g; \epsilon = Y$$

$$\text{o.w. } N$$

$$\text{Nvalid } g; (c, g_j) : \text{more} =$$

$$\text{if } \boxed{g_j} \boxed{e} \boxed{\delta(g_j, c)} \text{ then Ag Oracle}$$

$$\frac{\text{Nvalid } g; \text{ more}}{O.W. N}$$



0 1 00

1 1 1

1 1 0 1 0 0

- Y

0 0 0

1 0 0 0

1 0 1 1

- N

(0, A)(1, A)(0, A)(0, A) ✓

str( $\Sigma \times Q)^* = \Sigma^*$

(0, A)(1, B)(0, C)(0, D) ✓

str  $\epsilon = \epsilon$

(0, B)(1, C)(1, D)(0, D) X

str ( $C, \perp$ ): move  $\in$

$\delta(A, 0) = \Sigma A \}$

$\delta(D, 0) = \emptyset$

$C \circ$  str more

accepts :  $\Sigma^* \rightarrow Y/N$

accepts  $w = Y$  iff  $\exists t \in \text{traces. } str(t) = w$

valid  $\Leftrightarrow t \in F$

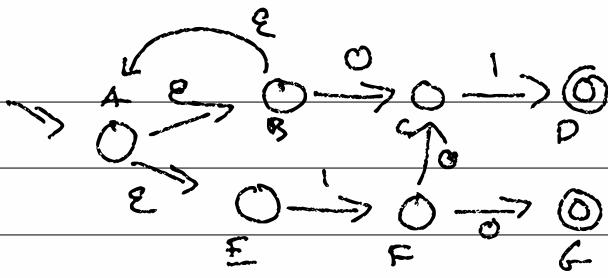
and last-state( $t$ )  $\in F$

NFA-accepts :  $\Sigma^* \rightarrow Y/N$

figure all possible traces

check if valid and if strings match

check if past is in  $\epsilon F$



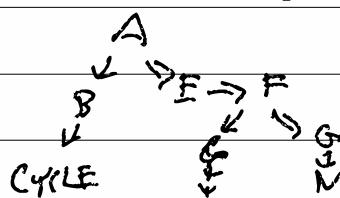
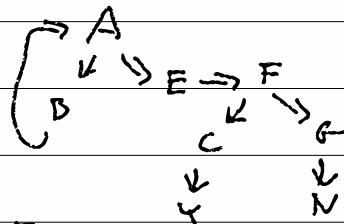
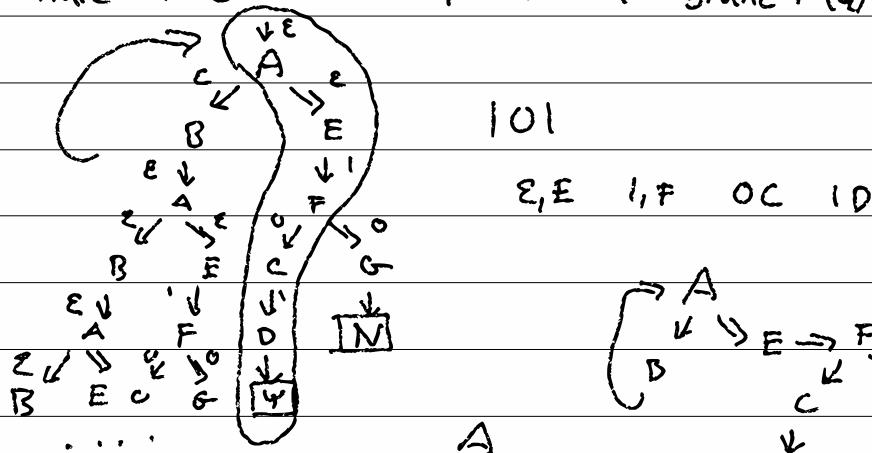
$(\epsilon, B)(0, C)(1, D) \quad 01 = 0001$

$(\epsilon, E)(1, F)(0, G) \quad 101 = 01001$

$(\epsilon, E)(1, F)(0, G) \quad 10 = 0100$

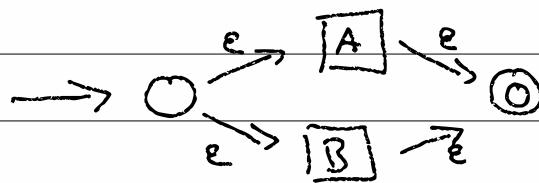
$(\epsilon, B)(\epsilon, A) \times \text{ where } \times \text{ is valid}$   
 $\rightarrow \text{valid}$

Trace Tree = T | N | Branch ( $\epsilon, Q$ ) (List TTI)



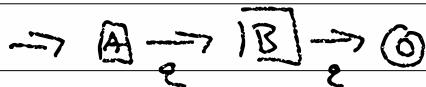
Forking model of NFAs (make TT)  
 Backtracking model (explores TT)

4-5)  $A \cup B$



$x \in A$   
 $\square \rightarrow \square$   
State  $X$  transitions

$A \circ B$



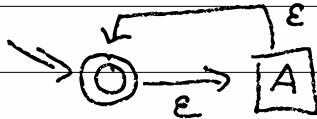
state of  $A$

$\square \rightarrow \square$

All accepting states

of  $A$  transition to  $\square$

$A^*$

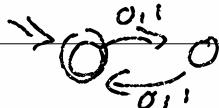


4-6)   $\forall A$ ,  $A \in \text{DFA} \Leftrightarrow A \in \text{NFA}$

$\Rightarrow$

$\Leftarrow$

DFA  $\Rightarrow$  NFA



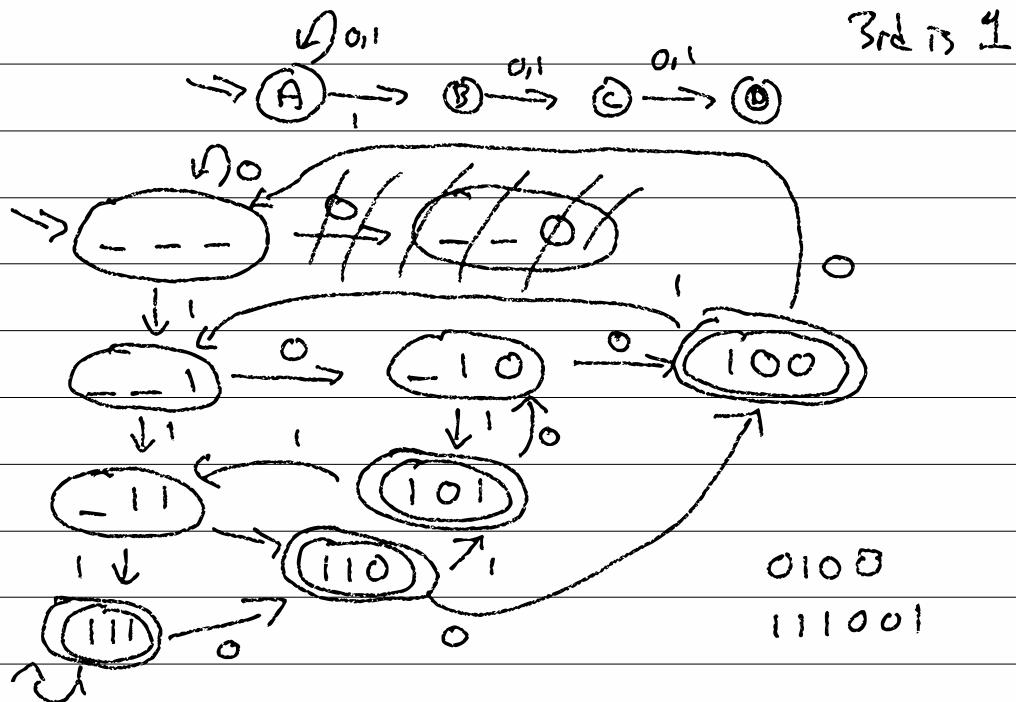
DFA  $\delta: Q \times \Sigma \rightarrow Q$

NFA  $\delta': Q \times \Sigma_c \rightarrow P(Q)$

$$\delta'(q_i, \epsilon) = \emptyset$$

$$\delta'(q_i, c \in \Sigma) = \{\delta(q_i, c)\}$$

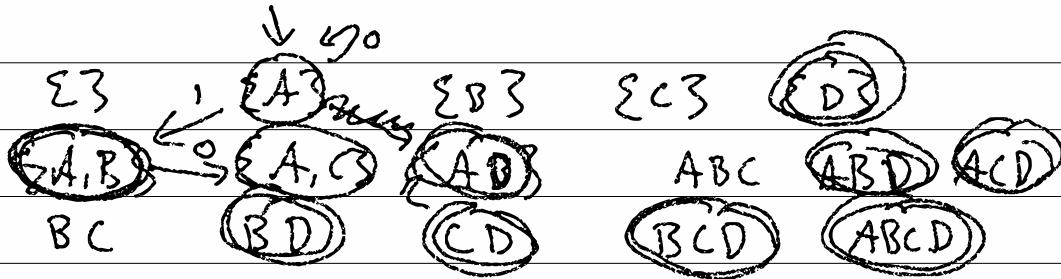
NFA  $\Rightarrow$  DFA



4-7)  $\text{NFA} = (\mathbb{Q}, \Sigma, q_0, \delta; \mathbb{P}(\mathbb{Q}) \xrightarrow{\Sigma} \mathbb{P}(\mathbb{Q}))$

$\text{DFA}^{\text{out}} = (\mathbb{Q}', \Sigma, q'_0, \delta': \mathbb{Q}' \times \Sigma \rightarrow \mathbb{Q}', F' \subseteq \mathbb{Q}')$

$$\mathbb{Q}' = \mathbb{P}(\mathbb{Q})$$



$$q'_0 = \{q_0\}$$

$F'$  = any state where  $\cap F \neq \emptyset$

$$\delta'(\{q_1, \dots, q_n\}, c) = \bigcup \delta(q_i, c)$$

$$\underline{5-1} / A \cup B \quad \delta_A : Q_A \times \Sigma \rightarrow Q_A$$
$$\delta_B : Q_B \times \Sigma \rightarrow Q_B$$

$$\delta' : \overbrace{Q_A \times Q_B}^{\text{(Q}_A \times \text{Q}_B)} \times \Sigma \rightarrow Q$$

$$\delta'((q_a, q_b), c) = (\delta_A(q_a, c), \delta_B(q_b, c))$$

char

$$(p, c) \Rightarrow \text{new Pair } \left( \begin{array}{l} \downarrow \\ \text{pair} < \text{State}, \text{State} \end{array} \right) \left( \begin{array}{l} \nearrow \\ \text{fst} \end{array} \right) \left( \begin{array}{l} \nearrow \\ \text{snd} \end{array} \right) \left( \begin{array}{l} \text{delta a}(p, \text{fst}, c), \\ \text{delta b}(p, \text{snd}, c) \end{array} \right);$$

## S-2/ NFA $\rightarrow$ DFA

$(Q, \Sigma, q_0 \in Q,$

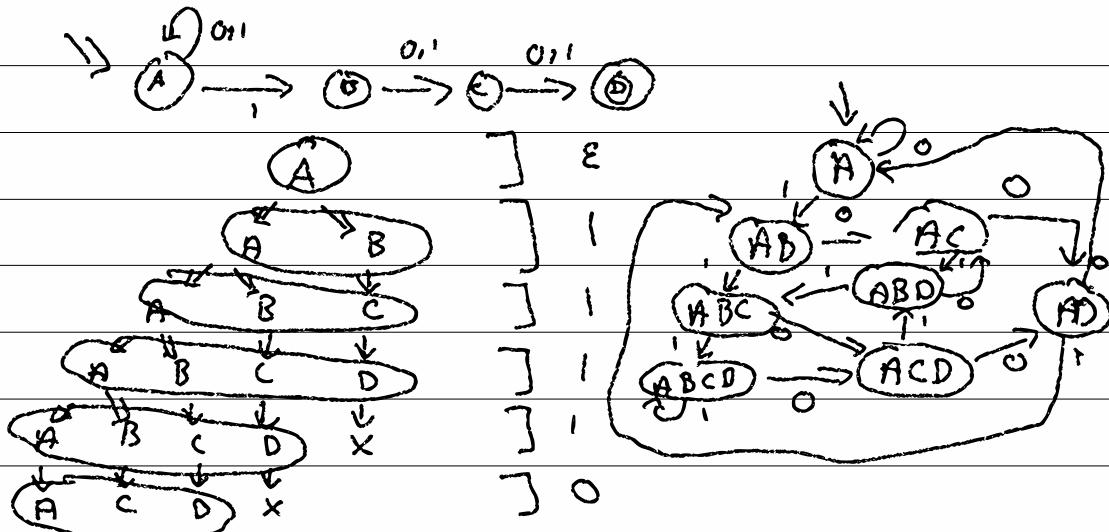
$\delta: Q \times \Sigma \rightarrow P(Q),$

