Stochastic Processes and Applications

Assignment - 2

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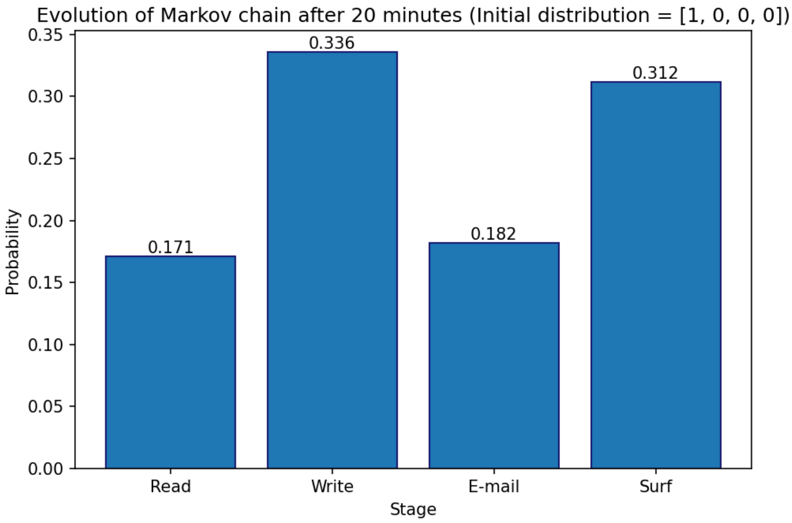
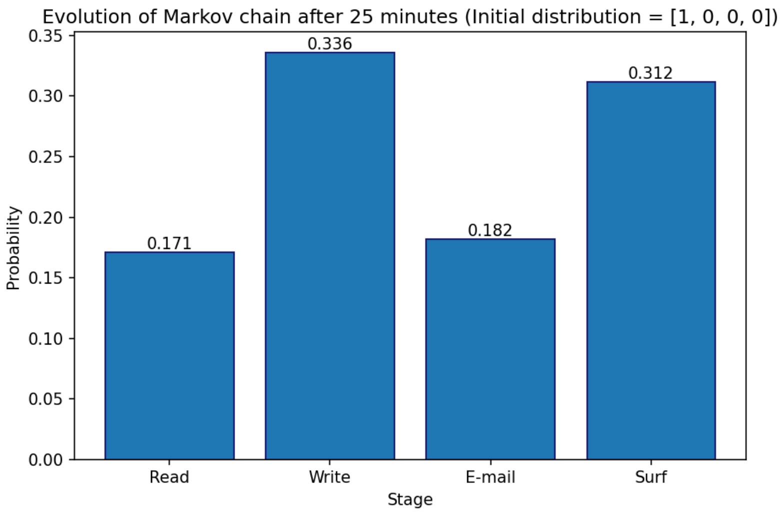
Question 1

We are given the 4 stages of writing a paper – read (r), write (w), e-mail (e) and surf (s), which act as the states of the Markov chain. We are also given a 1-step transition probability matrix (tpm) p.

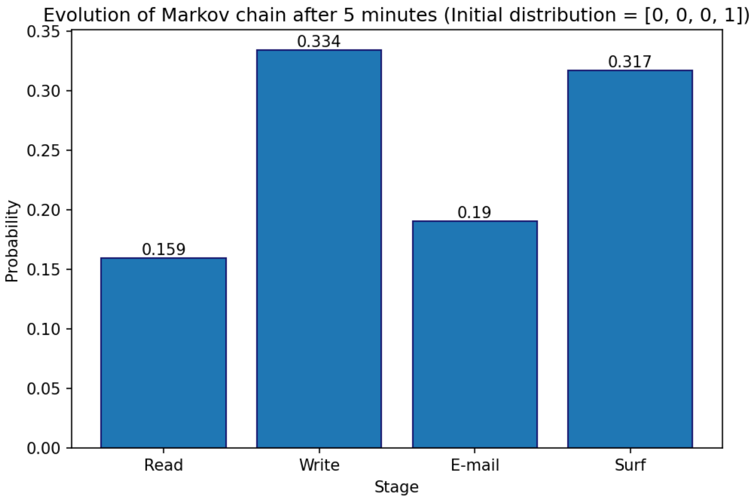
(a) To study how the Markov chain evolves after 20 minutes ($\pi^{(20)}$), we need information about the initial probability distribution ($\pi^{(0)}$). Assuming that reading is the initial stage while writing a paper ($X_0 = r$), $\pi^{(0)} = [1,\ 0,\ 0,\ 0]$.
We know that $\pi^{(i+1)} = \pi^{(i)}p\ \ \forall\, i \geq 0$. Hence, $\pi^{(20)} = \pi^{(0)}p^{20}$. From running the code, we get $\pi^{(20)} = [0.171,\ 0.336,\ 0.182,\ 0.312]$. Hence, we find that $P(X_{20} = s \mid X_0 = r) = 0.312$.



