Stochastic Processes and Applications

Assignment - 1

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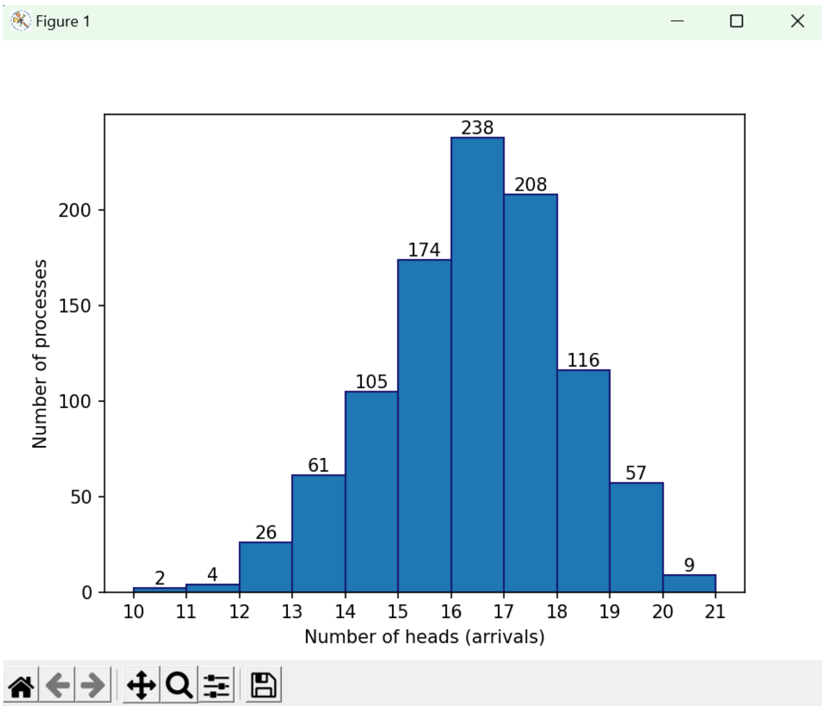
Question 1

We toss a coin 20 times and model it as a Bernoulli process. It follows the assumptions of a Bernoulli process – there are 2 possible outcomes (head, tails), outcome of each toss (trial) is independent of the other, and p = probability of arrival (say, getting a head) is constant across trials.

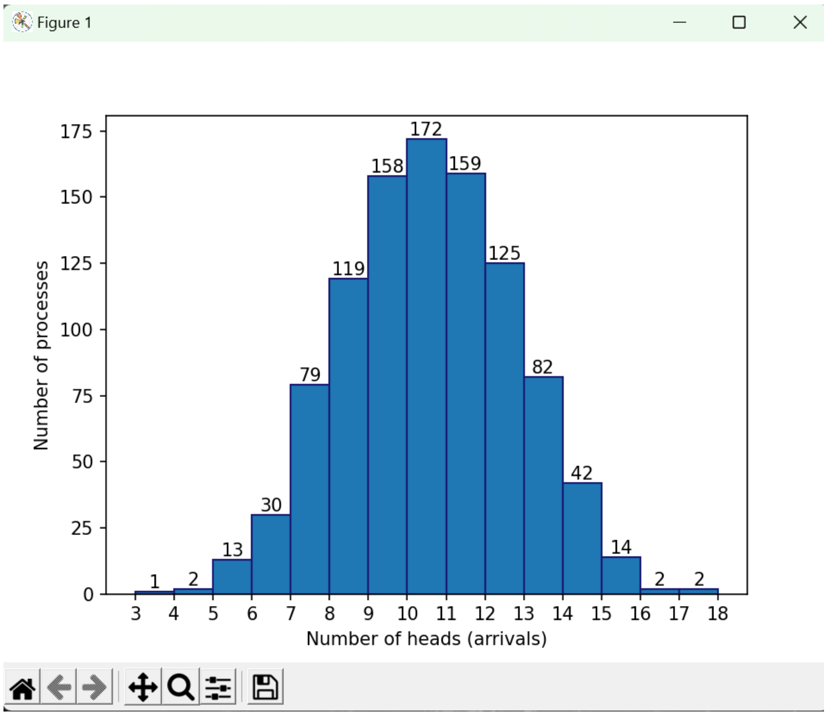
We simulate the process 1000 times taking different values of p. In each process, number of heads (arrivals) can take a value in {0, 1, …, 20}. Hence, we can plot a histogram where x-axis represents number of arrivals and y-axis represents number of processes with that many arrivals.

