FUZZY RELATIONS, FUZZY GRAPHS, AND FUZZY ARITHMETIC

INTRODUCTION

3 Important concepts in fuzzy logic

- **Fuzzy Relations**
- **Fuzzy Graphs**

} Form the foundation of fuzzy rules

- **Extension Principle --** basis of fuzzy Arithmetic

- This is what makes a fuzzy system tick!

Fuzzy Relations

- Generalizes classical relation into one that allows partial membership
  - Describes a relationship that holds between two or more objects
    - Example: a fuzzy relation "Friend" describe the degree of friendship between two person (in contrast to either being friend or not being friend in classical relation!)

Fuzzy Relations

- A fuzzy relation $\tilde{R}$ is a mapping from the Cartesian space X x Y to the interval [0,1], where the strength of the mapping is expressed by the membership function of the relation $\mu_{\tilde{R}}(x,y)$
- The “strength” of the relation between ordered pairs of the two universes is measured with a membership function expressing various “degree” of strength [0,1]

Fuzzy Cartesian Product

Let

$\tilde{A}$ be a fuzzy set on universe X, and

$\tilde{B}$ be a fuzzy set on universe Y, then

$$\tilde{A} \times \tilde{B} = \tilde{R} \subset X \times Y$$

Where the fuzzy relation R has membership function

$$\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A} x \tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

Fuzzy Cartesian Product: Example

Let

$\tilde{A}$ **defined on a universe of three discrete temperatures, X = {x_1,x_2,x_3}, and**

$\tilde{B}$ **defined on a universe of two discrete pressures, Y = {y_1,y_2}**

Fuzzy set $\tilde{A}$ represents the "ambient" temperature and

Fuzzy set $\tilde{B}$ the "near optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\left.\begin{aligned}\tilde{A} &= \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}\\ &\textit{and}\\ \tilde{B} &= \frac{0.3}{y_1} + \frac{0.9}{y_2}\end{aligned}\right\}\quad \tilde{A} \times \tilde{B} = \tilde{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy Composition

Suppose

$\tilde{R}$ **is a fuzzy relation on the Cartesian space X x Y,**

$\tilde{S}$ **is a fuzzy relation on the Cartesian space Y x Z, and**

$\tilde{T}$ **is a fuzzy relation on the Cartesian space X x Z; then fuzzy max-min and fuzzy max-product composition are defined as**

$$\tilde{T} = \tilde{R} \circ \tilde{S}$$

$$\max - \min$$

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\tilde{R}}(x,y) \wedge \mu_{\tilde{S}}(y,z))$$

$$\max - product$$

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\tilde{R}}(x,y) \bullet \mu_{\tilde{S}}(y,z))$$

Fuzzy Composition: Example (max-min)

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.5 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Using max-min composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_1, z_1) &= \bigvee_{y \in Y} (\mu_{\tilde{R}}(x_1, y) \wedge \mu_{\tilde{S}}(y, z_1)) \\ &= \max[\min(0.7, 0.9), \min(0.5, 0.1)] \\ &= 0.7 \end{aligned}\right\} \quad \tilde{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Fuzzy Composition: Example (max-Prod)

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.5 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Using max-product composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_2, z_2) &= \bigvee_{y \in Y} (\mu_{\tilde{R}}(x_2, y) \bullet \mu_{\tilde{S}}(y, z_2)) \\ &= \max[(0.8, 0.6), (0.4, 0.7)] \\ &= 0.48 \end{aligned}\right\} \quad \tilde{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} .63 & .42 & .25 \\ .72 & .48 & .20 \end{bmatrix} \end{array}$$

Application: Computer Engineering

Problem: In computer engineering, different logic families are often compared on the basis of their power-delay product. Consider the fuzzy set $\tilde{F}$ of logic families, the fuzzy set $\tilde{D}$ of delay times(ns), and the fuzzy set $\tilde{P}$ of power dissipations (mw).