$F \subseteq Q)$

$(Q', \Sigma, q'_0 \in Q'$

$\delta': Q' \times \Sigma \rightarrow Q',$

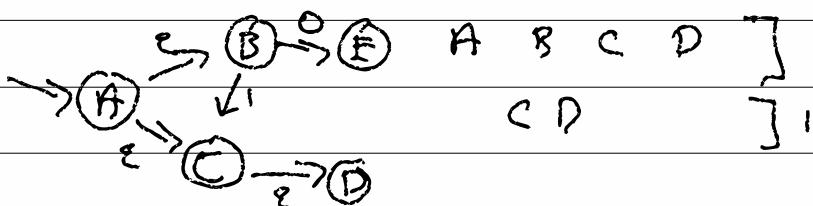
$F' \subseteq Q')$



$$\begin{array}{c} \leftarrow \downarrow \downarrow \downarrow \\ A \ B \\ \downarrow \downarrow \downarrow \\ B \ C \end{array} ]^0$$

$$\begin{array}{c} \leftarrow \downarrow \downarrow \downarrow \\ A \ B \\ \downarrow \downarrow \downarrow \\ B \ C \end{array} ]^0$$

$$\begin{array}{c} \leftarrow \downarrow \downarrow \downarrow \\ A \ B \\ \downarrow \downarrow \downarrow \\ B \ C \end{array} ]^0$$



5-3/

$$E: Q' \rightarrow Q' \quad P(Q) \xrightarrow{P(Q)} - \text{follow all C-transitions}$$

Trace Tree DFA

$Q'$  = things at the bottom of a tree  
set = a set of states of  
the NFA =  $P(Q)$

$g_0'$  = the top of the tree  
= the set that has only the first state  
 $= E(\{\}) \in P(Q)$

$\delta'$  = maps the bottom of the tree to the next level  
= set of all next states of each state in the level of the tree

$$\delta'(Q_i, c) = \bigcup_{q_i \in Q_i} \delta(q_i, c)$$

$F$  = any level of tree with some accepting state  
= any set with an element in  $F$   
=  $\{Q_i \mid \underbrace{Q_i \subseteq Q \text{ and } Q_i \cap F \neq \emptyset}\}_{Q_i \in P(Q) = Q'}$

= (set-of-gs  $\rightarrow$   
for each  $g_i$  in set-of-gs  
if dfa.F.apply( $g_i$ ) then  
return true  
return false)

5-y)

$E(\text{set } \langle Q \rangle g_i)$

queue  $\langle Q \rangle$  next = ~~empty~~  $g_i$

set  $\langle Q \rangle$  seen = empty

while  $(\text{not } \langle Q \rangle \text{ is empty})$

$\delta(\text{next}, \text{first}, \varepsilon)$  add those

to next unless in seen

return seen

$E(A) = \text{least fixed point of}$

$E^*(A)$

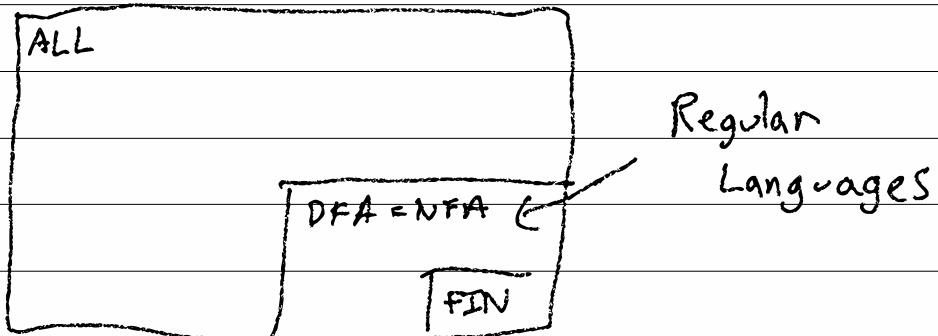
$E^*(A) = A \cup \bigcup_{g_i \in A} \delta(g_i, \varepsilon)$

6-1)  $\forall N \in \text{NFA}, \exists D \in \text{DFA}.$  compile :

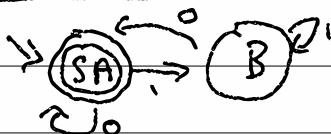
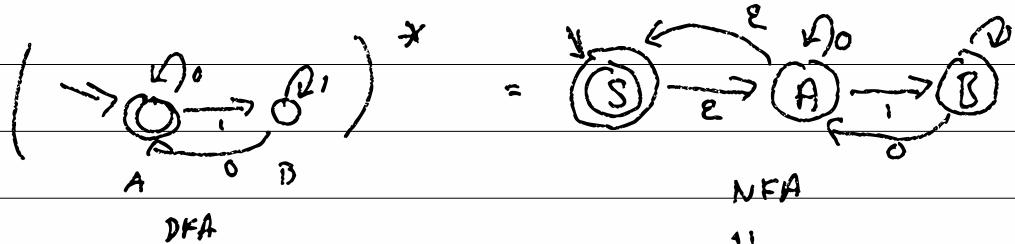
$$N = D$$

$\forall D \in \text{DFA}, \exists N \in \text{NFA}.$

$$D = N$$



Program f ...      'f; g'  
 program g ...      compositional



## 6-2 / Regular Expressions

re :=	$\epsilon$	EMPTY
	$\emptyset$	NULL
	c	Char
	re U re	CUP
	* re	STAR
	re o re	CIRC

interface RegEx { }

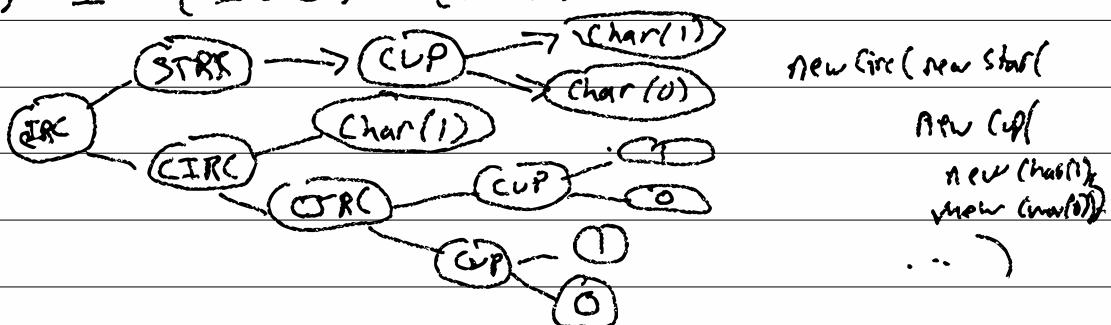
class RE\_EMPTY() impl REGEX { .. . }

RE\_NULL()

RE\_Char (char c)

RE\_Cup (REGEX lhs, REGEX rhs)

$(\text{I}\cup\text{O})^* \text{I} \circ (\text{I} \cup \text{O}) \circ (\text{I} \cup \text{O})$  - "3rd from end is I"



6-3 |  $L : RE \rightarrow \text{ALL} = P(\Sigma^*)$

fix the language if doesn't  
(or come)

$$L(\epsilon) = \{\epsilon\}$$

$$L(\emptyset) = \emptyset$$

$$L(c) = \{c\}$$

$$L(r \cup r') = L(r) \cup L(r')$$

$$L(r \circ r') = L(r) \circ L(r')$$

$$L(r^*) = L(r)^*$$

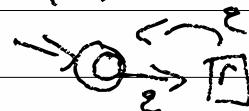
class Up {  
     $L()$   $\in$   
    return up(lhs, L(), rhs, L());  
}

compile : RE  $\rightarrow$  NFA

$$\text{compile } (\epsilon) = \xrightarrow{\epsilon} \textcircled{0}$$

$$\text{compile } (r^*) =$$

$$\text{compile } (\emptyset) = \xrightarrow{\epsilon} \textcircled{0}$$



$$\text{compile } (c) = \xrightarrow{\epsilon} \textcircled{0} \xrightarrow{c} \textcircled{0}$$

$$\text{compile } (r \cup r') = \xrightarrow{\epsilon} \textcircled{0} \xrightarrow{c} \boxed{r} \xrightarrow{\epsilon} \boxed{r'} \xrightarrow{\epsilon} \textcircled{0}$$

$$\text{compile } (r \circ r') = \xrightarrow{\epsilon} \boxed{r} \xrightarrow{\epsilon} \boxed{r'} \xrightarrow{\epsilon} \textcircled{0}$$

6-4

$$"\ " = \epsilon$$

$$\underline{\quad} = \emptyset$$

$$"\ c" = c$$

$$"\ xyz^*" = x \circ y \circ (z^*)$$

$$"\ (xyz)^*" = (x \circ y \circ z)^*$$

$$"\ xyz" = x \circ y \circ z$$

$$"\ [abc]" = (a \cup b \cup c) \quad (a, b, c \in \Sigma)$$

$$"\ (a \mid b \mid c)" = \Rightarrow \quad (a, b, c, \in \Sigma^*)$$

[012]

(zero | one | two)

$$\cdot = \epsilon$$

$$"\ .^* \backslash. m; s" = \Sigma^* \circ ' . ' \circ ' m' \circ ' ; ' \circ ' s'$$

8-5/

gen : RE  $\rightarrow \Sigma^*$  or false

gen  $\epsilon = \epsilon$

gen  $\emptyset = \text{FALSE}$

gen  $c = 'c'$  flip coin

gen  $x \cup y = \text{gen } x \sqcup \text{gen } y$

gen  $x \circ y = \text{gen } x \circ \text{gen } y$

gen  $x^* = \boxed{\square}$

= gen  $(\epsilon \cup x \circ x^*)$   
"mjs"

equal : RE  $\times$  RE  $\rightarrow$  Bool

equal  $x \ y =$

NFA2DFA(compile  $x$ )  $\xrightarrow{\text{?}} \text{def equality?}$   
NFA2DFA(compile  $y$ )

$$(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$$

G-6)

$$x + 0 = x$$

$$x = x$$

$$x \cdot 1 = x$$

$$\frac{a = b}{a+x = b+x}$$

$$\frac{a = b}{ax = bx} \quad x \neq 0$$

$$2(3x + 17) = 6x + 34 \quad \text{"algebra"}$$

$$3x + 17 = 3x + 17$$

$$3x = 3x$$

$$x = x$$

$$17 \cancel{x} \approx 17 \cancel{z}?$$

WZSS

$$\emptyset \cup x = x \cup \emptyset = x$$

$$\emptyset \circ x = x \circ \emptyset = \emptyset$$

$$\varepsilon \circ x = x \circ \varepsilon = x$$

$$\emptyset^* = \varepsilon \approx \varepsilon \cup \emptyset \circ \emptyset^* \quad x^* = \varepsilon \cup x \circ x^*$$
$$\approx \varepsilon \cup \emptyset = \varepsilon$$

$$x \circ (y \cup z) = x \circ y \cup x \circ z$$

NFA  $\approx$  DFA      in:  $N$  states      ( $Q$ )

out:  $z^N$  states      ( $P/Q$ )

