Notes on uniform convergence

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1 Numerical sequences

We begin by recalling some properties of *numerical sequences*. By a numerical sequence we simply mean a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$. We assume that the reader is familiar with the definition of convergence of such a sequence. We also assume that the reader is familiar with (numerical) series, $\sum_{k=1}^{\infty} a_k$. We say that the series converges if the sequence $\{S_n\}_{n=1}^{\infty}$, $S_n = \sum_{k=1}^{n} a_k$, of partial sums converges. The series $\sum_{k=1}^{\infty} a_k$ is called *absolutely convergent* if $\sum_{k=1}^{\infty} |a_k|$ converges. Recall that an absolutely convergent sequence is also convergent. We recall without proof some convergence criteria for series.

Proposition 1.1. *Assume that* $\sum_{k=1}^{\infty} a_k$ *converges. Then* $\lim_{k\to\infty} a_k = 0$.

Proposition 1.2 (Comparison test)**.** *Assume that* $0 \le |a_k| \le b_k$ *for all* k*. If the series* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent.*

Proposition 1.3 (Ratio test)**.** *Assume that* $\lim_{k\to\infty} |a_{k+1}/a_k| = q$.

i) If $q < 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

ii) If $q > 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Proposition 1.4 (Root test)**.** *Assume that* $\lim_{k\to\infty} |a_k|^{1/k} = r$.

i) If $r < 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

ii) If $r > 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *diverges.*

A sequence of complex numbers converges if and only if the real and imaginary parts both converge. To check the convergence of a complex series one can therefore check the real and imaginary parts separately. A complex series $\sum_{k=1}^{\infty} z_k$ is said to converge absolutely if $\sum_{k=1}^{\infty} |z_k|$ is absolutely convergent. From the inequalities $|\operatorname{Re} z|, |\operatorname{Im} z| \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z|$ it follows that $\sum_{k=1}^{\infty} z_k$ is absolutely convergent if and only if the series $\sum_{k=1}^{\infty} \operatorname{Re} z_k$ and $\sum_{k=1}^{\infty} \operatorname{Im} z_k$ are both absolutely convergent. Propositions 1.1–1.4 remain true for complex series if the absolute value is interpreted as the absolute value of a complex number. Finally, we remark that a sequence or a series may of course start at a different index than 1; the important thing is what happens for large indices.

Example 1.1. The series $\sum_{k=0}^{\infty} e^{-k}(\cos k + i \sin k)$ is absolutely convergent since

$$|e^{-k}(\cos k + i \sin k)| = e^{-k},$$

and $\sum_{k=0}^{\infty} e^{-k}$ is a convergent geometric series. We can in fact calculate the sum of the series using the formula for the sum of a geometric series (you should check that this extends to complex numbers!)

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-k}(\cos k + i \sin k) &= \sum_{k=0}^{\infty} e^{-k(1+i)} = \sum_{k=0}^{\infty} \left(e^{-(1+i)}\right)^k \\ &= \frac{1}{1-e^{-(1+i)}} = \frac{e(e-\cos 1) + ie \sin 1}{1+e^2-2e\cos 1}.\end{aligned}$$

This implies in particular that

$$\sum_{k=0}^{\infty} e^{-k} \cos k = \frac{e(e-\cos 1)}{1+e^2-2e\cos 1} \quad \text{and} \quad \sum_{k=0}^{\infty} e^{-k} \sin k = \frac{e \sin 1}{1+e^2-2e\cos 1}.$$

2 Function sequences

A *function sequence* is a sequence of functions $\{f_n\}_{n=1}^{\infty}$ with the same domain of definition and target domain. To be concrete we shall assume that $f_n \colon I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. The simplest form of convergence of a function sequence is *pointwise convergence.*

Definition 2.1. We say that $f_n \to f$ pointwise if $f_n(t) \to f(t)$ for each $t \in I$.

Pointwise convergence of function series is similarly defined by requiring that the sequence of partial sums converges pointwise.

Unfortunately, pointwise convergence is not sufficient for many purposes in analysis, in particular when one wants to interchange limits. As a motivating example and to show how function sequences appear when solving differential equations, we consider the definition of the exponential function.

