Complete Solutions to Exercises 3.1

1. We need to check all 10 axioms. Let $\mathbf{u}=\begin{pmatrix} a \\ b \end{pmatrix}$, $\mathbf{v}=\begin{pmatrix} c \\ d \end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix} e \\ f \end{pmatrix}$

Axiom 1. We first check closure under vector addition. We have

$$\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} a+c \\ b+d \end{pmatrix} \text{ and } \begin{pmatrix} a+c \\ b+d \end{pmatrix} \text{ is in } \mathbb{R}^2 \quad \surd$$

Hence we have closure under vector addition.

Axiom 2. Commutative.

$$\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} a+c \\ b+d \end{pmatrix}=\begin{pmatrix} c+a \\ d+b \end{pmatrix}=\begin{pmatrix} c \\ d \end{pmatrix}+\begin{pmatrix} a \\ b \end{pmatrix} \quad \surd$$

Hence $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

Axiom 3. Associative law:

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \left[\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c \\ d \end{pmatrix}\right]+\begin{pmatrix} e \\ f \end{pmatrix} \\ &= \begin{pmatrix} a+c \\ b+d \end{pmatrix}+\begin{pmatrix} e \\ f \end{pmatrix} \qquad \surd \\ &= \begin{pmatrix} a+c+e \\ b+d+f \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c+e \\ d+f \end{pmatrix}=\mathbf{u}+(\mathbf{v}+\mathbf{w})\end{aligned}$$

We have $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Axiom 4. Neutral element is $\mathbf{O}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and for any $\mathbf{u}$ we have

$$\mathbf{u}+\mathbf{O}=\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} a+0 \\ b+0 \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}=\mathbf{u} \quad \surd$$

Axiom 5. Additive inverse, we have

$$\mathbf{u}+(-\mathbf{u})=\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} -a \\ -b \end{pmatrix}=\begin{pmatrix} a-a \\ b-b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\mathbf{O} \quad \surd$$

Axiom 6. Let k be scalar then

$$k\mathbf{u}=k\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} ka \\ kb \end{pmatrix} \quad \surd$$

Hence $k\mathbf{u}$ is also in $\mathbb{R}^2$, therefore we have closure under scalar multiplication.

Axiom 7. Associative Law for scalar multiplication. Let k and c be real numbers then

$$k(c\mathbf{u})=k\left(c\begin{pmatrix} a \\ b \end{pmatrix}\right)=k\begin{pmatrix} ca \\ cb \end{pmatrix}=\begin{pmatrix} kca \\ kcb \end{pmatrix}=(kc)\begin{pmatrix} a \\ b \end{pmatrix}=(kc)\mathbf{u} \quad \surd$$

Axiom 8. Distributive Law for vectors. Let k be a real number then

$$\begin{aligned}k(\mathbf{u}+\mathbf{v}) &= k\left[\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c \\ d \end{pmatrix}\right] \\ &= k\begin{pmatrix} a+c \\ b+d \end{pmatrix} \qquad \surd \\ &= \begin{pmatrix} ka+kc \\ kb+kd \end{pmatrix}=\begin{pmatrix} ka \\ kb \end{pmatrix}+\begin{pmatrix} kc \\ kd \end{pmatrix}=k\begin{pmatrix} a \\ b \end{pmatrix}+k\begin{pmatrix} c \\ d \end{pmatrix}=k\mathbf{u}+k\mathbf{v}\end{aligned}$$

Axiom 9. Distributive Law for scalars. Let k and c be real numbers then

$$(k+c)\mathbf{u}=(k+c)\begin{pmatrix}a\\b\end{pmatrix}$$

$$=\begin{pmatrix}(k+c)a\\(k+c)b\end{pmatrix} \quad \surd$$

$$=\begin{pmatrix}ka+ca\\kb+cb\end{pmatrix}=\begin{pmatrix}ka\\kb\end{pmatrix}+\begin{pmatrix}ca\\cb\end{pmatrix}=k\begin{pmatrix}a\\b\end{pmatrix}+c\begin{pmatrix}a\\b\end{pmatrix}=k\mathbf{u}+c\mathbf{u}$$

Axiom 10. Identity Element. For every vector $\mathbf{u}$ in V we have

$$1\mathbf{u}=1\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\times a\\1\times b\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}=\mathbf{u} \quad \surd$$

Since all 10 axioms are satisfied, we conclude that $\mathbb{R}^2$ is a vector space.

