

1-1/

How do we know if a math formula
is true?

How do we know if an algorithm
(like Euclid's GCD) "works"?

↙ ↓

correct effective

Does an algorithm exist?

What is an algorithm?

Does a program exist? ← problems

What is a program? ← models

I-2] A set is "a bunch of stuff"

\emptyset - nothin' in it
 $\forall x, x \notin \emptyset$

$\{ \text{pen}, \text{phone} \}$ $\{ \text{phone}, \text{pen} \}$
 $\{\checkmark, \square\}$

$\nexists \text{ pen} \in \{ \text{pen}, \text{phone} \}$

$\forall x, x \in \{y\}$ iff $x = y$

union - \cup

$\forall x, x \in A \cup B$ iff $x \in A$ or $x \in B$

$\{ \text{pen}, \text{phone} \} = \{ \text{pen} \} \cup \{ \text{phone} \}$

[=3] "The set of all true math formulas"

A set IS its membership

" $1+1=2$ " $\in TS \uparrow ?$

"Is there a god?"

"Will Buffy be remade?"

All sets "constructed" via \emptyset , $\{\cdot\}$, \cup are finite.

$$x \in \{\emptyset\} \cup \{\{\cdot\}\}$$

The Universe (U)

$A \subseteq B$ iff $\forall x, x \in A \rightarrow x \in B$

↳ Our universe is made of strings
and strings are sequences of characters
and chars are elements of an alphabet
an alphabet is a finite set



$$\Sigma = \{0, 1\}$$

↑
chars

$$\{0, 1, \cup, \$, +\}$$

↑
chars

"0100001" = a string = s

length = 7

$$s(0) = 0$$

$$s(1) = 1 \quad s(2) = 0$$

U = Σ^* ← special notation

$$A^* = \{\epsilon\} \cup A \circ A^*$$

epsilon = " " = the string w/ no characters

$x \in A \circ B$ iff $x(0) \in A$ and
 $x(1..) \in B$

$$\{0, 1\} \circ \{0, 1\} = \{00, 01, 10, 11\}$$

$$\{1\} \circ \{0\} = \{10\}$$

LS / #1. Decide a data type to represent alphabets and characters.

Alphabet = List < Character >

Character = Object / void*
we need equality

#2. Decide a data type for strings

interface String { }

class MtString implements String { .. }

class OneString impl String { }

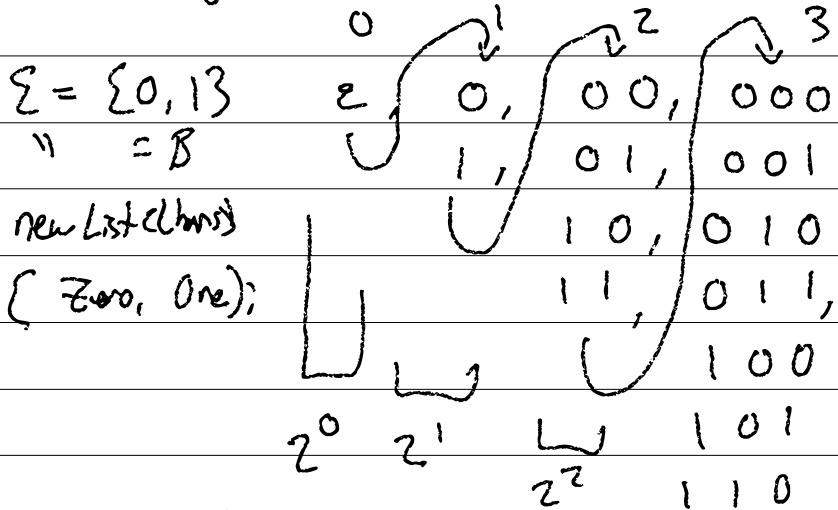
OneString (char c, String s) { ... }

Zero = new BasicChar('0'); One = new BC('1');

010 = new OneS(Zero, new OneS(One, new
OneS(Zero, new MtS())));

(EP)

1-6) Every alphabet has a lexicographical ordering of the strings in Σ^*



$|A|^i$ where i is layer 2^i
looking at

lexi : $\Sigma \times \mathbb{N} \rightarrow \Sigma^*$

lexi $\emptyset 0 = \emptyset$

lexi $\emptyset 1 = 0$

lexi $\emptyset 2 = 1$

lexi $\emptyset 6 = 101$

2-1 "1+1" \rightarrow "2"

"1+1 = 2" \in Truth

"1+1 = 3" \notin Truth

$\emptyset \quad \Sigma^3 \quad A \cup B$

Alphabet Σ Universe Σ^*

{0, 1}

{ε, 0110, 000001,



3

$P(A) \quad 2^A$

$x \in P(A)$ iff $x \subseteq A$ ($x \subseteq A$, iff
 $\forall y \in x, y \in A$)

$A = \{0, 1, 2, 3\}$

$\emptyset \in P(A) \quad \emptyset \subseteq A$

0110 {1, 2}

{0} \in $\emptyset \subseteq \{2, 3\} \in \{0, 1, 2, 3\} \quad \underline{\textcircled{1}} \quad 123$

$P(\Sigma^*) \quad \Sigma^* = \{ \epsilon, 0, 1, 00, 111111 \}$

$\emptyset \in P(\Sigma^*)$

...

{ε} $\in P(\Sigma^*)$

0011, ...

all even length strings $\in P(\Sigma^*) = \{ \epsilon, 00, 11, 01, \dots \}$

GIFS $\in P(\Sigma^*)$

{GIFS of me} $\in P(\Sigma^*)$

JPGs w/ a cat in them $\in P(\Sigma^*)$

2-2 $\text{ALL} = \mathcal{P}(\Sigma^*)$

$\text{FIN} =$ the set of
finite sets

ALL	- True math
	- G-ATs
	- Even strings
$\overline{\text{FIN}}$	

$\emptyset \in \text{FIN}$

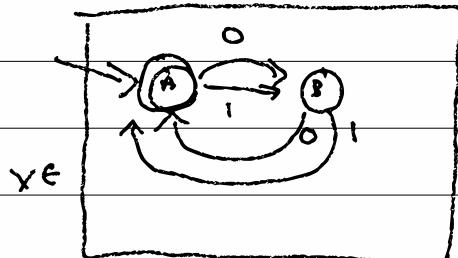
$\forall x \in \Sigma^*, \{x\} \in \text{FIN}$

$A \in \text{FIN} \wedge B \in \text{FIN}$

$\Rightarrow A \cup B \in \text{FIN}$

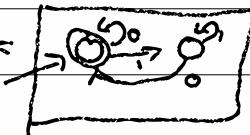
All even strings =

DFA - a deterministic finite automata



Σ or Σ^*

even numbers =



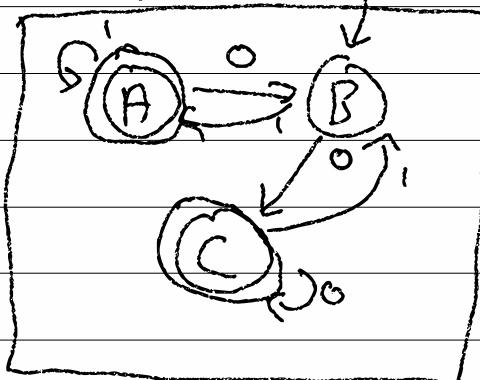
○ - states $\Sigma A, B \}$

→ ○ - start state A

○○ - accepting states $\Sigma A \}$

$\overset{x}{\rightarrow} \circ$ - transition $A \xrightarrow{x} B$

x - labels are Σ



$\Sigma A, B, C \}$

$\Sigma A, C \}$

$\Sigma = \Sigma_0, 1 \}$

A	0	1
B	C	A
C	C	B

2-3) $x \in \text{DFA} (\underbrace{\text{states}, \text{alphabet}, \text{start}, \text{accepting}}_{\text{states } Q, \Sigma, q_0 \in Q, F \subseteq Q}, \delta: Q \times \Sigma \rightarrow Q - \text{transitions})$

DFA configuration = $Q \times \Sigma^*$
 $\stackrel{\uparrow}{[q]} w^{\uparrow}$

config update function : config $\times \text{DFA} \rightarrow \text{config}$
 $[q]w \rightarrow [q']w'$

$[q_i]x \rightarrow [q_j]y \text{ iff } \delta(q_i, x) = q_j$
 $x \in \text{DFA} \text{ iff } [q_0]x \Rightarrow \Rightarrow \Rightarrow \Rightarrow [q_f] \in$
 and $q_f \in F$

0110 $\in \text{EvenLen}$;iff $[A]0110 \rightarrow [B]110 \rightarrow [A]10$
 $\rightarrow [B]0 \rightarrow [A] \in AF\{A\}$ ✓

class DFA Σ

.. $Q, \Sigma, F, q_0, \delta \dots$

public bool accepts (String x) {

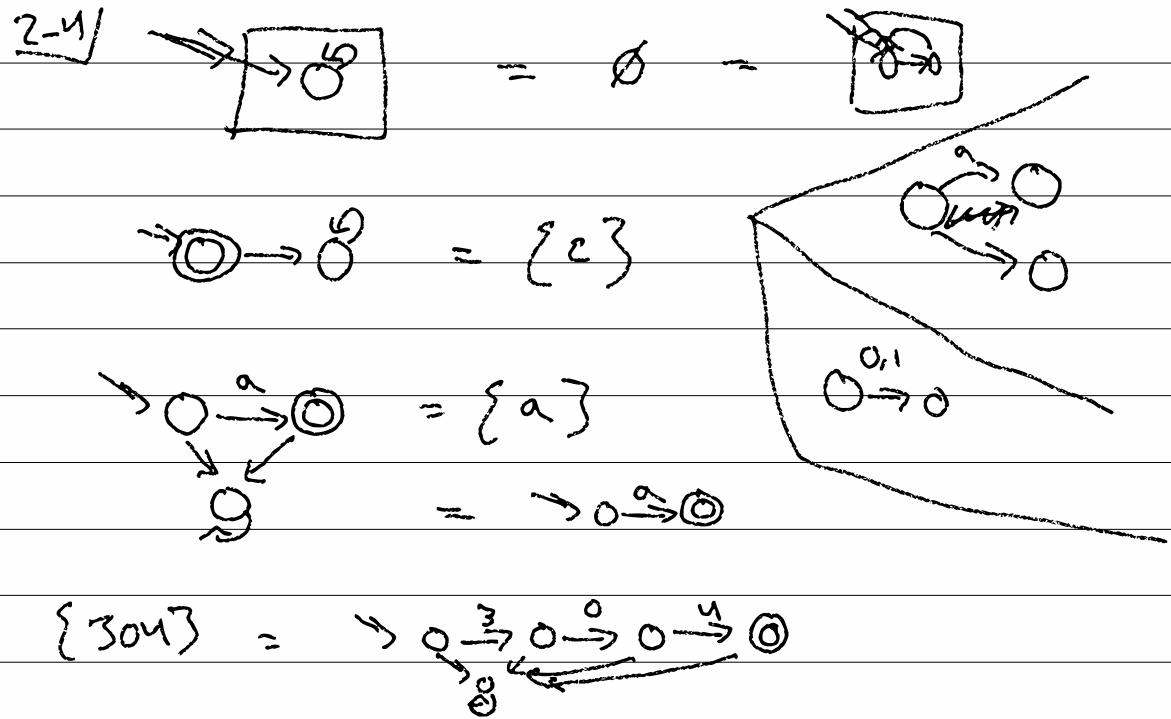
State $q_i = q_0;$

while ($(x, \cancel{\neq} \text{empty}) \Sigma$

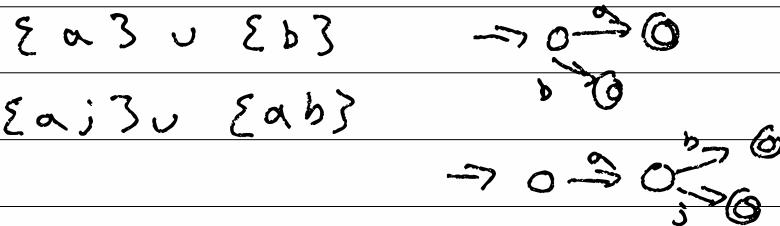
$q_i = \delta(q_i, x, \text{first}());$

$x = x, \text{rest}();$ } }

return $F, \text{in}(q_i);$ } }



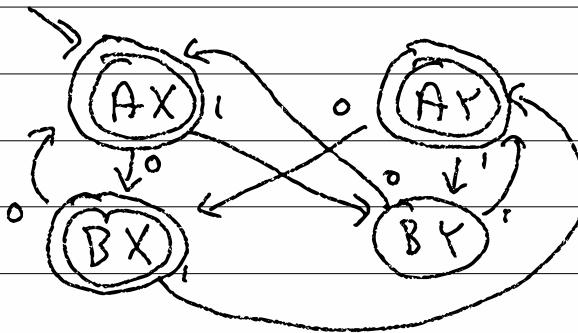
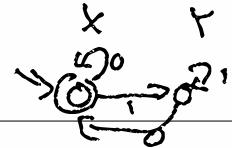
$A \cup B \leftarrow \text{DFA}$ (if $A \in \text{DFA}$ and $B \in \text{DFA}$)



2-5 Even Len



Is Even



ε	✓
00	✓
11	✓
0	✓
110	✓

n

$(x, y) \in A \times B$

; if $x \in A \wedge y \in B$

$$A = (Q_A, \Sigma, g_{0A}, \delta_A, F_A)$$

$$B = (Q_B, \Sigma, g_{0B}, \delta_B, F_B)$$

$$X = A \cup B$$

$$Q_X = Q_A \times Q_B \quad \delta_X = ((g_A, g_B), c) =$$

$$g_{0X} = (g_{0A}, g_{0B}) \quad (\delta_A(g_A, c),$$

$$F_X = F_A \times F_B - n \quad \delta_B(g_B, c))$$

$$F_A \times Q_B \cup Q_A \times F_B - V$$

$x \in A \cap B$; if $x \in A \wedge x \in B$

2-6) $x + A^c$ iff $x \notin A \quad (x \in u)$

Even Len odd Len
 $\rightarrow \textcircled{0} \rightarrow \textcircled{0}$ $\Rightarrow \rightarrow \textcircled{0} \rightarrow \textcircled{0}$

$$F = \{A\}$$

complement

$$F' = Q - F$$

or F^c (wrt Q)

Algorithm for $X \subseteq Y$ if X, Y are DFAs

3-11 DFA \Rightarrow example or false

DFA:

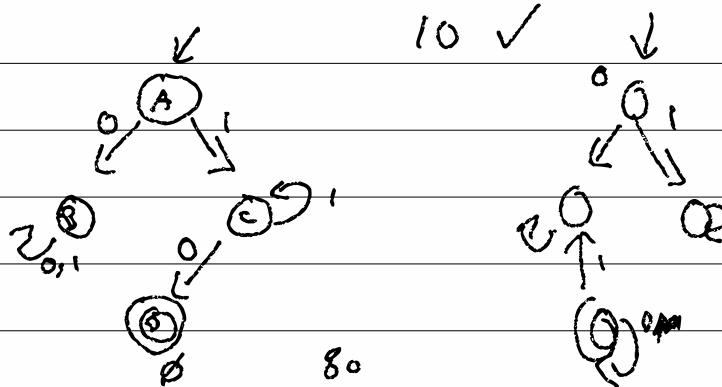
Q : Sstate \Rightarrow Bool

Σ : list of characters

s_0 : state

S : (state \times char) \rightarrow Sstate

F : State \Rightarrow Bool



$\Sigma = \{A, B, C, D\}$

$\Sigma = \{A\}$

$[A]$

$A \Rightarrow \text{?}$

$\Sigma = \{B, C, D\}$

$\Sigma = \{A\}$

$[B, C]$

$B \Rightarrow A, 0$
 $C \Rightarrow A, 1$

$\Sigma = \{C, D\}$

$\Sigma = \{A, B\}$

$[C]$

'Yes, it is possible.'

$\Sigma = \{D\}$

$\Sigma = \{A, B, C\}$

$[D]$

$D \Rightarrow C, 0$

$\Sigma = \{E\}$

$\Sigma = \{A, B, C\}$

$[E]$

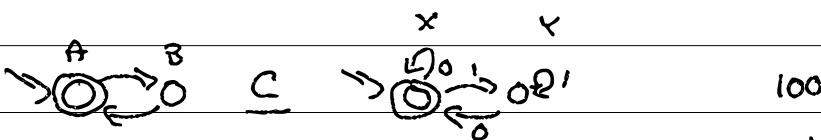
or, if not, No!

