On a Theorem of Beilinson and the Derived Category of Coherent Sheaves on $\mathbb{P}^1$

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Abstract

A celebrated theorem of Beilinson in the short note [Bei78] establishes an equivalence of derived categories between the coherent sheaves on $\mathbb{P}^n$ and modules over certain finite dimensional algebras. We give a gentle introduction to this result in the case of $\mathbb{P}^1$ using the more modern machinery of tilting theory. In particular, we ultimately show that this derived equivalence links the geometry of $\mathbb{P}^1$ to the representation theory of the Kronecker quiver, demonstrating a fruitful connection between the fields of algebraic geometry and representation theory.

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I am very fortunate to have met the people I have met in my five years here at UNSW, especially those for whom the word 'lemon' means more than an innocuous fruit. To the other honours and postgraduate students I have spent time with this year, thank you for being a great bunch all around.

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Contents

Chapter 0

Introduction

Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.'

– Michael F. Atiyah [Ati02]

0.1 Motivation and outline

The *coherent sheaves* on an algebraic variety X are of central significance in algebraic geometry, being closely linked to many geometric properties of X itself. In a manner of speaking, these are a better behaved generalisation of *vector bundles*. Originally the preserve of analytic geometry, the genesis of coherent sheaves and *coherent sheaf cohomology* in the algebro-geometric setting was Serre's revolutionary paper [FAC].

It had already been deduced by Serre that the cohomology of *algebraic* vector bundles on a smooth projective variety enjoyed a certain duality property, appropriately named *Serre duality*. Grothendieck sought to generalise this duality to coherent sheaves and for a wider class of algebraic spaces, ultimately using the nascent theory of *schemes* introduced by Grothendieck himself in [EGA1]. In so doing, he realised that whereas the *canonical line bundle* sufficed to give the usual Serre duality, what was needed in the general case was instead a *dualising complex* of sheaves to fulfil the same purpose. As a response to the shortcomings of homological algebra in accommodating this shift in viewpoint, Grothendieck and his student Verdier developed the notion of *derived categories* and a more unified approach to *derived functors*.

In the years since their development, derived categories have become a tool of vital importance in algebraic geometry, but are also of relevance further afield in the study of partial differential equations (e.g. *D-modules*) and string theory (e.g. the *homological mirror symmetry* program of Kontsevich). The fusion between derived categories and coherent sheaves in particular has led to some powerful results. For example, the (bounded) derived category $\mathbf{D}^b(\mathsf{Coh}(X))$ of coherent sheaves on a variety X sometimes captures all there is to know about said variety: Bondal and Orlov showed in [BO01] that under certain algebro-geometric hypotheses, the data of the category $\mathbf{D}^b(\mathsf{Coh}(X))$ is enough to uniquely reconstruct X. This makes it a useful category to try and understand.

An early example of work in this direction is due to Beilinson, who studied the coherent sheaves of $\mathbb{P}^n$ over an algebraically closed field k in [Bei78]. The main result, hereafter referred to as *Beilinson's theorem*, is that there is an equivalence of derived categories between $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^n))$ and $\mathbf{D}^b(\mathsf{mod}(\Lambda^{\mathrm{op}}))$, where Λ is a certain non-commutative finite dimensional k-algebra. Since one may identify Λ with the *path algebra* of a certain *quiver*, this theorem provides not only a link from algebraic geometry to the representation theory of finite dimensional algebras, but also to that of quivers.

There are two well-known proofs of Beilinson's theorem. The original method of proof in [Bei78] is by constructing a *resolution of the diagonal* of $\mathbb{P}^n$, a technique belonging to algebraic geometry. Eventually, the techniques of *tilting theory* emerged as giving a more satisfying proof instead using the methods of representation theory. An outcome of the seminal work of Brenner and Butler in 1979, tilting theory is a systematic means of constructing equivalences between the module categories of two finite dimensional algebras. The language of derived categories, which also proved to be useful in this context, was infused into tilting theory by Happel and Rickard in the 1980s.

The goal of this thesis is to give an accessible exposition of how one may prove Beilinson's theorem using tilting theory. To make this more tractable, we will restrict our attention to the case of $\mathbb{P}^1$ only. Actually understanding the statement of the theorem even in this simpler case is already a non-trivial task, so a necessary secondary goal, and one which occupies the majority of this thesis, is to give a solid and gentle introduction to all of the required background. Though this topic fits within the intersection of algebraic geometry, homological algebra and representation theory, the dominant themes of this thesis are the former two.

A brief synopsis of the flow of the material per chapter is as follows. We start with Chapter 1, which is a discussion of abelian categories. This turns out to be just the right level of categorical abstraction for many areas of modern mathematics due to it unlocking the machinery of homological algebra, which is in turn the focus of Chapter 2. The most fundamental tool in this subject is cohomology, and in considering its utility as a rich and very general method of producing algebraic invariants, we will see that an 'improvement' to the notion of abelian categories that is more inclined to the study of cohomology is in order. Derived categories end up fulfilling this purpose, so we discuss them at length in Chapter 3. Following this, we change course completely in Chapter 4 to study the theory of coherent sheaves on $\mathbb{P}^1$ and various aspects of its sheaf cohomology. The fact that we are working only in this simple case allows us to present this in a quite hands-on way than the more generic treatment one finds in the literature. The preceding chapters clear the path finally for Chapter 5, in which we set up the notion of a *tilting bundle* which yields Beilinson's equivalence. For the reader's convenience, Appendix A records all of the explicit conventions and various pieces of notation we use throughout the thesis.

There is a substantial amount of technical background that we must cover to set the appropriate scene in the first three chapters. We will emphasise breadth rather than depth when it comes to these topics, so we will be unapologetically light at times on details, deferring extensively to the relevant literature. The material in these chapters is to be treated principally as a tool for what is to come, and it is often the case that for many stated results, their proofs are not particularly insightful and are little more than *abstract nonsense* (i.e. category-theoretic formalities).

0.2 Assumed knowledge

The most serious prerequisite is a moderate background in category theory. In addition to being well-acquainted with its core ideas (categories, functors and natural transformations), we assume the reader is familiar with basic categorical relationships (duality, equivalences of categories and adjoint functors) and universal constructions (products, coproducts, kernels and cokernels). The standard reference for this material is [Mac98]. Some exposure to limits and colimits, which provide a unifying language for these universal constructions, is nice to have but will not be explicitly used.

A background in ring and module theory at the level of [DF03, Chapters 8 and 10] will also be assumed. We will bundle this with a single fairly light assumption from commutative algebra that one may encounter in the course of studying the aforementioned topics: a working knowledge of localisations of rings and modules. For details of this process, one may consult [AM69, Chapter 3] or [DF03, Section 15.4].

Finally, we assume familiarity with the basics of classical algebraic geometry. It is enough to be aware of the main aims of the subject, the definitions and main results concerning (quasi-)affine and (quasi-)projective varieties, and the basic study of regular functions and regular maps of varieties. For a reference, see [Har77, Sections I.1–4]. A very basic acquaintance with affine schemes would be nice to have but not essential.

0.3 Author's note

When I embarked on this project, my intended goal was to give a comprehensive treatment of Beilinson's theorem in the simplest case of $\mathbb{P}^1$, including the full details of its proof. This is a result which has received much attention among algebraic geometers and representation theorists alike, but as far as I can tell, there has been no formal literature aimed at a wider audience. A story like this, to me, is too good not to be told, and as is the hope of any honours student, I wanted my work to bridge that gap. This was certainly an ambitious goal, but as is my nature, I threw myself at it anyway.

There are at least two reasons why this thesis does not strictly meet this goal. It brings me no joy to admit this whatsoever, but the first is that I was simply not able to completely bridge that gap for myself this year. There is a tremendous amount of abstract background material that one has to learn to even begin to make sense of such a result, and the scale of this only becomes apparent once one starts to pour over the details. This ultimately consumed my attention for the entire year. This being the case, I also realise separately there is still immense value in having all of the necessary details leading up to some greater result being explicitly spelled out in one place. That is a worthwhile goal in and of itself, and the length of this document is a firm testament to the fact that one can make a whole thesis project out of it.

I therefore present this work as a complete coverage and background in the abstract machinery one needs *before* approaching this theorem, and an exposition of the mathematical beauty present within these background details themselves. After reading this work, one would be well prepared to venture into the more professional literature to see the finer details that I unfortunately cannot give you.

Chapter 1

Abelian categories

In this first chapter, we introduce *abelian categories*, a class that captures some essential properties of categories such as Ab and $\mathsf{Mod}(R)$. We will mostly follow [Mac98; Fre64]. The origin of abelian categories in contemporary form is Grothendieck's influential *Tōhoku paper* of 1957, though this work is predated by Buchsbaum's similar notion of an *exact category* (which now means something else entirely) defined two years prior.

The main selling point of working with abelian categories is that they are endowed with enough structure to make sense of exact sequences of morphisms, and in a way that is sufficiently general enough to describe many categories arising in practice. Owing to this flexibility, they are now a foundational abstraction in several branches of modern algebra, especially those areas which this thesis deals in.

1.1 Additive structures on categories

Using $\mathsf{Mod}(R)$ as our prototype, we will identify and develop some suitable analogues of these aforementioned essential properties. We begin with an observation about its morphisms. For any pair of R-modules M and N, the set $\mathrm{Hom}_R(M, N)$ is not only a set but also an abelian group, since we are able to add module homomorphisms together pointwise. This addition also interacts bilinearly with composition, i.e. we have

$$f(g+h) = fg + fh \quad \text{and} \quad (f+g)h = fh + gh$$

for any compatible module homomorphisms f, g and h. If R is commutative, then each hom-set has even more structure, becoming itself an R-module with the usual action.

Since similar structure is found on the hom-sets of other, more exotic categories and makes manipulating morphisms far more pleasant, this is a reasonable starting point.

Definition 1.1.1. A category $\mathcal{A}$ is said to be *preadditive* if every hom-set of $\mathcal{A}$ is an abelian group and the composition of morphisms is bilinear with respect to addition. A preadditive category is *R-linear* if every hom-set is also a module over some ring R.

It may also be possible to multiply and add objects themselves, which for R-modules is achieved by taking direct products and sums. Moreover, these two notions coincide if one only considers finite collections of modules. There is also a zero module 0 which, among other things, satisfies $M \oplus 0 \cong M$ for any R-module M. Phrased in the language of category theory, the direct product and sum constructions are the product and coproduct of the category $\mathsf{Mod}(R)$ respectively, and the zero module is its zero object.

Definition 1.1.2. A preadditive category $\mathcal{A}$ is said to be *additive* if has a zero object $0 \in \mathcal{A}$ and every pair of objects in $\mathcal{A}$ has a product in $\mathcal{A}$.

Proposition 1.1.1 ([Mac98, Theorem VIII.2.2])**.** *In an additive category, the product of two objects is also a coproduct of those objects, and hence a biproduct.*

If a category has all binary biproducts, then it must also have all finite biproducts, and the existence of a zero object allows one to make sense of empty biproducts in $\mathcal{A}$ too. This gives a slightly more precise working definition of additive categories. We shall refer to finite biproducts in $\mathcal{A}$ as direct sums and use the same notation as for modules.

Many common functors one works with in practice interact with this additive structure nicely. The most important examples in module theory, the Hom and tensor product functors, are group homomorphisms on hom-sets and preserve direct sums. This motivates the following class of functors which appear throughout Chapter 2 and Chapter 3.

Definition 1.1.3. A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is said to be *additive* if it is a group homomorphism on hom-sets.

Lemma 1.1.1 ([Mac98, Proposition VIII.2.4])**.** *A functor $F : \mathcal{A} \to \mathcal{B}$ is additive if and only if it commutes with direct sums, i.e. $F(A \oplus B) \cong F(A) \oplus F(B)$ for any A, $B \in \mathcal{A}$.*

The final properties left to look at are those which enable exact sequences to be defined. Recall that any module homomorphism $f : M \to N$ has a kernel and image, as well as a cokernel and coimage defined by the quotient modules

$$\operatorname{coker}(f) := N/\operatorname{im}(f) \quad \text{and} \quad \operatorname{coim}(f) := M/\ker(f).$$

By the first isomorphism theorem, there is a canonical isomorphism $\mathrm{coim}(f) \cong \mathrm{im}(f)$, and hence a commutative diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, hook] & M \arrow[r, "f"] \arrow[d, two heads] & N \arrow[r, two heads] & \mathrm{coker}(f). \\
& \mathrm{coim}(f) \arrow[r, "\cong"] & \mathrm{im}(f) \arrow[u, hook]
\end{tikzcd}$$

In particular, we observe that f factors through its image and coimage. We generalise this by way of the following pair of definitions.

Definition 1.1.4. An additive category $\mathcal{A}$ is said to be *preabelian* if any morphism in $\mathcal{A}$ has a kernel and a cokernel in $\mathcal{A}$. We say that $\mathcal{A}$ is *abelian* if every monic in $\mathcal{A}$ (resp. epic) is *normal*, i.e. is the kernel (resp. cokernel) of a morphism in $\mathcal{A}$.

Using the universal properties of kernels and cokernels, the abelian category condition may be refined to the more precise statement that any monic is the kernel of its cokernel, and dually any epic is the cokernel of its kernel. Applying the categorical definitions rather than the usual module-theoretic ones, one may verify that these statements are true in $\mathsf{Mod}(R)$, which provides some a posteriori justification for defining abelian categories in this way.

Let us work towards developing a more convincing intuition. In studying morphisms of preabelian categories, kernels of cokernels and cokernels of kernels appear already, making for good analogues of the image and coimage. Once again, we leave it to the reader to check that these agree with the module-theoretic concepts defined above.

Definition 1.1.5. Let f be a morphism in a preabelian category $\mathcal{A}$. Then

$$\mathrm{im}(f) := \ker(\mathrm{coker}(f)) \quad \text{and} \quad \mathrm{coim}(f) := \mathrm{coker}(\ker(f))$$

are the categorical *image* and *coimage* of f respectively.

The upshot of this definition is that similar to the above diagram in $\mathsf{Mod}(R)$, any morphism in a preabelian category canonically factors through its image and coimage.

Lemma 1.1.2. *Let f be a morphism in a preabelian category $\mathcal{A}$. Then there is a canonical morphism* $\mathrm{coim}(f) \to \mathrm{im}(f)$ *fitting into a commutative diagram*

$$\begin{tikzcd}
\ker(f) \arrow[r, hook] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, two heads] & \mathrm{coker}(f). \\
& \mathrm{coim}(f) \arrow[r, dashed] & \mathrm{im}(f) \arrow[u, hook]
\end{tikzcd}$$

Proof. By the universal property of cokernels, f must factor uniquely through a morphism $i : \operatorname{coim}(f) \to B$, so

$$\operatorname{coker}(f) \circ i \circ \operatorname{coim}(f) = \operatorname{coker}(f) \circ f = 0.$$

Recalling that all cokernels are epic, it follows that i factors uniquely through a morphism $u : \operatorname{coim}(f) \to \operatorname{im}(f)$. Arguing dually, f factors uniquely through a morphism $p : A \to \operatorname{im}(f)$ that in turn factors uniquely through a morphism $v : \operatorname{coim}(f) \to \operatorname{im}(f)$. By this uniqueness, u and v must coincide, so the morphism is indeed canonical. $\square$

If $\mathcal{A}$ is abelian, a tighter analogy with $\mathsf{Mod}(R)$ may be drawn: they are preabelian categories where the first isomorphism theorem holds. In many treatments, such as the original [Tōhoku], this observation is actually taken as the definition. It is instructive to show that these definitions are equivalent, since doing so requires the following lemmas which give familiar characterisations of morphisms.

Lemma 1.1.3 ([Fre64, Theorem 2.12])**.** *In an abelian category, any monic which is also an epic is an isomorphism.*

Lemma 1.1.4 ([Fre64, Theorems 2.17 and 2.17*])**.** *Let $f : A \to B$ be a morphism in an abelian category. Then f is monic if and only if $\ker(f) = 0$ as an object, i.e. $\operatorname{coim}(f) = A$. Dually, f is epic if and only if $\operatorname{coker}(f) = 0$, i.e. $\operatorname{im}(f) = B$.*

Proposition 1.1.2. *Let $\mathcal{A}$ be a preabelian category. Then $\mathcal{A}$ is abelian if and only if the morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ of Lemma 1.1.2 is an isomorphism for any $f \in \mathcal{A}$.*

Proof sketch. For the forward direction, one uses the abelian category condition and the universal properties of kernels and cokernels to show that in Lemma 1.1.2, i is monic. By dualising this argument, we may also conclude that p is an epic. This shows that $\operatorname{coim}(f) \to \operatorname{im}(f)$ is monic and epic, and hence an isomorphism by Lemma 1.1.3. For the reverse direction, it is enough to prove that any monic $f : A \to B$ in $\mathcal{A}$ is a kernel. To this end, one uses Lemma 1.1.4 and the assumption to rewrite the commutative diagram in Lemma 1.1.2 as

$$\begin{tikzcd} 0 \arrow[r, hook, "0"] & A \arrow[r, hook, "f"] \arrow[d, two heads, "\mathrm{id}"'] & B \arrow[r, two heads] & \operatorname{coker}(f), \\ & A \arrow[r, "\cong"] & \operatorname{im}(f) \arrow[u, hook] & \end{tikzcd}$$

and then argues by universal properties that $f = \ker(\operatorname{coker}(f))$. $\square$

1.2 Working with abelian categories

With the definition at hand, we now look at how an abelian category can be reasoned about, once again with $\mathsf{Mod}(R)$ serving as inspiration. Though several concepts that we will introduce here can be defined just as well for less strictly defined categories, it is often the case that they are most potent in the abelian setting. As a first example, we may consider analogues of submodules and quotient modules in any abelian category, which will be important for defining *(co)homology* in Section 2.1.

Definition 1.2.1. Let $\mathcal{A}$ be an abelian category and fix an object $A \in \mathcal{A}$. A *subobject* of A is the data of a monic $s : S \hookrightarrow A$ identified up to equivalence with all monics $t : T \hookrightarrow A$ such that $s = tu$ for some isomorphism $u : S \cong T$. We use the suggestive notation $S \leq A$. Moreover, we define $A/S := \operatorname{coker}(s)$ to be the *quotient object* of A by its subobject S.

Exactly as promised at the beginning of this chapter, the language of exact sequences can also be directly imported from $\mathsf{Mod}(R)$ to abelian categories.

Definition 1.2.2. An *exact sequence* in an abelian category $\mathcal{A}$ is a sequence

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

such that $\operatorname{im}(f_{i-1}) = \ker(f_i)$ for each i, and a *short exact sequence* is one of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

In particular, f is monic and g is epic.

Definition 1.2.3. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is *left exact* if it sends any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$ to the left exact sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

in $\mathcal{B}$. Similarly, F is *right exact* if it instead turns it into a right exact sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0.$$

We say that F is *exact* if it is both left and right exact.

Example 1.2.1. Module theory provides examples of each type of exactness. The functors $\mathrm{Hom}_R(M, -)$ and $- \otimes_R M$ are left exact and right exact respectively, which in the case where R is commutative follows from the tensor-hom adjunction (since left adjoint functors are right exact and vice versa). Given a multiplicatively closed subset $S \subseteq R$, the localisation functor $S^{-1}(-) : \mathsf{Mod}(R) \to \mathsf{Mod}(S^{-1}R)$ is exact.

Remark 1.2.1. An equivalent way to define left and right exact functors (as in [Mac98, Section VIII.3], say) are as those which preserve all finite *limits* and *colimits* respectively. We will not define these categorical concepts here, so instead we simply note some important consequences. Since products and kernels are finite limits, these constructions are preserved by any left exact functor. Similarly, coproducts and cokernels are finite colimits, and so are preserved by right exact functors.

Next, we introduce analogues of projective and injective modules. Just as they are powerful in the study of modules, such objects are indispensable in the study of abelian categories, as will be made abundantly clear in the two following chapters.

Definition 1.2.4. An object $P \in \mathcal{A}$ is said to be *projective* if for every epic $f : A \twoheadrightarrow B$, any morphism $g : P \to B$ factors through f:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "g"] \\ A \arrow[r, two heads, "f"'] & B. \end{tikzcd}$$

Dually, an object $I \in \mathcal{A}$ is said to be *injective* if for every monic $f : A \hookrightarrow B$, any morphism $g : A \to I$ factors through f:

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed] \\ I. & \end{tikzcd}$$

The category $\mathcal{A}$ clearly need not be abelian for the preceding definitions to work, but if it is, the following familiar characterisations become available. The proof is similar (though more pedantic) to the argument one gives for $\mathsf{Mod}(R)$, so we omit it.

Proposition 1.2.1. *Let $\mathcal{A}$ be an abelian category. Then $P \in \mathcal{A}$ is projective if and only if the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. Dually, $I \in \mathcal{A}$ is injective if and only if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact.*

Recall that in $\mathsf{Mod}(R)$, any R-module M admits a surjection $F \twoheadrightarrow M$ with F free (a fortiori, projective) and an injection $M \hookrightarrow I$ with I injective (a so-called *injective envelope* of M). These properties end up being quite special in the broader context of abelian categories.

Definition 1.2.5. An abelian category $\mathcal{A}$ has *enough projectives* if for any object $A \in \mathcal{A}$, there exists an epic $P \twoheadrightarrow A$ with P projective. Dually, $\mathcal{A}$ has *enough injectives* if for any A there exists a monic $A \hookrightarrow I$ with I injective.

We do not even have to leave the realm of module theory to find abelian categories which fail one or both of these conditions. This serves also as a preliminary warning that abelian categories can stray far from the categories of modules we originally used to motivate their definition, only guaranteeing we have similar exactness machinery.

Example 1.2.2. There are no non-zero finite abelian groups which are injective or projective, so the category FinAb certainly has neither enough projectives or injectives. Broadening our scope to finitely generated abelian groups (i.e. to $\mathsf{mod}(\mathbb{Z})$) allows us to recover enough projectives, but the lack of any injectives persists.

To conclude this chapter, we discuss common ways of proving statements about an abelian category $\mathcal{A}$. With exact sequences, one is also able to recover an ensemble of lemmas discussed in [Mac98, Section VIII.4], such as the *five lemma* and the *snake lemma*. These lemmas go hand in hand with the proof technique of *diagram chasing* (c.f. [Rie16, Section 1.6]), which is exploited to maximum effect in homological algebra.

Done purely at the level of categorical abstractions, many of these proofs, especially those involving diagram chases, are an obsessively technical sport. If $\mathcal{A}$ is *concrete* (i.e. there is a faithful, forgetful functor $\mathcal{A} \to \mathsf{Set}$), many of these categorical abstractions can be sidestepped in favour of working directly with elements of objects. Since $\mathsf{Mod}(R)$ is concrete, this is often the way one proves things in the module case. A deep theorem of abelian categories shows that this method of proof can still be valid in some instances, since morally such categories 'are' just a category of certain modules. The proof of this theorem is non-trivial, and we defer the details to its coverage in [Wei94, Section 1.6].

Definition 1.2.6. A category $\mathcal{C}$ is said to be *small* if its class of objects is a set, and *locally small* if every hom-set of $\mathcal{C}$ is a set.

Theorem 1.2.1 (Freyd-Mitchell embedding theorem)**.** *If $\mathcal{A}$ is a small abelian category, then there exists a ring R and a fully faithful, exact functor $\mathcal{A} \to \mathsf{Mod}(R)$. In particular, $\mathcal{A}$ is equivalent to a full subcategory of $\mathsf{Mod}(R)$.*

The most interesting abelian categories are rarely small, but some things one wishes to prove are 'local' in the sense that it is enough to consider a small abelian subcategory amenable to this theorem (e.g. one that contains all of the objects in some diagram). One case where this strategy potentially fails is when proving statements about projectives and injectives, since it may not be true that the projectives and injectives in a subcategory of $\mathsf{Mod}(R)$ are the usual projective and injective R-modules.

Chapter 2

Homological algebra

As we have attempted to communicate in Chapter 1, abelian categories are such a bountiful abstraction because they grant a category the power of exact sequences. At the same time, exactness is not a foregone conclusion for any sequence of morphisms arising in practice, and a mathematically interesting problem in its own right is to attempt to understand 'how far' from exactness a sequence is. This is the fundamental problem studied in *homological algebra*, which has proved to be an irreplaceable technical tool in the modern practice of algebraic geometry, algebraic topology and a whole host of branches of mathematics.

The goal of this chapter is to develop the required background in this area needed to delve into the derived category of an abelian category in Chapter 3, following [Wei94]. Homological algebra is a topic best served with a more tangible motivating application in mind, and this is traditionally sourced from algebraic topology. Since such material is outside of the scope of this thesis, we will continue to tether our intuitions to $\mathsf{Mod}(R)$.

In all that follows, $\mathcal{A}$ and $\mathcal{B}$ are always abelian categories.

2.1 The category of complexes

Homological algebra can be interpreted as the study of a certain category, so our first task of this chapter is to define it. We start with its objects.

Definition 2.1.1. A *chain complex* in $\mathcal{A}$ is a collection of objects $C^\bullet = \{C_i\}_{i\in\mathbb{Z}}$, called the *chains*, and morphisms $d_\bullet = \{d_i : C_i \to C_{i-1}\}_{i\in\mathbb{Z}}$, called the *differentials*, such that

$d_i d_{i+1} = 0$ for each $i \in \mathbb{Z}$:

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots .$$

Dually, a *cochain complex* in $\mathcal{A}$ is a collection of objects $C^\bullet = \{C^i\}_{i\in\mathbb{Z}}$, called the *cochains*, and morphisms $d^\bullet = \{d^i : C^i \to C^{i+1}\}_{i\in\mathbb{Z}}$, also called the *differentials*, such that $d^i d^{i-1} = 0$ for each $i \in \mathbb{Z}$:

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots .$$

A common topological slogan for the conditions imposed on differentials is that "the boundary of the boundary of a sphere is trivial". The subscripts and superscripts are often suppressed so that in either case the condition can be interpreted informally as saying $d^2 = 0$. Similar informalities within homological algebra abound.