Note – In histogram, the bar between x = x1 and x = x2 represents the number of processes with number of arrivals ∈ [x1, x2). Hence, it correctly represents frequency corresponding to number of arrivals = x1 (since x1, x2 are discrete).

(i) p = 0.8



(ii) p = 0.5



For a Bernoulli process with n trials and probability of arrivals = p, expected value (mean) of number of arrivals = n*p.

Proof –

Let Z_i = number of arrivals in i^{th} trial and N = total number of arrivals in n trials. Since it is a Bernoulli process, each Z_i ~ iid Bernoulli(p). Hence E[Z_i] = p ∀ i.

E[N] = E[$Z_1 + Z_2 + \ldots + Z_n$] = E[Z_1] + E[Z_2] + … + E[Z_n] = n*p

This is confirmed by the provided graphs. For n = 20 and p = 0.8, n*p = 16 and it was the mode value. Similarly when n = 20 and p = 0.5, n*p = 10 and it was the mode value.

Another observation is that compared to second simulation, processes in first simulation recorded more arrivals (in general). This is explained by the fact that p = probability of arrival is set at a higher value in first simulation compared to second one.

Question 2

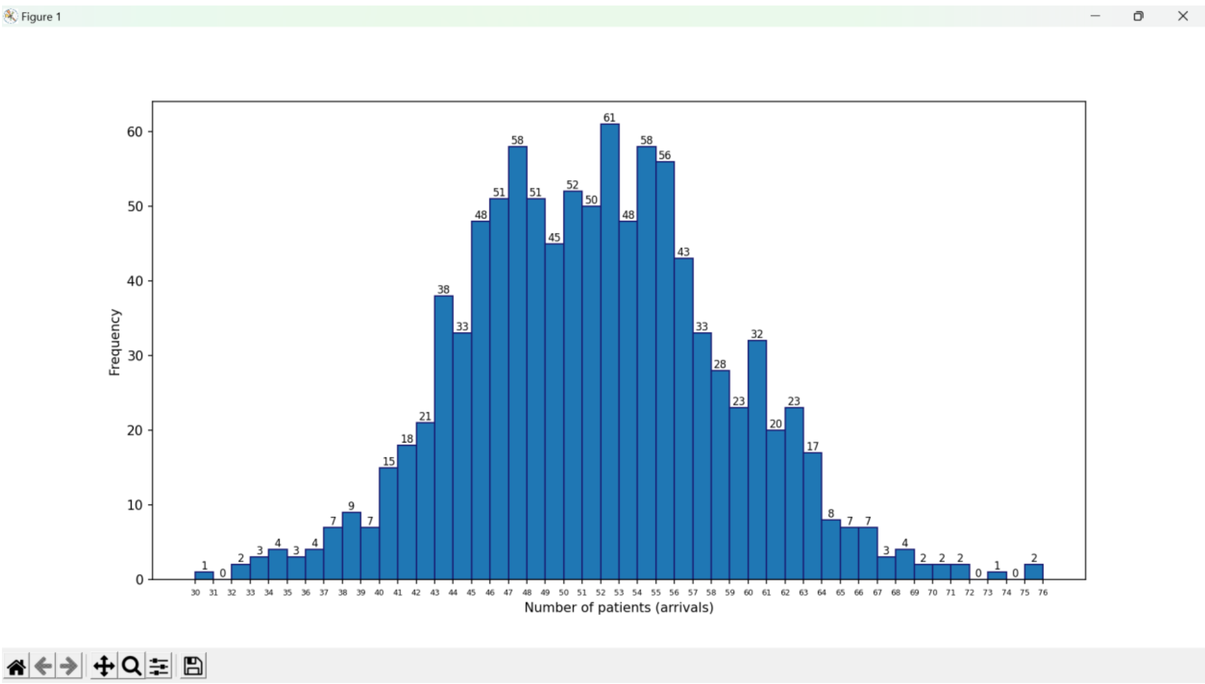
We study the number of patients arriving at a hospital's emergency room and model it as a Poisson process. It follows the assumptions of a Poisson process – interarrival times are iid (reneval process) and have an exponential distribution. We want to capture the number of arrivals in time (0, t]. For this, we start at time = 0, keep adding next inter-arrival time (as per exponential distribution) and incrementing the number of arrivals until time < t.

