Differential Equations (MAT 231)

Lecture Notes

(Ordinary Linear Differential Equations with constant coefficients)

Mayamma Joseph (mayamma.joseph@christuniversity.in)

Basic Results and Terminology
1. The Operator D

The operator D stands for $\frac{d}{dx}$. In general $D^n = \frac{d^n}{dx^n}$. Hence we have

$$\frac{dy}{dx} = Dy$$

$$\frac{d^2y}{dx^2} = D^2y$$

$$\frac{d^3y}{dx^3} = D^3y$$

and for any positive integer n

$$\frac{d^ny}{dx^n} = D^ny.$$

The following results are true for the operator D.

1. $D^m + D^n = D^n + D^m$
2. $D^mD^n = D^nD^m$
3. $D(u + v) = Du + Dv$ where u and v are functions of x.
4. $(D - \alpha)(D - \beta) = (D - \beta)(D - \alpha)$ where α and β are constants.

$D^{-1} = \frac{1}{D}$ stands for integration. In general $D^{-n} = \frac{1}{D^n}$. Hence

$$\frac{1}{D}2x = \int 2xdx$$
$$= x^2$$

and

$$\frac{1}{D^2}x = \int \left(\int xdx\right)$$
$$= \int \left(\frac{x^2}{2}\right) dx$$
$$= \frac{x^3}{6}.$$

In general, $F(D)$ represents a function in the operator D. Some examples for functions of D are given below.

$$F(D) = D^3 + 3D^2 - 2D + 1$$
$$\phi(D) = 2D^2 - 3D + 5 \text{ and}$$
$$\psi(D) = D^3 + 8$$

We observe that the following results are true for functions involving the operator D.:

$$\phi(D)\psi(D) = \psi(D)\phi(D) \text{ and}$$
$$\phi(kD) = k\phi(D).$$

It is to be remembered that the expression

$$\frac{1}{F(D)}y = \phi(x)$$

implies that

$$y = F(D)\phi(x).$$

The operator D is used conveniently to represent the ODE's. For example,

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x + sinx + x^2$$

is represented as

$$\left(D^3 + 3D^2 - 2D + 1\right)y = e^x + sinx + x^2$$

The following results are useful for further discussions.

$\boxed{\textit{Result I: } F(D)e^{ax} = e^{ax}F(a)}$

$\boxed{\textit{Result II: } F(D)(e^{ax}V) = e^{ax}F(D+a)V; where \text{ V } is\ a\ function\ of \text{ x .}}$

$\boxed{\textit{Result III: } F(D^2)(cosax) = F(-a^2)cosax}$

$\boxed{\textit{Result IV: } F(D^2)(sinax) = F(-a^2)sinax}$

All these results can be easily verified. For example, let us consider $F(D)e^{ax}$ when $F(D) = 3D^3 + 2D^2 + D + 5$. Then

$$\begin{aligned}
F(D)e^{ax} &= \left(3D^3 + 2D^2 + D + 5\right)e^{ax} \\
&= 3D^3\left(e^{ax}\right) + 2D^2\left(e^{ax}\right) + D\left(e^{ax}\right) + 5e^{ax} \\
&= 3a^3e^{ax} + 2a^2e^{ax} + ae^{ax} + 5e^{ax} \\
&= e^{ax}\left(3a^3 + 2a^2 + a + 5\right) \\
&= e^{ax}F(a).
\end{aligned}$$

Hence in general we have

$$\boxed{F(D)e^{ax} = e^{ax}F(a).} \text{——(Result I)}$$

We will also verify Result II when $F(D)$ is a polynomial of degree 3 in D. Let

$$F(D) = a_3D^3 + a_2D^2 + a_1D + a_0.$$

Then

$$\begin{aligned} F(D)e^{ax}V &= \left(a_3D^3 + a_2D^2 + a_1D + a_0\right) e^{ax}V \\ &= a_3D^3\left(e^{ax}V\right) + a_2D^2\left(e^{ax}V\right) + a_1D\left(e^{ax}V\right) + a_0e^{ax}V \end{aligned}$$

