APPLICATIONS OF GRAPH THEORY IN HUMAN LIFE

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ABSTRACT

The author presents some graph theoretical planning techniques which have been employed in the design of a GSM (Group Special Mobile) operated by the Bharat Sanchar Nigam Limited. Apart from a new variant of the by now classical application of graph coloring to frequency assignment, these techniques deal with the determination of the base station identity code (BSIC), hopping sequence number (HSN), and location area code (LAC). It is shown that GSM radio network planning involves a number of optimization problems and Time Table Scheduling Problems which can be treated by graph theoretical methods.

Key Words :*Bipartite Graph, Euler graph, Hamiltonian graph, Connected graph, planner graph*

1. Introduction:

The origin of graph theory started with the problem of Koinsber bridge, in 1735. This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits. In 1852, Thomas Gutherie found the famous four color problem. Then in 1856, Thomas. P. Kirkman and William R.Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once. In 1913, H.Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken. This time is considered as the birth of Graph Theory. Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory. Any how the term “Graph” was introduced by Sylvester in 1878 where he drew an analogy between “Quantic invariants” and covariants of algebra and molecular diagrams. In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory. In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

1.1 Definition: A graph – usually denoted G(V,E) or G = (V,E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set V={a,b,c,d,e.f} and edge set E = {(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)}.

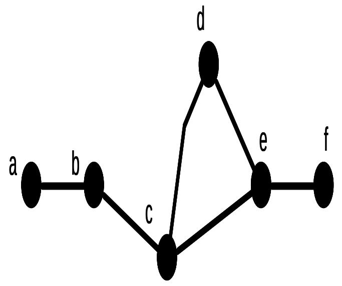


Figure 1.

1. 5 Definition: Two vertices *u* and *v* are *adjacent* if there exists an edge (*u,v*) that connects them.

1.6 Definition: An edge (u,v) is said to be *incident* upon nodes u and v.

1.7 Definition: An edge e = (u,u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

1.8 Definition: Every graph has associated with it an *adjacency matrix*, which is a binary *n*×*n* matrix A in which a_{ij} = 1 and a_{ji} = 1 if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Adjacency matrix for graph in Figure 1.

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

1.10 Definition : A *subgraph* of a graph *G* is a graph whose points and lines are contained in *G*. A complete subgraph of *G* is a section of *G* that is complete

1.11 Definition : While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

1.12 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.

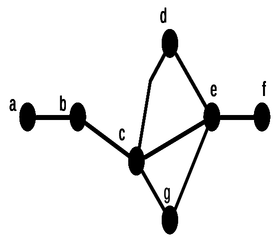


Figure: 2

1.13 Definition : A *component* of a graph is defined as a maximal subgraph in which a path exists from every node to every other (i.e., they are mutually reachable). The size of a component is defined as the number of nodes it contains. A connected graph has only one component.

1.14 Definition : A sequence of adjacent vertices $v_0, v_1, \ldots, v_n$ is known as a *walk*. In Figure 3, the sequence *a,b,c,b,c,g* is a walk. A walk can also be seen as a sequence of *incident* edges, where two edges are said to be incident if they share exactly one vertex.

1.15 Definition : A walk is closed if $v_o = v_n$.

1.16 Definition : A walk in which no vertex occurs more than once is known as a *path*. In Figure 3, the sequence *a,b,c,d,e,f* is a path.

1.17 Definition : A walk in which no edge occurs more than once is known as a *trail*. In Figure 3, the sequence *a,b,c,e,d,c,g* is a trail but not a path. Every path is a trail, and every trail is a walk.

1.18 Definition : A *cycle* can be defined as a closed path in which *n* >= 3. The sequence *c,e,d* in Figure 3 is a cycle.

1.19 Definition : A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

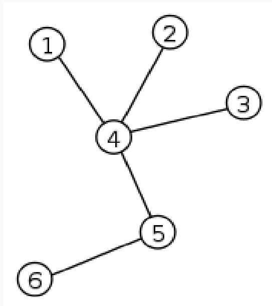


Figure 3: A labeled tree with 6 vertices and 5 edges

1.20 Definition : A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

1.21 Definition : The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a,b,c,d,e* has length 4.

1.22 Definition : The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.23 Definition : In the mathematical field of graph theory, a **bipartite graph** (or **bigraph**) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

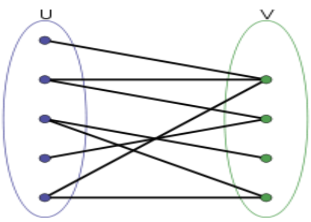


Figure 4: Example of a bipartite graph.