2. Very similar to solution 1.

3. The rules of matrix algebra established in Chapter 1 ensure that M_{22} is a vector space. However we can show these again as follows:

Let $\mathbf{u}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}e & f\\g & h\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}i & j\\k & l\end{pmatrix}$. Checking all 10 axioms.

Axiom 1. We check $\mathbf{u}+\mathbf{v}$ is also in the set M_{22}:

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}a & b\\c & d\end{pmatrix}+\begin{pmatrix}e & f\\g & h\end{pmatrix}=\begin{pmatrix}a+e & b+f\\c+g & d+h\end{pmatrix} \quad \surd$$

Since $\mathbf{u}+\mathbf{v}$ is a 2 by 2 matrix therefore it is in M_{22}.

Axiom 2. Commutative, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. We know from Chapter 1 that matrix addition is commutative. You may like to check this if you want.

Axiom 3. Similarly we have the associative law, $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for matrix addition.

Axiom 4. Neutral element $\mathbf{O}$ which satisfies

$$\mathbf{u}+\mathbf{O}=\begin{pmatrix}a & b\\c & d\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}=\mathbf{u} \quad \surd$$

Axiom 5. Additive inverse.

$$\mathbf{u}+(-\mathbf{u})=\begin{pmatrix}a & b\\c & d\end{pmatrix}+\begin{pmatrix}-a & -b\\-c & -d\end{pmatrix}$$

$$=\begin{pmatrix}a-a & b-b\\c-c & d-d\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\mathbf{O} \quad \surd$$

Axiom 6. Let k be scalar then

$$k\mathbf{u}=k\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}ka & kb\\kc & kd\end{pmatrix} \quad \surd$$

Since $\begin{pmatrix}ka & kb\\kc & kd\end{pmatrix}$ is a 2 by 2 matrix which means it is in M_{22} therefore we have closure under scalar multiplication.

Axiom 7. Associative Law for scalar multiplication. Let k_1 and k_2 be real numbers then

$$k_1(k_2\mathbf{u}) = k_1\begin{pmatrix} k_2a & k_2b \\ k_2c & k_2d \end{pmatrix} = \begin{pmatrix} k_1k_2a & k_1k_2b \\ k_1k_2c & k_1k_2d \end{pmatrix} = (k_1k_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (k_1k_2)\mathbf{u} \qquad \surd$$

Axiom 8. Distributive Law for vectors. Let k be a real number then

$$\begin{aligned} k(\mathbf{u}+\mathbf{v}) &= k\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] \\ &= k\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \\ &= \begin{pmatrix} ka+ke & kb+kf \\ kc+kg & kd+kh \end{pmatrix} \qquad \surd \\ &= \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} + \begin{pmatrix} ke & kf \\ kg & kh \end{pmatrix} = k\begin{pmatrix} a & b \\ c & d \end{pmatrix} + k\begin{pmatrix} e & f \\ g & h \end{pmatrix} = k\mathbf{u} + k\mathbf{v} \end{aligned}$$

Axiom 9. Distributive Law for scalars. Let k_1 and k_2 be scalars then

$$\begin{aligned} (k_1+k_2)\mathbf{u} &= (k_1+k_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} (k_1+k_2)a & (k_1+k_2)b \\ (k_1+k_2)c & (k_1+k_2)d \end{pmatrix} \\ &= \begin{pmatrix} k_1a+k_2a & k_1b+k_2b \\ k_1c+k_2c & k_1d+k_2d \end{pmatrix} \qquad \surd \\ &= \begin{pmatrix} k_1a & k_1b \\ k_1c & k_1d \end{pmatrix} + \begin{pmatrix} k_2a & k_2b \\ k_2c & k_2d \end{pmatrix} = k_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} + k_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = k_1\mathbf{u} + k_2\mathbf{u} \end{aligned}$$

Axiom 10. Identity element, 1. We have

$$1\mathbf{u} = 1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{u} \quad \surd$$

Since **all** 10 axioms are satisfied therefore we conclude that the set of 2 by 2 matrices M_{22} is a vector space.

4. Again the rules of matrix algebra established in chapter 1 prove that M_{23} is a vector space but you can justify it to yourself by checking the 10 axioms as we did for M_{22} in the above question.

