On Certain Contraction Mappings in a Partially Ordered Vector Space

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A report by

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1 Introduction

G. Birkhoff along with H. Samelson proved that a method to solve problems which combine the uniqueness, purpose, and existence of eigenvectors of positive operators is by giving a suitable metric on a subset in relevance to which the operators are contractions. These 2 American mathematicians have proved the Perron Theorem for matrices with the fact that positive elements intersect the positive quadrant with a hyperplane using the Hilbert metric. Birkhoff with his knowledge had the mindset to positive linear operators in a wide setting. This can be used for an array of nonlinear operators only because of the contraction mapping principle is non - linear. This paper uses a somewhat different distance function which is not confined to such a section and we come up with a theorem for a class of nonlinear mappings that contract this metric.
After giving a certain amount of introduction, the metric is explained in Theorem 2 and the finished product is derived in certain subsets. Then there is a theorem of nonlinear operators followed by 2 examples.

2 Preliminaries

Through the course of this paper, X will denote a real, normed linear space partially ordered with a means of a non - empty set K and a closed subset X with the following properties :

1. $x, y \in K \Rightarrow x + y \in K$
2. $x \in K, \alpha \geq 0 \Rightarrow \alpha x \in K$
3. $x \in K, -x \in K \Rightarrow x = 0$

and we denote x $\leq$ y if and only if y-x $\in$ K. K is normal if there exist a constant which is positive "λ", such that : 0 $\leq$ x $\leq$ y which tells us that $\| x \| \leq \lambda \| y \|$.

Lemma 1.

If K is normal with constant λ ; then x $\leq \lambda$ * y, y $\leq \lambda$ * x, $\| x \| \leq$ m, $\|$ y $\| \leq$ m together imply $\| x - y \| \leq$ m (1 + 2 λ)(λ - 1).

Proof
Given that x - y $\leq$ (λ- 1)y and y - x $\leq$ (λ - 1)x hence there exist z, z' $\in$ K such that :

$$x - y + z = (\lambda -1)y \text{ and } y - x + z' = (\lambda -1)x$$

Then,
$$\parallel z \parallel \leq \gamma \parallel z + z' \parallel = \gamma \parallel x - y + z + y - x + z' \parallel$$

=γ ‖ (λ- 1) y + (λ- 1)x ‖

$\leq$ 2 m $\lambda(\lambda - 1)$

Hence,
$$\parallel x - y \parallel = \parallel x - y + z - z \parallel$$

$$\leq \parallel x - y + z \parallel + \parallel z \parallel$$

$$\leq (\lambda - 1)m(1 + 2\lambda)$$

The variables x and y that belong to K but both are not zero are said to be linked iff there exists finite positive real numbers λ and μ with x $\leq$ λ*y and y $\leq$ μ*x. This is a relationship that splits K into a set of mutually exclusive constituents where each of which is called a "blunted" sub - cone that is a subset of K with their respective properties such as :

0 does not belong which means they are open.

3 The definition of an ordered metric

Let x and y be linked. Let us define α and β by the equations -

α= ∞ {λ: x $\leq \lambda y$} , $\beta = \infty$ {μ w : y $\leq \mu x$},

and since K is closed, x $\leq$ α y and y $\leq$ β x so that if either α= 0 or β= 0 then x = y = 0 which is not taken into consideration since x and y are linked. Let d(x,y) = log $\{max(\alpha, \beta)\}$.

Lemma 2

d(,) defines a metric on each constituent of K.

Proof Given d(x,y) = d(y,x) is clear from the definition. If y$\leq$x and x $\leq$y; therefore by property (1) (iii) of K,
x = y; so that if x not equal to y, it is either α or β that are strictly greater than 1.
Thus, d(x,y) $\geq$ 0 and d(x,y) = 0 if and only if x = y.
In conclusion, suppose x, y and z belong to a constituent; then

$$x \leq \alpha_1 y,\ y \leq \beta_1 x$$

$$x \leq \alpha_2 z,\ z \leq \beta_2 x$$

$$Z \leq \alpha_3 y,\ y \leq \beta_3 z$$

α(i) and β(i) satisfy the inequalities.

$$d(x,y) = \log \alpha(1) \leq \log(\alpha(2)^*\alpha(3)) = \log(\alpha(2) + \log(\alpha(3))$$

$$\leq \log \{max(\alpha(2), \beta(2)\ \} + \log \max\ (\alpha(3), \beta(3)\}$$

$$= d(x,z) + d(z,y) \text{ also if } \beta(1) \geq \alpha(1).$$

Lemma 3

Let K be a normal cone which is complete in the norm topology, then each constituent, C, is complete with respect to d(,).

Given that:
K= normal cone which is complete in the norm topology. C= Cauchy with respect to d(,).