Example 2.1. The exponential function $\exp t$ can be introduced in different ways. Here we do it by seeking a power series solution

$$x(t) = \sum_{k=0}^{\infty} a_k t^k$$

of the differential equation $x'(t) = x(t)$ with initial value $x(0) = 1$. Assuming that we can differentiate the series pointwise, we obtain

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k,$$

where we have replaced the index k by $k+1$ in the last series. Substituting this in the differential equation and identifying terms of the same power, we obtain the condition

$$(k+1)a_{k+1} = a_k$$

for all k. The initial condition implies that $a_0 = 1$ and thus $a_k = 1/k!$. This gives finally the formal solution

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

It is easy to see that this function series converges pointwise by the quotient test (check this!). We take this as our definition of $\exp t$. In order to verify that $\exp t$ solves the differential equation, we need to be able to differentiate the series termwise.

What is required here is the interchanging of two limits:

$$\lim_{h\to 0}\left(\lim_{n\to\infty}\sum_{k=1}^{n}\frac{f_k(t+h)-f_k(t)}{h}\right) = \lim_{n\to\infty}\left(\lim_{h\to 0}\sum_{k=1}^{n}\frac{f_k(t+h)-f_k(t)}{h}\right).$$

More generally, we may ask when

$$\lim_{t\to a}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to a} f_n(t).$$

This does not hold if one only has pointwise convergence.

Example 2.2. Let

$$f_n(t) = \frac{1}{n^2t^2+1}, \quad t\in\mathbb{R}.$$

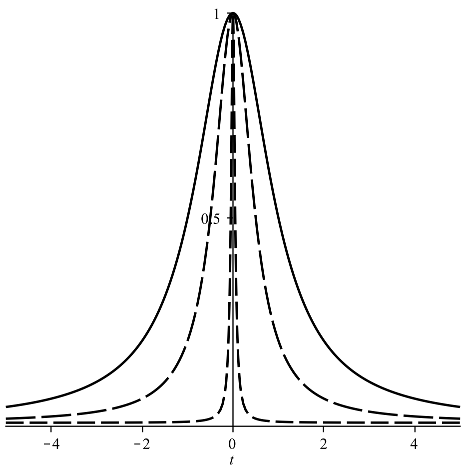
Then

$$\lim_{n\to\infty} f_n(t) = \begin{cases} 0, & t\neq 0, \\ 1, & t = 0. \end{cases}$$

Hence,

$$0 = \lim_{t\to 0}\lim_{n\to\infty} f_n(t) \neq \lim_{n\to\infty}\lim_{t\to 0} f_n(t) = 1.$$

The pictures below show f_n for $n = 1, 2$ and 20.



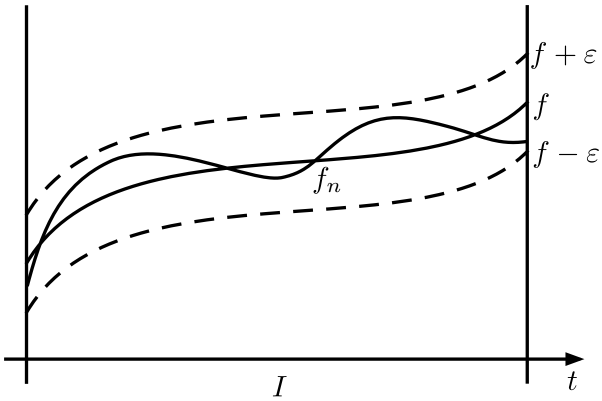
What is needed is a stronger form of convergence called *uniform convergence*.

3 Uniform convergence

Definition 3.1. We say that the sequence $\{f_n\}$ converges uniformly to f on I if

$$\sup_{t\in I} |f_n(t) - f(t)| \to 0 \text{ as } n \to \infty.$$

This means that for any $\varepsilon > 0$ we can find N such that $\sup_{t\in I} |f_n(t) - f(t)| < \varepsilon$ if $n \geq N$. In other words, the graph of f_n lies between the graphs of $f \pm \varepsilon$ for $t \in I$ and $n \geq N$.



We introduce the notation

$$\|f\| = \sup_{t\in I} |f(t)|$$

(the interval I is assumed to be fixed).

