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Using Action-Process-Object-Schema (APOS) Theory students' strategies while solving a linear transformations modelling problem in a Linear Algebra course are studied. Modelling cycles were complemented by conceptual activities designed with a previously developed genetic decomposition for this concept. The work of students during the modelling process in the classroom is described in terms of questions and knowledge emerging from their own strategies, and in terms of the difficulties they faced. Results show some affordances of the modelling situation and the use of activities, and the difficulties faced by students.

Keywords: APOS, linear algebra, linear transformations, modelling, functions.

INTRODUCTION

Research on the teaching and learning of Linear Algebra has been the focus of attention of several research groups in the last ten years. Researchers coincide that in spite of its many applications, this is a difficult subject for students and many of the difficulties students' face have been underlined and explained in terms of different theoretical frameworks (for example, Dorier,Robert, Robinet & Rogalski, 2000; Sierpinska, 2000). In the last years, an interest on the use of different teaching methodologies in the teaching of Linear Algebra has developed with the aim of helping students in developing their understanding of abstract concepts starting from real or realistic modeling situations (Martin, Loch, Cooley, Dexter & Vidakovic, 2010; Wawro, Larson, Zandieh & Rasmussen, 2012; Trigueros & Possani, 2013) or by the use of technology to help them to relate different representations in order to give meaning to concepts (Maracci, 2008; Gueudet, 2004; Klasa, Oktaç & Soto, 2006; Romero & Oktaç, 2015).

It is in the context of this growing research area that a research project was developed in Mexico. The project has a double aim. On the one hand, investigating the way students construct different Linear Algebra concepts, and, on the other hand, studying the results of the use of modeling situations in the introduction of those concepts, together with activities based on APOS theoretical framework in the classroom (Oktaç & Trigueros, 2010).

In this paper we present part of this work related to the teaching and learning of Linear Transformations. In particular we focus on the design and use of a teaching sequence designed to foster students' understanding of this concept by relating geometrical and algebraic representations through the use of a modeling situation in a geometrical context.

SOME ANTECEDENTS

Linear transformations have received a lot of attention of researchers because of their importance in applications and the difficulties students face when learning them. Some results obtained follow. Students struggle when asked to find a linear transformation in a geometric context starting from the images of basis vectors; they have difficulties using systemic reasoning and using visualization to determine the transformations; students' show a tendency to use intuitive models when working geometrically and conceptualizing transformations as functions (Uicab & Oktaç, 2006; Ellis, Henderson, Rasmussen, & Zandieh, 2012). Roa-Fuentes and Oktaç (2010) developed two genetic decompositions for the concept of linear transformation and used them to investigate the way students may construct this concept. After analyzing students' responses in an interview, they found evidence supporting one of them where construction starts by using specific examples of linear transformations. In a study about the teaching and learning of linear transformation using Cabri- Géomètre, it was found that the use of that tool helped students to find relations between the geometric and the matrix associated to a linear transformations (Karrer & Jahn, 2008). When studying the role of change of basis and matrix representation of a linear transformation Montiel and Batthi (2010) described with care the role that semantics and gestures play in classroom interactions, while Bagley, Rasmussen and Zandieh (2012) discussed that under specific conditions students are able to relate matrices and linear transformations and that they are capable to work with matrices but not to relate the concept of function with that of matrix. Wawro, Larson, Zandieh & Rasmussen (2012) designed a hypothetical collective progression (HCP) to support students' understanding of linear transformations defined in terms of matrix multiplication. Their results show the proposed HCP fostered students reasoning in productive ways and helped students to coordinate local and global views of linear transformations as functions and as matrix multiplication for particular geometric mappings.

THEORETICAL BACKGROUND

APOS Theory is based on Piaget's concept of reflective abstraction. (Arnon et al., 2013). Its main constructs can be defined as follows. An Action is defined as a transformation of a mathematical object memorized by the individual or perceived as driven by external stimuli. After reflecting on Actions, they can be interiorized into a Process; Actions are no longer perceived as external and the individual can use them omitting steps and anticipating the results without having to perform the process. A Process may be coordinated with other Processes, or be reverted as needed in a problem situation. When an individual can see a Process as a totality, and needs to apply Actions on it, the Process can be encapsulated into an Object and new Actions can be applied to it. A Schema for a mathematical topic is considered as a coherent collection of Actions, Processes, Objects, and previously constructed Schemas related to the mathematical topic.

