PREDICATE (FIRST-ORDER) LOGIC

$\wedge$	$\vee$	$\neg$	$\Rightarrow$	$\bot, \top$	$P, Q, R, \ldots$

Propositions?

$\forall$	$\exists$	$x, y, z, \ldots$

Predicates?

Ex. 1: When is $\varphi = \neg(A \Rightarrow B) \vee \neg B$ true?

Sol: By truth table, whenever B is false (so $\varphi = \neg B$).

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$\neg B$	φ
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	T

Ex. 2: Simplify $(A \Rightarrow B) \vee (B \Rightarrow A)$.

Sol: Using the laws of boolean algebra,

$$
\begin{aligned}
(A \Rightarrow B) \vee (B \Rightarrow A) &= (\neg A \vee B) \vee (\neg B \vee A) && \text{material imp.} \\
&= \neg A \vee B \vee \neg B \vee A && \text{assoc.} \\
&= (A \vee \neg A) \vee (B \vee \neg B) && \text{comm. + assoc.} \\
&= T \vee T && \text{law of excl mid} \\
&= T && \hookrightarrow A \vee \neg A = T
\end{aligned}
$$

So this formula is a tautology.

Ex. 4: $F(x)$ = "person x is my friend", $P(x)$ = "person x is perfect".
How can we say "none of my friends are perfect" in predicate logic?

Sol: There are 3 equivalent ways:

1. $\forall x.\ F(x) \Rightarrow \neg P(x)$
2. $\forall x.\ P(x) \Rightarrow \neg F(x)$
3. $\neg(\exists x.\ F(x) \wedge P(x))$

MATHEMATICAL INDUCTION (over $\mathbb{N}$)

To prove some predicate $P : \mathbb{N} \to \{\top, \bot\}$ true for all $n \in \mathbb{N}$, we prove

1. Base case: $P(0)$ holds.
2. Inductive case: assume $P(k)$ holds for some k, then $P(k+1)$ holds.

[Next week: generalise to inductively-def'd sets other than $\mathbb{N}$.]

Ex. Use induction to prove that $\forall n \in \mathbb{N}.\ \overbrace{f(n) = n^2}^{:= P(n)}$, where

$$f : \mathbb{N} \to \mathbb{N}, \qquad f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n - 1 + f(n-1) & \text{if } n > 0 \end{cases}.$$

Pf. Base case is true since $f(0) = 0$. Next, assume that $P(n)$ holds for some $n \in \mathbb{N}$, i.e. $f(n) = n^2$. Then

$$\begin{aligned} f(n+1) &= 2(n+1) - 1 + f(n) \\ &= 2n + 1 + f(n) \\ &= 2n + 1 + n^2 \\ &= (n+1)^2. \quad ✓ \end{aligned}$$

$\square$