1-1

How do we know if a math formula is true?

How do we know if an algorithm (like Euclid's GCD) works?

correct   effective

Does an algorithm exist?

What is an algorithm?

Does a program exist? ⟵ problems

What is a program? ⟵ models

1-2] A set is "a bunch of stuff"

$\emptyset$ – nothin' in it

$\forall x, x \notin \emptyset$

$\{$pen, phone$\}$ $\{$phone, pen$\}$

$\{$ ✓ ,  $\}$

pen $\in$ $\{$pen, phone$\}$

$\forall x,\ x \in \{y\}$ iff $x = y$

union – $\cup$ $\quad$ $\cup$ $\cup$

$\forall x.\ x \in A \cup B$ iff $x \in A$ or $x \in B$

$\{$pen, phone$\} = \{$pen$\} \cup \{$phone$\}$

1-3) "The set of all true math formulas"

A set IS its membership

"$1+1=2$" $\in$ TS ↑ ?

"Is there a god?"

"Will Buffy be remake?"

All sets "constructed" via $\emptyset$, $\{x\}$, $\cup$ are finite.

$x \in$ {  } $\cup$ {  }

The <u>Universe</u> ($\mathcal{U}$)

$A \subseteq B$ (subset) iff $\forall x,\ x \in A \Rightarrow x \in B$

1-4) Our universe is made of strings
and strings are sequences of characters
and chars are elements of an alphabet
an alphabet is a finite set

$\Sigma = \{0, 1\}$ $\{0, 1, \cup, \$, +\}$

chars chars

"0100001" = a string = s

length = 7 $s(0) = 0$ $s(1) = 1$ $s(2) = 0$

special notation

$U = \Sigma^*$

$A^* = \{\varepsilon\} \cup A \circ A^*$

epsilon = "" = the string w/ no
characters

$x \in A \circ B$ iff $x(0) \in A$ and
$x(1 \ldots) \in B$

$\{0,1\} \circ \{0,1\} = \{00, 01, 10, 11\}$

$\{1\} \circ \{0\} = \{10\}$

L5/ #1. Decide a data type to represent alphabets and characters.

Alphabet = List <Character>

Character = Object / void*

we need equality

#2. Decide a data type for strings

```
interface String { }
class MtString implements String { ... }
class OneString impl String {
    OneString ( Char c, String s ) { ... } }
Zero = new BasicChar('0');   One = new BC('1');
010 = new OneS( Zero, new OneS(One, new
          OneS( Zero, new MtS())));
```

$\{\Sigma\}$

1-6/ Every alphabet has a lexicographical ordering of the strings in Σ^*

$\Sigma = \{0, 1\}$

" $= B$

new List(Chars)

{ Zero, One};

0	1	2	3
ε	0,	00,	000
	1,	01,	001
		10,	010
		11,	011,
			100
			101
			110
			111
2^0	2^1	2^2	2^3

$|A|^i$ where i = layer I'm looking at

lexi : $\Sigma \times N \to \Sigma^*$

lexi B 0 = ε

lexi B 1 = 0

lexi B 2 = 1

lexi B 6 = 1 1

2-1 "1+1" $\longrightarrow$ "2"

"1+1 = 2" $\in$ Truth

"1+1 = 3" $\notin$ Truth

$\emptyset$ $\quad \{x\}$ $\quad A \cup B$

Alphabet Σ $\qquad$ Universe Σ^*

$\{0, 1\}$ $\qquad$ $\{$ ε, 0110, 000001,  $\}$

$\mathbb{P}(A)$ $\quad 2^A$

$x \in P(A)$ iff $x \subseteq A$ $\quad$ ($x \subseteq A$ iff $\forall y \in x.\ y \in A$)

$A = \{0, 1, 2, 3\}$

$\emptyset \in P(A)$ $\quad \emptyset \subseteq A$ $\qquad$ 0110 $\quad \{1,2\}$

$\{0\} \in$ " $\quad \{2,3\} \in$ $\quad \{0,2,3\}$ $\quad \downarrow$ $\underline{0}12\underline{3}$

$P(\Sigma^*)$ $\qquad \Sigma^* = \{$ ε, 0, 1, 00, 111111 $\}$ ....

