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EGR 191
Rocket Lab Final Report

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January 8, 2019

Abstract

For this experiment, I assembled an accelerometer to collect rocket launch data, then, using only mathematical analysis for a rocket launch with known initial conditions, I was able to produce an accurate and reliable mathematical model that closely represents the actual data measured from the accelerometer in the rocket. Furthermore, I used the same model to produce equations and graphs for the position and velocity of the rocket in both the vertical and horizontal directions that correctly models experimental data to less than 5% error.

1. INTRODUCTION

In this lab, I learned how to operate and read oscilloscopes and function generators, how to understand, construct, and solder a circuit board, and how to collect data on those boards, such as clock frequencies and propagation delays, using an oscilloscope. I learned how to process accelerometer data in MatLab and how to improve it and convert it to actual acceleration using the calibration constant and offset. Additionally, I learned how to operate Mathematica to model real world experiments mathematically in order to accurately describe those experiments. Finally, I learned how to systematically improve mathematical models in increasing detail to make them more accurate.

2. EXPERIMENTAL APPARATUS

2.1 Capabilities

The accelerometer assembly used to measure data during my rocket launch has the capacity to measure linear acceleration along three perpendicular axes using an accelerometer chip mounted on the circuit board, which contains small test masses suspended by springs, outputting accelerometer data based on the displacement of the springs. The accelerometer measures acceleration up to a maximum of approximately 160 m/s^2, or 16g. The accelerometer converts the accelerometer data to a digital format using a 13-bit Analog-to-Digital converter, which can record 2^{13} = 8192 output values ranging from -4096 to +4095. Because the oscilloscope has only an 8-bit vertical resolution and the maximum range is 16g, the smallest step of resolution in the raw data output is 16/4095 = 0.0039g.

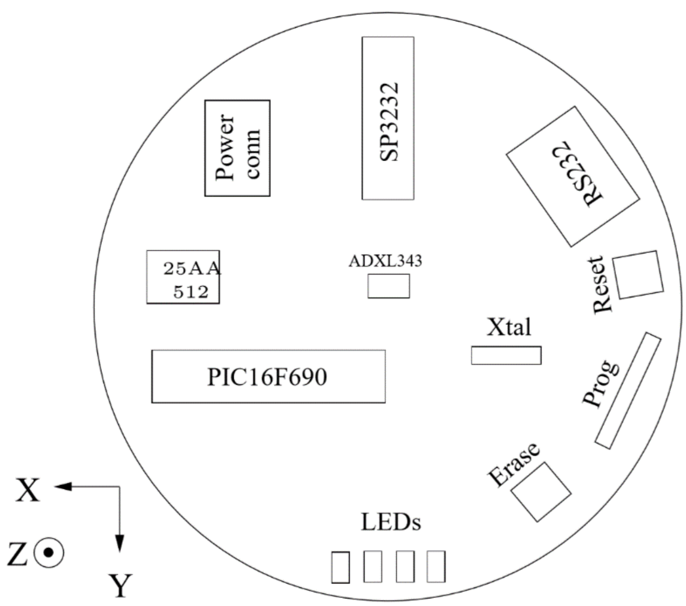


Figure 1: Layout of accelerometer with components labelled and coordinate system indicated. (From EGR Rocket Lab 3)

2.2 Hardware

The accelerometer contains a MicroChip PIC16F690 microcontroller that sends data from the accelerometer to memory or a computer and contains a highly accurate clock. The microcontroller contains both flash memory, used to store the program to be executed, and RAM, which stores intermediate results during calculations. It also contains timers and multiple input/output ports. The board also contains an additional MicroChip memory 25AA512 memory chip, which holds up to 2^{19} = 524,288 bits of data. The memory is recorded into two separate buffers, allowing for two rounds of data of approximately 65 seconds each before needing to be transferred to the computer.

The accelerometer uses serial communication through just one wire to send information, which requires a clock. The board uses a 2 MHz clock to communicate on the board and a 115.2 kHz clock to transfer data from the board to a computer. The board also contains a highly accurate Quartz Crystal oscillator to keep time for the board by mechanically oscillating at precisely 32,766.4 Hz, which differs a mere 0.01% from the official manufacture frequency of 32,768 Hz.

2.3 Stability

The calibration constant, C, of the accelerometer is more stable than the accelerometer offset, B, varying by a maximum of just 0.05% over a 10 minute interval. The offset, by comparison, differed by just 1.1% over the same 10 minute interval, remaining stable to 0.69 m/s^2. This variance in B, however small, is nonetheless a more significant variation than that observed for the calibration constant. Because of the inherent drift in the offset, calculated distance graphs based on acceleration also drift by 0.04 meters per second of recording data when used to record distance travelled.

3. Data Collection

3.1 Experiment

The main experiment consisted of the construction and launch of a simple water rocket, which housed an accelerometer. The rocket itself was built from an emptied soda bottle, three cardboard wing flaps, and a nose cone, as well has multiple foam components and packing peanuts used to protect the accelerometer. The rocket was filled with 0.5 L of water, pressurized to 55 psi = $4.7*10^5$ Pa, and placed on a lubricated launch tube. Upon launch, I pulled the arming pin in the accelerometer to begin recording data, then released the valve to initiate launch. After the rocket landed, I ran to it and repeatedly held it right-side-up then inverted for the remaining time before the buffer filled, about 65 seconds, to acquire stationary rocket data used to calibrate the accelerometer.

3.2 Calibration

Calibration is necessary to produce real accelerations from raw output, and involves solving for the offset and calibration constant in the accelerometer calibration equation (See Appendix A.1),

$$S = C * a + B,$$

where S is the accelerometer output, C is the calibration constant, a is the acceleration, and B is the offset, then using the equations and the known constants to convert accelerometer output to acceleration in m/s^2. To calculate the offset, I took the average of the right-side-up output and the inverted output. If there were no offset, the average of these would be zero, so the difference between the actual average and zero is the offset. Once B is known, I observed the right-side-up data, during which a = 9.8, and simply solved for C. To calibrate the accelerometer once B and C are known, I simply subtracted B from the accelerometer output and divided by C. By applying this equation to every data point collected, I calibrated the entire launch, yielding actual accelerations.

4. Basic Physics Model

4.1 Overview

To start on a mathematical model to describe the rocket launch, I began with simple equations derived from Newton's Second Law, F_{net} = m*a, that can be solved analytically for all phases of the flight. This simpler model accounted for the forces of gravity, pressure, water thrust, air resistance in freefall, and change in the rocket's mass during water thrust, but neglected friction on the launch tube, the launch angle with respect to the vertical, air resistance during thrust, air pressure change during water thrust, and the air exhaust phase.