3-2/ subset

$A \subseteq B \iff \forall x \in A. x \in B \rightarrow x \in B$

$$\{\alpha, \beta\} \subseteq \{\alpha, \beta, \gamma\} \quad U = \{\alpha, \beta, \gamma\}$$

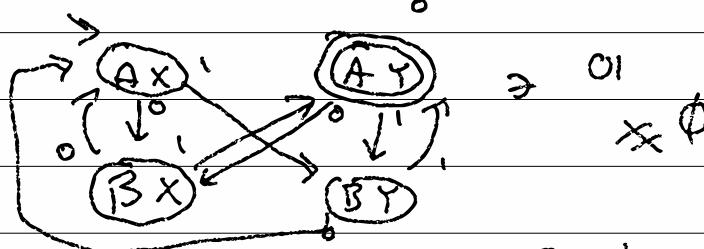
finite means naive works!



$$\boxed{\bar{A}} \subseteq \boxed{B} \cap \boxed{\bar{A}} = \boxed{\bar{A}}$$

$$\boxed{B} \cap \boxed{\bar{A}} = \boxed{\text{∅}}$$

$$\mathbb{B} \text{ EvenNum} = \rightarrow \xrightarrow{x} \xrightarrow{y} \xrightarrow{z} \text{01}$$



soundness: model \models theory

completeness: theory \vdash model

model = theory

$$3-3) \quad 0, 1, 2, -1, 5 \quad \mathbb{Z}, \mathbb{P}, \mathbb{N}$$

$$\{\mathbb{P}\} + \{\mathbb{N}\} = \{\mathbb{P}, \mathbb{Z}, \mathbb{N}\}$$

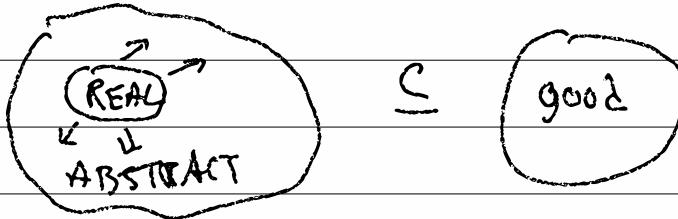
if $x > 0$ then

$$A \quad y = 5 \Rightarrow \{\mathbb{P}\}$$

o.w.

$$B \quad y = 0 \Rightarrow \{\mathbb{Z}\}$$

\Rightarrow assume $y = \{\mathbb{P}, \mathbb{Z}\}$



<u>3-w)</u>	Finite	=	\emptyset	Σ^3	$A \cup B$	EDFA
			A^c	$A \cap B$	$A \circ B$	

Infinite = A^*

* $x \in \Sigma^* \wedge y \in \Sigma^*$ then $xoy \in A \circ B$ iff
 $x \in A \wedge y \in B$

$$\varepsilon \circ y = y \quad \text{if } a \in \Sigma, (a \circ x) \circ y = a \circ (xoy)$$

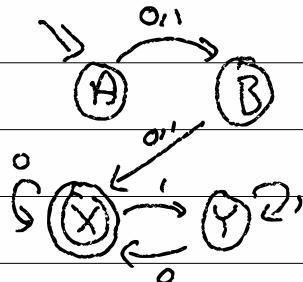
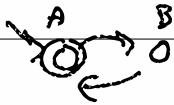
$$abcd = ab \circ cd$$

$$\{\text{jm}\} \circ \{\text{mj, nj}\} = \{\text{jim, jn}\}$$

$x \in A^*$ iff $x = x_0 \circ x_1 \circ \dots \circ x_n$ for $n \in N$
and $x_i \in A$

$$\{\text{jm}\}^* \ni \varepsilon, \text{ jm, jmjmjmjmjm}$$

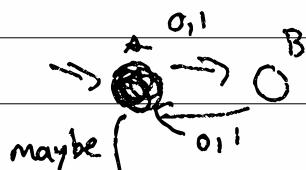
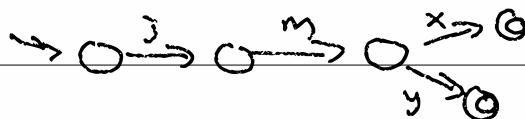
3-5/ Even Len \circ Even Num



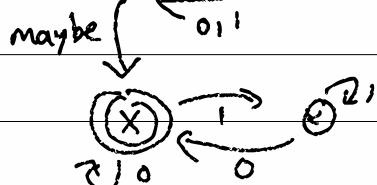
00110 ✓
0011 X

~~00110011~~

$\{ \text{im } 3 \circ \text{Ex, y} \}$

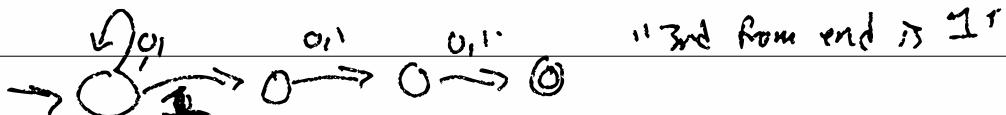


$\Sigma = \{0, 1\}$



$x \in \text{DFA}$ iff

There is some path
from q_0 to $q_f \in F$
labelled w/ x



"3rd from end is 1"

3.6) NFA = non-deterministic
finite automata

old world: the next step was obvious

$$\delta: Q \times \Sigma \rightarrow Q$$

new world: crazy options

- do you even read achar?
- which path do you take?

$$\delta': Q \times \{\text{maybe}\} \cup \Sigma \rightarrow P(Q)$$

$$\delta'(A, r) = \{A, B\}$$

$$\delta'(A, \text{maybe}) = \{C\}$$

epsilon

$$\epsilon \in \Sigma$$

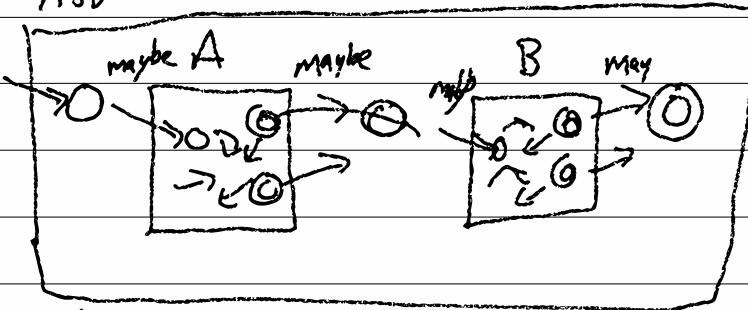
4-11 $A \circ B \in \text{DFA}$ iff $A \in \text{DFA}$
 A^* $\wedge B \in \text{DFA}$

NFA ($N - \underline{\text{non}} \text{ D-deterministic}$)

$$S: Q \times (\Sigma \cup \{\text{maybe}\}) \rightarrow P(Q)$$

$$S: Q \times \Sigma \rightarrow Q$$

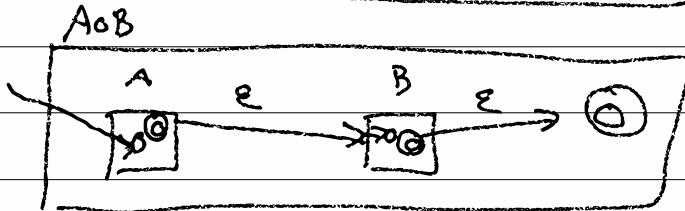
$A \circ B$



maybe written
as "ε"

what does (NFA)

this mean?



$\text{NFA} \leftrightarrow \text{DFA}$

4-2] what do NFAs mean?

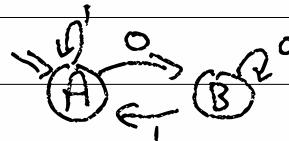
A DFA represents a set and
a set "is" a membership function

$$U \rightarrow \{0,1\}$$

$$\subseteq \Sigma^* \rightarrow \{\text{Y}, N\}$$

$$\text{config} = \Sigma^* \times Q$$

$$\Sigma^* \rightarrow Q^*$$



$$0110 \rightarrow \underline{ABAAB} \rightarrow \text{a trace}$$

$$\Sigma^* \rightarrow (\underline{Q}, \delta)^*$$

$$0110 \rightarrow \underbrace{(0, B)(1, A)(1, A)(0, B)}_{\text{a trace}} = \Sigma^* \cup \Sigma \epsilon^3$$

$$0A1A1A0B \rightarrow N$$

$$\text{valid? } : \delta(\boxed{\Sigma}, Q)^* \rightarrow \{\text{Y}, N\}$$

$$\text{valid } g; \epsilon = Y$$

$$\text{valid } g; (c, g_j) : \text{more} = \text{if } \delta(g_j, c) = \boxed{g_j}$$

$$\text{Nvalid? } : Q \times (\boxed{\Sigma} \times Q)^* \rightarrow B$$

$$\text{valid } g; \text{ more}$$

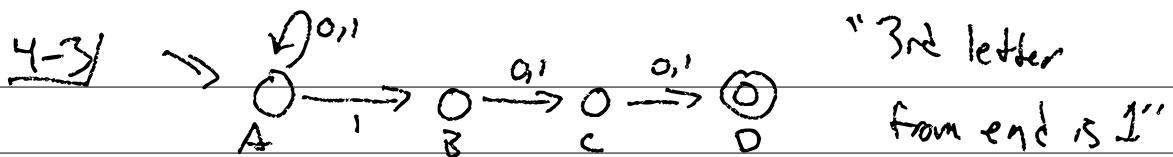
$$\text{Nvalid } g; \epsilon = Y$$

$$\text{o.w. } N$$

$$\text{Nvalid } g; (c, g_j) : \text{more} =$$

$$\text{if } \boxed{g_j} \boxed{e} \boxed{\delta(g_j, c)} \text{ then Ag Oracle}$$

$$\frac{\text{Nvalid } g; \text{ more}}{O.W. N}$$



0 1 00

1 1 1

1 1 0 1 0 0

- Y

0 0 0

1 0 0 0

1 0 1 1

- N

(0, A)(1, A)(0, A)(0, A) ✓

str($\Sigma \times Q)^* = \Sigma^*$

(0, A)(1, B)(0, C)(0, D) ✓

str $\epsilon = \epsilon$

(0, B)(1, C)(1, D)(0, D) X

str (C, -) : move \in

$\delta(A, 0) = \Sigma A \}$

$\delta(D, 0) = \emptyset$

C o str more

accepts : $\Sigma^* \rightarrow Y/N$

accepts $w = Y$ iff $\exists t \in \text{traces. } \text{str}(t) = w$.

valid $\Leftrightarrow t \in F$

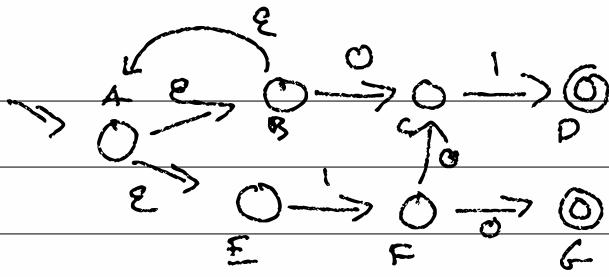
and last-state(t) $\in F$

NFA-accepts : $\Sigma^* \rightarrow Y/N$

figure all possible traces

check if valid and if strings match

check if past is in ϵF



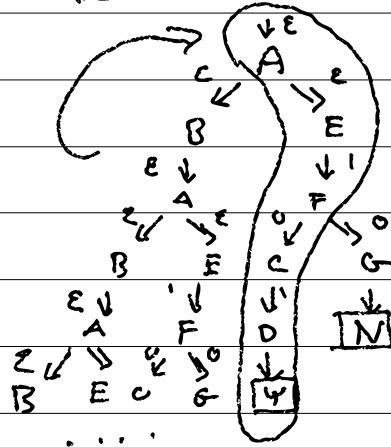
$$(\varepsilon, \beta)(0, c)(1, d) = 0.001$$

$$(\varepsilon, \Xi)(1, F)(0, C)(1, D) = 101 = \varepsilon_010001$$

$$(\varepsilon, E)(1, F)(0, G) \quad 10 = 20100$$

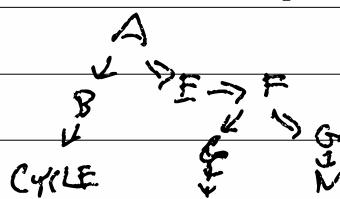
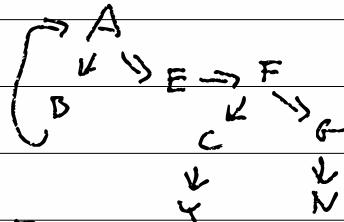
$(\mathcal{E}, \mathcal{B})(\varepsilon, A) \times$ where x is valid
 \rightarrow valid

Trace Tree = T | N | Branch (Σ, Q) (List TT)



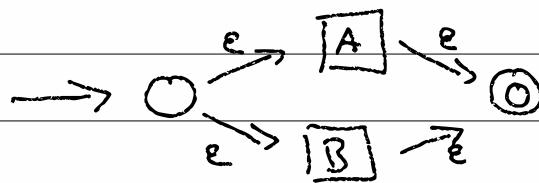
101

E, E - I, F OC ID



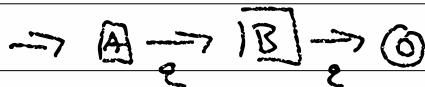
Forking model of NFAs (make TT)
Backtracking model (explores T?)

4-5) $A \cup B$



$x \in A$
 $\square \rightarrow \square$
State X transitions
to THE start

$A \circ B$



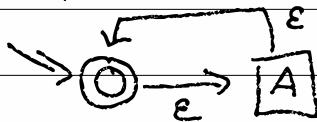
state of A

$\square \rightarrow \square$

All accepting states

of A transition to \square

A^*

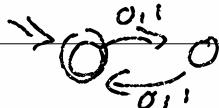


4-6) $\forall A$, $A \in \text{DFA} \Leftrightarrow A \in \text{NFA}$

\Rightarrow

\Leftarrow

DFA \Rightarrow NFA



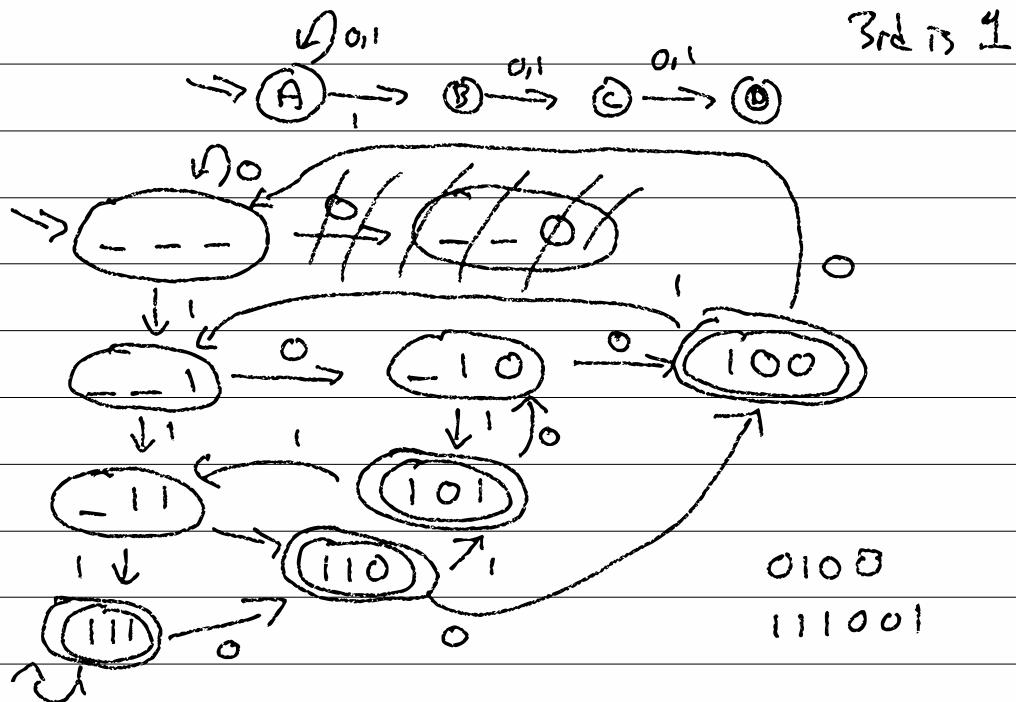
DFA $\delta: Q \times \Sigma \rightarrow Q$

NFA $\delta': Q \times \Sigma_c \rightarrow P(Q)$

$$\delta'(q_i, \epsilon) = \emptyset$$

$$\delta'(q_i, c \in \Sigma) = \{\delta(q_i, c)\}$$

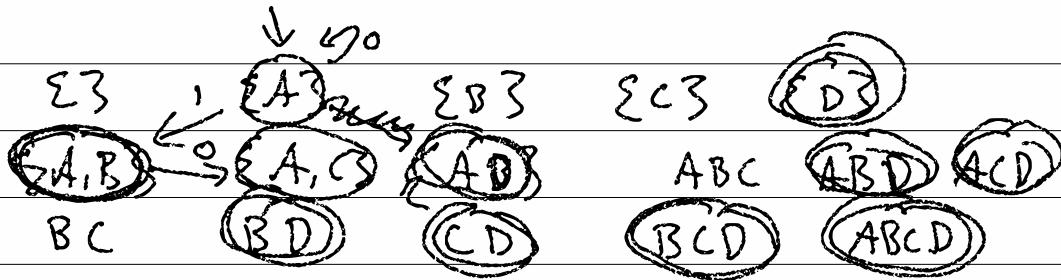
NFA \Rightarrow DFA



4-7) $\text{NFA} = (\mathbb{Q}, \Sigma, q_0, \delta; \mathbb{P}(\mathbb{Q}) \xrightarrow{\Sigma} \mathbb{P}(\mathbb{Q}))$

$\text{DFA}^{\text{out}} = (\mathbb{Q}', \Sigma, q'_0, \delta': \mathbb{Q}' \times \Sigma \rightarrow \mathbb{Q}', F' \subseteq \mathbb{Q}')$

$$\mathbb{Q}' = \mathbb{P}(\mathbb{Q})$$



$$q'_0 = \Sigma^{q_0}$$

F' = any state where $nF \neq \emptyset$

$$\begin{aligned} \delta'(\Sigma^{q_1}, \dots, \Sigma^{q_n}, c) &= \\ \cup \quad \delta(q_i, c) \end{aligned}$$

$$\underline{5-1} / A \cup B \quad \delta_A : Q_A \times \Sigma \rightarrow Q_A$$
$$\delta_B : Q_B \times \Sigma \rightarrow Q_B$$

$$\delta' : \overbrace{Q_A \times Q_B}^{\Sigma} \times \Sigma \rightarrow Q$$

$$\delta'((q_a, q_b), c) = (\delta_A(q_a, c), \delta_B(q_b, c))$$

char

$$(p, c) \Rightarrow \text{new Pair } \left(\begin{array}{l} \downarrow \\ \text{pair} < \text{State}, \text{State} \end{array} \right) \left(\begin{array}{l} \nearrow \\ \text{fst} \end{array} \right) \left(\begin{array}{l} \nearrow \\ \text{snd} \end{array} \right) \left(\begin{array}{l} \text{delta } a(p, \text{fst}, c), \\ \text{delta } b(p, \text{snd}, c) \end{array} \right);$$

