ORF435 / ORF535 / FIN535

Homework 4

Instructor: Professor John M. Mulvey

Due date: Thursday, 10/19/2017

Readings and notations

Read chapter 6.1-6.6 of "Investment Science", 2nd edition (the first edition is pretty much the same). We will use the notation for asset returns, expected returns, and portfolio weights in chapter 6.4, with the definition of covariance matrix $\mathbf{Q}$ here:

$$\mathbf{Q} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{bmatrix}.$$

Question 1: Matrix Form of Portfolio Optimization

1. Write $\mathbf{w} = (w_1, \dots, w_n)^T$. Show that the variance of the portfolio can be written as

$$\sigma^2_{\text{portfolio}} = \mathbf{w}^T \mathbf{Q} \mathbf{w}.$$

2. Write $\mathbf{1} = (1, 1, \dots, 1)^T$ (column vector with all 1's). Solve the (unconditional) minimum variance portfolio analytically by Lagrangian:

$$\begin{aligned} \underset{\mathbf{w}}{\text{minimize}} \quad & \mathbf{w}^T \mathbf{Q} \mathbf{w} \\ \text{subject to} \quad & \mathbf{1}^T \mathbf{w} = 1 \end{aligned}$$

Question 2: Quadratic Utility Function and CAPM Model
Solve questions 11 and 15 in chapter 11.

Question 3: Scenario Equivalent Version of Markowitz Model and More
1. A few clarifications on the previous homework:

(i) For the Sharpe ratio, you can use the historical average T-bill rate as risk-free. This shows a drawback of estimation with historical values: the period with 5-7% risk-free return may never happen again.

(ii) It seems that we cannot achieve 11% or 12% return. Then take 10.9% as the last point.

(iii) It is fine to work in groups, but please write your own codes (printed at the end of submission) for the problems. We will punish completely identical codes.

2. Compute the (monthly) covariance matrix $\mathbf{Q}$ and take the geometric average return $\hat{r}$ of the three assets. Generate 10000 one-period scenarios based on normal distribution $N(\hat{r}, \mathbf{Q})$.

3. Run the scenario equivalent version of Markowitz model with the same $\bar{r}$ as in homework 3 (take $\bar{r} = 10.5\%$ as the last point), and compare the two efficient frontiers.

4. Using the scenarios generated, find and report the portfolio that maximizes the expected log-utility of the investor. Mark this portfolio on your plot of (either) efficient frontier.

5. (**For graduate students only**) Find the portfolio that maximizes the Sortino ratio, which is given by

$$S = \frac{E[R] - R^T}{\sqrt{\mathrm{Var}^-(R - R^T)}}$$

Where $R^T = 0.4\%$ is the target rate of monthly return, and Var^- is the downside variance:

$$\mathrm{Var}^-(R - R^T) = \sum_{i=1}^{N} \frac{1}{N} (\max(0, R^T - R_i))^2.$$

Here each R_i represents the portfolio return in the ith scenario for $i = 1, 2, \ldots, N$.
Hint: Grid search may be the best way to solve this optimization problem since our non-convex solvers can stuck at some point. For a grid search, list all triplets that sum to 1 discretized at 0.01 or 0.001 level, and compare the Sortino ratio of every possible triplet.