5. We need to check all 10 axioms with vector addition and scalar multiplication defined as:

$$(f+g)(x) = f(x) + g(x)$$
$$(kf)(x) = kf(x)$$

Let f, g and h be members of the set $F[a,\ b]$ which means each of these are functions which are defined on the interval $[a,\ b]$.

Axiom 1. By definition we have $f+g$ is also in $F[a,\ b]$. Hence we have closure under vector addition.

Axiom 2. Commutative law:

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
&= g(x)+f(x)=(g+f)(x)
\end{aligned}
$$

Hence $f+g=g+f$.

Axiom 3. Associative law:

$$
\begin{aligned}
((f+g)+h)(x) &= (f+g)(x)+h(x) \\
&= f(x)+g(x)+h(x) \\
&= f(x)+(g+h)(x)=(f+(g+h))(x)
\end{aligned}
$$

We have $(f+g)+h=f+(g+h)$ which means that the associative law is established.

Axiom 4. Neutral element, which is $0(x)=0$ for **all** x in the interval $[a,\ b]$. For any function f in $F[a,\ b]$ we have

$$(f+0)(x)=f(x)+0(x)=f(x)+0=f(x)$$

Axiom 4 is satisfied because we have $f+0=f$.

Axiom 5. Additive inverse which is $-f$:

$$(f+(-f))(x)=f(x)+(-f(x))=f(x)-f(x)=0$$

Axiom 6. Let k be scalar then by definition of scalar multiplication we have

$$(kf)(x)=kf(x)$$

Since $kf(x)$ is a member of the set $F[a,\ b]$ therefore we have closure under scalar multiplication.

Axiom 7. Associative Law for scalar multiplication. Let k and c be real numbers then

$$[k(cf)](x)=kcf(x)=(kc)f(x)$$

Axiom 8. Distributive Law for vectors. Let k be a real scalar then

$$k(f+g)(x)=k(f(x)+g(x))=kf(x)+kg(x)=(kf+kg)(x)$$

We have $k(f+g)=kf+kg$.

Axiom 9. Distributive Law for scalars. Let k and c be real scalars, then

$$(k+c)f(x)=kf(x)+cf(x)=(kf+cf)(x)$$

We have $(k+c)f=kf+cf$.

Axiom 10. Identity element, for every f in $F[a,\ b]$ we have

$$1f(x)=f(x)$$

Hence Axiom 10 is satisfied.

Since we have established all 10 axioms therefore we conclude that the set $F[a,\ b]$ is a vector space.

6. Matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ do **not** form a vector space because there is **no neutral** element or there is **no zero** vector. We cannot have $\mathbf{O}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in this set therefore it does **not** form a vector space.

7. Checking the 10 axioms. The first 3 axioms are straightforward to check.

Axiom 4 we need the zero vector which in this case is $\mathbf{O}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$. We can get this by substituting $a=0$ and $b=0$ into $\begin{pmatrix}a & 0\\0 & b\end{pmatrix}$. Check that

$$\begin{pmatrix}a & 0\\0 & b\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}a & 0\\0 & b\end{pmatrix}$$

Axiom 5 is the existence of the additive inverse. For every vector $\begin{pmatrix}a & 0\\0 & b\end{pmatrix}$ there is $\begin{pmatrix}-a & 0\\0 & -b\end{pmatrix}$ such that

$$\begin{pmatrix}a & 0\\0 & b\end{pmatrix}+\begin{pmatrix}-a & 0\\0 & -b\end{pmatrix}=\begin{pmatrix}a-a & 0\\0 & b-b\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\mathbf{O}$$

Axiom 6. Let k be scalar then $k\begin{pmatrix}a & 0\\0 & b\end{pmatrix}=\begin{pmatrix}ka & 0\\0 & kb\end{pmatrix}$

(We have closure under scalar multiplication).
For Axiom 7 you just need to check:

$$k\left(c\begin{pmatrix}a & 0\\0 & b\end{pmatrix}\right)=(kc)\begin{pmatrix}a & 0\\0 & b\end{pmatrix}$$

Axiom 8. Distributive Law for vectors. Let k be a real number then

$$\begin{aligned}k\left(\begin{pmatrix}a & 0\\0 & b\end{pmatrix}+\begin{pmatrix}c & 0\\0 & d\end{pmatrix}\right)&=k\begin{pmatrix}a+c & 0\\0 & b+d\end{pmatrix}\\&=\begin{pmatrix}k(a+c) & 0\\0 & k(b+d)\end{pmatrix}\\&=\begin{pmatrix}ka+kc & 0\\0 & kb+kd\end{pmatrix}\\&=\begin{pmatrix}ka & 0\\0 & kb\end{pmatrix}+\begin{pmatrix}kc & 0\\0 & kd\end{pmatrix}\\&=k\begin{pmatrix}a & 0\\0 & b\end{pmatrix}+k\begin{pmatrix}c & 0\\0 & d\end{pmatrix}\end{aligned}$$

