DIFFERENTIAL CALCULUS (MAT 131)
Practice Problems (Unit I)

The following is a list of problems to help you practice well and prepare for the examinations. This set is not comprehensive. Try doing as many exercise problems as possible from Thomas' calculus. In particular solve all the problems from chapters 2 and 3 (all sections), Chapter 4 (sections 4.1 and 4.2), chapter 7 (section 7.5) and chapter 10 (section 10.8)

Limits

1. Evaluate (i) $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$ (ii)$\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$ (iii) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ (iv) $\lim_{x\to 1} \lfloor 1-x \rfloor$
2. Evaluate (i) $\lim_{x\to 0} \dfrac{1+x+sinx}{3cosx}$ (ii)$\lim_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$ (iii) $\lim_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$
3. State and prove the sum rule, the difference rule, the product rule and the quotient rule for limits.
4. State sandwich theorem. Use sandwich theorem to evaluate $\lim_{x\to 0} u(x)$ given that $1-\dfrac{x^2}{4} \leq u(x) \leq 1+\dfrac{x^2}{4}$.
5. Solve Questions 1 to 4 from section 2.2.
6. If $\lim_{x\to -2} f(x) = 1$, find (i) $\lim_{x\to -2} f(x)$ and (ii) $\lim_{x\to -2} \dfrac{f(x)}{x}$.
7. If $\lim_{x\to 2} \dfrac{f(x)-5}{x-2} = 3$, find (i) $\lim_{x\to 2} f(x)$ and (ii)$\lim_{x\to 2} f(x)+2$.
8. State the precise definition of *limit.*
9. Prove that the limit of a function if exists is unique.
10. Using the precise definition of limits, prove that (i) $\lim_{x\to 2} f(x) = 4$ where $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$
    (ii) $\lim_{x\to 1} f(x) = 1$ where $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$ (iii) $\lim_{x\to 1} f(x) = 2$ where $f(x) = \begin{cases} 4-2x & \text{if } x < 1 \\ 6x-4 & \text{if } \geq 1. \end{cases}$
11. Given a function $f(x)$ and numbers L, x_0 and $\epsilon > 0$ find an open interval (a, b) about x_0 in which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value of $\delta > 0$ such that $0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
    (i)$f(x) = 2x-2; L = 5; x_0 = 4; \epsilon = 0.01$
    (ii)$f(x) = \sqrt{x}; L = 1/2; x_0 = 1/4; \epsilon = 0.1$
    (iii)$f(x) = x^2; L = 3; x_0 = \sqrt{3}; \epsilon = 0.1$
    (iv) $f(x) = x^2; L = 4; x_0 = -2; \epsilon = 0.5$
    (v) $f(x) = x^2-5; L = 11; x_0 = 4; \epsilon = 1$
12. Evaluate (i) $\lim_{x\to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$ (ii) $\lim_{x\to 0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

13. Evaluate the following using the result $\lim_{\theta \to 0} \frac{sin\theta}{\theta}$. (Without using LHospital's rule) (i) $\lim_{\theta \to 0} \frac{sin\sqrt{2}\theta}{\sqrt{2}\theta}$. (ii) $\lim_{\theta \to 0} \frac{x + xcosx}{sinxcosx}$. (iii) $\lim_{\theta \to 0} \frac{1 - cos\theta}{sin2\theta}$. (iv) $\lim_{\theta \to 0} \frac{sin5\theta}{sin4\theta}$. (v)$\lim_{\theta \to 0} \frac{tan3\theta}{sin8\theta}$. (v) $\lim_{\theta \to 0} \frac{x - xcosx}{sin^3 3x}$.

14. Give an example of a function for which **LHL** and **RHL** exists at a point, but at that point. Also sketch the graph of this function .

15. Give an example of a function having removable discontinuity at a point. Also sketch the graph of this function.

16. Give an example of a function for which **LHL** and **RHL** does not exists at a point. Also sketch the graph of this function.

17. Let $f(x) = \begin{cases} 2x + a & for x \geq 0 \\ 3 & for x < 0. \end{cases}$. Find a so that $\lim_{x \to 0} f(x)$ exists.