6-7/ size : RE  $\Rightarrow$  Nat

$$\text{size } \emptyset = 1$$

$$\text{size } \varepsilon = 1$$

$$\text{size } c = 1$$

$$\text{size } (x \cup y) = \text{sz } x + \text{sz } y + 2$$

$$\text{size } (x \circ y) = \text{sz } x + \text{sz } y + 1$$

$$\text{size } (x^*) = \text{sz } x + 1$$

$$x \circ (y \cup z) = x \circ y \cup x \circ z$$



emacs \* .c

↓ bash

emacs .c

↓  $\Sigma^* \circ \omega C$

compile

→ o → o → o →

+

→ o → o → o

↓ N2D

Yf S

304.c

← ↗ ↘ ↙

304.c main.c

dir

304.c

awesome-quotes.txt

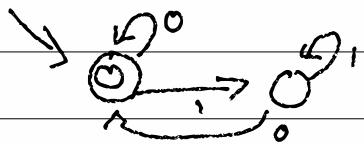
mycat.jpg

main.c

+

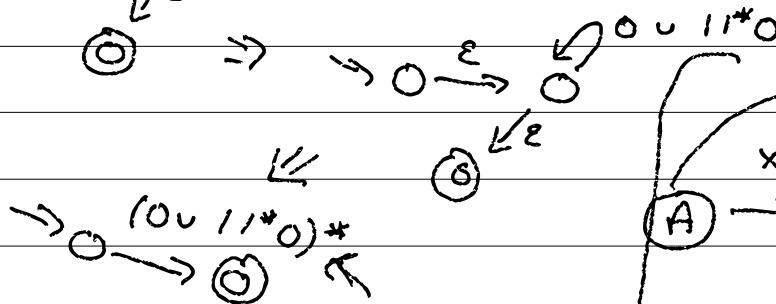
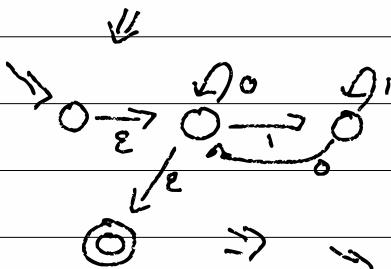
emacs

6-8] decompire : DFA  $\xrightarrow{(\text{NFA}) \text{ or }} \text{RE}$



$$\Sigma^* = 0 \cup \epsilon$$

$$(1 \cup 0)^* \cup 0 \cup \epsilon$$

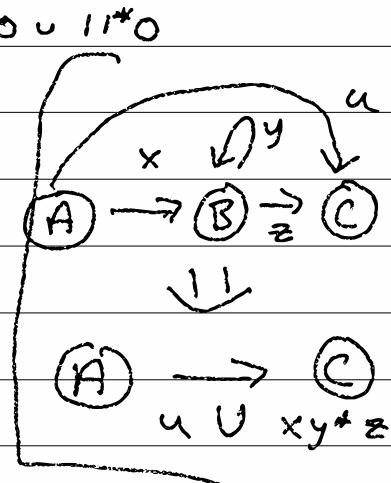


$$(0 \cup 11^* 0)^*$$

$\epsilon \quad 0 \quad 10 \quad 111110$

$011101111100$

11100011101100101010



G-9 / decompile : N-state NFA  
→ RE

START : N-NFA → (N+2)-GNFA

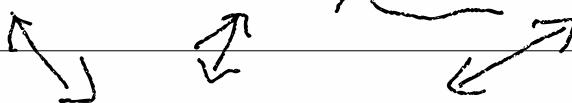
RIP : (N+1)-GNFA → N-GNFA

END : 2-GNFA → RE

decompile m = end ∘ rip<sup>n</sup> ∘ start (n)

7-1) DFA/NFAs  $\rightarrow$  RE

DFA<sub>s</sub>  $\leftrightarrow$  NFA<sub>s</sub>  $\leftrightarrow$  RE



Regular  
Languages

NFA  $\rightarrow$  RE

IN: n-NFA  $\rightarrow$  (n+2)-GNFA

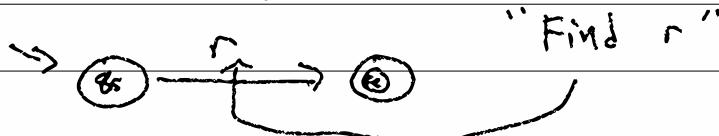
RIP<sup>n</sup>: (n+1)-GNFA  $\Rightarrow$  n-GNFA

OUT: 2-GNFA  $\rightarrow$  RE

GNFA =  $(Q, \Sigma, g_s, g_e, \Delta : (Q-g_e) \times (Q-g_s) \xrightarrow{e_Q} e_Q)$   
 $S : Q \times \Sigma \rightarrow Q \Rightarrow RE(\Sigma)$

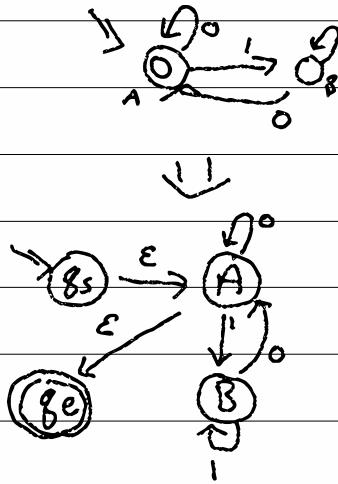
OUT: 2-GNFA  $\rightarrow$  RE

$(\{\Sigma, g_s, g_e\}, \Sigma, g_s, g_e, \{(g_s, g_e), \uparrow\})$   
 $= r = \Delta(g_s, g_e)$



7-2) IN: NFA  $\Rightarrow$  GNFA (n+2) -  
 $(Q, \Sigma, g_0, \delta: Q \times \Sigma \rightarrow P(Q))$   $(Q', \Sigma, g_s, g_e,$   
 $F, \Delta: (Q' - g_e) \times (\Sigma - \{g_e\}) \rightarrow R_E)$

$$Q' = Q \cup \{g_e, g_s\}$$



$$\Delta(g_i, g_j) = r$$

$$\Delta(g_s, g_0) = \epsilon$$

$$\Delta(g_s, g_j \neq g_0) = \emptyset$$

$$\Delta(g_s, g_f \in F, g_e) = \epsilon$$

$$\Delta(g_i \notin F, g_e) = \emptyset$$

$$\Delta(g_i, g_j) = \cup \{\epsilon\}$$

$$\delta(g_i, c) \ni g_j \}$$

7-3/ RIP =  $(n+1)$ -GNFA  $\rightarrow$   $n$ -GNFA  
 main  $'(Q, \Sigma, g_s, g_e, \Delta)'$   $\downarrow$   $'(Q', \Sigma, g_s, g_e, \Delta')$

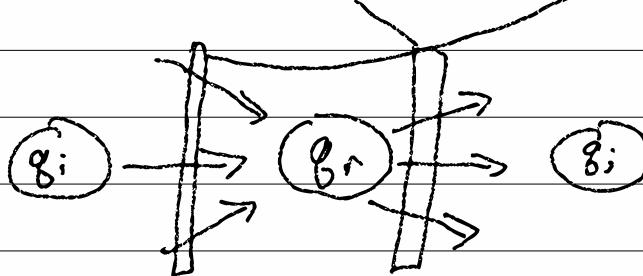
↓  
 jay  
 main

$$Q = Q' \cup \{ q \text{ gonna be killed} \}$$

↓  
 jay  
 main  
 ↓  
 exit  
 ↓  
 exit

$$\Delta' : \underbrace{(Q' - g_e)}_{g_r \in} \times \underbrace{(Q' - g_s)}_{g_r \in} \rightarrow \text{RE}$$

$$\Delta : \underbrace{(Q - g_e)}_{g_r \in} \times \underbrace{(Q - g_s)}_{g_r \in} \rightarrow \text{RE}$$

