

# Combinatorial game theory

**Combinatorial game theory (CGT)** is a mathematical theory that studies two-player games which have a *position* in which the players take turns changing in defined ways or *moves* to achieve a defined winning condition. CGT does not study games of chance (like poker). It restricts itself to games whose position is public to both players, and in which the set of available moves is also public (see perfect information). CGT principles can be applied to games like chess, checkers, Go, Arimaa, Hex, and Connect6 but these games are mostly too complicated to allow complete analysis (although the theory has had some recent successes in analyzing Go endgames).

Applying CGT to a *position* attempts to determine the optimum sequence of moves for both players until the game ends, and by doing so discover the optimum move in any position. In practice, this process is tortuously difficult unless the game is very simple.

CGT should not be confused with another mathematical theory, traditionally called "classical" game theory, used in the theory of economic competition and cooperation. Game theory includes games of chance, games of imperfect knowledge and games in which players move simultaneously, and they tend to represent real-life decision making situations.

## History

CGT arose in relation to the theory of impartial games, in which any play available to one player must be available to the other as well. One very important such game is nim, which can be solved completely. Nim is an impartial game for two players, and subject to the *normal play condition*, which means that a player who cannot move loses. In the 1930s, the Sprague-Grundy theorem showed that all impartial games are equivalent to heaps in nim, thus showing that major unifications are possible in games considered at a combinatorial level (in which detailed strategies matter, not just pay-offs).

In the 1960s, Elwyn R. Berlekamp, John H. Conway and Richard K. Guy jointly introduced the theory of a partisan game, in which the requirement that a play available to one player be available to both is relaxed. Their results were published in their book *Winning Ways for your Mathematical Plays* in 1982. However, the first book published on the subject was Conway's *On Numbers and Games*, also known as ONAG, which introduced the concept of surreal numbers and the generalization to games. *On Numbers and Games* was also a fruit of the collaboration between Berlekamp, Conway, and Guy.

Combinatorial games are generally, by convention, put into a form where one player wins when the other has no moves remaining. It is easy to convert any finite game with only two possible results into an equivalent one where this convention applies. One of the most important concepts in the theory of combinatorial games is that of the sum of two games, which is a game where each player may choose to move either in one game or the other at any point in the game, and a player wins when his opponent has no move in either game. This way of combining games leads to a rich and powerful mathematical structure.

John Conway states in ONAG that the inspiration for the theory of partisan games was based on his observation of the play in go endgames, which can often be decomposed into sums of simpler endgames isolated from each other in different parts of the board.



Mathematicians playing Konane at a Combinatorial game theory workshop (for technical content, see external link)

## Examples

The introductory text *Winning Ways* introduced a large number of games, but the following were used as motivating examples for the introductory theory:

- Blue-Red Hackenbush - At the finite level, this partisan combinatorial game allows constructions of games whose values are dyadic rational numbers. At the infinite level, it allows one to construct all real values, as well as many infinite ones which fall within the class of surreal numbers.
- Blue-Red-Green Hackenbush - Allows for additional game values that are not numbers in the traditional sense, for example, star.
- Domineering - Various interesting Games, such as hot games, appear in Domineering, due to the fact that there is sometimes an incentive to move, and sometimes not. This allows discussion of a game's temperature.
- Nim - An impartial game. This allows for the construction of the numbers. (It can also be seen as a green-only special case of Blue-Red-Green Hackenbush.)

The classic game Go was influential on the early combinatorial game theory, and Berlekamp and Wolfe subsequently developed an endgame and *temperature* theory for it (see references). Armed with this they were able to construct plausible Go endgame positions from which they could give expert Go players a choice of sides and then defeat them either way.

## Overview

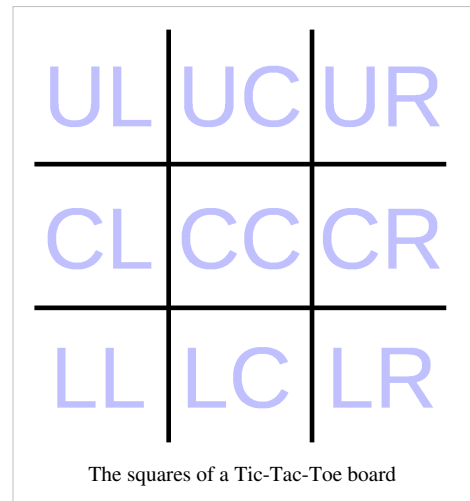
A game, in its simplest terms, is a list of possible "moves" that two players, called *left* and *right*, can make. The game position resulting from any move can be considered to be another game. This idea of viewing games in terms of their possible moves to other games leads to a recursive mathematical definition of games that is standard in combinatorial game theory. In this definition, each game has the notation  $\{L|R\}$ .  $L$  is the set of game positions that the left player can move to, and  $R$  is the set of game positions that the right player can move to; each position in  $L$  and  $R$  is defined as a game using the same notation.

