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ORF 435/535 Take Home Midterm Exam
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November 11, 2017

This is the take home part of the midterm exam, which also serves as review questions for the in-class portion of the exam. The exam is due exactly on 12 pm, November 16th. Collaboration is not allowed on the exam. You are allowed to access the IS book and any or your notes from the course (but not notes from others).

1. Utility Functions (25%)

A. Decide if the functions below can serve as a risk-averse utility function for wealth $x > 0$:

1) $U(x) = 3\sqrt{x}$
2) $U(x) = \ln(5x + 10)$
3) $U(x) = \frac{1}{10000}x^2$
4) $U(x) = -10e^{\frac{1}{x}}$

Show some work to support your argument.

B. Suppose a gambler enters a casino with \$1000. The dice-rolling game costs \$100 and rolls three dices with equal probability showing numbers 1-6. If there is just one "6", the game pays back the bet -- \$100; if there are exactly two "6"s, the game pays \$500 (which means a \$400 gain); if there are three "6"s, the jackpot will pay \$5000. For the utility functions (1) - (4) above, will the gambler take the bet? (Hint: first consider-is this a fair bet?)

C. Now we have selected a risk-averse utility function U for an investor. The constraint set is:

$$\boldsymbol{\Omega} = \{w \mid g_1(w) \leq 0,\ g_2(w) \leq 0, \dots,\ g_n(w) \leq 0\}.$$

- Show that $\boldsymbol{\Omega}$ is a convex set if $g_1, g_2, \dots, g_n$ are all convex functions.
- Use an example (drawing is fine) to illustrate that the reverse is not true.

2. More on Convex Programming (10%)

Suppose that you are attempting to find the shape of a rectangle that maximizes the area. The sole constraint is that the total length of the four sides is less than or equal to 40 meters.

A. Set up this optimization model.

B. Do you see any issues with this small optimization model?

3. Free Cash Flow Analysis (15%)

Reconsider example 5.8 (XX corporation). Assume the original settings: current earnings, initial capital, and depreciation factor. Now what will the maximum present value of the firm be if:

A. Current tax rate = 34%, interest rate r = 9% and $g(u) = 0.05\left[1 - e^{5(\alpha-u)}\right]$?

B. Tax rate goes down to 20% as proposed by the current Trump tax plan and the other parameters in A are assumed (r and g(u))?

C. A number of the Republicans in the U.S. Congress have suggested that lower taxes will increase innovation and the growth of earnings. Assume that growth is increased by 20% and $g(u) = 0.06\left[1 - e^{5(\alpha-u)}\right]$ and assume that the tax rate is 20%, r = 9%?

Compare cases (A), (B), and (C).

4. Growth Rate of Portfolios (20%)

We have shown that if the stock price follows geometric Brownian motion with mean μ and volatility σ, the growth rate of the stock is

$$\nu_{\text{stock}} = \mu - \frac{1}{2}\sigma^2.$$

We further assume that if all stock prices follow geometric Brownian motion, then any portfolio of these stocks will also follow geometric Brownian motion. Therefore, the above equation will also hold with the growth rate, mean and standard deviation of your portfolio.

- **A.** Show that if we equally invest in N uncorrelated stocks with the same mean μ and volatility σ, the growth rate of our portfolio will be $\nu_{\text{port}} = \mu - \frac{1}{2N}\sigma^2$.
- **B.** Now we take N=10. Empirical studies show that the stocks are correlated, and for simplicity we assume that the pairwise correlation $\rho_{ij} = 0.5$ for any two stocks $i \neq j$. We have also estimated that the growth rate of an individual stock is ν=10% and the volatility of each stock is σ=30%. What is the growth rate of the equally weighted portfolio in this case?
- **C.** Why do we care growth rate of the portfolio, not only the expected return? Be brief.

5. Financial Planning and Risk Analysis (30%)

Several of the most important decisions for an individual or family involve savings and investment policy for achieving a target wealth goal in the future. Consider a financial planning problem with periods $t = \{1, 2, \dots, T\}$, where the horizon T indicates the desired date for the future purchase. Rather than discounting cash flows, we advocate that the investor focus on the future, and the probability of achieving the target wealth at time T. We model uncertainty by means of a finite set of scenarios s $\in \{S\}$, with probability p_s for each scenario.

Suppose that the investor begins with capital/wealth = W_0. For periods t = {1, ..., T-1} we also have a (scenario-dependent) savings inflow denoted by $y_{t,s}$ to our investment portfolio. Assume that the savings are fixed for each scenario, occur at the end of each period, and are endogenous (given). Assets are defined by the set, $j \in \{1, 2, \ldots, m\}$ with returns $r_{j,t,s}$ (return from period t to period t+1) for each period and scenario. Assume no transaction costs, no taxes, no shorting, no borrowing and no intermediate outflow cash flows. You should consider cash to be an asset. Assume that the amount of funds (dollar value) invested in asset j, at time t, under scenario s = $x_{j,t,s}$. Be sure to include the non-anticipating constraints in questions below. We do not assume any particular policy rule in your stochastic programming model.

Note: No solving is needed for the questions; we will program the model later in the course.

A. Carefully set up a multi-stage stochastic program for this problem. In scenario s, denote the wealth at time t = $W_{s,t}$ (you will need to derive for this in your program), and asset allocation variables $x_{j,t,s}$ (dollar amounts, not weights), and the target wealth =W^{target}. First, we assume that the investor is interested in minimizing variance while aiming at the target wealth.

B. Revise the risk part of the objective function in part A to minimize downside risk relative to the target wealth W^{target}. Consider both the quadratic and the absolute version of downside risk.

C. Revise the model to minimize the two tail risk measures VaR_h and $CVaR_h$ (h=5%) for wealth at time T. Assume that the scenarios are granular enough such that the probability of having a loss exceeding the VaR is always exactly equal to 5% (rather than at most 5%). Hint: the VaR model can be restated as a non-convex model with binary variables.

The CVaR model (extra credit for undergraduate students, required for graduate students) can be restated as a convex program. Think about breaking up the losses in the CVaR expectation as $L_s = VaR_{95\%} + d_s$ where d_s is some sort of deviation variable. Now write the conditional expectation (conditional expectations are linear) as two parts. Your $VaR_{95\%}$ should be a decision variable in this model.

D. Revise the model to maximize the probability of achieving the target wealth at the horizon T.