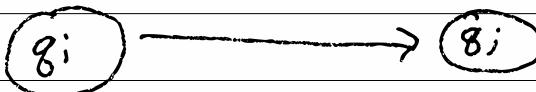


$$x \circ y^* \circ z \cup \alpha$$

$$\rightarrow x \circ z$$

$$\Delta'(q_i, q_j)$$

$$=$$

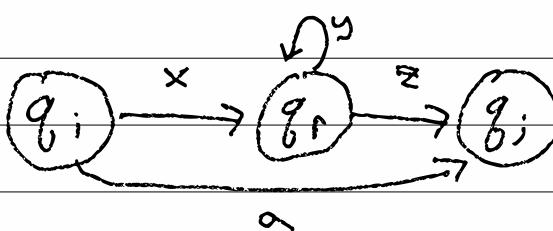


$$x = \Delta(q_i, q_r)$$

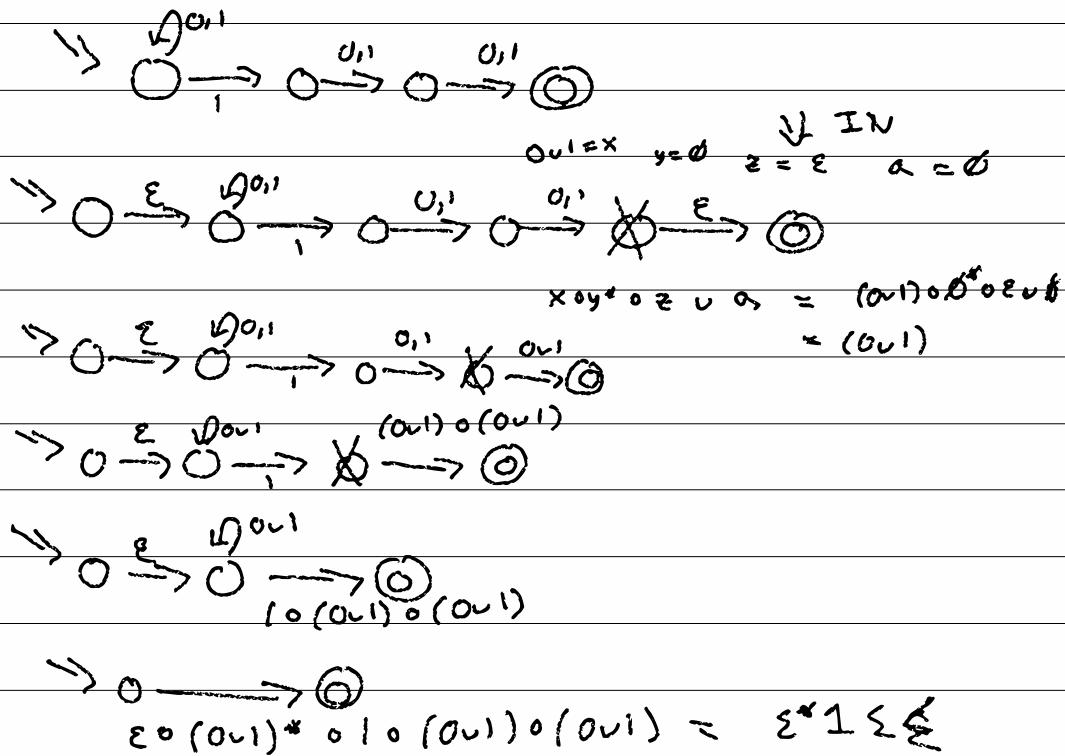
$$y = \Delta(q_r, q_r)^*$$

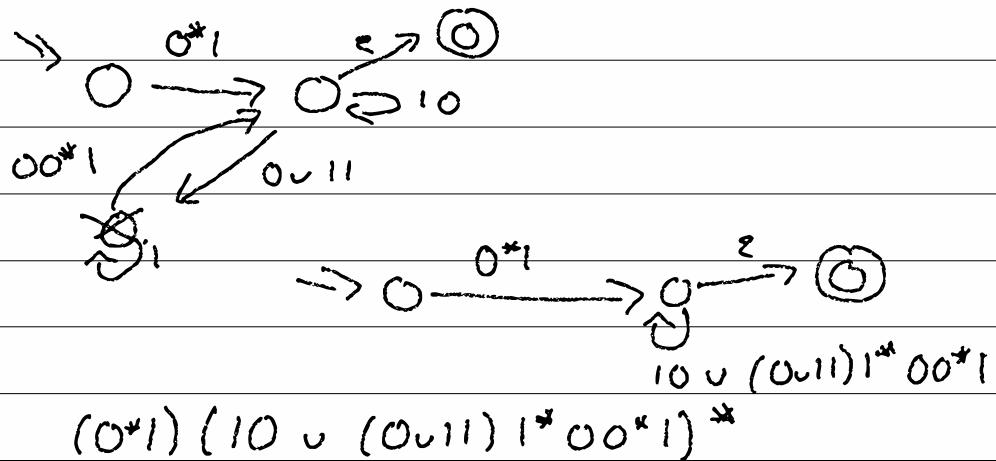
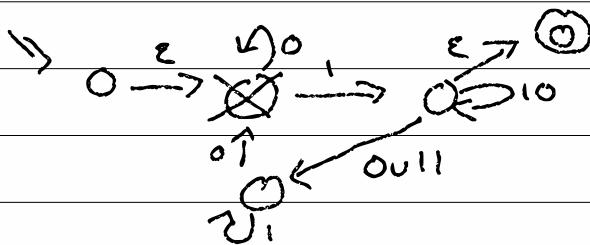
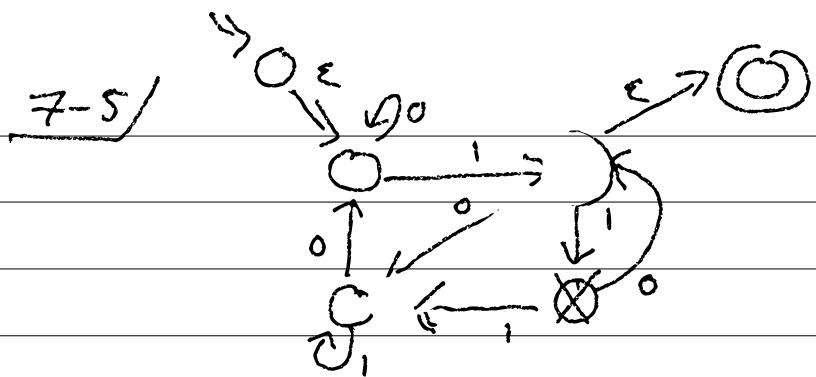
$$z = \Delta(q_r, q_j)$$

$$\alpha = \Delta(q_i, q_j)$$



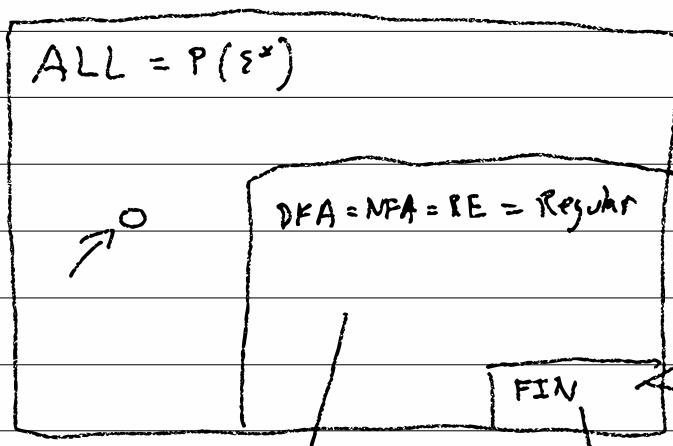
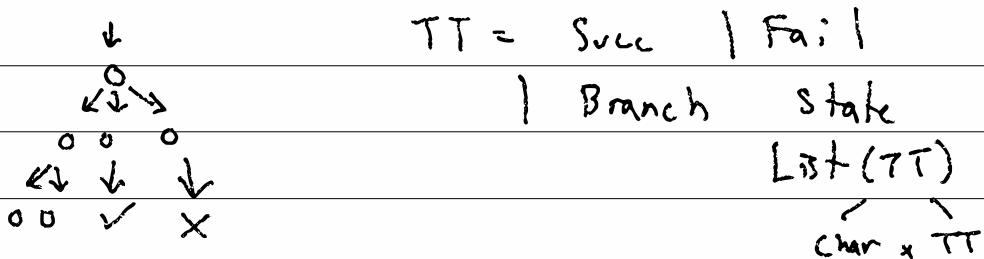
~~7-4/ int main () {~~  
 int ~~x = 8;~~  
~~int y = f(x, z);~~  
 ...  
 int y = x + 2 \* 2;  
 ↓  
 int f(int ~~z~~){  
 return ~~z + 2 \* 8;}~~





8-1 ε Object → Char (char c)  
 Upsilon () Epsilon ()  
 UTF-8

TT



$\{\text{all strings ending in } 0\}$

$\{\emptyset, 00\}$

$A \in REG \iff \exists d \text{ DFA. } L(d) = A$

$\neg A \in REG \iff \neg (\exists d) \iff \forall d. d \text{ DFA} \wedge$

$\neg \exists x. P(x) \iff \forall x. \neg P(x) \quad L(d) \neq A$

$\neg \forall x. P(x) \iff \exists x. P(x)$

8-2/ How can we know stuff about  
infinite sets?

$\forall x \in A, P(x)$

$P: DFA \rightarrow \text{Prop}$

$\Rightarrow z_0 \in Q$

$P: IP(\Sigma^*) \rightarrow \text{Prop}$

$\neg P(B)$  (where  $B \in IP(\Sigma^*)$  and we "hope"  
 $B$  isn't in DFA)

Q. What is P?

1. Prove  $\nexists RFG, P(A)$

Pumping  
lemma

2. Prove  $\exists B \in \text{RELL}, \neg P(B)$

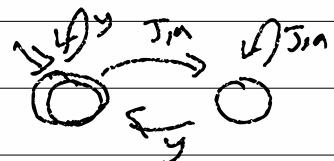
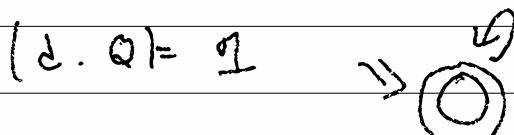
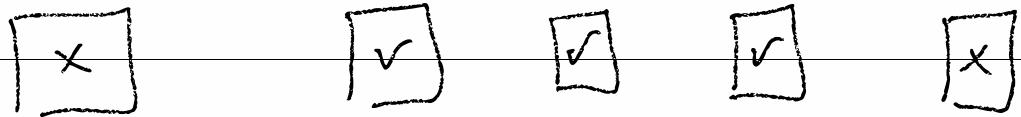
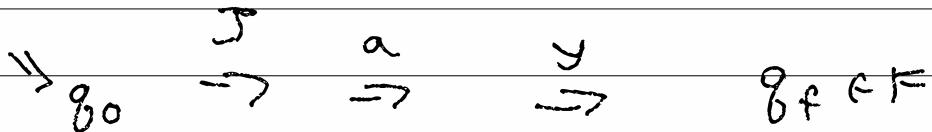
conclude  $B \notin \text{REG}$ .

$$\Sigma = \dots$$

8-3)

"Jay" & d

$\Sigma \ni \{s, a, y\}$



Daphne wins

pick a number of states ; 4

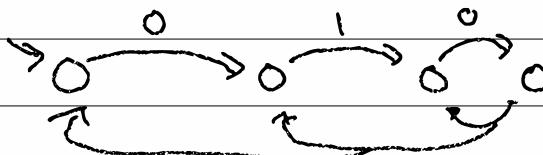
she picks a char

I say what state we go to

I win if I never say same state

she wins if I repeat

How many turns to win?

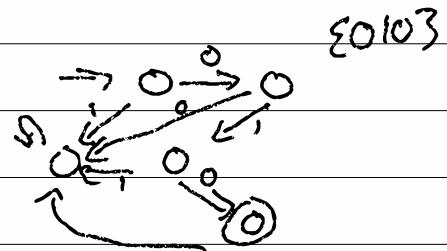
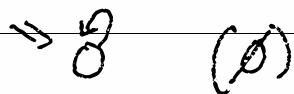
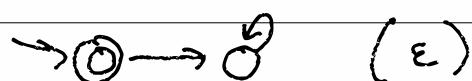


$4 \Rightarrow N \Rightarrow$

|@|

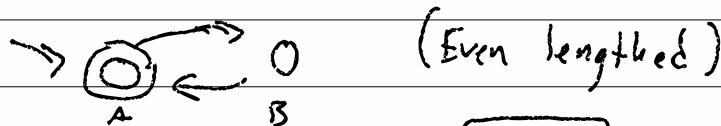
$\delta: Q \times \Sigma \rightarrow Q$  Total Fun  
 $\xrightarrow{\text{from}} \xrightarrow{\text{to}}$

All DFAs have a loop



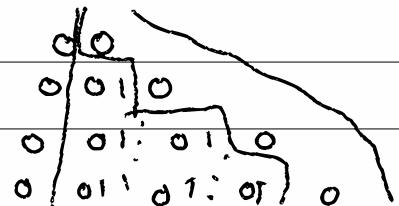
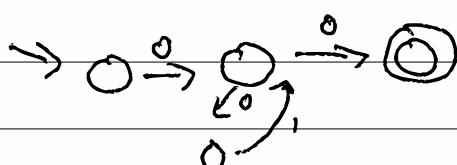
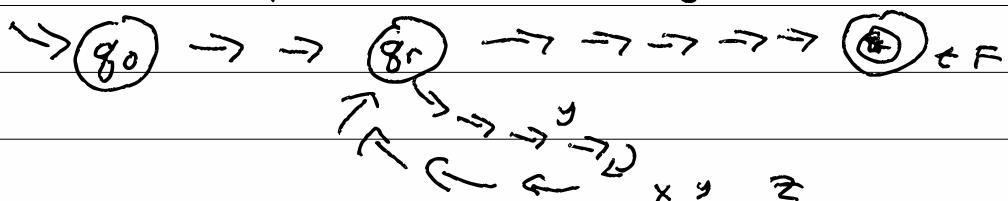
Some have "exciting" loops

$\exists s \in (\text{DFA})$ ,  $s = \dots \dots \dots \dots \dots$



$$\begin{aligned} \epsilon &= A \\ &= \epsilon \circ \epsilon \circ \epsilon \end{aligned}$$

$$0110 = \overbrace{ABA\bar{A}}^z = \epsilon \circ 01010$$



8-5/ If a machine hasn't an existing loop ...

They are all finite  
 $L(m) \in FIN$

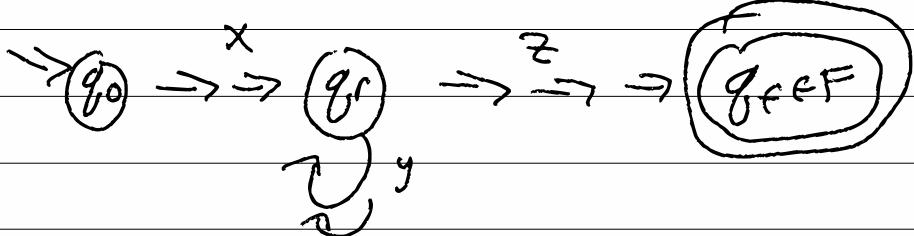
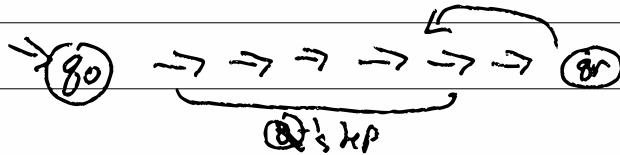
If it does have one ...

then it must be infinite (regular)

If there is an existing loop ...

what is the shortest string to  
"find" ;+ m?

$$|s| = |Q| \quad (s \in L(\delta))$$



$\in \text{ALL}(\text{a language}) (\mathcal{P}(\Sigma^*))$

8-6/ RPP(A) = Regular Pumping Property

? ( $\exists p \in \mathbb{N}, \quad / / \quad p = |Q|$

$\forall s \in A \quad | \quad |s| \geq p \quad )$

$\exists (x, y, z \in \Sigma^*) \quad | \quad s = xyz \quad \wedge$

$|xy| \leq p$

$|y| > 0 \quad )$

$\forall i \in \mathbb{N},$

$x \circ y^i \circ z \in A$

)

$\forall d \in \text{DFA}, \exists r \in \text{RE}. \quad L(d) = L(r) \quad - \text{Cof}$