Using Leibniz's Theorem we get

$$\begin{aligned} D^3\left(e^{ax}V\right) &= a^3e^{ax}V + 3a^2e^{ax}DV + 3ae^{ax}D^2V + e^{ax}D^3V \\ &= e^{ax}\left(a^3 + 3a^2D + 3aD^2 + D^3\right)V \\ &= e^{ax}\left(D+a\right)^3 V \end{aligned}$$

Similarly we have

$$D^2\left(e^{ax}V\right) = e^{ax}\left(D+a\right)^2 V$$

and

$$D\left(e^{ax}V\right) = e^{ax}\left(D+a\right)V$$

Hence

$$F(D)e^{ax}V = e^{ax}\left[a_3(D+a)^3 + a_2(D+a)^2 + a_1(D+a) + a_0\right]V$$

Thus we have

$$\boxed{F(D)e^{ax}V = e^{ax}F(D+a)V.}\text{——(Result II)}$$

Try verifying the other results yourself.

2. Linear Equations with constant coefficients

General form:

$$\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_0y = \phi(x) \tag{1}$$

where $a_{n-1}, a_{n-2}, \ldots, a_0$ are constants and $\phi(x)$ is a function of x. Using the operator D, the above equation can be written as

$$D^ny + a_{n-1}D^{n-1}y + \ldots + a_0y = \phi(x)$$

That is

$$\left(D^n + a_{n-1}D^{n-1} + \ldots + a_0\right)y = \phi(x).$$

or

$$F(D)y = \phi(x) \tag{2}$$

where $F(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_0$.

Case 1: F(D)y=0.

First let us consider the homogeneous differential equation

$$F(D)y = 0 \ or \ \left(D^n + a_{n-1}D^{n-1} + \ldots + a_0\right) y = 0 \tag{3}$$

Our aim is to find the general solution of

$$F(D)y = 0$$

First note that $F(D)$ can be considered as a polynomial in D of degree n. Therefore the equation $F(D) = 0$ will have n roots, say $m_1, m_1, \ldots, m_n$. Hence we have

$$F(D) = (D - m_1)(D - m_2) \ldots (D - m_n).$$

Thus Equation (3) can be expressed as

$$(D - m_1)(D - m_2) \ldots (D - m_n)y = 0. \tag{4}$$

That means any y that satisfies the equation $(D - m_i)y = 0$, for $i = 1, 2, \ldots, n$ satisfies Equation (3).

In general let us consider the equation

$$(D - m)y = 0 \text{ or } \frac{dy}{dx} - my = 0$$

This is a first order ODE in variable separable form. Separating the variables and solving, we get $y = ke^{mx}$ where k is a constant.This being the general solution of the equation of the form $(D - m)y = 0$, we conclude that $y = e^{mx}$ is definitely a solution of Equation (3).Let us therefore put $y = e^{mx}$ in (3). First we note that when $y = e^{mx}$ we have

$$\begin{aligned}
F(D)y &= (D^n + a_{n-1}D^{n-1} + \ldots a_n)y \\
&= (D^n + a_{n-1}D^{n-1} + \ldots a_n)e^{mx} \\
&= D^n(e^{mx}) + a_{n-1}D^{n-1}(e^{mx}) + \ldots + a_0e^{mx} \\
&= m^n e^{mx} + a_{n-1}m^{n-1}e^{mx} + \ldots + a_0e^{mx} \\
&= e^{mx}(m^n + a_{n-1}m^{n-1} + \ldots + a_0)
\end{aligned}$$

Hence $F(D)y = 0$ implies that

$$e^{mx}(m^n + a_{n-1}m^{n-1} + \ldots + a_0) = 0.$$

From this it follows that $(m^n + a_{n-1}m^{n-1} + \ldots + a_0) = 0$ or

$$F(m) = 0 \tag{5}$$

Now $F(m) = 0$ being a polynomial in m of degree n, there would be n roots; say $m_1, m_2, \ldots, m_n$ for Equation (5). Using these n roots we will get the general solution of Equation (3). We refer to Equation (5) as the *Auxiliary Equation(A.E)*.

Now let us consider the various cases depending upon the nature of the roots of $F(m) = 0$..