The index i of an individual chain C_i or cochain C^i is called its *degree.* With this, we see that the key difference between chain and cochain complexes is that chain differentials lower degree while cochain differentials raise it. For the most part, we will emphasise cochain complexes in this thesis in order to reflect the historical origins of derived categories and the preferences that have developed in the literature as a result. There is no loss of generality in this choice, since to a certain extent the decision to work with either type of complex over the other is an aesthetic one. The following proposition makes this more formal.

Proposition 2.1.1 (Chain-cochain duality)**.** *Any chain complex $C_\bullet$ may be regarded as a cochain complex $C^\bullet$ by inverting degrees, i.e. by setting $C^i = C_{-i}$ and $d^i = d_{-i}$.*

In practice, the easiest complexes to work with are those which satisfy some boundedness condition. This will be a key theme in this thesis, since the theory of derived categories without such assumptions in place becomes much more involved.

Definition 2.1.2. A complex $C^\bullet$ is said to be

- *bounded above* (resp. *bounded below*) if $C^i = 0$ for all sufficiently large (resp. sufficiently small) degrees i;
- *bounded* if it is both bounded above and bounded below;
- *positively* (resp. *negatively*) *graded* if $C^i = 0$ for all $i < 0$ (resp. $i > 0$).

For a cochain complex $C^\bullet$, we call $Z^i(C^\bullet) := \ker(d^i)$ and $B^i(C^\bullet) := \mathrm{im}(d^{i-1})$ the ith *cocycle* and *coboundary* objects. It is not difficult to show that the condition $d^2 = 0$ implies that $B^i(C^\bullet) \leq Z^i(C^\bullet)$, and so we may form the ith *cohomology* object

$$H^i(C^\bullet) := Z^i(C^\bullet)/B^i(C^\bullet).$$

The same considerations naturally apply to any chain complex $C_\bullet$, and we call

$$Z_i(C_\bullet) := \ker(d_i), \quad B_i(C_\bullet) := \operatorname{im}(d_{i+1}) \quad \text{and} \quad H_i(C_\bullet) := Z_i(C_\bullet)/B_i(C_\bullet)$$

the ith *cycle*, *boundary* and *homology* objects of $C_\bullet$ respectively.

The most interesting examples of chain and cochain complexes arise from applications, and we will see one in Section 4.5. For now, we collect some more remedial examples in order to introduce some useful terminology.

Example 2.1.1. Each $A \in \mathcal{A}$ may be regarded as a bounded complex $A^\bullet$ *concentrated in degree* n, i.e. such that $A^n = A$, $A^i = 0$ for all $i \neq n$ and $d = 0$. In this case, we have $H^n(A^\bullet) \cong A$ and $H^i(A^\bullet) = 0$ whenever $i \neq n$. Sometimes, we will write $A^\bullet = A[n]$.

Example 2.1.2. Since $\operatorname{im}(f) = \ker(g)$ implies that $gf = 0$, any exact sequence in $\mathcal{A}$ can be defined as a complex. More generally, a complex $C^\bullet$ is *exact* if $Z^i(C^\bullet) = B^i(C^\bullet)$ or *acyclic* if $H^i(C^\bullet) = 0$ for all $i \in \mathbb{Z}$, and it is immediate that these two notions are in fact equivalent. Studying the cohomology objects of $C^\bullet$ thus provides insight into the extent to which $C^\bullet$ fails to be exact degreewise.

We now define a suitable notion of a morphism of complexes.

Definition 2.1.3. Let $C^\bullet$ and $D^\bullet$ be cochain complexes in $\mathcal{A}$ with associated differentials d_C and d_D. A *morphism of cochain complexes* (or *cochain map*) $f : C^\bullet \to D^\bullet$ is a collection of morphisms $\{f^i : C^i \to D^i\}_{i \in \mathbb{Z}}$ such that $d_D f = f d_C$, i.e. there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r, "d_C"] \arrow[d, "f"'] & C^i \arrow[r, "d_C"] \arrow[d, "f"] & C^{i+1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r, "d_D"'] & D^i \arrow[r, "d_D"'] & D^{i+1} \arrow[r] & \cdots .
\end{tikzcd}$$

We say that f is an *isomorphism* if each f^i is an isomorphism.

Definition 2.1.4. The *category of cochain complexes* in $\mathcal{A}$ is the category $\mathbf{C}(\mathcal{A})$ whose objects and morphisms are as defined in Definition 2.1.1 and Definition 2.1.3, with composition defined by the degreewise composition of cochain maps. Its full subcategories

$\mathbf{C}^-(\mathcal{A})$, $\mathbf{C}^+(\mathcal{A})$ and $\mathbf{C}^b(\mathcal{A})$ are the categories of *bounded above*, *bounded below* and *bounded* cochain complexes in $\mathcal{A}$ respectively.

It turns out that $\mathbf{C}(\mathcal{A})$ is also an abelian category, and it inherits much of this structure from the base category $\mathcal{A}$. A partial account of the details can be found in [Wei94, Section 1.2], but we shall state the main ideas. The preadditive structure comes from the fact that each hom-set is an abelian group under degreewise addition of cochain maps, and the composition of cochain maps is clearly bilinear with respect to this addition. This upgrades into an additive structure upon noticing that the zero object of $\mathcal{A}$ concentrated in degree 0 plays the role of the zero object (denoted $0^\bullet$) of $\mathbf{C}(\mathcal{A})$, and that the degreewise direct sum of any finite collection of complexes $\{C_i^\bullet\}$ is a suitable biproduct for complexes. Given a cochain map $f : C^\bullet \to D^\bullet$, the complex

$$\cdots \longrightarrow \ker(f^{i-1}) \xrightarrow{d_C^i \circ \ker(f^{i-1})} \ker(f^i) \xrightarrow{d_C^{i+1} \circ \ker(f^i)} \ker(f^{i+1}) \longrightarrow \cdots$$

is a suitable kernel of f, with cokernels constructed similarly. This shows that $\mathbf{C}(\mathcal{A})$ is abelian. Due to this fact, it is possible to consider complexes of complexes in $\mathcal{A}$, which will play a role in Chapter 3. We give only the cochain definition.

Definition 2.1.5. A *double cochain complex* $C^{\bullet\bullet}$ in $\mathcal{A}$ consists of a collection of objects $\{C^{p,q}\}_{p,q\in\mathbb{Z}}$ together with *horizontal* and *vertical* differentials

$$d_h^{p,q} : C^{p,q} \to C^{p+1,q} \quad \text{and} \quad d_v^{p,q} : C^{p,q} \to C^{p,q+1}$$

subject to the anticommutativity relation $d_v d_h + d_h d_v = 0$, and such that each row and column in the following lattice diagram is a chain complex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{p,q-1} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{p+1,q-1} \arrow[r] \arrow[d, "d_v"'] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[r, "d_h"'] \arrow[d, "d_v"'] & C^{p,q} \arrow[r, "d_h"'] \arrow[d, "d_v"'] & C^{p+1,q} \arrow[r] \arrow[d, "d_v"'] & \cdots \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[r, "d_h"'] \arrow[d] & C^{p,q+1} \arrow[r, "d_h"'] \arrow[d] & C^{p+1,q+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Remark 2.1.1 (The sign trick)**.** For a fixed row q, the vertical differentials $d_v^{\bullet,q}$ in a double complex do not form a morphism of the complexes $C^{\bullet,q}$ to $C^{\bullet,q+1}$. This can be remedied by introducing a sign, i.e. by replacing $d_v^{p,q}$ with $(-1)^p d_v^{p,q}$. This has the effect of making squares in the above lattice alternate between commutativity and anticommutativity.

2.2 Cohomology and homotopy

Cohomology is perhaps the most valuable data associated to a cochain complex, so we now introduce a circle of related ideas.

Proposition 2.2.1. *A cochain map $f : C^\bullet \to D^\bullet$ induces morphisms $Z^i(C^\bullet) \to Z^i(D^\bullet)$ and $B^i(C^\bullet) \to B^i(D^\bullet)$. Thus, cohomology is an additive functor $H^i : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$.*

Proof. The argument for abelian categories is a technical use of universal properties, but it is instructive to at least see the explicit construction for $\mathsf{Mod}(R)$. Using the commutative diagram for f, it is not hard to see that each map f^i sends cocycles to cocycles and coboundaries to coboundaries in degree i. Thus the map

$$z + B^i(C^\bullet) \mapsto f^i(z) + B^i(D^\bullet)$$

is the desired module homomorphism $H^i(f) : H^i(C^\bullet) \to H^i(D^\bullet)$, and it follows that

$$H^i(gf) = H^i(g)H^i(f) \quad \text{and} \quad H^i(f+g) = H^i(f) + H^i(g)$$

for all compatible cochain maps f and g, so H^i is an additive functor as claimed. $\square$

Since $\mathbf{C}(\mathcal{A})$ is abelian, it makes sense to consider short exact sequences of complexes. This leads to one of the cornerstone theorems of homological algebra.

Theorem 2.2.1 (Long exact cohomology sequence)**.** *Consider a short exact sequence*

$$0^\bullet \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0^\bullet$$

in $\mathbf{C}(\mathcal{A})$. Then there exist connecting morphisms $\partial^i : H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ *such that*

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H^i(A^\bullet) \arrow[r, "H^i(f)"] & H^i(B^\bullet) \arrow[r, "H^i(g)"] & H^i(C^\bullet) \arrow[dll, dashed, "\partial^i"'] & \\
& H^{i+1}(A^\bullet) \arrow[r, "H^{i+1}(f)"'] & H^{i+1}(A^\bullet) \arrow[r, "H^{i+1}(g)"'] & H^{i+1}(A^\bullet) \arrow[r, dashed] & \cdots
\end{tikzcd}$$

is an exact sequence in $\mathcal{A}$. Moreover, this construction is natural in the sense that given a morphism of short exact sequences of complexes

$$\begin{tikzcd}
0^\bullet \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0^\bullet \\
0^\bullet \arrow[r] & (A')^\bullet \arrow[r] & (B')^\bullet \arrow[r] & (C')^\bullet \arrow[r] & 0^\bullet,
\end{tikzcd}$$

in $\mathbf{C}(\mathcal{A})$, there are commutative squares

$$\begin{tikzcd}
H^i(C^\bullet) \arrow[r, "\partial^i"] \arrow[d] & H^{i+1}(A^\bullet) \arrow[d] \\
H^i((C')^\bullet) \arrow[r, "(\partial')^i"'] & H^{i+1}((A')^\bullet).
\end{tikzcd}$$

Proof. The construction of the connecting morphisms requires the snake lemma. We defer the technical details to the discussion in [Wei94, Section 1.3]. □

For computational purposes, it is often useful to be able to detect when two complexes have the same cohomology. There are two related ways to do this, the latter of which is inspired by the concept in topology of the same name.

Definition 2.2.1. A cochain map $f : C^\bullet \to D^\bullet$ is called a *quasi-isomorphism* (or *qis*) if each of the maps $H^i(f) : H^i(C^\bullet) \to H^i(D^\bullet)$ induced by cohomology is an isomorphism. In case such a map f exists, $C^\bullet$ and $D^\bullet$ are said to be *quasi-isomorphic* complexes.

Definition 2.2.2. A cochain map $f : C^\bullet \to D^\bullet$ is said to be *null-homotopic* if there is a collection of morphisms $\{s^i : C^i \to D^{i-1}\}_{i\in\mathbb{Z}}$, called a *contraction* of f, such that $f = d_D s + s d_C$. We say that f is *homotopic* to another cochain map $g : C^\bullet \to D^\bullet$ if their difference $f - g$ is null-homotopic. We write $f \sim g$ and call a contraction of $f - g$ a *homotopy* from f to g:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r, "d_C"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl] & C^i \arrow[r, "d_C"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "s"'] & C^{i+1} \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "s"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & D^{i-1} \arrow[r, "d_D"'] & D^i \arrow[r, "d_D"'] & D^{i+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Moreover, we say that f is a *homotopy equivalence* if f and g are *homotopy inverses* of each other, i.e. $gf \sim \mathrm{id}_{C^\bullet}$ and $fg \sim \mathrm{id}_{D^\bullet}$.

Corollary 2.2.1. *Let $f, g : C^\bullet \to D^\bullet$ be cochain maps. If f is null-homotopic, then* *each morphism $H^i(f)$ is zero, and if $f \sim g$, then $H^i(f) = H^i(g)$.*

In other words, homotopic cochain maps induce the same morphisms at the level of cohomology. This implies that any homotopy equivalence is also a quasi-isomorphism. Though the converse is generally false, we have the following important partial result in this direction. This takes considerable effort to prove, but one can find a thorough account of the details in [Alu09, Section IX.5.4].

Proposition 2.2.2 ([Alu09, Theorem IX.5.9])**.** *If $C^\bullet$ and $D^\bullet$ are both bounded below complexes of injectives or both bounded above complexes of projectives, then any quasi-isomorphism $f : C^\bullet \to D^\bullet$ is a homotopy equivalence.*

2.3 Operations on complexes

Aside from taking cohomology, there are other useful things which one can do with complexes. We first introduce some more functors on $\mathbf{C}(\mathcal{A})$ which we will need later. The fact that these define additive functors is immediate by construction.

Definition 2.3.1. Let $n \in \mathbb{Z}$. The degreewise (left) shift of complexes and cochain maps by n defines an additive endofunctor $[n] : \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$, called the *$n$th shift* (or *$n$th translation*) functor. Explicitly, $[n]$ sends $C^\bullet$ to the complex $C^\bullet[n]$ with

$$C[n]^i = C^{i+n} \quad \text{and} \quad d[n]^i = (-1)^n d^{i+n},$$

and sends $f : C^\bullet \to D^\bullet$ to the cochain map $f[n] : C^\bullet \to D^\bullet$ with $f[n]^i = f^{i+n}$.

For technical reasons that we shall not explain, it turns out that introducing the sign $(-1)^n$ in the differential like this is generally convenient when setting up the theory. Though it seems like a small observation now, the shift complex has a convenient property in the context of Section 3.2.

Proposition 2.3.1. *The shift functor $[n]$ is autoequivalence for all $n \in \mathbb{Z}$.*

This shift functor is related to another common construction. Associated to any cochain map is its *mapping cone*, once again directly inspired by topology.

Definition 2.3.2. The *mapping cone* of a cochain map $f : C^\bullet \to D^\bullet$ is the cochain complex $\mathrm{cone}(f)$ whose cochains and differentials are given by

$$\mathrm{cone}(f)^i = C^{i+1} \oplus D^i \quad \text{and} \quad d^n = \begin{bmatrix} -d_C^{i+1} & 0 \\ -f^{i+1} & d_D^i \end{bmatrix},$$

with the differential acting as if by componentwise matrix multiplication on $C^{i+1} \oplus D^i$.

The mapping cone has two uses. The first is that it allows one to insert any cochain map $f : C^\bullet \to D^\bullet$ into a short exact sequence in $\mathbf{C}(\mathcal{A})$. In particular, we have that

$$0^\bullet \longrightarrow D^\bullet \longrightarrow \mathrm{cone}(f) \longrightarrow C^\bullet[1] \longrightarrow 0^\bullet$$

is exact, where the two middle cochain maps are the canonical injection and projection of direct sums respectively. This leads to its second use, detecting quasi-isomorphisms.

Corollary 2.3.1. *A cochain map is a quasi-isomorphism if and only if its mapping cone is an exact (equivalently, acyclic) complex.*

Proof. Using the identification $H^i(C^\bullet[1]) \cong H^{i+1}(C^\bullet)$, the long exact cohomology sequence for the above short exact sequence is of the form

$$\cdots \longrightarrow H^i(D^\bullet) \longrightarrow H^i(\mathrm{cone}(f)) \longrightarrow H^{i+1}(C^\bullet) \xrightarrow{\partial^i} H^{i+1}(D^\bullet) \longrightarrow \cdots,$$

where ∂ is the connecting map. It is therefore enough to show that $\partial^i = H^{i+1}(f)$, which is the content of [Wei94, Lemma 1.5.3]. $\square$

An even more simple-minded but crucial operation on complexes is applying an additive functor $F : \mathcal{A} \to \mathcal{B}$ degreewise. This induces a functor $\mathbf{C}(F) : \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{B})$, though for notational ease we will typically refer to $\mathbf{C}(F)$ simply by F itself.

Lemma 2.3.1. *The functor $\mathbf{C}(F)$ is additive and preserves homotopies, homotopy equivalences and mapping cones. Moreover, it commutes with shift functors.*

Proof. These properties follow from functoriality and Lemma 1.1.1. $\square$

In the case of double complexes, the only operation we will concern ourselves with is *totalisation*, which is a construction that collapses a double complex down into a normal complex in one of two ways.

Definition 2.3.3. Given a double complex $C^{\bullet\bullet}$, the *total complexes* $\mathrm{Tot}^\Pi(C^{\bullet\bullet})$ and $\mathrm{Tot}^\oplus(C^{\bullet\bullet})$ are those obtained by taking products and direct sums along each lattice diagonal (c.f. Definition 2.1.5), i.e.

$$\mathrm{Tot}^\Pi(C^{\bullet\bullet})^i = \prod_{i=p+q} C^{p,q} \quad \text{and} \quad \mathrm{Tot}^\oplus(C^{\bullet\bullet})^i = \bigoplus_{i=p+q} C^{p,q}.$$

In both cases, the differential is defined componentwise by $d = d_h^{p,q} + d_v^{p,q}$.

Remark 2.3.1. A priori, totalising a double complex involves taking infinite products or direct sums of objects in $\mathcal{A}$, which may not be possible if the base category $\mathcal{A}$ is not *(co)complete*. However, many double complexes arising in practice, including any we will encounter in this thesis, have only finitely many objects along each lattice diagonal. A sufficient condition for this to occur is that the double complex to be totalised is constructed out of bounded complexes.

2.4 Resolutions

A particularly useful philosophy in homological algebra is to try and replace a single object by an exact sequence of objects that are somehow easier to study. We will see in the next section that this has certain computational benefits, but a more honest motivation is as follows. For a given R-module M, the length of an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each P_i is a projective module is the maximal index i such that $P_i \neq 0$ but $P_j = 0$ for all $j \geq i$. The minimal length of any such sequence is called the *projective dimension* of M. This quantity provides a handle on how far M is from being projective, since any projective module has projective dimension 0. Moreover, the supremum over the projective dimensions of all R-modules is called the *global dimension* of the ring R and has important applications in the dimension theory of Noetherian rings.

Definition 2.4.1. A *left resolution* of an object $A \in \mathcal{A}$ is a negatively graded complex $C^\bullet$ extended by a morphism $\varepsilon : C^0 \to A$ to an exact sequence

$$\cdots \longrightarrow C^{-2} \xrightarrow{\ d\ } C^{-1} \xrightarrow{\ d\ } C^0 \xrightarrow{\ \varepsilon\ } A \longrightarrow 0.$$

We denote this by $C^\bullet \to A$. Similarly, a *right resolution* $A \to C^\bullet$ is a positively graded complex $C^\bullet$ extended by a morphism $\varepsilon : A \to C^0$ to an exact sequence

$$0 \longrightarrow A \xrightarrow{\ \varepsilon\ } C^0 \xrightarrow{\ d\ } C^1 \xrightarrow{\ d\ } C^2 \longrightarrow \cdots .$$

Remark 2.4.1. Left resolutions are typically chain complexes so that the grading of both left and right resolutions are positive, but this is just an aesthetic choice.

Remark 2.4.2. Resolutions of an object are equivalent to quasi-isomorphisms to or from that object concentrated in degree 0. For example, a right resolution $A \to C^\bullet$ is a quasi-isomorphism to some positively graded cochain complex $C^\bullet$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varepsilon"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r] & C^1 \arrow[r] & C^2 \arrow[r] & \cdots
\end{tikzcd}$$

In particular, this implies that $C^\bullet$ is exact except in degree 0.

The most significant class of resolutions is the following.

Definition 2.4.2. A *projective resolution* of $A \in \mathcal{A}$ is a left resolution $P^\bullet \to A$ such that each P^i is projective and an *injective resolution* of A is a right resolution $A \to I^\bullet$ such that each I^i is injective.

In the case of $\mathsf{Mod}(R)$, the existence of injective envelopes shows that every module has an injective resolution, and separately one can also find free (a fortiori, projective and flat) resolutions. The situation for other abelian categories can be drastically different, as the case of FinAb from Example 1.2.2 shows.

Proposition 2.4.1 ([Wei94, Lemma 2.2.5])**.** *If $\mathcal{A}$ has enough projectives (resp. injectives), then every object of $\mathcal{A}$ has a projective (resp. injective) resolution.*

Injective and projective resolutions of an object need not be unique whenever they exist, but any two of the same kind are always homotopy equivalent. This is an immediate corollary of the following result, which is so useful that it is commonly known as the *fundamental theorem/lemma of homological algebra.* We present only the statement for projective resolutions, but the dual statement for injective resolutions also holds.

Theorem 2.4.1 ([Wei94, Theorem 2.2.6])**.** *Let $f : A \to B$ be a morphism in $\mathcal{A}$. Then for any projective resolution $P^\bullet \to A$ and left resolution $C^\bullet \to B$, there is a lift of f to a chain map $\widetilde{f} : P^\bullet \to C^\bullet$, unique up to homotopy equivalence, making the following diagram commute:*

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-2} \arrow[r] \arrow[d, dashed, "\widetilde{f}"] & P^{-1} \arrow[r] \arrow[d, dashed, "\widetilde{f}"] & P^0 \arrow[r] \arrow[d, dashed, "\widetilde{f}"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & C^{-2} \arrow[r] & C^{-1} \arrow[r] & C^0 \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

2.5 Classical derived functors

Now that we have a systematic way to study the failure of a complex to be exact, we finish by discussing the same problem for a covariant functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories. Specifically, we will see that when F is left (resp. right) exact, there exist *classical right* (resp. *left*) *derived functors* that measure the failure of F to be exact. The reason for the term 'classical' will become clear in Section 3.4.

Our experience in the previous sections show that cohomology can be a useful tool in these situations. Replacing any object $A \in \mathcal{A}$ with a right resolution $A \to C^\bullet$ and applying a left exact functor $F : \mathcal{A} \to \mathcal{B}$ gives a cochain complex

$$\cdots \longrightarrow 0 \longrightarrow F(C^0) \longrightarrow F(C^1) \longrightarrow F(C^2) \longrightarrow \cdots$$

in $\mathcal{B}$. This will typically have non-trivial cohomology, but it is always acyclic when F is exact. While this does seem to have the measuring capabilities we seek, a problem is that the cohomology objects we obtain are sensitive to our choice of resolution. If we instead work with injective resolutions, then any two will be homotopy equivalent by Theorem 2.4.1, and so we will always recover the same cohomology objects. The intuitive reason for considering right resolutions here to begin with is that left exact functors have issues extending left exact sequences to the right (c.f. Definition 1.2.3).

Definition 2.5.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and suppose that $\mathcal{A}$ has enough injectives. For each $A \in \mathcal{A}$, fix an injective resolution $A \to I^\bullet$ and define

$$R^iF(A) := H^i(F(I^\bullet))$$

for $i \in \mathbb{Z}_{\geq 0}$. The functor $R^iF : \mathcal{A} \to \mathcal{B}$ is called the *ith right derived functor* of F.

We have already seen that this is a well-defined map on objects, but we have yet to specify what R^iF does to morphisms. Given a morphism $f : A \to B$, fix a pair of injective resolutions $A \to I^\bullet$ and $B \to J^\bullet$ and then lift f using Theorem 2.4.1 to a cochain map $\widetilde{f} : I^\bullet \to J^\bullet$. This defines a unique map

$$R^iF(f) := H^n(F(\widetilde{f})) : H^i(F(I^\bullet)) \to H^i(F(J^\bullet)),$$

since any lift of f is unique up to homotopy equivalence. It is now clear that R^iF is actually a functor to begin with and is moreover additive.

The key properties that the functors R^iF have are best expressed by the following framework due to Grothendieck. As the name implies, the model for this will be the collection of cohomology functors of Proposition 2.2.1, with the key property being that there is an analogue of the long exact cohomology sequence.

Definition 2.5.2. A *cohomological δ-functor* $T : \mathcal{A} \to \mathcal{B}$ consists of a collection of additive functors $\{T^i : \mathcal{A} \to \mathcal{B}\}_{i \in \mathbb{Z}_{\geq 0}}$ subject to the following two conditions:

1. For each short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, there exist morphisms $\delta^i : T^i(C) \to T^{i+1}(A)$ and a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T^0(A) \arrow[r] & T^0(B) \arrow[r] & T^0(C) \arrow[dll, dashed, "\delta^0"'] \\ & T^1(A) \arrow[r] & T^1(B) \arrow[r] & T^1(C) \arrow[dll, dashed, "\delta^1"'] \\ & T^2(A) \arrow[r] & T^2(B) \arrow[r] & T^2(C) \arrow[r, dashed] & \cdots \end{tikzcd}$$

in $\mathcal{B}$. (In particular, T^0 is left exact.)

