Tutorial 7: The E-machine

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The question

Exercise

Evaluate the following expression[a] using E-machine rules given in the lecture:

$$(\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ \texttt{If}\ (\texttt{Lt}\ x\ (\texttt{N}\ 2))\ (\texttt{N}\ 0)\ (\texttt{Ap}\ f\ (\texttt{Minus}\ x\ (\texttt{N}\ 1))))))\ (\texttt{N}\ 3)$$

[a] using some slightly more succinct notation as compared to the lecture slides

This will get incredibly long very quickly, so we're going to skip some busywork along the way. We start with

$$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ \texttt{If}\ (\texttt{Lt}\ x\ (\texttt{N}\ 2))\ (\texttt{N}\ 0)\ (\texttt{Ap}\ f\ (\texttt{Minus}\ x\ (\texttt{N}\ 1))))))\ (\texttt{N}\ 3).$$

The first function application

We use the application rule and work on the function being applied next:

$\longmapsto_E (\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright \circ \mid \bullet \succ \mathtt{Fun}\ (f.\, x.\ \mathtt{If}\ (\mathtt{Lt}\ x\ (\mathtt{N}\ 2))\ (\mathtt{N}\ 0)\ (\mathtt{Ap}\ f\ (\mathtt{Minus}\ x\ (\mathtt{N}\ 1)))).$

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Whenever we evaluate a function in the E-machine, we return back a closure which bundles the function value along with the current environment $\bullet$ in which it was originally used:

$$\longmapsto_E (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright \circ \mid \bullet \prec \texttt{Fun}\ \langle\!\langle \bullet; f\,.\,x.\ \cdots \rangle\!\rangle.$$

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$$\stackrel{*}{\longmapsto}_E (\mathtt{Ap}\ \langle\!\langle \bullet; f\,.\,x.\ \cdots \rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \prec 3.$$

We are returning a value with an Ap frame on the stack, so we pop off the frame, store the current environment $\bullet$ onto the stack, replace it with the new environment $\eta_1 = [x = 3, f = \langle\!\langle \bullet; f\,.\,x.\ \cdots \rangle\!\rangle]$ and evaluate the function body next. Symbolically, this transition looks something like:

$$\longmapsto_E \bullet \triangleright \circ \mid \eta_1 \succ \mathtt{If}\ (\mathtt{Lt}\ x\ (\mathtt{N}\ 2))\ (\mathtt{N}\ 0)\ (\mathtt{Ap}\ f\ (\mathtt{Minus}\ x\ (\mathtt{N}\ 1))).$$

Evaluating the `If`

The next thing to evaluate is the `If` expression, focusing on evaluating the conditional first:

$$\longmapsto_E (\mathtt{If}\ \Box\ (\mathtt{N}\,0)\,(\mathtt{Ap}\,f\,(\mathtt{Minus}\,x\,(\mathtt{N}\,1)))) \triangleright \bullet \triangleright \circ \mid \eta_1 \succ (\mathtt{Lt}\,x\,(\mathtt{N}\,2)).$$

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It should be reasonably clear that this condition evaluates to `False` since the environment asserts that $x = 3$ and $3 \nless 2$, so we'll fast forward a few steps until we reach that conclusion:

$$\overset{*}{\longmapsto}_E (\texttt{If}\ \square\ (\texttt{N}\,0)\ (\texttt{Ap}\ f\ (\texttt{Minus}\ x\ (\texttt{N}\,1)))) \rhd \bullet \rhd \circ \mid \eta_1 \prec \texttt{False}.$$

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Conditional evaluation rules tell us we should take the "else" branch of the `If` expression now, so we need to evaluate that expression:

$$\longmapsto_E \bullet \rhd \circ \mid \eta_1 \succ (\texttt{Ap}\ f\ (\texttt{Minus}\ x\ (\texttt{N}\,1))).$$