Case 1:All the n roots $m_1, m_2, \ldots, m_n$ are distinct.
In this case $y = e^{m_1 x}, y = e^{m_2 x}, \ldots, y = e^{m_n x}$ are n linearly independent solutions of Equation (3). Therefore the general solution of Equation(3)could be written as

$$G.S. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}.$$

Case 2: Some of the roots $m_1, m_2, \ldots, m_n$ are repeated.
Let us assume that $m_1 = m_2 = m$, say. Then

$$(D - m_1)(D - m_2)y = 0 \Rightarrow (D - m)(D - m)y = 0 \tag{6}$$

Let

$$(D - m)y = v \tag{7}$$

Then Equation (6) can be written as $(D - m)v = 0$.As we have seen earlier, the general solution of this equation would be

$$v = c_1 e^{mx}.$$

Substituting this value of v in equation (9) we get

$$(D - m)y = c_1 e^{mx} \tag{8}$$

That is

$$\frac{dy}{dx} - my = c_1 e^{mx} \tag{9}$$

This is a first order linear equation with $I.F. = e^{\int -mdx} = e^{-mx}$. Hence its solution is

$$ye^{-mx} = \left[\int c_1 e^{mx} e^{-mx} dx\right] + c_2 = c_1 x + c_2. \tag{10}$$

Therefore

$$y = (c_1 x + c_2)e^{mx}. \tag{11}$$

In general if m is a root repeated r times $(r \leq n)$ then $y = (c_1 x^{r-1} + c_2 x^{r-2} + \ldots + c_r)e^{mx}$ will be the solution corresponding to these r repeated roots and the required general solution would be

$$y = (c_1 x^{r-1} + c_2 x^{r-2} + \ldots + c_r)e^{mx} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}.$$

Case 3: Some of the roots $m_1, m_2, \ldots, m_n$ are complex.

First we observe that complex roots appear in conjugate pairs. Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ be a pair of complex roots. Then the solution corresponding to these roots is given by

$$\begin{aligned} k_1 e^{(\alpha+i\beta)x} + k_2 e^{(\alpha-i\beta)x} &= e^{\alpha x}\left(k_1 e^{i\beta x} + k_2 e^{-i\beta x}\right) \\ &= e^{\alpha x}\left(k_1(cos\beta x + isin\beta x) + k_2(cos\beta x - isin\beta x)\right) \\ &= e^{\alpha x}\left(c_1 cos\beta x + c_2 sin\beta x\right) \end{aligned}$$

where $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$.

Thus the general solution for $F(D)y = 0$ would be

$$y = e^{\alpha x}\left(c_1 cos\beta x + c_2 sin\beta x\right) + c_3 e^{m_3 x} + c_4 e^{m_4 x} \ldots + c_n e^{m_n x}.$$

Case 2: $F(D)y = \phi(x)$.

To find the general solution of $F(D)y = \phi(x)$ we proceed as follows.
Suppose $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of Equation (3) , then their linear combination $c_1y_1 + c_2y_2 + \ldots + c_ky_k$ also is a solution it. Therefore if Equation (3) is of order n, then $u = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ can be considered as a general solution of Equation (3) and hence we get

$$F(D)u = 0. \tag{12}$$

Suppose we are successful in finding a particular integral or particular solution $y = v$ of Equation (2). Then we have

$$F(D)v = \phi(x) \tag{13}$$

so that

$$\begin{aligned} F(D)(u+v) &= F(D)u + F(D)v \\ &= 0 + \phi(x) \\ &= \phi(x). \end{aligned}$$

That means $y = u + v$ is a general solution of $F(D)y = \phi(x)$ as it contains n arbitrary constants. The solution u is called the *Complementary Function (C.F.)* and the solution v is called a *Particular Integral (P.I.)*.
Hence the General Solution of $F(D)y = \phi(x)$ is of the form

$$General\ Solution = Complementary\ Function + Particular\ Integral.$$

Or

$$G.S. = C.F. + P.I.$$

where $C.F.$ is the solution of $F(D)y = 0$ and $P.I. = \frac{1}{F(D)}\phi(x)$.

Now let us proceed to find the Particular integral for $F(D)y = \phi(x)$.

Method of finding the Particular Integral (P.I) for $F(D)y = \phi(x)$

First we consider five special cases when $\phi(x)$ is a standard function of x.

Case 1:$\phi(x) = e^{mx}V$ where V is a function of x.
Case 2:$\phi(x) = e^{mx}$
Case 3:$\phi(x) = cosmx$
Case 4:$\phi(x) = sinmx$ and
Case 5:$\phi(x) = x^m$

Case 1:$\phi(x) = e^{mx}V$ where V is a function of x.