S-2/ NFA \rightarrow DFA

$(Q, \Sigma, q_0 \in Q,$

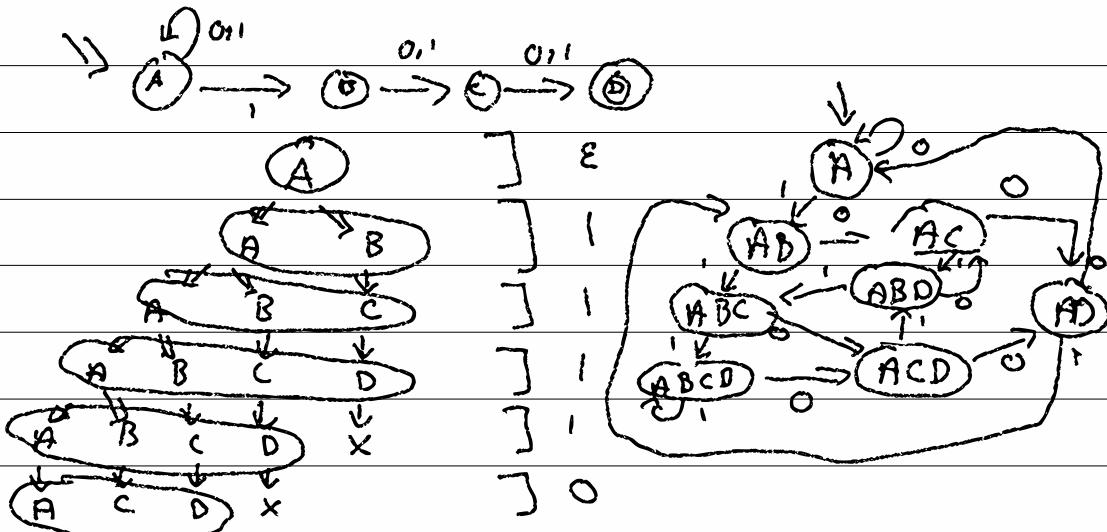
$\delta: Q \times \Sigma \rightarrow P(Q),$

$F \subseteq Q)$

$(Q', \Sigma, q'_0 \in Q'$

$\delta': Q' \times \Sigma \rightarrow Q',$

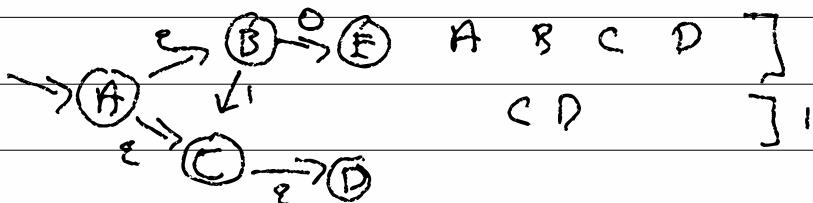
$F' \subseteq Q')$



$\begin{array}{c} \nwarrow \swarrow \\ A \quad B \\ \downarrow \quad \downarrow \\ C \end{array}]^0$

$\begin{array}{c} A \\ \downarrow \quad \downarrow \\ A \quad B \end{array}]^0$

$\begin{array}{cccc} A & C & C & A \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ DLAH & DLAH & DLAH & DLAH \end{array}]^x$



$\begin{array}{ccccc} A & B & C & D &] \\ CD & & & &]^1 \end{array}$

5-3/

$$E: Q' \rightarrow Q' \quad P(Q) \xrightarrow{P(Q)} - \text{follow all C-transitions}$$

Trace Tree DFA

Q' = things at the bottom of a tree
set = a set of states of
the NFA = $P(Q)$

g_0' = the top of the tree
= the set that has only the first state
 $= E(\{\}) \in P(Q)$

δ' = maps the bottom of the tree to the next level
= set of all next states of each state in the level of the tree

$$\delta'(Q_i, c) = \bigcup_{q_i \in Q_i} \delta(q_i, c)$$

F = any level of tree with some accepting state
= any set with an element in F
= $\{Q_i \mid \underbrace{Q_i \subseteq Q \text{ and } Q_i \cap F \neq \emptyset}\}_{Q_i \in P(Q) = Q'}$

= (set-of-gs \rightarrow
for each g_i in set-of-gs
if dfa.F.apply(g_i) then
return true
return false)

5-y)

$E(\text{set } \langle Q \rangle g_i)$

queue $\langle Q \rangle$ next = ~~empty~~ g_i

set $\langle Q \rangle$ seen = empty

while $(\text{not } \langle Q \rangle \text{ is empty})$

$\delta(\text{next}, \text{first}, \varepsilon)$ add those

to next unless in seen

return seen

$E(A) = \text{least fixed point of}$

$E^*(A)$

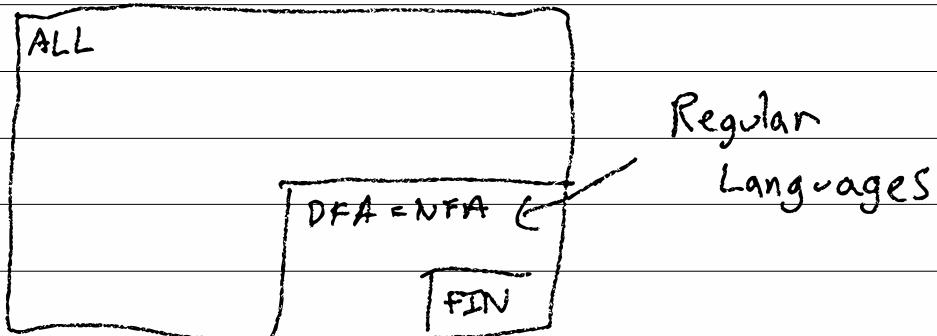
$E^*(A) = A \cup \bigcup_{g_i \in A} \delta(g_i, \varepsilon)$

6-1) $\forall N \in \text{NFA}, \exists D \in \text{DFA}.$ compile :

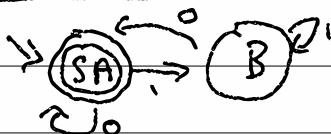
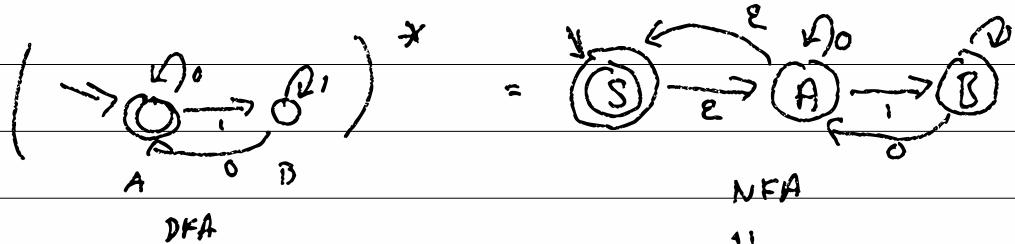
$$N = D$$

$\forall D \in \text{DFA}, \exists N \in \text{NFA}.$

$$D = N$$



Program f ... 'f; g'
 program g ... compositional



6-2 | Regular Expressions

$r ::=$	ϵ	$ $	EMPTY
	\emptyset	$ $	NULL
	c	$ $	Char
$r \cup r$		$ $	CUP
$\star r$		$ $	STAR
$r \circ r$		$ $	CIRC

interface Register { }

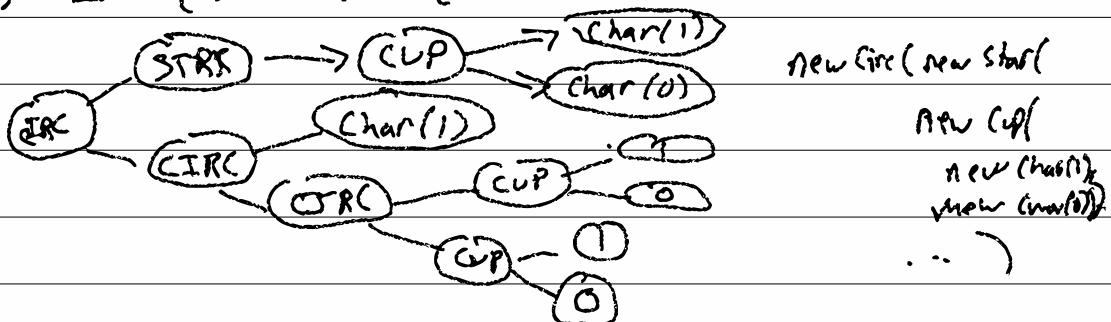
class RE_<empty> impl REGEX { . . . }

RE_NULL()

Re-char (char c)

Re-cup (Regex lhs , Regex rhs)

$(1\cup 0)^* \circ 1 \circ (1\cup 0) \circ (1\cup 0)$ - "3rd formal B.F."



6-3 | $L : RE \rightarrow \text{ALL} = P(\Sigma^*)$

fix the language if doesn't
(or come)

$$L(\epsilon) = \{\epsilon\}$$

$$L(\emptyset) = \emptyset$$

$$L(c) = \{c\}$$

$$L(r \cup r') = L(r) \cup L(r')$$

$$L(r \circ r') = L(r) \circ L(r')$$

$$L(r^*) = L(r)^*$$

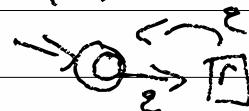
class Up {
 $L()$ \in
 return up(lhs, L(), rhs, L());
}

compile : RE \rightarrow NFA

$$\text{compile } (\epsilon) = \xrightarrow{\epsilon} \textcircled{0}$$

$$\text{compile } (r^*) =$$

$$\text{compile } (\emptyset) = \xrightarrow{\epsilon} \textcircled{0}$$



$$\text{compile } (c) = \xrightarrow{\epsilon} \textcircled{0} \xrightarrow{c} \textcircled{0}$$

$$\text{compile } (r \cup r') = \xrightarrow{\epsilon} \textcircled{0} \xrightarrow{c} \boxed{r} \xrightarrow{\epsilon} \boxed{r'} \xrightarrow{\epsilon} \textcircled{0}$$

$$\text{compile } (r \circ r') = \xrightarrow{\epsilon} \boxed{r} \xrightarrow{\epsilon} \boxed{r'} \xrightarrow{\epsilon} \textcircled{0}$$

6-4

$$"\ " = \epsilon$$

$$\underline{\underline{\quad}} = \emptyset$$

$$"\ c" = c$$

$$"\ xyz^*" = x \circ y \circ (z^*)$$

$$"\ (xyz)^*" = (x \circ y \circ z)^*$$

$$"\ xyz" = x \circ y \circ z$$

$$"\ [abc]" = (a \cup b \cup c) \quad (a, b, c \in \Sigma)$$

$$"\ (a \mid b \mid c)" = \Rightarrow \quad (a, b, c, \in \Sigma^*)$$

[012]

(zero | one | two)

$$\bullet = \epsilon$$

$$"\ .^* \backslash. m; s" = \Sigma^* \circ ' . ' \circ ' m' \circ ' ; ' \circ ' s'$$

8-5/

gen : RE $\rightarrow \Sigma^*$ or false

gen $\epsilon = \epsilon$

gen $\emptyset = \text{FALSE}$

gen $c = 'c'$ flip coin

gen $x \cup y = \text{gen } x \sqcup \text{gen } y$

gen $x \circ y = \text{gen } x \circ \text{gen } y$

gen $x^* = \boxed{\square}$

= gen $(\epsilon \cup x \circ x^*)$
"mjs"

equal : RE \times RE \rightarrow Bool

equal $x \ y =$

NFA2DFA(compile x) $\xrightarrow{\text{?}} \text{def equality?}$
NFA2DFA(compile y)

$$(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$$

G-6)

$$x + 0 = x$$

$$x = x$$

$$x \cdot 1 = x$$

$$\frac{a = b}{a+x = b+x}$$

$$\frac{a = b}{ax = bx} \quad x \neq 0$$

$$2(3x + 17) = 6x + 34 \quad \text{"algebra"}$$

$$3x + 17 = 3x + 17$$

$$3x = 3x$$

$$x = x$$

$$17 \cancel{x} \approx 17 \cancel{z}?$$

WZSS

$$\emptyset \cup x = x \cup \emptyset = x$$

$$\emptyset \circ x = x \circ \emptyset = \emptyset$$

$$\varepsilon \circ x = x \circ \varepsilon = x$$

$$\emptyset^* = \varepsilon \approx \varepsilon \cup \emptyset \circ \emptyset^*$$

$$\approx \varepsilon \cup \emptyset = \varepsilon$$

$$x \circ (y \cup z) = x \circ y \cup x \circ z$$

NFA \approx DFA in: N states (Q)

out: 2^N states (P/Q)

6-7/ size : RE \Rightarrow Nat

$$\text{size } \emptyset = 1$$

$$\text{size } \varepsilon = 1$$

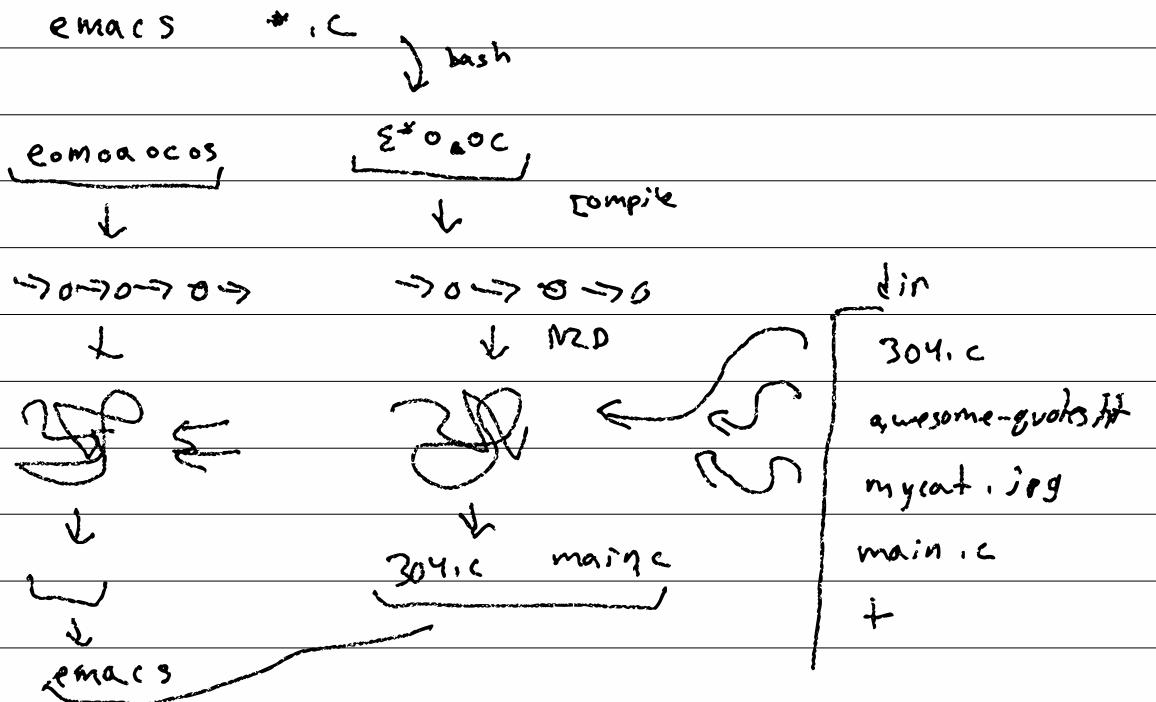
$$\text{size } c = 1$$

$$\text{size } (x \cup y) = \text{sz } x + \text{sz } y + 2$$

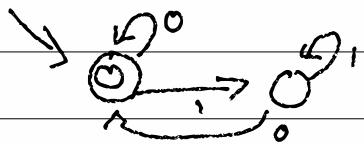
$$\text{size } (x \circ y) = \text{sz } x + \text{sz } y + 1$$

$$\text{size } (x^*) = \text{sz } x + 1$$

$$x \circ (y \cup z) = x \circ y \cup x \circ z$$

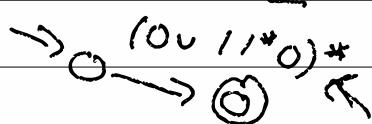
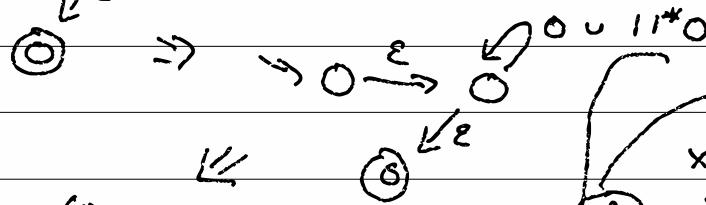
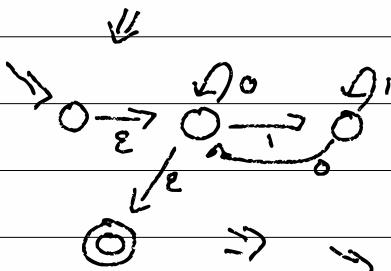


6-8] decompire : DFA $\xrightarrow{(\text{NFA}) \text{ or }} \text{RE}$



$$\Sigma^* \circ 0 \cup \epsilon$$

$$(1 \cup 0)^* \circ 0 \cup \epsilon$$

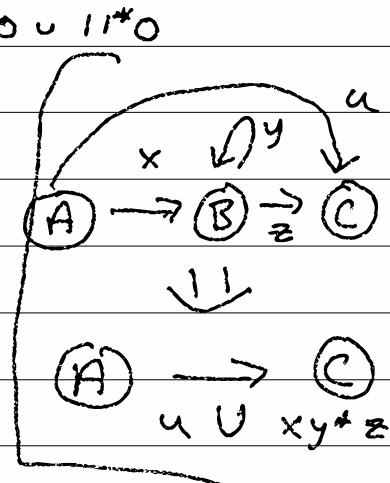


$$(0 \cup 11^* 0)^*$$

$\epsilon \quad 0 \quad 10 \quad 111110$

011101111100

11100011101100101010



G-9 / decompile : N-state NFA
→ RE

START : N-NFA → (N+2)-GNFA

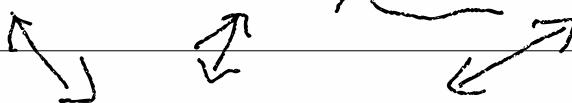
RIP : (N+1)-GNFA → N-GNFA

END : 2-GNFA → RE

decompile m = end ∘ ripⁿ ∘ start (n)