$\emptyset \in P(\Sigma^*)$

$\{\varepsilon\} \in P(\Sigma^*)$ $\qquad$ 0011, ...

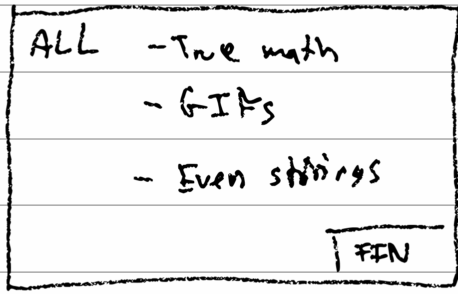
all even lengthed strings $\in P(\Sigma^*)$ $= \{$ε, 00, 11, 01, $\}$

GIFS $\in P(\Sigma^*)$ $\qquad$ {GIFs of me} $\in P(\Sigma^*)$

JPGs w/ a cat in them $\in P(\Sigma^*)$

2-2/ ALL $= P(\Sigma^*)$

FIN = the set of finite sets



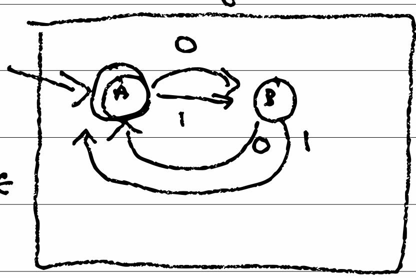
$\emptyset \in$ FIN

$\forall x \in \Sigma^*.\ \{x\} \in$ FIN

$A \in$ FIN $\wedge$ $B \in$ FIN $\rightarrow A \cup B \in$ FIN

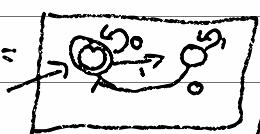
All even strings =

$x \in$



ε 01 11

even nums =



DFA - a deterministic finite automata

○ - states $\{A, B\}$

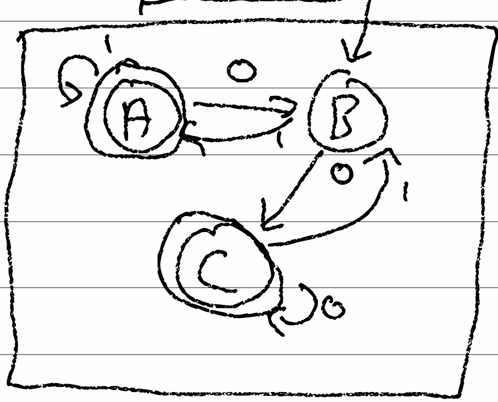
→□ - start state A

◎ - accepting states $\{A\}$

$\circ \xrightarrow{x} \circ$ - transition

x - labels are Σ

	0	1
A	B	B
B	A	A



$\{A, B, C\}$ B

$\{A, C\}$ $\Sigma = \{0, 1\}$

	0	1
A	B	A
B	C	A
C	C	B

2-3) $x \in DFA\,(\underbrace{Q}_{\text{states}},\ \underbrace{\Sigma}_{\text{alphabet}},\ \underbrace{q_0 \in Q}_{\text{start}},\ \underbrace{F \subseteq Q}_{\text{accepting}},$
$\delta : Q \times \Sigma \to Q$ – transitions)

DFA configuration = $\underset{[q]}{Q} \times \underset{w}{\Sigma^*}$

config update function : config × DFA → config

$[q]w \to [q']w'$

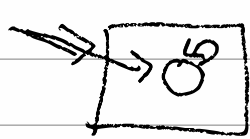
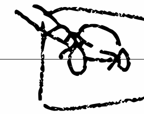
$[q_i]\,x\,y \to [q_j]\,y$ iff $\delta(q_i, x) = q_j$