First let us recall the Result II pertaining to the operator D that

$$F(D)(e^{mx}V) = e^{mx}F(D+m)V.$$

Therefore by considering $\dfrac{1}{F(D+m)}V$ in place of V and using Result II we get

$$F(D)\left\{e^{mx}.\left[\frac{1}{F(D+m)}V\right]\right\} = e^{mx}F(D+m)\frac{1}{F(D+m)}V = e^{mx}V.$$

Hence

$$\boxed{\frac{1}{F(D)}e^{mx}V = e^{mx}.\frac{1}{F(D+m)}V} \tag{14}$$

Case 2:$\phi(x) = e^{mx}$.

Result I on operator D is that

$$F(D)e^{mx} = F(m)e^{mx}.$$

Operating with $\frac{1}{F(D)}$ on both sides of the above result we get

$$\frac{1}{F(D)}.F(D)e^{mx} = \frac{1}{F(D)}.F(m)e^{mx}$$

.

That is,

$$e^{mx} = F(m).\frac{1}{F(D)}e^{mx}$$

or

$$\boxed{\frac{1}{F(D)}e^{mx} = \frac{e^{mx}}{F(m)}} \tag{15}$$

provided $F(m) \neq 0$.

If $F(m) = 0$, then $F(D) = (D-m)^r\psi(D)$ where $\psi(m) \neq 0$.

Therefore

$$\frac{1}{F(D)}e^{mx} = \frac{1}{(D-m)^r\psi(D)}e^{mx} = \frac{1}{(D-m)^r}\left[\frac{1}{\psi(m)}e^{mx}\right]$$

as $\psi(m) \neq 0$.

Now

$$\begin{aligned}\frac{1}{(D-m)^r}\left[\frac{1}{\psi(m)}e^{mx}\right] &= \frac{1}{\psi(m)}.\frac{1}{(D-m)^r}.(e^{mx}.1)\\ &= \frac{1}{\psi(m)}.e^{mx}.\frac{1}{(D+m-m)^r}.1 \text{ (using Result II)}\\ &= \frac{e^{mx}}{\psi(m)}.\frac{1}{D^r}.1\\ &= \frac{e^{mx}}{\psi(m)}.\frac{x^r}{r!}\end{aligned}$$

That is, when $F(D) = (D-m)^r\psi(D)$ with $\psi(m) \neq 0$,

$$\boxed{\frac{1}{F(D)}e^{mx} = \frac{e^{mx}}{\psi(m)}.\frac{x^r}{r!}} \tag{16}$$

Cases 3 and 4:$\phi(x) = cosmx$ or $\phi(x) = sinmx$

We know from Result III on operator D that

$$F(D^2)cosmx = F(-m^2)cosmx.$$

As in the above case, operating on both sides of the above result with $\frac{1}{F(D^2)}$ we get

$$\begin{aligned}\frac{1}{F(D^2)}.F(D^2)cosmx &= \frac{1}{F(D^2)}.F(-m^2)cosmx \\ &= F(-m^2).\frac{1}{F(D^2)}cosmx\end{aligned}$$

so that

$$cosmx = F(-m^2).\frac{1}{F(D^2)}cosmx$$

Hence when $F(-m^2) \neq 0$,

$$\boxed{\frac{1}{F(D^2)}cosmx = \frac{cosmx}{F(-m^2)}} \tag{17}$$

and similarly we can prove that

$$\boxed{\frac{1}{F(D^2)}sinmx = \frac{sinmx}{F(-m^2)}} \tag{18}$$

It is to be noted the above results are applicable only when $F(D)$ is expressed in terms of powers of D^2. But in general $F(D)$ contains both odd and even powers of D. In such cases we will proceed as follows. Let

$$F(D) = F_1(D) + F_2(D)$$

where $F_1(D)$ consists of even powers of D and $F_2(D)$ consists of odd powers of D. Then we have $F(D) = F_1(D^2) + DF_3(D^2)$. Replacing D^2 by $-m^2$ we get

$$P.I. = \frac{1}{p + qD}$$

where $p = F_1(-a^2) \neq 0$ and $q = F_3(-a^2) \neq 0$.