Estimating the next inter-arrival time –

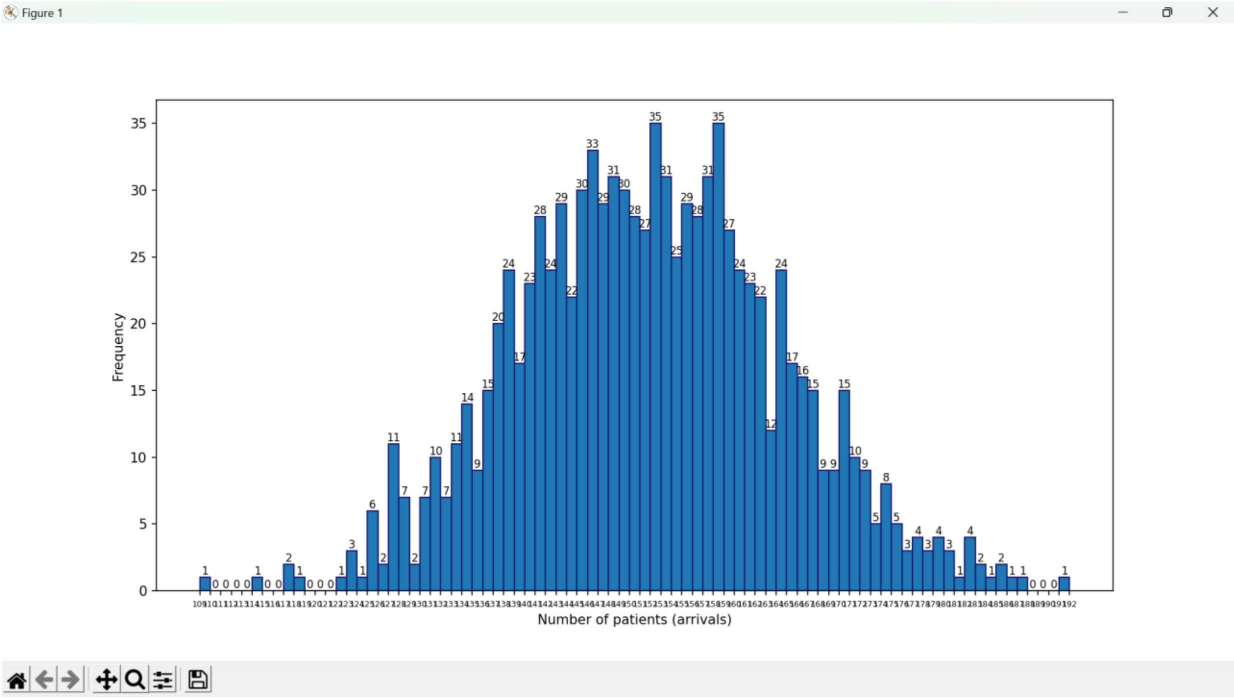
Generate a random number r ∈ [0, 1]. For exponential distribution, PDF is $f_{X_i}(x_i) = \lambda e^{-\lambda x_i}$ and CDF is $F_{X_i}(x_i) = 1 - e^{-\lambda x_i}$. We will map r to next inter-arrival time x_i (as per exponential distribution) using r = $1 - e^{-\lambda x_i}$. For example, r = 0 maps to $x_i = 0$ and r = 1 maps to $x_i \to \infty$. Solving the equation, we get $x_i = \frac{1}{\lambda} \ln \frac{1}{1-r}$.