$\neg \text{RPP}(B) :=$

$\forall p \in \mathbb{N},$

$\exists (s \in B \quad | \quad |s| \geq p \quad )$

$\forall (x, y, z \in \Sigma^*) \quad | \quad s = xyz \wedge |xy| \leq p \wedge |y| > 0 \quad \text{and} \quad$

$\exists i \in \mathbb{N},$

$x \circ y^i \circ z \notin B$

8-7) Need: an infinite space problem

```
O* 1 { while (getc() == '0') {  
    ungetc()  
    if (getc() == '1') { net false }  
    return getc() == EOF;  
} = 232 vint2_+
```

$\forall n \in \mathbb{N}$ ,

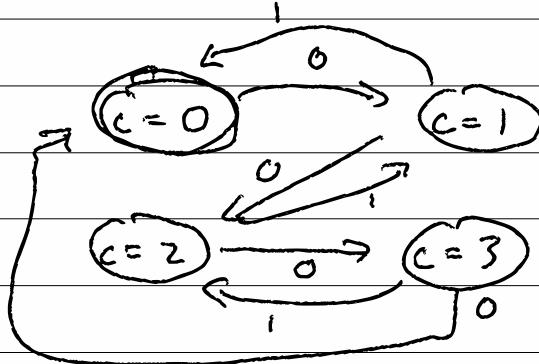
$0^n 1^n \in B$

vint count = 0;

while (in == 0) count++

while (in == 1) count--  
if (in == EOF) < count--  
return count == 0 ;

{ net false } }



q-1) RPP(A)  $\rightarrow$  Language =  $P(\Sigma^*) \approx A\bar{A}$

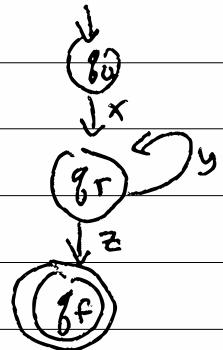
$$\exists p \in N, \quad |P| = |Q|$$

$$\forall s \in A \mid |s| > p \}.$$

$$\exists (x, y, z \in \Sigma^*) \mid s = xyz$$

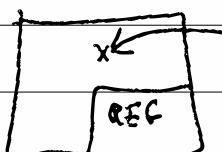
$$\forall i \in N, \quad |xy| < p$$

$$xyiz \in A \quad |y| > 0$$



$\neg RPP(A) =$

ALL



$$\forall p \in N,$$

$$\exists (s \in A \mid |s| > p)$$

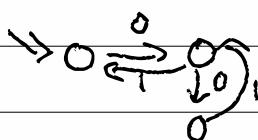
$$\forall (x, y, z \in \Sigma^*) \mid s = xyz \wedge |y| < p$$

$$\exists i \in N$$

$$xyiz \notin A$$

$$0^n 1^n \quad \text{ie} \quad x \in 0^n 1^n$$

$$\text{iff } \exists n \in N, \quad x = 0 \overbrace{0 \dots 0}^{\text{long part}} 1 \dots 1$$

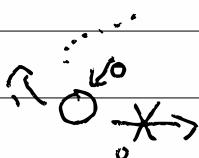


$$\text{int count} = 0$$

while (see 0) count++

while (see 1) count--

return count == 0



9-2)  $\neg \text{RPP}(\mathcal{O}^n, \mathcal{O}^n) =$

$\forall p \in \mathbb{N},$

$\exists (s \in \mathcal{O}^n)^n \mid |s| \geq p)$

choose  $s: s = x = z$

$s = \mathcal{O}^{p/2} \cup \mathcal{O}^{p/2}$

$s = \mathcal{O}^p, |s| = 2p \geq p$

$\forall (x, y, z \in \Sigma^*) \mid s = xyz \text{ and}$

$|xy| \leq p \text{ and}$

$|y| > 0$ )

$x = \mathcal{O}^a \quad a+b+c = p \quad a+b < p$

$y = \mathcal{O}^b \quad d = p \quad b > 0$

$z = \mathcal{O}^c \cup d$

$\exists i \in \mathbb{N}, xyiz \in \mathcal{O}^n, \mathcal{O}^n$

$xyiz = \mathcal{O}^a \mathcal{O}^b \cup \mathcal{O}^c \cup d = \mathcal{O}^{a+b+c} \cup d$

$a+b+c = d$

$b(i-1) \stackrel{-p}{=} \frac{0}{b} \quad i=0 \quad \boxed{i=1}$

fun  $n:$

if  $n = 0 : \text{ret } p \circ$

$p \circ : \mathbb{P} \circ$

else: let  $m = n-1$

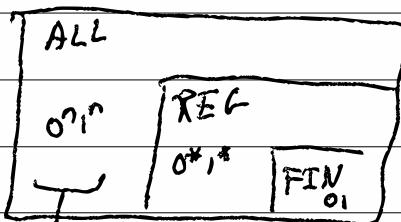
$p \circ : \text{th. } p_n \rightarrow p_{(n+1)}$

let  $p_m = \text{rec } m \Rightarrow \forall n: p_n$

$p \circ p_m$

induction on  $N:$

9-3/ There is stuff outside REG?



All computers  
are DFAs.

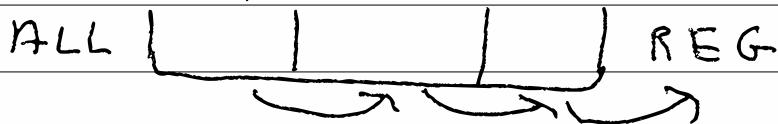
infinitely big

if  $0^n 1^n \in \text{REG}$

~~then~~<sup>and</sup>  $B \in \text{REG}$

then  $0^n 1^n \circ B \in \text{REG}$

$\Sigma_1 \quad \Sigma_0 \quad \text{CFL}$



$0^n 1^n \in \text{REG}$

w where  $\text{count}(0,w) = \text{count}(1,w)$

ww where  $w \in \Sigma^*$

010101  $\in \Sigma^*$

wwR where  $w \in \Sigma^*$

$$\underline{q-y} / \quad 0^x 1 0^y 1 0^{x+y}$$

$$01\ 0100 \Rightarrow "1+1=2" \quad \checkmark$$

$$001\ 0001\ 00000 \Rightarrow "2+3=5" \quad \checkmark$$

$$001\ 0010 \Rightarrow "2+2=1" \quad \times$$

A.p.

$$\exists s . \quad 0^p 1 0^p 1 0^{2p}$$

$$\nexists x y z . \quad x = 0^a \quad y = \underline{0^b} \quad z = 0^c 1 0^p 1 0^{2p}$$

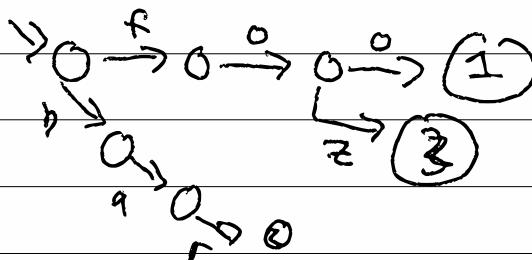
$$\exists ; . \quad \begin{matrix} \text{change this} \\ \text{can't change} \end{matrix}$$

$$\underbrace{z+z=4}$$

$$\Rightarrow \Rightarrow$$

$$\underbrace{g+z=10}$$

12-1 / trie       $\text{foo} \leftrightarrow 1$       digital  
                    $\text{bar} \leftrightarrow 2$       search  
                    $\text{baz} \leftrightarrow 3$       tree



CFGs

start variable       $0^n 1^n$       & REG

#       $S \rightarrow \epsilon$       ← rule, productions, transitions  
 D       $S \rightarrow OSI$

variables  
 non-terminals  
 Symbols

rule =  $\overbrace{V}^{\text{lhs}} \rightarrow \overbrace{V \cup \epsilon}^{\text{rhs}}$   
 $\text{rhs} = (\text{V} \cup \epsilon)^*$   
 $\xrightarrow{\text{terminal}}$

$\overset{B}{S} \rightarrow \overset{B}{OSI} \rightarrow \overset{B}{0OSII} \rightarrow \overset{B}{000SIII}$   
 "derivation"       $\overset{B}{000III}$

$w \in L$       iff       $\exists d. S \xrightarrow{*} w$

$$\frac{L(2)}{M} \left( \begin{array}{l} S \Rightarrow 01S \\ S \Rightarrow SS \end{array} \right) = \emptyset$$

Context-free grammar

$$0101 \in \xrightarrow{\quad} \text{alphabet } (V, \Sigma, R, S)$$

$\downarrow$  finite set       $\downarrow \in V$

$$B \Rightarrow \epsilon$$

$$B \Rightarrow B \cup N \cup B$$

$$N \Rightarrow \epsilon \qquad P(V \times (V \cup \Sigma)^*)$$

$$N \Rightarrow 0N \qquad (V \Rightarrow P((V \cup \Sigma)^*))$$

$\Sigma(S, 01S),$

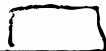
$$R = \{S, SS\} \quad V = \{\Sigma\} \quad \Sigma = \{0, 1\} \quad S = S$$

REG

DFA

REX  $(\cup, \cdot, ^*)$

CFL



CFG

12-3 / A B

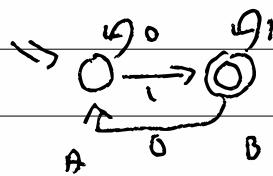
$\rightarrow A \cup B : S \Rightarrow A, S$

$S \Rightarrow B, S$

$A \circ B : S \Rightarrow A, S \quad B, S$

$A \cap B : \times$

DFA  $\Rightarrow$  CFG

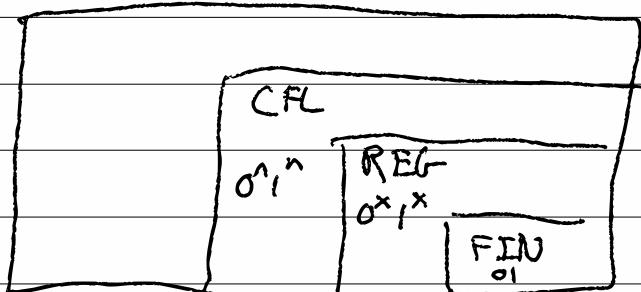
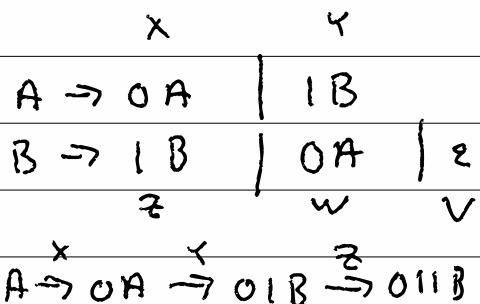


01101

$S = g_0$

$g_i \rightarrow c g_j \text{ iff } \delta(g_i, c) = g_j$

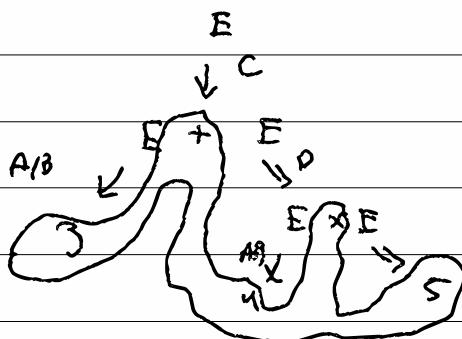
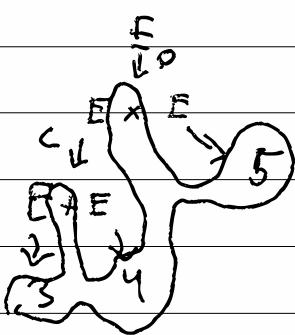
$g_f \rightarrow \epsilon \text{ iff } g_f \in F$



P D C D

$$12 - 4 / E \Rightarrow O \quad | \quad I \quad | \quad E + E \quad | \quad E \times E$$

$$3 + 4 \times 5$$



fowim  $E \Rightarrow n$

fowin  $O = O$

$I = I$

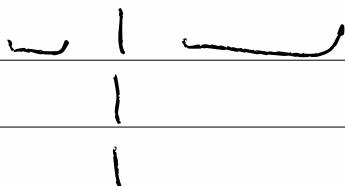
Plus(L, R)  $\Leftarrow$  fowim L + fowim R