Now

$$\frac{1}{p+qD}sinmx = \frac{(p-qD)}{(p-qD)(p+qD)}sinmx$$

which can be evaluated.

When $F(-m^2) = 0$ we need to treat the problem in a different way. For the time being we will illustrate two special cases:

$$\frac{1}{D^2+m^2}sinmx$$

and

$$\frac{1}{D^2+m^2}cosmx.$$

First let us consider $\frac{1}{D^2+m^2} sinmx$. Using Euler's formula

$$e^{i\theta} = cos\theta + isin\theta$$

we observe that

$$e^{imx} = cosmx + isinmx$$

Hence

$$sinmx = \text{ Imaginary part of } e^{imx}$$

Using this result we find that

$$\begin{aligned}
\frac{1}{D^2+m^2} sinmx &= \frac{1}{(D+im)(D-im)} sinmx \\
&= \frac{1}{(D+im)(D-im)} \text{Imaginary part of } e^{imx} \\
&= \text{Imaginary part of } \frac{1}{(D+im)(D-im)} e^{imx} \\
&= \text{Imaginary part of } \frac{1}{(im+im)} \frac{x}{1!} e^{imx} \\
&= \text{Imaginary part of } \frac{1}{(2im)} x(cosmx + isinmx) \\
&= \text{Imaginary part of } \frac{x}{(2im)} cosmx + \frac{x}{(2im)} isinmx \\
&= \text{Imaginary part of } \frac{x(-i)}{(2im)(-i)} cosmx + \frac{x}{(2m)} sinmx \\
&= \text{Imaginary part of } \frac{(-i)x}{2m} cosmx + \frac{x}{2m} sinmx \\
&= \left(\frac{-x}{2m}\right) cosmx
\end{aligned}$$

$$\boxed{\frac{1}{D^2+m^2} sinmx = \frac{-x}{2m} cosmx = \frac{x}{2} \int sinmxdx} \tag{19}$$

Similarly we can prove that

$$\boxed{\frac{1}{D^2+m^2} cosmx = \frac{x}{2m} sinmx = \frac{x}{2} \int cosmxdx} \tag{20}$$

Case 5:$\phi(x) = x^m$

We have to evaluate $\frac{1}{F(D)} x^m$. In this case we take out the lowest degree term in D from $F(D$ so that we can express $F(D) = \frac{1}{D^k}[1 \pm \psi(D)]^{-1}$. The term $[1 \pm \psi(D)]^{-1}$ could be expanded as a series in terms of D using one of the following results.

1. $(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$
2. $(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$

3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots$
4. $(1-D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots$

It is to be remembered that $\frac{1}{D}$ stands for integration and that $D^n x^m = 0$ when $n > m$.

Let us now summarize the above results to solve linear ODE with constant coefficients $F(D)y = \phi(x)$ as follows.

Working Rule:

Step I: Solve the *Auxiliary Equation* $F(m) = 0$ of the given differential equation $F(D)y = 0$.

Step II: Based on the nature of the roots of the Auxiliary Equation, find the *Complementary Function (C.F.)* using the following rules:

- If all the roots are real and distinct, then

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}.$$

- If m is root that is repeated r times, say $m_1 = m_2 = \ldots m_r = m$; then

$$C.F. = (c_1 x^{r-1} + c_2 x^{r-2} + \ldots + c_r)e^{mx} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}.$$

- If $\alpha \pm i\beta$ is a pair of complex roots and the remaining roots are $m_3, m_4, \ldots, m_n$, then

$$C.F. = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}.$$

Step III: Find the *Particular Integral (P.I.)* using the following rules:

- $\boxed{\frac{1}{F(D)} e^{mx} = \frac{e^{mx}}{F(m)}}$; where $F(m) \neq 0$.
- $\boxed{\frac{1}{F(D)} e^{mx} = \frac{1}{(D-m)^r \psi(D)} e^{mx} = \frac{1}{(D-m)^r} \left[\frac{1}{\psi(m)} e^{mx}\right]}$;where m is a root repeated r times and $\psi(m) \neq 0$.
- $\boxed{\frac{1}{F(D^2)} cosmx = \frac{cosmx}{F(-m^2)}}$; provided $F(-m^2) \neq 0$
- $\boxed{\frac{1}{F(D^2)} sinmx = \frac{sinmx}{F(-m^2)}}$; provided $F(-m^2) \neq 0$
- $\boxed{\frac{1}{D^2 + m^2} sinmx = \frac{-x}{2m} cosmx = \frac{x}{2} \int sinmx dx}$
- $\boxed{\frac{1}{D^2 + m^2} cosmx = \frac{x}{2m} sinmx = \frac{x}{2} \int cosmx dx}$
- $\boxed{\frac{1}{F(D)} e^{mx} V = e^{mx} . \frac{1}{F(D+m)} V}$