Proof
Let us consider xn to be a sequence in C. Also, let α_{pq}=∞\{λ: xp $\leq \lambda xq$\} where (p,q=1,2,...) As proved before, $\{x_n\}$ is a Cauchy sequence that is bounded in norm and hence, it converges to an element $\mu \in$K which then converges to μ where $\mu \in$C.
A sequence can be termed as Cauchy sequence if the terms of sequence in the long run all become subjectively near each other.
Applying the definition of Cauchy sequence in xn, for every $\varepsilon > 0$, there exists an N such that d(xp,xq) ¡1 for all p,q $\geq$ N. This means that the maximum value of α_{pq} and αqp is less than exp(1), where p,q$\geq$N . By elaborating the statement above, αpn ¡ exp(1) where p$\geq$N thereby xp $\leq$ exp(1) xN$\leq$3xN. Considering that K is normal, $\|$xp$\| \leq 3\gamma\|$xn$\|$. Henceforth, $\|$xn$\|$ is bounded m where m= max$\|$x1$\|$, $\|$x2$\|$, $\ldots$,$\| x_n \|$, 3$\lambda\|$xn$\|$.
Relative on ε, there exists δ¿0 given that ε¿0 in such a manner that exp(δ) $\leq$ 1+(ε/M) where M=m(1+2γ). In similar manner, since xn is a Cauchy sequence, for every $\varepsilon > 0$, there exists an N such that: d(xp,xq) ¡δ for all p,q $\geq$ N That is, max(α_{pq}, αqp)¡1+(ε/m), therefore, xp $\leq$ 1+(ε/m)=ε for every p,q $\geq$ Nε

Therefore, it can be stated that xn being a cauchy sequence in norm and as K is complete, there exists an element $\mu \in$ K such that: lim(n-¿∞) $\|$xn-$\mu\|$=0

Precedently, d(xp,xq)¡ ε for all such sufficiently large p and q that is xp $\leq$ exp(ε)xq and xq $\leq$ exp(ε)xp. However, lim (q-¿ ∞) $\|$xq-$\mu\|$=0 and K is closed in the norm topology, hence:

Xp$\leq$ exp(ε)μ and $\mu\leq$ exp(ε)xp for all sufficiently large p.

The above statement validates that $\mu \in$ C, given that it is connected with xp. Also, d(xp, μ) $\leq \varepsilon$ which holds true for all large value of p. However, ε is chosen arbitrarily which is as similar as saying that the sequence xn converges to μ in respect to the metric.

Remarks: It is not mandatory for X to be complete. Lemma 3 and the following theorem holds true even for "locally convex spaces" given that "normal" holds a suitable meaning. Given that A is a vector space with a locally convex topology created by a system of semi-norms pa K which is said to be normal if: For each α, there exists a positive real number $\gamma\alpha$ such that: 0$\leq$x$\leq$y, which implies pα(x) $\leq \gamma\alpha$ pα(y)

Hence, lemma is then proved by replacing $\|\bullet\|$ by pα ($\bullet$) everywhere and similarly, "norm topology" by "locally convex topology."

See Bonsall [2] and also Schaefer [S].

4 Contraction Mapping

The statement for contraction mapping is as follows :
Let K be a complete and normal cone and let us take T, a mapping of X with the following properties :

There exists p with 0$\leq$p<1 such that x , y $\in$ K, x $\leq$ αy and y $\leq$ β x together imply that Tx $\leq$ α'Ty and Ty$\leq$. β'Tx with There exists a variable x0 such that Tx0 are linked Then a vector is said to exist, belonging to the constituent containing x0 such that Tu=u where u is unique in that particular constituent (constituent containing x0 in this case). The iterative sequence defined by xn=Txn-1 (where n=1,2,3,.....) converges to u.

Proof

Let us take C to be the constituent containing x0, then T(C) is a subset of C. Let y$\in$C, then x0 and y are said to be linked. By property (1) Tx0 and Ty are also linked, therefore by the transitive nature of the relation and property (2), x0 and Ty are linked. This means that Ty $\in$ C and implies that if property (2) is satisfied by one x0, it is also satisfied for every other point in the constituent C.
From property (1), T is a contraction of C with reference to d(,) and C is said to be complete by lemma 3 so that a unique fixed point u in C follows the contraction mapping principle. xn converges in norm follows as it is

Cauchy in the metric, thereby it is Cauchy in the norm. (from lemma 3 proof).

COROLLARY. Suppose $\lambda > 0$, then there exists u$\lambda \in$ C (where uλ is unique in C) such that Tu$\lambda = \lambda$uλ

Proof
Apply the theorem to the operator T $\lambda = \lambda$-1T.

Remarks

1. The vector u belongs to the constituent C and is therefore, not 0.
2. It is possible to prove without any appeal to the subsidiary metric (With the help of the hypothesis of the theorem) that xn is Cauchy in norm. Its limit u is fixed point of T and that u is unique in the constituent containing x0 (constituent C). The proof is essentially the proof of contraction mapping principle.
3. Condition (1) on T can be replaced by a criterion which is slightly more restrictive and but more natural. There exists p with 0$\leq$p¡1 such that x $\in$ K and x$\leq \alpha$ y $\rightarrow$ Tx$\leq \alpha$PTy
4. It is not always true to assume T to be positive. (that is it maps K into itself). Though certain assumptions show that T maps any constituent to K which contains an x0 with property (2) into itself.
5. Condition (2) is satisfied if for example an order unit e is mapped into an order unit e'.
6. The case where p=1 is critical for this method and it generally requires more conditions to obtain results. This is particularly true for linear operators which map K into itself. An example for the same would be Samelson's method.

5 References

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