Theorem 3.1. *Suppose that $\{f_n\}$ is a sequence of continuous functions which converges uniformly to f on I. Then f is continuous.*

Proof. Let $x \in I$ and $\varepsilon > 0$ and choose N such that $\|f_n - f\| < \varepsilon/3$ when $n \geq N$. The function f_N is continuous, and hence there exists a $\delta > 0$ such that $|f_N(x) - f_N(y)| < \varepsilon/3$ when $|x - y| < \delta$. Thus,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq 2\|f_n - f\| + |f_N(x) - f_N(y)| \\ &< \varepsilon, \end{aligned}$$

when $|x - y| < \delta$. □

Theorem 3.2. *Suppose that I is a compact interval and that $\{f_n\}$ is a sequence of continuous functions which converges uniformly to f on I. Then*

$$\lim_{n\to\infty} \int_I f_n(t)\,dt = \int_I f(t)\,dt.$$

Proof. Note first that f is continuous and hence Riemann integrable by Theorem 3.1. From the triangle inequality for integrals, we obtain that

$$\left|\int_I f_n(t)\,dt - \int_I f(t)\,dt\right| \le \int_I |f_n(t) - f(t)|\,dt \le |I|\,\|f_n - f\|,$$

where $|I|$ denotes the length of the interval I. Since the right hand side converges to 0, the result follows. $\square$

Corollary 3.1. *Suppose that $\{f_n\}$ is a sequence of continuously differentiable functions on I. Assume that the sequence $\{f_n'\}$ converges uniformly to g and that $\{f_n(a)\}$ converges for some $a \in I$. Then $\{f_n\}$ converges pointwise, the limit function f is C^1 and $f' = g$.*

Proof. We have

$$f_n(t) = f_n(a) + \int_a^t f_n'(s)\,ds$$

for each n. By Theorem 3.2 the right hand side converges to

$$f(t) = f(a) + \int_a^t g(s)\,ds$$

for each $t \in I$. We leave it as an exercise for the reader to show that the convergence is in fact uniform on compact subsets of I. By the fundamental theorem of calculus we obtain that $f'(t) = g(t)$. $\square$

Example 3.1. The sequence $f_n(t) = 1/(n^2t^2+1)$ from Example 2.2 converges pointwise to

$$f(t) = \begin{cases} 0, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Since the limit function is not continuous at $t = 0$, the convergence is not uniform on any interval containing 0. This can also be seen directly, since

$$|f_n(t) - f(t)| = \frac{1}{n^2t^2+1}$$

when $t \neq 0$, and hence

$$\sup_{t\in I} |f_n(t) - f(t)| = 1$$

if $0 \in I$. On the other hand, if we consider an interval of the form $I = [a, \infty)$, with $a > 0$, we find that

$$\sup_I |f_n(t) - f(t)| = \frac{1}{n^2a^2+1} \to 0$$

as $n \to \infty$, so that the convergence is uniform.

4 Function series

Theorem 4.1 (Weierstrass' M-test)**.** *Let $\{f_k\}$ be a sequence of functions on I and assume that there exists a sequence $\{M_k\}$ of positive numbers such that $|f_k(t)| \le M_k$ for all k and*

$$\sum_{k=1}^{\infty} M_k < \infty.$$

The series $\sum_{k=1}^{\infty} f_k(t)$ converges absolutely and uniformly on I.

Proof. It follows from Proposition 1.2 (Comparison test) that the series converges absolutely for each t. Let $S(t)$ be the limit. Then

$$|S_n(t) - S(t)| = \left| \sum_{k=n+1}^{\infty} f_k(t) \right| \le \sum_{k=n+1}^{\infty} M_k.$$

Since the right hand side converges to 0 by assumption, it follows that $\|S_n - S\| \to 0$. □

Definition 4.1. A series of the form

$$\sum_{k=0}^{\infty} a_k (t - t_0)^k,$$

where the a_k are real (or complex) numbers is called a power series. The number

$$R = \sup\{r \ge 0 \colon \{a_k r^k\}_{k=0}^{\infty} \text{ is bounded}\}$$

is called the *radius of convergence* of the series.

The usual way in which power series appear is through Taylor expansion.