2. Given a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$, each of the below squares in $\mathcal{B}$ commute:

$$\begin{tikzcd} T^i(C) \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A) \arrow[d] \\ T^i(C') \arrow[r, "(\delta')^i"'] & T^{i+1}(A'). \end{tikzcd}$$

Definition 2.5.3. Let $S, T : \mathcal{A} \to \mathcal{B}$ be cohomological δ-functors. A *morphism of δ-functors* $\mu : S \to T$ is a collection of natural transformations $\{\mu^i : S^i \Rightarrow T^i\}_{i \in \mathbb{Z}_{\geq 0}}$ which commute with δ_S and δ_T. That is, for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, each of the below squares in $\mathcal{B}$ commute:

$$\begin{tikzcd} S^i(C) \arrow[r, "\delta_S^i"] \arrow[d, "\mu_C^i"'] & S^{i+1}(A) \arrow[d, "\mu_A^{i+1}"] \\ T^i(C) \arrow[r, "\delta_T^i"'] & T^{i+1}(A). \end{tikzcd}$$

Moreover, T is *universal* if for any choice of S and natural transformation $\mu^0 : S^0 \Rightarrow T^0$

there exists a unique extension of μ^0 to a morphism of δ-functors $\mu : S \to T$.

Proposition 2.5.1. *The collection of all right derived functors of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ form a cohomological universal δ-functor.*

Proof. Both parts of this claim are non-trivial. The proofs in the dual case of the *left* derived functors $L_i F$ of a *right* exact F are the content of [Wei94, Theorems 2.4.6 and 2.4.7], but with our initial hypotheses it will turn out that

$$R^i F(A) = (L_i F^{\mathrm{op}})^{\mathrm{op}}(A)$$

for all $A \in \mathcal{A}$, so this is indeed enough. $\square$

There are several other benefits to this construction. First, the information of the original functor F is captured by $R^0 F$. Indeed, since F is left exact, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(I^0) \longrightarrow F(I^1)$$

is exact for any injective resolution $A \to I^\bullet$, so

$$R^0 F(A) = H^0(F(I^\bullet)) \cong \ker(F(I^0) \to F(I^1)) = \mathrm{im}(F(A) \to F(I^0)) = F(A),$$

and moreover $R^0 F(f) = F(f)$ for any morphism f. If I is an injective object, then computing $R^i F(I)$ using the injective resolution

$$0 \longrightarrow I \xrightarrow{\mathrm{id}} I \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

shows that $R^i F(I) = 0$ for $i > 0$. It is also clear that if F is exact, then $R^i F = 0$ for $i > 0$. The upshot is therefore that the right derived functors of F provide a convenient way of measuring how far F is from being exact as intended, and the fact that they are a cohomological universal δ-functor shows that they are the most natural way to do so.

Remark 2.5.1. Even with enough injectives, actually obtaining a injective resolution for an object can be difficult. With reference to a specific functor F, an object $X \in \mathcal{A}$ is said to be F-*acyclic* if $R^i F(X) = 0$ for $i > 0$, and it turns out that it is sufficient to use F-acyclic resolutions to compute $R^i F(A)$, since $R^i F(A) \cong H^i(F(X^\bullet))$ for any F-acyclic resolution $A \to X^\bullet$. This can be seen by the *dimension shifting* argument outlined (albeit for the left derived case again) in [Wei94, Exercise 2.4.3].

We now summarise the analogous construction of left derived functors when F is right exact. What we will end up with is a *homological* δ-functor, which is defined in a dual way to cohomological δ-functors (c.f. [Wei94, Section 2.1]). The strategy is the same as last time, except we now use projective resolutions and take negative cohomology.

Proposition 2.5.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor and suppose that $\mathcal{A}$ has enough projectives. For each $A \in \mathcal{A}$, fix a projective resolution $P^\bullet \to A$ and define*

$$L_i F(A) := H^{-i}(F(P^\bullet))$$

*for each $i \in \mathbb{Z}_{\geq 0}$. The corresponding functor $L_i F : \mathcal{A} \to \mathcal{B}$ is called the n*th classical left derived functor *of F, and it satisfies the following properties:*

1. *Each functor $L_i F$ is well-defined up to natural isomorphism and additive.*
2. *$L_0 F(A) \cong F(A)$ and $L_0 F(f) = F(f)$ for each object $A \in \mathcal{A}$ and morphism $f \in \mathcal{A}$.*
3. *If P is projective, then $L_i F(P) = 0$ for $i > 0$.*
4. *The left derived functors of F are a homological universal δ-functor.*

Remark 2.5.2. The above discussion assumes covariant functors, but the construction of left and right derived functors can easily be carried out for a contravariant functor $F : \mathcal{A} \to \mathcal{B}$. Since $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is covariant and an injective resolution in $\mathcal{A}^{\mathrm{op}}$ becomes a projective resolution in $\mathcal{A}$, one can compute right derived functors when F is left exact and $\mathcal{A}$ has enough projectives, and dually for left derived functors.

We close out this chapter by introducing the two most important examples of these derived functors, followed by some final remarks.

Definition 2.5.4. Let $\mathcal{A}$ be an abelian category with enough injectives, and fix an object $A \in \mathcal{A}$. Then the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ is left exact, and its suite of classical right derived functors are the associated *Ext functors*

$$\mathrm{Ext}^i_{\mathcal{A}}(A, -) := R^i \,\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathsf{Ab}.$$

Definition 2.5.5. Fix an R-module N. Then the functor $- \otimes_R N$ is right exact, and its suite of classical left derived functors are the associated *Tor functors*

$$\mathrm{Tor}^R_i(-, N) := L_i(- \otimes_R N) : \mathsf{Mod}(R) \to \mathsf{Ab}.$$

The names given to these derived functors are suggestive of some of their applications. Given an R-module M, the Ext groups $\mathrm{Ext}^1_R(M, N)$ are related to the *module extensions* of M by another R-module N. If $r \in R$ is not a zero divisor, the Tor group

$\mathrm{Tor}_1^R(R/(r), M)$ is the *r-torsion subgroup* of M. We will not discuss these applications any further, but these are the subject of [Wei94, Chapter 3].

A useful fact about Ext that we will use later is the following.

Lemma 2.5.1. *Let $\mathcal{A} \to \mathcal{B}$ be an equivalence of abelian categories both with enough projectives. Then for all $A, B \in \mathcal{A}$ and $i \in \mathbb{Z}_{\geq 0}$, there is an isomorphism*

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \cong \mathrm{Ext}^i_{\mathcal{B}}(F(A), F(B)).$$

Proof. Fix an injective resolution $B \to I^\bullet$. Since F is fully faithful, we have

$$\mathrm{Hom}_{\mathcal{A}}(A, I^j) \cong \mathrm{Hom}_{\mathcal{B}}(F(A), F(I^j))$$

for each $j \in \mathbb{Z}_{\geq 0}$. As in Remark 1.2.1, it turns out that an equivalences of categories also preserves limits and colimits, and from this it follows that F is an exact functor that preserves injectives. This shows that $F(B) \to F(I^\bullet)$ is a projective resolution, so

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = H^i \mathrm{Hom}_{\mathcal{A}}(A, I^\bullet) \cong H^i \mathrm{Hom}_{\mathcal{B}}(F(A), F(I^\bullet)) = \mathrm{Ext}^i_{\mathcal{A}}(F(A), F(B))$$

for $i \in \mathbb{Z}_{\geq 0}$ as claimed. $\square$

Remark 2.5.3. Since Hom_R and $\otimes_R$ are *bifunctors*, one also could have decided to construct Ext and Tor by instead taking the derived functors $R^i \mathrm{Hom}_R(-, N)$ and $L_i(M \otimes_R -)$. Through a careful process of *balancing*, one sees that in fact these functors will be naturally isomorphic to Ext and Tor as we have defined them above. The details are lengthy and immaterial to the goals of this thesis, so we refer the interested reader to [Wei94, Section 2.7].

Remark 2.5.4. The Tor functors provide the most straightforward example of a functor which benefits from being computed by acyclic resolutions as described in Remark 2.5.1. Retaining the hypotheses of Proposition 2.5.2, an object $X \in \mathcal{A}$ is F-acyclic if $L_iF(X) = 0$ for $i > 0$, and similarly we have $L_iF(A) \cong H^{-i}(F(X^\bullet))$ for an F-acyclic resolution $X^\bullet \to A$. In particular, $\mathrm{Tor}_i^R(A, N) = 0$ for $i > 0$ whenever A is a flat R-module, so Tor can be computed using flat resolutions.

Chapter 3

Derived categories

We are now sufficiently prepared to establish the notion of the *derived category*, following the texts [Wei94; Huy06; GM03]. At its core, this technical apparatus provides a more natural computational setting for the study of homological algebra than the category of cochain complexes. In comparison to the level of abstractions we have encountered thus far, the material in this chapter will at times become rather technical, so it will be beneficial to first outline the main motivations for derived categories and collect some desiderata to hold onto.

The central conceit is that from a computational standpoint, it is convenient to be able to treat complexes that have the same cohomology (i.e. those which are quasi-isomorphic, homotopy equivalent or both) as if they are isomorphic. Indeed, we have seen a concrete instance of this already in Section 2.5, which suggests that we should be able to freely interchange the objects of an abelian category with any of its resolutions when computing derived functors. At the same time, a complex is more than just its cohomology, and so on the face of it, associating complexes to a mathematical object of interest rather than just its cohomology leads to a more fruitful source of algebraic invariants. In much the same way, the historical viewpoint of Grothendieck was that the suite of classical derived functors should be the information extracted from a single derived functor existing at the level of complexes in the derived category.

In all that follows, $\mathcal{A}$ will be an abelian category.

3.1 The homotopy category

Recall that in Section 2.1, we introduced $\mathbf{C}(\mathcal{A})$, the category of cochain complexes with objects in $\mathcal{A}$, and noted that it itself was abelian. In the first phase of constructing the derived category of $\mathcal{A}$, we wish to alter $\mathbf{C}(\mathcal{A})$ to identify cochain maps up to homotopy equivalence. The justification for this should be clear, as we have already seen that homotopy equivalent maps preserve cohomology.

Definition 3.1.1. Let $\mathcal{C}$ be a category. A *congruence relation* $\sim$ on $\mathcal{C}$ consists of an equivalence relation $\sim_{A,B}$ on each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ which is compatible with composition, that is, if $f \sim_{A,B} f'$ and $g \sim_{B,C} g'$, then $gf \sim_{A,C} g'f'$. We define the *quotient category* $\mathcal{C}/\sim$ to have the same objects as $\mathcal{C}$, but whose hom-sets are equivalence classes of morphisms in hom-sets of $\mathcal{C}$, i.e.

$$\mathrm{Hom}_{\mathcal{C}/\sim}(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim$$

for each pair of objects A, $B \in \mathcal{C}$. The full *quotient functor* $\pi : \mathcal{C} \to \mathcal{C}/\sim$ is the identity on objects and sends morphisms to their equivalence class under $\sim$.

Definition 3.1.2. The *homotopy category* of an abelian category $\mathcal{A}$ is the quotient category $\mathbf{K}(\mathcal{A}) := \mathbf{C}(\mathcal{A})/\sim$ under homotopy equivalence.

Eventually we will consider the *bounded* homotopy categories $\mathbf{K}^*(\mathcal{A})$, which are the full subcategories of $\mathbf{K}(\mathcal{A})$ corresponding to the categories $\mathbf{C}^*(\mathcal{A})$ for each $* \in \{b, +, -\}$. Under certain hypotheses on $\mathcal{A}$, considering the full subcategories of $\mathbf{K}^*(\mathcal{A})$ consisting of complexes of injectives and projectives is sufficient to describe its bounded derived category, which are the particular flavour of derived category we will concern ourselves with. These details will be made precise at the end of Section 3.3.

By construction, the image of any homotopy equivalence in $\mathbf{C}(\mathcal{A})$ under π is an isomorphism in $\mathbf{K}(\mathcal{A})$. It is possible to formulate a universal property for $\mathbf{K}(\mathcal{A})$ along these lines, which provides a useful way of thinking about the homotopy category.

Proposition 3.1.1 ([Wei94, Proposition 10.1.2])**.** *Any functor $F : \mathbf{C}(\mathcal{A}) \to \mathcal{D}$ with the property that it sends homotopy equivalences in $\mathbf{C}(\mathcal{A})$ to isomorphisms in $\mathcal{D}$ factors uniquely through $\mathbf{K}(\mathcal{A})$.*

$$\begin{tikzcd} \mathbf{C}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{D}. \\ \mathbf{K}(\mathcal{A}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proposition 3.1.2. *The category $\mathbf{K}(\mathcal{A})$ is additive and has additive quotient functor.*

After identifying homotopy equivalent maps, we unfortunately lose the abelian structure of $\mathbf{C}(\mathcal{A})$ in general. Derived categories suffer the same fate, and at a glance this seems rather damning for any hopes of doing homological algebra, as we no longer have access to exact sequences. However, the shift functor on complexes gives $\mathbf{K}(\mathcal{A})$ some additional structure beyond being additive, and it turns out that there is a principled way to think of certain sequences of maps in a way that mimics exact sequences.

Definition 3.1.3. Let $u : A^\bullet \to B^\bullet$ be a map of complexes in $\mathbf{C}(\mathcal{A})$. The *strict triangle* on u is the ordered triple of maps (u, v, δ) in $\mathbf{K}(\mathcal{A})$, where v and δ are maps in $\mathbf{C}(\mathcal{A})$ arising from the short exact sequence

$$0^\bullet \longrightarrow B^\bullet \xrightarrow{v} \operatorname{cone}(u) \xrightarrow{\delta} A^\bullet[1] \longrightarrow 0^\bullet.$$

Definition 3.1.4. Given maps

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

in $\mathbf{K}(\mathcal{A})$, the ordered triple (u, v, w) is called an *exact triangle* on $(A^\bullet, B^\bullet, C^\bullet)$ if there is a strict triangle (u', v', w') and a commutative diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d, "f"'] & B^\bullet \arrow[r, "v"] \arrow[d, "g"'] & C^\bullet \arrow[r, "w"] \arrow[d, "h"] & A^\bullet[1] \arrow[d, "f[1]"] \\
(A')^\bullet \arrow[r, "u'"'] & (B')^\bullet \arrow[r, "v'"'] & \operatorname{cone}(u') \arrow[r, "\delta"'] & (A')^\bullet[1]
\end{tikzcd}$$

in $\mathbf{K}(\mathcal{A})$ such that the maps f, g and h are isomorphisms.

One should think of exactness for triangles as saying that there is an isomorphism between (u, v, w) and the strict triangle (u', v', δ). The main feature of short exact sequences in $\mathbf{C}(\mathcal{A})$ is that the cohomology functor induces long exact sequences in $\mathcal{A}$. When one instead works in $\mathbf{K}(\mathcal{A})$, cohomology remains a well-defined family of functors since homotopic maps induce the same maps on cohomology, and an exact triangle still induces a long exact cohomology sequence in $\mathcal{A}$.

Proposition 3.1.3 ([Wei94, Corollary 10.1.4])**.** *Given an exact triangle (u, v, w) in $\mathbf{K}(\mathcal{A})$, there is a long exact sequence*

$$\cdots \longrightarrow H^i(A^\bullet) \xrightarrow{H^i(u)} H^i(B^\bullet) \xrightarrow{H^i(v)} H^i(C^\bullet) \xrightarrow{H^i(w)} H^{i+1}(A^\bullet) \longrightarrow \cdots$$

in $\mathcal{A}$, where we make the identification $H^{i+1}(A^\bullet) \cong H^i(A^\bullet[1])$.

Remark 3.1.1. Exact triangles in $\mathbf{K}(\mathcal{A})$ are not a perfect substitute for exact sequences in $\mathbf{C}(\mathcal{A})$. To wit, not every short exact sequence

$$0^\bullet \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0^\bullet$$

in $\mathbf{C}(\mathcal{A})$ can be completed to an exact triangle (u, v, w) in $\mathbf{K}(\mathcal{A})$, in the sense that the last map $w : C^\bullet \to A^\bullet[1]$ may not exist. For an example, see [Wei94, Exercise 10.1.2].

Remark 3.1.2. With the exception of Proposition 3.1.3 (which requires that mapping cones exist), everything we have discussed in this section only depends on the fact that $\mathcal{A}$ is additive. This allows us to make sense of $\mathbf{K}(\mathcal{C})$ whenever $\mathcal{C}$ is a full additive subcategory of $\mathcal{A}$, which will have important implications for the derived category later. It is also clear that everything we have discussed works with $\mathbf{K}^*(\mathcal{A})$ in place of $\mathbf{K}(\mathcal{A})$.

3.2 Triangulated categories

This particular structure of triangles, including the fact that exact triangles give rise to long exact cohomology sequences, can be found in categories other than the homotopy category. Since this will apply also to derived categories, it is worth taking an intermission to discuss the general case, which is that of Verdier's *triangulated categories*.

We start with a general definition of triangles and morphisms of triangles. These concepts are similar to what we have already seen for $\mathbf{K}(\mathcal{A})$, but will make sense in any category with a functor similar to the shift of complexes.

Definition 3.2.1. A *translation functor* on a category $\mathcal{C}$ is an autoequivalence (i.e. an equivalence of categories $T : \mathcal{C} \to \mathcal{C}$). For notational ease, we shall write $[n] = T^n$ and $[-n] = T^{-n}$ for each $n > 0$, where T^n and T^{-n} are the n-fold applications of T and T^{-1} respectively.

Definition 3.2.2. Let $\mathcal{C}$ be a category equipped with a *translation functor* T. Given an ordered triple of objects (A, B, C) in $\mathcal{C}$, a *triangle* over (A, B, C) is an ordered triple (u, v, w) corresponding to a sequence

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1],$$

so called as this sequence may suggestively be 'folded' into a triangular diagram

$$
\begin{tikzcd}
 & C \arrow[dl, "w"'] & \\
A \arrow[rr, "u"'] & & B. \arrow[ul, "v"']
\end{tikzcd}
$$

Definition 3.2.3. Given two triangles (u, v, w) and (u', v', w') over (A, B, C) and (A', B', C') respectively, a *morphism of triangles* is an ordered triple of morphisms (f, g, h) such that the diagram

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1]
\end{tikzcd}
$$

commutes in $\mathcal{C}$, and an *isomorphism of triangles* if f, g and h are isomorphisms.

In a manner of speaking, a triangulated category is one in which all triangles 'play nicely' with a certain family of triangles analogous to the exact triangles in $\mathbf{K}(\mathcal{A})$. This is made more precise by the following definition, rather infamous for its complexity.

Definition 3.2.4. A *triangulated category* is an additive category $\mathcal{D}$ equipped with a translation functor T and a family of triangles, called the *exact* (or *distinguished*) triangles in $\mathcal{D}$, subject to the following four axioms:

(TR1) For each object A, the triangle $(\mathrm{id}_A, 0, 0)$ is exact over $(A, A, 0)$. For each morphism $u : A \to B$, there exists an object C together with morphisms v and w so that the triangle (u, v, w) is exact over (A, B, C). Any triangle isomorphic to an exact triangle is also exact.

(TR2) *(The rotation axiom.)* If the triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is exact, then so are the rotated triangles

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1],$$

$$C[-1] \xrightarrow{-w[-1]} A \xrightarrow{u} B \xrightarrow{v} C.$$

(TR3) *(The morphism axiom.)* Given exact triangles (u, v, w) and (u', v', w') over

(A, B, C) and (A', B', C') respectively, as well as morphisms $f : A \to A'$ and $g : B \to B'$ such that $gu = u'f$, there exists a morphism $h : C \to C'$ so that (f, g, h) is a morphism of triangles, i.e.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, dashed, "h"] & A[1] \arrow[d, "{f[1]}"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1] \end{tikzcd}$$

may be completed to a commutative diagram in $\mathcal{D}$.

(TR4) *(The octahedral axiom.)* Omitted, but the interested reader may refer to [Wei94, Definition 10.2.1] for a full statement.

Our reasons for omitting (TR4) are twofold. It is not needed in this thesis, since we will not prove that any categories are triangulated, and the axiom itself is rather complicated to print due to the octahedral diagram from which it gets its name.

Example 3.2.1. We claimed that $\mathbf{K}(\mathcal{A})$ is triangulated, although so far we have only checked that (TR1) holds. For the remaining details, see [Wei94, Proposition 10.2.4]. Naturally, the same is true for the categories $\mathbf{K}^*(\mathcal{A})$, and it will even be true for $\mathbf{K}^*(\mathcal{C})$ when $\mathcal{C}$ is some full additive subcategory of $\mathcal{A}$.

Remark 3.2.1. A triangulated category $\mathcal{D}$ is usually not abelian, and it turns out that a necessary and sufficient condition for this to occur is that $\mathcal{D}$ is a *semisimple* category. For details, see [GM03, Section III.2.3 and Exercise IV.1.1]. In the particular case of $\mathbf{K}(\mathcal{A})$, one notable obstruction is that homotopy equivalent maps most likely do not share the same kernels or cokernels.

Of course, what we would really like in order to make the substitution most effective is a generalisation the result of Proposition 3.1.3 to arbitrary triangulated categories.

Definition 3.2.5. Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is said to be a *cohomological functor* if, for any exact triangle (u, v, w) on (A, B, C), there is a long exact sequence

$$\cdots \longrightarrow H(A[i]) \xrightarrow{u_i^*} H(B[i]) \xrightarrow{v_i^*} H(C[i]) \xrightarrow{w_i^*} H(A[i+1]) \longrightarrow \cdots$$

in $\mathcal{A}$, where the induced maps are given by $f_i^* = H(f[i])$. In this long exact sequence, we often write $H^i = H \circ [i]$ for $i > 0$ and $H^0 = H$ for notational ease.

In other words, cohomological functors send exact triangles to long exact sequences, but we may also consider functors that instead send them to exact triangles. This fills the remaining hole left by the loss of exact sequences, which is a substitute for the notion of exact functors between triangulated categories, and will prove useful in Section 3.4. Though some authors also use the term 'exact' to refer to such a functor, we will opt to make the terminological distinction between usual exactness and triangulated exactness clear in the following definition.

Definition 3.2.6. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories, and denote by $T_{\mathcal{D}}$ and $T_{\mathcal{D}'}$ their respective translation functors. We say that an additive functor $F : \mathcal{D} \to \mathcal{D}'$ is an *exact triangulated functor* if it satisfies the following two conditions:

1. F commutes with translation up to natural isomorphism, i.e. there exists a natural isomorphism of composite functors $\eta : F \circ T_{\mathcal{D}} \Rightarrow T_{\mathcal{D}'} \circ F$.

2. F sends exact triangles to exact triangles, in the sense that for each exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

in $\mathcal{D}$, the following is an exact triangle in $\mathcal{D}'$:

$$F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \xrightarrow{\eta_A \circ F(w)} F(A)[1].$$

3.3 Localising the homotopy category

We now have a triangulated category $\mathbf{K}(\mathcal{A})$ that turns homotopy equivalences into isomorphisms. While it is true that homotopy equivalences are quasi-isomorphisms, we have noted previously that the converse is generally false. If we are yet to fully achieve our goal of identifying complexes with the same cohomology, this observation suggests that we need a way to make the quasi-isomorphisms actual isomorphisms too. Taking cues from the method in commutative algebra of the same name, the second and final phase of constructing the derived category is to take a *localisation* of $\mathbf{K}(\mathcal{A})$.

Definition 3.3.1. Let S be a collection of morphisms in a category $\mathcal{C}$. A *localisation of $\mathcal{C}$ with respect to S* is a category $S^{-1}\mathcal{C}$ and a *localisation functor* $Q_S : \mathcal{C} \to S^{-1}\mathcal{C}$ such that $Q_S(s)$ is an isomorphism for every $s \in S$, and which is universal among all functors with this property. That is, any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism in $\mathcal{D}$ for every $s \in S$ factors uniquely through Q_S:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q_S"'] & \mathcal{D}. \\ S^{-1}\mathcal{C} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

The universal property shows that if a localisation exists, it is unique up to an equivalence of categories. We will follow the explicit construction given by the *calculus of fractions* proposed by Gabriel and Zisman in 1967. The mild hypothesis that this calculus requires is that S forms a *multiplicative system.* There are some more foundational concerns (i.e. hom-sets potentially no longer being sets) that are technically relevant in what follows, but which we will ignore.

Definition 3.3.2. A *multiplicative system* in a category $\mathcal{C}$ is a collection of morphisms S satisfying the following three axioms:

1. *(Closure.)* S is closed under composition and contains all identity morphisms.

2. *(The Ore condition.)* Given $t : D \to B$ in S and $g : A \to B$ in $\mathcal{C}$, there exist $s : C \to A$ in S and $f : C \to D$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} C \arrow[r, dashed, "f"] \arrow[d, dashed, "s"'] & D \arrow[d, "t"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

   commutes. Symmetrically, given the morphisms f and s as in the above diagram, one may find such t and g. This is expressed informally as $t^{-1}g = fs^{-1}$.

3. *(Cancellation.)* If $f, g : A \to B$ are morphisms in $\mathcal{C}$, then $sf = sg$ for some $s : B \to C$ in S if and only if $ft = gt$ for some $t : D \to A$ in S.

Proposition 3.3.1. *Let S be the collection of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$. Then S is a multiplicative system in $\mathbf{K}(\mathcal{A})$.*

Proof. By definition, S is the collection of morphisms such that $H^i(s)$ is an isomorphism for all $i \in \mathbb{Z}$ and $s \in S$, so we say that S *arises from* the functor $H^0 : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$. By the first claim of [Wei94, Proposition 10.4.1], any collection arising from a cohomological functor must be a multiplicative system. □

Proceeding according to the calculus of fractions, localising $\mathbf{K}^*(\mathcal{A})$ does not require changing the set of objects. It therefore remains to say what the morphisms should be and how to compose them, which is the actual namesake of this construction.

Definition 3.3.3. An arrangement of cochain maps $A^\bullet \overset{s}{\leftarrow} X^\bullet \overset{f}{\rightarrow} B^\bullet$ in $\mathbf{K}(\mathcal{A})$ is called a *fraction* (or *roof*) if s a quasi-isomorphism. We use the more compact and suggestive notation fs^{-1} whenever the underlying complexes are understood.

