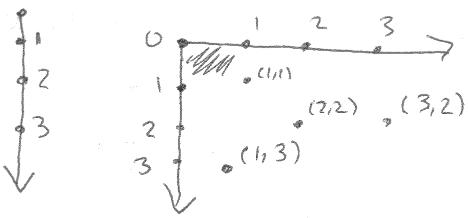
25-1

$\mathbb{N}$ and E are the same size
$\downarrow$ naturals $\quad \downarrow$ evens

$\mathbb{N}$ and $\mathbb{Q}$ $\nearrow$ rationals $\longrightarrow$ $1/4,\ 4/8,\ 8/12,\ 1/3,$ etc
$\hookrightarrow 0, 1, 2, 3, 4,$ etc

$$\mathbb{Q} = \mathbb{N} \times \mathbb{N}$$

$\mathbb{N}$ $\quad$ $\mathbb{Q}$ $\quad$ $\frac{a}{b} \in \mathbb{Q}$ iff $(a, b) \in \mathbb{N} \times \mathbb{N}$



$\mathbb{N}$ ss $\mathbb{Q}$?
$\exists f$ where f is a bijection from $\mathbb{N}$ to $\mathbb{Q}$
where f^{-1} is a bijection from $\mathbb{Q}$ to $\mathbb{N}$

✗ $f(n) = (0, n) : \mathbb{N} \to \mathbb{Q}$

✗ $f^{-1}(x, y) = x + y : \mathbb{Q} \to \mathbb{N}$

✓ $f^{-1}(x, y) =$ assume $x = x_0 x_1 \dots x_n$ where $x_i \in \{0, 1\}$
$y = y_0 y_1 \dots y_m$ $\quad$ $x_i = 0$ if $i > n$
$\ell = \max(n, m) \times 2$ $\quad$ $y_i = 0$ if $i > m$
return $z = z_0 \dots z_\ell$ where $z_{2 \cdot n} = x_n$, $z_{2 \cdot n + 1} = y_n$

✓ $f^{-1}(x, y) =$ cut diagonal strips $\longleftarrow$ Cantor's Function
$$\frac{1}{2}(x + y)(x + y + 1) + y$$

$\mathbb{N}$ ss $\underbrace{\underbrace{\mathbb{N} \times \mathbb{N}}_{\mathbb{N}} \times \mathbb{N}}_{\mathbb{N}}$ $\qquad$ $\underbrace{\underbrace{\mathbb{N} \times \mathbb{N}}_{\mathbb{N}} \times \underbrace{\mathbb{N} \times \mathbb{N}}_{\mathbb{N}}}_{\mathbb{N}}$ $\qquad$ $\underbrace{\underbrace{\underbrace{\mathbb{N} \times \mathbb{N}}_{\mathbb{N}} \times \mathbb{N}}_{\mathbb{N}} \times \mathbb{N}}_{\mathbb{N}}$

$\mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$\mathbb{N}$ ss $\mathbb{N}^k$, $\forall k$ $\qquad$ $\mathbb{Z} = \{+, -\} \times \mathbb{N}$

if X is countable or finite
and Y is countable
then $X \times Y$ is countable

Turing Machines are COUNTABLE. $\quad$ $TM = (Q, \Sigma, \Gamma, q_0, \delta, q_a, q_r)$
$\mathcal{P}(F) \times F \times F \times Q \times F \times Q \times Q$

25-2/ Is there a set that's infinite and NOT the same size as $\mathbb{N}$?

$\mathbb{R}$ = real numbers

$0 \in \mathbb{R}$ $\quad 42 \in \mathbb{R}$ $\quad \pi \in \mathbb{R}$

$e \in \mathbb{R}$ $\quad \sqrt{2} \in \mathbb{R}$

↳ irrational (ie $\neg \exists x, y.\ \pi = x/y$)

↳ infinite digits

$\mathbb{R} \neq \mathbb{N} \times \mathbb{N}$ (not rational))

$\mathbb{R} \neq \mathbb{N}^*$ (not finite)

$\mathbb{R} = P(\mathbb{Q}) \times P(\mathbb{Q})$ $\quad$ if $(S, G) \in \mathbb{R}$, then $\exists$ real r

Dedekind Cut $\quad$ where $\forall s \in S.\ s < r$

"Those $\delta - \varepsilon$ proofs, huh?" $\quad \forall g \in G.\ r < g$

$\mathbb{R}$ = Cauchy sequence (infinite series of $\mathbb{Q}$, converging to the r)

C.S-y way $\quad$ Rs between $[0, 1)$ = $\mathbb{R}_{01}$ (write number in binary)

$\mathbb{R}_{01} = \mathbb{N} \to \{0, 1\}$

$.5 \doteq .10\ldots = \lambda pos.$ if $pos = 0$, then 1
o.w, then 0

$.75 \doteq .110\ldots = \lambda pos.$ if $pos = 0$, then 1
$pos = 1$, then 1
o.w, 0

$.\overline{10}$
$= \lambda pos.$ if pos is even, 1
o.w., 0

$\pi/4 = \lambda pos. \ \ldots ?$

$\exists f \in$ bijection from $\mathbb{N}$ to $\mathbb{R}_{01}$?

$f(n) = r_n$ $\quad r_n = \lambda pos.\ \ldots ?$

$f^{-1}(r_n) = n$

onto: $\forall r \in \mathbb{R}_{01}.\ \exists n \in \mathbb{N}.\ f(n) = r$

$f(22) = .\overline{10}$ $\quad$!onto: $\exists r \in \mathbb{R}_{01}.\ \forall n \in \mathbb{N}.\ f(n) \neq r$

$f(2048) = .101101$

$f(99) = \pi/4$

$\boxed{\exists r \in \mathbb{R}_{01}.\ \forall n \in \mathbb{N}.\ \cancel{\forall pos}\ \exists pos.\ f(n)(pos) \neq r(pos)}$

$r = \lambda pos.\ \neg f(pos)(pos)$

$f(pos)$ = some r (r_J)

$r_J(pos)$

5-3

$\mathbb{N}$ is $\aleph_0$ (aleph 0)

$\mathbb{R}$ is $\aleph_1$

$\mathbb{R}$ is bigger than $\mathbb{N}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$\mathbb{R}_{01}$ $\qquad\qquad$ $\mathbb{N} \times \mathbb{N}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

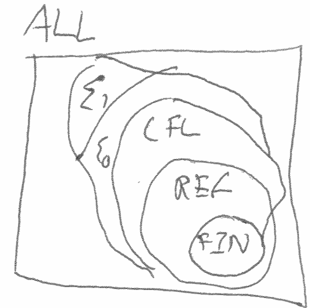
Infinite binary sequence $\qquad\qquad$ $F \times \mathbb{N}$

$\mathbb{N} \Rightarrow \{0,1\}$ $\qquad\qquad$ $\downarrow$

$\qquad\qquad$ $F^k \times \mathbb{N}^j$

$\downarrow$ $\qquad\qquad$ $\downarrow$

ALL = $\boxed{?}$ is bigger than TM



ALL = Σ_1

$\Rightarrow$ all problems are solvable by TMs

Next time!