# Rotations of Quaternions using Complex Analysis

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## 1 Introduction

The concept of quaternions are going to be explained in this article and also the different operations that can be applied on the same.
For fully understanding the beauty of quaternions, it is necessary to know it's roots and where they come from. The complex number system is the basis for which quaternions have been researched and developed.
This complex number system teaches of a whole new of numbers known as Imaginary numbers. These numbers are used in equations which have no solution. For example -

$$x^2 + 1 = 0$$

To fully solve the above equation, we have to give

$$x^2 = -1$$

which is not possible since the square of any number cannot be negative. All mathematicians cannot settle with the fact that equations do not have solutions and hence they came up with the concept of imaginary numbers.
The imaginary number is represented in the form

$$i^2 = -1$$

The set of complex numbers which contains the sum of the real and imaginary number is of the form -

$$z = a + bi \quad a, b \in \mathbb{R}, \quad i^2 = -1$$

Complex numbers are said to be purely real when

$$b = 0$$

and purely imaginary when

$$a = 0$$

Complex numbers can be added and subtracted as well by adding and subtracting the real and imaginary parts respectively.

Addition:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

Subtraction:

$$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

Multiplication of complex numbers is also possible by multiplying each term of the complex number by the scalar.

$$\lambda(a + bi) = \lambda a + \lambda bi$$

We can find the product of 2 complex numbers by applying the normal standard algebraic equations.

$$\begin{aligned} z_1 &= (a_1 + b_1 i) \\ z_2 &= (a_2 + b_2 i) \\ z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i \end{aligned}$$

The square of a complex number can be found out by multiplying the complex number by itself.

$$\begin{aligned} z &= (a + bi) \\ z^2 &= (a + bi)(a + bi) \\ &= (a^2 - b^2) + 2abi \end{aligned}$$

The conjugate of a complex number is basically the imaginary part negated.

$$\begin{aligned} z &= (a + bi) \\ z^* &= (a - bi) \end{aligned}$$

We get a special result when we multiply the complex number with it's conjugate -

$$\begin{aligned} z &= (a + bi) \\ z^* &= (a - bi) \\ zz^* &= (a + bi)(a - bi) \\ &= a^2 - abi + abi + b^2 \\ &= a^2 + b^2 \end{aligned}$$

To find the modulus (magnitude, absolute value) of the complex number, we can use the conjugate to find the answer. The absolute value of a complex number is the square root of the complex number multiplied by its conjugate.

$$
\begin{aligned}
z &= (a+bi) \\
|z| &= \sqrt{zz^*} \\
&= \sqrt{(a+bi)(a-bi)} \\
&= \sqrt{a^2+b^2}
\end{aligned}
$$

To find the quotient of 2 complex numbers, we multiple  divide the denominator by it's conjugate.

$$
\begin{aligned}
z_1 &= (a_1+b_1i) \\
z_2 &= (a_2+b_2i) \\
\frac{z_1}{z_2} &= \frac{a_1+b_1i}{a_2+b_2i} \\
&= \frac{(a_1+b_1i)(a_2-b_2i)}{(a_2+b_2i)(a_2-b_2i)} \\
&= \frac{a_1a_2-a_1b_2i+b_1a_2i-b_1b_2i^2}{a_2^2+b_2^2} \\
&= \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}i
\end{aligned}
$$

i can be raised to other powers as well -

$$
\begin{array}{rcrcrcr}
i^0 & = & & & & & 1 \\
i^1 & = & & & & & i \\
i^2 & = & & & & & -1 \\
i^3 & = & ii^2 & = & & & -i \\
i^4 & = & i^2i^2 & = & & & 1 \\
i^5 & = & ii^4 & = & & & i \\
i^6 & = & ii^5 & = & i^2 & = & -1
\end{array}
$$

$$
\begin{aligned}
i^0 &= 1 \\
i^{-1} &= -i \\
i^{-2} &= -1 \\
i^{-3} &= i \\
i^{-4} &= 1 \\
i^{-5} &= -i \\
i^{-6} &= -1
\end{aligned}
$$

## 2 The Complex Plane

The Complex Plane(also known as Argand Plane) is a way to represent complex numbers in a two dimensional plane. Complex Numbers can be represented by plotting real numbers on the horizontal axis($x$) and plotting imaginary numbers on the vertical axis($y$).