(b) Similarly, to study how the Markov chain evolves after 25 minutes ($\pi^{(25)}$), we perform the calculation $\pi^{(25)} = \pi^{(0)}p^{25}$. From running the code, we get $\pi^{(25)} = [0.171,\ 0.336,\ 0.182,\ 0.312]$. An interesting observation is that $\pi^{(25)} = \pi^{(20)}$ which means that we reach a limiting distribution.



We need to find the value of $P(X_{25} = s \mid X_{20} = s)$. Given $\pi^{(20)} = [0,\ 0,\ 0,\ 1]$, we need to find $\mathrm{P}(X_{25} = s)$. Using $\pi^{(25)} = \pi^{(20)}p^5$, we get $\pi^{(25)} = [0.159,\ 0.334,\ 0.19,\ 0.317]$. Hence, $P(X_{25} = s \mid X_{20} = s) = 0.317$.



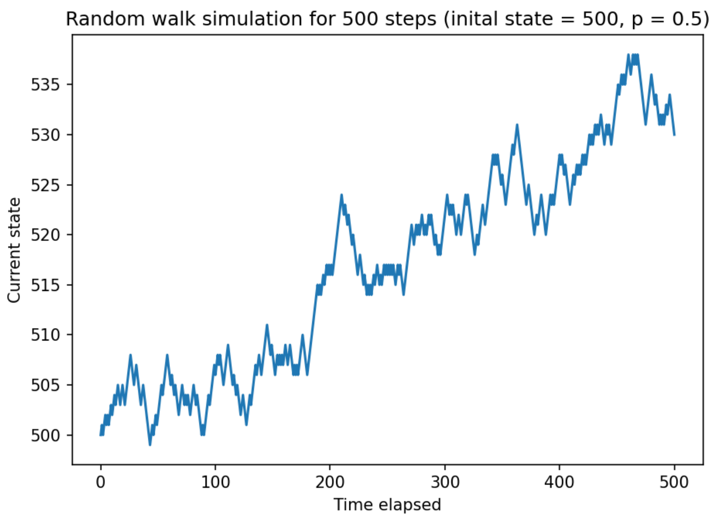
(c) In order to find stationary distribution π^*, we need to solve the system of linear equations represented by matrix equation $\pi^* p = \pi^*$. An additional condition is that sum of values in π^* is equal to 1 (total probability = 1). In code, we compute this by finding eigenvalues and eigenvectors for matrix p^T. Then, we select eigenvector corresponding to eigenvalue 1 and normalize it to make its sum = 1. We get the stationary distribution as $\pi^* = [0.171,\ 0.336,\ 0.182,\ 0.312]$.

(d) Based on our observation in part (b), we know that a limiting distribution exists. Since the limiting distribution does not depend on the initial distribution, we can take a different $\pi^{(0)} = [0.25,\ 0.25,\ 0.25,\ 0.25]$. In order to verify our observation and find the limiting distribution using code, we follow an iterative procedure. In each iteration, we get the next distribution by multiplying the current distribution with tpm. Then, we calculate the 2-norm of the difference of these distributions. To account for floating point inaccuracies, we check whether it is < 10^{-5}. The condition becomes true when we obtain our limiting distribution. By running the code, we find that the limiting distribution is found after 11 iterations. Limiting distribution = $[0.171,\ 0.336,\ 0.182,\ 0.312]$, which verifies our observation. Furthermore, it is same as stationary distribution computed in part (c).

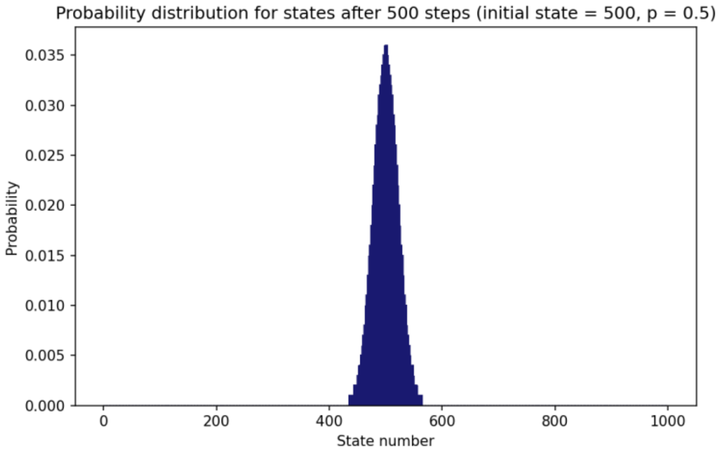
Question 2

We are given a 1-D random walk with initial state i. From each state, probability of moving to state on the right is p and state on the left is 1-p. In each part, we simulate a random walk for 500 steps. Hence, the set of possible states is {i-500, .. , i-1, i, i+1, .. , i+500} and initial probability distribution (for X_0) is [0, .. , 0, 1, 0, .. , 0]. Finally, proability of being in state i on step n is calculated as $P(X_n = i) = \ p * \ P(X_{n-1} = i - 1) + (1 - p) * P(X_{n-1} = i + 1)\ \forall\ n \geq 1.$