Theorem 4.2. *Assume that the power series*

$$\sum_{k=0}^{\infty} a_k (t - t_0)^k$$

has positive radius of convergence. The series converges uniformly and absolutely in the interval $[t_0 - r, t_0 + r]$ whenever $0 < r < R$ and diverges when $|t - t_0| > R$. The limit is infinitely differentiable and the series can be differentiated termwise.

Proof. If $|t - t_0| > R$ the sequence $a_k |t - t_0|^k$ is unbounded, so the series diverges. Consider an interval $|t - t_0| \le r$, where $r \in (0, R)$. Choose $r < \tilde{r} < R$. Then

$$|a_k (t - t_0)^k| \le |a_k| \tilde{r}^k \left(\frac{r}{\tilde{r}}\right)^k \le C \left(\frac{r}{\tilde{r}}\right)^k,$$

when $|t - t_0| \le r$ for some constant C since $\{a_k \tilde{r}^k\}$ is bounded. Since $r/\tilde{r} < 1$ the corresponding sequence converges and it follows from Weierstrass' M-test that the power series converges uniformly when $|t - t_0| \le r$ to some function S. If we formally differentiate termwise, we obtain the series

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}(t-t_0)^k.$$

This is a again a power series. The radius of convergence for the new series is the same as for the original series (show this!). It therefore follows from Corollary 3.1 that S is C^1 and that the power series can be differentiated termwise. Continuing inductively we find that $S \in C^\infty$. $\square$

Example 4.1.

- The power series defining the exponential function has infinite radius of convergence, $R = \infty$.
- The Taylor series for $1/(1-t)$, $\sum_{k=0}^{\infty} t^k$, has $R = 1$.

A power series with $R > 0$ can be extended to the complex plane. Let us assume for simplicity that $t_0 = 0$. We substitute $z \in \mathbb{C}$ for t and obtain the series

$$\sum_{k=0}^{\infty} a_k z^k.$$

The series converges uniformly (and absolutely) in the disc $\{z \in \mathbb{C} \colon |z| \le r\}$ in the complex plane, whenever $r < R$. It can also be differentiated with respect to z when $|z| < R$. The complex derivative is defined using difference quotients, just as in the real case. This is properly discussed in a course in complex analysis.

Exercises

Exercise 1. Let

$$f_n(t) = \arctan(nt).$$

Determine $\lim_{n\to\infty} f_n(t)$.

a) Is the convergence uniform on $\mathbb{R}$?

b) Is the convergence uniform on the interval $[1, \infty)$?

Exercise 2. What is the radius of convergence of the following power series?

a) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^k} t^k$

b) $\displaystyle\sum_{k=0}^{\infty} k!\, t^k$

c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}$

d) $\displaystyle\sum_{k=0}^{\infty} k t^{k^2}$

Exercise 3. The equation

$$t^2 x'' + t x' + (t^2 - m^2) x = 0$$

is called *Bessel's equation* of order m. Find a solution of Bessel's equation of order 0 satisfying $x(0) = 1$ by making a power series Ansatz. What is the radius of convergence of the power series?

Exercise 4. This exercise concerns the exponential function

$$\exp t = \sum_{k=0}^{\infty} \frac{t^k}{k!},$$

which solves the initial value problem

$$\begin{cases} x'(t) = x(t), \\ x(0) = 1. \end{cases} \tag{4.1}$$

a) Prove that $\exp t \cdot \exp(-t) = 1$ by differentiating the product on the left and using (4.1). In particular this shows that $\exp(t) \neq 0$ for all t and by continuity that $\exp t > 0$. Moreover, it shows that $\exp(-t) = 1/\exp t$.

b) Show that $\exp t$ is the only solution of (4.1) by differentiating $\exp(-t) \cdot x(t)$, where $x(t)$ is an arbitrary solution.

c) Prove that $\exp a \cdot \exp b = \exp(a+b)$ by differentiating $\exp(ta) \cdot \exp(tb) \cdot \exp(-t(a+b))$.

d) Let $e = \exp 1$. Prove that $\exp x = e^x$ for all x by using c) (you can first prove it for integers, then for rational numbers and finally for arbitrary real numbers by approximation with rational numbers).