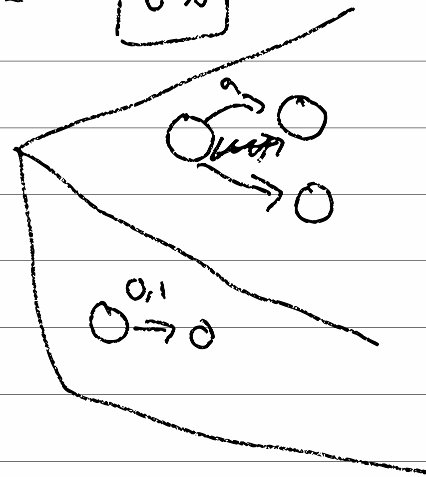
$x \in DFA$ iff $[q_0]x \to \to \to \to \to [q_f]\varepsilon$ and $q_f \in F$

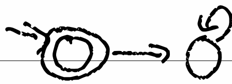
$0110 \in EvenLen$ iff $[A]0110 \to [B]110 \to [A]10 \to [B]0 \to [A]\varepsilon$ $A \in \{A\}$ ✓

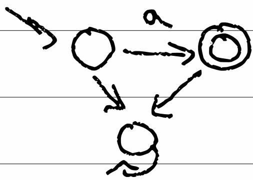
```
class DFA {
  ... Q, Σ, F, q0, δ ...
  public bool accepts (String x) {
    State qi = q0;
    while (!x.~~isEmpt~~ empty()) {
      qi = δ(qi, x.first());
      x = x.rest(); }
    return F.in(qi); } }
```

2-4

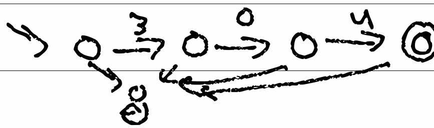
 $= \emptyset =$ 



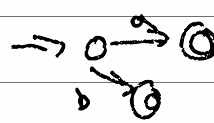
 $= \{\varepsilon\}$

 $= \{a\}$

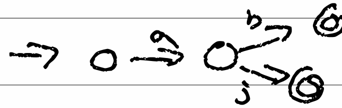
$=$ 

$\{304\} =$ 

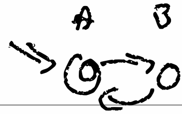
$A \cup B \in DFA$ (if $A \in DFA$ and $B \in DFA$)

$\{a\} \cup \{b\}$ 

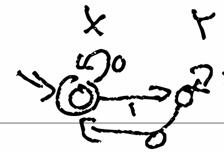
$\{a;\} \cup \{ab\}$



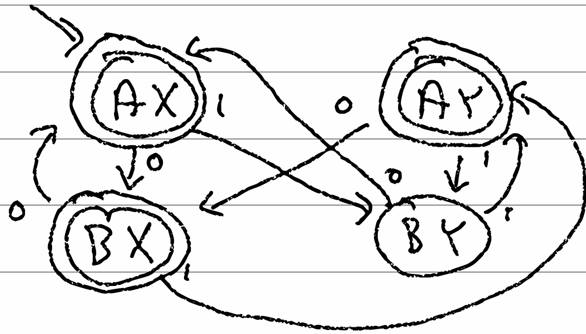
2-5 EvenLen



IsEven



$\cup$



ε ✓

00 ✓

11 ✓

0 ✓

110 ✓

$\cap$

$(x, y) \in A \times B$

iff $x \in A \land y \in B$

$A = (Q_A, \Sigma, q_{0A}, \delta_A, F_A)$

$B = (Q_B, \Sigma, q_{0B}, \delta_B, F_B)$

$X = A \cup B$

$Q_X = Q_A \times Q_B$

$q_{0X} = (q_{0a}, q_{0b})$

$F_X = F_A \times F_B$ — $\cap$

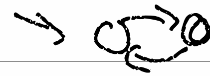
$F_A \times Q_B \cup Q_A \times F_B$ — $\cup$

$\delta_X((q_a, q_b), c) = (\delta_a(q_a, c), \delta_b(q_b, c))$

$x \in A \cap B$ iff $x \in A \land x \in B$

2-6/ $x \in A^c$ iff $x \notin A$ $(x \in U)$

Even Len | odd len

 $\Rightarrow$ 

$F = \{A\}$ | $F' = Q - F$

or F^c (wrt Q)