2. Euler Path and Example, Hamiltonian Path and Hamiltonian Circuit

2.1Definition : An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

2.2Example : Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

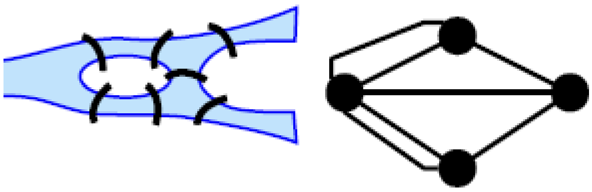


Figure 5: K¨onigsberg problem

2.3 Definition : Another closely related problem is finding a **Hamilton path** in the graph (named after an Irish mathematician, Sir William Rowan Hamilton). Whereas an Euler path is a path that visits every edge exactly once, a Hamilton path is a path that visits every vertex in the graph 4 exactly once.
A **Hamilton circuit** is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example **Hamilton Circuit** would be

2.4 Example :

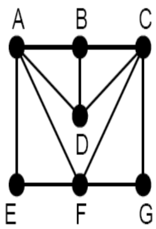


Figure 6: Hamilton Circuit would be AEFGCDBA.

3. Applications of GSM and Time Table Scheduling :

Graph theoretical concepts are widely used to study and model various applications, in different areas. They include, study of molecules, construction of bonds in chemistry and the study of atoms. Similarly, graph theory is used in sociology for example to measure actors

prestige or to explore diffusion mechanisms. Graph theory is used in biology and conservation efforts where a vertex represents regions where certain species exist and the edges represent migration path or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites and to study the impact of migration that affect other species. Graph theoretical concepts are widely used in Operations Research. For example, the traveling salesman problem, the shortest spanning tree in a weighted graph, obtaining an optimal match of jobs and men and locating the shortest path between two vertices in a graph. It is also used in modeling transport networks, activity networks and theory of games. The network activity is used to solve large number of combinatorial problems. The most popular and successful applications of networks in OR is the planning and scheduling of large complicated projects. The best well known problems are PERT(Project Evaluation Review Technique) and CPM (Critical Path Method). Next, Game theory is applied to the problems in engineering, economics and war science to find optimal way to perform
certain tasks in competitive environments. To represent the method of finite game a digraph is used. Here, the vertices represent the positions and the edges represent the moves.

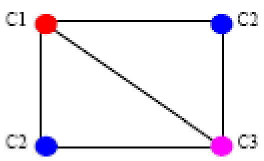
3.1 Traveling Salesman Problem :

TSP is a very well-known problem which is based on Hamilton cycle. The problem statement is: Given a number of cities and the cost of traveling from any city to any other city, find the cheapest round-trip route that visits every city exactly once and return to the starting city.
In graph terminology, where the vertices of the graph represent cities and the edges represent the cost of traveling between the connected cities (adjacent vertices), traveling salesman problem is just about trying to find the Hamilton cycle with the minimum weight. This problem has been shown to be NP-Hard. Even though the problem is computationally difficult, a large number of heuristics and exact methods are known, so that some instances with tens of thousands of cities have been solved. The most direct solution would be to try all permutations and see which one is cheapest (using brute force search). The running time for this approach is $O(V\,!)$, the factorial of the number of cities, so this solution becomes impractical even for only 20 cities. A dynamic programming solution solves the problem with a runtime complexity of $O(V\,22V\,)$ by considering
dp[end][state] which means the minimum cost to travel from start vertex to end vertex using the vertrices stated in the state (start vertex can be any vertex chosen at the start). As there are

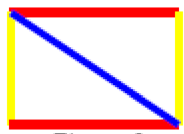
$V\ 2V$ subproblems and the time complexity to solve each sub-problems is $O(V)$, the overall runtime complexity $O(V\ 22V)$.