Given compatible fractions $A^\bullet \overset{s}{\leftarrow} X^\bullet \overset{f}{\rightarrow} B^\bullet$ and $B^\bullet \overset{t}{\leftarrow} Y^\bullet \overset{g}{\rightarrow} C^\bullet$, let us attempt to compose them. Using the Ore condition, choose a quasi-isomorphism $r : Z^\bullet \to X^\bullet$ and a cochain map $h : Z^\bullet \to Y^\bullet$ fitting into a commutative diagram

$$\begin{tikzcd}
 & & B^\bullet & & \\
 & X^\bullet \arrow[ur, "f"] \arrow[dl, "s"'] & & Y^\bullet \arrow[ul, "t"'] \arrow[dr, "g"] & \\
A^\bullet & & Z^\bullet \arrow[ul, dashed, "r"] \arrow[ur, dashed, "h"'] & & C^\bullet.
\end{tikzcd}$$

We declare the fraction

$$A^\bullet \overset{sr}{\leftarrow} Z^\bullet \overset{gh}{\rightarrow} C^\bullet$$

to be 'the' composite of fs^{-1} and gt^{-1}. It is clear that this composition operation is associative, and that the fraction

$$A^\bullet \overset{\mathrm{id}}{\leftarrow} A^\bullet \overset{\mathrm{id}}{\rightarrow} A^\bullet$$

acts as a two-sided identity. However, one issue is that since r and h are not guaranteed to be unique, neither in general is the composition of two fractions. To remedy this, we need a notion of fraction equivalence.

Definition 3.3.4. Two fractions

$$A^\bullet \overset{s_1}{\leftarrow} X_1{}^\bullet \overset{f_1}{\rightarrow} B^\bullet \quad \text{and} \quad A^\bullet \overset{s_2}{\leftarrow} X_2{}^\bullet \overset{f_2}{\rightarrow} B^\bullet$$

are *equivalent* if they are *dominated* by a third fraction $A^\bullet \overset{s_3}{\leftarrow} X_3{}^\bullet \overset{f_3}{\rightarrow} B^\bullet$, i.e. there are cochain maps $X_3{}^\bullet \to X_1{}^\bullet$ and $X_3{}^\bullet \to X_2{}^\bullet$ completing a commutative diagram

$$\begin{tikzcd}
 & & X_3{}^\bullet \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dll, dashed, bend right, "s_3"'] \arrow[drr, dashed, bend left, "f_3"] & & \\
 & X_1{}^\bullet \arrow[dl, "s_1"'] \arrow[drrr, "f_1"' near end] & & X_2{}^\bullet \arrow[dlll, "s_2" near end] \arrow[dr, "f_2"] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}$$

in $\mathbf{K}(\mathcal{A})$. The collection of equivalence classes of fractions is denoted $\mathrm{Hom}_S(A^\bullet, B^\bullet)$.

To amend the composition procedure, we now choose a representative fraction for both classes, composing them as before and then declare the class of the result to be the composite. We omit the verification that this is independent of the choice of representatives and thus a well-defined operation, since it is just an exercise in setting up the appropriate diagram and checking commutativity.

In view of our earlier observations, the upshot is that the collections $\mathrm{Hom}_S(A^\bullet, B^\bullet)$ fulfil the properties needed to be the hom-sets of a category. Thus at long last, everything is now in place to define the derived category. By the Gabriel-Zisman theorem, the definition below will precisely be a localisation of the homotopy category with respect to the quasi-isomorphisms, and so has the property we have sought after. Moreover, thanks to [Wei94, Proposition 10.4.1], we do not sacrifice the triangulated structure.

Definition 3.3.5. Let $\mathcal{A}$ be an abelian category, and S the multiplicative system of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$. The *derived category* of $\mathcal{A}$ is the triangulated category $\mathbf{D}(\mathcal{A})$ with the same objects as $\mathbf{K}(\mathcal{A})$ and hom-sets

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_S(A^\bullet, B^\bullet).$$

The localisation functor $Q_{\mathcal{A}} : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is the identity on objects and sends a cochain map $f : A^\bullet \to B^\bullet$ to the fraction

$$A^\bullet \overset{\mathrm{id}}{\leftarrow} A^\bullet \overset{f}{\rightarrow} B^\bullet.$$

The *bounded* derived categories $\mathbf{D}^*(\mathcal{A})$ and localisation functors $Q^*_{\mathcal{A}} : \mathbf{K}^*(\mathcal{A}) \to \mathbf{D}^*(\mathcal{A})$ are defined similarly, and are all full subcategories of $\mathbf{D}(\mathcal{A})$.

Proposition 3.3.2 ([GM03, Proposition III.5.2])**.** *The composite functor*

$$\iota_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathbf{C}^b(\mathcal{A}) \xrightarrow{\pi} \mathbf{K}^b(\mathcal{A}) \xrightarrow{Q^*_{\mathcal{A}}} \mathbf{D}^b(\mathcal{A})$$

which concentrates objects and morphisms in degree 0 is fully faithful.

With the derived category now at hand, it is worth pausing to reflect on the goals we established at the start of this chapter. By Remark 2.4.2, any resolution of an object in $\mathcal{A}$ is quasi-isomorphic to the image of that object under the above embedding functor, and by the universal property of $Q_{\mathcal{A}}$ this becomes an isomorphism of objects in $\mathbf{D}(\mathcal{A})$. Thus in the derived category, the passage from an object to any of its resolutions is completely natural, which was precisely the main motivation for trying to invert quasi-isomorphisms in the first place. Another benefit of $\mathbf{D}(\mathcal{A})$ over $\mathbf{K}(\mathcal{A})$ is that the triangulated structure of the derived category is better behaved.

Proposition 3.3.3 ([Wei94, Example 10.4.9])**.** *Every short exact sequence*

$$0^\bullet \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0^\bullet$$

in $\mathbf{C}(\mathcal{A})$ fits into an exact triangle (u, v, w) on $(A^\bullet, B^\bullet, C^\bullet)$ in $\mathbf{D}(\mathcal{A})$ isomorphic to the strict triangle on u.

The price to pay for all of this on-paper computational convenience is that compared to $\mathbf{K}(\mathcal{A})$, the morphisms in $\mathbf{D}(\mathcal{A})$ are more elaborate and can be difficult to actually work with in general, especially in unbounded derived categories. This may seem like a perverse outcome, but in practice one is most interested in the bounded derived categories anyway, and it is often the case that $\mathcal{A}$ has certain properties which allow for simplifications to be made. We now discuss what can be done in such cases.

The need to localise the homotopy category arose from the observation that not all quasi-isomorphisms are homotopy equivalences. However, we have also seen that a partial converse for bounded below complexes of injectives holds by Proposition 2.2.2. On the face of it, this suggests that whenever $\mathcal{A}$ has enough injectives and $\mathcal{I}$ is the full additive subcategory of $\mathcal{A}$ consisting of injectives, the triangulated subcategory $\mathbf{K}^+(\mathcal{I})$ of $\mathbf{K}(\mathcal{A})$ already suffices in order to invert quasi-isomorphisms.

Proposition 3.3.4 ([Wei94, Theorem 10.4.8])**.** *Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{I}$ the full additive subcategory of injectives in $\mathcal{A}$. Then*

$$T^+ : \mathbf{K}^+(\mathcal{I}) \hookrightarrow \mathbf{K}^+(\mathcal{A}) \xrightarrow{Q^+_{\mathcal{A}}} \mathbf{D}^+(\mathcal{A})$$

is an equivalence of categories.

We will not prove this. The argument in [Wei94] uses the machinery of *Cartan-Eilenberg resolutions* (i.e. double complexes acting as 'resolutions' for complexes) and formal properties of localisation. Alternatively, the (longer) proof of [Huy06, Proposition 2.40] shows more directly that the composite is fully faithful and essentially surjective, i.e.

$$\mathrm{Hom}_{\mathbf{K}^+(\mathcal{I})}(I^\bullet, J^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet) \tag{3.3.1}$$

is bijective and every complex $A^\bullet \in \mathbf{D}^+(\mathcal{A})$ is isomorphic in $\mathbf{D}^+(\mathcal{A})$ to some complex $I^\bullet \in \mathbf{K}^+(\mathcal{I})$. Unfortunately, this equivalence of categories does not immediately descend to $\mathbf{D}^b(\mathcal{A})$ without the inclusion of an extra hypothesis.

Corollary 3.3.1. *The functor of Proposition 3.3.4 is an equivalence of categories*

$$T^b : \mathbf{K}^b(\mathcal{I}) \hookrightarrow \mathbf{K}^b(\mathcal{A}) \xrightarrow{Q^b_{\mathcal{A}}} \mathbf{D}^b(\mathcal{A})$$

if all objects of $\mathcal{A}$ have injective resolutions of finite length.

The dual statements of Proposition 3.3.4 and Corollary 3.3.1 for projective objects and $\mathbf{D}^-(\mathcal{A})$ also hold, but this case will be of less interest to us.

3.4 Total derived functors

Aside from allowing one to identify an object with all of its resolutions, another tangible sense in which derived categories are 'better' for homological algebra is their relationship with the classical derived functors introduced in Section 2.5. In this section, we will explore this and show that, in certain cases, we may consolidate these functors into one *total derived functor* between derived categories.

To begin, we shall consider the obstructions to an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories inducing a functor between derived categories. Any such F extends to an additive functor $\mathbf{C}(F) : \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{B})$, and thanks to Lemma 2.3.1, it follows that $\mathbf{C}(F)$ extends further to an additive, exact triangulated functor $\mathbf{K}(F) : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ in the obvious way. As before, we will commit a slight abuse of notation by conflating this functor with F. The difficulties begin when we try to extend F to a functor $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$, that is, to find a functor G which fits into a commutative diagram

$$\begin{tikzcd} \mathbf{K}(\mathcal{A}) \arrow[r, "\mathbf{K}(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathbf{K}(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ \mathbf{D}(\mathcal{A}) \arrow[r, dashed, "G"'] & \mathbf{D}(\mathcal{B}). \end{tikzcd}$$

If we are fortunate enough that F preserves quasi-isomorphisms, then this is possible: the composite functor $Q_{\mathcal{B}} \circ F : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ also preserves quasi-isomorphisms, so the universal property of $Q_{\mathcal{A}}$ ensures that such a functor G exists. By Corollary 2.3.1 and Lemma 2.3.1, we indeed find ourselves in this situation when F is exact, and moreover it turns out that G will be an exact triangulated functor. Under more practical assumptions, we cannot expect that this naïve construction will go through, nor that such a simple commutative diagram will characterise the resulting functor.

In light of our partial progress, we may as well assume we are given an exact triangulated functor $F : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ as a starting point, which subsumes the typical case of a left and right exact functor defined at the level of abelian categories. Going forward, we will also always need to impose some kind of boundedness condition on input complexes.

Definition 3.4.1. Let $F : \mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an exact triangulated functor. A *total right derived functor* is an exact triangulated functor $\mathbf{R}F : \mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ equipped with a natural transformation $\xi : Q_{\mathcal{B}} \circ F \Rightarrow \mathbf{R}F \circ Q^*_{\mathcal{A}}$ satisfying the following universal property: if $G : \mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ is another exact triangulated functor, then any natural transformation $\zeta : Q_{\mathcal{B}} \circ F \Rightarrow G \circ Q^*_{\mathcal{A}}$ factors uniquely through ξ. Explicitly, this means that there is a unique natural transformation $\eta : \mathbf{R}F \Rightarrow G$ such that

$$\zeta_{A^\bullet} = (\eta \circ Q^*_{\mathcal{A}})_{A^\bullet} \circ \xi_{A^\bullet}$$

for each $A^\bullet \in \mathbf{K}^*(\mathcal{A})$, or equivalently that the functor diagram

$$\begin{tikzcd} Q_{\mathcal{B}} \circ F \arrow[r, Rightarrow, "\zeta"] \arrow[d, Rightarrow, "\xi"'] & G \circ Q^*_{\mathcal{A}}. \\ \mathbf{R}F \circ Q^*_{\mathcal{A}} \arrow[ur, Rightarrow, "\eta \circ Q^*_{\mathcal{A}}"'] & \end{tikzcd}$$

commutes. Dually, a *total left derived functor* of F is an exact triangulated functor $\mathbf{L}F : \mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ and a natural transformation $\xi : \mathbf{L}F \circ Q^*_{\mathcal{A}} \Rightarrow Q_{\mathcal{B}} \circ F$ satisfying the universal property that for any exact triangulated functor $G : \mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$, any natural transformation $\zeta : G \circ Q^*_{\mathcal{A}} \Rightarrow Q_{\mathcal{B}} \circ F$ factors through ξ.

In the case where we are initially only given an additive functor $F : \mathcal{A} \to \mathcal{B}$, we will also write $\mathbf{R}F$ and $\mathbf{L}F$ for the total derived functors of its induced exact triangulated functor $\mathbf{K}^*(F) : \mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ as above.

Once again, this is a non-constructive definition. At the very least, the universal property implies that if a total left or right derived functor of F exists, it is unique up to natural isomorphism. In the presence of stronger assumptions, it turns out that there is a principled way to go about actually constructing these functors.

Proposition 3.4.1. *Let $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an exact triangulated functor, and suppose that $\mathcal{A}$ has enough injectives. Fix a quasi-inverse U^+ of the equivalence T^+ in Proposition 3.3.4. Then the composite functor*

$$\mathbf{R}^+F : \mathbf{D}^+(\mathcal{A}) \xrightarrow{U^+} \mathbf{K}^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{F} \mathbf{K}^+(\mathcal{B}) \xrightarrow{Q^+_{\mathcal{B}}} \mathbf{D}^+(\mathcal{B})$$

is a total right derived functor of F on $\mathbf{D}^+(\mathcal{A})$.

Proof. The most important data for our purposes is the functor $\mathbf{R}^+F$ itself, but it technically remains to specify a natural transformation $\xi : Q^+_{\mathcal{A}} \circ F \Rightarrow (Q^+_{\mathcal{B}} \circ F \circ U^+) \circ Q^+_{\mathcal{A}}$. We defer the details of this to [Wei94, Theorem 10.5.6]. □

Corollary 3.4.1. *Retain the hypotheses of Proposition 3.4.1. If $I^\bullet$ is a bounded below complex of injectives, then $\mathbf{R}^+F(I^\bullet) \cong (Q^+_{\mathcal{B}} \circ F)(I^\bullet)$.*

The dual statements for total left derived functors naturally hold. This construction provides two great technical advantages over the classical approach to derived functors. The first that we have already mentioned is that in the right circumstances, the total derived functors capture the ensemble of classical derived functors from Section 2.5. In fact, some authors (e.g. [Huy06]) take the following as definitions.

Proposition 3.4.2. *Suppose that $F : \mathcal{A} \to \mathcal{B}$ is additive. Then under suitable conditions on F and $\mathcal{A}$ ensuring that $\mathbf{R}^+F$ or $\mathbf{L}^-F$ exist, the ith classical derived functors of F at an object $A \in \mathcal{A}$ can be recovered as the cohomology*

$$R^iF(A) \cong H^i\mathbf{R}^+F(A) \quad \text{and} \quad L_iF(A) \cong H^{-i}\mathbf{L}^-F(A)$$

when F is left or right exact respectively. Naturally, one first embeds objects from $\mathcal{A}$ into the derived category using the fully faithful functor described in Section 3.3.

Proof. Given an injective resolution $A \to I^\bullet$, we have $A \cong I^\bullet$ as objects in $\mathbf{D}^+(\mathcal{A})$. Then by Corollary 3.4.1 and the fact that $Q^+_{\mathcal{B}}$ is the identity on objects, one has

$$H^i\mathbf{R}^+F(A) \cong H^i\mathbf{R}^+F(I^\bullet) \cong H^i(F(I^\bullet)) = R^iF(A).$$

One argues similarly for the claim in the left derived functor case. □

The other technical advantage is that in suitable situations, derived functors at the level of derived categories may be composed in a far simpler manner than derived functors at the level of abelian categories, which normally requires the use of *spectral sequences*. The following result shows the necessary hypotheses for composing total right derived functors, but the dual assertion for total left derived functors naturally holds.

Proposition 3.4.3. *Suppose that $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}^+(\mathcal{B})$ and $G : \mathbf{K}^+(\mathcal{B}) \to \mathbf{K}^+(\mathcal{C})$ are exact triangulated functors for abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Let $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{B}}$ be the subcategories of injective objects in $\mathcal{A}$ and $\mathcal{B}$. Then if $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and F sends $\mathcal{I}_{\mathcal{A}}$ into $\mathcal{I}_{\mathcal{B}}$, there is a natural isomorphism $\mathbf{R}^+(G \circ F) \Rightarrow \mathbf{R}^+G \circ \mathbf{R}^+F$.*

Proof. Let ξ_F, ξ_G and $\xi_{G\circ F}$ be the natural transformations associated to the derived functors $\mathbf{R}^+F$, $\mathbf{R}^+G$ and $\mathbf{R}^+(G\circ F)$ respectively. The commutative functor diagram

$$\begin{tikzcd} Q^+_{\mathcal{C}} \circ G \circ F \arrow[r, Rightarrow, "\xi_G \circ F"] \arrow[d, Rightarrow, "\xi_{G\circ F}"'] & \mathbf{R}^+G \circ Q^+_{\mathcal{B}} \circ F \arrow[d, Rightarrow, "\mathbf{R}^+G\circ\xi_F"] \\ \mathbf{R}^+(G\circ F)\circ Q^+_{\mathcal{A}} \arrow[r, Rightarrow, dashed, "\eta\circ Q^+_{\mathcal{A}}"'] & (\mathbf{R}^+G)\circ(\mathbf{R}^+F)\circ Q^+_{\mathcal{A}}, \end{tikzcd}$$

is completed by a natural transformation $\eta : \mathbf{R}^+(G\circ F) \Rightarrow (\mathbf{R}^+G)\circ(\mathbf{R}^+F)$ thanks to the universal property of $\mathbf{R}^+(G\circ F)$. Now, take any $A^\bullet \in \mathbf{D}^+(\mathcal{A})$ and use Proposition 3.3.4 to find a complex of injectives $I^\bullet \in \mathbf{K}^+(\mathcal{I}_{\mathcal{A}})$ isomorphic to $A^\bullet$ in $\mathbf{D}^+(\mathcal{A})$. That the component $(\eta\circ Q^+_{\mathcal{A}})_{I^\bullet}$ is an isomorphism follows from the calculation

$$\begin{aligned} \mathbf{R}^+(G\circ F)(Q^+_{\mathcal{A}}(I^\bullet)) &= (Q^+_{\mathcal{C}}\circ G\circ F)(I^\bullet) \\ &\cong (\mathbf{R}^+G\circ Q^+_{\mathcal{B}}\circ F)(I^\bullet) \\ &\cong (\mathbf{R}^+G\circ\mathbf{R}^+F)(Q^+_{\mathcal{A}}(I^\bullet)) \end{aligned}$$

by Corollary 3.4.1. The component $\eta_{A^\bullet}$ factors through the isomorphisms $A^\bullet \cong I^\bullet$ and $(\eta\circ Q^+_{\mathcal{A}})_{I^\bullet}$, so we conclude that $\eta_{A^\bullet}$ is also an isomorphism. □

Remark 3.4.1. Similar to Remark 2.5.1, the total derived functor of an exact triangulated functor F can also be constructed instead using complexes of F-acyclic objects $X^\bullet$, i.e. such that $H^i(F(X^\bullet)) = 0$ for all i. As usual, complexes of projectives or injectives are F-acyclic for every choice of F. Assuming that every object of $\mathbf{K}^*(\mathcal{A})$ is quasi-isomorphic to some F-acyclic complex, one can modify the proofs of the preceding results so that injective assumptions can be suitably replaced with F-acyclic assumptions. In the statement of Proposition 3.4.3 in particular, it is sufficient that F sends complexes of F-acyclics to G-acyclics.

We now turn our attention to upgrading Ext and Tor to total derived functors. It is perfectly valid to take derived functors of the usual Hom and tensor product functors directly, but it would be most sensible to work with a more general variant of these functors. To see why, recall that $\mathrm{Hom}_{\mathcal{A}}$ is a covariant bifunctor

$$\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$$

and notice that we may think of the induced triangulated exact functor of $\mathrm{Hom}_{\mathcal{A}}(A,-)$

on the homotopy category $\mathbf{K}^+(\mathcal{A})$ also as a covariant bifunctor

$$\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathsf{Ab}).$$

A similar observation holds for the tensor product functor. We are thus compelled us to define variants of these functors that instead take complexes in *both* arguments, and which of course also induces a triangulated exact functor on $\mathbf{K}^*(\mathcal{A})$. We will do this by means of constructing an appropriate double complex and then totalising it.

Definition 3.4.2. Let $A^\bullet$ and $B^\bullet$ be cochain complexes in $\mathcal{A}$. Consider the double complex $C^{\bullet\bullet}$ of abelian groups defined by

$$C^{p,q} = \mathrm{Hom}_{\mathcal{A}}(A^{-p}, B^q),$$

with horizontal and vertical differentials defined for each $f : A^{-p} \to B^q$ by

$$d_h^{p,q}(f) = d_B \circ f \quad \text{and} \quad d_v^{p,q}(f) = (-1)^{p+q+1} f \circ d_A.$$

The totalisation $\mathrm{Hom}^\bullet_{\mathcal{A}}(A^\bullet, B^\bullet) = \mathrm{Tot}^\Pi(C^{\bullet\bullet})$ is the *internal Hom* of $A^\bullet$ and $B^\bullet$.

The sign in d_v is convenient, because with it the cohomology of internal Hom classifies cochain maps up to homotopy equivalence and shifts. That is,

$$H^n(\mathrm{Hom}^\bullet_{\mathcal{A}}(A^\bullet, B^\bullet)) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet[n]).$$

This is explained in more detail in [Wei94, Section 10.7]. Moreover, since the functor $\mathrm{Hom}_{\mathcal{A}}(-, B)$ is contravariant, it makes sense for the internal Hom to mimic the same contravariance in first argument, which here will turn a cochain complex into a chain complex. As a consequence, one must invert the degree of $A^\bullet$. In any case, we obtain a functor which induces an exact triangulated functor on the homotopy category, so we may take the total derived functor.

Definition 3.4.3. Suppose that $\mathcal{A}$ has enough injectives, and fix a complex $A^\bullet$ in $\mathcal{A}$. The *derived Hom functor* is the total right derived functor

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(A^\bullet, B^\bullet) := \mathbf{R}^+\mathrm{Hom}^\bullet_{\mathcal{A}}(A^\bullet, -)(B^\bullet).$$

We now repeat the above for the tensor product functor.

Definition 3.4.4. Let $M^\bullet$ and $N^\bullet$ be cochain complexes of right and left R-modules respectively. Consider the double complex $C^{\bullet\bullet}$ of abelian groups defined by

$$C^{p,q} = M^p \otimes_R N^q,$$

with horizontal and vertical differentials defined for each $m \otimes n \in M^p \otimes_R N^q$ by

$$d_h^{p,q}(m \otimes n) = d_M(m) \otimes n \quad \text{and} \quad d_v^{p,q}(m \otimes n) = (-1)^p m \otimes d_N(n).$$

The totalisation $M^\bullet \otimes_R N^\bullet = \mathrm{Tot}^{\oplus}(C^{\bullet\bullet})$ is the *internal tensor product* of $M^\bullet$ and $N^\bullet$.

Definition 3.4.5. Let $M^\bullet$ be a complex of right R-modules. The *derived tensor product functor* is the total left derived functor $M^\bullet \otimes_R^{\mathbf{L}} N^\bullet := \mathbf{L}^-(M^\bullet \otimes_R -)(N^\bullet)$.

Remark 3.4.2. The cohomology of the derived Hom and tensor product functors as we have now defined them obviously do not recover the usual Ext and Tor groups, but rather the *hyperext* and *hypertor* groups of the *hyper-derived functors* of $\mathrm{Hom}_{\mathcal{A}}$ and $\otimes_R$. These are a notion that sits between classical and total derived functors that are unnecessary for our purposes, so we defer further discussion to [Wei94, Chapter 5].

To close out this chapter, we show that the usual tensor-hom adjunction has a generalisation to the derived category, which is key to giving the derived equivalence we seek. As always, we will assume a commutative ring R, in which case both $\mathrm{Hom}_R(M, N)$ and $M \otimes_R N$ have the structure of an R-module. It follows that $\mathbf{R}\mathrm{Hom}$ and $\otimes_R^{\mathbf{L}}$ map into $\mathbf{D}(\mathsf{Mod}(R))$. For simplicity, we will write $\mathbf{D}^*(R) = \mathbf{D}^*(\mathsf{Mod}(R))$.