7-1) DFA/NFAs \rightarrow RE

DFA_s \leftrightarrow NFA_s \leftrightarrow RE



Regular
Languages

NFA \rightarrow RE

IN: n-NFA \rightarrow (n+2)-GNFA

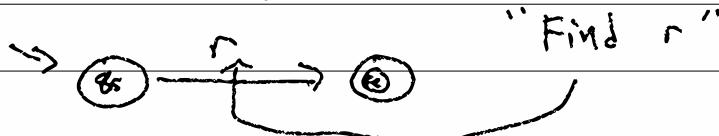
RIPⁿ: (n+1)-GNFA \Rightarrow n-GNFA

OUT: 2-GNFA \rightarrow RE

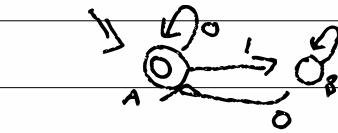
GNFA = $(Q, \Sigma, g_s, g_e, \Delta : (Q-g_e) \times (Q-g_s) \xrightarrow{e_Q} e_Q \rightarrow RE(\Sigma))$
 $S: Q \times \Sigma \rightarrow Q$

OUT: 2-GNFA \rightarrow RE

$(\{\Sigma, g_s, g_e\}, \Sigma, g_s, g_e, \{(g_s, g_e), \uparrow\})$
 $= r = \Delta(g_s, g_e)$



7-2) IN: NFA \Rightarrow GNFA
 $(Q, \Sigma, g_0, \delta: Q \times \Sigma \rightarrow P(Q))$ $(Q', \Sigma, g_s, g_e,$
 $F, \Delta: (Q' - g_e) \times (\Sigma - \epsilon) \rightarrow$
 $Q' = Q \cup \{g_e, g_s\} \rightarrow RE)$



$$\Delta(g_i, g_j) = r$$

$$\Delta(g_s, g_0) = \epsilon$$



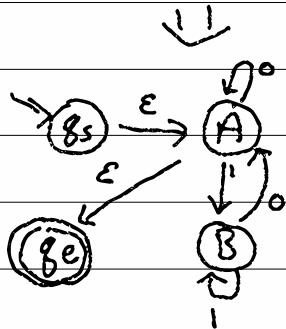
$$\Delta(g_s, g_j \neq g_0) = \emptyset$$

$$\Delta(g_f \in F, g_e) = \epsilon$$

$$\Delta(g_i \notin F, g_e) = \emptyset$$

$$\Delta(g_i, g_j) = \cup \{\epsilon\}$$

$$\delta(g_i, c) \ni g_j \}$$



7-3/ RIP = $(n+1)$ -GNFA \rightarrow n -GNFA
 main $'(Q, \Sigma, g_s, g_e, \Delta)'$ \downarrow $'(Q', \Sigma, g_s, g_e, \Delta')$

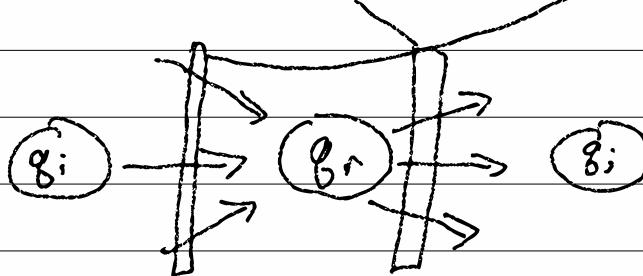
↓
 jay
 main

$$Q = Q' \cup \{ q \text{ gonna be killed} \}$$

↓
 jay
 main
 ↓
 exit
 ↓
 exit

$$\Delta' : \underbrace{(Q' - g_e)}_{g_r \in} \times \underbrace{(Q' - g_s)}_{g_r \in} \rightarrow \text{RE}$$

$$\Delta : \underbrace{(Q - g_e)}_{g_r \in} \times \underbrace{(Q - g_s)}_{g_r \in} \rightarrow \text{RE}$$

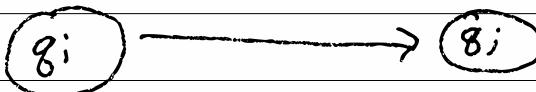


$$x \circ y^* \circ z \cup \alpha$$

$$\rightarrow x \circ z$$

$$\Delta'(q_i, q_j)$$

$$=$$

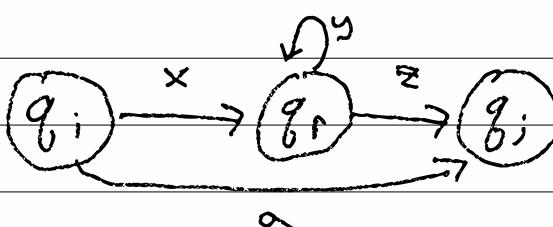


$$x = \Delta(q_i, q_r)$$

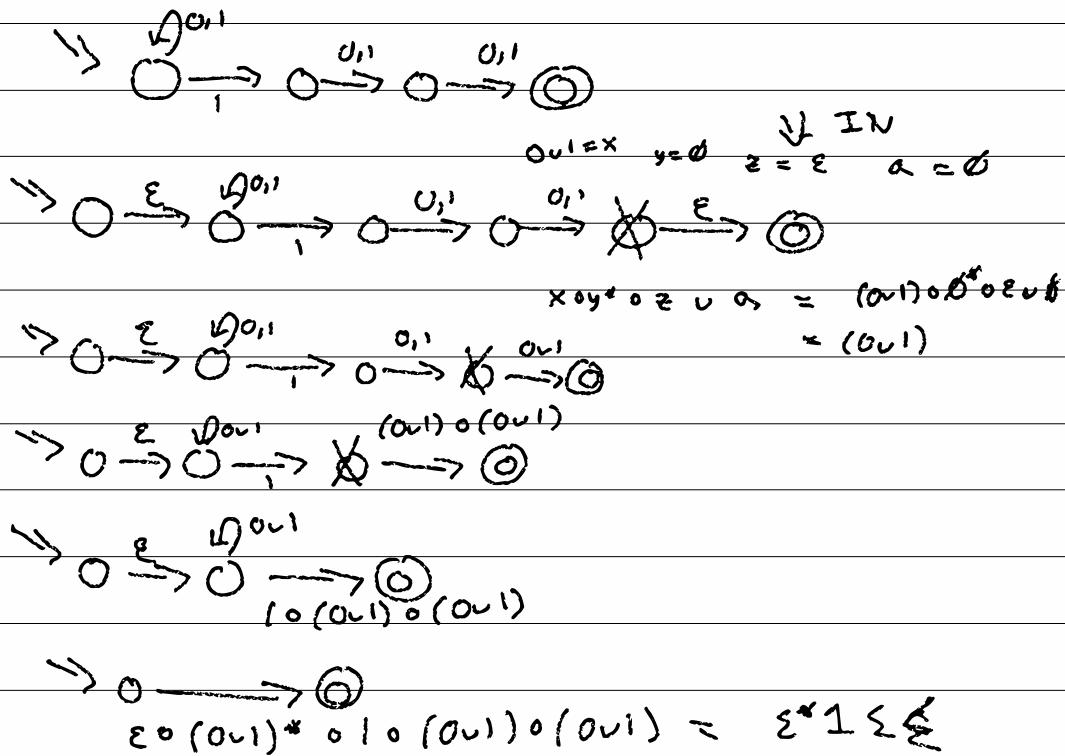
$$y = \Delta(q_r, q_r)^*$$

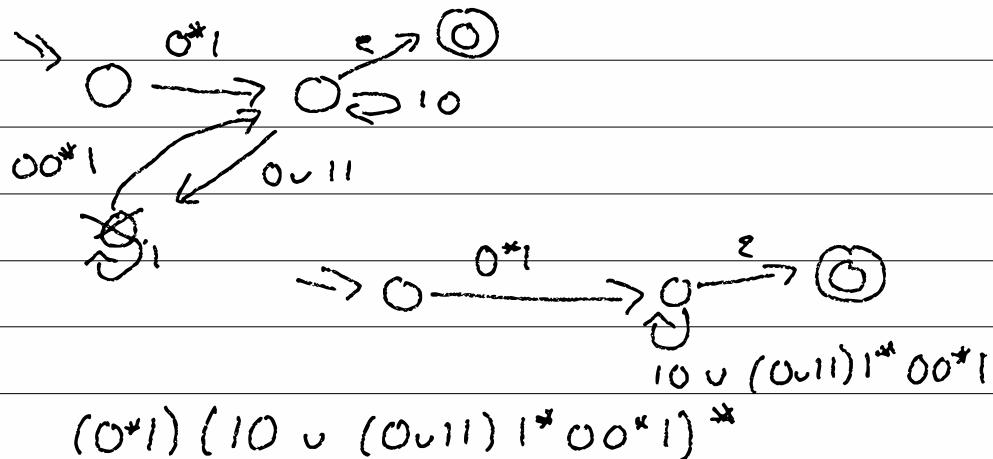
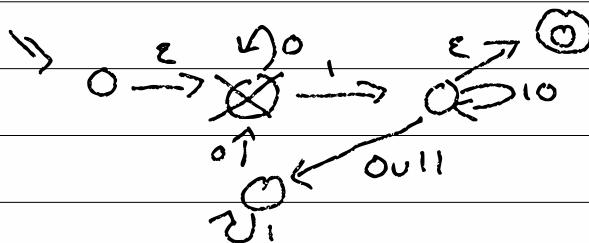
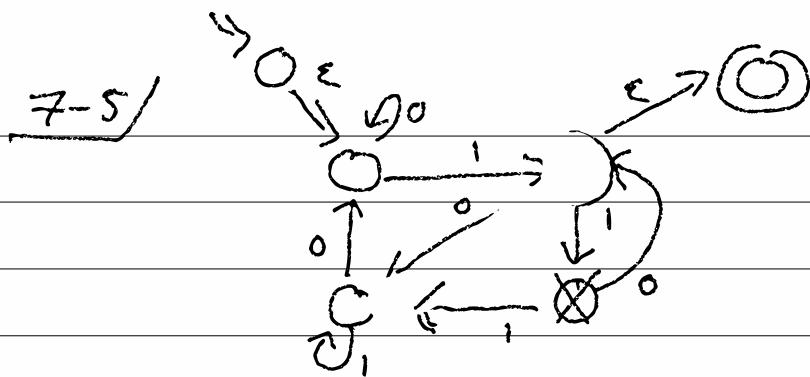
$$z = \Delta(q_r, q_j)$$

$$\alpha = \Delta(q_i, q_j)$$



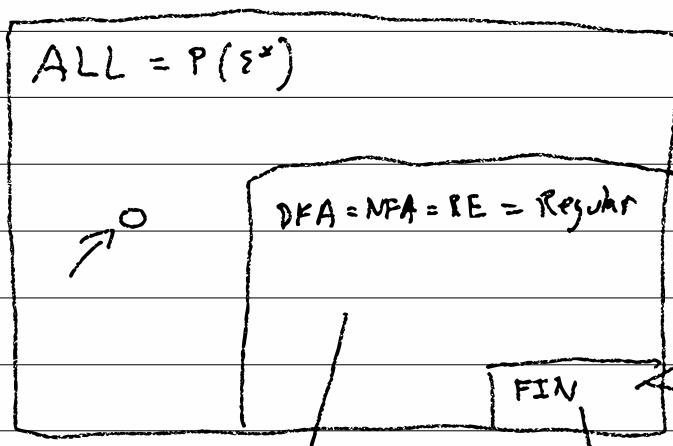
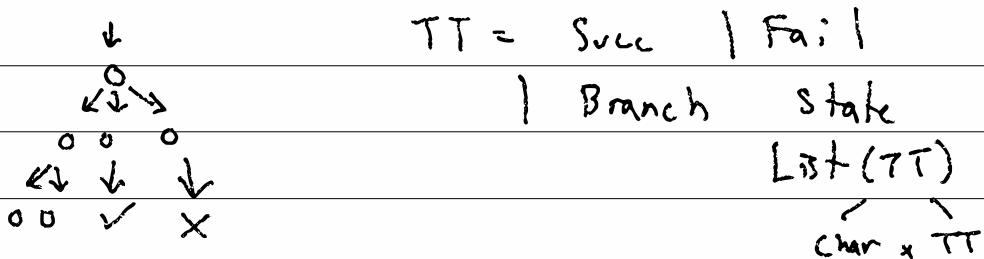
~~7-4/ int main () {~~
 int ~~x = 8;~~
~~int y = f(x, z);~~
 ...
 int y = x + 2 * 2;
 ↓
 int f(int ~~z~~){
 int ~~z~~;
 return ~~z + 2 * 8;}~~





8-1 ε Object → Char (char c)
 Upsilon () Epsilon ()
 UTF-8

TT



$\{\text{all strings ending in } 0\}$

$\{\emptyset, 00\}$

$A \in REG \iff \exists d \text{ DFA}, L(d) = A$

$\neg A \in REG \iff \neg (\exists d) \iff \forall d, d \text{ DFA} \wedge$

$\neg \exists x, P(x) \iff \forall x, \neg P(x) \quad L(d) \neq A$

$\neg \forall x, P(x) \iff \exists x, P(x)$

8-2/ How can we know stuff about
infinite sets?

$\forall x \in A, P(x)$

$P: DFA \rightarrow \text{Prop}$

$\Rightarrow z_0 \in Q$

$P: IP(\Sigma^*) \rightarrow \text{Prop}$

$\neg P(B)$ (where $B \in IP(\Sigma^*)$ and we "hope"
 B isn't in DFA)

Q. What is P?

1. Prove $\nexists RFG, P(A)$

Pumping
lemma

2. Prove $\exists B \in \text{RELL}, \neg P(B)$

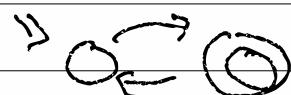
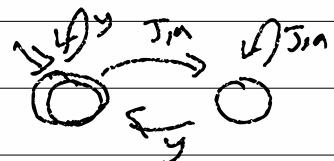
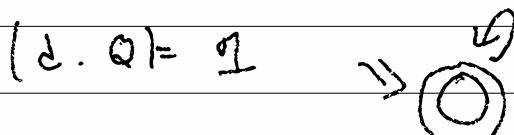
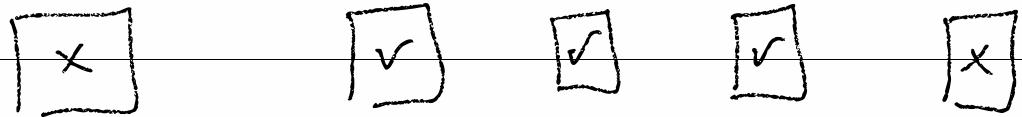
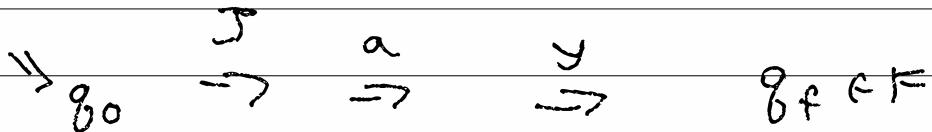
conclude $B \notin \text{REG}$.

$$\Sigma = \dots$$

8-3)

"Jay" & d

$\Sigma \ni \{s, a, y\}$



Daphne wins

pick a number of states ; 4

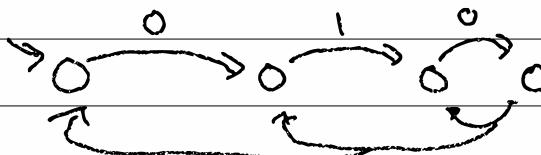
she picks a char

I say what state we goto

I win if I never say same state

she wins if I repeat

How many turns to win?

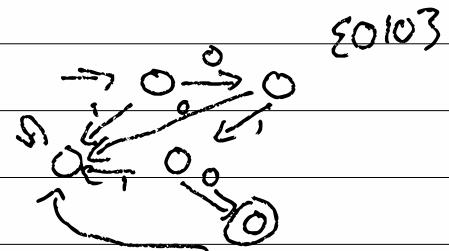
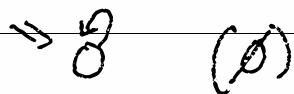
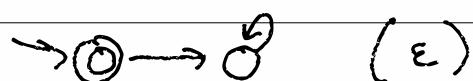


$4 \Rightarrow N \Rightarrow$

|@|

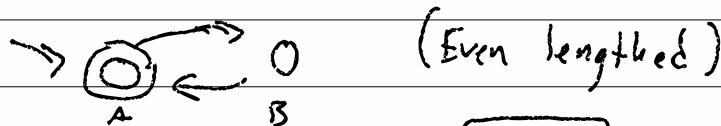
$\delta: Q \times \Sigma \rightarrow Q$ Total Fun
 $\xrightarrow{\text{from}} \xrightarrow{\text{to}}$

All DFAs have a loop



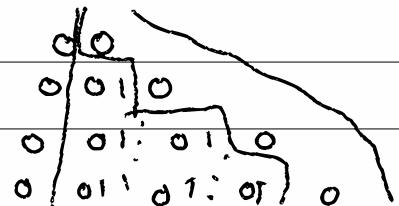
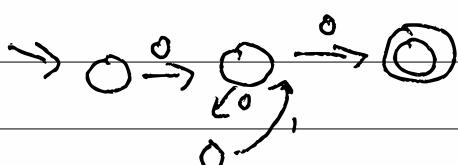
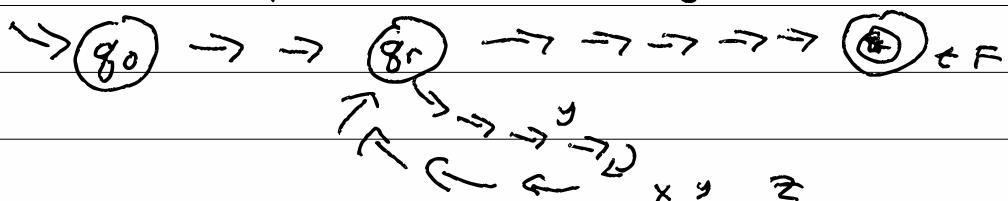
Some have "exciting" loops

$\exists s \in (\text{DFA})$, $s = \dots \dots \dots \dots \dots$



$$\begin{aligned} \epsilon &= A \\ &= \epsilon \circ \epsilon \circ \epsilon \end{aligned}$$

$$0110 = \overbrace{ABA\bar{A}}^z = \epsilon \circ 01010$$



8-5/ If a machine hasn't an existing loop ...