3.2 Vertex Coloring:

Vertex coloring is one of the most important concepts in graph theory and is used in many real time applications in computer science. Various coloring methods are available and can be used on requirement basis. The proper coloring of a graph is the coloring of the vertices and edges with minimal number of colors such that no two vertices should have the same color. The minimum number of colors is called as the chromatic number and the graph is called properly colored graph.



Proper vertex coloring with Chromatic number 3



Proper edge coloring with Chromatic number 3

3.3 Map coloring and GSM mobile phone networks:

Groups Special Mobile (GSM) is a mobile phone network where the geographical area of this network is divided into hexagonal regions or cells. Each cell has a communication tower which connects with mobile phones within the cell. All mobile phones connect to the GSM network by searching for cells in the neighbours. Since GSM operate only in four different frequency ranges, it is clear by the concept of graph theory that only four colors can be used to color the cellular regions. These four different colors are used for proper coloring of the regions. Therefore, the vertex coloring algorithm may be used to assign at most four different frequencies for any GSM mobile phone network. The authors have given the concept as follows:

Given a map drawn on the plane or on the surface of a sphere, the four color theorem assets that it is always possible to color the regions of a map properly using at most four distinct colors such that no two adjacent regions are assigned the same color. Now, a dual graph is constructed by putting a vertex inside each region of the map and connect two distinct vertices by an edge iff their respective regions share a whole segment of their boundaries in common. Then proper coloring of the dual graph gives proper coloring of the original map. Since, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual graph and vice versa. By coloring the map regions using four color theorem, the four frequencies can be assigned to the regions accordingly.

3.4Time table scheduling:

Allocation of classes and subjects to the Teachers is one of the major issues if the constraints are complex. Graph theory plays an important role in this problem. For ‘t’ Teachers with ‘n’ subjects the available number of ‘p’ periods timetable has to be prepared. This is done as follows. A bipartite graph (or bigraph is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets) G where the vertices are the number of Faculty say t_1, t_2, t_3, t_4, ……. t_k and n number of subjects say n_1, n_2, n_3, n_4, ……. n_m such that the vertices are connected by ‘pi’ edges. It is presumed that at any one period each Teacher can teach at most one subject and that each subject can be taught by maximum one Teacher. Consider the first period. The timetable for this single period corresponds to a matching in the graph and conversely, each matching corresponds to a possible assignment of Teacher to subjects taught during that period. So, the solution for the timetabling problem will be obtained by partitioning the edges of graph G into minimum number of matching. Also the edges have to be colored with minimum number of colors. This problem can also be solved by vertex coloring algorithm. “ The line graph L(G) of G has equal number of vertices and edges of G and two vertices in L(G) are connected by an edge iff the corresponding edges of G have a vertex in common. The line graph L(G) is a simple graph and a proper vertex coloring of L(G) gives a proper edge coloring of G by the same number of colors. So, the problem can be solved by

finding minimum proper vertex coloring of L(G).” For example, Consider there are 4 Teachers namely t_1, t_2, t_3, t_4,. and 5 subjects say n_1, n_2, n_3, n_4, n_5 to be taught. The teaching requirement matrix $p = [p_{ij}]$ is given as.

P	n_1	n_2	n_3	n_4	n_5
t_1	2	0	1	1	0
t_2	0	1	0	1	0
t_3	0	1	1	1	0
t_4	0	0	0	1	1

Figure – 7: The teaching requirement matrix for four Teachers and five subjects

The bipartite graph is constructed as follows.

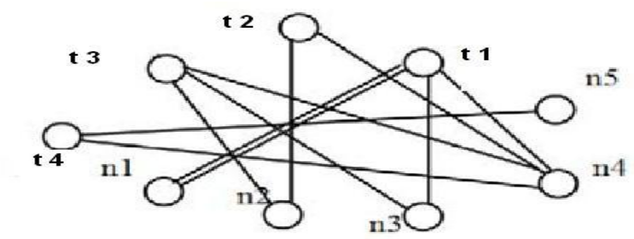


Figure – 8: Bipartite graph with 4 Teachers and 5 subjects

Finally, the authors found that proper coloring of the above mentioned graph can be done by 4 colors using the vertex coloring algorithm which leads to the edge coloring of the bipartite multigraph G. Four colors are interpreted to four periods

….	1	2	3	4
t_1	n_1	n_2	n_3	n_4

Figure – 9: The schedule for the four subjects

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