18. Evaluate the following limits if exists:(Hint: Determine LHL and RHL and verify if they are equal) (i) $\lim_{x \to 0} xsin(1/x)$ (ii) $\lim_{x \to 0} \left(\frac{e^{1/x}}{1 + e^{1/x}} \right)$ (iii) $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} 3x - 2 & if\ x < 1 \\ 4x^2 - 3x & if\ x > 1. \end{cases}$ (iv) $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} \frac{x - 1}{|x - 1|} & if\ x \neq 1 \\ 0 & if\ x = 1. \end{cases}$ (v) $\lim_{x \to 1} g(x)$ where $g(x) = x - \lceil x \rceil$

Continuity

19. Define the term continuity.

20. Describe with an example types of discontinuities.

21. Discuss the continuity of the following functions:
(i) $f(x) = \begin{cases} x^2 + 2 & when\ x > 1 \\ 2x + 1 & when\ x = 1 \\ 3 & when\ x < 1 \end{cases}$ at $x = 1$. (ii) $f(x) = \begin{cases} xsin(1/x) & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ at $x = 0$.
(iii) $f(x) = \begin{cases} \frac{1}{1 + e^{1/x}} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ at $x = 0$.(iv) $f(x) = \begin{cases} \frac{e^{1/x} - 1}{e^{1/x} + 1} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$ at $x = 0$. (v) $f(x) = sin(1/x)$ at $x = 0$.

22. State the extreme value theorem for continuous functions.

23. State the intermediate value theorem for continuous functions.

24. For what value of a is $f(x) = \begin{cases} x^2 - 1 & ifx < 3 \\ 2ax & if\ x \geq 3 \end{cases}$ is continuous at every x?

25. For what values of a and b is $f(x) \begin{cases} -2 & if\ x \leq -1 \\ ax - b & if -1 < x < 1 \\ 3 & ifx \geq 1 \end{cases}$ continuous at every x?

26. Let $f(x) = \begin{cases} 1 & if x \leq 3 \\ ax+b & if 3 < x < 5 \\ 7 & if x \geq 5. \end{cases}$
Determine the values of a and b so that $f(x)$ is continuous for all values of x.

Differentiability

27. Define derivative of a function at a point.

28. Prove that if a function $f(x)$ is differentiable at a point then it is continuous at that point but not conversely.

29. Examine the differentiability of the following functions at the specified points: (Hint: Find the left hand derivative and right hand derivative and check if they are equal)
(i) $f(x) = \begin{cases} x^2 & if x \leq 3 \\ 6x-9 & if x > 3 \end{cases}$ at $x = 3$. (ii) $f(x) = \begin{cases} x^2 sin(1/x) & when\ x \neq 0 \\ 0 & when x = 0. \end{cases}$ at x=0.
(iii) $f(x) = \begin{cases} 1+2x & if -1 \leq x \leq 0 \\ 1-3x & if 0 < x < 1 \\ x-3 & if 1 < x \leq 2 \end{cases}$ at $x = 0$ and $x = 1$.

30. For what choices of a and b will $f(x) = \begin{cases} x^2 & if x \leq 1 \\ 2ax+b & if x > 1. \end{cases}$ be differentiable at $x = 1$?

Mean Value Theorems

31. State and prove Rolle's theorem.

32. State and prove Lagrange's mean value theorem

33. State and prove Cauchy's mean value theorem.

34. State Rolle's theorem and prove it geometrically.

35. State Lagrange's mean value theorem and prove it geometrically.

36. State Cauchy's mean value theorem and obtain Lagrange's mean value theorem as a reduction from it.

37. State the two forms of Lagrange's theorem.

38. Verify Rolle's theorem for the following functions:
(i) $f(x) = (x-a)^3(x-b)^4$ in $[a, b]$. (ii) $f(x) = log\left[\dfrac{x^2+ab}{x(a+b)}\right]$ in $[a, b]$ where a and b are positive numbers. (iii) $f(x) = \begin{cases} x^2+1 & if 0 \leq x \leq 1 \\ 3-x & if 1 < x \leq 2. \end{cases}$ in $[0, 2]$ (iv) $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$.

39. Discuss the applicability of Rolle's theorem for the following functions. Give reasons for your answer:
(i) $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$. (ii) $f(x) = |x|$ in $[-1, 1]$ (iii) $f(x) = secx$ in $[0, 2\pi]$.