4.2 Phases

Phase 1 of launch describes the rocket as it moves up the launch tube with constant acceleration, pushed up by the force of air pressure difference between the inside the bottle on the tube and the atmosphere. Using Newton's Second Law, I derived (Derivation - See Appendix B.1) an equation describing the acceleration of the rocket during this phase,

$$\frac{dv}{dt} = \frac{P_0 a}{m_0} - g,$$

where $\frac{dv}{dt}$ is acceleration, P_0 is the air pressure, a is the area of the rocket nozzle, m_0 is the mass of the full rocket, and g is the acceleration due to gravity. Because all of these variables are constant for phase 1, the acceleration is therefore also constant over time, equaling 26.17 m/s^2. Phase one lasts 0.12 seconds and terminates when the rocket reaches the top of the tube, 0.22 m off the ground. In the basic model, this equation does not yet account for friction on the launch tube.

Phase 2 of the basic model describes motion driven by the water exhaust as the rocket accelerates upward. Beginning again with Newton's Second Law, I derived the rocket equation for thrust of constant velocity and change in mass (See Appendix B.2). Then, I solved for $\frac{dv}{dt}$ (i.e. acceleration) (See Appendix B.3) to produce a differential equation for velocity that Mathematica can solve,

$$\frac{dv}{dt} = \frac{u\frac{dm}{dt}}{m(t)} - g,$$

where $\frac{dv}{dt}$ is acceleration, u is the exhaust velocity, $\frac{dm}{dt}$ is the mass loss rate, and $m(t)$ is the mass as a function of time. From that differential, I produced equations for acceleration, velocity, and position in terms of t using Mathematica. In this model, phase 2 lasts 0.32 seconds, during which the acceleration increases from 46.6 m/s^2 to 172.8 m/s^2 and the rocket travels from a height of 0.22 m to 4.95 m. The basic model for phase 2 neglected air resistance, the launch angle of the rocket, and the entire air exhaust phase.

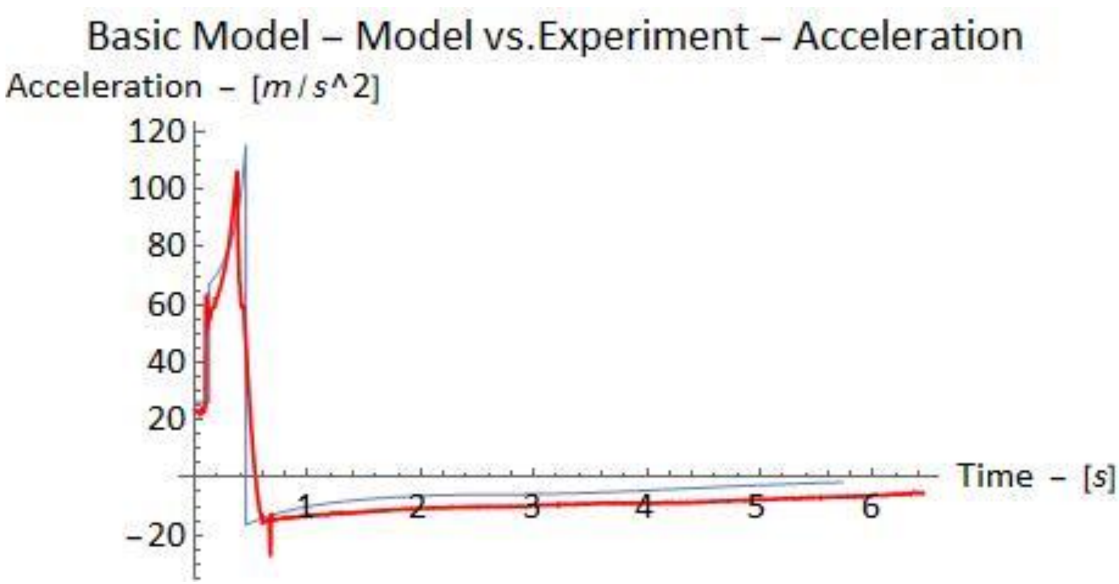
Phase 3 describes the motion of the rocket in freefall before reaching the maximum height, affected by air resistance, given by

$$F_d = \frac{1}{2} c \rho_{air} A v(t)^2,$$

and gravity, where c is the air drag coefficient, ρ_{air} is the density of air, A is the cross-sectional area of the rocket, and $v(t)$ is the velocity as a function of time. Thus, the differential equation for $\frac{dv}{dt}$ for this phase (Derivation - See Appendix B.4), also derived from Newton's Second Law, is given by,

$$\frac{dv}{dt} = -\frac{c\rho_{air}Av(t)^2}{2m_f} - g,$$

where m_f is the mass of the empty rocket. The force of air resistance, given by the first term in the equation, is determined by the air drag coefficient, the density of air, the cross-sectional area of the rocket, and the velocity. The force of gravity, the second term, is simply determined by the mass and the acceleration due to gravity. This model for phase three, however, assumes that the rocket flies directly perpendicular to the ground and does not account for the flight angle. This model's phase 3 lasts 2.25 seconds, during which time the acceleration approaches -9.8 m/s^2 from the negative direction. When the rocket reaches its maximum height and the velocity is zero, the acceleration is exactly g.



Key:

● – Experiment

● – Model

Figure 2:

Time vs. Acceleration graph in the y-direction for experiment and model

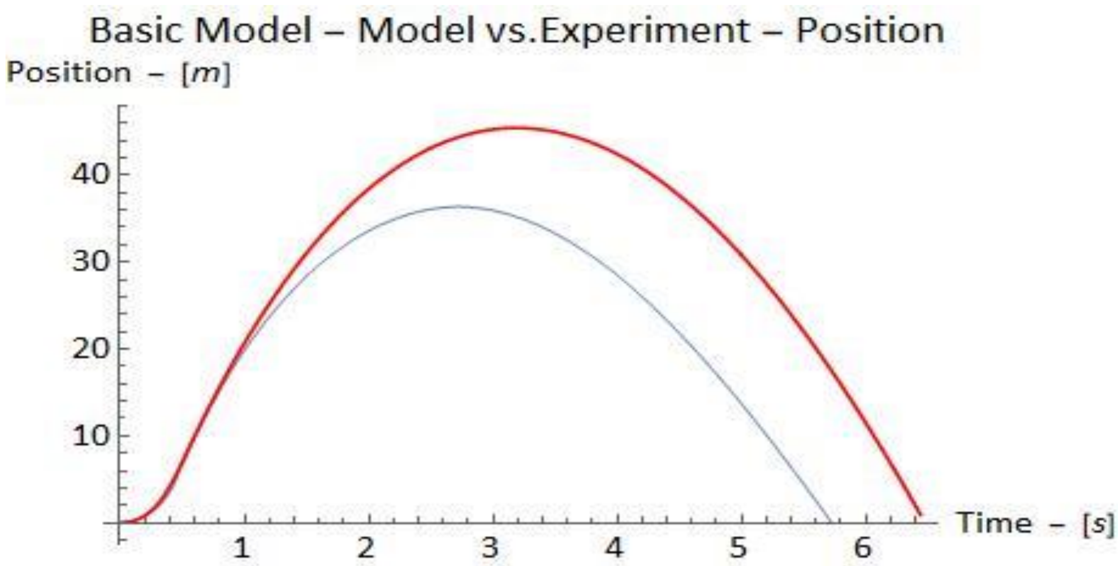


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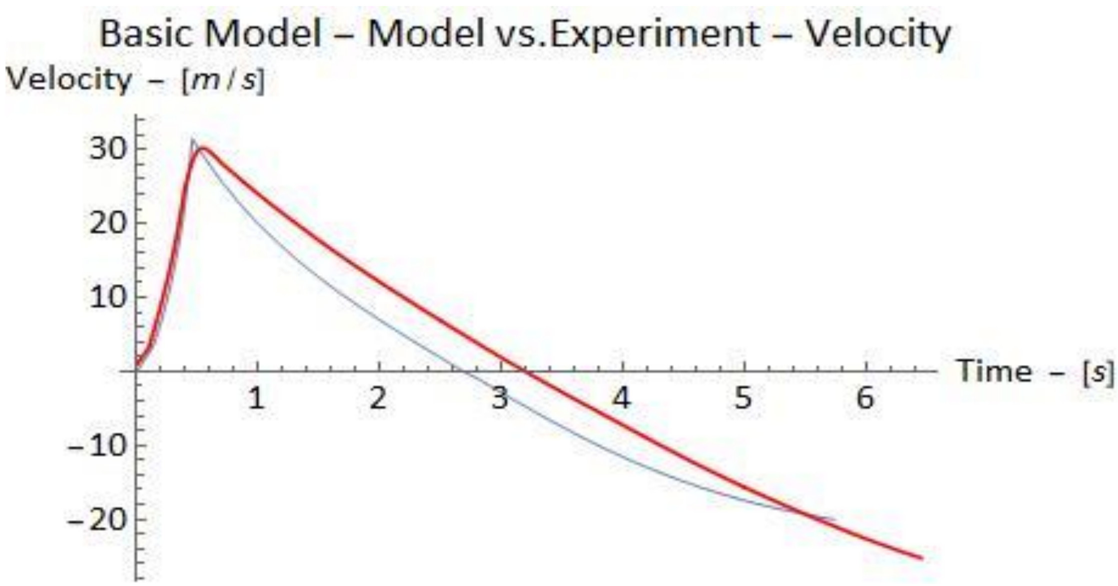
Figure 4:

Time vs. Position graph in the y-direction for experiment and model



Key:

● – Experiment

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Figure 3:

Time vs. Velocity graph in the y-direction for experiment and model

Phase 4 is very similar to phase 3, describing the rocket in freefall after passing the maximum height. The differential equation $\frac{dv}{dt}$ (See Appendix B.5),

$$\frac{dv}{dt} = \frac{c\rho_{air}Av(t)^2}{2m_f} - g,$$

also derived from Newton's Second Law, is identical to that of phase 3, except that the term describing the drag force is positive. In this simpler model, phase 4, like the others, neglects the flight angle and all horizontal motion. Phase 4 lasts 3.03 seconds. During the fall, the acceleration approaches zero from -9.8 m/s^2 as it nears terminal velocity.

4.3 Conclusion

The entire modelled rocket flight lasted 5.73 seconds and reached a maximum height of 36.4 m. Ultimately, however, these values are not accurate enough to adequately model the flight as this model neglects too many smaller factors which, when combined, will seriously improve the accuracy of the model. Discrepancies like the inconsistent acceleration in freefall (See Figure 2), offset of lines in the velocity figure (See Figure 3), and the dramatic difference in height reached (See Figure 4) all demonstrate that an improved model is necessary to accurately reflect the experiment. These differences are due to the neglected factors such as friction, flight angle, air exhaust, and drag force during thrust.

5. Complete Physics Model

5.1 Overview

In order to create a more accurate physics model for the experimental rocket flight, I reevaluated every equation from the basic model to incorporate friction on the launch tube, the launch angle relative to the vertical, the flight angle relative to the vertical, the air exhaust phase, and drag force during thrust. The complete model required the use of Bernoulli's equation for incompressible flow, Bernoulli's equation for compressible flow, and trigonometry, all of which improved but complicated the mathematical model for each phase.

5.2 Phases

Phase 1 saw the smallest change from the basic model, in which only friction was missing. Because the rocket's relationship to the launch tube does not have an easily defined normal force, the frictional force was assumed to be a small fraction of the overall weight of the rocket. This contribution, when added to the original differential equation, yields the final equation for $\frac{dv}{dt}$ for phase 1 (Derivation - See Appendix C.1),

$$\frac{dv}{dt} = \frac{P_0 a}{m_0} - g(1 + k),$$

where k is the tube friction coefficient. The modelled phase 1 lasts 0.12 seconds, during which time the acceleration remains constant at 26.56 m/s^2. Phase 1 ends when the rocket reaches a height of 0.22 m, the top of the launch tube.

Modelling phase 2 for the complete model required a significantly higher level of sophistication than for its basic counterpart. In the basic model, it was sufficient to assume that the air pressure within the rocket remained constant as the water was ejected, when in reality the loss of water, and thus increase in air volume inside, decreased the air pressure. I used Bernoulli's equation for incompressible flow to derive an expression for the exhaust velocity, $u(t)$ of the water in terms of air pressure $P(t)$, atmospheric pressure P_{atm}, and the density of water ρ_{water} (Derivation - See Appendix C.2).

$$u(t) = \sqrt{\frac{2(P(t) - P_{atm})}{\rho_{water}}}$$

Then, I substituted this expression for exhaust velocity into an expression for $\frac{dV}{dt}$, or change in volume as a function of time (See Appendix C.3), which is determined by the product of the exhaust velocity and the area of the rocket nozzle. This yields an expression for $\frac{dV}{dt}$ in terms of pressure, which I replaced with volume using the known relationship between pressure and volume for adiabatic flow (See Appendix C.4) to produce a solvable differential equation for $\frac{dV}{dt}$ (See Appendix C.5),

$$\frac{dV}{dt} = a\sqrt{\frac{2\left(P_0\left(\frac{V_0}{V(t)}\right)^{\gamma} - P_{atm}\right)}{\rho_{water}}}$$

in terms of $V(t)$, ρ_{water}, P_{atm}, the initial air pressure P_0, the initial volume of air V_0, and the adiabatic gas constant γ.