They are all finite
 $L(m) \in FIN$

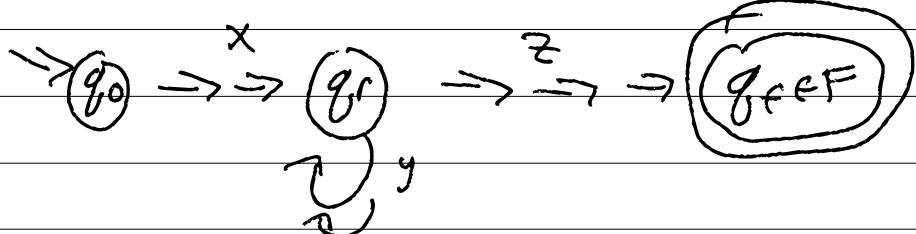
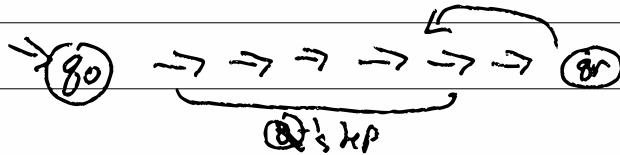
If it does have one ...

then it must be infinite (regular)

If there is an existing loop ...

what is the shortest string to
"find" ;+ m?

$$|s| = |Q| \quad (s \in L(\delta))$$



$\in \text{ALL}(\text{a language}) (\mathcal{P}(\Sigma^*))$

8-6/ RPP(A) = Regular Pumping Property

? ($\exists p \in N, \quad / / \quad p = |Q|$

$\forall s \in A \quad | \quad |s| \geq p \quad)$

$\exists (x, y, z \in \Sigma^*) \quad | \quad s = xyz \quad \wedge$

$|xy| \leq p$

$|y| > 0 \quad)$

$\forall i \in N,$

$x \circ y^i \circ z \in A$

)

$\forall d \in \text{DFA}, \exists n \in \mathbb{N}, L(d) = L(n) \rightarrow \text{Coq}$

$\neg \text{RPP}(B) :=$

$\forall p \in N,$

$\exists (s \in B \quad | \quad |s| \geq p \quad)$

$\forall (x, y, z \in \Sigma^*) \quad | \quad s = xyz \wedge |xy| \leq p \wedge |y| > 0 \quad \text{and} \quad$

$\exists i \in N,$

$x \circ y^i \circ z \notin B$

8-7) Need: an infinite space problem

```
O* 1 {  
    while (getc() == '0') {}  
    ungetc()  
    if (getc() == '1') { net false }  
    return getc() == EOF;
```

$$= 2^{32} \text{ uint2_t}$$

$\forall n \in \mathbb{N},$

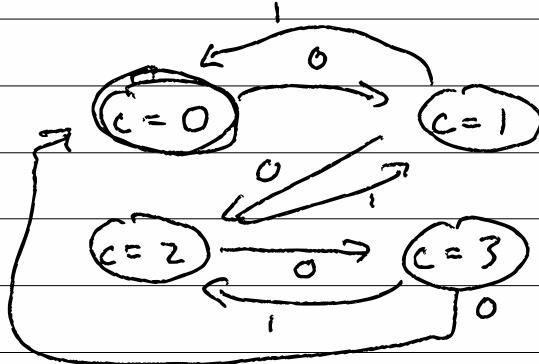
$0^n 1^n \in B$

$\text{uint} \text{ count} = 0;$

while ($in == 0$) count++

while ($in == 1$) count--
~~if (in == EOF) count--~~
return count == 0

{ net false } }



q-1) RPP(A) \rightarrow Language = $P(\Sigma^*) \approx A\bar{A}$

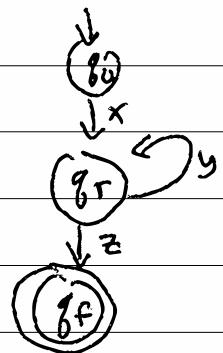
$$\exists p \in N, \quad |P| = |Q|$$

$$\forall s \in A \mid |s| > p \}.$$

$$\exists (x, y, z \in \Sigma^*) \mid s = xyz$$

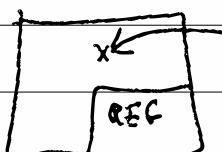
$$\forall i \in N, \quad |xy| < p$$

$$xyiz \in A \quad |y| > 0$$



$\neg RPP(A) =$

ALL



$$\forall p \in N,$$

$$\exists (s \in A \mid |s| > p)$$

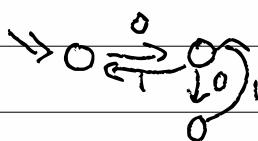
$$\forall (x, y, z \in \Sigma^*) \mid s = xyz \wedge |y| < p$$

$$\exists i \in N$$

$$xyiz \notin A$$

$$0^n 1^n \quad \text{ie} \quad x \in 0^n 1^n$$

$$\text{iff } \exists n \in N, \quad x = 0 \overbrace{0 \dots 0}^{\text{long part}} 1 \dots 1$$

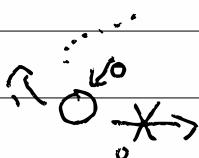


$$\text{int count} = 0$$

while (see 0) count++

while (see 1) count--

return count == 0



9-2) $\neg \text{RPP}(\mathcal{O}^n, \mathcal{O}^n) =$

$\forall p \in \mathbb{N},$

$\exists (s \in \mathcal{O}^n)^n \mid |s| \geq p)$

choose $s: s = x = z$

$s = \mathcal{O}^{p/2} \cup \mathcal{O}^{p/2}$

$s = \mathcal{O}^p, |s| = 2p \geq p$

$\forall (x, y, z \in \Sigma^*) \mid s = xyz \text{ and}$

$|xy| \leq p \text{ and}$

$|y| > 0$)

$x = \mathcal{O}^a \quad a+b+c = p \quad a+b < p$

$y = \mathcal{O}^b \quad d = p \quad b > 0$

$z = \mathcal{O}^c \cup d$

$\exists i \in \mathbb{N}, xyiz \in \mathcal{O}^n, \mathcal{O}^n$

$xyiz = \mathcal{O}^a \mathcal{O}^b \cup \mathcal{O}^c \cup d = \mathcal{O}^{a+b+c} \cup d$

$a+b+c = d$

$\frac{b(i-1)}{b} = \frac{-p}{b} \quad i=0 \quad \boxed{i=1}$

fun $n:$

if $n = 0 : \text{ret } p \circ$

$p \circ : \mathbb{P} \circ$

else: let $m = n-1$

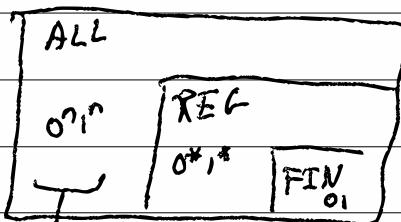
$p \circ : \text{th. } p_n \rightarrow p_{(n+1)}$

let $p_m = \text{rec } m \Rightarrow \forall n: p_n$

$p \circ p_m$

induction on $N:$

9-3/ There is stuff outside REG?



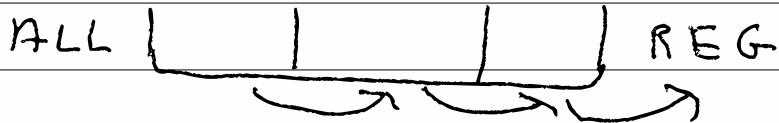
All computers
are DFAs.

if $0^n 1^n \in \text{REG}$

~~then~~^{and} $B \in \text{REG}$

then $0^n 1^n \circ B \in \text{REG}$

$\Sigma_1 \quad \Sigma_0 \quad \text{CFL}$



$0^n 1^n \in \text{REG}$

w where $\text{count}(0,w) = \text{count}(1,w)$

ww where $w \in \Sigma^*$

010101 $\in \Sigma^*$

wwR where $w \in \Sigma^*$

$$\underline{q-y} / \quad 0^x 1 0^y 1 0^{x+y}$$

$$01\ 0100 \Rightarrow "1+1=2" \quad \checkmark$$

$$001\ 0001\ 00000 \Rightarrow "2+3=5" \quad \checkmark$$

$$001\ 0010 \Rightarrow "2+2=1" \quad \times$$

A.p.

$$\exists s . \quad 0^p 1 0^p 1 0^{2p}$$

$$\nexists x y z . \quad x = 0^a \quad y = \underline{0^b} \quad z = 0^c 1 0^p 1 0^{2p}$$

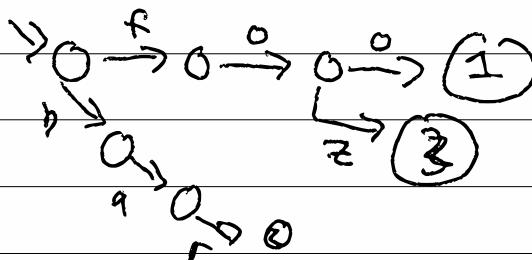
$$\exists i . \quad \begin{matrix} \text{change this} \\ \text{can't change} \end{matrix}$$

$$\underbrace{z+z=4}$$

$$\Rightarrow \Rightarrow$$

$$\underbrace{g+z=10}$$

12-1 / trie $\text{foo} \leftrightarrow 1$ digital
 $\text{bar} \leftrightarrow 2$ search
 $\text{baz} \leftrightarrow 3$ tree



CFGs

start variable $0^n 1^n$ & REG

$S \rightarrow \epsilon$ ← rule, productions, transitions
 D $S \rightarrow OSI$

variables
non-terminals

Symbols

rule = $\overbrace{V}^{\text{lhs}} \rightarrow \overbrace{V \cup \epsilon}^{\text{rhs}}$
 $\text{rhs} = (\text{V} \cup \epsilon)^*$
 $\xrightarrow{\text{terminal}}$

$\overset{B}{S} \rightarrow \overset{B}{OSI} \rightarrow \overset{B}{0OSII} \rightarrow \overset{B}{000SIII}$
 "derivation" $\overset{B}{000III}$

$w \in L$ iff $\exists d. S \xrightarrow{*} w$

$$\frac{L(2)}{M} \left(\begin{array}{l} S \Rightarrow 01S \\ S \Rightarrow SS \end{array} \right) = \emptyset$$

Context-free grammar

$$0101 \in \xrightarrow{\quad} \text{alphabet} \xrightarrow{\quad} (V, \Sigma, R, S)$$

$$B \Rightarrow \epsilon \qquad \qquad \downarrow \text{finite set}$$

$$B \Rightarrow B \cup N \cup B \qquad \qquad \downarrow \in V$$

$$N \Rightarrow \epsilon \qquad \qquad P(V \times (V \cup \Sigma)^*)$$

$$N \Rightarrow 0N \qquad \qquad (V \Rightarrow P(V \cup \Sigma)^*)$$

$$\Sigma(S, 01S),$$

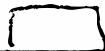
$$R = \{S, SS\} \quad V = \{\Sigma\} \quad \Sigma = \{0, 1\} \quad S = S$$

REG

DFA

REX ($\cup, \cdot, ^*$)

CFL



CFG

12-3 / A B

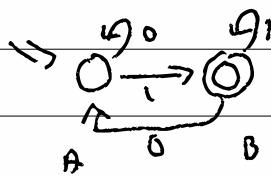
$\rightarrow A \cup B : S \Rightarrow A, S$

$S \Rightarrow B, S$

$A \circ B : S \Rightarrow A, S \quad B, S$

$A \cap B : \times$

DFA \Rightarrow CFG

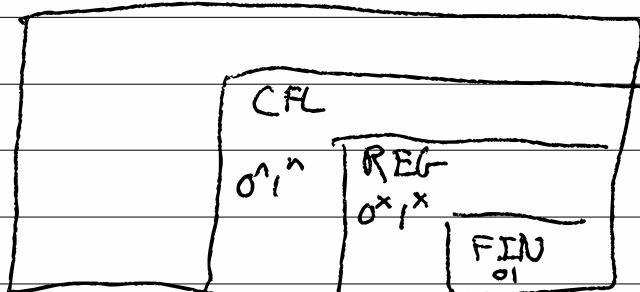
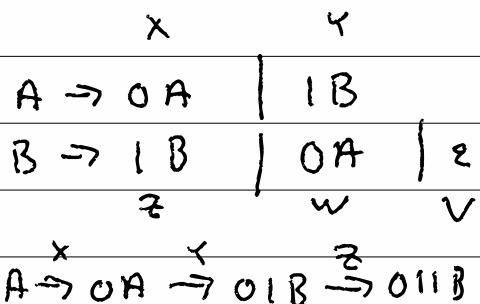


01101

$S = g_0$

$g_i \rightarrow c g_j \text{ iff } \delta(g_i, c) = g_j$

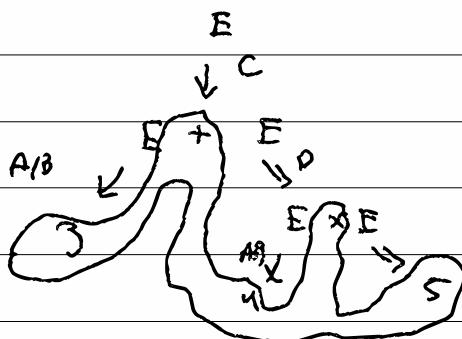
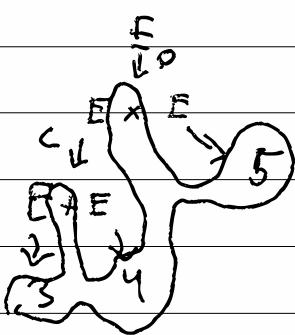
$g_f \rightarrow \epsilon \text{ iff } g_f \in F$



P D C D

$$12 - 4 / E \Rightarrow O \quad | \quad I \quad | \quad E + E \quad | \quad E \times E$$

$$3 + 4 \times 5$$



fowim $E \Rightarrow n$

fowin $O = O$

$I = I$

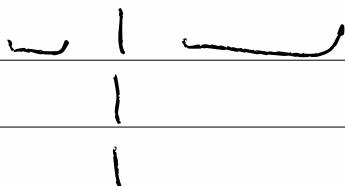
Plus(L, R) \Leftarrow fowim L + fowim R

Mult(L, R) \Leftarrow fowim L \times fowim R

ambiguous = There are multiple
trees (derivations)

if cond

for the same string



(2-5) Nice! amb \rightarrow unamb

not possible

V, Σ, R, S

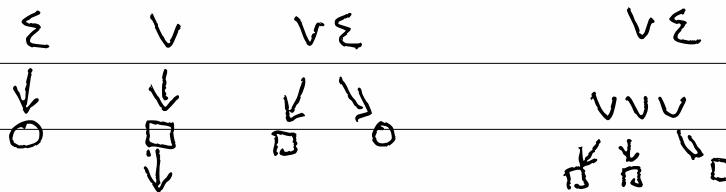
$$V \Rightarrow (V \cup \Sigma)^*$$

ϵ

$\epsilon \Sigma V$

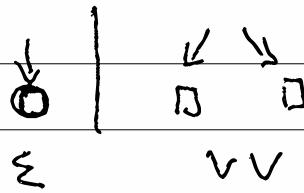
V

VVVVV



want: "pure" binary trees

Noam



Chomsky-Normal Form

$$\begin{array}{c|c} V \rightarrow \Sigma & A \rightarrow C \\ V \rightarrow VV & A \rightarrow BC \\ S \rightarrow \epsilon & \#S \end{array}$$

GFG \rightarrow CNF

(2-5) $S \rightarrow \epsilon \mid OSI$

— Add a new start

$R \rightarrow S$

$S \rightarrow \epsilon \mid OSI$

— remove ϵ -rules

$R \rightarrow S \mid \epsilon$

$S \rightarrow OSI \mid OI$

remove unit rules

$R \rightarrow OSI \mid OI \mid \epsilon$

$S \rightarrow OSI \mid OI$

add extra vars for ≥ 2

$R \rightarrow TI \mid OI \mid \epsilon$

$S \rightarrow TI \mid OI$

$T \rightarrow OS$

add terminals "names"