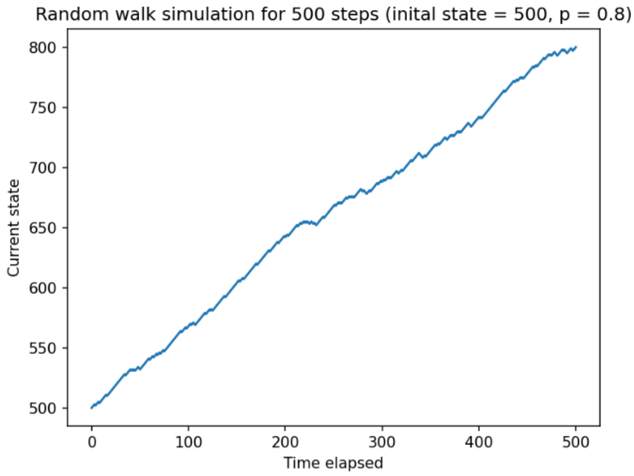
(a) Given p = 0.5, following is a simulation of random walk.



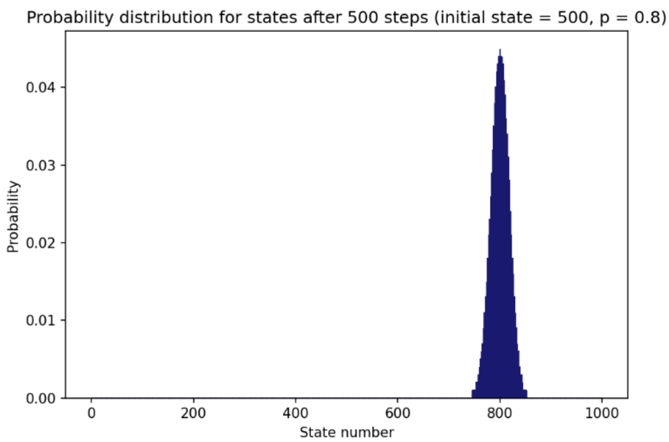
We can also calculate the probability distribution for states after 500 steps, using the formula mentioned earlier.



(b) Given p = 0.8, following is a simulation of random walk.

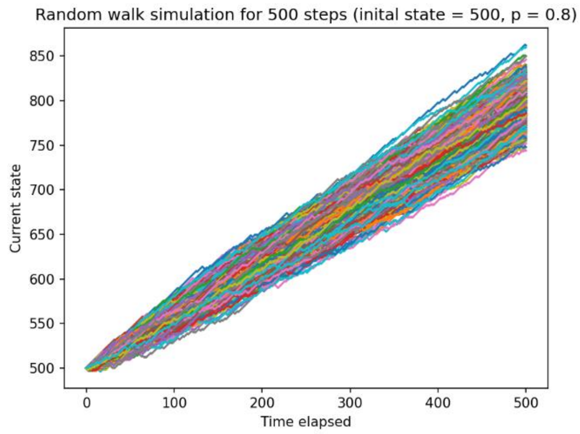


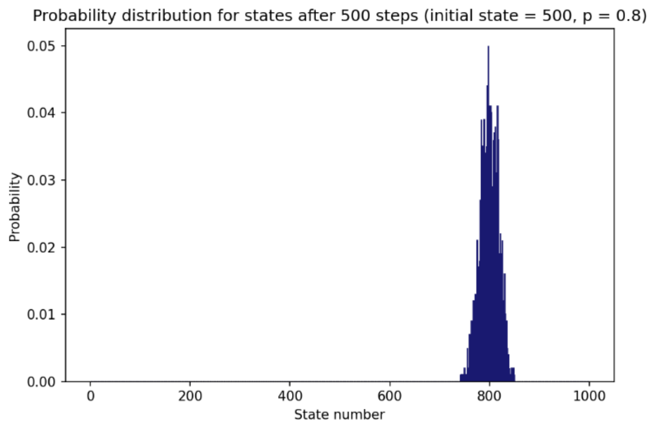
Similarly, we can calculate the probability distribution for states after 500 steps.



In part (b), we took a larger value of p compared to part (a). A larger value of p increases the probability of moving to state on the right on each step. The graphs confirm this phenomenon. Comparing the first graphs, we observe that a random walk with p = 0.8 (on average) ends up at a state much to the right of the final state when p = 0.5. The same is depicted formally using the probability distribution in second graphs.

(c) Given p = 0.8, we simulate the random walk similar to previous parts. But this time, we perform 1000 simulations and obtain the frequency for each state becoming the final state. Then we divide these values by number of simulations (= 1000) to get a probability estimate as follows.





We observe that the results in part (c) look similar to part (b). This is because we take the same setup of 1-D random walk in both cases – including the initial state, number of steps and value of p. The only difference is that values in (b) are computed using mathematical relation and values in (c) are computed by sampling data of 1000 random walk simulations.