After solving the differential equation for $\frac{dV}{dt}$ to yield a volume function in phase 2, I derived an expression for the thrust of the rocket in terms of volume starting from the formula for rocket thrust then substituted in the expression for exhaust velocity (See Appendix C.6). Additionally, I derived an expression describing the mass of the rocket over time in terms of volume (See Appendix C.7). After deriving the equation for thrust and mass, I derived expressions for $\frac{dv}{dt}$ in both the x and y directions using Newton's Second Law and a launch angle θ, measured with respect to the vertical (See Appendix C.8). These equations for $\frac{dv_x}{dt}$ and $\frac{dv_y}{dt}$ allow me to produce equations and graphs for the acceleration, velocity, and position of the rocket in both the x and y directions. This model's phase 2 lasts 0.31 seconds, and the y-acceleration increases from 58.31 m/s^2 to 114.52 m/s^2. By the end of phase 2, the rocket has travelled just 5.2m up from the top of the launch tube and 1.2 m horizontally, but has increased vertical speed from 4.25 m/s to 28.94 m/s.

Phase 3 describes the motion of the rocket during the air exhaust phase, which was omitted entirely from the basic model due to its brevity and complexity, but is included in the complete model. The complexity of the air exhaust phase is due to the fact that the air is compressed in the process, making Bernoulli's equation for incompressible flow inaccurate. Instead, I used Bernoulli's equation for compressible flow (See Appendix C.9) to model the air exhaust. That equation allows me to derive an expression for $\frac{dm}{dt}$ of the air exhaust (See Appendix C.10) and the air exhaust velocity (See Appendix C.11). Also necessary to evaluate both of those expressions was an expression for the density of the air in the rocket as a function of time, which I derived from the adiabatic relationship between density, the unknown, and pressure, a known quantity (See Appendix C.12).

However, when the ratio of pressures between the inside of the bottle and the atmosphere is below 0.528, the actual flow rate does not match the theoretical rate given by Bernoulli's equation (See Figure 5). Because the flow rate would be faster than the speed of sound below that ratio and because the maximum speed that information can propagate through a gas is the speed of sound, I had to split phase 3 into two parts. During part one, the flow rate is constant at 343 m/s and $\frac{dm}{dt}$ is expressed by maximizing the ratio of pressures from the previous expression (See Appendix C.13), yielding a range of $\frac{dm}{dt}$ from 0.067 kg/s to 0.0046 kg/s in the span of 0.05 s. The flow rate and $\frac{dm}{dt}$ for part 2 are simply given by the expressions previously derived from Bernoulli's equation, ranging from 343 m/s to 228.8 m/s and 0.05 kg/s to 0.038 kg/s in the span of 0.08 s, respectively. Using the known expressions for exhaust velocity and $\frac{dm}{dt}$ for both part 1 and part 2 and Newton's Second Law, I derived expressions for thrust then $\frac{dv}{dt}$ in both the x and y directions (See Appendix C.14). I used these expressions in Mathematica to create expressions for acceleration, velocity, and position for both parts 1 & 2. In the complete model, the entire air exhaust phase lasts just 0.1 seconds, during which time the rocket has a maximum acceleration of 93.7 m/s^2 at start of the phase. In total, the rocket increases speed from 42.6 m/s at the start to 55.5 m/s once all excess air is expelled.

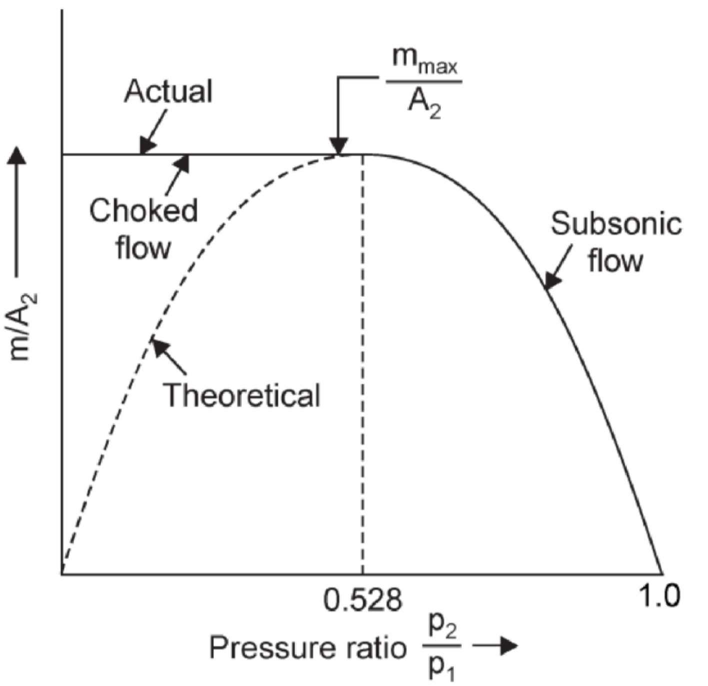


Figure 5: Mass exhaust rate as a function of $\rho(t)$. Displays constant flow rate below pressure ratio 0.528 (From EGR Rocket Lab 9)

Phase 4 combines phases 3 & 4 from the basic model, but accounts for the varying flight angle by integrating a new equation for the flight angle, ϕ, as a function of time relative to the vertical and integrating it into the model with trigonometry. To evaluate ϕ at any given time, I simply took the inverse cosine of the y component of velocity divided by the hypotenuse of the x and y components, given by

$$\phi(t) = \arccos\left(\frac{v_y(t)}{\sqrt{v_x^2 + v_y^2}}\right).$$

The x and y components of velocity are in turn, however, dependent of ϕ, so I derived a system of equations for $\frac{dv_x}{dt}$, $\frac{dv_y}{dt}$, and $\phi(t)$ (See Appendix C.15). I used Mathematica to solve this system, yielding equations for velocity in both the x and y directions, which I used to produce equations and graphs for acceleration and position for phase 4. Phase 4 lasts 5.9 seconds, during which time the rocket follows a parabolic-like trajectory, reaching a maximum vertical altitude of 46.27 m. The acceleration is -16.18 m/s^2

at the start of the phase and decreases to -5.88 m/s^2 as the drag force increases and the rocket approaches terminal velocity.