Theorem 3.4.1 (Derived tensor-hom adjunction)**.** *Fix a complex $B^\bullet \in \mathbf{D}^-(R)$. Then for any $A^\bullet \in \mathbf{D}^-(R)$ and $C^\bullet \in \mathbf{D}^+(R)$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(R)}(A^\bullet, \mathbf{R}\mathrm{Hom}(B^\bullet, C^\bullet)) \cong \mathrm{Hom}_{\mathbf{D}(R)}(A^\bullet \otimes_R^{\mathbf{L}} B^\bullet, C^\bullet).$$

Proof. We claim instead that there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}(A^\bullet, \mathbf{R}\mathrm{Hom}(B^\bullet, C^\bullet)) \cong \mathbf{R}\mathrm{Hom}(A^\bullet \otimes_R^{\mathbf{L}} B^\bullet, C^\bullet), \tag{\dagger}$$

as this yields the original claim upon taking H^0. For complexes concentrated in degree 0 (i.e. for R-modules L, M and N), one can check after unfolding the definitions of $\mathbf{R}\mathrm{Hom}$ and $\otimes_R^{\mathbf{L}}$ that this degenerates to the usual tensor-hom adjoint isomorphism

$$\mathrm{Hom}(L, \mathrm{Hom}_R(M, N)) \cong \mathrm{Hom}(L \otimes_R M, N), \tag{$\star$}$$

so we now seek to use Proposition 3.4.3 to upgrade this to the derived category. Since we are working in the derived category, it suffices to assume that $A^\bullet$ and $C^\bullet$ are complexes of projectives and injectives respectively. Consider the functors

$$\mathrm{Hom}^\bullet(A^\bullet, \mathrm{Hom}^\bullet(B^\bullet, C^\bullet)) \quad \text{and} \quad \mathrm{Hom}^\bullet(A^\bullet \otimes_R B^\bullet, C^\bullet)$$

in $B^\bullet$ in the homotopy category. The left-hand side is the composite of the functors

$$F_1(B^\bullet) = \mathrm{Hom}^\bullet(B^\bullet, C^\bullet) \quad \text{and} \quad G_1(X^\bullet) = \mathrm{Hom}^\bullet(A^\bullet, X^\bullet)$$

while the right-hand side is the composite of

$$F_2(B^\bullet) = A^\bullet \otimes_R B^\bullet \quad \text{and} \quad G_2(X^\bullet) = \mathrm{Hom}^\bullet(X^\bullet, C^\bullet).$$

Since each cochain of $A^\bullet$ is projective, one can check that F_1 sends complexes of projectives (i.e. F_1-acyclic complexes) to complexes of injectives (i.e. G_1-acyclic complexes). Considering Remark 3.4.1, we may apply Proposition 3.4.3 to obtain the natural isomorphism $(\mathbf{R}G_1 \circ \mathbf{R}F_1)(B^\bullet) \cong \mathbf{R}(G_1 \circ F_1)(B^\bullet)$, which is to say that

$$\mathbf{R}\mathrm{Hom}(A^\bullet, \mathbf{R}\mathrm{Hom}(B^\bullet, C^\bullet)) \cong \mathbf{R}(\mathrm{Hom}^\bullet(A^\bullet, \mathrm{Hom}^\bullet(-, C^\bullet)))(B^\bullet).$$

Similarly, one can check that F_2 sends complexes of projectives (i.e. F_2-acyclic complexes) to complexes of projectives (i.e. G_2-acyclic complexes). Examining the proof of Remark 3.4.1 more closely, we see that the conclusion still holds even for the mixed total derived functors $(\mathbf{R}G_2 \circ \mathbf{L}F_2)(B^\bullet) \cong \mathbf{R}(G_2 \circ F_2)(B^\bullet)$, which is to say that

$$\mathbf{R}\mathrm{Hom}(A^\bullet \otimes_R^{\mathbf{L}} B^\bullet, C^\bullet) \cong \mathbf{R}(\mathrm{Hom}^\bullet(A^\bullet \otimes_R -, C^\bullet))(B^\bullet).$$

By taking appropriate injective and projective resolutions, $(\star)$ therefore shows that both sides of (†) are naturally isomorphic to the same total right derived functor. $\square$

Chapter 4

The coherent sheaf theory of $\mathbb{P}^1$

With our abstract preliminaries from the previous chapters all set up, this chapter marks a change of course, as we will now move into the realm of algebraic geometry and ultimately study the *coherent sheaves* on the projective line and some rudimentary coherent sheaf cohomology. These sheaves are characterised by their connection to modules, which make them amenable to the techniques of commutative algebra. Moreover, they are sufficiently restrictive as a type of sheaf to be tractable to study while also encompassing many examples frequently arising in practice.

We will present a heavily specialised treatment that is sufficient to obtain a working definition for $\mathbb{P}^1$. Our main source for this are the set of mini-lectures [Cha22; Cha23]. This treatment is neither novel nor particularly unusual beyond this, as we will also integrate material from the trio of sources [CK10; HSM13; Cra18] that view these sheaves in much the same way.

In all that follows, we fix an algebraically closed field k.

4.1 A primer on sheaves

This section is intended to provide a brief overview of *sheaves* in the abstract with a view toward algebraic geometry. We will strictly consider generalities here, but our needs for the more specific and forthcoming study of coherent sheaves can be met with little more than a few basic and well-motivated definitions.

Recall that one of the main ways to study an algebraic variety is to look at the rings of regular functions defined locally and globally. An important consequence of the way the Zariski topology is constructed is that these regular functions enjoy a certain local-to-global relationship: two regular functions on the variety X which agree locally on an open subset $U \subseteq X$ agree globally on X. Regular functions are far from the only data on X that behaves similar in spirit to this. As one of many examples, one can develop a cogent theory of *(Kähler) differentials* when X is smooth which, among other things, can be used to define the *geometric genus* of X, an important numerical invariant. There is an abundance of examples of this phenomena beyond just algebraic geometry, such as the continuous functions on any topological space. The language of sheaves arises as a convenient organisational tool in all of these contexts.

Definition 4.1.1. Let X be a topological space. A *presheaf of abelian groups* $\mathcal{F}$ on X is the data of abelian group $\mathcal{F}(U)$ for each open set U, and for each inclusion of opens $V \subseteq U$ a group homomorphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$. These homomorphisms are moreover required to be functorial in the sense that $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ for all open sets U and $\rho_{U,W} = \rho_{V,W}\rho_{U,V}$ for all inclusions of open sets $W \subseteq V \subseteq U$.

Though in this definition presheaf data takes values in the category Ab, one can substitute for other concrete categories, such as Set and Ring; more usefully and most commonly, one considers an abelian category. The elements of $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U, and the sections over X itself are called the *global sections*. It is also common to denote the space of sections of $\mathcal{F}$ over U by $\Gamma(U, \mathcal{F})$ or $H^0(U, \mathcal{F})$. The homomorphisms $\rho_{U,V}$ are called *restriction maps* due to how they are typically defined, and for a section $s \in \mathcal{F}(U)$ we introduce the suggestive shorthand $s|_V = \rho_{U,V}(s)$.

As we have already alluded to above, the most fundamental example of a presheaf in algebraic geometry is the following.

Example 4.1.1. The regular functions of a quasi-projective variety X define a presheaf of rings $\mathcal{O}_X$ called the *structure sheaf* of X. For each open set U, we let $\mathcal{O}_X(U)$ be the ring $\mathcal{O}(U)$ of regular functions defined on U, and take as restriction maps the usual contravariant inclusions $\mathcal{O}(U) \hookrightarrow \mathcal{O}(V)$.

The definition of a presheaf captures the organisational aspects of the data of $\mathcal{O}_X$, but we have already noted that there is more going on in the form of local-to-global relationships. Our next definition aims to make this more formal.

Definition 4.1.2. A presheaf $\mathcal{F}$ on X is called a *sheaf* if for any open set U and open cover $U = \bigcup_{i \in I} U_i$, the following two conditions are satisfied:

1. *(Identity.)* Given $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for all $i \in I$, we have $s = 0$.

2. *(Gluability.)* Given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$$

for all $i, j \in I$, there is a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

Example 4.1.2. As the name implies, the structure sheaf $\mathcal{O}_X$ is also a sheaf. Aside from the importance of regular functions in general, what makes $\mathcal{O}_X$ so fundamental is that many other useful sheaves occurring in algebraic geometry are 'compatible' with it. More precisely, we call a sheaf $\mathcal{F}$ an *$\mathcal{O}_X$-module* if each $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$, and for each inclusion of open sets $V \subseteq U \subseteq X$ we have

$$(s+t)\big|_V = s\big|_V + t\big|_V \quad \text{and} \quad (f \cdot s)\big|_V = f\big|_V \cdot s\big|_V$$

for all $s, t \in \mathcal{F}(U)$ and $f \in \mathcal{O}_X(U)$. Naturally, $\mathcal{O}_X$ is itself an $\mathcal{O}_X$-module.

With the following notion of morphisms, one obtains the pair of categories $\mathsf{Psh}(X)$ and $\mathsf{Sh}(X)$ of all (pre)sheaves of abelian groups on some fixed space X.

Definition 4.1.3. Let $\mathcal{F}$ and $\mathcal{G}$ be (pre)sheaves of abelian groups on X. A *morphism of (pre)sheaves* $f : \mathcal{F} \to \mathcal{G}$ is the data of group homomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subseteq X$ compatible with restriction, i.e. there is a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "-|_V"'] & \mathcal{G}(U) \arrow[d, "-|_V"] \\
\mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

for each inclusion of opens $V \subseteq U \subseteq X$. The composition of morphisms is defined in the obvious way. We say that f is an *isomorphism* if there is a morphism $g : \mathcal{G} \to \mathcal{F}$ such that $fg = \mathrm{id}_{\mathcal{G}}$ and $gf = \mathrm{id}_{\mathcal{F}}$.

Finally, we define a basic but useful operation on sheaves that we will need later.

Definition 4.1.4. Let $\mathcal{F}$ be a sheaf on X. For each open $U \subseteq X$, the *restricted sheaf* $\mathcal{F}|_U$ is the sheaf on U such that $(\mathcal{F}|_U)(V) = \mathcal{F}(V)$ for each open $V \subseteq U$, and $(\rho|_U)_{V,W} = \rho_{V,W}$ for each inclusion of opens $W \subseteq V \subseteq U$.

4.2 Modules as sheaves on $\mathbb{A}^1$

As an important precursor to coherent sheaves, we show in this section that any module over $\mathcal{O}(\mathbb{A}^1) \cong k[x]$ can be suitably interpreted as a sheaf on $\mathbb{A}^1$. This also gives an explicit algebro-geometric interpretation of modules and localisation.

We start with some preliminary results concerning open sets in $\mathbb{A}^1$. Recall that a *principal open subset* of an affine variety X is an open subset of the form

$$D(f) := X \setminus V(f) = \{x \in X : f(x) \neq 0\}$$

for some $f \in k[X]$. The collection of all principal opens forms a base for the Zariski topology on X, but even stronger statements can be made about the topology on $\mathbb{A}^1$.

Lemma 4.2.1. *Every open subset of $\mathbb{A}^1$ is principal.*

Proof. If U is open, then $U = \mathbb{A}^1 \setminus V(I)$ for some ideal $I \trianglelefteq k[x]$. But $k[x]$ is a principal ideal domain, so $I = \langle f \rangle$ for some $f \in k[x]$ and hence $U = D(f)$. $\square$

Lemma 4.2.2. *Let $f \in k[x]$, and consider the principal open set $D(f)$ in $\mathbb{A}^1$. Then every principal open set within $D(f)$ is of the form $D(fg)$ for some $g \in k[x]$. Moreover, the intersection of principal opens $D(f)$ and $D(g)$ is $D(fg)$.*

Proof. If $D(h) \subseteq D(f)$, then $V(f) \subseteq V(h)$ and hence $\langle h \rangle \subseteq \langle f \rangle$ by standard properties of affine varieties, which implies that f divides h. The second claim is equivalent to the fact that $V(f) \cup V(g) = V(fg)$. $\square$

The moral of these results is that to specify a sheaf on $\mathbb{A}^1$, it is entirely sufficient to consider only principal open sets determined by some polynomial. Intuitively, the same conclusions ought to hold mutatis mutandis for any open $U \subseteq \mathbb{A}^1$. To make this idea more formal, if $U = D(f)$ for some $f \in k[x]$, then by [Ful08, Proposition 6.3.5] we can take the definition of the ring of regular functions for U to be the localisation

$$\mathcal{O}(U) := \mathcal{O}(\mathbb{A}^1)_f \cong k[x, f^{-1}],$$

and it is not hard to see now that Lemma 4.2.2 holds with $\mathbb{A}^1$ replaced by U and $k[x]$ replaced by $\mathcal{O}(U)$. For reasons that will become clear in due course, we will broaden the scope of the following definition to any open subset of $\mathbb{A}^1$ rather than just $\mathbb{A}^1$ itself.

Definition 4.2.1. Let U be an open subset of $\mathbb{A}^1$ and M a module over the ring $\mathcal{O}(U)$. We define a presheaf of abelian groups $\widetilde{M}$ on U, called the *sheaf associated to M*, by declaring that the sections over each principal open $D(f) \subseteq U$ are $\Gamma(D(f), \widetilde{M}) = M_f$, and equipping $\widetilde{M}$ with restriction maps

$$\rho_{f,fg} : M_f \to M_{fg}, \qquad \frac{m}{f^r} \mapsto \frac{mg^r}{(fg)^r}.$$

Some a posteriori justification for this definition is as follows. Algebraically, $D(f) \subseteq U$ is the region in which f is non-zero and hence invertible in $\mathcal{O}(U)$, so to specify local sections it is natural to consider the *extension of scalars*

$$\mathcal{O}(U)_f \otimes_{\mathcal{O}(U)} M \cong M_f.$$

Also, while the sheaf $\widetilde{M}$ only considers this space of sections as an abelian group, each M_f has by definition the structure of an $\mathcal{O}(U)_f$-module.

Example 4.2.1. The structure sheaf $\mathcal{O}_{\mathbb{A}^1}$ is the associated sheaf of the $k[x]$-module $k[x]$. By our remarks above, it follows that every associated sheaf $\widetilde{M}$ on $\mathbb{A}^1$ is also an example of an $\mathcal{O}_{\mathbb{A}^1}$-module.

Example 4.2.2. Let $a \in k$, and consider the $k[x]$-module $M = k[x]/\langle x - a\rangle$. If $U = D(f)$ is an open set in $\mathbb{A}^1$, then it follows from polynomial division that $f = f(a)$ in M. If $a \in U$, then $f(a)$ is non-zero in k and hence invertible in M, so we have $M_f \cong M \cong k$ as $k[x]$-modules. If instead $a \notin U$, then $f(a) = 0$ in k, and so M_f must be the zero module. We say that the sheaf $\widetilde{M}$ is *supported* at the point $x = a$ in the sense that one only obtains non-zero sections over open neighbourhoods of a. For this reason, we call $\widetilde{M}$ a *skyscraper sheaf* on $\mathbb{A}^1$.

Let us now check that the presheaf $\widetilde{M}$ is deserving of being called a sheaf. We will adapt the proofs of a related result for affine schemes in [EH00, Proposition I.18] and [Vak24, Theorem 4.1.2], originally due to Serre. By virtue of the fact that we are working within $\mathbb{A}^1$, we will be allowed to sidestep some sheaf-theoretic technicalities that one encounters in this more general setting. The following lemma is key.

Lemma 4.2.3. *Every open cover of an open subset of $\mathbb{A}^1$ has a finite subcover.*

Proof. In light of our observations above, it suffices to consider a cover by principal opens $U = \bigcup_i D(f_i)$, with $f_i \in \mathcal{O}(U)$. It is immediate that the zero locus of the ideal $\mathfrak{a}$ generated by the f_i must be empty, i.e. $\mathfrak{a} = \mathcal{O}(U)$. Each element of $\mathfrak{a}$ is by definition a finite linear combination of the f_i, so in particular there is a *partition of unity* $\sum_{i\in I} a_i f_i = 1$ where all but finitely many of the $a_i \in \mathcal{O}(U)$ are non-zero. We may thus take as a finite subcover the sets $D(f_i)$ such that $a_i \neq 0$. $\square$

Proposition 4.2.1. *The presheaf $\widetilde{M}$ in Definition 4.2.1 is a sheaf on any open $U \subseteq \mathbb{A}^1$.*

Proof. Let $V \subseteq U$ be open. Again, in light of the previous observations, we may write $V = D(f)$ for some $f \in \mathcal{O}(U)$ and consider an arbitrary open cover $V = \bigcup_{i\in I} D(f_i)$

for some $f_i \in \mathcal{O}(U)$. We claim that is that it is in fact enough to show that the sheaf property holds for global sections, i.e. in the case $V = U$. To see why, note that V is open in $\mathbb{A}^1$, so by Lemma 4.2.2 we have

$$V = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} V \cap D(f_i) = \bigcup_{i \in I} D(ff_i).$$

Via the inclusion of rings $\mathcal{O}(U) \hookrightarrow \mathcal{O}(V)$, each function ff_i may be regarded as a function in $\mathcal{O}(V)$. That is, the open cover of V above can be surreptitiously rewritten as an open cover $V = \bigcup_{i \in I} D(f_i')$ for some $f_i' \in \mathcal{O}(V)$, which shows that the general case can indeed be reduced to this special case. Thus for the remainder of the proof, we may assume that $V = U$ so that $\widetilde{M}(V) \cong M$. Before we begin the proof proper, we will also fix a finite subcover $U = \bigcup_{\alpha \in A} D(f_\alpha)$ by Lemma 4.2.3.

To start, let us check that the identity property holds. Suppose that the global section $s \in M$ vanishes upon restriction in M_{f_i} for all $i \in I$. Then s is annihilated in M by some power of each f_α on the finite subcover. By finiteness, we may choose a sufficiently large power N such that every f_α^N annihilates s. It follows that the ideal $\mathfrak{b}$ generated by the f_α^N annihilates s. But via the proof of Lemma 4.2.3, the ideal $\mathfrak{a}$ generated by the f_α is $\mathcal{O}(U)$, and it is not hard to see that $\mathcal{O}(U) = \mathfrak{a}^{2N} \subseteq \mathfrak{b}$, so we conclude that s is annihilated by the entire ring $\mathcal{O}(U)$. Hence $s = 0$ in M.

Next, we check gluability. Suppose we are given sections $s_i \in M_{f_i}$ such that $s_i = s_j$ in $M_{f_i f_j}$ for all $i, j \in I$. As before, this certainly descends to our finite subcover, so let us first prove it on that. We can write each section as $s_\alpha = m_\alpha / f_\alpha^{r_\alpha}$ for some $m_\alpha \in M$ and integer r_α. Since $D(f_\alpha) = D(f_\alpha^{r_\alpha})$, for ease of notation we will let $g_\alpha = f_\alpha^{r_\alpha}$ such that each s_α is naturally an element of M_{g_α}. By clearing denominators in $M_{g_\alpha g_\beta}$, we may appeal once more to finiteness to choose a sufficiently large power N such that

$$(g_\alpha g_\beta)^N (g_\beta m_\alpha - g_\alpha m_\beta) = 0$$

in M for every $\alpha, \beta \in A$. For further ease of notation, write $n_\alpha = g_\alpha^N m_\alpha$ and $h_\alpha = g_\alpha^{N+1}$ so that the above condition reads $h_\beta n_\alpha = h_\alpha n_\beta$. Moreover, since $D(h_\alpha) = D(g_\alpha)$, it follows as in Lemma 4.2.3 that there is a partition of unity $\sum_{\alpha \in A} a_\alpha h_\alpha = 1$ for some $a_\alpha \in \mathcal{O}(U)$. Now, considering the global section $s = \sum_{\alpha \in A} a_\alpha n_\alpha$, we see that

$$sh_\alpha = \sum_{\beta \in A} a_\beta h_\alpha n_\beta = \sum_{\beta \in A} a_\beta h_\beta n_\alpha = n_\alpha$$

in M for each $\alpha \in A$. It is not difficult to see that $M_{h_\alpha} \cong M_{f_\alpha}$, and by construction $n_\alpha / h_\alpha = s_\alpha$ in M_{h_α}, so we conclude from this that $s = s_\alpha$ in M_{f_α} as needed. It remains to upgrade gluability to infinite open covers. The previous argument holds generically for any finite open cover of U, so for any choice of $i \in I \setminus A$, we may similarly construct a global section $s' \in M$ such that $s' = s_\alpha$ in each M_{f_α} and $s' = s_i$ in M_{f_i}. But by the identity property, we must have $s = s'$, and so we are done. □

We now see that the passage to open subsets of $\mathbb{A}^1$ in Definition 4.2.1 was necessary to ensure that we may reduce the claim to something easier to prove.

4.3 Coherent sheaves on $\mathbb{P}^1$

With our preparations out of the way, we are now in a position to introduce *coherent sheaves* on $\mathbb{P}^1$. It will be instructive to give the usual definitions upfront in order to provide some sense of where our hands-on definitions we obtain later come from.

Definition 4.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on some quasi-projective variety X over k. Then $\mathcal{F}$ is *quasi-coherent* if there is an affine open cover $X = \bigcup_i U_i$ (i.e. with each U_i isomorphic to an affine variety) and an $\mathcal{O}(U_i)$-module M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We say that $\mathcal{F}$ is *coherent* if furthermore each module M_i is finitely generated over $\mathcal{O}(U_i)$.

There is another equivalent definition, which we package up in the following result.

Proposition 4.3.1 ([Har77, Proposition II.5.4])**.** *An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for each affine open $U \subseteq X$, there is an $\mathcal{O}(U)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. Moreover, $\mathcal{F}$ is coherent if and only if M is finitely generated over $\mathcal{O}(U)$.*

Though Definition 4.3.1 and Proposition 4.3.1 are standard fare in algebraic geometry, the reader should feel some trepidation here. The issue is that as we have constructed it, Definition 4.2.1 only applies to quasi-affine varieties in $\mathbb{A}^1$ rather than arbitrary affine varieties, so the assertion that $\mathcal{F}|_U \cong \widetilde{M}$ for some affine open $U \subseteq X$ is ill-defined. We will only concern ourselves with the varieties $\mathbb{A}^1$ and $\mathbb{P}^1$ for which this issue is not a concern. The intention of presenting these definitions in spite of this imprecision is to convey the important idea that quasi-coherent sheaves on X are 'locally just modules'.

The category of quasi-coherent sheaves on X is denoted $\mathsf{QCoh}(X)$, and the coherent sheaves form a full subcategory $\mathsf{Coh}(X)$. In the case of $\mathbb{A}^1$, these categories end up being as nice as one could realistically ask for.

Corollary 4.3.1. *The category $\mathsf{QCoh}(\mathbb{A}^1)$ is equivalent to $\mathsf{Mod}(k[x])$ via the pair of naturally inverse functors $\mathcal{F} \mapsto \Gamma(\mathbb{A}^1, \mathcal{F})$ and $M \mapsto \widetilde{M}$. Moreover, this restricts to an equivalence of categories between $\mathsf{Coh}(\mathbb{A}^1)$ and $\mathsf{mod}(k[x])$.*

Proof. This is a specialisation and restatement of [Har77, Corollary II.5.5], but the main contention follows from Proposition 4.3.1. □

In other words, the coherent sheaves on $\mathbb{A}^1$ are little more than finitely generated $k[x]$-modules. Technical concerns notwithstanding, the same claim is true mutatis mutandis for all affine varieties, which is a significant reduction in the complexity of studying these sheaves. However, one can go even further specifically in the case of $\mathbb{A}^1$. In particular, the structure theorem for finitely generated modules over a principal ideal domain can be appropriated as a classification theorem for coherent sheaves on $\mathbb{A}^1$. For convenience, we state this as the following theorem in its primary decomposition form.

Theorem 4.3.1. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{A}^1$, i.e. a finitely generated $k[x]$-module M, then there are $r, s \in \mathbb{Z}_{\geq 0}$, powers $n_1, n_2, \ldots, n_s \in \mathbb{Z}_{\geq 0}$ and points $a_1, \ldots, a_r \in k$, not necessarily distinct, such that $\mathcal{F}$ is the sheaf associated to the module*

$$M \cong k[x]^{\oplus r} \oplus \frac{k[x]}{\langle x - a_1 \rangle^{n_1}} \oplus \cdots \oplus \frac{k[x]}{\langle x - a_s \rangle^{n_s}}.$$

Put another way, this theorem also shows that any coherent sheaf on $\mathbb{A}^1$ splits as a sum of a torsion-free and a torsion summand, an observation we will return to later.

The next natural question to ask is whether there is also an elegant theory for coherent sheaves on non-affine quasi-projective varieties. Immediately, $\mathbb{P}^1$ makes for a good case study, since $\mathbb{P}^1$ has a standard affine open cover by two copies of $\mathbb{A}^1$ (i.e. we are again free of prior technical concerns).

We first fix some conventions for $\mathbb{P}^1$ that we will use throughout. Recall that the aforementioned affine open cover is given by $\mathbb{P}^1 = U_x \cup U_y$, where

$$U_x = \{(x : y) : x \neq 0\} \quad \text{and} \quad U_y = \{(x : y) : y \neq 0\}.$$

Each point $(x : y) \in U_x$ is the same as $(1 : y/x)$, so there is a clear homeomorphism $\phi_x : U_x \to \mathbb{A}^1_z$ identifying U_x with the affine line $\mathbb{A}^1_z$ with coordinate $z = y/x$. Similarly, there is a clear homeomorphism $\phi_y : U_y \to \mathbb{A}^1_w$ where $\mathbb{A}^1_w$ has coordinate $w = x/y$. We call the maps ϕ_x and ϕ_y *affine charts* for $\mathbb{P}^1$, and similarly we call the open subsets U_x and U_y *affine patches*. We can convert between affine charts on $U_{xy} = U_x \cap U_y$ using the *transition map* $\tau : \mathbb{A}^1_z \setminus \{0\} \to \mathbb{A}^1_w \setminus \{0\}$ defined by

$$\tau(z) := \phi_y \phi_x^{-1}(z) = \phi_y(1 : z) = \phi_y(z^{-1} : 1) = z^{-1}.$$

Because of this, we can also view $\mathbb{P}^1$ as the result of gluing together the affine lines $\mathbb{A}^1_z$ and $\mathbb{A}^1_w$ according to the rule $w = z^{-1}$ whenever $z, w \neq 0$. For algebraic convenience, we will replace the affine coordinate w with z^{-1} so that, formally, $\mathbb{P}^1 = \mathbb{A}^1_z \sqcup \mathbb{A}^1_{z^{-1}}$.