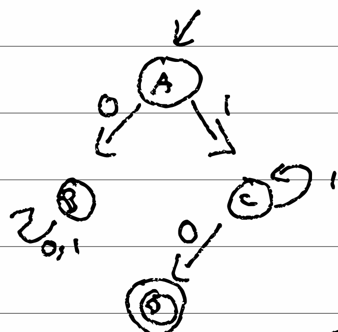
complement

Algorithm for $X \subseteq Y$ if X & Y are DFAs

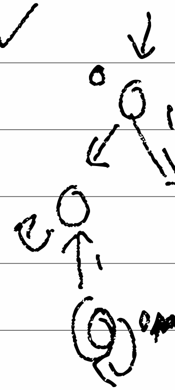
3-1 // DFA ⇒ example or false

DFA:

- Q: State → Bool
- Σ: list of characters
- q_0: state
- δ: (state × char) → State
- F: State → Bool



10 ✓



Q	$\emptyset$	q_0
$\{A, B, C, D\}$	$\{\}$	[A]
$\{B, C, D\}$	$\{A\}$	[B, C]
$\{C, D\}$	$\{A, B\}$	[C]
$\{D\}$	$\{A, D, C\}$	[D]
$\{\}$	$\{A, B, C, D\}$	[ ]

A ⇒ ε
B ⇒ A, 0
C ⇒ A, 1
Yes, it's possible.'
D ⇒ C, 0

or if not, No!

10

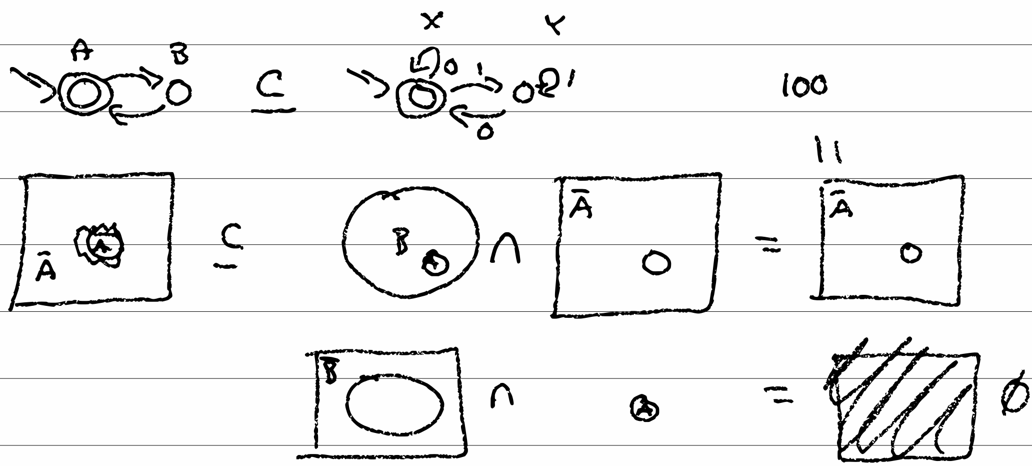
3-2/ subset

$A \subseteq B$ iff $\neg\, \forall x \in U.\ x \in A \Rightarrow x \in B$

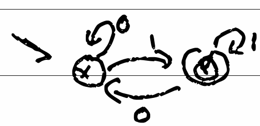
$\exists x \in U.\ x \in A \wedge x \notin B$

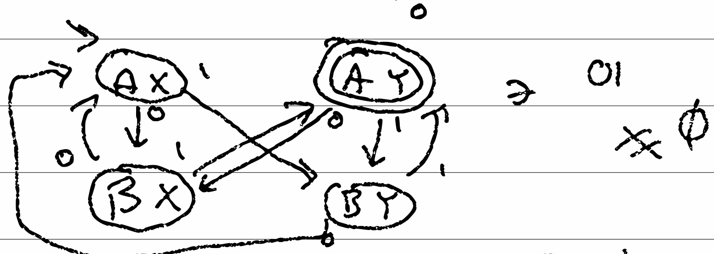
$\{a, b\} \subseteq \{a, b, c\}$ $\quad U = \{a, b, c\}$

finite means naive works!