Research using APOS Theory starts by designing a model that intends to predict the mental constructions involved in the construction of the studied concept. This model is called a genetic decomposition (GD). It specifies the mental constructions in terms of the constructs of the theory needed in the understanding of that concept. A GD, as a model, is not unique, different models may be proposed, but it is important that a GD can be supported by experimental data from students. Usually, this is not the case, some of the predicted constructions are not found in students' work, while students show other constructions not predicted by the model. The GD is then refined. This process can be repeated many times until a model predicting students' constructions is found. The GD is also used to design activities to guide students' constructions of the concepts of interest. Students work collaboratively in teams with these activities and whole group discussions are organized in order to promote students' reflection on what they had done.

Although modelling is not included in APOS theoretical framework it is consistent with APOS structures (Trigueros, 2008): When students face a modelling problem, they use the mathematical Schemas and Schemas constructed in other disciplines or in their daily life to approach the problem they face. They take elements of those Schemas to choose variables, and to formulate some hypothesis about the behavior of the expected solution. Through Actions and Processes on some of the components of the Schema, and through coordination of Processes, a mathematical model emerges. This mathematical model is encapsulated into an Object, and new Actions, Processes, coordinations and relations are applied on it to determine its properties and to respond the questions posed by the modelling problem or to pose new questions.

METHOD

We first present the GD used for this study. We did not propose a new GD, but used a slightly modified version of the refined GD proposed by Roa-Fuentes and Oktaç (2010) which is schematically described in the Figure 1. It was used in the design of the conceptual activities designed for the course and the research instruments.

A design context based on an illustration shown in a textbook (Nakos & Joyner, 1999) was selected to present a problem situation designed by the authors:

A cartoonist needs to show the figure of a man on a bicycle, he has drawn, in motion and in different positions to appear in a film (Figure 2). *He has contacted you to help him by making the necessary calculations to program the drawings on the computer. He asks you to send the calculations together with each figure, so that he is able to write a program.*

The problem was used in two occasions accompanied by activities designed with the GD. These activities guided students' constructions emerging from discussion on the problem towards the construction of Linear Transformations. The first experience took place in Brazil with 8 students who were finishing their studies and had taken a

Linear Algebra course. They volunteered to participate in a 4 sessions of 4 hours each