- For P.I. $\frac{1}{F(D)}x^m$, expand the term $\frac{1}{F(D)}$ by re-arranging appropriately and using the following results:
  1. $(1-D)^{-1} = 1 + D + D^2 + D^3 + \ldots$
  2. $(1+D)^{-1} = 1 - D + D^2 - D^3 + \ldots$
  3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots$
  4. $(1-D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots$

Step IV: Write the final answer as General Solution=Complementary Equation +Particular Integral. That is

$$\textbf{G.S.=C.F.+P.I.}$$

General Method to find the Particular Integral

What we have discussed so far are methods to determine particular integrals for standard functions like e^{mx}; $sinmx$; $cosmx$ and x^m. The procedure we have followed is more convenient to solve problems where $\phi(x)$ is one of the standard functions mentioned above.

In general, to find the particular integral for any function X of x, we proceed as follows.

Remember that our objective is to determine

$$\frac{1}{F(D)}X.$$

First we will evaluate

$$\frac{1}{(D-m)}X$$

where $(D-m)$ is a factor of $F(D)$.
Let $y = \frac{1}{(D-m)}X$. Then

$(D-m)y = X$ or,

$(\frac{d}{dx} - m)y = X$

That is, $\frac{dy}{dx} - my = X$;which is a linear equation with $I.F. = e^{\int -mdx} = e^{-mx}$

Hence we get $y.e^{-mx} = \int X.e^{-mx}dx$

Thus $y = e^{mx} \int X.e^{-mx}dx$

That is, $\frac{1}{(D-m)}X = e^{mx} \int X.e^{-mx}dx$

Thus we have

$$\boxed{\frac{1}{(D-m)}X = e^{mx} \int X.e^{-mx}dx}. \tag{21}$$

Similarly,

$$\boxed{\frac{1}{(D+m)}X = e^{-mx}\int X.e^{mx}dx} \tag{22}$$

*Note:*When

$$\frac{1}{F(D)} = \frac{1}{(D-m_1)(D-m_2)\dots(D-m_n)}$$

we can resolve it into partial fractions so that

$$\begin{aligned} P.I. &= \frac{1}{F(D)}X \\ &= \left[\frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + \dots + \frac{An}{(D-m_n)}\right]X \\ &= \frac{A_1}{(D-m_1)}X + \frac{A_2}{(D-m_2)}X + \dots + \frac{An}{(D-m_n)}X \\ &= A_1e^{m_1x}\int Xe^{-m_1x} + A_2e^{m_2x}\int Xe^{-m_2x} + \dots + A_ne^{m_nx}\int Xe^{-m_nx}. \end{aligned}$$

Exercise

Solve the Following:

1. $(D^3 - 13D - 12)y = 0.$
2. $(D^3 + 6D^2 + 11D + 6)y = 0.$
3. $(D^3 - 2D^2 - 4D + 8)y = 0.$
4. $(D^4 - a^4)y = 0.$
5. $(D^2 + 4D + 3)y = e^{-2x}.$
6. $(2D^3 - 3D^2 + 1)y = e^x + 1.$
7. $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}.$
8. $(2D^2 + 2D + 3)y = x^2 + 2x - 1.$
9. $(D^2 + D + 1)y = sin2x.$
10. $(D^3 - 3D^2 + 4D - 2)y = e^x + cosx.$
11. $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x.$
12. $(D^2 - 2D + 1)y = x^3e^{3x}.$
13. $(D^2 - 7D - 6)y = e^{2x}x^2.$
14. $(D^3 - 2D + 4)y = e^xcosx.$
15. $(D^3 - 3D^2 - 6D + 8)y = xe^{-3x}.$
16. $(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x} + 4sinx.$