Mult(L, R)  $\Leftarrow$  fowim L  $\times$  fowim R

ambiguous = There are multiple  
trees (derivations)

if cond

for the same string



(Z-S) Nice! amb  $\rightarrow$  unamb

not possible

$V, \Sigma, R, S$

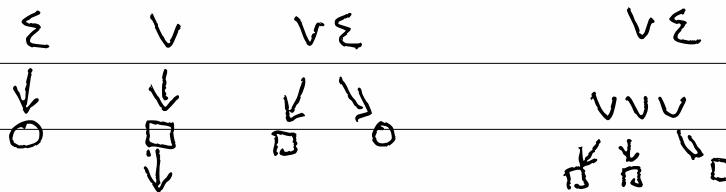
$$V \Rightarrow (V \cup \Sigma)^*$$

$\epsilon$

$\epsilon \Sigma V$

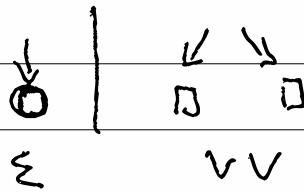
$V$

VVVVV



want: "pure" binary trees

Noam



Chomsky-Normal Form

$$\begin{array}{c|c} V \rightarrow \Sigma & A \rightarrow C \\ V \rightarrow VV & A \rightarrow BC \\ S \rightarrow \epsilon & \#S \end{array}$$

GFG  $\rightarrow$  CNF

(2-5)  $S \rightarrow \epsilon \mid OSI$

— Add a new start

$R \rightarrow S$

$S \rightarrow \epsilon \mid OSI$

— remove  $\epsilon$ -rules

$R \rightarrow S \mid \epsilon$

$S \rightarrow OSI \mid OI$

remove unit rules

$R \rightarrow OSI \mid OI \mid \epsilon$

$S \rightarrow OSI \mid OI$

add extra vars for  $\geq 2$

$R \rightarrow TI \mid OI \mid \epsilon$

$S \rightarrow TI \mid OI$

$T \rightarrow OS$

add terminals "names"

$R \rightarrow TB \mid AB \mid \epsilon$

$S \rightarrow TB \mid AB$

$T \rightarrow AS$

$A \rightarrow O$

$B \rightarrow I$

12-6 / 00111 + 0<sup>n</sup>1<sup>n</sup>

S  $\Rightarrow \epsilon$

S  $\Rightarrow 0S1$

A

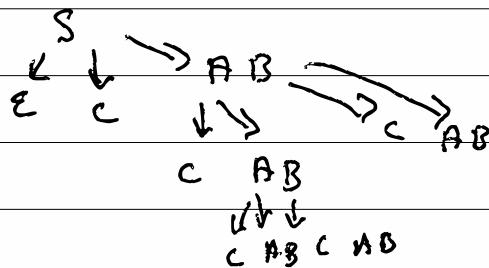
B

A  $\Rightarrow c$  + 1

B  $\Rightarrow$  + 2

A  $\Rightarrow BC$  + 2

A  $\Rightarrow$  done



S  $\Rightarrow XYZ$

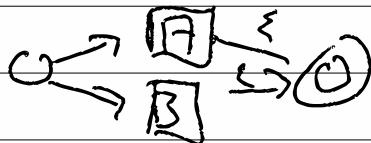
X  $\Rightarrow ZZS$  | 0

Y  $\Rightarrow XYX$  | 1

Z  $\Rightarrow YSX$

$$A = \{x, y, z\}$$

$$B = \{x, y, z\}$$



$$Q = \{S, E, (0, x), (0, y), (0, z), (1, x), (1, y), (1, z)\}$$

$$= \{S, E\} \cup 0 \times A \cup 1 \times B$$

unionstate = start | end | fromA  $g_A$   
| fromB  $g_B$

$$\delta(\text{start}, \epsilon) = \{ \text{fromA } g_0, \text{ fromB } g_0 \}$$

$$\delta(\text{fromA } g_i, c) = \text{fromA } (\delta_B(g_i, c))$$

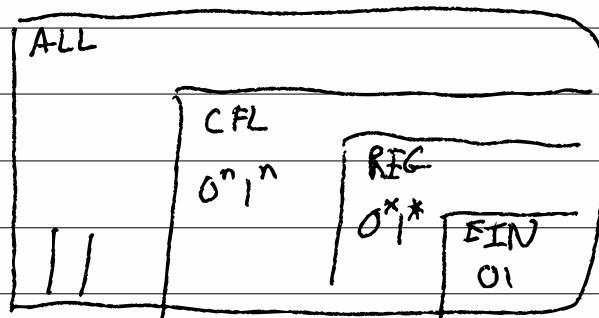
12-7 / new DFA ("blat",  
 $q_i \rightarrow q_i = 0 \text{ if } q_i = 1,$   
sigma, delta,  
 $F$ )



complement (DFA  $\perp$ )  $\Sigma$

new DFA ( $\perp$ ,  $q_i \rightarrow !d.F(q_i)$ )  
( $d.Q$ ,  $d.\Sigma$ ,  $d.\delta$ ,  $d.F$ )

(5-1)



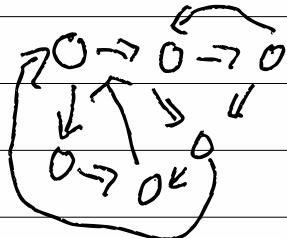
REGULAR

$$\text{DFAs} \longleftrightarrow \text{Regex}$$
$$\epsilon \qquad \qquad \qquad \{\}$$

CFL

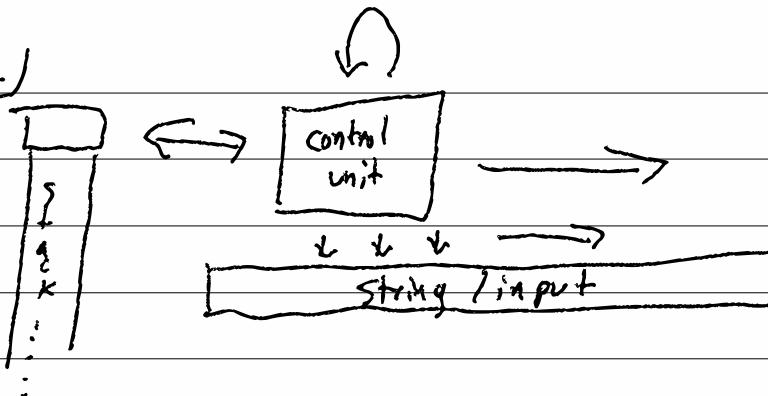
$$\boxed{\text{PDA}} \longleftrightarrow \text{CFG}$$

push-down n automata



$\Sigma^* 1 \Sigma \Sigma$

15-2)



$$\text{DFA} : Q \times \Sigma \rightarrow Q$$

$$\text{Deterministic PDA} : Q \times \Sigma \times \Gamma \rightarrow Q \times \Gamma$$

$$\text{PDA} : Q \times \Sigma \times \Gamma \rightarrow P(Q \times \Gamma)$$

$\Sigma \cup \{\epsilon\}$

$$Q \times \Sigma \times \Gamma \xrightarrow{\quad} Q \times \Gamma \quad (\text{ignored stack})$$

$$Q \times \Sigma \times \Gamma \rightarrow Q \times \Gamma \quad (\text{pop})$$

$$\times \Sigma \rightarrow Q \times \Gamma \quad (\text{push})$$

$$\times \Gamma \rightarrow \times \Gamma \quad (\text{replace})$$

$$\epsilon \times \Gamma \rightarrow \times \Gamma$$

$$\delta(q_i, c) = q_j$$

$$[q_i]_c w \rightarrow [q_j]_w$$

$$c \in \Sigma \quad w \in \Sigma^*$$

$$\delta(q_i, c, \alpha) \ni (q_j, \beta)$$

$$\beta \alpha [q_i]_c w \rightarrow \beta \gamma [q_j]_w$$

$$\alpha \in \Gamma_\Sigma \quad c \in \Sigma$$

$$\beta \in \Gamma^* \quad w \in \Sigma^*$$

config

15-3/ simulate : PDA  $\times \cancel{S} \rightarrow \text{config}$   
 $\text{sim } (\mathcal{Q}, \Sigma, \Gamma, q_0, \delta, F) \quad (\Gamma^*, q_f, \Sigma^*) =$

let  $\alpha : \beta \in g$  in

let  $c = w \in S$  in

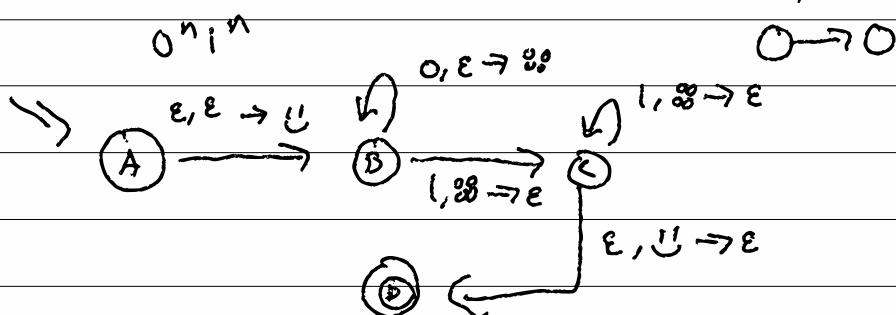
let  $(q_j, \alpha) \in \delta(q_i, c, \alpha)$  in  
 $(\alpha = \beta, q_j, w)$

accepts : PDA  $\times \Sigma^* \rightarrow \text{bool}$

accepts  $p \; s = \text{while } c, s \notin \Sigma \text{ do}$

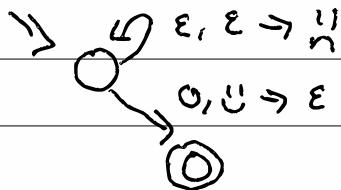
return  $\left( \begin{array}{l} \text{sim } q_f \\ c, q_j \in p, c \end{array} \right)$   
 where  $c_0 = (\Sigma, p, q_0, S)$

$c, \alpha \Rightarrow \beta$



$\epsilon[A]0011 \rightarrow \cup[B]0011 \rightarrow \cup^{00}[B]011 \rightarrow \cup^{0000}[B]11$   
 $\rightarrow \cup^{00}[C]1 \rightarrow \cup[C]\epsilon \rightarrow \epsilon[D]\epsilon \rightarrow \text{YES}$

15-4/



CFG  $\rightarrow$  PDA

$$\begin{array}{l} S \Rightarrow \epsilon \\ S \Rightarrow 0S1 \\ S \Rightarrow Xyz \ 011\ 3yx \end{array} \Rightarrow \begin{array}{c} 0 \rightarrow 0 \rightarrow 0 \\ \downarrow \\ 0 \end{array}$$

$$\Gamma = \Sigma \cup V$$

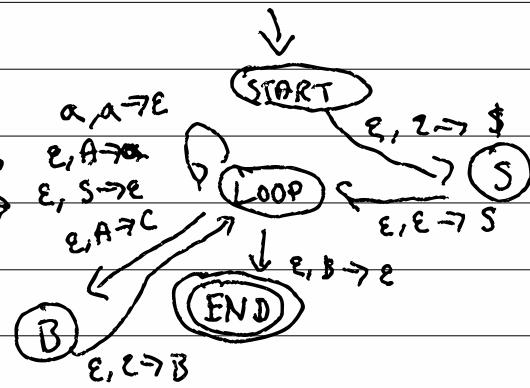
CNF  $\rightarrow$  PDA

$$A \Rightarrow BC \quad (B, C \neq S)$$

$$S \Rightarrow \epsilon$$

$$A \Rightarrow a$$

$$A[\text{Loop}] \Rightarrow C[B] \Rightarrow CB[\text{Loop}]$$



$$\$[L]0011 \rightarrow \$[S]0011 \rightarrow \$S[L]0011 \rightarrow \$ISO[L]0011$$