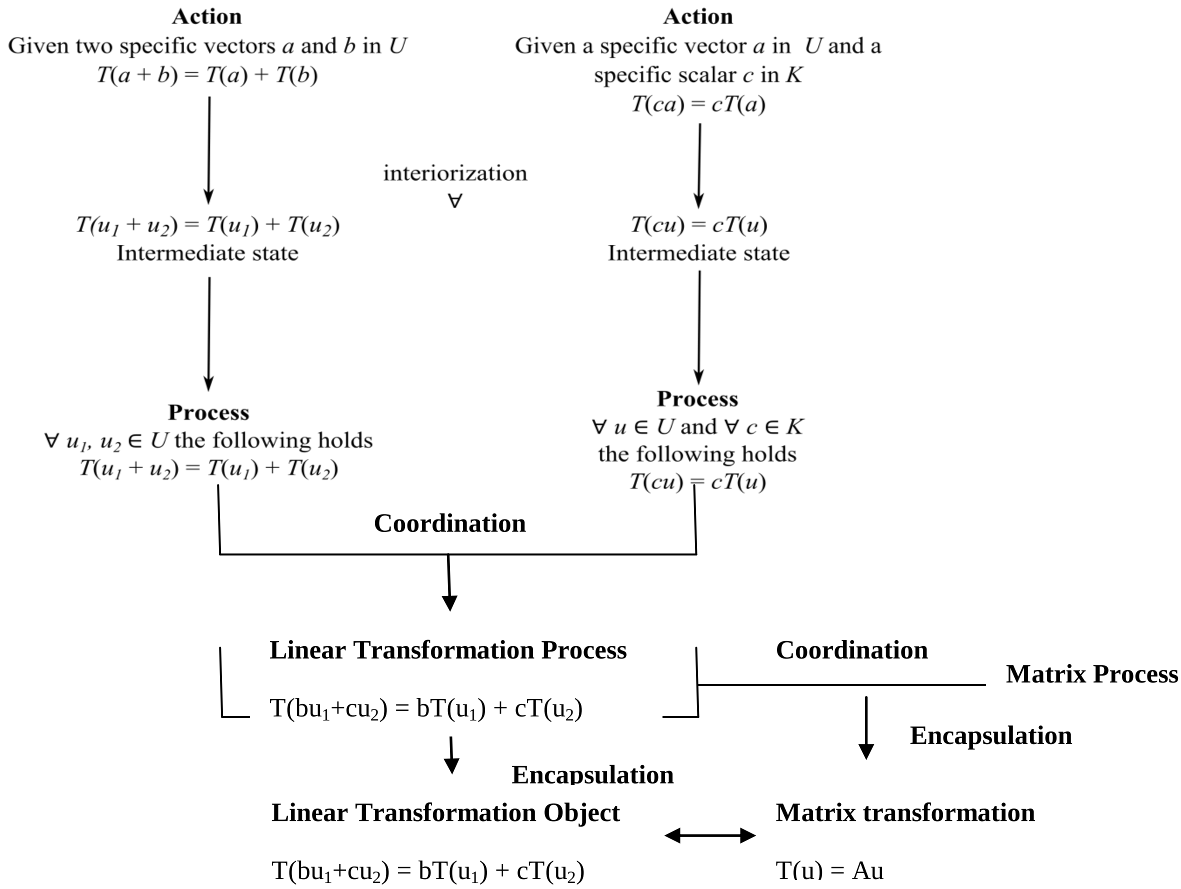

Figure 1: Genetic Decomposition for Linear Transformations

modeling workshop, 3 of them worked in teams during the four sessions while 5 worked individually. The aim of this experience was to test the modeling situation and the conceptual activities. All the work of the students was kept and their dialogues were recorded. The second experience took place during five two hours sessions at the classroom in a Mexican university during an Introduction to Linear

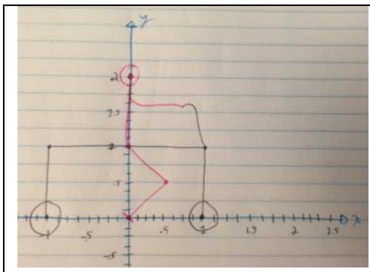	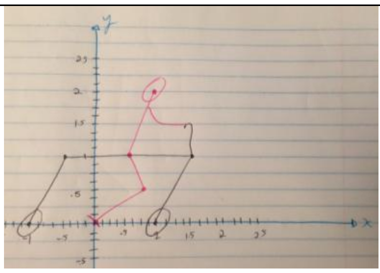	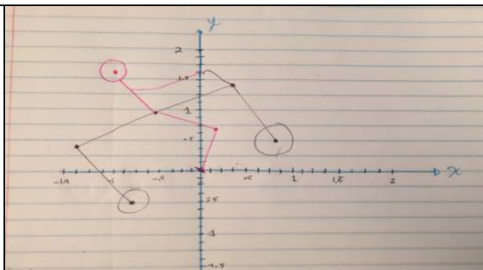
Figure given for problem.	Example of a needed movement: cyclist appears as moving.	Example of a another needed movement: cyclist appears on a ramp.

Figure 2: Examples of drawings needed in the problem situation presented

Algebra course attended by 23 engineering and applied mathematics students. Students worked in cycles including collaborative work in teams of three students, followed by whole group discussion. All sessions were audio recorded and all the work of students was kept. Homework problems related to both types of activities were used. After two weeks of the final session 8 students from different teams were invited to a semi-structured interview. Data obtained was analyzed using the GD by both researchers and results were negotiated between them. In this paper we present only results from the work done in the classroom. In excerpts and examples, Brazilian students will be labeled as S1, S2… and Mexican students with a letter describing their team and a number identifying a student.

RESULTS

Students' main strategy during the first cycle, in both experiences, was the use of their knowledge about vectors to explore the given data and making tables to try to find a rule for the transformation. This strategy was successful for them in the case of simple transformations such as shear transformations and translation, but was not easy to apply in the case of more complex transformations, as rotation, where students of both countries showed difficulties.

Most students participating in the first experience, in Brasil, did not recognize, at the beginning of the workshop, that a linear transformation is a function. They did observe some analogy between transformations and functions, but were not clear about their relation. When introduced to the modeling problem they were able to make this relation clear. This was evident when students were surprised when facing situations that were different from those they had encountered before and the conclusions they were making from them:

> S1: they are functions, since transformations depend on the initial vectors. The domain in calculus was R, and now we are studying domain in R2.