40. If $f'(x) > 0$ at all points $x \in [a, b]$, show that $f(x)$ is strictly increasing in $[a, b]$.

41. If $x > o$ show that (i) $\frac{x}{1+x} < log(1+x) < x$ and (ii) $0 < \frac{1}{log(1+x)} - \frac{1}{x} < 1$.f

42. Find the value or values of c that satisfy the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of Mean Value Theorem for the functions and intervals in the following problems:
(i) $f(x) = x^2 + 2x - 1, [0, 1]$ (ii) $f(x) = \sqrt{x-1}, [1, 3]$ (iii) $g(x) = \begin{cases} x^3 & , -2 \le x \le 0 \\ x^2 & , 0 < x \le 2. \end{cases}$

43. Show that the following functions have exactly one root in the given interval:
(i) $f(x) = x^4 + 3x + 1, [-2, -1]$ (ii) $g(t) = \sqrt{t} + \sqrt{1+t} - 4, [-\infty, 0]$ (iii) $r(\theta) = 2\theta - cos^2\theta + \sqrt{2}$.

44. Suppose $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x ? Give reasons for your answer.

45. It took 14 seconds for a mercury thermometer to rise from -19^oC to 100^oC when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of $8.5^oC/sec$.

46. Verify Lagrange's Mean Value Theorem for $f(x) = (x-1)(x-2)(x-3)$ in $[0, 4]$.

47. Verify Cauchy's Mean Value Theorem for : (i) $f(x) = e^x$ and $g(x) = e^{-x}$. in $[a, b]$. (ii) $\phi(x) = sinx$ and $\psi(x) = cosx$ in $[a, b]$. (iii) $F(x) = logx$ and $G(x) = \frac{1}{x}$ in $[1, e]$.

Indeterminate Forms

48. State L'Hôpital's rule.

49. Give an example each for the indeterminate forms: $\frac{0}{0}$; $\frac{\infty}{\infty}$; $\infty.0$; $\infty - \infty$; 1^∞; 0^0 and ∞^0.

50. Find the limits of the following:

(i) $\lim_{\theta\to 0} \frac{3^{sin\theta}-1}{\theta}$ (ii) $\lim_{x\to 0^+} \frac{ln(x^2+2x)}{lnx}$ (iii) $\lim_{\theta\to 0} \frac{\theta - sin\theta\ cos\theta}{tan\theta - \theta}$ (iv) $\lim_{\theta\to 0} \frac{\theta - sin\theta}{\theta^3}$
(v) $\lim_{x\to b} \frac{x^b - b^x}{x^x - b^b}$. (vi) $\lim_{x\to 0} \frac{a^x - 1 - xloga}{x^2}$ (vii) $\lim_{x\to\pi/2} \frac{tanx}{tan3x}$ (viii) $\lim_{x\to 0} \frac{log(sin2x)}{log(sinx)}$
(ix) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{xtanx}\right)$ (x) $\lim_{x\to 0} \left(\frac{1}{x^2} - -cot^2 x\right)$ (xi) $\lim_{x\to 0} (xlog(tanx))$ (xii) $\lim_{x\to 0^+} (1+x)^{1/x}$
(xiii) $\lim_{x\to 0} \left(\frac{tanx}{x}\right)^{1/x^2}$ (xiv) $\lim_{x\to 0} (1+sinx)^{cotx}$ (xv) $\lim_{x\to 0} \left(\frac{1}{x}\right)^{tanx}$ (vi) $\lim_{x\to 0} (e^x + x)^{1/x}$.

Taylor and Maclaurin Series

51. State Taylor's theorem and Maclaurin's theorem

52. Expand the function e^{sinx} upto the term containing x^4 using Maclaurin's expansion.

53. Show that $\log_e(secx) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$

54. Using Maclaurin's expansion obtain the first 4 terms of the following functions: (i) $\log(1+sinx)$ (ii) $e^{ax}sin(bx)$ (iii) $\log\left(x + \sqrt{1+x^2}\right)$ (iv) $sinx$ (v) e^x (vi) $\left(sin^{-1}(x)\right)^2$