Axiom 9. Distributive Law for scalars. Let k and c be real numbers then we need to show that $(k+c)\mathbf{u}=k\mathbf{u}+c\mathbf{u}$:

$$\begin{aligned}(k+c)\begin{pmatrix}a & 0\\0 & b\end{pmatrix}&=\begin{pmatrix}(k+c)a & 0\\0 & (k+c)b\end{pmatrix}\\&=\begin{pmatrix}ka+ca & 0\\0 & kb+cb\end{pmatrix}\\&=\begin{pmatrix}ka & 0\\0 & kb\end{pmatrix}+\begin{pmatrix}ca & 0\\0 & cb\end{pmatrix}=k\begin{pmatrix}a & 0\\0 & b\end{pmatrix}+c\begin{pmatrix}a & 0\\0 & b\end{pmatrix}\end{aligned}$$

Axiom 10. Identity element is 1 and we have

$$1\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

ALL 10 axioms are satisfied, therefore we conclude that matrices of the type $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ do form a vector space.

8. Matrices of size 2 by 2 do **not** form a vector space because we do **not** have closure under vector addition.

Suppose $(a+d) \neq (b+c)$ and let $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+1 & b+1 \\ c+1 & d+1 \end{pmatrix}$$

The determinant of $\begin{pmatrix} a+1 & b+1 \\ c+1 & d+1 \end{pmatrix}$ is equal to

$$\begin{aligned}(a+1)(d+1)-(c+1)(b+1) &= ad+a+d+1-(cb+c+b+1) \\ &= \underbrace{ad-cb}_{=0 \text{ by } (\dagger)}+a+d+1-c-b-1 \\ &= (a+d)-(b+c) \neq 0 \qquad \left[\text{Because } (a+d) \neq (b+c)\right]\end{aligned}$$

Hence $\mathbf{u}+\mathbf{v}$ is **not** a 2 by 2 matrix whose determinant is equal to zero.

If $a+d=b+c$ then let $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. In this case we can show that the matrix obtained by $\mathbf{uv}$ is **not** in the given set because the determinant is **not zero**. This set of matrices **does not** form a vector space.

9. We need to check all 10 axioms to show that the given set P_2 is a vector space. Let $\mathbf{p}(x) = ax^2+bx+c$, $\mathbf{q}(x) = dx^2+ex+f$ and $\mathbf{r}(x) = gx^2+hx+i$. Checking the axioms is very similar to Example 1.

10. The last axiom 10 fails because

$$1\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1\times 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Since $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ therefore axiom 10 fails. This means that $\mathbb{R}^2$ with the given scalar multiplication definition is **not** a vector space.

11. The set $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ where $a \geq 0$ and $b \geq 0$ is **not** a vector space because axiom 6 fails. Axiom 6 states that we have closure under scalar multiplication. The scalar multiplication

$$(-1)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1\times a \\ -1\times b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

which is **not** in the given set V because $-a<0$ and $-b<0$. Therefore the given set is **not a vector** space.

12. The set of rationals is **not** a vector space because when we multiply by a scalar, which is any real number, then the result is **not** a rational number. For example if we consider the rational 1 and multiply this by the scalar ® then the result is 1® = ® which is **not** a rational number. Remember ® is **irrational**.
We do **not** have closure under scalar multiplication therefore the set of rationals $\mathbb{Q}$ is **not** a vector space.
13. This is easy to check that all 10 axioms of a vector space are satisfied.
The zero vector $\mathbf{O}$ is the normal number 0.
14. Axiom 6 which states that we have closure under scalar multiplication, that is if we multiply a vector by any scalar then the result is also in the given set. In this case axiom 6 fails because for example $-1(5)=-5$ but -5 is **not** a positive real number therefore $\mathbb{R}^+$ is **not** a vector space.

15. Vector $-\mathbf{u}$ is also in the vector space *V* provided **u** is in *V*.
Since *V* is a vector space so by Axiom 5:

> Additive Inverse. For every vector **u** there is a vector $-\mathbf{u}$ (minus **u**) which satisfies $\mathbf{u}+(-\mathbf{u})=\mathbf{O}$.