$R \rightarrow TB \mid AB \mid \epsilon$

$S \rightarrow TB \mid AB$

$T \rightarrow AS$

$A \rightarrow O$

$B \rightarrow I$

12-6 / 00111 + 0ⁿ1ⁿ

S $\Rightarrow \epsilon$

S $\Rightarrow 0S1$

A

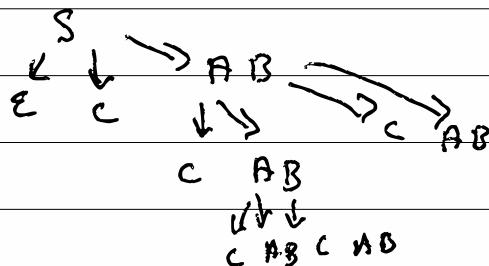
B

A $\Rightarrow c$ + 1

B \Rightarrow + 2

A $\Rightarrow BC$ + 2

A \Rightarrow done



S $\Rightarrow XYZ$

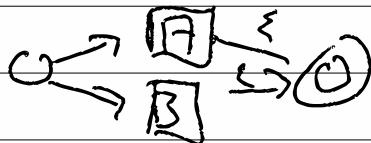
X $\Rightarrow ZZS$ | 0

Y $\Rightarrow XYX$ | 1

Z $\Rightarrow YSX$

$$A = \{x, y, z\}$$

$$B = \{x, y, z\}$$



$$Q = \{S, E, (0, x), (0, y), (0, z), (1, x), (1, y), (1, z)\}$$

$$= \{S, E\} \cup 0 \times A \cup 1 \times B$$

unionstate = start | end | fromA g_A
| fromB g_B

$$\delta(\text{start}, \epsilon) = \{ \text{fromA } A_0, \text{ fromB } B_0 \}$$

$$\delta(\text{fromA } g_i, c) = \text{fromA } (\delta_B(g_i, c))$$

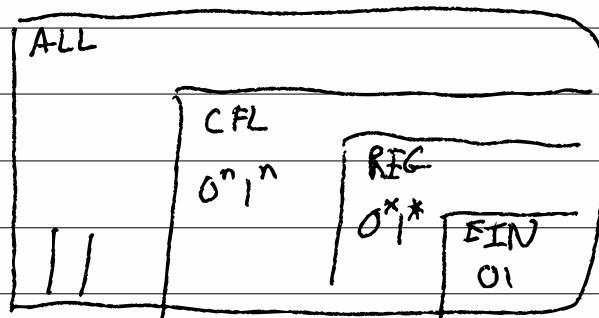
12-7 / new DFA ("blat",
 $q_i \rightarrow q_i = 0 \text{ if } q_i = 1,$
sigma, delta,
 F)



complement (DFA \perp) Σ

new DFA (\perp , $q_i \rightarrow !d.F(q_i)$)
($d.Q$, $d.\Sigma$, $d.\delta$, $d.F$)

(5-1)



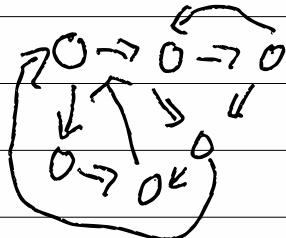
REGULAR

$$\text{DFAs} \longleftrightarrow \text{Regex}$$
$$\epsilon \qquad \qquad \qquad \{\}$$

CFL

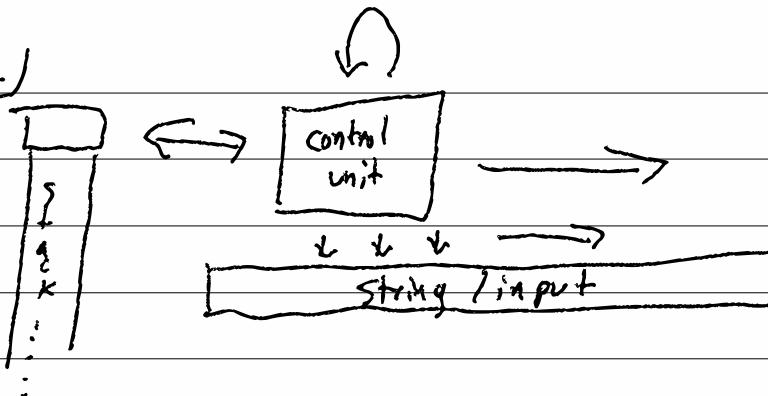
$$\boxed{\text{PDA}} \longleftrightarrow \text{CFG}$$

push-down n automata



$\Sigma^* 1 \Sigma \Sigma$

15-2)



$$\text{DFA: } Q \times \Sigma \rightarrow Q$$

$$\text{Deterministic PDA: } Q \times \Sigma \times \Gamma \rightarrow Q \times \Gamma$$

$$\text{PDA: } Q \times \Sigma \times \Gamma \rightarrow P(Q \times \Gamma)$$

$\Sigma \cup \{\epsilon\}$

$$Q \times \Sigma \times \Gamma \xrightarrow{\quad} Q \times \Gamma \quad (\text{ignored stack})$$

$$Q \times \Sigma \times \Gamma \rightarrow Q \times \Gamma \quad (\text{pop})$$

$$\times \Sigma \rightarrow Q \times \Gamma \quad (\text{push})$$

$$\times \Gamma \rightarrow \times \Gamma \quad (\text{replace})$$

$$\epsilon \times \Gamma \rightarrow \times \Gamma$$

$$\delta(q_i, c) = q_j$$

$$[q_i]_c w \rightarrow [q_j]_w$$

$$c \in \Sigma \quad w \in \Sigma^*$$

$$\delta(q_i, c, \alpha) \ni (q_j, \beta)$$

$$\beta \in [q_j]_w \rightarrow \beta \gamma [q_j]_w$$

$$\alpha \in \Gamma_\Sigma \quad c \in \Sigma$$

$$\beta \in \Gamma^* \quad w \in \Sigma^*$$

config

15-3/ simulate : PDA $\times \cancel{S} \rightarrow \text{config}$
 $\text{sim } (\mathcal{Q}, \Sigma, \Gamma, q_0, \delta, F) \quad (\Gamma^*, q_f, \Sigma^*) =$

let $\alpha : \beta \in g$ in

let $c = w \in S$ in

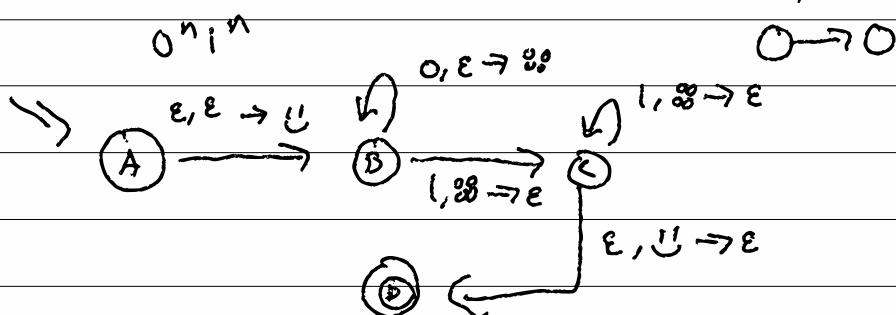
let $(q_j, \alpha) \in \delta(q_i, c, \alpha)$ in
 $(\alpha = \beta, q_j, w)$

accepts : PDA $\times \Sigma^* \rightarrow \text{bool}$

accepts $p \; s = \text{while } c, s \notin \Sigma \text{ do}$

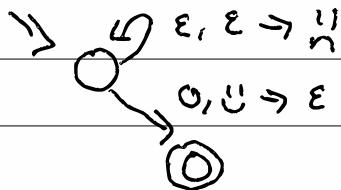
return $\left(\begin{array}{l} \text{sim } c, q_i \in p, F \\ \text{where } c_0 = (\Sigma, p, q_0, S) \end{array} \right)$

$c, \alpha \Rightarrow \alpha$



$\epsilon[A]0011 \rightarrow \cup[B]0011 \rightarrow \cup^{00}[B]011 \rightarrow \cup^{0000}[B]11$
 $\rightarrow \cup^{00}[C]1 \rightarrow \cup[C]\epsilon \rightarrow \epsilon[D]\epsilon \rightarrow \text{YES}$

15-4/



CFG \rightarrow PDA

$$\begin{array}{l} S \rightarrow \epsilon \\ S \rightarrow 0S1 \\ S \rightarrow Xyz011syz \end{array} \Rightarrow \begin{array}{c} 0 \rightarrow 0 \rightarrow 0 \\ \downarrow \\ 0 \end{array}$$

$$\Gamma = \Sigma \cup V$$

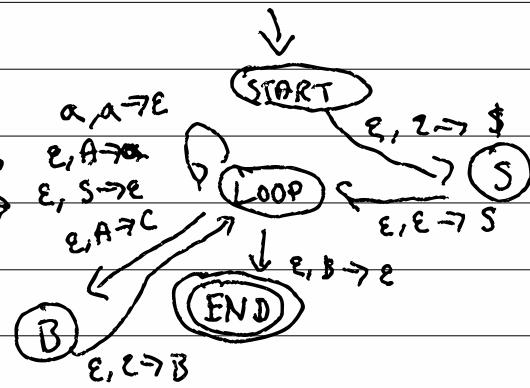
CNF \rightarrow PDA

$$A \rightarrow BC \quad (B, C \neq S)$$

$$S \rightarrow \epsilon$$

$$A \rightarrow a$$

$$A[\text{Loop}] \rightarrow C[B] \rightarrow CB[\text{Loop}]$$



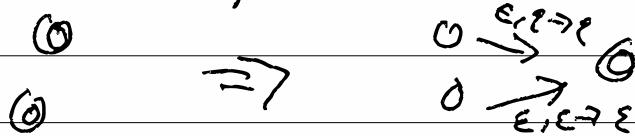
$$\epsilon[\text{ST}]0011 \rightarrow \$[S]0011 \rightarrow \$\$[L]0011 \rightarrow \$\$SO[L]0011 \rightarrow \$\$SO[L]0011$$

$$\Downarrow$$
$$\$1\$[L]0011 \leftarrow \$1\$SO[L]0011 \leftarrow \$1\$S[L]0011$$
$$\$1\$[L]0011 \rightarrow \$1[L] \rightarrow \$[L] \rightarrow [\text{END}] \rightarrow \checkmark$$

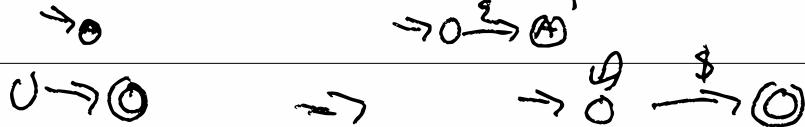
15-5/ PDAs are too complicated

so, we'll simplify with some rules

- Make a single accept



- Guarantee stack is empty on accept



- Always push or pop

push : ϵ , Γ

pop : Γ , ϵ

X ignore : ϵ , ϵ

X replace : Γ , Γ

Every symbol pushed is eventually popped

15-6/ $V = (\mathbb{Q} \times \mathbb{Q})$ $S = (g_0, g_f)$

$$(r, +) \in \delta(p, a, \varepsilon)$$

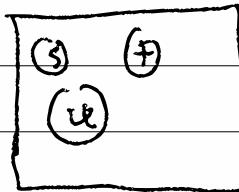
$$(g, \varepsilon) \in (s, b, +)$$

$$\overline{(p, g) \rightarrow a \underset{\in \Sigma}{\in} (r, s) \underset{\in V}{\in} b \underset{\in \Sigma}{\in}}$$

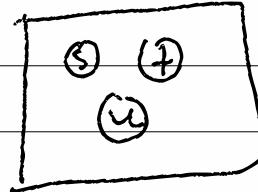
$$(p, p) \rightarrow \varepsilon$$

$$(p, g) \Rightarrow (p, r) (r, g) \quad \forall p, g, r$$

A \leftarrow cast



B \leftarrow bst



shown cast, bst

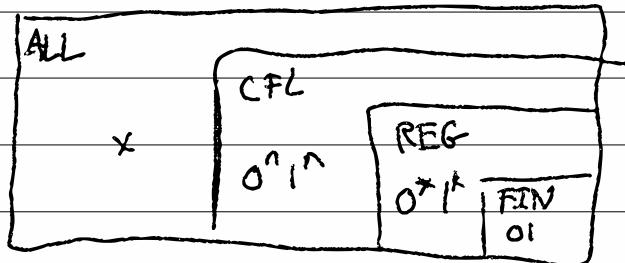
A \cup B \leftarrow ~~cast, bst~~

shown $\langle x, y \rangle$ = start

| and A x

| and B y

16-1



DFA_s \longleftrightarrow REG

$0^n 1^n \in \text{REG}$

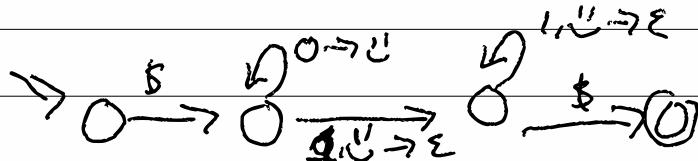
PDA_s \longleftrightarrow CFG

$x \notin \text{CFL}$

CFL = ALL

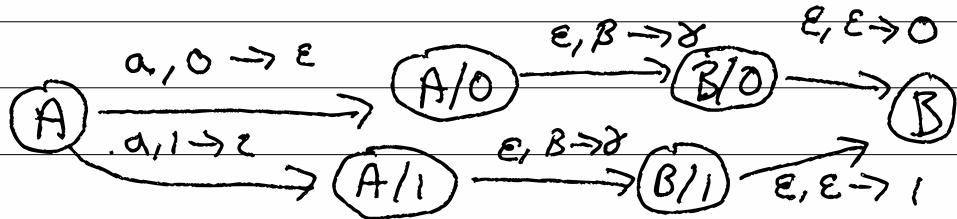
1.	$P \dots \forall x \in \text{REG}, P(x)$	$\forall c \in \text{CFL}, P(c)$
2.	$\exists x \in \text{ALL}, \neg P(x)$	$\exists x \in \text{ALL}, \neg P(x)$
3.	$\Rightarrow x \notin \text{REG}$	$\Rightarrow x \notin \text{CFL}$

$0^n 1^n \rightarrow \xrightarrow{\quad} \textcircled{0} \xrightarrow{\quad} 1 \xrightarrow{\quad} 2 \xrightarrow{\quad} 3$



16-2 Look at 2nd th, by

$\alpha, ?\beta \rightarrow ?\gamma$



$$y \beta Q[A] \omega \Rightarrow y/B[A/0] \omega \Rightarrow y\gamma[B/0] \omega \\ \Rightarrow y\gamma 0[B] \omega$$

$$0^n 1^n \qquad 0^n 1^* 0^n \qquad 0^n 1^n 0^n$$

$$0^n 1^* 0^y \text{ s.t } x+y=2n$$

$$\overbrace{0^n 1^n 0^n}^{0^n 1^* 0^n} \xrightarrow{\epsilon [] 0^n \rightarrow \epsilon [] \epsilon} \begin{cases} \epsilon [] 0^n 1^n 0^n \\ 0^n [] 1^n 0^n \\ \epsilon [] 0^n \end{cases}$$

$$\epsilon [] 0^n 1^n 0^n \rightarrow 0^n [] 1^n 0^n \rightarrow 0^{n-1} 00 [] 1^n 0^n \\ \rightarrow 00 [] 0^n \rightarrow$$

16-3/ RPP

$\forall A \in \text{REG},$

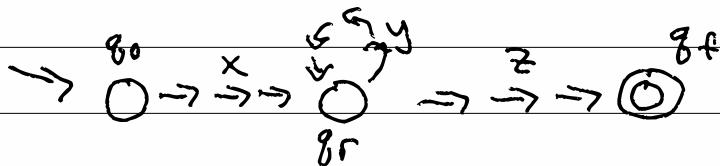
$\exists p \in \mathbb{N},$

$\forall (s \in A \mid |s| \geq p)$

$\exists (x, y, z \in \Sigma^* \mid |xy| \leq p \wedge |y| > 0)$

$\forall i \in \mathbb{N},$

$xy^iz \in A.$

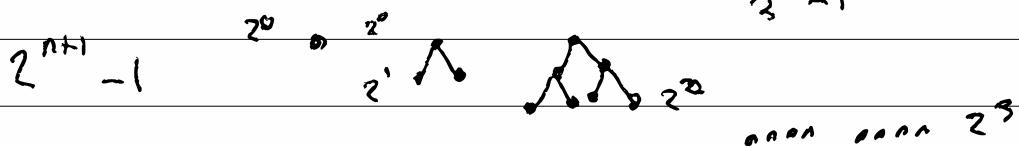


CFG $G(\text{CNF}) = (V, \Sigma, R, S) \quad S \Rightarrow \epsilon \in R$

$V = \{v_0, \dots, v_n\} \quad S = v_0 \quad \forall i \Rightarrow v_i \in R$

$v_i \Rightarrow v_x v_y$

v_0	v_1	v_2	v_n	\rightarrow	v_r	$v_r \in V$
v_1	v_2	v_2		\vdots		\vdots	
v_2				v_n		v_r	



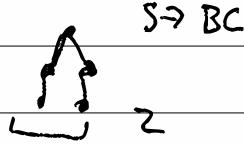
(6-y) minimum length of tree of height n

$S \rightarrow E$

$$\bullet = 0$$

$S \rightarrow \alpha$

$$= 1$$



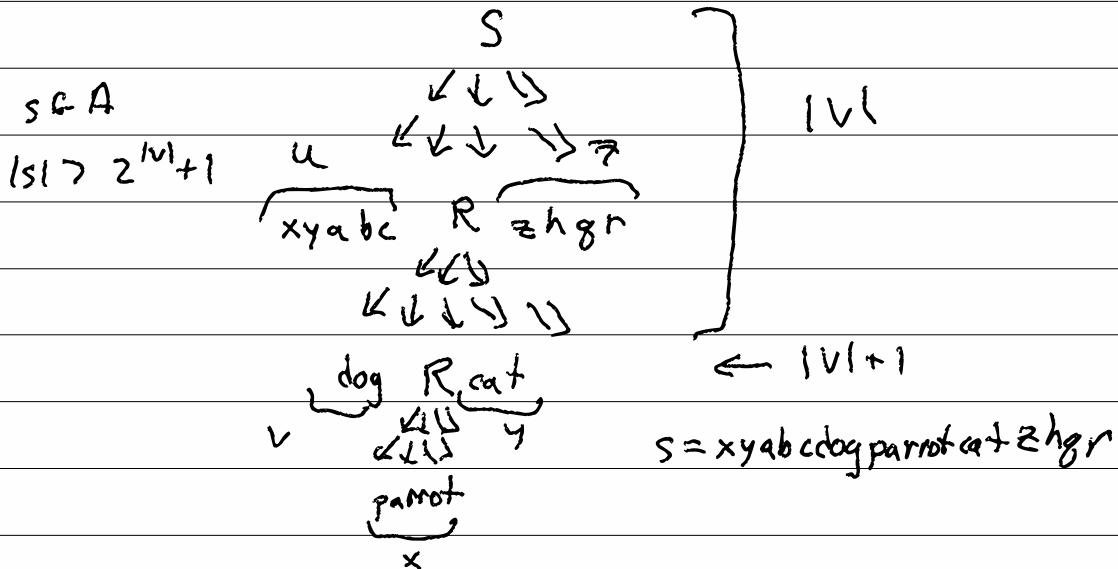
$S \rightarrow BC$

$$n=4$$

$$\Rightarrow$$

$$\text{len} \geq 3$$

If the len of a string is $> 2^{|V|} + 1$
then the height must be $|V| + 1$



$$S \rightarrow u R z$$

$$R \rightarrow v R y$$

$$R \rightarrow x$$

16-5) Context-free pumping property

$$\forall A \in \text{CFL}, \quad \exists p \in \mathbb{N}, \quad \forall s \in A \quad |s| \geq p \quad \exists (u, v, x, y, z \in \Sigma^*) \quad |s = uvxyz| \quad |vxy| \geq p \quad |vy|^i > 0$$