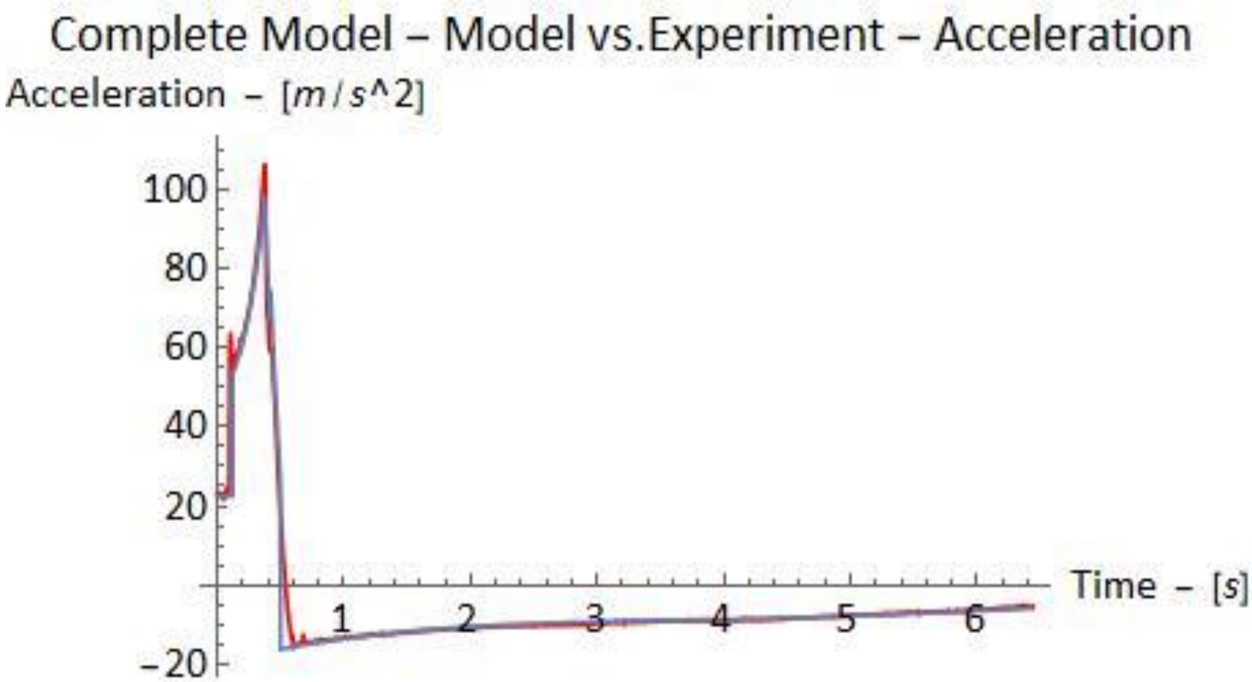
5.3 Conclusion

The complete model projects a total flight time of 6.4 seconds, reaching a maximum altitude of 46.27 m. This model is significantly more complex than the previous basic model, integrating Bernoulli's equations for compressible and incompressible flow, trigonometry, launch and flight angles, air resistance during thrust, friction, and the air exhaust phase, all of which were missing from the basic model.

6. MODEL VS. EXPERIMENT COMPARISON

Figure 6: Table compares duration, initial acceleration, and final acceleration for every phase of rocket flight.

	Duration (sec)		Initial Acceleration (m/s^2)		Final Acceleration (m/s^2)	
	Experiment	Model	Experiment	Model	Experiment	Model
Phase 1	0.095	0.12	23.51	22.56	24.9	22.56
Phase 2	0.28	0.25	57.07	53.5	106.5	96.96
Phase 3	0.12	0.13	81.82	93.7	17.375	25.49
Phase 4	5.94	5.90	-15.17	-16.18	-5.39	-5.88



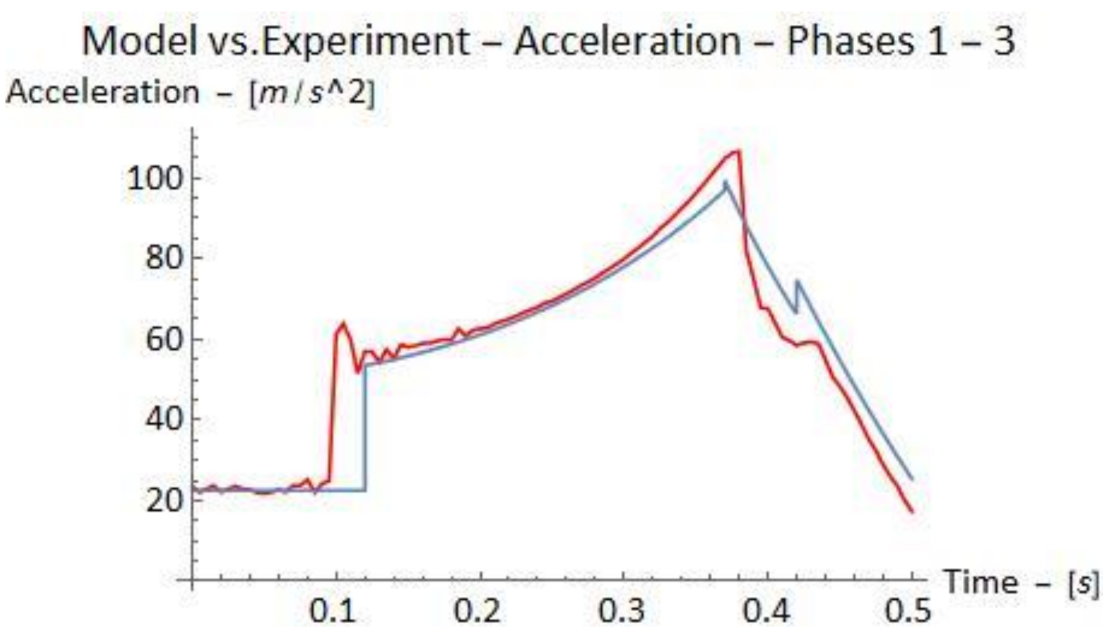
Key:

● – Experiment

● – Model

Figure 7:

Time vs. Acceleration graph in the y-direction for experiment and model



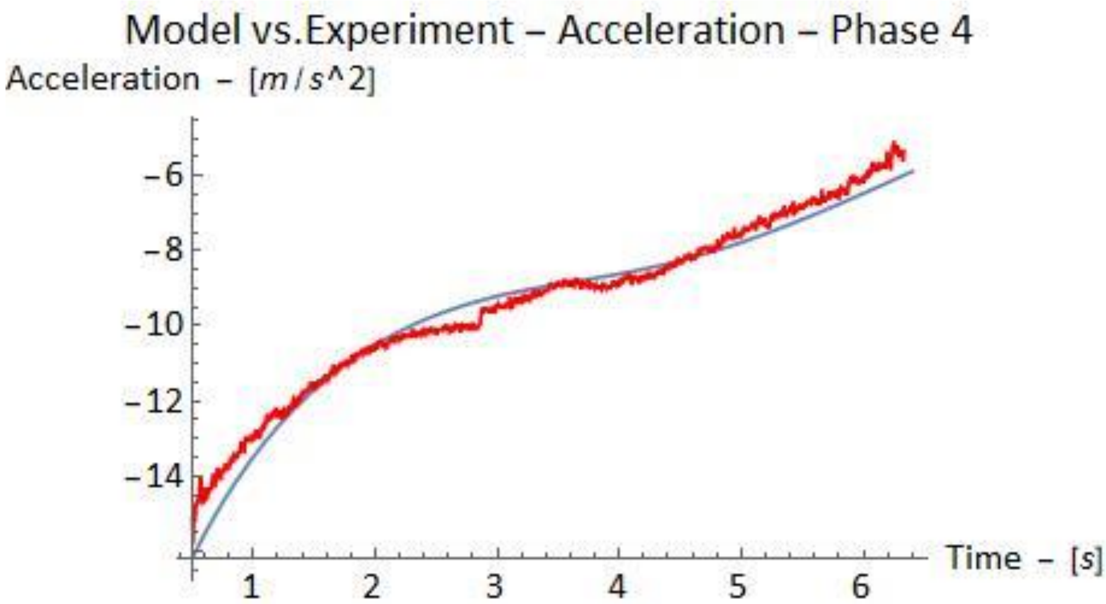
Key:

● – Experiment

● – Model

Figure 8:

Time vs. Acceleration graph in the y-direction for experiment and model for phases 1 - 3



Key:

● – Experiment

● – Model

Figure 9:

Time vs. Acceleration graph in the y-direction for experiment and model

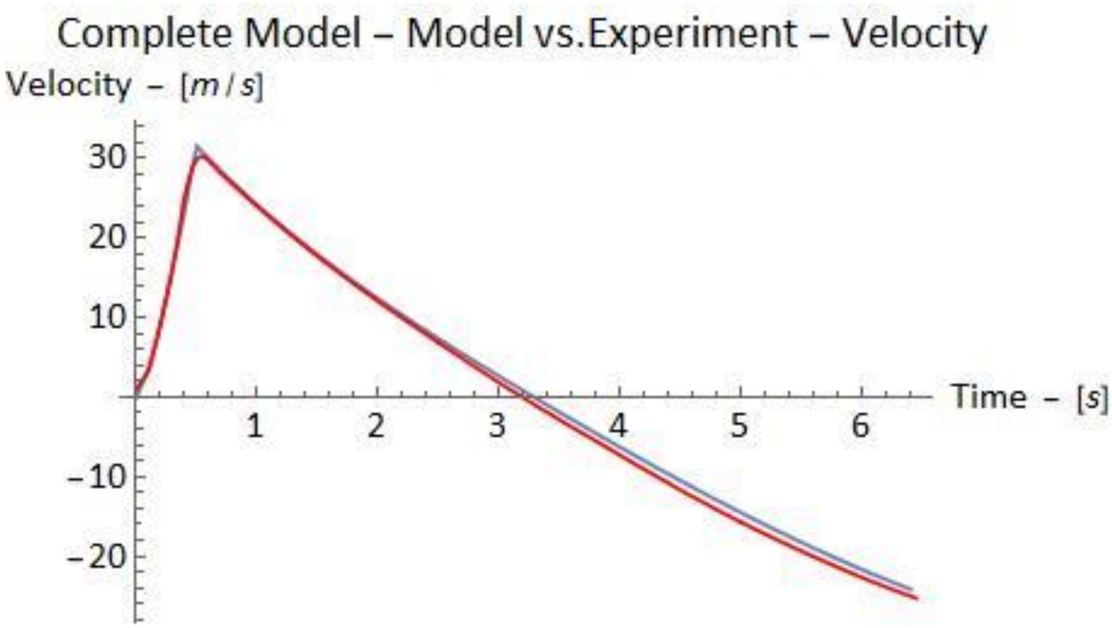


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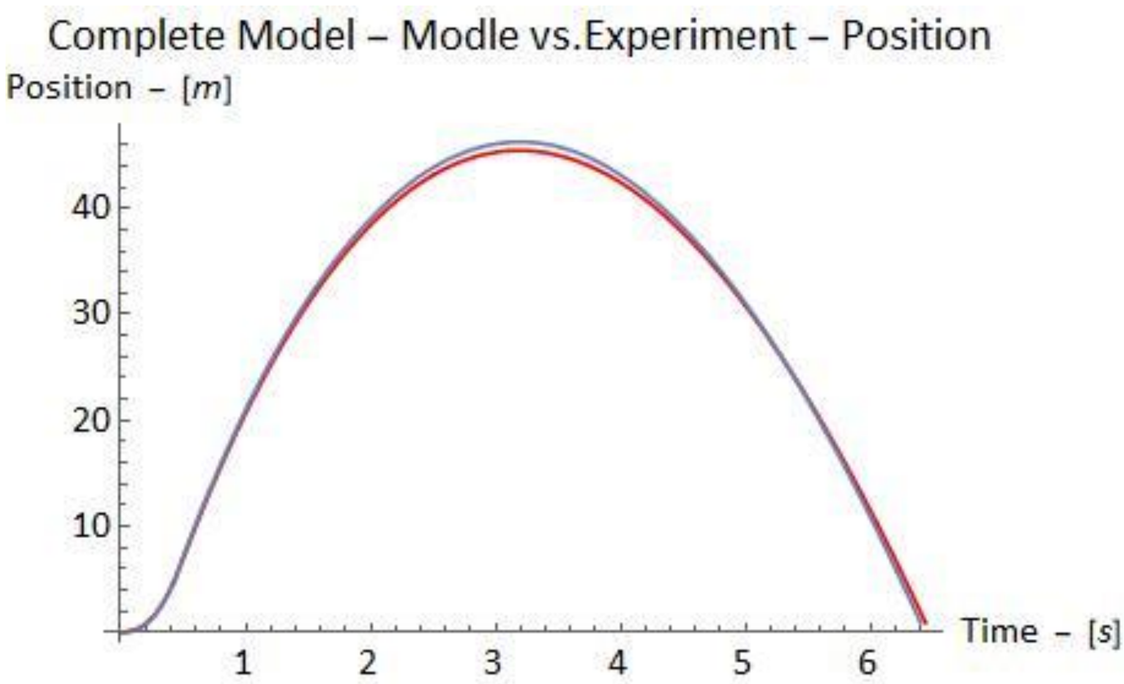
Figure 10:

Time vs. Velocity graph in the y-direction for experiment and model



Key:

● – Experiment

● – Model

Figure 11:

Time vs. Position graph in the y-direction for experiment and model

7. Discussion

7.1 Overview

Overall, the complete model is an excellent representation of the experimental rocket flight, as illustrated by the agreement between not only the acceleration graph (See Figure 7), but also the resulting velocity and position graphs (See Figures 10 & 11). Two of the most accurate phases of flight with respect to the complete model were phases 1 and 4.

7.2 Phases

In phase 1, which is characterized by almost constant acceleration, the model differed from the experimental data by as much as 9.4%, but as little as 1%, which is largely due to inherent noise in the data and shaking of the accelerometer during launch instead of an inaccuracy of the model, as is visible in the acceleration graph for phases 1 – 3 (See Figure 8). On average, the model differed from the data by less than 1 m/s^2 over the entire phase.