We simulate the process 1000 times taking different values of λ (rate of arrival). In each process, number of arrivals can take a value in {0, 1, …}. Hence, we can plot a histogram where x-axis represents number of arrivals and y-axis represents number of processes with that many arrivals. We can use value of t = 10.

(i) $\lambda = 5$



(ii) $\lambda = 15$



For a Poisson(λ, t) process, mean value of number of arrivals is $\lambda * t$. This was confirmed by the provided graphs. For $\lambda = 5$ and t = 10, λ * t = 50 and the mean value of number of arrivals was computed to be 51.173. Similarly, for $\lambda = 15$ and t = 10, λ * t = 150 and the mean value of number of arrivals was computed to be 151.376. Clearly, compared to first simulation, processes in second simulation had more arrivals (in general). This is explained by the fact that λ = rate of arrival is set at a higher value in the second simulation while keeping time t constant.

Question 2 (c)

Simulating first-inter arrival time of process – It can be done using the process of estimating inter-arrival time described earlier. We simulate the process 1000 times and record the value of first inter-arrival time in each process. Since the values of inter-arrival time are continuous, we can use a scatter plot to represent the data. x-axis represents the process number and y-axis represents the value of first inter-arrival time.

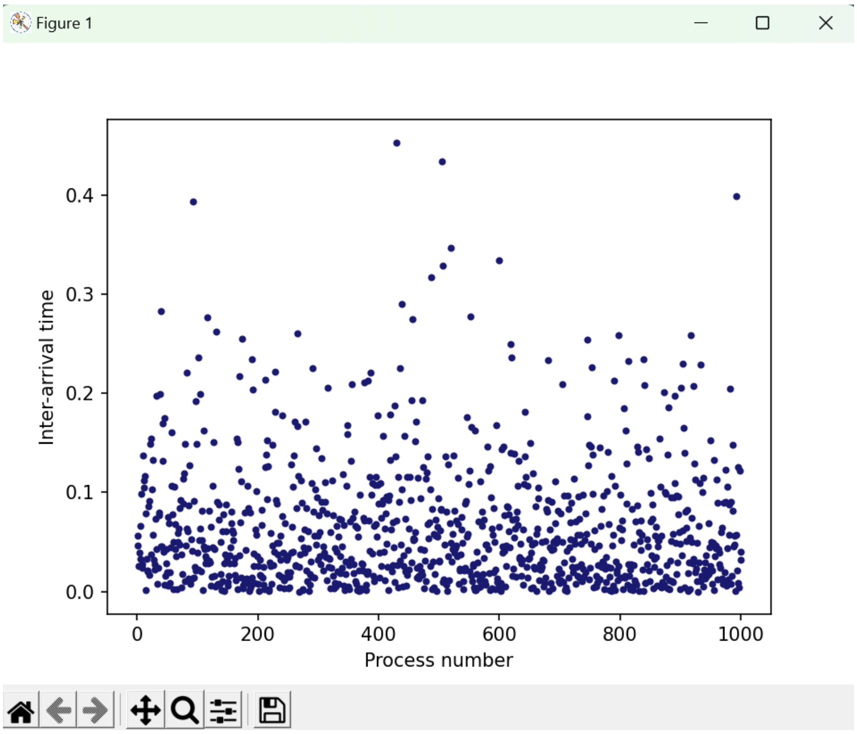


Figure 1
Inter-arrival time
Process number
0.0
0.1
0.2
0.3
0.4
0
200
400
600
800
1000