Use tic-tac-toe as an example, label each of the nine boxes of the standard Tic-Tac-Toe board by *UL* for *Upper Left*, *CC* for *Center Center*, and *LR* for *Lower Right* (and so on), and suppose that each box may contain an X or O symbol. We use e.g. *XUL* to stand for the game position in which an X has been placed in the upper left box. Then, the initial position can be described in combinatorial game theory notation as

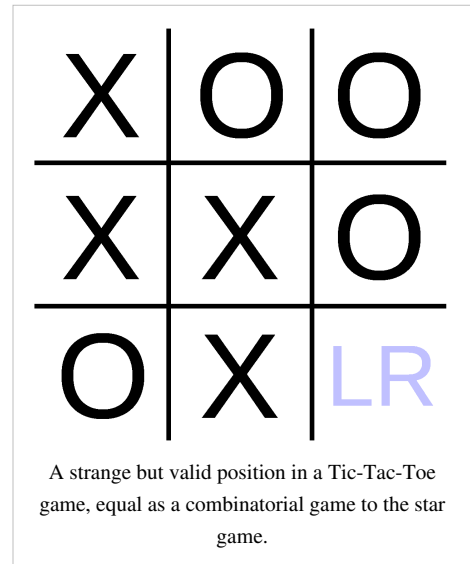
$$\{XUL, XUC, XUR, XCL, XCC, XCR, \dots | OUL, OUC, OUR, OCL, OCC, OCR, \dots\}.$$

Note that, in standard Tic-Tac-Toe play, the players alternate turns, but this alternation is handled implicitly by the definitions of combinatorial game theory rather than being encoded within the game states. The Tic-Tac-Toe game labeled *XUL* above could in turn be described by the following notation

$$XUL = \{XUL_XUC, XUL_XUR, XUL_XCL, \dots | XUL_OUC, \dots\}.$$



Moving on down the chain, eventually the game might come to this state (a very strange game indeed, but still valid):



$$XUL\_OUR\_XCC\_OCR\_XLC\_OLL\_XCL\_OUC = \{\{\}\}|\{\}\}.$$

The above game describes a scenario in which there is only one move left for either player, which is the Lower Right corner, and if either player makes that move, that player wins. The  $\{\}$  in each player's move list is called the zero game, and can actually be abbreviated 0. In the zero game, neither player has any valid moves; thus, the player whose turn it is when the zero game comes up automatically loses.

Additionally, the game which is labeled the rather complex "XUL\_OUR\_XCC\_OCR\_XLC\_OLL\_XCL\_OUC" above also has a much simpler notation, and is called the star game, which can also be abbreviated \*. In the star game, the only valid move leads to the zero game, which means that whoever's turn comes up during the star game automatically wins.

An additional type of game, not found in Tic-Tac-Toe, is a *loopy* game, in which a valid move of either *left* or *right* is a game which can then lead back to the first game. A game that does not possess such moves is called *nonloopy*.

## Game Abbreviations

### Numbers

Numbers represent the number of free moves, or the move advantage of a particular player. By convention positive numbers represent an advantage for Left, while negative numbers represent an advantage for Right. They are defined recursively with 0 being the base case.

$$0 = \{\}$$

$$1 = \{0\}, 2 = \{1\}, 3 = \{2\}$$

$$-1 = \{0\}, -2 = \{-1\}, -3 = \{-2\}$$

The zero game is a loss for the first player.

The sum of number games behaves like the integers, for example  $3 + -2 = 1$ .

## Star

*Star*, written as  $*$  or  $\{0|0\}$ , is a first-player win since either player must (if first to move in the game) move to a zero game, and therefore win.

$$* + * = 0$$

## Up

*Up*, written as  $\uparrow$ , is a position in combinatorial game theory.<sup>[1]</sup> In standard notation,  $\uparrow = \{0|*\}$ .

$$-\uparrow = \downarrow \text{ (down)}$$

*Up* is strictly positive ( $\uparrow > 0$ ), but is infinitesimal. *Up* is defined in *Winning Ways for your Mathematical Plays*.

## Down

*Down*, written as  $\downarrow$ , is a position in combinatorial game theory.<sup>[1]</sup> In standard notation,  $\downarrow = \{*\mid 0\}$ .

$$-\downarrow = \uparrow \text{ (up)}$$

*Down* is strictly negative ( $\downarrow < 0$ ), but is infinitesimal. *Down* is defined in *Winning Ways for your Mathematical Plays*.

## Nimbers

An impartial game is one where, at every position of the game, the same moves are available to both players. For instance, Nim is impartial, as any set of objects that can be removed by one player can be removed by the other. However, tic-tac-toe is not impartial, because a move by one player leaves a different position (more crosses) than a move by the other player (more circles). For any ordinal number, one can define an impartial game generalizing Nim in which, on each move, either player may replace the number with any smaller ordinal number; the games defined in this way are known as nimbers. The Sprague–Grundy theorem states that every impartial game is equivalent to a nimber.

## Notes

- [1] E. Berlekamp, J. H. Conway, R. Guy (1982). *Winning Ways for your Mathematical Plays*. Academic Press. ISBN 0-12-091101-9, ISBN 0-12-091102-7.

## References

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## External links

- List of combinatorial game theory links (<http://www.ics.uci.edu/~eppstein/cgt/>) at the homepage of David Eppstein
  - An Introduction to Conway's games and numbers (<http://arxiv.org/pdf/math/0410026>) by Dierk Schleicher and Michael Stoll
  - Combinational Game Theory terms summary (<http://senseis.xmp.net/?CGTPath#toc1>) by Bill Spight
  - Combinatorial Game Theory Workshop, Banff International Research Station, June 2005 ([http://www.pims.math.ca/birs/birspages.php?task=displayevent&event\\_id=05w5048](http://www.pims.math.ca/birs/birspages.php?task=displayevent&event_id=05w5048))
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