$\overline{\text{EvenNum}} =$





model checking

Soundness: model $\subseteq$ theory

completeness: theory $\subseteq$ model

model = theory

3-3/ $0, 1, 2, -1, 5$ Z, P, N

$\{P\} + \{N\} = \{P, Z, N\}$

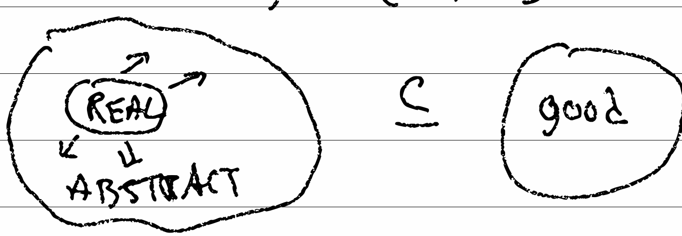
if $x > 0$ then

A $\quad y = 5 \Rightarrow \{P\}$

o.w

B $\quad y = 0 \Rightarrow \{Z\}$

$\Rightarrow$ assume $y = \{P, Z\}$



3-4) Finite = $\boxed{\emptyset \quad \{x\} \quad A \cup B \quad A^c \quad A \cap B}$ $\in$ DFA $\quad A \circ B$

Infinite = A^*

$x \in \Sigma^* \wedge y \in \Sigma^*$ then $x \circ y \in A \circ B$ iff $x \in A \wedge y \in B$

$\varepsilon \circ y = y \qquad$ if $a \in \Sigma$, $(a \circ x) \circ y = a \circ (x \circ y)$

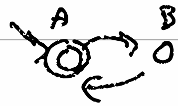
$abcd = a \circ b \circ c \circ d$

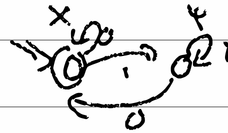
$\{j\} \circ \{m, n\} = \{jm, jn\}$

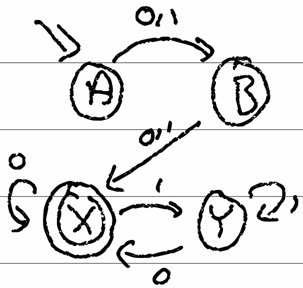
$x \in A^*$ iff $x = x_0 \circ x_1 \circ \ldots \circ x_n$ for $n \in \mathbb{N}$ and $x_i \in A$

$\{jm\}^* \ni \varepsilon,\ jm,\ jmjmjmjm$

3-5 / Even Len ∘ Even Num



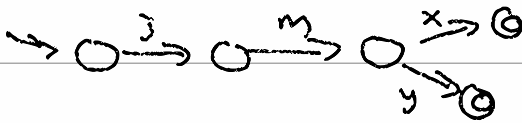


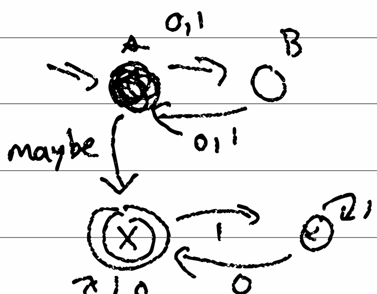


$\underbrace{00}\underbrace{110}$ ✓

0011 ×

$\{jm\} \circ \{x,y\}$

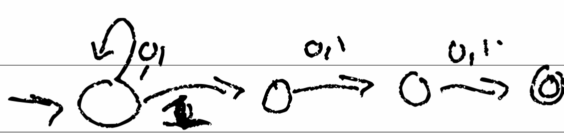




$\Sigma = \{0,1\}$

$x \in$ DFA iff

there is some path from q_0 to $q_f \in F$ labelled w/ x



"3rd from end is 1"

3-6/ NFA = non-deterministic
finite automata

old world: the next step was obvious
$\delta: Q \times \Sigma \Rightarrow Q$

new world: crazy options
- do you even read a char?
- which path do you take?