Some recursive function calls

Just like before, we're evaluating a function application, which usually means we have to evaluate the function expression itself though. Here, however, the function expression is a variable, and its value is that of the function closure put into η_1 earlier. Skipping these few steps, we fast-forward to the point where we're evaluating the argument in the application expression:

$$\overset{*}{\longmapsto}_E (\texttt{Ap}\ \langle\!\langle \bullet; f.\, x.\ \cdots \rangle\!\rangle\ \Box) \triangleright \bullet \triangleright \circ \mid \eta_1 \succ (\texttt{Minus}\ x\ (\texttt{N}\ 1)).$$

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We know that $x = 3$ in η_1, so it should be reasonably clear this argument will evaluate to 2:

$$\stackrel{*}{\longmapsto}_E (\texttt{Ap}\ \langle\!\langle \bullet; f\,.\,x.\ \cdots \rangle\!\rangle\ \square) \triangleright \bullet \triangleright \circ \mid \eta_1 \prec 2.$$

Once again, we're now in a situation where we're returning a value with an `Ap` frame on the stack, so we pop off the frame, add the current environment η_1 to the stack, replace it with the new environment $\eta_2 = [x = 2, f = \langle\!\langle \bullet; f\,.\,x.\,\cdots \rangle\!\rangle]$ and evaluate the function body next. Symbolically,

$$\longmapsto_E \eta_1 \triangleright \bullet \triangleright \circ \mid \eta_2 \succ \mathtt{If}\ (\mathtt{Lt}\ x\ (\mathtt{N}\ 2))\ (\mathtt{N}\ 0)\ (\mathtt{Ap}\ f\ (\mathtt{Minus}\ x\ (\mathtt{N}\ 1))).$$

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With the exception of a slightly different stack, this looks like something we have done previously! Because we've already been through that once, we can tell what's going to happen: x is currently 2 according to η_2, so the If conditional will be False once again since $2 \nless 2$. Thus, we will be evaluating the "else" branch again, which turns out to applying f once again with the argument $x - 1$, and it should be reasonably clear that $x - 1 = 1$. So, skipping quite a number of steps until the evaluation of $x - 1$ has finished, we have:

$$\overset{*}{\longmapsto}_E (\texttt{Ap}\ \langle\!\langle \bullet; f\,.\,x.\,\cdots \rangle\!\rangle\ \square) \rhd \eta_1 \rhd \bullet \rhd \circ \mid \eta_2 \prec 1.$$

This is another situation with an `Ap` frame on the stack and a value being returned, so we do as we've done twice before now: set $\eta_3 = \left[x = 1, f = \langle\!\langle \bullet; f\,.\,x.\ \cdots \rangle\!\rangle\right]$, and we obtain

$$\longmapsto_E \eta_2 \rhd \eta_1 \rhd \bullet \rhd \circ \mid \eta_3 \succ \texttt{If}\ (\texttt{Lt}\ x\ (\texttt{N}\ 2))\ (\texttt{N}\ 0)\ (\texttt{Ap}\ f\ (\texttt{Minus}\ x\ (\texttt{N}\ 1))).$$

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We can play the same game as before and predict what this expression we're now on the hook to evaluate will become. According to η_3, we have $x = 1$, and $1 < 2$, so the conditional actually evaluates to True in this case. Taking the "then" branch of the If expression and evaluating it, we get a final expression value of 0, so no more recursive function applications to do. Skipping all of those explicit steps, we now have:

$$\overset{*}{\longmapsto}_E \eta_2 \rhd \eta_1 \rhd \bullet \rhd \circ \mid \eta_3 \prec 0.$$

Unwinding

We're returning a value and an environment is atop the stack. In such a situation, we take that environment on the stack, replace the current environment with it, and continue propagating the returned value. That is, we'll get something looking like this:

$$\longmapsto_E \eta_1 \triangleright \bullet \triangleright \circ \mid \eta_2 \prec 0.$$

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We'll repeat this twice (for the other environments η_1 and $\bullet$ on the stack). At long last, then, we end up with a final state: empty stack, empty environment, and returning a value!

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$$\overset{*}{\longmapsto}_E \circ \mid \bullet \prec 0.$$

Observe that we had a bunch of environments sitting on the stack adjacent to each other, which was induced by tail-recursive calls to f. We'll see how to optimise that situation soon.