ORF435 / ORF535 / FIN535

Homework 6

Instructor: Professor John M. Mulvey

Due date: Thursday, 11/9/2017

Question 1: Linear Regression Models with LASSO

We will use the following notation for linear regression model:

$$y_t = \boldsymbol{X_t}\boldsymbol{\beta} + \epsilon_t, \tag{1}$$

where t is used to index each observation, y_t is called the **dependent variable** and the vector $\boldsymbol{X_t} = (X_t^1, X_t^2, \ldots, X_t^m)$ is called the **explanatory variable** (or **independent variable**) for observation t. $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)^T$ are the unknown parameters.

We know that the Ordinary Least Squares (OLS) estimator can be expressed as

$$\hat{\boldsymbol{\beta}}^{\text{OLS}} = \underset{\beta}{\text{argmin}} \left\{ \sum_{t=1}^{n} (y_t - \boldsymbol{X_t}\boldsymbol{\beta})^2 \right\} = \underset{\beta}{\text{argmin}} \left\{ (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) \right\}. \tag{2}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$ is a column vector of all observations and $\boldsymbol{X}$ is an n-by-m matrix.

1. Solve this problem analytically.

Now we introduce the LASSO regression:

$$\hat{\boldsymbol{\beta}}^{\text{LASSO}} = \underset{\beta}{\text{argmin}} \left\{ \sum_{t=1}^{n} (y_t - \boldsymbol{X_t}\boldsymbol{\beta})^2 + \lambda \sum_{j=1}^{m} |\beta_j| \right\}. \tag{3}$$

Here $\lambda > 0$ is called a **tuning parameter** or **hyper-parameter**.

2. Show that this is a convex problem. (Hint: Employ the $[\text{dev}^+, \text{dev}^-]$ conversion)

3. Show that $\sum_{j=1}^{m} |\hat{\boldsymbol{\beta}}_j^{\text{LASSO}}| \leq \sum_{j=1}^{m} |\hat{\boldsymbol{\beta}}_j^{\text{OLS}}|$. (Hint: prove by contradiction.)

Question 2: Our First Investment Planning Problem

This part involves constructing a forward-looking planning model for your client that inherits $100,000 and is planning to save this money for a $120,000 down payment on a condo. We further assume that her salary will not be used on this goal. For this effort, we focus on a 10-year model and set T = 120 (with monthly simulation). As discussed in class, we will evaluate a policy rule simulation fit for this model. The objective is to purchase the condo in **five** years, if possible.

1. Load the historical data of four assets (US equity, treasury bond, corporate bond, T-bill) into your computing software. Report the usual return and risk profile of the assets. (Note: From now on, this report set will include geometric mean, annualized mean, volatility, annualized volatility and correlation matrix.)

2. We have learned that the estimate of the variance/covariance matrix can be fairly accurate based on historical data, but the expected return may not. Briefly explain why this happens, using our historical data on US equity. (Read chapter 9 of the IS book)

3. Now simulate 10000 scenarios such that each scenario depicts the returns for the four assets over a 10-year time horizon (i.e. 120 periods). Each period (monthly return) follows the $N(r, \mathbf{Q})$ distribution where r is given below and $\mathbf{Q}$ is your historical covariance matrix.

Asset	Projected Annual Return	Equivalent Monthly Return
US Equity	7%	0.565%
Government Bonds	2.50%	0.206%
Corporate Bonds	4%	0.368%
Cash	0.25%	0.021%

4. (For graduate students only) Take the first period of each scenario (10000 samples). Compute the simulated VaR and CVaR and compare with historical VaR and CVaR, for each asset. Write a short paragraph about your findings. Keep in mind that simulated data have different average returns with historical data. (Hint: the sort function will do the trick.)

Part 5 (on the next page) is for all students.

5. Simulate the wealth path for the scenarios, and report the mean value and standard deviation of the final wealth at the end of 5th year for the policy rules below. What is the probability that your client reaches her goal?

A) If she plays "safe" and invests 90% in government bond and the rest in cash?
B) If she chooses a 40% stock, 30% government bond, 20% corporate bond, and 10% cash investment?
C) If she follows your recommendation? (Suggest a portfolio for her, no need for perfection at this point).

For each of these mixes, what will the answer be for the end of 10th year? What is the VaR and CVaR of the portfolio at the end of 10th year, assuming that she waits for 10 years to purchase the condo? Here the VaR and CVaR should be evaluated with respect to the goal.

Notes: rebalance your portfolio each month (such that the return of your portfolio each month will just be $\sum_{i=1}^{4} w_i * r_{t,i,k}$ where t is time, i is asset number and k is scenario number) to the target mixes: ignore taxes and other transaction costs, no intermediate cash flows.