$\delta': Q \times \{maybe\} \cup \Sigma \rightarrow P(Q)$

$\delta'(A, 1) = \{A, B\}$

$\delta'(A, maybe) = \{C\}$

epsilon

ε Σ e ε

4-1

$A \circ B \in DFA$ iff $A \in DFA$

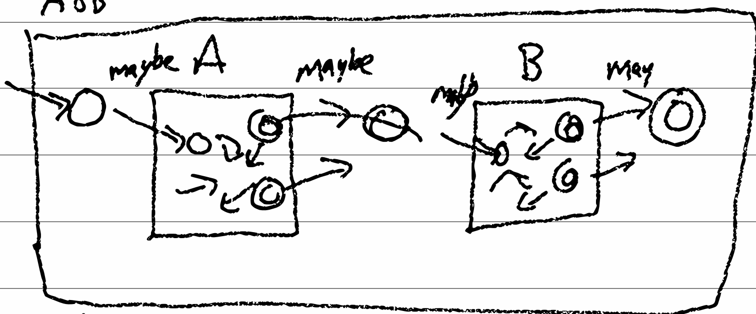
A^* $B \in DFA$

NFA (N - non D- deterministic)

$\delta : Q \times (\Sigma \cup \{maybe\}) \to P(Q)$

$\delta : Q \times \Sigma \to Q$

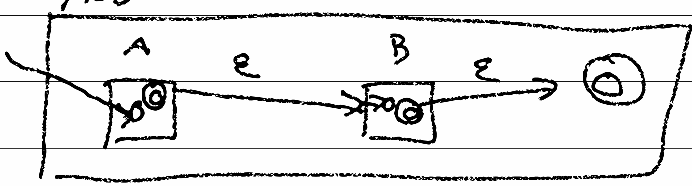
$A \circ B$



maybe written as "ε"

What does (NFA) this mean?

$A \circ B$



NFA $\Leftrightarrow$ DFA

4-2 | What do NFAs mean?

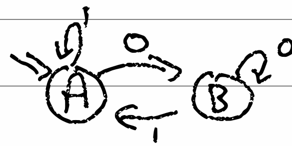
A DFA represents a set and a set "is" a membership function

$U \to \text{Bool}$

$= \Sigma^* \to \{Y, N\}$

config $= \Sigma^* \times Q$

$\Sigma^* \to Q^*$



0110 $\to \underbrace{ABAAB}$ $\to$ a trace

$\Sigma^* \to (\Sigma, Q)^*$

0110 $\to \underbrace{(0,B)\,(1,A)\,(1,A)\,(0,B)}$

Σ_e ~~make~~ $= \Sigma \cup \{\epsilon\}$

0A 1A 1A 0B $\to$ N

valid? : $Q \times (\boxed{\Sigma}, Q)^* \to \{Y, N\}$ DFA

valid q_i ε = Y

valid q_i (c, q_j) : more = if $\delta(q_i, c) \boxed{==} q_j$ valid q_j more

o.w. N

Nvalid? : $Q \times (\boxed{\Sigma_e} \times Q)^* \to B$

Nvalid q_i ε = Y

Nvalid q_i (c, q_j) : more =

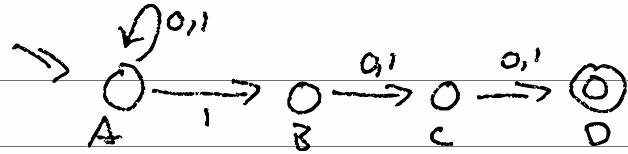
if $\boxed{q_j \boxed{\in} \delta(q_i, c)}$ then

Nvalid q_j more

o.w. N

An Oracle

4-3/



"3rd letter from end is 1"