In view of Definition 4.3.1, a quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$ is locally determined on this open cover by $\mathcal{F}|_{U_x}$ and $\mathcal{F}|_{U_y}$, i.e. by a module M over $\mathcal{O}(U_x) \cong k[z]$ and a module M' over $\mathcal{O}(U_y) \cong k[z^{-1}]$ respectively. One complication in simply forgetting the sheaf structure and retaining only the pair (M, M') is that between them, they encode a significant amount of overlapping sheaf data on U_{xy}. To resolve this problem, it is enough to specify an isomorphism of sheaves

$$\vartheta : \left(\mathcal{F}|_{U_x}\right)\Big|_{U_{xy}} \to \left(\mathcal{F}|_{U_y}\right)\Big|_{U_{xy}}.$$

Algebraically, U_{xy} is the region in which both z and z^{-1} are invertible in $k[z]$ and $k[z^{-1}]$ respectively, so it stands to reason that ϑ can equivalently be specified as an isomorphism between $\Gamma(D(z), \widetilde{M}) = M_z$ and $\Gamma(D(z^{-1}), \widetilde{M'}) = M'_{z^{-1}}$ as modules over $k[z, z^{-1}]$, the ring of *Laurent polynomials* in z. This is the outline for the alternative definition of quasi-coherent sheaves for $\mathbb{P}^1$ we will use in this thesis. We will also pair it with a more tailored notion of a morphism than Definition 4.1.3.

Definition 4.3.2. A *quasi-coherent sheaf* on $\mathbb{P}^1$ is the data of a triple $\mathcal{F} = (M, M', \vartheta)$ consisting of a $k[z]$-module M, a $k[z^{-1}]$-module M' and a $k[z^{-1}]$-module isomorphism $\vartheta : M_z \to M'_{z^{-1}}$, called the *descent datum* of $\mathcal{F}$. We say that $\mathcal{F}$ is *coherent* if M and M' are both finitely generated over $k[z]$ and $k[z^{-1}]$ respectively.

Definition 4.3.3. Let $\mathcal{F} = (M, M', \vartheta)$ and $\mathcal{G} = (N, N', \varrho)$ be quasi-coherent sheaves on $\mathbb{P}^1$. A *morphism* $\Phi : \mathcal{F} \to \mathcal{G}$ is a pair (φ, φ') consisting of a $k[z]$-module homomorphism $\varphi : M \to N$ and a $k[z^{-1}]$-module homomorphism $\varphi' : M' \to N'$, subject to the condition that there is a commutative diagram of $k[z, z^{-1}]$-modules

$$\begin{tikzcd}
M_z \arrow[r, "\varphi_z"] \arrow[d, "\vartheta"'] & N_z \arrow[d, "\varrho"] \\
M'_{z^{-1}} \arrow[r, "\varphi'_{z^{-1}}"'] & N'_{z^{-1}}.
\end{tikzcd}$$

If $\Psi : \mathcal{G} \to \mathcal{H}$ is another morphism of quasi-coherent sheaves with data (ψ, ψ'), then we define the *composite* $\Psi \circ \Phi : \mathcal{F} \to \mathcal{H}$ to be the morphism with data $(\psi \circ \varphi, \psi' \circ \varphi')$.

Choosing to regard quasi-coherent sheaves in this manner is somewhat ‘unnatural’, insofar as it is often more advantageous to try and work in coordinate-free settings. It is often necessary to fix coordinates for actual computations though, and this presents no significant obstruction in practice.

Proposition 4.3.2. *Let $\mathcal{Q}$ be the category of quasi-coherent sheaves on $\mathbb{P}^1$ in the sense of Definition 4.3.2 and Definition 4.3.3, and let $\mathcal{C}$ be the full subcategory of $\mathcal{Q}$ consisting of coherent sheaves. Then $\mathsf{QCoh}(\mathbb{P}^1)$ is equivalent to $\mathcal{Q}$, and this restricts to an equivalence of categories between $\mathsf{Coh}(\mathbb{P}^1)$ and $\mathcal{C}$.*

Proof. We give only a brief idea of the proof. The functor $\mathsf{QCoh}(\mathbb{P}^1) \to \mathcal{Q}$ in this equivalence is easy to construct, as given $\mathcal{F} \in \mathsf{QCoh}(\mathbb{P}^1)$ we have

$$\Gamma\left(\mathcal{F}\big|_{U_x}, U_{xy}\right) = \Gamma(\mathcal{F}, U_{xy}) = \Gamma\left(\mathcal{F}\big|_{U_y}, U_{xy}\right),$$

so we simply take $\mathcal{F} \mapsto \left(\Gamma(\mathcal{F}, U_x), \Gamma(\mathcal{F}, U_y), \mathrm{id}_{\Gamma(\mathcal{F}, U_{xy})}\right)$. The functor $\mathcal{Q} \to \mathsf{QCoh}(\mathbb{P}^1)$ naturally inverse to this is harder to construct, but in essence will follow from the forthcoming description of the sections and restriction maps of some $\mathcal{F} \in \mathcal{Q}$. $\square$

From this point forward, we therefore identify $\mathsf{QCoh}(\mathbb{P}^1)$ and $\mathsf{Coh}(\mathbb{P}^1)$ with $\mathcal{Q}$ and $\mathcal{C}$ and dispense with the general definitions from the start of this section. Though we will stop short of an explicit verification that these triples are actually sheaves in the technical sense of Definition 4.1.2, it will be worthwhile to at least describe what the sections and restriction maps of some $\mathcal{F} \in \mathsf{QCoh}(\mathbb{P}^1)$ are to get a feel for how to work with these sheaves computationally. We first recall the following facts about $\mathbb{P}^1$.

Lemma 4.3.1. *Every open subset $U \subseteq \mathbb{P}^1$ is of the form $D(F) := \mathbb{P}^1 \setminus V(F)$ for some homogeneous polynomial $F \in k[x, y]$. Moreover, any open subset $V \subseteq U$ is of the form $V = D(FG)$ for some other homogeneous polynomial $G \in k[x, y]$.*

This result is the projective analogue of Lemma 4.2.1 and Lemma 4.2.2. Since the proof is similar, we shall omit it and mention only that one instead needs to use the fact that $k[x, y]$ is Noetherian for the first claim. We may always arrange that F and G have minimal degree here, since for example we can take

$$F(x, y) = (b_1 x - a_1 y)(b_2 x - a_2 y) \cdots (b_d x - a_d y),$$

where the $(a_i : b_i)$ are the distinct points of the (finite) set $\mathbb{P}^1 \setminus U$. At this juncture, it will be convenient to introduce some notation and an important fact about dehomogenisation of polynomials.

Definition 4.3.4. Let $F \in k[x, y]$ be a homogeneous polynomial. Setting $z = y/x$, the *dehomogenisations* of F separately with respect to x and y produces the pair of univariate polynomials which, whenever they are defined, are denoted by

$$F_x(z) := F(1, z) \in k[z] \quad \text{and} \quad F_y(z^{-1}) := F(z^{-1}, 1) \in k[z^{-1}].$$

Lemma 4.3.2. *The dehomogenisations of a homogeneous polynomial $F \in k[x, y]$ are associates in the ring of Laurent polynomials $k[z, z^{-1}]$, where $z = y/x$.*

Proof. This follows from homogenity, since if $d = \deg F$, then

$$F_x(z) = F(1, z) = F(zz^{-1}, z) = z^d F(z^{-1}, 1) = z^d F_y(z^{-1}). \qquad \square$$

Now, suppose that $U \subseteq \mathbb{P}^1$ is the open set $D(F)$ as in Lemma 4.3.1. The dehomogenisations of F identify U in each affine patch, i.e. we have homeomorphisms

$$U \cap U_x \cong D(F_x) \quad \text{and} \quad U \cap U_y \cong D(F_y),$$

so a section of $\mathcal{F}$ over U is a pair of affine sections $(s, s') \in M_{F_x} \times M'_{F_y}$, say

$$s = m/F_x^i \quad \text{and} \quad s' = m'/F_y^j.$$

It remains to describe how to glue s and s' on $U \cap U_{xy}$. Algebraically, this is the region in which both F_x and F_y are invertible in $k[z, z^{-1}]$, so we consider the modules $M_{F_x,z}$ and $M'_{F_y,z^{-1}}$ over the ring $k[z, z^{-1}, F_x^{-1}, F_y^{-1}]$. Since $F_x = z^{\deg F} F_y$ in this ring, the fraction $\vartheta(m)/F_x^i$ is a well-defined element of $M'_{F_y,z^{-1}}$, and we will insist for the sake of compatibility that it agrees with s', i.e. we impose the condition

$$\vartheta(m)/F_x^i = m'/F_y^j$$

in $M'_{F_y,z^{-1}}$. By definition of equality in a localised module, this holds if and only if

$$F_y^n(F_y^j \vartheta(m) - F_x^i m') = 0 \tag{4.3.1}$$

in $M'_{z^{-1}}$ for some power $n \in \mathbb{Z}_{\geq 0}$. The restriction maps at this point are clear, so we present the following definition to summarise this discussion.

Definition 4.3.5. Let $\mathcal{F} = (M, M', \vartheta)$ be a quasi-coherent sheaf on $\mathbb{P}^1$ and $U = D(F)$ an open subset of $\mathbb{P}^1$. A *local section* of $\mathcal{F}$ over U is a pair

$$(s, s') = (m/F_x^i, m'/F_y^j) \in M_{F_x} \times M'_{F_y}$$

satisfying Eq. (4.3.1), and a *global section* of $\mathcal{F}$ is a pair $(m, m') \in M \times M'$ satisfying

$\vartheta(m) = m'$. For any open subset $V = D(FG) \subseteq U$, the *restriction map* $\rho_{U,V}$ of $\mathcal{F}$ sends the section $(s, s') \in \Gamma(U, \mathcal{F})$ to the section

$$\left(s\big|_{D(F_x G_x)}, s'\big|_{D(F_y G_y)}\right) \in \Gamma(V, \mathcal{F})$$

via the affine restriction maps for $\widetilde{M}$ and $\widetilde{M'}$ on each component.

We will study the two most fundamental families of coherent sheaves on $\mathbb{P}^1$ in depth for the remainder of this section, but it will be convenient to introduce some more specific terminology and definitions as we go.

Definition 4.3.6. A quasi-coherent sheaf $\mathcal{F} = (M, M', \vartheta)$ on $\mathbb{P}^1$ is *locally free* of rank r if $M \cong k[z]^r$ and $M' \cong k[z^{-1}]^r$. A locally free sheaf of rank 1 is called a *line bundle*.

Example 4.3.1. The structure sheaf $\mathcal{O} := \mathcal{O}_{\mathbb{P}^1}$ can be described as the line bundle $(k[z], k[z^{-1}], \mathrm{id})$. This is a special case of a more general family of line bundles on $\mathbb{P}^1$, as for each $n \in \mathbb{Z}$ we define a line bundle $\mathcal{O}(n) = (k[z], k[z^{-1}], \vartheta_n)$, called the nth *twisting sheaf*, where ϑ_n is the multiplication map given by z^{-n} on $k[z, z^{-1}]$. These are all examples of $\mathcal{O}_{\mathbb{P}^1}$-modules.

To see just how much varying the degree n of the twist changes $\mathcal{O}(n)$, we consider its global sections. These are pairs $(f, g) \in k[z] \times k[z^{-1}]$ such that $g = z^{-n} f$ in $k[z, z^{-1}]$, so each section is determined by the choice of f. This immediately implies that there are no non-zero global sections when $n < 0$, since $z^{-n} f$ consists of monomials of non-negative degree in z while the opposite is true for g. By a similar degree argument, we see that the only possible choices of f when $n \geq 0$ are of the form $f(z) = \sum_{i=0}^{n} \lambda_i z^i$ for scalars $\lambda_i \in k$, so $\Gamma(\mathbb{P}^1, \mathcal{O}(n)) \cong k^{n+1}$ as vector spaces over k. In the case $n = 0$, this recovers the well-known fact that all global regular functions on $\mathbb{P}^1$ are constant.

Example 4.3.2. Let us determine all morphisms $\mathcal{O}(m) \to \mathcal{O}(n)$, i.e. all pairs of endomorphisms $\varphi : k[z] \to k[z]$ and $\varphi' : k[z^{-1}] \to k[z^{-1}]$ such that

$$\begin{tikzcd}
k[z, z^{-1}] \arrow[r, "\varphi_z"] \arrow[d, "\times z^{-m}"'] & k[z, z^{-1}] \arrow[d, "\times z^{-n}"] \\
k[z, z^{-1}] \arrow[r, "\varphi'_{z^{-1}}"'] & k[z, z^{-1}]
\end{tikzcd}$$

commutes. Since $k[z]$ is a free $k[z]$-module of rank 1, φ is multiplication by some $f \in k[z]$. Similarly, φ' is multiplication by some $g \in k[z^{-1}]$, so the commutativity of the above diagram requires that $z^{-n} f = z^{-m} g$, which in turn holds if and only if $f = z^{n-m} g$. Since f is determined for a particular choice of g by this equation, it suffices to find all possible g. By the same logic we used in studying the global sections above, the degree (in z^{-1}) of g must be at most $n-m$, so the only morphism when $m>n$ is zero. On the other hand, any $g \in \langle 1, z^{-1}, \ldots, z^{-(n-m)} \rangle \trianglelefteq k[z^{-1}]$ determines a non-trivial morphism when $m \leq n$. Hence as a vector space over k,

$$\operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n)) \cong \begin{cases} k^{n-m+1} & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases}$$

This also shows that $\mathcal{O}(m) \cong \mathcal{O}(n)$ if and only if $m=n$.

In the case $m \leq n$, it follows that $\operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n)) \cong \Gamma(\mathbb{P}^1, \mathcal{O}(n-m))$. A result that generalises this observation is the following useful fact.

Lemma 4.3.3. *For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$, we have an isomorphism*

$$\operatorname{Hom}_{\mathsf{QCoh}(\mathbb{P}^1)}(\mathcal{O}, \mathcal{F}) \cong \Gamma(\mathbb{P}^1, \mathcal{F})$$

as vector spaces over k.

Proof. This is the sheaf analogue of the group isomorphism $\operatorname{Hom}_R(R, M) \cong M$, so we can expect the proof to mimic it. Given a sheaf morphism $(\varphi, \varphi') : \mathcal{O} \to \mathcal{F}$, it follows from the commutativity of the diagram

$$\begin{tikzcd} k[z,z^{-1}] \arrow[r, "\varphi_z"] \arrow[d, equal] & M_z \arrow[d, "\vartheta"] \\ k[z,z^{-1}] \arrow[r, "\varphi'_{z^{-1}}"'] & M'_{z^{-1}} \end{tikzcd}$$

that $(\varphi(1), \varphi'(1))$ is a global section of $\mathcal{F}$. This defines a k-linear map whose inverse is the map sending any global section (m, m') of $\mathcal{F}$ to the pair (φ, φ'), where $\varphi : k[z] \to M$ is the unique $k[z]$-module homomorphism with $\varphi(1) = m$, and $\varphi' : k[z^{-1}] \to M'$ is similarly determined over $k[z^{-1}]$ by $\varphi'(1) = m'$. Since $\vartheta(m) = m'$, it follows that the above diagram commutes, so we indeed have a morphism $(\varphi, \varphi') : \mathcal{O} \to \mathcal{F}$. $\square$

With the twisting sheaves studied sufficiently for now, we move on to the next key class of coherent sheaves. These arise as instances of the following construction.

Definition 4.3.7. Given quasi-coherent sheaves $\mathcal{F} = (M, M', \vartheta)$ and $\mathcal{G} = (N, N', \varrho)$ on $\mathbb{P}^1$, we say that $\mathcal{G}$ is a *subsheaf* of $\mathcal{F}$ if $N \leq M$, $N' \leq M'$ and ϱ is a restriction of ϑ.

The *quotient* of $\mathcal{F}$ by some subsheaf $\mathcal{G}$ is defined as the quasi-coherent sheaf

$$\mathcal{F}/\mathcal{G} := (M/N, M'/N', \overline{\vartheta})$$

whose descent datum is the induced isomorphism on quotient modules

$$\overline{\vartheta} : M_z/N_z \to M'_{z^{-1}}/N'_{z^{-1}}.$$

Example 4.3.3. Let D be a closed subset of $\mathbb{P}^1$, say $D = V(F)$ for some homogeneous polynomial $F \in k[x,y]$ by Lemma 4.3.1. Then the dehomogenisations of F precisely give the ideals of vanishing for $D \cap U_x$ and $D \cap U_y$, i.e.

$$I(D \cap U_x) = \langle F_x \rangle \trianglelefteq k[z] \quad \text{and} \quad I(D \cap U_y) = \langle F_x \rangle \trianglelefteq k[z^{-1}].$$

Since F_x and F_y are associates in $k[z, z^{-1}]$, the identity map induces an isomorphism $\langle F_x \rangle_z \cong \langle F_y \rangle_{z^{-1}}$, so the structure sheaf admits the coherent subsheaf

$$\mathcal{O}(-D) := (\langle F_x \rangle, \langle F_y \rangle, \mathrm{id}) \leq \mathcal{O},$$

called an *ideal sheaf.* Moreover, the quotient $\mathcal{O}_D := \mathcal{O}/\mathcal{O}(-D)$ can be seen as a generalisation of the skyscraper sheaf from Example 4.2.2 for $\mathbb{P}^1$ in the case $D = \{p\}$ for some $p \in \mathbb{P}^1$. We denote the skyscraper sheaf at p by $\mathcal{O}_p$.

Let us study the sections of $\mathcal{O}_p$ over U_x for each projective point $p = (a : b)$. By computing the dehomogenisations of the polynomial $bx - ay$, it follows that

$$\mathcal{O}(-p) = (\langle b - az \rangle, \langle bz^{-1} - a \rangle, \mathrm{id}),$$

so the sheaf $\mathcal{O}_p$ is determined by the modules

$$M = \frac{k[z]}{\langle b - az \rangle} \quad \text{and} \quad M' = \frac{k[z^{-1}]}{\langle bz^{-1} - a \rangle}$$

over $k[z]$ and $k[z^{-1}]$ respectively. Since z^{-1} is not torsion in M', the sections over U_x will be pairs $(\overline{f}, \overline{g}/z^{-j}) \in M \times M'_{z^{-1}}$ such that $z^{-j}\overline{f} = \overline{g}$ in $M'_{z^{-1}}$. Each section is therefore determined by the choice of $\overline{f}$, which is a global section of the skyscraper sheaf $\widetilde{M}$ on $\mathbb{A}^1_z$. It follows from the discussion in Example 4.2.2 that

$$\Gamma(U_x, \mathcal{O}_p) \cong \begin{cases} k & \text{if } p \in U_x \\ 0 & \text{if } p \notin U_x. \end{cases}$$

More generally, one can show the same holds with U_x replaced by any open subset U

of $\mathbb{P}^1$. This agrees with our intuition in the affine case.

What makes these examples so compelling is that any coherent sheaf on $\mathbb{P}^1$ is built from twisting sheaves and skyscraper sheaves at a point. This is the content of *Grothendieck's splitting theorem* for $\mathbb{P}^1$, and serves as the projective analogue of Theorem 4.3.1. Before this, we must extend Example 4.3.3 by considering *thickenings* of skyscraper sheaves.

Definition 4.3.8. Fix some $m \in \mathbb{Z}_{\geq 0}$ and a projective point $p = (a : b)$. Then the *thickened skyscraper sheaf* $\mathcal{O}_{mp} := \mathcal{O}/\mathcal{O}(-mp)$, where

$$\mathcal{O}(-mp) := \mathcal{O}(-p)^m = (\langle b - az \rangle^m, \langle bz^{-1} - a \rangle^m, \mathrm{id}).$$

This is the geometric analogue of raising an ideal of a ring to a power. Applying the argument in Example 4.3.3, it is not difficult to see that

$$\Gamma(U, \mathcal{O}_{mp}) \cong \begin{cases} k^m & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

The sheaf $\mathcal{O}_{mp}$ is a torsion $\mathcal{O}_{\mathbb{P}^1}$-module, and it will play the same role as the module $k[x]/\langle x - a \rangle^n$ does in Theorem 4.3.1. In particular, it suffices to classify the torsion part of any coherent sheaf.

Theorem 4.3.2 (Grothendieck's splitting theorem)**.** *Every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$ decomposes as the direct sum*

$$\mathcal{F} \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r) \oplus \mathcal{O}_{m_1 p_1} \oplus \cdots \oplus \mathcal{O}_{m_s p_s}.$$

The direct sum of quasi-coherent sheaves is defined exactly as one intuitively expects, though an explicit definition will be given in the next section. A proof of the splitting theorem is beyond the scope of this thesis. For our purposes, the upshot is more important, namely that to understand any coherent sheaf on $\mathbb{P}^1$, one only really needs to understand the twisting sheaves and thickened skyscraper sheaves.

4.4 Operations on coherent sheaves

Since we have explicitly defined quasi-coherent sheaves in terms of module data, one should intuitively expect that some of the structure and operations on the category of modules will carry over to $\mathsf{QCoh}(\mathbb{P}^1)$ and $\mathsf{Coh}(\mathbb{P}^1)$. This indeed ends up being the case, so the goal of this section is to explicitly spell out the most important aspects.

A key technical detail that we will implicitly use throughout this discussion is that localisation is exact, so it preserves subobjects, quotients, direct sums, kernels and cokernels at the level of modules (c.f. Remark 1.2.1). In particular, we can use this fact to describe the descent data of the sheaves in the constructions below in a way that is more suited to computations. This will also lead to many of the definitions we give in this section being quite straightforward, such as the following.

Definition 4.4.1. Let $\mathcal{F} = (M, M', \vartheta)$ and $\mathcal{G} = (N, N', \varrho)$ be quasi-coherent sheaves on $\mathbb{P}^1$. The *direct sum* of $\mathcal{F}$ and $\mathcal{G}$ is the quasi-coherent sheaf

$$\mathcal{F} \oplus \mathcal{G} := (M \oplus N, M' \oplus N', \vartheta \oplus \varrho).$$

Here, we have used the fact that $(M \oplus N)_f \cong M_f \oplus N_f$ to simplify the descent datum. It is easy to see that $\mathsf{QCoh}(\mathbb{P}^1)$ and $\mathsf{Coh}(\mathbb{P}^1)$ are additive categories, since there is an obvious way to bilinearly add morphisms and an even more obvious candidate for the zero object. We leave the task of checking these small technical details to the reader.

Next, we treat the abelian structure. Retaining the notation of Definition 4.3.3, a quasi-coherent sheaf morphism $\Phi : \mathcal{F} \to \mathcal{G}$ on $\mathbb{P}^1$ gives rise to a commutative diagram

$$\begin{tikzcd}
\ker(\varphi)_z \arrow[r, hook] & M_z \arrow[r, "\varphi_z"] \arrow[d, "\vartheta"'] & N_z \arrow[r, two heads] \arrow[d, "\varrho"] & \operatorname{coker}(\varphi)_z \\
\ker(\varphi')_{z^{-1}} \arrow[r, hook] & M'_{z^{-1}} \arrow[r, "\varphi'_{z^{-1}}"'] & N'_{z^{-1}} \arrow[r, two heads] & \operatorname{coker}(\varphi')_{z^{-1}}.
\end{tikzcd}$$

The commutativity of the middle square implies that the composites

$$\varphi'_{z^{-1}} \circ \vartheta \circ \ker(\varphi)_z = \varrho \circ \varphi_z \circ \ker(\varphi)_z$$

are zero, so the restriction of ϑ to $\ker(\varphi)_z$ is a map into $\ker(\varphi')_{z^{-1}}$.

Definition 4.4.2. The *kernel* of a quasi-coherent sheaf morphism $\Phi : \mathcal{F} \to \mathcal{G}$ on $\mathbb{P}^1$ is the quasi-coherent subsheaf $\ker(\Phi) := (\ker(\varphi), \ker(\varphi'), \vartheta) \leq \mathcal{F}$. Here, we introduce a slight abuse of notation by identifying ϑ with the restriction $\vartheta|_{\ker(\varphi)_z}$. One defines $\operatorname{coker}(\Phi)$ using the above commutative diagram similarly, and hence also $\operatorname{im}(\Phi)$.

Since $k[z]$ and $k[z^{-1}]$ are Noetherian rings, we see that kernels and cokernels of coherent sheaf morphisms must in turn be coherent. As before, we leave it to the reader to check the category-theoretic details necessary for the following claim.

Proposition 4.4.1. *The categories* $\mathsf{QCoh}(\mathbb{P}^1)$ *and* $\mathsf{Coh}(\mathbb{P}^1)$ *are abelian.*

With this, we may consider all of the usual notions for an abelian category. We gave subobjects and quotient objects more palatable, explicit definitions in Definition 4.3.7. By Lemma 1.1.4, the monics, epics and isomorphisms in $\mathsf{QCoh}(\mathbb{P}^1)$ are also explicit: we call the morphism $\Phi = (\varphi, \varphi)$ *injective* if φ and φ' are injective module homomorphisms. *Surjective* and *bijective* morphisms are defined similarly. To see these tools in action, we consider the following result that will be useful in the next section.

Lemma 4.4.1. *For any $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{P}^1$, we have $\mathcal{O}(-mp) \cong \mathcal{O}(-m)$.*

Proof. Let $p = (a : b)$, and consider the maps $\varphi : k[z] \to k[z]$ and $\varphi' : k[z^{-1}] \to k[z^{-1}]$ given by multiplication by $(b - az)^m$ and $(bz^{-1} - a)^m$ respectively. As one easily checks, this determines an injective sheaf morphism $\mathcal{O}(-m) \to \mathcal{O}$ whose image is $\mathcal{O}(-mp)$. $\square$

Of course, the most significant tool in an abelian category are short exact sequences, so we consider two important examples for coherent sheaves.