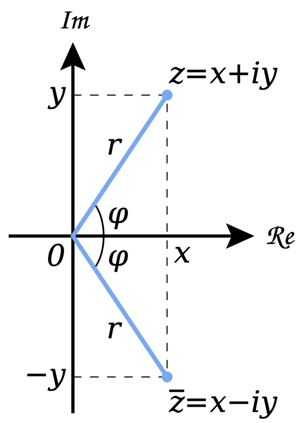


Figure 1: Example of Complex Plane.

A prominent application of complex numbers is two dimensional transformations and rotations. If we multiply a complex number by $i$, we can rotate the complex number through the complex plane at 90 increments.

Let us consider an arbitrary point $p$ in the complex plane:

$$p = 2 + i$$

and if we multiply it by $i$, it gives us:

$$\begin{array}{rcl} p & = & 2 + i \\ q & = & pi \\ & = & (2 + i)i \\ & = & 2i + i^2 \\ & = & -1 + 2i \end{array}$$

Similarly, Multiplying $q$ by $i$:

$$\begin{array}{rcl} q & = & -1 + 2i \\ r & = & qi \\ & = & (-1 + 2i)i \\ & = & -i + 2i^2 \\ & = & -2 - i \end{array}$$

repeating the same for $r$:

$$\begin{aligned} r &= -2 - i \\ s &= ri \\ &= (-2 - i)i \\ &= -2i - i^2 \\ &= 1 - 2i \end{aligned}$$

And finally $s$ by $i$ gives us $t$:

$$\begin{aligned} s &= 1 - 2i \\ t &= si \\ &= (1 - 2i)i \\ &= i - 2i^2 \\ &= 2 + i \end{aligned}$$

$\therefore p = q = r = s = (2 + i)$

After performing the above operations on $p, q, r$ and $s$, it gives us the value we of p=$2 + i$, the same value we started with.

By multiplying the complex number by $-i$, we can rotate it clock-wise.

# 3 Rotations

Arbitrary rotations can also be performed on the complex plane by operating on the complex numbers in the form(polar) :

$$q = \cos\theta + i \sin\theta$$

Multiplying any complex number by $q$(known as the rotor) produces the general formula:

$$\begin{aligned} p &= a + bi \\ q &= \cos\theta + i\sin\theta \\ pq &= (a + bi)(\cos\theta + i\sin\theta) \\ a' + b'i &= a\cos\theta - b\sin\theta + (a\sin\theta + b\cos\theta)i \end{aligned}$$

Which can be represented by using matrices as:

$$\begin{bmatrix} a' & -b' \\ b' & a' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The above notation is the method to rotate a point in the complex plane anti-clockwise about the origin.

# 4 Quaternions

Using the above operations, we can apply them on a three dimensional complex plane with the inclusion of 2 other imaginary numbers, other than $i$.

A Quaternion can be represented in its general form as :

$$q = s + xi + yj + zk \quad s, x, y, z \in \mathbb{R}$$

where $i, j$, and $k$ are the imaginary numbers.

As given by Hamilton's famous expression:

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$\begin{array}{ccc} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

From the above equations, we notice that there is a relationship between the imaginary numbers $i, j$, and $k$ and the cross product rules for the unit cartesian vectors.

$$\begin{array}{ccc} \mathbf{x} \times \mathbf{y} = \mathbf{z} & \mathbf{y} \times \mathbf{z} = \mathbf{x} & \mathbf{z} \times \mathbf{x} = \mathbf{y} \\ \mathbf{y} \times \mathbf{x} = -\mathbf{z} & \mathbf{z} \times \mathbf{y} = -\mathbf{x} & \mathbf{x} \times \mathbf{z} = -\mathbf{y} \end{array}$$

Hamilton also recognized that the $i$, $j$, and $k$ imaginary numbers could be used to represent three cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ with the same properties of imaginary numbers, with the constraint : $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$.