> S2: In general, in Differential and Integral Calculus courses, domain and codomain of function were real numbers...

The problem made them reflect on what they had studied before and helped them to relate two concepts that had remained compartmentalized in their previous studies; they were able to assimilate transformations into their function's schema. In the Mexican experience, students had not studied transformations before, and students discussions showed that most students were thinking about transformations as functions with domain and range in R^2 when working with the problem. This was evidenced in comments such as C3:"So it is a function but instead of real numbers you have vectors in the domain and range" or "Yes, and the rule is x+2y for the first component and the second is always the same, y, so you only apply it". They talked about functions in terms of input-output: H1:"…you can check here, if you use this point [writing (-1, 1)] with this rule, you get this [writing (-1/2,1)] … and doing twice the first comma the second, it works for all the points." They evidenced, in general, a process conception of transformation as a function. They were able to relate the function process they had constructed through algebra and calculus to consider a new type of functions that work in the same way but in R^2.

In both countries students presented difficulties related to the use of mathematical language. For example, a Brazilian student wrote:

S7 a) T(x,y)=(x,0,y) $\neq$ R3 ==> T is not surjective. $\therefore$ T is not an isomorphism.

Although we consider that the appropriation of mathematical language occurs gradually, it was surprising that students in the last semester of a Mathematics program could still compare a specific vector to a vector space. This issue was taken into account in the Mexican experience by paying a lot of attention to students' productions and pointing out mistakes to be corrected.

Working on the modeling situation made students of both countries reflect on transformations' properties. For example, after doing two transformations, shear transformation and translation, students in two teams, C and E did action to compare the transformations in terms of their properties:

E2: "This one changes the form of the bicycle (*shear transformation*), but this one (*translation*) does not, it only changes its position in space"

E1: Yes, one is like a deformation, it doesn't change the position of the cyclist, but this other only moves it around".

E2: When this changes this form the points move, but the origin stays in the same place. If you move it around, all the points change, including the origin. How can we say this?

E3: What I see is that in both all the points go to a new position, in the first the origin does not change and the points are in different relative positions, while in the other all the points change and all change by the same amount... but, well they are different transformations.

Later on, when trying to do the rotation one student of the same group notices:

E3: This transformation does not change the form, it does not change the origin either…so, some transformations don't move the origin, other leave the drawing as it was.

During whole group discussion the teacher in Mexico recovered this discussion and asked the group about differences they could find about transformations. The teacher defined isometries. She asked students to verify which transformations were isometries. Then gave them activities based on the GD where linear transformations were introduced. Students worked without difficulty with those activities. Students demonstrated they had constructed Linear Transformations as processes.

Students in Brazil had no difficulty while solving activities related to decide if a transformation was or not linear, we can say that all of them had constructed a process conception of linear transformation. They made comments such as:

> S3 If we multiply a scalar by a vector or a scalar multiplying by its transformation, the result will be the same.

Or, when discussing if a linear transformation is an isomorphism, they wrote:

> S4 and S5: T(0,0,0) = (0,0); T(1,0,1)=(1,0); T(1,0,2)=(1,0). As T(1,0,1)=T(1,0,2)=(1,0) ==> T is not injective. T is not an isomorphism.

Most students in both experiences showed difficulties when facing rotations since the rule was not easy to find from the picture or from a table of values. Most students in Brazil had not been introduced to the matrix form of linear transformations, when the teacher introduced it, they considered it as a novelty and used it without problems, showing again encapsulation of Linear Transformations. Mexican students struggled with the rule using trigonometric functions; they had not been introduced to the matrix representation of the transformation. After some time, students in team A found a possible rule (Figure 3a), and only students in group C realized those equations could be written as the product of a matrix and a vector (Figure 3b). These students also showed encapsulation when they realized that a composition of transformations was needed:

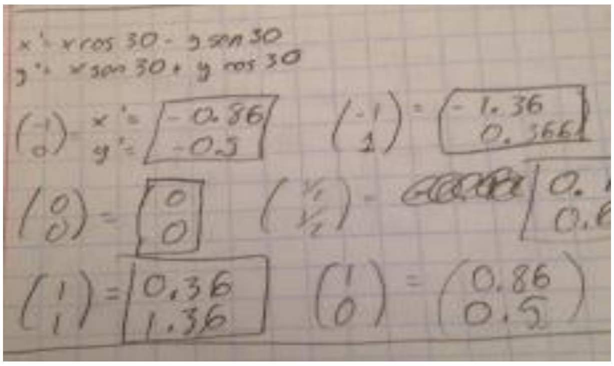

Figure 3a Work on rotation

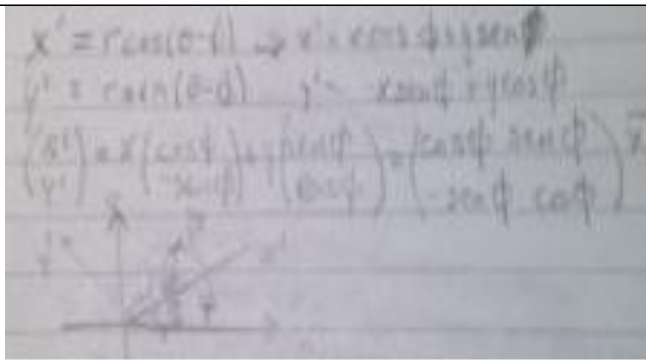

Figure 3b Finding a matrix

> C2: If we do this rotation, it is weird; the tires would be underground, because, look the origin is here at the cyclist feet. So I think we need to do first the rotation but then a translation so that the tires are on the ground. It is a composition of transformations.

The second cycle was devoted to work with the composition of transformations and activities related to proving if different transformations given in different representations were or not linear. Students in both countries worked directly, without difficulties, with composition of transformations, demonstrating, again, that they had included the concept of transformation into their function schema, and encapsulation of linear transformations as shown by the following discussion of students in Mexico:

A1: It has to look as if he is going uphill and moving.... so we should use the rotation first and then do the shear transformation, is that OK?

A2: Yes, I guess it is like the composition of functions. We do one and then we apply the second to the vector obtained. OK... but since we have the matrices we can do the matrix product, I think.

Students, in general worked well doing the proofs in different representations and in finding associated matrices to transformations or images of transformations, and doing composition of transformations. At times, work with the modeling problem or with the activities was difficult for students and the teacher had to help. This happened more frequently in Mexico than in Brazil. For example some students in Brazil struggled with an activity involving the kernel of a linear transformation; also they had problems remembering the conditions that made the existence of the inverse of a transformation possible. Although these students had studied these topics, they showed they had constructed an action conception of transformation; work on the activities helped them reflect on their constructions and possibly to interiorize them into a process. Mexican students also showed difficulties with these activities and some of them struggled with finding matrices associated to transformations because they had not constructed a process conception for the concept of basis of a vector space.

DISCUSSION AND CONCLUSION

Results of this study show students' constructions during the modeling process and the work with activities designed with the genetic decomposition. This tool proved to be effective in the analysis of students' constructions during the whole experience in both countries; it can be said that it describes the basic constructions involved in the construction of the concept.

Students in both countries found the modeling experience interesting and important in the understanding of the linear transformation concept. They showed their interest by being involved in their work and even Brazilian students who had already studied this topic found novelties in it which made them re-think about this concept.

The use of the modeling situation mobilized students previously constructed schemas and their development in ways that were not predicted by the researchers. It also favored the emergence of some ideas that previous literature showed to be difficult for students. One important result of this study is students' reconstruction of their function schema and their reflection on properties of different transformations. The problem elicited a group of mathematical models that were used by students as objects to explore what they expected from their predictions, to combine them through compositions and to find their properties. The activities designed with the genetic decomposition played an important role in introducing the notion of linear transformations and in guiding students' exploration towards its construction. They were also important in focusing students' attention in some aspects of the model that they had not previously taken into account.

The analysis of students' productions and discussions made the recognition of those constructions that seem to play a fundamental role in the learning of linear transformations such as making sense of what linearity means in the context of Linear Algebra and the relation of linear transformation with functions, matrices and with the concept of basis, although not all of this has been described in this paper. The concept of transformation emerged quite easily from work with the modeling problem. However, linear transformations had to be introduced in the activities so that students were able to conclude that translations are not linear transformations. It is important to underline, how, while working on the model, students were able to develop on their own powerful conceptual tools, such as a way to determine the difference between rigid and non rigid transformations; and a relation to the concept of matrix. The emergence of these ideas gives evidence that the use of modeling situations in the classroom promotes the construction of knowledge. The complementary use of activities designed with the genetic decomposition played an important role in the development of students' schema for function and for linearity.

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