Example 4.4.1. There is a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

associated to any closed $D \subseteq \mathbb{P}^1$, sometimes called the *ideal sheaf sequence*. This is a geometric analogue of the short exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

associated to any ideal $I \trianglelefteq R$.

Proposition 4.4.2 (Euler sequence for $\mathbb{P}^1$)**.** *There is a short exact sequence*

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)^{\oplus 2} \longrightarrow \mathcal{O} \longrightarrow 0.$$

Proof. As is easy to check, the pairs $(\times 1, \times z^{-1})$ and $(\times z, \times 1)$ both determine morphisms $\mathcal{O}(-1) \to \mathcal{O}$. This induces maps $\varphi : k[z]^{\oplus 2} \to k[z]$ and $\varphi' : k[z^{-1}]^{\oplus 2} \to k[z^{-1}]$ given explicitly by $\varphi(f, g) = f + zg$ and $\varphi'(f, g) = z^{-1}f + g$. These assemble to determine a morphism $\Phi : \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}$, since the diagram

$$\begin{tikzcd} k[z,z^{-1}]^{\oplus 2} \arrow[r, "\varphi_z"] \arrow[d, "\times z"'] & k[z,z^{-1}] \arrow[d, equal] \\ k[z,z^{-1}]^{\oplus 2} \arrow[r, "\varphi'_{z^{-1}}"'] & k[z,z^{-1}] \end{tikzcd}$$

commutes. Both φ and φ' are surjective module homomorphisms, and it follows that Φ is surjective. To obtain the Euler exact sequence, it therefore suffices to show that $\mathcal{O}(-2) \cong \ker(\Phi)$. The map $\psi : 1 \mapsto (-z, 1)$ induces a $k[z]$-module isomorphism

$$k[z] \cong \{(-zg, g) : g \in k[z]\} = \ker(\varphi),$$

and similarly the map $\psi' : 1 \mapsto -(1, -z^{-1})$ induces a $k[z^{-1}]$-module isomorphism

$$k[z^{-1}] \cong \{(f, -z^{-1}f) : f \in k[z^{-1}]\} = \ker(\varphi').$$

The introduction of the sign in ψ' ensures that the diagram

$$\begin{tikzcd} k[z, z^{-1}] \arrow[r, leftrightarrow, "\psi_z"] \arrow[d, "\times z^2"'] & \ker(\varphi)[z^{-1}] \arrow[d, "\times z"] \\ k[z, z^{-1}] \arrow[r, leftrightarrow, "\psi'_{z^{-1}}"'] & \ker(\varphi')[z] \end{tikzcd}$$

commutes both ways, and so the pair (ψ, ψ') is the desired sheaf isomorphism. $\square$

Since the tensor product of two modules produces another module over commutative rings, we can also reasonably expect to have an equivalent notion for quasi-coherent sheaves on $\mathbb{P}^1$. We once again base our definition on the R_f-module isomorphism $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$.

Definition 4.4.3. Given quasi-coherent sheaves $\mathcal{F} = (M, M', \vartheta)$ and $\mathcal{G} = (N, N', \varrho)$ on $\mathbb{P}^1$, the *tensor product* of $\mathcal{F}$ and $\mathcal{G}$ is the quasi-coherent sheaf

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{G} := (M \otimes_{k[z]} N, M' \otimes_{k[z^{-1}]} N', \vartheta \otimes_{k[z,z^{-1}]} \varrho).$$

Just as it does for modules, $\otimes_{\mathcal{O}_{\mathbb{P}^1}}$ defines a bifunctor on $\mathsf{QCoh}(\mathbb{P}^1)$ which is right exact in both arguments, though we will not explicitly verify this claim. As is a recurring theme at this point, tensor products involving twisting sheaves are easy to describe.

Proposition 4.4.3. *For $m, n \in \mathbb{Z}$ we have $\mathcal{O}(m) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(n) \cong \mathcal{O}(m + n)$.*

Proof. The most crucial step is to understand why the sheaf $\mathcal{O}(m + n)$ appears in this isomorphism. But this is just an easy consequence of the algebra of tensors, since

$$z^{-m} f \otimes_{k[z,z^{-1}]} z^{-n} g = f \otimes_{k[z,z^{-1}]} z^{-(m+n)} g$$

for any $f, g \in k[z, z^{-1}]$, i.e. the descent datum of $\mathcal{O}(m) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(n)$ and the map $1 \otimes \vartheta_{m+n}$ are the same. The proof is now clear, as one simply lifts the R-module isomorphism

$R \otimes_R R \cong R$ given by $r \otimes s \mapsto rs$ to an appropriate coherent sheaf isomorphism. $\square$

Our interest in the interaction between $\otimes_{\mathcal{O}_{\mathbb{P}^1}}$ and these twisting sheaves stems from the following important class of quasi-coherent sheaves motivated by module theory.

Definition 4.4.4. A quasi-coherent sheaf $\mathcal{F} = (M, M', \vartheta)$ on $\mathbb{P}^1$ is called *flat* if M and M' are flat modules over $k[z]$ and $k[z^{-1}]$ respectively.

The flat modules are equivalently those for which the tensor product of modules is an exact functor, and we can check that the analogous statement naturally holds for a flat quasi-coherent sheaf on $\mathbb{P}^1$ in our setup.

Proposition 4.4.4. *If $\mathcal{G} = (N, N', \varrho)$ is a flat quasi-coherent sheaf on $\mathbb{P}^1$, then the functor $- \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{G} : \mathsf{QCoh}(\mathbb{P}^1) \to \mathsf{QCoh}(\mathbb{P}^1)$ is exact.*

Proof. Consider a short sequence of sheaves

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{\Phi_1} \mathcal{F}_2 \xrightarrow{\Phi_2} \mathcal{F}_3 \longrightarrow 0$$

with $\Phi_1 = (\varphi_1, \varphi_1')$ and $\Phi_2 = (\varphi_2, \varphi_2')$. It is matter of checking definitions to see that this sequence is exact if and only if the short sequences of modules

$$0 \longrightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0,$$

$$0 \longrightarrow M_1' \xrightarrow[\varphi_1']{} M_2' \xrightarrow[\varphi_1']{} M_3' \longrightarrow 0$$

are exact over $k[z]$ and $k[z^{-1}]$ respectively. Since N and N' are flat modules, the claim is now immediate upon tensoring the above short exact sequence of sheaves by $\mathcal{G}$. $\square$

The most basic class of flat sheaves are the twisting sheaves, and in light of this, we define a new operation on quasi-coherent sheaves.

Definition 4.4.5. Let $n \in \mathbb{Z}$ and define

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(n)$$

for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$. We call $\mathcal{F}(n)$ the *degree n twist* of $\mathcal{F}$, and it induces an exact *twisting functor* $(n) : \mathsf{QCoh}(\mathbb{P}^1) \to \mathsf{QCoh}(\mathbb{P}^1)$.

Proposition 4.4.5. *Twisting is an autoequivalence of* $\mathsf{QCoh}(\mathbb{P}^1)$.

Proof. Let $n \in \mathbb{Z}$. The key observation is that by Proposition 4.4.3, we have

$$\mathcal{F}(-n)(n) = (\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(-n)) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(n) \cong \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O} = \mathcal{F}(0),$$

and there is an obvious isomorphism $\mathcal{F}(0) \cong \mathcal{F}$. Similarly, $\mathcal{F}(n)(-n) \cong \mathcal{F}$. From these facts, it is not difficult to see that the functors (n) and $(-n)$ must be naturally inverse to each other, so the degree n twist is an equivalence of categories. $\square$

4.5 Coherent sheaf cohomology for $\mathbb{P}^1$

To conclude this chapter, we discuss some aspects of *coherent sheaf cohomology* for $\mathbb{P}^1$. In its full form, this is an indispensable technique in modern algebraic geometry that applies even more generally than just for quasi-coherent sheaves. In keeping with the spirit of this chapter so far, we present only what is needed for our goals.

As a prelude to this discussion, now seems like a fitting time to address the issue of injectives and projectives in $\mathsf{QCoh}(\mathbb{P}^1)$ and $\mathsf{Coh}(\mathbb{P}^1)$, for which there is mixed news.

Proposition 4.5.1. *The category* $\mathsf{QCoh}(\mathbb{P}^1)$ *has enough injectives, but not enough projectives. The category* $\mathsf{Coh}(\mathbb{P}^1)$ *has neither enough injectives nor enough projectives.*

We are not adequately equipped to treat these claims, so we only make brief remarks. The first claim is true because $\mathbb{P}^1$ is an example of a *Noetherian scheme*, and the category of quasi-coherent sheaves on any such scheme will always have enough injectives (c.f. [Har77, Exercise III.3.6(a)]). This does not descend to $\mathsf{Coh}(\mathbb{P}^1)$ since coherent sheaves are intuitively "too small" to be injective, in much the same way that the injective envelope of a finitely generated module is not usually finitely generated. As for the lack of projectives in both categories, this can be seen to fail already for the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, which is the upshot of [Har77, Exercise III.6.2]. This will be a more significant observation for Chapter 5, but the more immediate task of setting up coherent sheaf cohomology is contingent only on the first claim.

The motivation for this technology is to study potential problems in lifting global sections. It turns out that the *global sections functor* $\Gamma(\mathbb{P}^1, -) : \mathsf{QCoh}(\mathbb{P}^1) \to \mathsf{Vect}_k$ is left exact, so given a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

on $\mathbb{P}^1$, we have only the left exact sequence of vector spaces

$$0 \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{F}_1) \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{F}_2) \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{F}_3).$$

The failure of the map $\Gamma(\mathbb{P}^1, \mathcal{F}_2) \to \Gamma(\mathbb{P}^1, \mathcal{F}_3)$ to be surjective is exactly the obstruction to lifting a global section from $\mathcal{F}_3$ to $\mathcal{F}_2$, and there are two main ways to qualitatively measure this. The more sophisticated view is given by Grothendieck in [Tōhoku], who suggests that we should study the *derived functor cohomology groups*

$$H^i(\mathcal{F}) = H^i(\mathbb{P}^1, \mathcal{F}) := R^i\Gamma(\mathbb{P}^1, \mathcal{F}) \cong \operatorname{Ext}^i(\mathcal{O}, \mathcal{F}).$$

The last isomorphism follows by Lemma 4.3.3, and it is for this reason that one has the additional notation $H^0(X, \mathcal{F})$ for global sections. While this is certainly powerful for answering theoretical questions, this version of sheaf cohomology is inconvenient to compute in practice since we must explicitly construct an injective resolution of $\mathcal{F}$. Serre's original version of sheaf cohomology for algebraic geometry as defined in [FAC] instead uses the easier to compute *Čech cohomology groups* $\check{H}^i(\mathcal{F})$ defined below.

A priori, these two cohomology theories may not yield isomorphic cohomology groups. Fortunately, they are always isomorphic for quasi-coherent sheaves on *Noetherian separated schemes* (c.f. [Har77, Theorem III.4.5]) such as $\mathbb{A}^1$ and $\mathbb{P}^1$. In our discussion, we therefore mostly consider sheaf cohomology a la Serre.

Definition 4.5.1. Fix an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of a topological space X and a well-order on the index set I. For ease of notation, we will write the intersection of the subsets corresponding to the ordered $(p+1)$-tuple $\boldsymbol{i} = (i_0, i_1, \ldots, i_p) \in I^{p+1}$ as

$$U_{i_0 i_1 \cdots i_p} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p},$$

where $i_0 < i_1 < \cdots < i_p$. Then for any sheaf of abelian groups $\mathcal{F}$ on X, one defines the pth *Čech cochains* to be the abelian group

$$\check{C}^p(\mathcal{U}, \mathcal{F}) := \bigoplus_{\boldsymbol{i} \in I^{p+1}} \Gamma(U_{i_0 i_1 \cdots i_p}, \mathcal{F})$$

for all $p \in \mathbb{Z}_{\geq 0}$. We define the pth differential map $d : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$d(s_{i_0 i_1 \cdots i_p})_{\boldsymbol{i} \in I^{p+1}} := \left(\sum_{j=0}^{p+1} (-1)^j s_{i_0 i_1 \cdots \widehat{i_j} \cdots i_{p+1}} \Big|_{U_{i_0 i_1 \cdots \widehat{i_j} \cdots i_{p+1}}} \right)_{\boldsymbol{i} \in I^{p+2}},$$

where $i_0 i_1 \cdots \widehat{i_j} \cdots i_{p+1}$ is the list with the index i_j omitted. From this, one obtains the

Čech cochain complex $\check{C}^\bullet(\mathcal{U},\mathcal{F})$,

$$0 \longrightarrow \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{\ d\ } \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\ d\ } \check{C}^2(\mathcal{U},\mathcal{F}) \xrightarrow{\ d\ } \cdots,$$

and its cohomology are the *Čech cohomology groups* $\check{H}^p(\mathcal{F}) = \check{H}^p(\mathcal{U},\mathcal{F})$ for $p \geq 0$.

Mercifully, this setup is much more tractable for $\mathbb{P}^1$. The Čech cohomology groups are generally dependent on the choice of open cover, but not in the aforementioned case of Noetherian separated schemes, so we can proceed using our standard affine open cover $\mathcal{U} = \{U_x, U_y\}$ with impunity. For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$, we have

$$\check{C}^0(\mathcal{U},\mathcal{F}) = \Gamma(U_x,\mathcal{F}) \oplus \Gamma(U_y,\mathcal{F}) \quad \text{and} \quad \check{C}^1(\mathcal{U},\mathcal{F}) = \Gamma(U_{xy},\mathcal{F}),$$

and clearly $\check{C}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p \geq 2$. The only non-trivial differential in the Čech complex is the map $d : \Gamma(U_x,\mathcal{F}) \oplus \Gamma(U_y,\mathcal{F}) \to \Gamma(U_{xy},\mathcal{F})$ given by

$$d(\boldsymbol{s},\boldsymbol{t}) = \boldsymbol{s}\big|_{U_{xy}} - \boldsymbol{t}\big|_{U_{xy}},$$

where $\boldsymbol{s} = (s, s')$ and $\boldsymbol{t} = (t, t')$ in the notation of Definition 4.3.5. A technical point that is unique to the particular way we have set up quasi-coherent sheaves on $\mathbb{P}^1$ is that one may need to use the descent datum for $\mathcal{F}$ first to ensure that these restrictions can be subtracted in this manner (i.e. they must be compatible with each other first).

It follows that the only non-zero cohomology groups for $\mathbb{P}^1$ are

$$\check{H}^0(\mathcal{F}) = \ker(d) \quad \text{and} \quad \check{H}^1(\mathcal{F}) = \operatorname{coker}(d),$$

and it is not hard to see that $\check{H}^0(\mathcal{F})$ indeed corresponds to $\Gamma(\mathbb{P}^1,\mathcal{F})$ by the sheaf property. The equivalent statement about the vanishing of the higher derived functor cohomology groups is known as *Grothendieck vanishing*, and is rather non-trivial in comparison (c.f. [Stacks, Proposition 20.20.7]). With the advanced knowledge that Čech and derived functor cohomology agree in this situation, we will simply write $H^0(\mathcal{F})$ and $H^1(\mathcal{F})$ for both cohomology theories. We will also persist in calling these spaces cohomology *groups* even though they are in fact vector spaces over k.

The value in introducing Čech cohomology in general upfront rather than giving only the specific setup for $\mathbb{P}^1$ is that we see that the Čech cohomology groups are the cohomology of a cochain complex, so all of the usual notions of Section 2.2 are available. In particular, each $H^p : \mathsf{QCoh}(\mathbb{P}^1) \to \mathsf{Vect}_k$ is an additive functor which commutes with direct sums, so as a consequence of Grothendieck's splitting theorem, the cohomology groups of any coherent sheaf on $\mathbb{P}^1$ reduces to a direct sum of the cohomology groups of $\mathcal{O}(n)$ and $\mathcal{O}_{mp}$. In this vein, we present two fundamental computations.

Lemma 4.5.1. *For any $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{P}^1$, $H^1(\mathcal{O}_{mp}) = 0$.*

Proof. Put another way, the claim is that the differential map d is surjective. The sections of $\mathcal{O}_{mp}$ are supported only in open neighbourhoods of p, so if p is either of the degenerate points $(1:0)$ or $(0:1)$, then $p \notin U_{xy}$ and $d = 0$, so the claim is trivial in this case. In the non-degenerate case, we have the isomorphisms of vector spaces

$$\Gamma(U_x, \mathcal{O}_{mp}) \cong \Gamma(U_y, \mathcal{O}_{mp}) \cong \Gamma(U_{xy}, \mathcal{O}_{mp}) \cong k^m.$$

Since the descent datum of $\mathcal{O}_{mp}$ acts as the identity, the restriction maps to U_{xy} are simply the identity maps on k^m, so we see that d is once again surjective. $\square$

Lemma 4.5.2. *For any $n \in \mathbb{Z}$, we have*

$$H^1(\mathcal{O}(n)) \cong \begin{cases} k^{|n|-1} & \text{if } n \leq -2 \\ 0 & \text{if } n \geq -1. \end{cases}$$

Proof. As should be familiar at this point, we can characterise the sections of $\mathcal{O}(n)$ over U_x as pairs $(f, g) \in k[z] \times k[z, z^{-1}]$ such that $g = z^{-n} f$. Each section is determined by the choice of f, so there is an isomorphism $\Gamma(U_x, \mathcal{O}(n)) \cong k[z]$. By similar arguments, we have $\Gamma(U_y, \mathcal{O}(n)) \cong k[z^{-1}]$, and similarly

$$\Gamma(U_{xy}, \mathcal{O}(n)) \cong k[z]_z \cong k[z, z^{-1}] \cong k[z^{-1}]_{z^{-1}}.$$

The corresponding restriction maps from U_x and U_y to U_{xy} are the canonical inclusions $k[z] \hookrightarrow k[z]_z$ and $k[z^{-1}] \hookrightarrow k[z]_{z^{-1}}$ respectively, but to ensure compatibility we shall postcompose the second restriction map by $\vartheta_n^{-1} : k[z^{-1}]_{z^{-1}} \to k[z]_z$, i.e. by multiplying by z^n. Thus the Čech differential is ultimately the map $d(f, g) = f - z^n g$.

It remains to find $\operatorname{coker}(d)$. We see that $\operatorname{im}(d)$ is a vector subspace of $k[z, z^{-1}]$ over k, and clearly it contains the subspace generated by all non-negative powers of z. Moreover, $\operatorname{im}(d)$ contains $z^i = z^n z^{-(n-i)}$ for a given $i \leq -1$ if and only if $n - i \geq 0$. This is trivially true when $n \geq -1$, so in this case $\operatorname{im}(d) = k[z, z^{-1}]$ and $H^1(\mathcal{O}(n)) = 0$. When $n \leq -2$, we have $n - i = -|n| - i < 0$ for all $-(|n| - 1) \leq i \leq -1$, so in this case $\operatorname{im}(d)$ is the (set-theoretic) complement of the subspace

$$\left\langle z^{-1}, z^{-2}, \ldots, z^{-(|n|-1)} \right\rangle \cong k^{|n|-1}.$$

It follows that $H^1(\mathcal{O}(n)) \cong k^{|n|-1}$. $\square$

The final pair of cohomological facts we will collect relate to the Ext groups. The second of these will rely on an additional result in coherent sheaf cohomology that is slightly beyond our reach in this thesis.

Lemma 4.5.3. *Let $\mathcal{G}$ be a coherent sheaf on $\mathbb{P}^1$. Then for all $n \in \mathbb{Z}$ and $i \in \mathbb{Z}_{\geq 0}$,*

$$\operatorname{Ext}^i(\mathcal{O}(n), \mathcal{G}) \cong H^i(\mathcal{G} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(-n)).$$

In particular, $\operatorname{Ext}^i(\mathcal{O}(n), \mathcal{G}) = 0$ *for* $i \geq 2$.

Proof. Since the degree $-n$ twist is an autoequivalence, by Lemma 2.5.1 we have

$$\begin{aligned}\operatorname{Ext}^i(\mathcal{O}(n), \mathcal{G}) &\cong \operatorname{Ext}^i(\mathcal{O}, \mathcal{G} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(-n)) \\ &\cong H^i(\mathcal{G} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(-n))\end{aligned}$$

by construction of the derived functor cohomology. These cohomology groups vanish for $i \geq 2$, so the second part of this claim is immediate. $\square$

Proposition 4.5.2. *The category* $\mathsf{Coh}(\mathbb{P}^1)$ *is* hereditary. *That is,* $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ *for all $i \geq 2$ and coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{P}^1$.*

Proof. Since Ext^i is an additive functor, it is enough to prove the claim when $\mathcal{F}$ is either a twisting sheaf or a skyscraper sheaf by Grothendieck's splitting theorem. We have treated the former case already in Lemma 4.5.3, so suppose that $\mathcal{F} = \mathcal{O}_{mp}$. Recalling Lemma 4.4.1, we have the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}(-m) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{mp} \longrightarrow 0,$$

Applying a degree n twist for some $n \in \mathbb{Z}$ yields a new short exact sequence

$$0 \longrightarrow \mathcal{O}(n-m) \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}_{mp}(n) \longrightarrow 0.$$

First, we note that $\mathcal{O}_{mp}(n) \cong \mathcal{O}_{mp}$. Indeed, the sections are still supported only at the point p, and the algebraic effect of the twist on the descent datum in neighbourhoods of p is scaling by some negative power of a unit in k, which is an invertible operation. Next, since the contravariant functor $\operatorname{Hom}(-, \mathcal{I})$ is exact for any injective quasi-coherent sheaf $\mathcal{I}$, taking an injective resolution $\mathcal{G} \to \mathcal{I}^\bullet$ yields a short exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_{mp}, \mathcal{I}^\bullet) \longrightarrow \operatorname{Hom}(\mathcal{O}(n), \mathcal{I}^\bullet) \longrightarrow \operatorname{Hom}(\mathcal{O}(n-m), \mathcal{I}^\bullet) \longrightarrow 0$$

of complexes with the direction reversed. Now, for each $i \geq 2$, we consider the exact portion of the corresponding long exact cohomology sequence

$$\operatorname{Ext}^{i-1}(\mathcal{O}(n-m), \mathcal{G}) \longrightarrow \operatorname{Ext}^{i}(\mathcal{O}_{mp}, \mathcal{G}) \longrightarrow \operatorname{Ext}^{i}(\mathcal{O}(n), \mathcal{G}).$$

The right term vanishes for all $i \geq 2$ by Lemma 4.5.3, and so does the left term when $i > 2$, so we conclude that $\operatorname{Ext}^{i}(\mathcal{O}_{mp}, \mathcal{G}) = 0$ in this case. Thus it remains to show that

$$\operatorname{Ext}^{1}(\mathcal{O}(n-m), \mathcal{G}) \cong \operatorname{Ext}^{1}(\mathcal{O}, \mathcal{G}(m-n)) \cong H^{1}(\mathcal{G}(m-n))$$

also vanishes. This may not be true a priori for any choice of n, but by a vanishing theorem of Serre (c.f. [Har77, Theorem III.5.2(b)]), the cohomology group $H^{1}(\mathcal{G}(m-n))$ is guaranteed to vanish when $m-n$ is sufficiently large, i.e. when n is sufficiently negative. Taking such an n therefore completes the proof of the claim. □

Chapter 5

Beilinson's theorem for $\mathbb{P}^1$

It is now time to deliver on the main goal of this thesis, that being to state and prove *Beilinson's theorem* for $\mathbb{P}^1$. We will follow the methods of *tilting theory*, ultimately realising this result as a consequence of the existence of a certain *tilting bundle* in $\mathsf{Coh}(\mathbb{P}^1)$. This will bring to bear the considerable arsenal of mathematical tools we have worked hard to assemble so far.

5.1 The tilting bundle and the Kronecker quiver

The goal of this section is to motivate why the coherent sheaves on $\mathbb{P}^1$ connect to the representation theory of algebras and quivers on a more foundational level before we move to the derived category. We start from the algebra side.

Definition 5.1.1. The *tilting bundle* of $\mathbb{P}^1$ is the sheaf $\mathcal{T} := \mathcal{O} \oplus \mathcal{O}(1)$. Its endomorphism algebra $\Lambda := \mathrm{End}(\mathcal{T}) = \mathrm{Hom}(\mathcal{T}, \mathcal{T})$ is called the *tilted algebra* of $\mathbb{P}^1$.

This choice is not strictly unique, as it will turn out that one can take any twist of $\mathcal{T}$ and the results of this chapter will still go through. At the outset, we immediately note that Λ is a non-commutative, finite dimensional k-algebra. To begin to understand Λ further, we recall an important decomposition arising in the theory of idempotents.

Definition 5.1.2. Let A be a k-algebra and e an *idempotent* element of A (i.e. $e^2 = e$). Then the *(two-sided) Peirce decomposition* of A with respect to e as a k-algebra is the direct sum decomposition

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

For details, see [Lam01, §21]. So that we may compute the Peirce decomposition of Λ, we first need to identify an idempotent endomorphism in $\mathcal{T}$. Given that $\mathcal{T}$ is formed as a direct sum, a straightforward choice seems to be the *first projection* endomorphism $\Pi_1 := (\pi_1, \pi_1')$ defined by the projection maps

$$\pi_1 : k[z]^{\oplus 2} \to k[z]^{\oplus 2} \quad \text{and} \quad \pi_1' : k[z^{-1}]^{\oplus 2} \to k[z^{-1}]^{\oplus 2}$$

onto the respective first summands. It is not difficult to see that Π_1 is indeed an endomorphism of $\mathcal{T}$, and clearly $\Pi_1^2 = \Pi_1$. Similarly, one obtains the *second projection* endomorphism $\Pi_2 := (\pi_2, \pi_2')$ with analogous properties and such that $\mathrm{id}_{\mathcal{T}} - \Pi_1 = \Pi_2$. Thus the Peirce decomposition for Λ with respect to Π_1 reads

$$\Lambda = \Pi_1 \Lambda \Pi_1 \oplus \Pi_1 \Lambda \Pi_2 \oplus \Pi_2 \Lambda \Pi_1 \oplus \Pi_2 \Lambda \Pi_2.$$

In the following series of lemmas, we identify these direct summands.