## 4.1 Addition and Subtraction

Quaternions can be added and subtracted similar to the way complex numbers are added and subtracted.

$$\begin{aligned} q_a &= [s_a, \mathbf{a}] \\ q_b &= [s_b, \mathbf{b}] \\ q_a + q_b &= [s_a + s_b, \mathbf{a} + \mathbf{b}] \\ q_a - q_b &= [s_a - s_b, \mathbf{a} - \mathbf{b}] \end{aligned}$$

## 4.2 Multiplication

The product of 2 quaternions can be expressed as follows :

$$\begin{aligned} q_a &= [s_a, \mathbf{a}] \\ q_b &= [s_b, \mathbf{b}] \\ q_a q_b &= [s_a, \mathbf{a}][s_b, \mathbf{b}] \\ &= (s_a + x_a i + y_a j + z_a k)(s_b + x_b i + y_b j + z_b k) \\ &= (s_a s_b - x_a x_b - y_a y_b - z_a z_b) \\ &\quad +(s_a x_b + s_b x_a + y_a z_b - y_b z_a)i \\ &\quad +(s_a y_b + s_b y_a + z_a x_b - z_b x_a)j \\ &\quad +(s_a z_b + s_b z_a + x_a y_b - x_b y_a)k \end{aligned}$$

### 4.3 Real Quaternion

A quaternion is said to be a real quaternion if it contains the vector term $\mathbf{0}$:

$$q = [s, \mathbf{0}]$$

### 4.4 Pure Quaternion

A pure quaternion is similar to a real quaternion, where its scalar term is zero.

$$q = [0, \mathbf{v}]$$

## 5 Rotation

As discussed above, in a two dimensional plane, a point can be rotated with the help of a complex number called the Rotor. Similarly, we can express a quaternion to be a rotor that is used to rotate a point in a three dimensional plane.

$$q = [\cos\theta, \sin\theta\mathbf{v}]$$

To test the theory, let us consider a quaternion $q$ and a vector $\mathbf{p}$. We begin by computing the product of $q$ and $\mathbf{p}$. $\mathbf{p}$ can be represented as :

$$p = [0, \mathbf{p}]$$

And $q$ is a unit-norm quaternion in the form:

$$q = [s, \lambda\hat{\mathbf{v}}]$$

Then,

$$\begin{aligned} p' &= qp \\ &= [s, \lambda\hat{\mathbf{v}}][0, \mathbf{p}] \\ &= [-\lambda\hat{\mathbf{v}} \cdot \mathbf{p}, s\mathbf{p} + \lambda\hat{\mathbf{v}} \times \mathbf{p}] \end{aligned}$$

We see that the resulting equations is a general quaternion with both scalar and a vector parts.

Let's first consider the "special" case where $\mathbf{p}$ is perpendicular to $\hat{\mathbf{v}}$ in which case, the dot-product term $-\lambda\hat{\mathbf{v}} \cdot \mathbf{p} = 0$ and the result becomes the Pure quaternion:

$$p' = [0, s\mathbf{p} + \lambda\hat{\mathbf{v}} \times \mathbf{p}]$$

In this case, to rotate $\mathbf{p}$ about $\hat{\mathbf{v}}$ we just substitute $s = \cos\theta$ and $\lambda = \sin\theta$.

$$p' = [0, \cos\theta\mathbf{p} + \sin\theta\hat{\mathbf{v}} \times \mathbf{p}]$$

As an example, let's rotate a vector $\mathbf{p}$ 45 about the z-axis then our quaternion $q$ is:

$$
\begin{aligned}
q &= [\cos\theta, \sin\theta\mathbf{k}] \\
&= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right]
\end{aligned}
$$

And let's take a vector **p** that adheres to the special case that **p** is perpendicular to **k**:

$$p = [0, 2\mathbf{i}]$$

Now let's find the product of $qp$:

$$
\begin{aligned}
p' &= qp \\
&= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right][0, 2\mathbf{i}] \\
&= \left[0, 2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i}\right] \\
&= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}]
\end{aligned}
$$

Which results in a Pure quaternion that is rotated 45 about the **k** axis. We can also confirm that the magnitude of the resulting vector is maintained:

$$
\begin{aligned}
|\mathbf{p}'| &= \sqrt{\sqrt{2}^2 + \sqrt{2}^2} \\
&= 2
\end{aligned}
$$

Therefore we can say that magnitude of the resultant vector is maintained after the rotation.

# 6 Conclusion

To summarise our report, quaternions provide us advantages over using matrices for representing rotations.
Rotation concatenation using quaternions is faster than using combined rotations in matrix form.
Converting quarternions to matrices is faster than for Euler angles. Therefore the inherent speed and efficiency increase helps game engines to manipulate and render 3D objects more effectively.
Without the use of complex numbers and complex analysis, three dimensional transformations would be a herculean task to accomplish. Quaternion Interpolation greatly enhances rotations of 3D objects in a complex plane. Using traditional methods such as Transformations, do not work well while working with 3D objects.