Hence $-\mathbf{u}$ is in *V*.

16. Need to prove that $(-k)\mathbf{u}=-(k\mathbf{u})=k(-\mathbf{u})$.
Proof. Since $-k$ is a real number we can write $-k=-1k$. Using this we have

$$\begin{aligned}(-k)\mathbf{u}&=(-1k)\mathbf{u}\\&=(-1)(k\mathbf{u})=-(k\mathbf{u}) \qquad \left[\text{Because } (-1)\mathbf{u}=-\mathbf{u}\right]\end{aligned}$$

Similarly we show that $(-k)\mathbf{u}=k(-\mathbf{u})$:

$$\begin{aligned}(-k)\mathbf{u}&=(-1k)\mathbf{u}\\&=\left(k(-1)\right)\mathbf{u}\\&=k(-1\mathbf{u})=k(-\mathbf{u})\end{aligned}$$

Hence combining both results we have $(-k)\mathbf{u}=-(k\mathbf{u})=k(-\mathbf{u})$.

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17. Need to prove if $k\mathbf{u}=k\mathbf{w}$ then $\mathbf{u}=\mathbf{w}$.
Proof. We have $k\mathbf{u}=k\mathbf{w}$ which means that

$$k\mathbf{u}-k\mathbf{w}=\mathbf{O}$$
$$k(\mathbf{u}-\mathbf{w})=\mathbf{O}$$

Since $k\neq 0$ therefore $\mathbf{u}-\mathbf{w}=\mathbf{O}$ which gives $\mathbf{u}=\mathbf{w}$.

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18. Required to prove if $k_1\mathbf{u}=k_2\mathbf{u}$ then $k_1=k_2$.
Proof. We have

$$k_1\mathbf{u}=k_2\mathbf{u}$$
$$k_1\mathbf{u}-k_2\mathbf{u}=\mathbf{O}$$
$$(k_1-k_2)\mathbf{u}=\mathbf{O}$$

Since $\mathbf{u}\neq\mathbf{O}$ therefore $k_1-k_2=0$ which gives $k_1=k_2$.

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19. Need to prove $-(\mathbf{u}+\mathbf{w}) = -\mathbf{u}-\mathbf{w}$.

Proof. We have

$$\begin{aligned}-(\mathbf{u}+\mathbf{w}) &= (-1)(\mathbf{u}+\mathbf{w})\\ &= (-1)\mathbf{u}+(-1)\mathbf{w}\\ &= -\mathbf{u}+(-\mathbf{w})\\ &= -\mathbf{u}-\mathbf{w}\end{aligned}$$

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20. Required to prove $\underbrace{\mathbf{u}+\mathbf{u}+\mathbf{u}+\cdots+\mathbf{u}}_{n \text{ copies}} = n\mathbf{u}$. We use mathematical induction.

Remember the procedure is to prove it for a base case such as $n=1$. Assume it is true for $n=k$ and then prove it for $n=k+1$.
Proof. We have $\mathbf{u} = 1\mathbf{u} = \mathbf{u}$ √.
Assume it is true for $n=k$, that is

$$\underbrace{\mathbf{u}+\mathbf{u}+\mathbf{u}+\cdots+\mathbf{u}}_{k \text{ copies}} = k\mathbf{u} \qquad (*)$$

Need to prove it for $n=k+1$. We have to prove that

$$\underbrace{\mathbf{u}+\mathbf{u}+\mathbf{u}+\cdots+\mathbf{u}+\mathbf{u}}_{k+1 \text{ copies}} = (k+1)\mathbf{u}$$

Consider the $k+1$ additions of **u**:

$$\begin{aligned}\underbrace{\mathbf{u}+\mathbf{u}+\mathbf{u}+\cdots+\mathbf{u}}_{k \text{ copies}}+\mathbf{u} &= \underbrace{k\mathbf{u}}_{\text{by } (*)}+\mathbf{u}\\ &= k\mathbf{u}+1\mathbf{u} = (k+1)\mathbf{u}\end{aligned}$$

Hence by mathematical induction we have our result, $\underbrace{\mathbf{u}+\mathbf{u}+\mathbf{u}+\cdots+\mathbf{u}}_{n \text{ copies}} = n\mathbf{u}$.

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