Lemma 5.1.1. *We have* $\Pi_1 \Lambda \Pi_1 = k\Pi_1$ *and* $\Pi_2 \Lambda \Pi_2 = k\Pi_2$.

Proof. It will be sufficient to prove only the first claim, as the proof of the second is virtually identical. Given any endomorphism $\Phi = (\varphi, \varphi')$ of $\mathcal{T}$, we can write

$$\varphi(f, g) = (\varphi_1(f, g), \varphi_2(f, g)),$$

where $\varphi_1, \varphi_2 : k[z]^2 \to k[z]$ are necessarily $k[z]$-module homomorphisms. Since

$$(\pi_1 \circ \varphi \circ \pi_1)(f, g) = f \cdot (\pi_1 \circ \varphi \circ \pi_1)(1, g) = f \cdot (\varphi_1(1, 0), 0),$$

we see that the map $\pi_1 \circ \varphi \circ \pi_1$ is determined by the polynomial $\lambda = \varphi_1(1, 0)$. Similarly, there are $k[z^{-1}]$-module homomorphisms $\varphi_1', \varphi_2' : k[z^{-1}]^2 \to k[z^{-1}]$ so that

$$(\pi_1' \circ \varphi' \circ \pi_1')(f, g) = f \cdot (\varphi'(1, 0), 0),$$

so $\pi_1' \circ \varphi' \circ \pi_1'$ is determined by the polynomial $\lambda' = \varphi_1'(1, 0)$. The diagram

$$\begin{tikzcd}
k[z, z^{-1}]^{\oplus 2} \arrow[r, "\pi_1 \circ \varphi \circ \pi_1"] \arrow[d, "(\times 1, \times z^{-1})"'] & k[z, z^{-1}]^{\oplus 2} \arrow[d, "(\times 1, \times z^{-1})"] \\
k[z, z^{-1}]^{\oplus 2} \arrow[r, "\pi_1' \circ \varphi' \circ \pi_1'"'] & k[z, z^{-1}]^{\oplus 2}
\end{tikzcd}$$

associated to $\Pi_1 \Phi \Pi_1$ commutes, so it follows by chasing the image of the element $(1, 0)$ that $\lambda = \lambda'$. Since $\lambda \in k[z]$ and $\lambda' \in k[z^{-1}]$, this forces $\lambda \in k$. Thus $\pi_1 \circ \varphi \circ \pi_1 = \lambda \pi_1$ and $\pi_1' \circ \varphi' \circ \pi_1' = \lambda \pi_1'$. $\square$

Lemma 5.1.2. *We have* $\Pi_1 \Lambda \Pi_2 = 0$.

Proof. Given an endomorphism $\Phi = (\varphi, \varphi')$ of $\mathcal{T}$, it is easy to compute that

$$\pi_1 \circ \varphi \circ \pi_2 = 0 \quad \text{and} \quad \pi_1' \circ \varphi' \circ \pi_2' = 0,$$

so $\Pi_1 \Phi \Pi_2 = 0$ for all such Φ. □

Before tackling the final summand, we define a type of endomorphism that will help us completely characterise the elements of $\Pi_2 \Lambda \Pi_1$.

Definition 5.1.3. Let $(\lambda, \lambda') \in k[z] \times k[z^{-1}]$ be a pair of *weight polynomials* for $\mathcal{T}$, i.e. such that $\lambda' = z^{-1}\lambda$ in $k[z, z^{-1}]$. Then the *right shift of* $\mathcal{T}$ *with weight* (λ, λ') is the endomorphism $S_{\lambda,\lambda'} := (s_\lambda, s'_{\lambda'})$, where $s_\lambda : k[z]^2 \to k[z]^2$ and $s'_{\lambda'} : k[z^{-1}]^2 \to k[z^{-1}]^2$ are the module homomorphisms

$$s_\lambda(f, g) = (0, \lambda f) \quad \text{and} \quad s'_{\lambda'}(f, g) = (0, \lambda' f).$$

The verification that any right shift of $\mathcal{T}$ is indeed an endomorphism of $\mathcal{T}$ is straightforward and left to the reader. This is a non-standard notion and somewhat restrictive in its setup, since any choice of weights is determined by the choice of λ, and it is easy to see from a degree argument that λ can only be a constant or linear polynomial in z, say $\lambda(z) = a + bz$ for some $a, b \in k$. This is strictly to our advantage, however. The construction of right shifts is linear with respect to the weight polynomials, by which we mean that

$$s_\lambda = a s_1 + b s_z \quad \text{and} \quad s'_{\lambda'} = a s'_{z^{-1}} + b s'_1,$$

which is to say that $S_{\lambda,\lambda'} = a S_{1,z^{-1}} + b S_{z,1}$. Since the composition of any two right shifts is trivially 0 in Λ, the set of all right shifts forms the subalgebra of Λ generated as a vector space over k by $S_{1,z^{-1}}$ and $S_{z,1}$.

Lemma 5.1.3. *The subalgebra of right shifts of* $\mathcal{T}$ *is precisely* $\Pi_2 \Lambda \Pi_1$.

Proof. The argument is similar to Lemma 5.1.1, so we provide fewer details this time. Consider an endomorphism $\Phi = (\varphi, \varphi')$ of $\mathcal{T}$ and, in the prior notation, let $\lambda = \varphi_2(1, 0)$ and $\lambda' = \varphi_2'(1, 0)$. Then $\Pi_2 \Phi \Pi_1$ is the endomorphism of $\mathcal{T}$ determined by the maps

$$(\pi_2 \circ \varphi \circ \pi_1)(f, g) = (0, \lambda f) \quad \text{and} \quad (\pi_2' \circ \varphi' \circ \pi_1')(f, g) = (0, \lambda' f).$$

Chasing the image of $(1, 0)$ in the corresponding commutative diagram for $\Pi_2 \Phi \Pi_1$, one sees that $\lambda' = z^{-1}\lambda$ in $k[z, z^{-1}]$, so $\Pi_2 \Phi \Pi_1$ is indeed the right shift $S_{\lambda,\lambda'}$. □

Putting these results together, we have

$$\Lambda = k\Pi_1 \oplus kS_{1,z^{-1}} \oplus kS_{z,1} \oplus k\Pi_2.$$

The algebra structure is determined by the idempotent relations $\Pi_1^2 = \Pi_1$ and $\Pi_2^2 = \Pi_2$, as well as the non-zero products

$$S_{1,z^{-1}}\Pi_1 = S_{1,z^{-1}}, \quad S_{z,1}\Pi_1 = S_{z,1}, \quad \Pi_2 S_{1,z^{-1}} = S_{1,z^{-1}} \quad \text{and} \quad \Pi_2 S_{z,1} = S_{z,1}.$$

We now turn to interpreting this algebra from the point of view of *quivers*. These may seem decidedly less complicated objects than coherent sheaves on $\mathbb{P}^1$, but have a rich and interesting representation theory in their own with many surprising connections to other areas of mathematics.

Definition 5.1.4. A *quiver* is a directed multigraph $Q = (Q_0, Q_1, \mathrm{hd}, \mathrm{tl})$, where Q_0 is its set of *vertices*, Q_1 is its set of *arrows*, and $\mathrm{hd}, \mathrm{tl} : Q_1 \to Q_0$ are set functions which assign to each arrow its *head* and *tail* vertex respectively.

This definition allows in principle for quivers consisting of infinitely many vertices or arrows, but the theory is understandably best behaved under finiteness assumptions on both Q_0 and Q_1. We shall consider the single example relevant to our goals.

Example 5.1.1. The *Kronecker quiver* is the graph

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2.$$

Formally, this is the quiver $\Delta = (\{1,2\}, \{\alpha, \beta\}, \mathrm{hd}, \mathrm{tl})$, where

$$\mathrm{hd}(\alpha) = \mathrm{hd}(\beta) = 1 \quad \text{and} \quad \mathrm{tl}(\alpha) = \mathrm{tl}(\beta) = 2.$$

The origin of Δ traces back to a classical problem (c. 1890) of linear algebra studied by its namesake, as Kronecker was interested in classifying all pairs of linear maps

$$\mathbb{C}^m \underset{T}{\overset{S}{\rightrightarrows}} \mathbb{C}^n$$

of vector spaces over $\mathbb{C}$ up to a change of basis, i.e. $(S,T) \sim (S',T')$ if

$$S' = B^{-1}SA \quad \text{and} \quad T' = B^{-1}TA$$

for some linear isomorphisms $A : \mathbb{C}^m \to \mathbb{C}^m$ and $B : \mathbb{C}^n \to \mathbb{C}^n$. Though this problem predates quivers, this is in fact a question about the *finite dimensional representations* of Δ over $\mathbb{C}$. Just as one is able to do in the representation theory of finite groups as modules over its group algebra, we can equivalently view representations of a quiver as modules over its *path algebra*. This will also reveal where the relationship between the quiver Δ and the tilting bundle $\mathcal{T}$ comes from.

Definition 5.1.5. Given a quiver Q, a *path* is a (possibly empty) sequence of arrows $p = a_1a_2\cdots a_n$ in Q such that $\mathrm{tl}(a_i) = \mathrm{hd}(a_{i+1})$ for all $1 \le i < n$. We shall write $\mathrm{hd}(p) = \mathrm{hd}(a_1)$ and $\mathrm{tl}(p) = \mathrm{tl}(a_n)$. There is an empty path for each vertex $i \in Q_0$, called the *trivial path* e_i, such that $\mathrm{hd}(e_i) = \mathrm{tl}(e_i) = i$.

Definition 5.1.6. Fix a field k. Given a quiver Q, let kQ be the vector space over k freely generated by paths in Q, i.e. the set of formal k-linear combinations $\sum \lambda_i p_i$ with $\lambda_i \in k$ and p_i ranging over all paths in Q. For any two paths $p = a_1a_2\cdots a_n$ and $q = b_1b_2\cdots b_m$, we define the *concatenation* of p and q to be the element

$$pq := \begin{cases} a_1a_2\cdots a_nb_1b_2\cdots b_m & \text{if } \mathrm{tl}(p) = \mathrm{hd}(q) \\ 0 & \text{otherwise.} \end{cases}$$

This extends to a bilinear product, and we call kQ the *path algebra* of Q over k.

Example 5.1.2. For the Kronecker quiver, the path algebra over k is the vector space

$$k\Delta = ke_1 \oplus ke_2 \oplus k\alpha \oplus k\beta.$$

To describe the algebra structure, we find the relations on these basis paths. It is clear that the trivial paths are always idempotents of the path algebra, so $e_1^2 = e_1$ and $e_2^2 = e_2$. The only other non-zero concatenations of paths in $k\Delta$ are

$$e_1\alpha = \alpha, \quad e_1\beta = \beta, \quad \alpha e_2 = \alpha \quad \text{and} \quad \beta e_2 = \beta.$$

Moreover, as one easily checks, the map

$$e_1 \mapsto \Pi_1, \quad e_2 \mapsto \Pi_2, \quad \alpha \mapsto S_{1,z^{-1}} \quad \text{and} \quad \beta \mapsto S_{z,1}$$

determines a k-algebra isomorphism $k\Delta \cong \Lambda^{\mathrm{op}}$, the *opposite algebra* of Λ.

Remark 5.1.1. An alternative way to present the data of these k-algebras is as a certain *generalised matrix algebra* of lower-triangular matrices. The set

$$\begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix} := \left\{ \begin{bmatrix} a & 0 \\ (b,c) & d \end{bmatrix} : a,b,c,d \in k \right\}$$

is a vector space over k with addition defined in the obvious way, and becomes a k-algebra via the multiplication operation

$$\begin{bmatrix} a & 0 \\ (b,c) & d \end{bmatrix} \begin{bmatrix} e & 0 \\ (f,g) & h \end{bmatrix} = \begin{bmatrix} ae & 0 \\ (be+df, ce+dg) & dh \end{bmatrix}.$$

This matrix algebra is isomorphic to Λ, as witnessed by the map

$$\Pi_1 \mapsto \begin{bmatrix} 1 & 0 \\ (0,0) & 0 \end{bmatrix}, \qquad \Pi_2 \mapsto \begin{bmatrix} 0 & 0 \\ (0,0) & 1 \end{bmatrix},$$
$$S_{1,z^{-1}} \mapsto \begin{bmatrix} 0 & 0 \\ (1,0) & 0 \end{bmatrix}, \qquad S_{z,1} \mapsto \begin{bmatrix} 0 & 0 \\ (0,1) & 0 \end{bmatrix}.$$

With a more general view toward the representation theory of finite dimensional algebras, path algebras of quivers occupy a key role. They prove to be a rich source of (often non-commutative) algebras, and as we have seen with the Kronecker quiver, any finite quiver with no oriented cycles will yield a finite dimensional path algebra. In fact, one can say much more.

Theorem 5.1.1 (Gabriel)**.** *Let A be a finite dimensional algebra over a field k. Then there exists a quiver Q and an ideal $I \trianglelefteq kQ$ such that A is* Morita equivalent *to kQ/I, i.e. the categories $\mathsf{mod}(A)$ and $\mathsf{mod}(kQ/I)$ are equivalent.*

We can generalise the discussion of the tilted algebra by considering $\mathrm{Hom}(\mathcal{T}, \mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$. This is a finite dimensional vector space over k on which Λ acts on the right by precomposition, so all told we obtain a functor

$$\mathrm{Hom}(\mathcal{T}, -) : \mathsf{Coh}(\mathbb{P}^1) \to \mathsf{mod}(\Lambda^{\mathrm{op}}).$$

We can tell right away that this is not an equivalence of categories for two reasons. By Example 4.3.2, we see that $\mathrm{End}(\mathcal{O}(-1)) \neq 0$ and

$$\mathrm{Hom}(\mathcal{T}, \mathcal{O}(-1)) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}(-1)) \oplus \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(-1)) = 0$$

in $\mathsf{mod}(\Lambda^{\mathrm{op}})$, so the functor $\mathrm{Hom}(\mathcal{T}, -)$ is not fully faithful. It is also true that an equivalence of abelian categories is necessarily exact, and we see by Lemma 4.5.3 that

$$\mathrm{Ext}^1(\mathcal{T}, \mathcal{O}(-1)) \cong H^1(\mathcal{O}(-1)) \oplus H^1(\mathcal{O}(-2)) \cong k \neq 0.$$

From the point of view of this thesis and everything we have done so far, the latter is the more interesting observation: it suggests that there is additional cohomological information we are not seeing at the level of the abelian categories.

5.2 Tilting in the derived category

We are now justified in wondering if anything more interesting happens in the derived category where this information ought to be more visible. The logical conclusion of this inquiry is Beilinson's theorem. The proof via *tilting theory* for this result first emerged in the literature with the publication of [Bae88], though in the context of smooth *weighted projective varieties*. Rather than using this original work as our source, we instead adapt the simpler exposition given in [Cra07] for the details to sketch how this result follows.

In the interest of keeping an already lengthy and multidisciplinary thesis focused, we will not discuss tilting theory as a subject area, and rather just give an overview of how its methods apply in this case. This is an area best approached and motivated from the point of view of representation theory, and we defer the interested reader to [HHK07], the peerless source on the topic.

The key notion is that of a *tilting object*. We will present only the definition relevant to this geometric case.

Definition 5.2.1. Let $\mathcal{A}$ be an abelian category and $\mathcal{D} = \mathbf{D}^b(\mathcal{A})$ its bounded derived category. Call a subcategory of $\mathcal{D}$ *épaisse* if it is closed under isomorphisms and shifts, as well as taking cones of morphisms and direct summands of objects. An object $A \in \mathcal{A}$ is said to *classically generate* $\mathcal{D}$ if the *épaisse envelope* of A, i.e. the smallest épaisse subcategory of $\mathcal{D}$ containing A (as a complex concentrated in degree 0, say), is $\mathcal{D}$ itself.

Definition 5.2.2. A coherent sheaf $\mathcal{T}$ is a *tilting sheaf* on $\mathbb{P}^1$ if

(T1) $\mathrm{Ext}^i(\mathcal{T}, \mathcal{T}) = 0$ for all $i > 0$;

(T2) the k-algebra $\mathrm{End}(\mathcal{T})$ has finite global dimension (c.f. Section 2.4);

(T3) $\mathcal{T}$ classically generates $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$.

We call a locally free tilting sheaf a *tilting bundle* on $\mathbb{P}^1$.

With the following series of lemmas, we check that $\mathcal{O} \oplus \mathcal{O}(1)$ satisfies all three of the properties of a tilting bundle, showing it deserves its name.

Lemma 5.2.1. *The sheaf $\mathcal{O} \oplus \mathcal{O}(1)$ satisfies (T1).*

Proof. Since Ext^1 is an additive functor, it follows that

$$\mathrm{Ext}^1(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(1)) \cong H^1(\mathcal{O}) \oplus H^1(\mathcal{O}(1)) \oplus H^1(\mathcal{O}(-1)) \oplus H^1(\mathcal{O}) = 0$$

by Lemma 4.5.3 and Lemma 4.5.2. □

Lemma 5.2.2. *The sheaf $\mathcal{O} \oplus \mathcal{O}(1)$ satisfies (T2).*

Proof. As we have seen, $\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(1))$ is the algebra opposite to the path algebra $k\Delta$. But it is well known that any quiver without oriented cycles has a *hereditary* path algebra, i.e. of global dimension 1. We will not prove this, and instead defer to [ARS97, Proposition III.1.4]. □

Lemma 5.2.3. *The sheaf $\mathcal{O} \oplus \mathcal{O}(1)$ satisfies (T3).*

Proof. This is the most challenging property to establish. Let $\mathcal{E}$ be the *épaisse envelope* of $\mathcal{O} \oplus \mathcal{O}(1)$ in $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$. Throughout the proof, we implicitly treat sheaves as complexes concentrated in degree 0 in $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$ and write $\mathcal{F} \in \mathcal{E}$ if $\mathcal{E}$ contains $\mathcal{F}$ in this manner. Since the cone of $0 : \mathcal{F}[-1] \to \mathcal{G}$ is $\mathcal{F} \oplus \mathcal{G}$, it follows that $\mathcal{E}$ is also closed under taking direct sums. Thus to show that $\mathcal{E} = \mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$, it is enough to show that we can build every coherent sheaf on $\mathbb{P}^1$ in degree 0, as taking direct sums of shifted complexes then recovers all other bounded complexes. Moreover, Grothendieck's splitting theorem further reduces this to obtaining the twisting sheaves and thickened skyscraper sheaves. To facilitate this, we establish one last closure property. Given coherent sheaves $\mathcal{F}, \mathcal{G} \in \mathcal{E}$, any short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\Phi} \mathcal{G} \xrightarrow{\Psi} \mathcal{H} \longrightarrow 0.$$

induces an exact triangle in $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$ by Proposition 3.3.3 isomorphic to the strict triangle on Φ. In particular, we have a quasi-isomorphism $\mathrm{cone}(u) \to \mathcal{H}$ which becomes an isomorphism in $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$, so closure under cones and isomorphisms gives $\mathcal{H} \in \mathcal{E}$. Using the rotation axiom (TR2) for triangulated categories, this argument shows by extension that given that any two of the three terms in the above short exact sequence belong to $\mathcal{E}$, the third also necessarily belongs to $\mathcal{E}$.

Since $\mathcal{E}$ contains $\mathcal{O}$ and $\mathcal{O}(1)$ by construction, we can obtain $\mathcal{O}(2) \in \mathcal{E}$ by considering the degree 2 twist of the Euler exact sequence, i.e.

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus 2} \longrightarrow \mathcal{O}(2) \longrightarrow 0.$$

Taking successive degree 1 twists of the above sequence inductively, we also obtain $\mathcal{O}(n) \in \mathcal{E}$ for all $n > 0$. Similarly, to obtain $\mathcal{O}(-1)$, we instead take the degree 1 twist of the Euler exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

and then negatively twist this exact sequence inductively to obtain $\mathcal{O}(n) \in \mathcal{E}$ for all $n < 0$ too. Finally, we now take the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}(-m) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{mp} \longrightarrow 0$$

shows that $\mathcal{O}_{mp} \in \mathcal{E}$ for all $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{P}^1$. This completes the proof. $\square$

At long last, we can now state and prove Beilinson's theorem reimagined by Baer.

Theorem 5.2.1 (Beilinson, Baer)**.** *Let $\mathcal{T}$ be the tilting bundle of $\mathbb{P}^1$ as defined in Definition 5.1.1 and Λ the corresponding tilted algebra. Then the total derived functor*

$$\mathbf{R}\mathrm{Hom}(\mathcal{T}, -) : \mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1)) \to \mathbf{D}^b(\mathsf{mod}(\Lambda^{\mathrm{op}}))$$

induces an equivalences of bounded derived categories whose quasi-inverse is

$$- \otimes^{\mathbf{L}}_{\Lambda} \mathcal{T} : \mathbf{D}^b(\mathsf{mod}(\Lambda^{\mathrm{op}})) \to \mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1)).$$

We will stop short of giving the proof of this result, which is found in [Cra07, Theorem 2.1] and instead briefly comment on the necessity of the tilting bundle assumptions in its proof. The requirement that the tilted algebra has finite global dimension is necessary to ensure that the image of $- \otimes^{\mathbf{L}}_{\Lambda} \mathcal{T}$ produces bounded complexes. The property (T2) shows that

$$\mathbf{R}\mathrm{Hom}(\mathcal{T}, \Lambda \otimes^{\mathbf{L}}_{\Lambda} \mathcal{T}) = A,$$

i.e. the composite is the identity on Λ. Finally, the property (T3) shows that the image of $- \otimes^{\mathbf{L}}_{\Lambda} \mathcal{T}$, which is the triangulated subcategory of $\mathbf{D}^b(\mathsf{Coh}(\mathbb{P}^1))$ generated by $\mathcal{T}$. This is then where (T3) applies to close out the proof.

The tilting theory literature expands more upon the significance of this theorem. In particular, a classical theorem of Happel is that any hereditary abelian category with tilting object is, up to equivalence, the category of coherent sheaves on some *weighted projective line* or the category of left modules over some finite dimensional algebra. We defer to [HHK07] for more details.

Appendix A

Conventions

For the avoidance of doubt, the following is a reference for the conscious conventions and general notational choices we have made in this thesis:

- The set of non-negative integers is denoted $\mathbb{Z}_{\geq 0}$. Similarly, the set of non-positive integers is denoted $\mathbb{Z}_{\leq 0}$.

- Unless otherwise stated, all rings should be assumed to be commutative. Most algebras will not be, but will always be associative and unital.

- If A is a ring or an algebra, the category of left A-modules is denoted by $\mathsf{Mod}(A)$. The full subcategory of finitely generated left A-modules is denoted $\mathsf{mod}(A)$. We do not consider right A-modules at any point in this thesis.

- If R is a ring and $f \in R$, the ring R_f denotes the localisation of R at the multiplicatively closed subset $S = \{1, f, f^2, \dots\} \subseteq R$. More generally, if M is an R-module, the R_f-module M_f denotes the localisation of M at S. We will often abuse notation by conflating an element $m \in M$ with the fraction $m/1 \in M_f$. Via the canonical ring monomorphism $R \hookrightarrow R_f$, one may also think of M_f as an R-module, and it will be convenient to implicitly apply this principle at times.

- Given $f, g \in R$, we denote the *iterated* localisation of an R-module M at f and then at g by $M_{f,g}$, which is an $R[f^{-1}, g^{-1}]$-module. Manifestly, the order in which one does these localisations does not matter. (Note that this is *not* the same thing as the localisation of M at fg, as the elements of $M_{f,g}$ are fractions $m/(f^i g^j)$ for some $m \in M$ and usually different powers $i, j \in \mathbb{Z}_{\geq 0}$.)

- Given a category $\mathcal{C}$, we write $A \in \mathcal{C}$ and $f \in \mathcal{C}$ to denote that an object A and a morphism f belong to $\mathcal{C}$, rather than the common Ob and Hom notation.
- We denote both the object K and the morphism $k : K \hookrightarrow A$ constituting a kernel of a morphism $f : A \to B$ in some category by $\ker(f)$, with it being clear from context the way in which a particular usage should be read. We treat all cokernels, images and coimages similarly.
- Given functors F, $G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$ as well as natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \Rightarrow H$, we denote by $\varepsilon \circ \eta : F \Rightarrow H$ the *vertical composition* of ε with η having components $(\varepsilon \circ \eta)_X = \varepsilon_X \circ \eta_X$ for each $X \in \mathcal{C}$. This is the implied composition operation for all commutative diagrams of functors.
- There are two distinct ways of composing a natural transformation with a functor. Fix functors G, $H : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta : G \Rightarrow H$. Given functors $F : \mathcal{B} \to \mathcal{C}$ and $K : \mathcal{D} \to \mathcal{E}$, we define the natural transformations

  $$F \circ \eta : F \circ G \Rightarrow F \circ H \quad \text{and} \quad \eta \circ K : G \circ K \Rightarrow H \circ K$$

  to have components

  $$(F \circ \eta)_X = F \circ \eta_X \quad \text{and} \quad (\eta \circ K)_Y = \eta_{K(Y)}$$

  for each $X \in \mathcal{B}$ and $Y \in \mathcal{D}$.

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