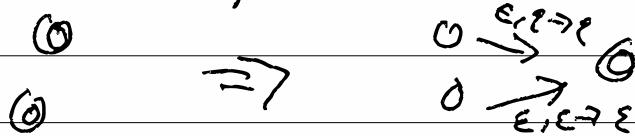
$$\$1\$R[L]11 \leftarrow \$1ISO[L]011 \leftarrow \$1\$[L]011$$

$$\$1\$[L]11 \rightarrow \$1[L]1 \rightarrow \$[L] \rightarrow [\text{END}] \rightarrow \checkmark$$

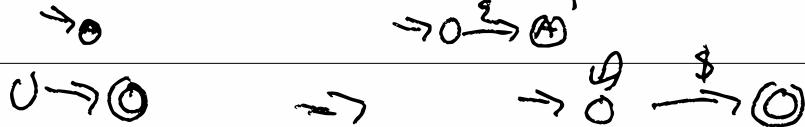
15-5/ PDAs are too complicated

so, we'll simplify with some rules

- Make a single accept



- Guarantee stack is empty on accept



- Always push on pop

push :  $\epsilon$ ,  $\Gamma$

pop :  $\Gamma$ ,  $\epsilon$

X ignore :  $\epsilon$ ,  $\epsilon$

X replace :  $\Gamma$ ,  $\Gamma$

Every symbol pushed is eventually popped

15-6/  $V = (\mathbb{Q} \times \mathbb{Q})$   $S = (g_0, g_f)$

$$(r, +) \in \delta(p, a, \varepsilon)$$

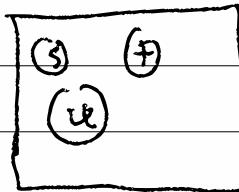
$$(g, \varepsilon) \in (s, b, +)$$

$$\overline{(p, g) \rightarrow a \underset{\in \Sigma}{\in} (r, s) \underset{\in V}{\in} b \underset{\in \Sigma}{\in}}$$

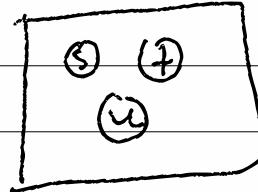
$$(p, p) \rightarrow \varepsilon$$

$$(p, g) \Rightarrow (p, r) (r, g) \quad \forall p, g, r$$

A  $\leftarrow$  cast



B  $\leftarrow$  bst



shown cast, bst

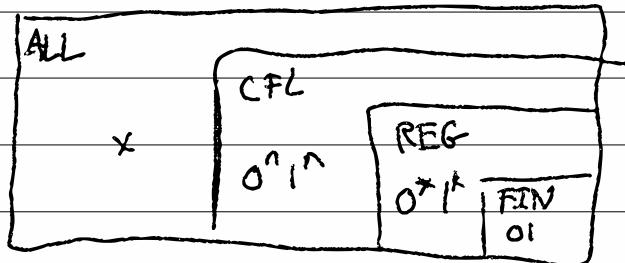
A  $\cup$  B  $\leftarrow$  ~~cast, bst~~

shown  $\langle x, y \rangle$  = start

| and A x

| and B y

16-1



DFA<sub>s</sub>  $\longleftrightarrow$  REG

$0^n 1^n \in \text{REG}$

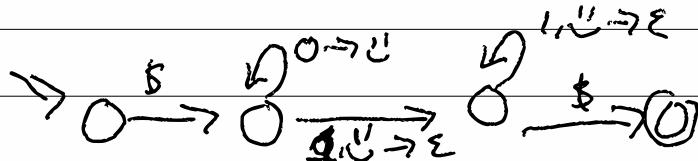
PDA<sub>s</sub>  $\longleftrightarrow$  CFG

$x \notin \text{CFL}$

CFL = ALL

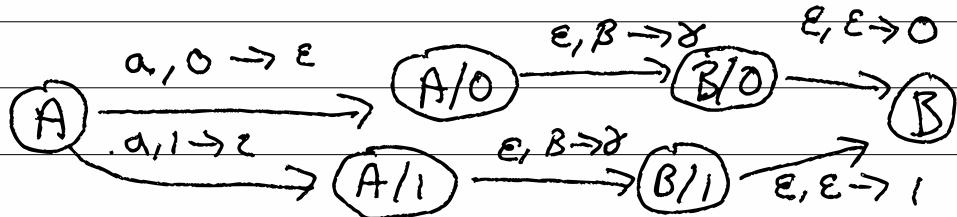
1.	$P \dots \forall x \in \text{REG}, P(x)$	$\forall c \in \text{CFL}, P(c)$
2.	$\exists x \in \text{ALL}, \neg P(x)$	$\exists x \in \text{ALL}, \neg P(x)$
3.	$\Rightarrow x \notin \text{REG}$	$\Rightarrow x \notin \text{CFL}$

$0^n 1^n \rightarrow \xrightarrow{\quad} \textcircled{0} \xrightarrow{\quad} 1 \xrightarrow{\quad} 2 \xrightarrow{\quad} 3$



16-2 Look at 2nd th, by

$\alpha, ?\beta \rightarrow ?\gamma$



$$y \beta Q[A] \omega \Rightarrow y/B[A/0] \omega \Rightarrow y\gamma[B/0] \omega \\ \Rightarrow y\gamma 0[B] \omega$$

$$0^n 1^n \qquad 0^n 1^* 0^n \qquad 0^n 1^n 0^n$$

$$0^n 1^* 0^y \text{ s.t } x+y=2n$$

$$\overbrace{0^n 1^n 0^n}^{0^n 1^* 0^n} \xrightarrow{\epsilon [ ] 0^n \rightarrow \epsilon [ ] \epsilon} \begin{cases} \epsilon [ ] 0^n 1^n 0^n \\ 0^n [ ] 1^n 0^n \\ \epsilon [ ] 0^n \end{cases}$$

$$\epsilon [ ] 0^n 1^n 0^n \rightarrow 0^n [ ] 1^n 0^n \rightarrow 0^{n-1} 00 [ ] 1^n 0^n \\ \rightarrow 00 [ ] 0^n \rightarrow$$

16-3/ RPP

$\forall A \in \text{REG},$

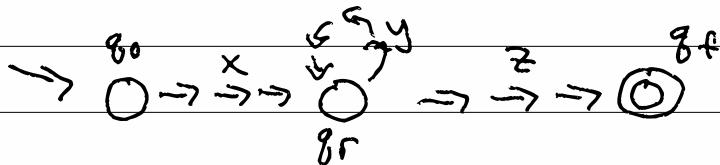
$\exists p \in \mathbb{N},$

$\forall (s \in A \mid |s| \geq p)$

$\exists (x, y, z \in \Sigma^* \mid |xy| \leq p \wedge |y| > 0)$

$\forall i \in \mathbb{N},$

$xy^iz \in A.$

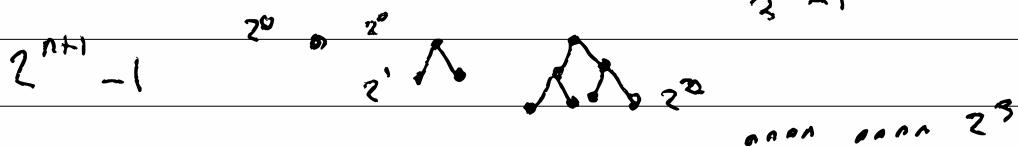


CFG  $G$  (CNF) =  $(V, \Sigma, R, S)$   $S \Rightarrow \epsilon \in R$

$V = \{v_0, \dots, v_n\}$   $S = v_0$   $v_i \Rightarrow \alpha \in R$

$v_i \Rightarrow v_x v_y$

$v_0$	$v_1$	$v_2$	....	$v_n$	$\rightarrow$	$v_r$	$v_r \in V$
$v_1$	$v_2$	$v_2$		$\vdots$		$\vdots$	
$v_2$				$v_n$		$v_r$	



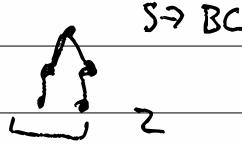
(6-y) minimum length of tree of height  $n$

$S \rightarrow E$

$$\bullet = 0$$

$S \rightarrow a$

$$= 1$$



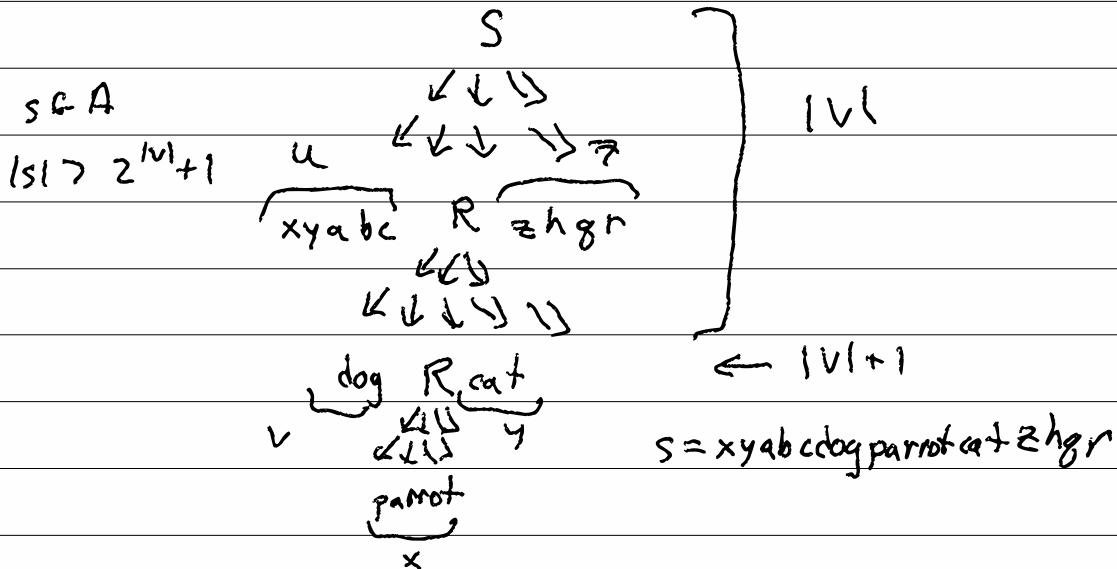
$S \rightarrow BC$

$$n=4$$

$$\Rightarrow$$

$$\text{len} \geq 3$$

If the len of a string is  $> 2^{|V|} + 1$   
then the height must be  $|V| + 1$



$$S \rightarrow u R z$$

$$R \rightarrow v R y$$

$$R \rightarrow x$$

16-5) Context-free pumping property

$$\forall A \in \text{CFL}, \quad \exists p \in \mathbb{N}, \quad \forall s \in A \quad |s| \geq p \quad \exists (u, v, x, y, z \in \Sigma^*) \quad |s = uvxyz| \quad |vxy| \geq p \quad |vy|^i > 0$$

$\forall i \in \mathbb{N}$ ,

$$uv^ixy^iz \in A$$

$$\begin{array}{ll} S \Rightarrow uRz & S \Rightarrow uRz \rightarrow uxz \\ R \Rightarrow vRy & S \Rightarrow uRz \rightarrow uvRyz \rightarrow uv^2Ry^2z \\ R \Rightarrow x & \rightarrow uv^2xy^2z \quad (i=2) \end{array}$$

$$0^n, n \in \mathbb{N} \in \text{CFL} \quad \overset{p=4}{\sim} \quad s = 0011 \quad |vxy| = 2 \leq 4$$
$$u=0 \quad v=0 \quad x=\epsilon \quad y=1 \quad z=1 \quad |vy|=2>0$$
$$uv^ixy^iz \in A? \quad 00^{i+1}1^{i+1} = 0^{i+1}1^{i+1} \in A$$

$$16-6 \quad 0^n 1^n 0^n = A$$

given  $p$

$$0^p 1^p 0^p$$

given  $u, v, x, y, z$

$$\begin{aligned} u &= 0^{\vec{w}} & v &= 0^{\vec{v}} & x &= 0^{\vec{x}} & y &= 0^{\vec{y}} \\ z &= 0^{\vec{z}} 1^n 0^p \end{aligned}$$