0100	111	110100	— Y
000	1000	1011	— N

(0, A)(1, A)(0, A)(0, A) ✓ $\quad$ str $(\Sigma \times Q)^* \to \Sigma^*$

(0, A)(1, B)(0, C)(0, D) ✓ $\quad$ str $\varepsilon = \varepsilon$

$\underbrace{(0, B)}_{\delta(A,0) = \{A\}}$ (1, C)(1, D) $\underbrace{(0, D)}_{\delta(D,0) = \emptyset}$ X $\quad$ str $(c, \_)$::more $= c \circ$ str more

accepts : $\Sigma^* \to$ Y/N

accepts w = Y iff $\exists t \in$ traces. str$(t) = w$

valid q_0 t = Y

and last-state$(t) \in F$

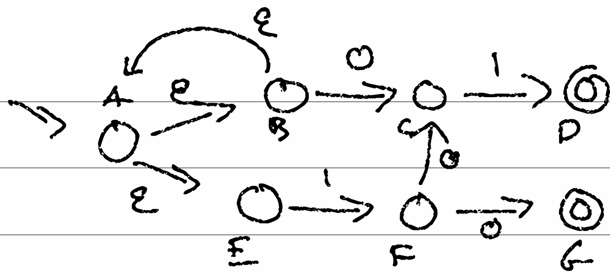
NFA-accepts : $\Sigma^* \to$ Y/N

figure all possible traces

check if valid and is strings match

check if last is in $\in F$

4-4



(ε, B) (0, C) (1, D) 01 = ε0001

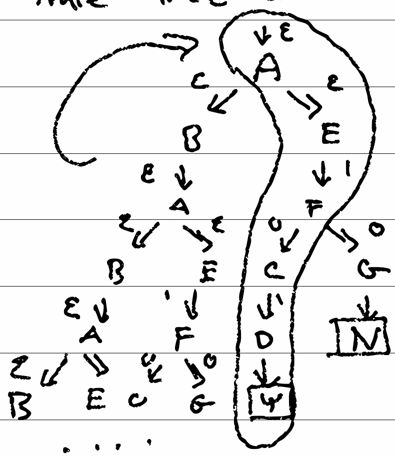
(ε, E) (1, F) (0, C) (1, D) 101 = ε010001

(ε, E) (1, F) (0, G) 10 = ε0100

(ε, B) (ε, A) x where x is valid

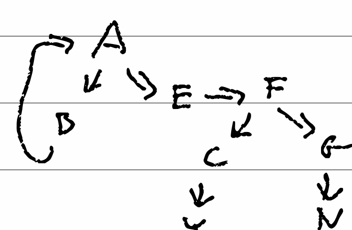
⇒ valid

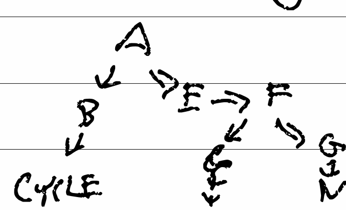
Trace Tree = Y | N | Branch (Σ, Q) (List TT)



101

ε,E 1,F 0C 1D



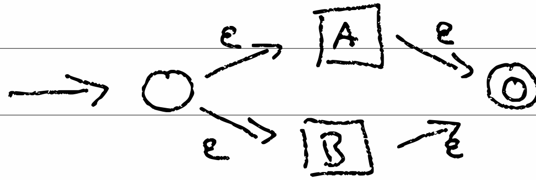


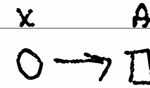
Forking model of NFAs (make TT)

Back-tracking model (explores TT)

4-5/

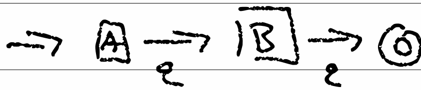
$A \cup B$





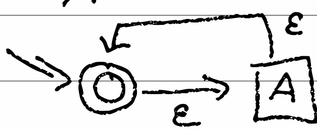
state X transitions to THE start state of A