In phase 2, the largest error between the experiment and the model occurs at the end of the phase as it transitions to phase 3 (See Figure 8). This error is a difference of 9.54 m/s^2, or 8.96% error, though the error during the majority of the phase remains below 5%. The model projects that the rocket runs out of fuel just 0.03 seconds earlier than the actual duration given by the experimental data. According to the diameter of the nozzle, the volume of water, and the air pressure recorded during the experiment, the water thrust phase should only last as long as the model predicts, so the slight disagreement with the experimental data that caused the largest error in this phase is likely due to a small discrepancy in one of those values on launch day.

Phase 3 contains the largest model deviation from recorded data, reaching an error of 14.5% at the start of the phase and as much as 31.8% at the end (See Figure 8). The large error observed at the end of this phase is not surprising, as this phase is the most complicated, unstable, and brief part of the launch. Due to its complexity, the offset between the experimental data and the model during this phase could be caused by a number of factors, the most likely reason being either the omission of air resistance during the phase due to its brevity and complexity, or a small inaccuracy in the exact time when the rocket switched from choked to free air flow, which would alter the air exhaust speed at that moment and thus result in a different thrust calculation.

Phase 4 is a very close match between the model and the experiment for the majority of freefall, achieving the largest error at the end of the phase (See Figure 9). Immediately following the transition from phase 3 to 4, the model is accurate to 1.01 m/s^2, or 6.2%, then remains below 5% error for the majority of the flight. At the end of the phase, however, the accelerations differ by 0.49 m/s^2, or 9.1%, the largest error observed over the entire phase, though it represents a difference of less than half of 1 m/s^2, a relatively small error. Phase 4, like phase 1, was easier to model than phases 2 and 3 due to its relative simplicity, allowing for a much more accurate model. The error that is observed in phase 4, however small, could be attributed to instability of the rocket during flight caused by shaking, wind, or rotation of the rocket.

7.3 Conclusion

Although the complete model is not perfect, it is nevertheless an excellent model for rocket flight, as illustrated in the velocity and position graphs (See Figures 10 & 11). The errors observed in the acceleration model were small enough to allow for extremely accurate projections of position and velocity such that these models differed from actual launch data by at most 1.8% and 4.1% respectively.

8. CONCLUSIONS

The complete model proved to be an accurate and reliable method for predicting the motion of a rocket using given preconditions such as mass, air pressure, rocket dimensions, etc. By integrating into the model factors such as friction on the launch tube, the launch angle, the flight angle, the air exhaust phase, and drag force during thrust as well as utilizing tools such as Bernoulli's equations for compressible and incompressible flow and trigonometry, I assembled a model that closely models actual data. The accelerometer, on the other hand, proved to be reliable and accurate tool to collect real-world data for analysis and comparison without suffering an excessive amount of drift or noise that would cloud measurements. Mathematica allowed me to produce solutions, integrations, and derivations for complex differential equations that resulted from starting points as simple as Newton's Second Law. In the end, the model was accurate enough to produce graphs for position and velocity with maximum errors of less than 5%.

APPENDIX - KEY

$$A = cross - sectional\ area\ of\ rocket$$
$$a = area\ of\ rocket\ nozzle$$
$$a(t) = acceleration\ as\ a\ function\ of\ time$$

$B = accelerometer\ offset$
$C = accelerometer\ calibration\ constant$
$c = air\ drag\ coefficient$
$\frac{dm}{dt} = change\ in\ mass\ as\ a\ function\ of\ time$
$\frac{dv}{dt} = change\ in\ velocity\ (acceleration)\ as\ a\ function\ of\ time$
$\frac{dV}{dt} = change\ in\ volume\ as\ a\ function\ of\ time$
$F_{drag} = air\ drag\ force$
$F_{ext} = sum\ of\ all\ net\ external\ forces$
$F_f = force\ of\ friction$
$F_{net} = sum\ of\ all\ net\ forces$
$F_P = force\ of\ pressure$
$F_T = force\ of\ thrust$
$g = acceleration\ due\ to\ gravity$
$\gamma = adiabatic\ gas\ constant$
$h = height$
$k = tube\ friction\ coefficient$
$m_f = mass\ of\ empty\ rocket$
$m_0 = mass\ of\ full\ rocket$
$m(t) = mass\ of\ rocket\ as\ a\ function\ of\ time$
$P_0 = initial\ rocket\ air\ pressure$
$P(t) = pressure\ as\ a\ function\ of\ time$
$P_{atm} = atmospheric\ air\ pressure$
$\phi(t) = flight\ angle\ relative\ to\ the\ vertical\ as\ a\ function\ of\ time$
$\rho_{air} = density\ of\ air$
$\rho(t) = density\ of\ air\ inside\ rocket\ as\ a\ function\ of\ time$
$\rho_{water} = density\ of\ water$
$S = accelerometer\ output$
$T(t) = thrust\ as\ a\ function\ of\ time$
$t = time$
$\theta = launch\ angle\ relative\ to\ the\ vertical$
$u = exhaust\ velocity$
$u(t) = exhaust\ velocity\ as\ a\ function\ of\ time$
$V(t) = volume\ as\ a\ function\ of\ time$
$V_0 = initial\ volume$
$v_0 = initial\ velocity$

APPENDIX A – DATA COLLECTION

A.1:

$$S = C * a(t) + B$$

APPENDIX B – BASIC MODEL

B.1:

Line 2: Two forces: Air pressure and gravity

$$F_{net} = m\frac{dv}{dt}$$
$$m\frac{dv}{dt} = F_P - F_g$$
$$m\frac{dv}{dt} = P_0 a - gm$$
$$\frac{dv}{dt} = \frac{P_0 a}{m_0} - g$$

B.2:

Line 2: Two forces: Thrust and gravity

$$F_{net,ext} = m\frac{dv}{dt}$$
$$F_T - F_g = m\frac{dv}{dt}$$
$$-u\frac{dm}{dt} - F_g = m\frac{dv}{dt}$$
$$u\frac{dm}{dt} + m(t)\frac{dv}{dt} = F_g$$

B.3:

$$\frac{dv}{dt} = \frac{u\frac{dm}{dt}}{m(t)} - g$$

B.4:

Line 2: Two forces: drag force and gravity

$$F_{net} = m\frac{dv}{dt}$$
$$-F_{drag} - F_g = m\frac{dv}{dt}$$
$$m * \frac{dv}{dt} = -\frac{c\rho_{air}Av(t)^2}{2} - mg$$

$$\frac{dv}{dt} = -\frac{c\rho_{air}Av(t)^2}{2m_f} - g$$

B.5:

$$\frac{dv}{dt} = \frac{c\rho_{air}Av(t)^2}{2m_f} - g$$

APPENDIX C – COMPLETE MODEL

C.1:

Line 1: Three forces: Pressure, gravity, and friction

Line 3: Combines the two "mg" terms

$$m\frac{dv}{dt} = F_P - F_g - F_f$$
$$m\frac{dv}{dt} = P_0 a - mg - kmg$$
$$\frac{dv}{dt} = \frac{P_0 a}{m_0} - g(1+k)$$