$\forall i \in \mathbb{N}$,

$$uv^ixy^iz \in A$$

$$\begin{array}{ll} S \Rightarrow uRz & S \Rightarrow uRz \rightarrow uxz \\ R \Rightarrow vRy & S \Rightarrow uRz \rightarrow uvRyz \rightarrow uv^2Ry^2z \\ R \Rightarrow x & \rightarrow uv^2xy^2z \quad (i=2) \end{array}$$

$$0^n, n \in \mathbb{N} \in \text{CFL} \quad \overset{p=4}{\sim} \quad s = 0011 \quad |vxy| = 2 \leq 4$$
$$u=0 \quad v=0 \quad x=\epsilon \quad y=1 \quad z=1 \quad |vy|=2>0$$
$$uv^ixy^iz \in A? \quad 00^{i+1}1^{i+1} = 0^{i+1}1^{i+1} \in A$$

$$16-6 \quad 0^n 1^n 0^n = A$$

given p

$$0^p 1^p 0^p$$

given u, v, x, y, z

$$\begin{aligned} u &= 0^{\vec{w}} & v &= 0^{\vec{v}} & x &= 0^{\vec{x}} & y &= 0^{\vec{y}} \\ z &= 0^{\vec{z}} 1^n 0^p \end{aligned}$$

$u x y$ has repetition

$$\boxed{0^p 1^p 0^p} \rightarrow u x y = 1 \text{ symbol}$$

$$\boxed{0^p 1^p 0^p} \rightarrow u x y = 2 \text{ symbols}$$

$$\boxed{0^p 1^p 0^p} \rightarrow 3 \text{ symbols}$$

$$0^{p+i} 1^p 0^p$$

$$0^p 1^{p+i} 0^{p+i}$$

$$u x y = 0^a 1^p 0^b \quad | \quad | = a + p + b \leq p$$

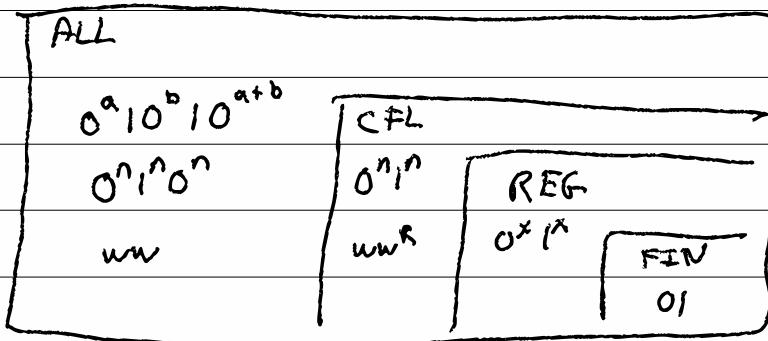
DFAunion = pair $\langle X, Y \rangle$

NFAunion = start

state from left X ("x")

state from right Y ("y")

END



17-1 Finite : 01

Regular: $0^* 1^*$

Context-free: $0^n 1^n$

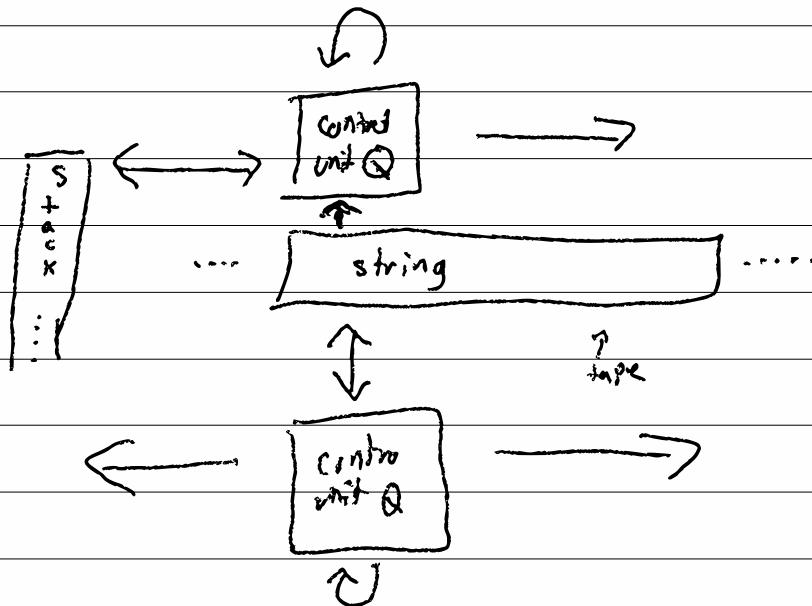
Context-free pumping property
CFPP

context-free pumping property

\Rightarrow CFG, PDS

Turing Machine Alan Turing

Goal: Effective ~~math~~ algorithm for true math.



$$\boxed{172} \quad \delta: \mathbb{Q} \times \mathbb{X} \rightarrow \mathbb{Q} \times \mathbb{P} \times \{L, R\}$$

states : Q

config :

input : Σ

L&S

$$P^* \times Q \times P^*$$

tape : Γ ($\Sigma \subseteq \Gamma$) $\omega \in \Gamma$

$$g_0 \in Q$$

$g_a \in Q$ (accepting)

$g_r \leftarrow Q$ (rejecting)

$$c_0 = \varepsilon [g_0] w$$

for $w \in \Sigma^*$ input

$w \in L(m)$ iff

$$E[g_0]w \Rightarrow^* x[g_a]y$$

$$S(g_i, a) = (g_i, b, R)$$

$$\delta(g_i, a) = (g_i, b, \angle)$$

$$x [g_i] ay \Rightarrow x b [g_i] y$$

$$x \in [g_i]_{\alpha y} \Rightarrow x[g_j]_{\alpha b y}$$

$$x[g_i]y \Rightarrow \omega x[g_i]y\omega$$

left : tape \rightarrow tape

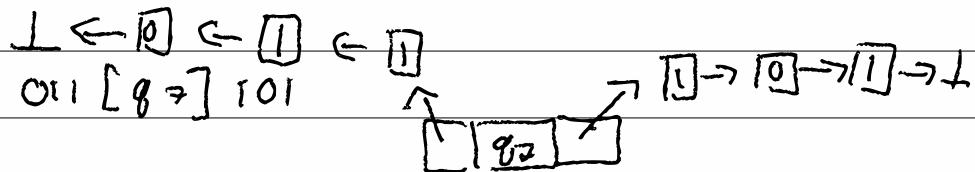
right : tape \rightarrow tape

$$\text{left } \varepsilon = \varepsilon : \omega$$

right $\varepsilon = \omega - \epsilon$

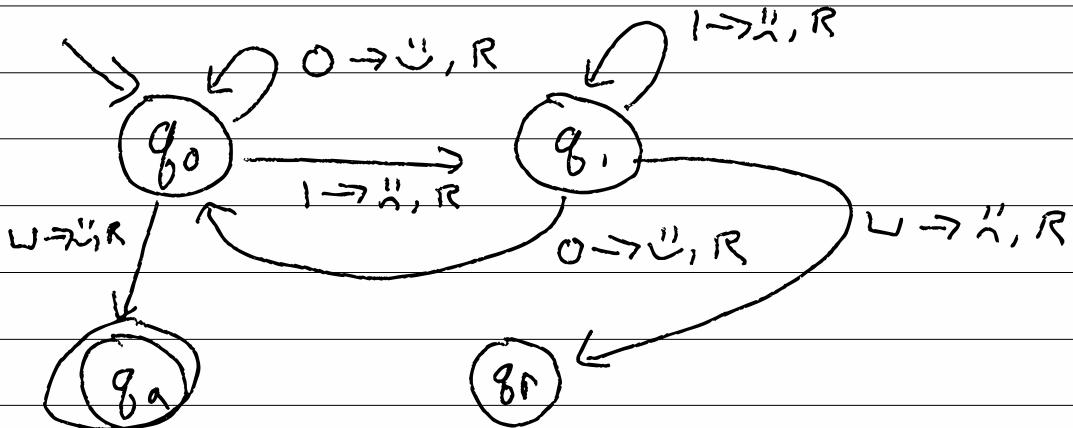
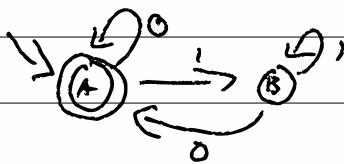
$$\text{left}(x:c) = x$$

$$\text{right}(c \cdot y) = y$$



(DFA)

173)



$[A]^{110} \rightarrow [B]_{10} \rightarrow [B]_0 \rightarrow [A]_\varepsilon \rightarrow \checkmark$

$\varepsilon [q_0]^{110} \rightarrow 1 [q_1]_{10} \rightarrow 1 1 [q_1]_0 \rightarrow$
 $\rightarrow 1 1 1 [q_1]_\varepsilon \rightarrow 0 1 1 1 [q_0] 0 \rightarrow 0 1 1 1 0 [q_0]$

input: $(Q, \Sigma, q_0, \delta: Q \times \Sigma \rightarrow Q, F \subseteq Q)$

output: $Q' = Q \cup \{q_a, q_r\}$

$\Sigma' = \Sigma$

$\Gamma = \varepsilon \cup \{\omega\}$

$q'_0 = q_0$

$\delta'(q_i, c) = (\delta(q_i, c), \omega, R)$ if $c \notin \omega$

$\delta'(q_i, \omega) = (q_a, \omega, R)$ if $q_i \in F$

(q_r, ω, R) if $q_i \notin F$

$$\Sigma = \{0, 1, \#\}$$

17-4 / $w \# w$ where $w \in \Sigma^*$

I saw a zero

$[] 01 \# 01 \rightarrow \sim [] 1 \# 01 \rightarrow \sim 1 [] \# 01$

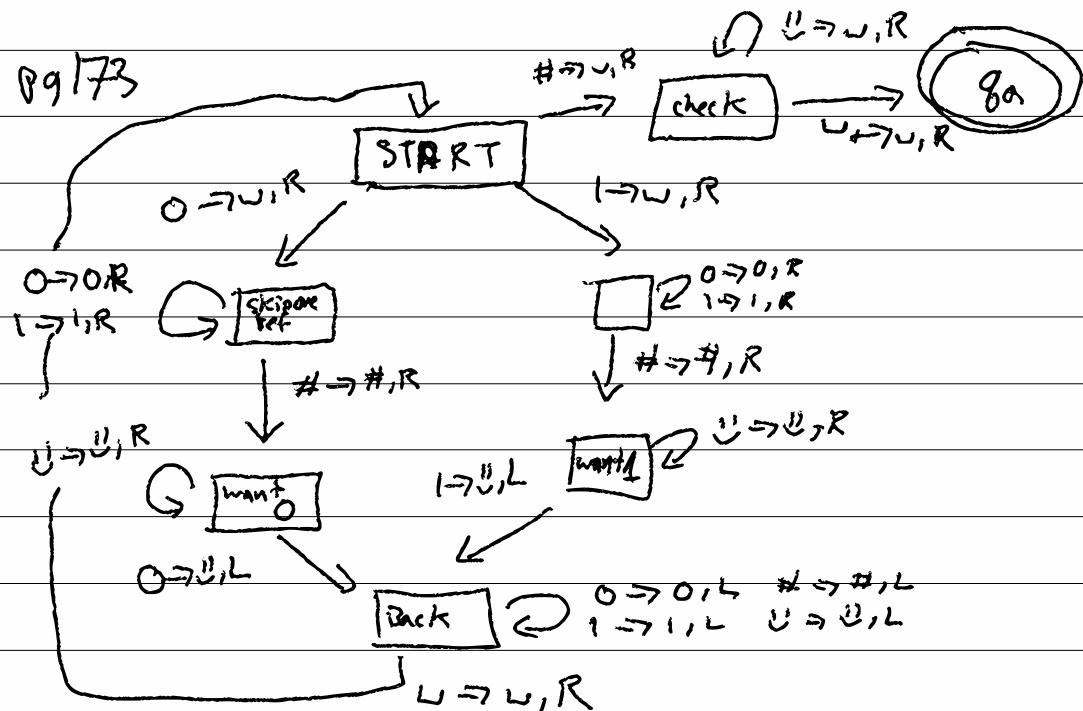
$\sim [] \# 01 \rightarrow \sim [I \text{ expect a zero}] 01 \rightarrow \sim 1 [I \text{ am happy}] \# 01$

$\rightarrow \sim [H] 1 \# 01 \rightarrow [H] \sim 1 \# 01$

$\rightarrow \sim [B] 1 \# 01 \Rightarrow \sim [I \text{ saw a } 1] \# 01$

$\sim \# [I \text{ expect a } 1] 01 \quad \sim \# 0 [\text{ looks like } 1] 1$

Pg 173



17-5/ A computable function
 f

is a Turing - machine
and

$$f(x) = y$$

if f

$$\in [g_0]x \Rightarrow^* w[g_a]y$$

$$\text{add} \begin{matrix} 1 \\ 0 \end{matrix} = \begin{matrix} 1 \end{matrix}$$

$$\begin{matrix} 1 \end{matrix} = \begin{matrix} 10 \end{matrix}$$

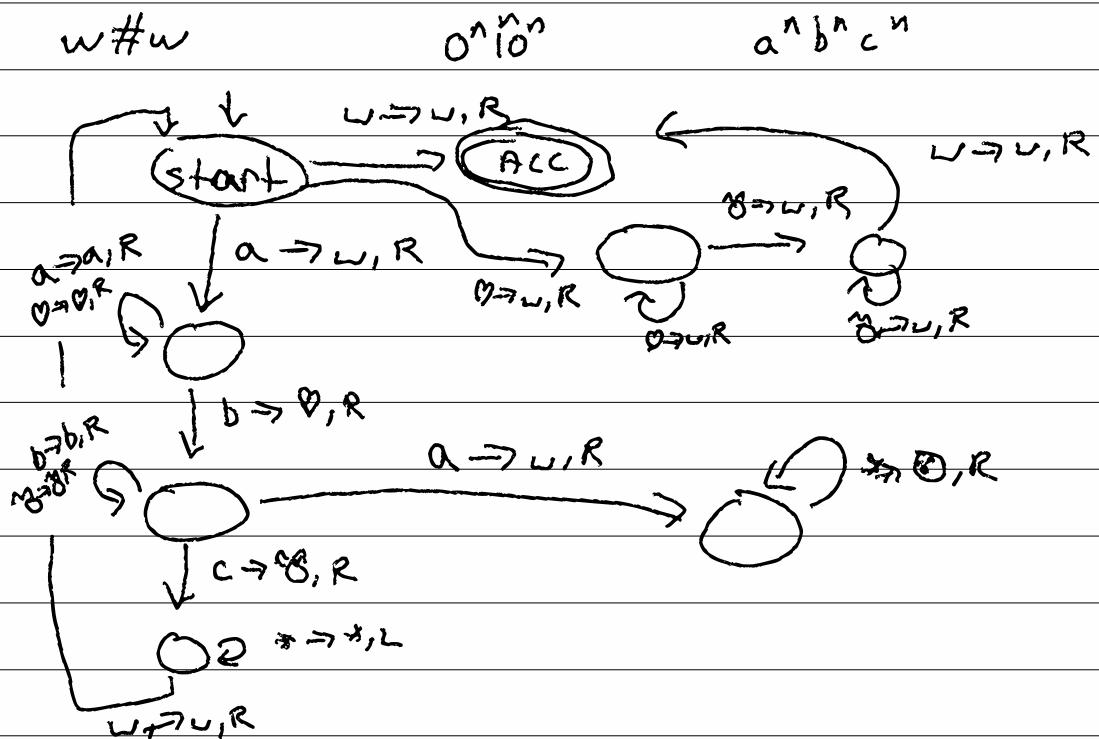
$$\begin{matrix} 10 \end{matrix} = \begin{matrix} 11 \end{matrix}$$

$$\begin{matrix} 11 \end{matrix} = \begin{matrix} 100 \end{matrix}$$

($\geq \Sigma$)

[18-1] $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

$Q = \{q_0, q_f\}$



$$O^x + O^y = O^{x+y}$$

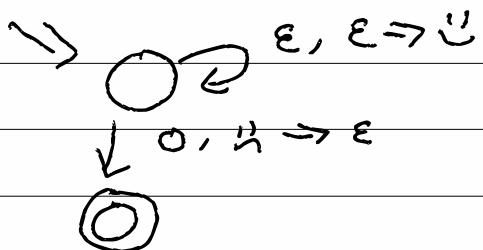
$$f(O^x + O^y) = O^{x+y}$$

18-2) When a DFA runs, how long will it take on input w ?

$|w|$

When a PDA on input w ?