$A \circ B$



All accepting states of A transition to X

A^*

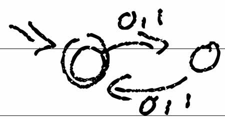


4-6/ $\forall A.\ A \in DFA \iff A \in NFA$

$\Rightarrow$

$\Leftarrow$

DFA $\Rightarrow$ NFA



DFA $\delta: Q \times \Sigma \to Q$

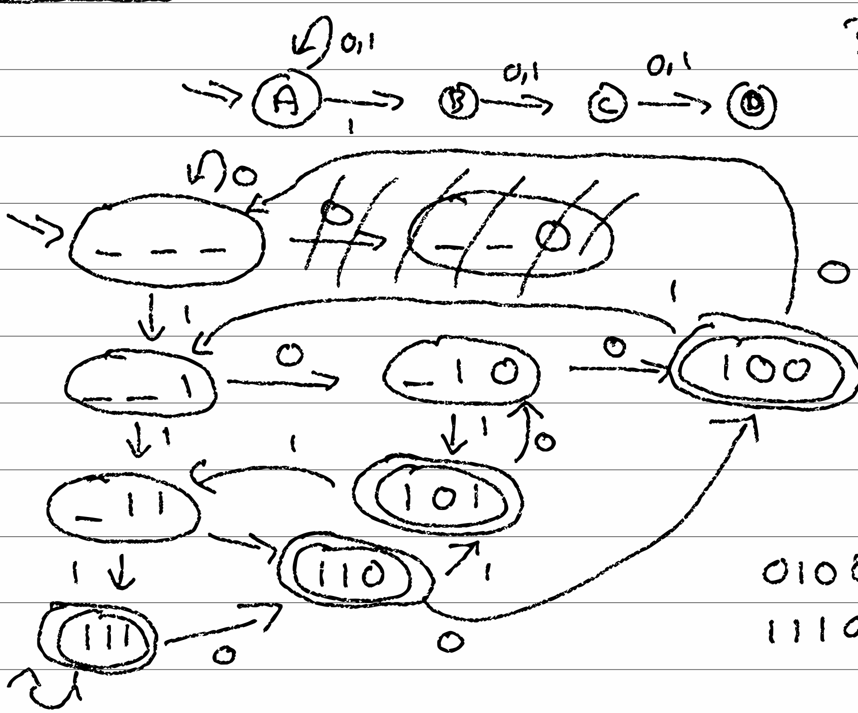
NFA $\delta': Q \times \Sigma_\varepsilon \to P(Q)$

$$\delta'(q_i, \varepsilon) = \emptyset$$

$$\delta'(q_i, c \in \Sigma) = \{ \delta(q_i, c) \}$$

NFA $\Rightarrow$ DFA

3rd is 1



0100

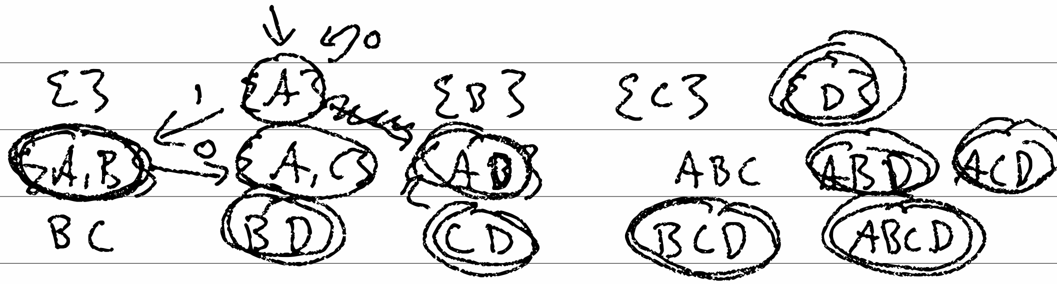
111001

4–7/

in NFA $= (Q, \Sigma, q_0, \delta: Q \times \Sigma_\varepsilon \to P(Q), F)$

out DFA $= (Q', \Sigma, q_0', \delta': Q' \times \Sigma \to Q', F' \subseteq Q')$

$Q' = P(Q)$



$q_0' = \{q_0\}$

F' = any state where $\cap F \neq \emptyset$

$\delta'(\{q_i \ldots q_n\}, c) =$
$\bigcup \delta(q_i, c)$