C.2:

Line 1: Bernoulli's Equation for Incompressible flow

Line 2: Eliminate all zero terms

Line 3: Solve for $u(t)$

$$P(t) + \frac{1}{2}\rho_{water}v_{0,water}^2 + \rho_{water}gh = P_{atm} + \frac{1}{2}\rho_{water}u(t)^2 + \rho_{water}gh$$
$$P(t) = P_{atm} + \frac{1}{2}\rho_{water}u(t)^2$$
$$u(t) = \sqrt{\frac{2(P(t) - P_{atm})}{\rho_{water}}}$$

C.3:

Line 1: Change in volume given by area * velocity

Line 2: Substitute equation for $u(t)$ (See Appendix C.2)

$$\frac{dV}{dt} = u(t) * a$$
$$\frac{dV}{dt} = a\sqrt{\frac{2(P(t) - P_{atm})}{\rho_{water}}}$$

C.4:

Line 1: Given relationship for adiabatic flow

Line 2: Solve for $P(t)$

$$V(t) = V_0\left(\frac{P(t)}{P_0}\right)^{\gamma}$$
$$P(t) = P_0\left(\frac{V_0}{V(t)}\right)^{\gamma}$$

C.5:

Line 1: See Appendix C.3

Line 2: Substitute in $V(t)$ for $P(t)$ using relationship (See Appendix C.4)

$$\frac{dV}{dt} = a\sqrt{\frac{2(P(t) - P_{atm})}{\rho_{water}}}$$
$$\frac{dV}{dt} = a\sqrt{\frac{2\left(P_0\left(\frac{V_0}{V(t)}\right)^{\gamma} - P_{atm}\right)}{\rho_{water}}}$$

C.6:

Line 1: Given equation for rocket thrust

Line 2: Given equation for change in mass

Line 3: Substitution of line 2 into line 1

Line 4: Substitute $u(t)$ (See Appendix C.2)

Line 5: Substitute relationship between $P(t)$ and $V(t)$ (See Appendix C.4)

$$T(t) = u(t)\frac{dm}{dt}$$
$$\frac{dm}{dt} = u(t)\rho_{water}a$$

$$T(t) = u(t)^2 \rho_{water} a$$
$$T(t) = 2a(P(t) - P_{atm})$$
$$T(t) = 2a(P_0 \left(\frac{V_0}{V(t)}\right)^{\gamma} - P_{atm})$$

C.7:

Line 1: Linear equation for mass loss rate

Line 2: Substitute $\frac{dm}{dt}$ (See Appendix C.6 Line 2)

Line 3: Substitute $u(t)$ (See Appendix C.2)

Line 4: Substitute $P(t)$ (See Appendix C.4)

$$m(t) = m_0 - \frac{dm}{dt} t$$
$$m(t) = m_0 - \left(a\rho_{water} u(t)\right)t$$
$$m(t) = m_0 - a\rho_{water} \sqrt{\frac{2(P(t) - P_{atm})}{\rho_{water}}}\, t$$
$$m(t) = m_0 - a\rho_{water} \sqrt{\frac{2\left(P_0 \left(\frac{V_0}{V(t)}\right)^{\gamma} - P_{atm}\right)}{\rho_{water}}}\, t$$

C.8:

Line 2: Three forces: Thrust, drag force, and gravity

Lines 3 & 4: x and y direction expansions of forces

$$F_{net} = m\frac{dv}{dt}$$
$$F_T - F_{drag} - F_g = m\frac{dv}{dt}$$
$$\frac{dv_y}{dt} = \frac{T(t)\cos\theta - \frac{1}{2} cA\rho_{air}\left(v_x^2 + v_y^2\right) cos\theta - m(t)g}{m(t)}$$
$$\frac{dv_x}{dt} = \frac{T(t)\sin\theta - \frac{1}{2} cA\rho_{air}\left(v_x^2 + v_y^2\right) sin\theta}{m(t)}$$

C.9:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{P(t)}{\rho(t)} = \left(\frac{\gamma}{\gamma-1}\right)\frac{P_{atm}}{\rho_{air}} + \frac{u(t)^2}{2}$$

C.10:

Line 1: Given equation for $\frac{dm}{dt}$

Line 2: Substitute $u(t)$ (See Appendix C.11)

$$\frac{dm}{dt} = \rho(t) * a * u(t)$$

$$\frac{dm}{dt} = a\sqrt{\frac{2\gamma P(t)\rho(t)}{\gamma-1}\left[\left(\frac{P_{atm}}{P(t)}\right)^{\frac{2}{\gamma}} - \left(\frac{P_{atm}}{P(t)}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

C.11:

Line 1: Solve for $u(t)$ in Appendix C.9

$$u(t) = \sqrt{\frac{2\gamma}{\gamma-1}\left(\frac{P(t)}{\rho(t)} - \frac{P_{atm}}{\rho_{air}}\right)}$$

C.12:

Line 1: Given relationship between pressure and density

Line 2: Solve for $\rho(t)$

$$\frac{P(t)}{\rho(t)^{\gamma}} = \frac{P_{atm}}{\rho_{air}^{\gamma}}$$

$$\rho(t) = \left(\frac{P_{atm}}{\rho_{air}^{\gamma}P(t)}\right)^{\gamma}$$

C.13:

Line 1: (See Appendix C.10) Ratio of pressures set to maximize function

$$\frac{dm}{dt} = a\sqrt{\frac{2\gamma P(t)\rho(t)}{\gamma-1}\left[\left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} - \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]}$$

C.14:

Line 2: Three forces: Thrust, drag force, and gravity

Lines 3 & 4: x and y direction expansions of forces

$$F_{net} = m\frac{dv}{dt}$$

$$T(t) - F_{drag} - F_g = m\frac{dv}{dt}$$

$$\frac{dv_y}{dt} = \frac{u(t)\frac{dm}{dt}cos\theta - \frac{1}{2}ca\rho_{air}\left(v_x^2 + v_y^2\right)cos\theta - mg}{m}$$

$$\frac{dv_x}{dt} = \frac{u(t)\frac{dm}{dt}sin\theta - \frac{1}{2}ca\rho_{air}\left(v_x^2 + v_y^2\right)sin\theta}{m}$$

C.15:

Lines 1 – 3: System of equations in freefall. Two forces: drag force and gravity

$$\frac{dv_x}{dt} = -\frac{\frac{1}{2}ca\rho_{air}\left(v_x^2 + v_y^2\right)\sin\left(\phi(t)\right)}{m}$$

$$\frac{dv_y}{dt} = -\frac{\frac{1}{2}ca\rho_{air}\left(v_x^2 + v_y^2\right)\cos\left(\phi(t)\right)}{m} - g$$

$$\phi(t) = \arccos\left(\frac{v_y(t)}{\sqrt{v_x^2 + v_y^2}}\right)$$