$2^{|w|}$ (loop on the stack)



On input w , what is the destiny of a Turing machine?

ε(ACC) & (REJ, LOOP, DIVERGE)

ACC $\epsilon [q_0] w \Rightarrow \Rightarrow \Rightarrow \Rightarrow x [q_a] y$

REJ $\epsilon [q_0] w' \Rightarrow \Rightarrow \Rightarrow \Rightarrow x' [q_r] y'$

LOOP $\epsilon [q_0] w'' \Rightarrow \Rightarrow z [q_i] \Leftarrow \Rightarrow \Rightarrow z [q_i] u$

D
IVE
RGING $\forall x, q_i, y, \epsilon [q_0] w \Rightarrow^* x [q_i] y$

implies $x [q_i] y \Rightarrow x' [q_i] y'$
 $\exists x' q_i y'.$

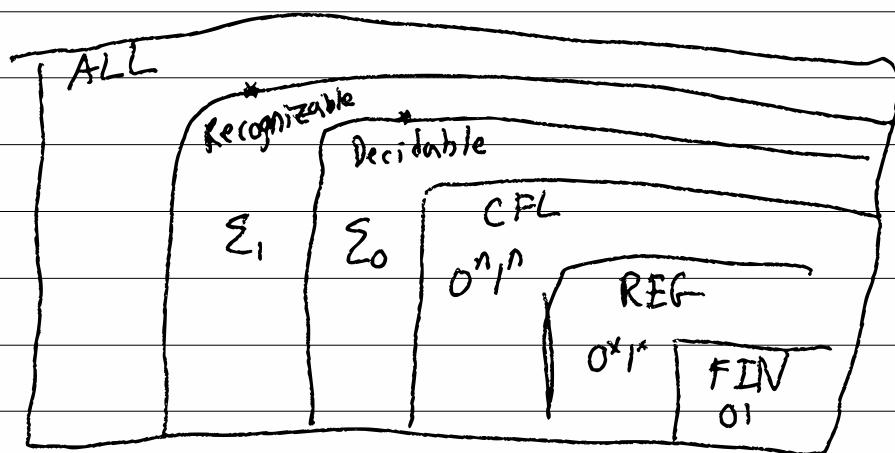
18-3 / A TM is either a

recognizer — may LOOP on
some input

decider — always ACC or REJ
never LOOP

a recognizable language, A , means $\exists m \in \text{recognizers}$
 $L(m) = A$

a decidable language, B , means $\exists m \in \text{deciders}$
 $L(m) = B$



Cog

CoC

CiC

(8-4) A Turing enumerator ...

DFA_s

REX_s

PDA_s

CFG_s

TM_s

enumerators

$(Q, \Sigma, \Gamma, q_0, \delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}, g_{P \in \overline{A}})$

really Q
not $Q - \{q_0\}$

$w \in L(e)$ iff $e[q_0] \xrightarrow{\epsilon} \star \xrightarrow{g_p} w$

"recognizer" enumerator \Rightarrow Outback

"decider" enumerator \Rightarrow shortest-to-longest
(lexicographic)

18-5) DFA union

NFA union

R&G v

\xrightarrow{NFA}

$(Q_x, \Sigma, \delta_{0x}, \delta_x, F_x)$

$(Q_y, \Sigma, \delta_{0y}, \delta_y, F_y)$

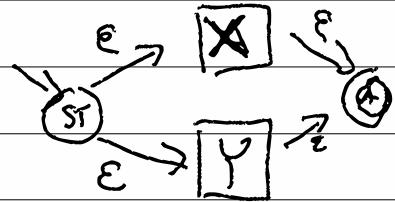
\Rightarrow Pair $\langle Q_x, Q_y \rangle$

$(Q_x \times Q_y, \Sigma)$

$(\delta_{0x}, \delta_{0y})$,

$$\delta((q_x, q_y), c) = (\delta_x(q_x, c), \delta_y(q_y, c))$$

$Q_x \times F_y \cup F_x \times Q_y$)



$(Q_x, \Sigma, \delta_{0x}, \delta_x, F_x)$
 $(Q_y, \Sigma, \delta_{0y}, \delta_y, F_y)$

$\emptyset = \{\epsilon \text{ START}, \text{Acc}\}$ ~~if $c \in \emptyset$~~
 $\cup \quad \epsilon 03 \times X$
 $\cup \quad \epsilon 13 \times Y$

$$\delta(\text{START}, \epsilon) = \{ (0, \delta_{0x}), (1, \delta_{0y}) \}$$

$$c = \emptyset$$

$$\delta(\text{ACC}, \epsilon) = \emptyset$$

$$\delta((0, q_x), c) = \{ 03 \times \delta_x(q_x, c) \cup$$

(if $q_x \in F_x$ then Acc) if $c = \epsilon$)

interface NFAUnionState $\langle X, Y \rangle$

NUS_START () = $\langle \text{nat} \rightarrow X \text{ or } Y \rangle$

NUS_ACC ()

NUS_FromX (X)

NUS_FromY (Y)

19-1/ Closure properties

A set A

and operation $f : A \rightarrow \underline{A}$

" A is closed under f "

The regular languages are closed under

$C, \cup, \cap, \circ, ^*$

The CFLs

$\cup, \circ, ^*$

What are Σ_0 (decidable)

Σ_1 (recognizable) languages
closed under?

Is Σ_0 closed under complement? ✓

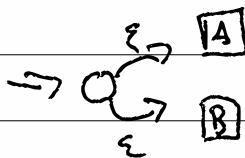
Σ_1

19-2 / Union

DFA_s $Q = Q_A \times Q_B$

(did both at same time)

NFA_s



TMs — can't do NFA (deterministic)

$$\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

$Q \rightarrow Q_A \times Q_B$ tape \rightarrow two tapes

TMs — simple (like assembly) DFA_s

↑

↑

↑

MTMs

C

NFA



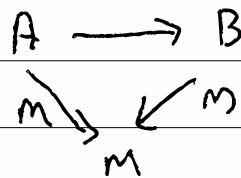
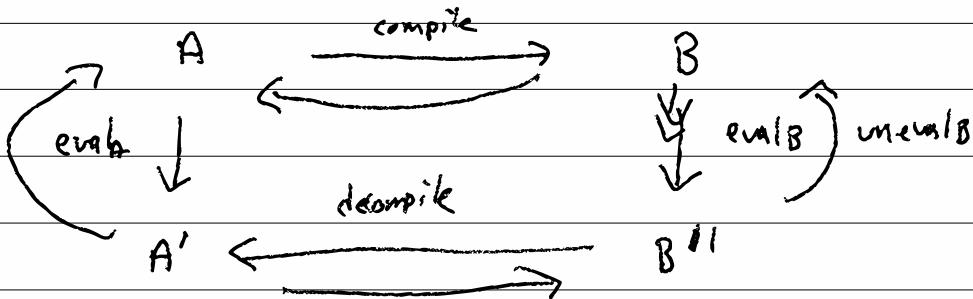
x 86

19-3/ Compiler correctness

$\forall A \in \text{input. } \exists B \in \text{output. } m(A) = m(B)$

$\xrightarrow{\text{C, assembly}}$ $\xrightarrow{\text{asm, binary}}$
 NFA DFA

bi-simulation



Galois connection

(a-y) TM w/ "stay"

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

$$\delta_{SPTM} : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

$$\delta(q_i, a) = (q_j, b, S)$$

$$u[q_i]av \Rightarrow u[q_j]bv$$

$$\delta(q_i, a) = (q_j, b, S)$$

$$\Rightarrow$$

$$(q_k, c, L/R)$$

$$\delta(q_i, a) = (q_{aj}, b, R)$$

where $\delta(q_i, b) = ?$

$$u[q_i]av$$

$$ua[q_{almost j}]v$$

$$\leftarrow$$

$$u[q_i]bv$$

$$\forall x \in \Gamma$$

$$\delta(q_{aj}, x) = (q_j, x, L)$$

19-5) MTMs (exist to do v/n)

$$\delta_A : Q_A \times \Gamma_A \rightarrow Q_A \times \Gamma_A \times \{L, R\}$$

$$\delta_B : Q_B \times \Gamma_B \rightarrow Q_B \times \Gamma_B \times \{L, R\}$$

$$\delta_{A \cup B} : (Q_A \times Q_B) \times \Gamma_A \times \Gamma_B$$

$$\rightarrow (Q_A \times Q_B) \times (\Gamma_A \times \{L, R\}) \times (\Gamma_B \times \{L, R\})$$

$$\delta_{A \cup B} ((q_a, q_b), (t_a, t_b)) =$$

$$\text{let } (q'_a, t'_a, d'_a) = \delta_A (q_a, t_a)$$

$$(q'_b, t'_b, d'_b) = \delta_B (q_b, t_b) \text{ in}$$

$$((q'_a, q'_b), (t'_a, d'_a), (t'_b, d'_b)) \text{ if } q'_a = \text{Acc}$$

then Acc

Multi-tape Turing Machine

$$\delta : Q \times \Gamma^k \rightarrow Q \times (\Gamma \times \{L, R\})^k$$

$$\delta(q_i, a, \alpha) =$$

$$(q_j, (b, L), (\beta, R))$$

$$\begin{matrix} u \\ x \end{matrix} \left[\begin{matrix} q_i \\ \alpha \end{matrix} \right] \begin{matrix} a \\ v \end{matrix} \Rightarrow \begin{matrix} u \\ x \beta \end{matrix} \left[\begin{matrix} q_j \\ \beta \end{matrix} \right] \begin{matrix} cbv \\ y \end{matrix}$$

19-6 MTM config $\xrightarrow{\text{complete}}$ STM config

eval \downarrow $\Gamma = \{\#\} \cup$ \downarrow
 step/ \Rightarrow $\cup \Gamma_A \cup (\Gamma_A \times \Sigma^*)$ \downarrow
 new config $\cup \Gamma_B \cup (\Gamma_B \times \Sigma^*)$ new s config

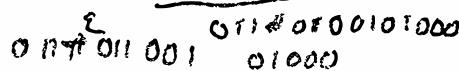
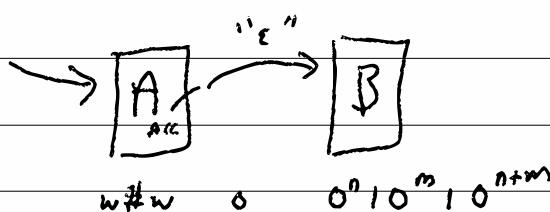
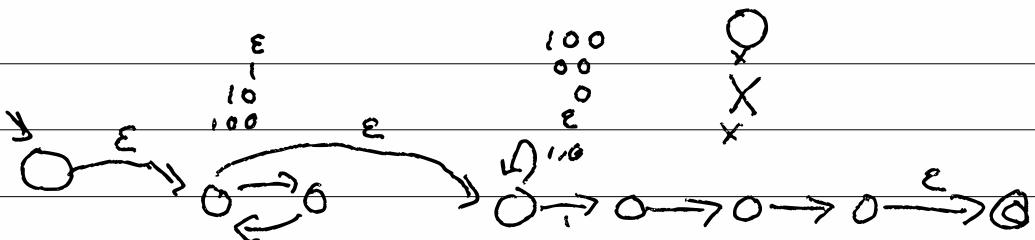
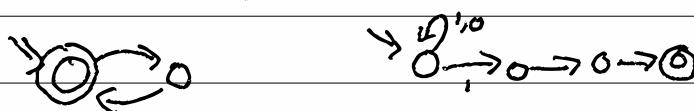
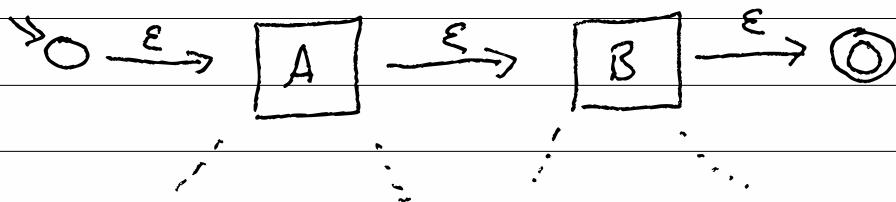
$a \in [q_i]_{\alpha y}$ $\xrightarrow{\quad}$ $[q_i]_{\alpha y} \neq x \circ y$
 x (q_i, α) (q_i, α, α)
 \downarrow $(q_i, (b, L))$ $(q_i, (L, L), (B, R))$
 $a \in [q_i]_{\beta y}$ \leftarrow (q_i)
 $x \in [q_i]_{\beta y} \neq x \circ y$

Σ_0 is closed under \cup and \cap ✓
 Σ_1 ✓



20+1 $\Sigma_0 : C, n, v, \circ, *, \epsilon$
 $\Sigma_1 : \square, n, v,$

$f(g)$ is a comp fun & $g(x)$ is a comp
 $f(g(x))$ is comp



20-2) Given TM M for lang A $(L(M) = A)$
N for lang B

Then

type 0: concat (M, N) \Leftarrow Given input x,
input (x) split input into $x = uv$ in all possible
ways (nondeterministically) try next

type 1: u run M(u) (if REJ, then REJ)

type 2: v run N(v) (if REJ, then try next)

type 3: sim type not Acc if both Acc

not REJ

ϵ abc - REJ

a bc - M.ACCEPT

ab c - N.Loops

abc ϵ - Acc

$\Sigma_0 : 0 \rightarrow \checkmark$

$\Sigma_1 : 0 \rightarrow \checkmark$

Z03/ Non-deterministic Turing Machine

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

$$\text{Non-det} : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{L, R\})$$

$$2\text{-choices} : Q \times \Gamma \rightarrow (Q \times Q) \xrightarrow{\text{fork()}} \begin{cases} L \\ R \end{cases} \Rightarrow \text{rel(config)}$$

$$\underbrace{\delta(q_i, a) = (q_j, b, R)}_{x[q_i]ay \Rightarrow x[b]q_jy} \xrightarrow{(a, b)} \begin{cases} \omega \\ R \end{cases} \quad P(\text{config}_q, \text{config})$$

$$\underbrace{\delta(q_i, a) = (q_j, q_k)}_{x[q_i]ay \Rightarrow x[q_j]ay} \xrightarrow{(a \in h)} \begin{cases} 0=0 \\ x=y \\ a=b \\ ax=by \\ a(x+y)=b(y+ax) \end{cases}$$

= : rel (num-sentences)

$$0=0 \quad \begin{matrix} a \neq 0 \\ x \neq y \end{matrix} \quad \frac{x=y}{a+b \neq 0+y}$$

$$\begin{matrix} x=y \\ a=b \\ ax=by \end{matrix} \quad \begin{matrix} a \neq b \\ x \neq y \end{matrix} \quad \frac{a=b}{a(x+y)=b(y+ax)}$$

$$\underbrace{\delta(q_i, a) = (q_j, q_k)}_{x[q_i]ay \Rightarrow x[q_k]ay} \xrightarrow{(1 \in h)} \text{"oracle"-model}$$

↔ ↔ ↔

reject if

$\Rightarrow^x []$

20-4] Forking semantics

F-config = Sequence of config

acc if

for input w

F-config₀ = $["\varepsilon [q_0]w"]$

$\Rightarrow^* ["x[q_0]y", c_2 \dots c_n]$

$$S(q_i, a) = (q_j, b, R)$$

$["x[q_j]y", c_2 \dots c_n]$

$$["x[q_i]ay", c_2 \dots c_n]$$

\Rightarrow

$[c_2 \dots c_n]$

$$[c_2 \dots c_n, "xb[q_j]y"]$$

$$S(q_i, a) = (q_j, q_k)$$

$$["x[q_i]ay", c_2, \dots, c_n]$$

$$\Rightarrow [c_2, \dots, c_n, "x[q_i]ay", "x[q_k]ay"]$$

$[c_1, \dots, c_n]$

Fconfig₀ \longrightarrow

config

$c_1 \# c_2 \# \dots \# c_n$

\downarrow
run

\downarrow
 $x[q_i]y$

\wedge $\Rightarrow x[q_j]y$ $\Rightarrow Q$

Fconfig₁ \leftarrow

config

$\Gamma' = \Gamma \cup Q$

O \rightarrow ND \rightarrow 2ND \rightarrow BT

\rightarrow Forking

✓

20-5/

Acc

$$- \quad \delta(q_i, a) = (\text{qAcc}, \cup, \omega)$$

REJ

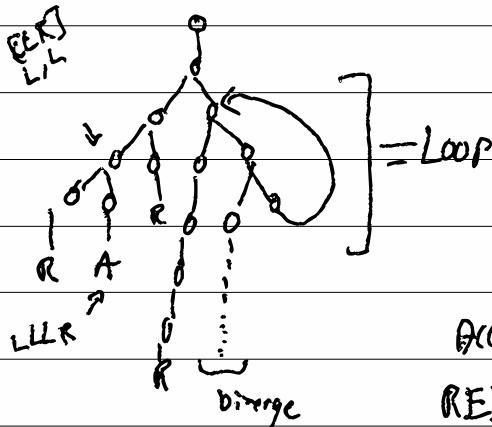
$$- \quad \delta(q_i, a) = (\text{qREJ}, \cup, \cup)$$

?

$$- \quad \delta(q_i, a) = (q_j, b, L/R)$$

?

$$- \quad \delta(q_i, a) = (q_j, q_k)$$



$$\text{map} = \{LR\}^*$$

explain in

lexicographic
order

Acc if some map leads to Acc
REJ if entire "level" (length) is REJ

if ans on level k $2^k + 2^{k-1} + 2^{k-2} + \dots = \sum_{i=0}^k 2^i$

$$k2^k$$

tape 0: w

tape 1: current map \rightarrow add 1

tape 2: simulation tape

20-6

0 \leftarrow B

$w \in A^*$ iff

\downarrow

$w = w_1 \dots w_n$

ND

where $w_i \in A$

\downarrow
2 ND

\swarrow \searrow
fork BT

\swarrow \leftarrow
MT

\downarrow
TM