$u x y$  has repetition

$$\boxed{0^p 1^p 0^p} \rightarrow u x y = 1 \text{ symbol}$$

$$\boxed{0^p 1^p 0^p} \rightarrow u x y = 2 \text{ symbols}$$

$$\boxed{0^p 1^p 0^p} \rightarrow 3 \text{ symbols}$$

$$0^{p+i} 1^p 0^p$$

$$0^p 1^{p+i} 0^{p+i}$$

$$u x y = 0^a 1^p 0^b \quad | \quad | = a + p + b \leq p$$

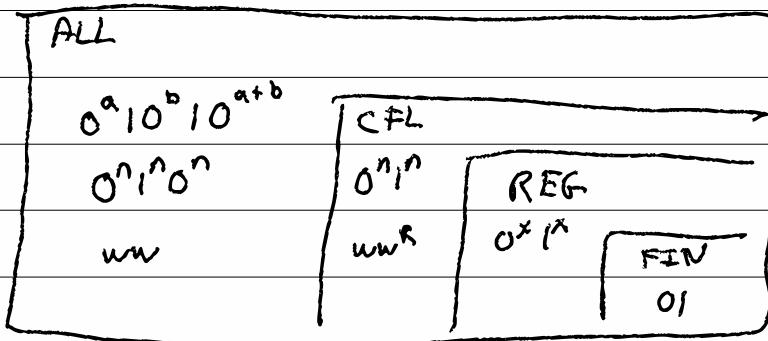
DFAunion = pair  $\langle X, Y \rangle$

NFAunion = start

state from left  $X$  ("x")

state from right  $Y$  ("y")

END



17-1 Finite : 01

Regular:  $0^* 1^*$

Context-free:  $0^n 1^n$

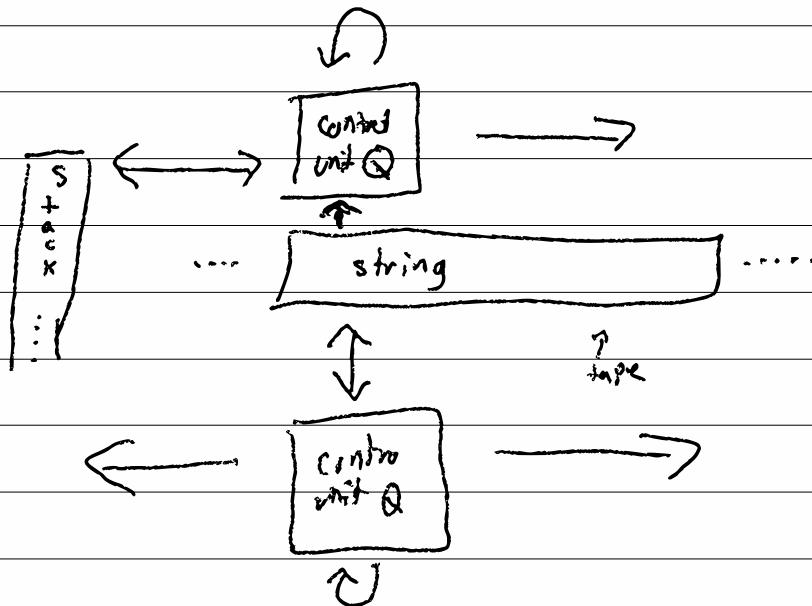
Context-free pumping property  
CFPP

context-free pumping property

$\Rightarrow$  CFG, PDS

Turing Machine      Alan Turing

Goal: Effective ~~math~~ algorithm for true math.



17-2  $\delta: Q \times \Sigma \rightarrow Q \times \Gamma \times \{L, R\}$

$\Sigma = \{q_0, q_f\}$

states:  $Q$

input:  $\Sigma$

$\sqsubset \in \Sigma$

config:

$\Gamma^* \subset Q \times \Gamma^*$

tape:  $\Gamma$  ( $\Sigma \subseteq \Gamma$ )  $\sqsubset \in \Gamma$

$q_0 \in Q$

$c_0 = \varepsilon[q_0] \sqsubset$

$q_a \in Q$  (accepting)

for  $w \in \Sigma^*$  input

$q_r \in Q$  (rejecting)

$w \in L(m)$  iff

$\varepsilon[q_0]w \Rightarrow^* x[q_a]y$

$\delta(q_i, a) = (q_j, b, R)$

$\delta(q_i, a) = (q_j, b, L)$

$x[q_i]ay \Rightarrow x[b][q_j]y$

$x[c][q_i]ay \Rightarrow x[q_j]cb[y]$

$x[q_i]y \Rightarrow \sqsubset x[q_i]y \sqsubset$

left: tape  $\rightarrow$  tape

right: tape  $\rightarrow$  tape

left  $\varepsilon = \varepsilon : \sqsubset$

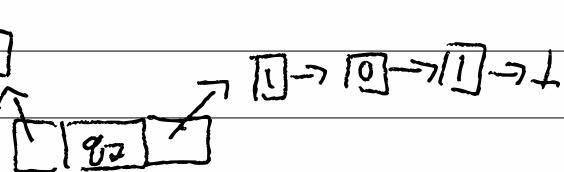
right  $\varepsilon = \sqsubset : \varepsilon$

left  $(x : c) = x$

right  $(c : y) = y$

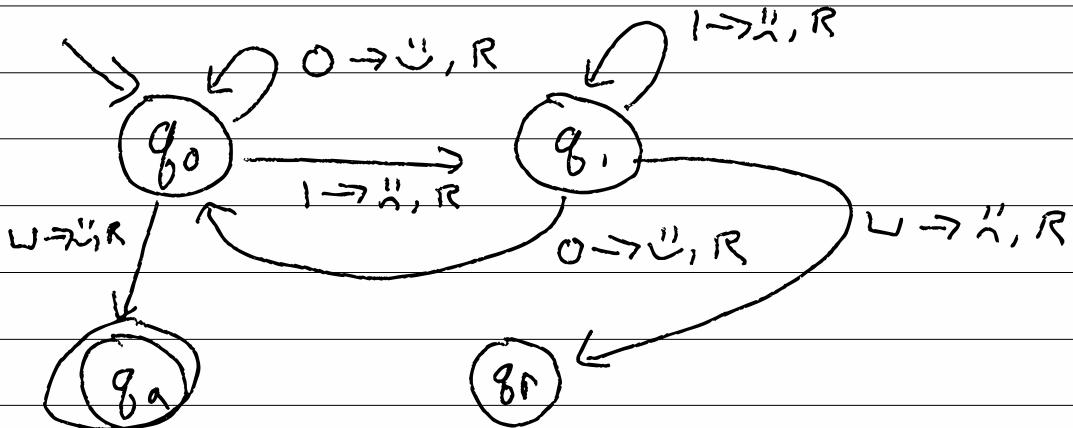
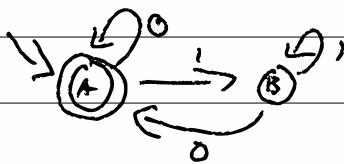
$\perp \leftarrow \overline{0} \leftarrow \boxed{1} \leftarrow \boxed{0}$

011 [q<sub>2</sub>] 101



(DFA)

173)



$[A]^{110} \rightarrow [B]_{10} \rightarrow [B]_0 \rightarrow [A]_\varepsilon \rightarrow \checkmark$

$\varepsilon [q_0]^{110} \rightarrow 1 [q_1]_{10} \rightarrow 1 1 [q_1]_0 \rightarrow \dots$   
 $\rightarrow 1 1 1 [q_1]_\varepsilon \rightarrow \omega 1 1 1 [q_1] \omega \rightarrow \omega 1 1 1 1 [q_1]$

input:  $(Q, \Sigma, q_0, \delta: Q \times \Sigma \rightarrow Q, F \subseteq Q)$

output:  $Q' = Q \cup \{q_a, q_r\}$

$\Sigma' = \Sigma$

$\Gamma = \varepsilon \cup \{\omega\}$

$q'_0 = q_0$

$\delta'(q_i, c) = (\delta(q_i, c), \omega, R)$  if  $c \notin \omega$

$\delta'(q_i, \omega) = (q_a, \omega, R)$  if  $q_i \in F$

$(q_r, \omega, R)$  if  $q_i \notin F$

$$\Sigma = \{0, 1, \#\}$$

17-4 /  $w \# w$  where  $w \in \Sigma^*$

I saw a zero

$[] 01 \# 01 \rightarrow \sim [] 1 \# 01 \rightarrow \sim 1 [ ] \# 01$

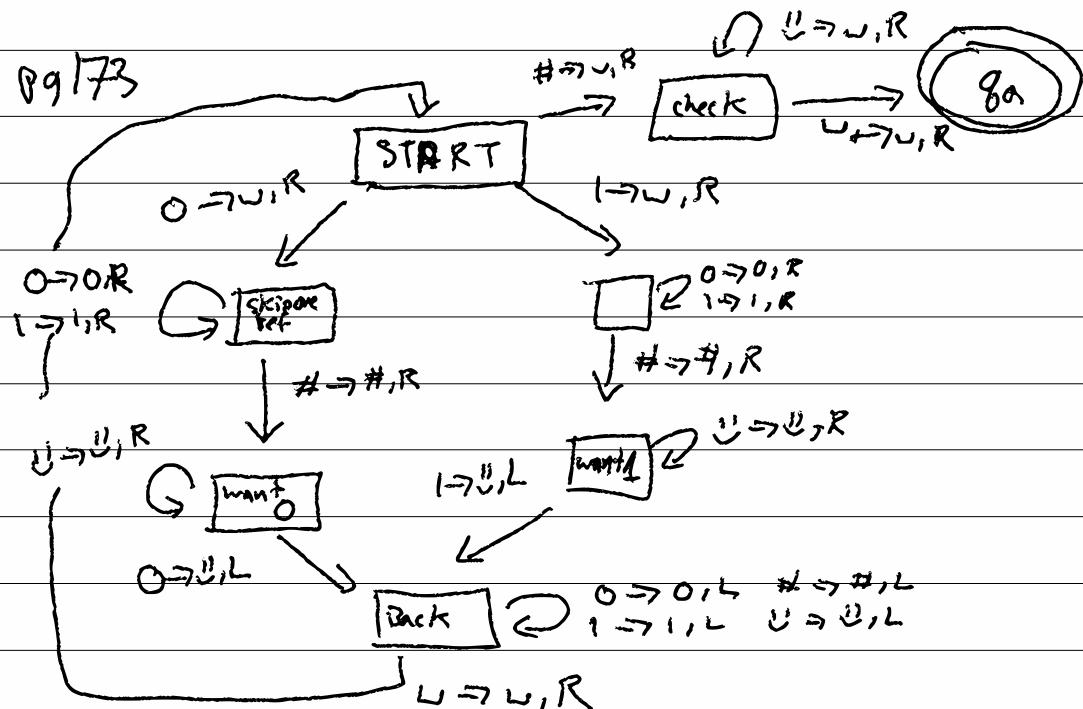
$\sim [ ] \# 01 \rightarrow \sim [ I \text{ expect a zero} ] 01 \rightarrow \sim 1 [ I \text{ am happy} ] \# 01$

$\rightarrow \sim [ H ] 1 \# 01 \rightarrow [ H ] \sim 1 \# 01$

$\rightarrow \sim [ B ] 1 \# 01 \Rightarrow \sim [ I \text{ saw a } 1 ] \# 01$

$\sim \# [ I \text{ expect a } 1 ] 01 \quad \sim \# 0 [ \text{ looked } 1 ] 1$

Pg 173



17-5/ A computable function  
 $f$

is a Turing - machine  
and

$$f(x) = y$$

if  $f$

$$\in [g_0]x \Rightarrow^* w[g_a]y$$

$$\text{add} \begin{matrix} 1 \\ 0 \end{matrix} = \begin{matrix} 1 \end{matrix}$$

$$\begin{matrix} 1 \end{matrix} = \begin{matrix} 10 \end{matrix}$$

$$\begin{matrix} 10 \end{matrix} = \begin{matrix} 11 \end{matrix}$$

$$\begin{matrix} 11 \end{matrix} = \begin{matrix} 100 \end{matrix}$$