Christ University
Dept. of Mathematics

B.Sc. II Semester- MAT 131

Practice Problems: Set-III

A. Partial Differentiation

1. Find the first order partial derivatives with respect to x and y for the following functions

   a. $U = x^2 + y^2$
   b. $U = y/x$ and $U = x/y$
   c. $U = \sqrt{xy}$
   d. $U = e^{xy}$
   e. $U = \sqrt{\left(x^2 + y^2\right)}$ and $u = 1/\sqrt{\left(x^2 + y^2\right)}$
   f. $U = \sin(xy)$
   g. $U = \cos^{-1}(x^3 + y^3)$
   h. $U = \tan^{-1}(y/x)$
   i. $U = \cos^{-1}(x/y)$
   j. $U = \log\sqrt{\left(x^2 + y^2\right)}$

2. Show that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ for the following functions.

   a. $U = ax^2 + 2hxy + by^2$
   b. $U = \tan^{-1}\left(\dfrac{x}{y}\right)$
   c. $U = \tan^{-1}(y/x)$
   d. $U = x \sin y + y \sin x$
   e. $U = x^y + y^x$
   f. $U = \log\left(\dfrac{x^2 + y^2}{xy}\right)$
   g. $U = x \tan y + y \tan x$
   h. $U = e^{ax} \sin by$
   i. $U = \sin^{-1}(y/x)$

j. $U = \log(x^2 + y^2)$
k. $U = \log[\tan(y/x)]$
l. $U = ye^{-(x/y)}$
m. $U = (1-2xy+y^2)^{-1/2}$
n. $U = \log(e^x+e^y)$
o. $U = \dfrac{x^2+y^2}{x+y}$
p. $U = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$

3. If $u = \tan^{-1}\left(\dfrac{y}{x}\right)$ show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.
4. If $x = r\cos\theta$ and $y = r\sin\theta$ show that
   a. $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1.$
   b. $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right].$
   c. $\left(\dfrac{\partial^2 r}{\partial x^2}\right)\left(\dfrac{\partial^2 r}{\partial y^2}\right) = \left(\dfrac{\partial^2 r}{\partial x \partial y}\right)^2.$
5. If $U = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, *show that* $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$.
6. If $z = \left(1 - 2xy + y^2\right)^{-1/2}$ show that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = y^2 z^3$.
7. If $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$ verify whether $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$.
8. If $u = e^x \sin y$ show that $u_{xx} + u_{yy} = 0$.
9. If $U = \log(x^2 + y^2 + z^2)$, show that $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$.
10. if $u = \tan(y+ax) + (y-ax)^{3/2}$, show that $\dfrac{\partial^2 u}{\partial x^2} - a^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

11. If u = $x^2(y-z) + y^2(z-x)+ z^2(x-y)$, show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

12. If u =$(x^2 + y^2+z^2)^{-1/2}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u$.

13. If u = $\log\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$.

14. If u = $\log(x^3+y^3+z^3-3xyz)$, show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$.

15. If u = $\log(x^3+y^3-x^2y-xy^2)$, show that $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=-\left(x+y\right)^{-2}$.

16. If u = sinx coshy and v = cosx sinhy verify $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$.

17. If u = e^x(x cosy – y sin y), show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

18. If u = $\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$, then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$.

19. If u = log (tan x + tan y), show that $\left(\sin 2x\right)\frac{\partial u}{\partial x}+\left(\sin 2y\right)\frac{\partial u}{\partial y}=2$.

20. If f(x, y) = tan ($\tan^{-1}x + \tan^{-1}y$), show that $(1 + x^2)\ \frac{\partial f}{\partial x}=(1+y^2)\frac{\partial f}{\partial y}$.

21. If u = e^{xy} show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$.

22. If u = $\sqrt{x^2+y^2+z^2}$, show that $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$.

23. If u = x f(x+y) + y g(x+y), show that $\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=o$.

24. If u = $\varphi\left(y+ax\right)+\phi(y-ax)$ show that $\frac{\partial^2 u}{\partial x^2}=a^2\frac{\partial^2 u}{\partial y^2}$.

25. If U = $\dfrac{xy}{x+y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0$.

26. If Z = cos(x+y) + sin (x-y), show that $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial y^2}$.

27. If U = $\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$ show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

28. If $u=\tan^{-1}\left(\dfrac{xy}{\sqrt{1+x^2+y^2}}\right)$, show that $\dfrac{\partial^2 u}{\partial x\partial y}=\left(1+x^2+y^2\right)^{-3/2}$.

29. If U = $\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$.

30. If U = log $\sqrt{x^2+y^2+z^2}$ show that $(x^2+y^2+z^2)$ $\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)=1$.

31. If $u=e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x\partial y\partial z}=(1+3xyz+x^2y^2z^2)$.

32. If u = f(r) where $r^2 = x^2 + y^2$, show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=f''(r)+\dfrac{1}{r}f'(r)$.

B. Euler’s Theorem on Homogeneous Functions

1. State and prove Euler’s theorem on homogeneous functions on two variables.
2. Verify Euler’s theorem for the following functions.
   a. $U = ax^2-2hxy + by^2$
   b. $u=\dfrac{x^5+y^5}{x^2-y^2}$
   c. $U = x^4 \log (y /x)$
   d. $u=x^n\sin\left(\dfrac{y}{x}\right)$
   e. $u=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$
   f. $U = x^3-2x^2y + 3xy^2+y^3$

g. $u = \dfrac{x^2 + y^2}{x - y}$

h. $u = \dfrac{x^{3/2} - y^{3/2}}{x + y}$

i. $u = \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

j. $u = \left(x^{1/2} + y^{1/2}\right)\left(x^n + y^n\right)$

3. If u = $\sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ = tan u.

4. If $V = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} + \dfrac{1}{2}\cot V = 0$.

5. If u = $\tan^{-1}\left(\dfrac{x^3 + y^3}{x + y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ = sin 2u.

6. If $z = xyf\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$.

7. If $z = f\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = o$.

8. If $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

9. If $z = \sin^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}\tan z$.

10. If u = $\tan^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ = ½ sin 2u.

11. If u = log $\left(\dfrac{x^4 + y^4}{x - y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ =3.

12. If u = $\sin^{-1}(x/y) + \tan^{-1}(y/x)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

13. If $f(x,y) = \sqrt{x^2 - y^2}\sin^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = f(x,y)$.

14. If $u = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

15. If u = $\frac{xy}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

16. If $u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

17. If $u = \log\frac{x^4+y^4+x^2y^2}{x+y+\sqrt{xy}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

18. State and prove extension of Euler's theorem on homogeneous functions

19. If u = $\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$.

20. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}.$$

21. If u = $\frac{xy}{x+y}$ show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

22. If $u = \sin^{-1}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$ show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right).$$

23. If $u = \frac{xy}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\frac{\partial u}{\partial x}$.

24. If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

25. Find $\frac{dy}{dx}$ using partial differentiation in the following cases:

a. $x^3 - 3axy + y^3 = 0$

b. $x^y = y^x$

c. $2x^2 + 5xy + 2y^2 = 1$

d. $y^x = x$

e. $\sin y = x \sin(a+y)$

f. $e^x + e^y = 2xy$

g. $x^3 + 3x^2 y + 6xy^2 + y^3 = 1$

h. $(\sin x)^y - y^{\sin x} = a$

i. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

26. If $u = f(r,s,t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

27. If $u = f(x-y, y-z, z-x)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

28. If $u = \varphi(y+ax) + \psi(y-ax)$ show that $\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}$.