If $\tilde{F} = \{NMOS, CMOS, TTL, ECL, JJ\}$,

$\tilde{D} = \{0.1, 1, 10, 100\}$,

$\tilde{P} = \{0.01, 0.1, 1, 10, 100\}$

Suppose $\tilde{R}_1 = \tilde{D} \times \tilde{F}$ and $\tilde{R}_2 = \tilde{F} \times \tilde{P}$

$$\tilde{R}_1 = \begin{array}{c} \\ 0.1 \\ 1 \\ 10 \\ 100 \end{array} \begin{array}{c} \begin{array}{ccccc} N & C & T & E & J \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & .6 & 1 \\ 0 & .1 & .5 & 1 & 0 \\ .4 & 1 & 1 & 0 & 0 \\ 1 & .2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and

$$\tilde{R}_2 = \begin{array}{c} \\ N \\ C \\ T \\ E \\ J \end{array} \begin{array}{c} \begin{array}{ccccc} .01 & .1 & 1 & 10 & 100 \end{array} \\ \begin{bmatrix} 0 & .4 & 1 & .3 & 0 \\ .2 & 1 & 0 & 0 & 0 \\ 0 & 0 & .7 & 1 & 0 \\ 0 & 0 & 0 & 1 & .5 \\ 1 & .1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Application: Computer Engineering (Cont)

We can use max-min composition to obtain a relation between delay times and power dissipation: i.e., we can compute $\tilde{R}_3 = \tilde{R}_1 \circ \tilde{R}_2$ or $\mu_{\tilde{R}_3} = \vee(\mu_{\tilde{R}_1} \wedge \mu_{\tilde{R}_2})$

$$\tilde{R}_3 = \begin{array}{c} \\ 0.1 \\ 1 \\ 10 \\ 100 \end{array} \begin{array}{c} \begin{array}{ccccc} .01 & .1 & 1 & 10 & 100 \end{array} \\ \begin{bmatrix} 1 & .1 & 0 & .6 & .5 \\ .1 & .1 & .5 & 1 & .5 \\ .2 & 1 & .7 & 1 & 0 \\ .2 & .4 & 1 & .3 & 0 \end{bmatrix} \end{array}$$

Application: Fuzzy Relation Petite

Fuzzy Relation Petite defines the degree by which a person with a specific height and weight is considered petite. Suppose the range of the height and the weight of interest to us are {5’, 5’1”, 5’2”, 5’3”, 5’4”,5’5”,5’6”}, and {90, 95,100, 105, 110, 115, 120, 125} (in lb). We can express the fuzzy relation in a matrix form as shown below:

$$\tilde{P} = \begin{array}{c} \\ 5' \\ 5'1'' \\ 5'2'' \\ 5'3'' \\ 5'4'' \\ 5'5'' \\ 5'6'' \end{array} \begin{array}{c} \begin{array}{cccccccc} 90 & 95 & 100 & 105 & 110 & 115 & 120 & 125 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & .5 & .2 \\ 1 & 1 & 1 & 1 & 1 & .9 & .3 & .1 \\ 1 & 1 & 1 & 1 & 1 & .7 & .1 & 0 \\ 1 & 1 & 1 & 1 & .5 & .3 & 0 & 0 \\ .8 & .6 & .4 & .2 & 0 & 0 & 0 & 0 \\ .6 & .4 & .2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Application: Fuzzy Relation Petite

$$\tilde{P} = \begin{array}{c} \\ 5' \\ 5'1'' \\ 5'2'' \\ 5'3'' \\ 5'4'' \\ 5'5'' \\ 5'6'' \end{array} \begin{array}{c} \begin{array}{cccccccc} 90 & 95 & 100 & 105 & 110 & 115 & 120 & 125 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & .5 & .2 \\ 1 & 1 & 1 & 1 & 1 & .9 & .3 & .1 \\ 1 & 1 & 1 & 1 & 1 & .7 & .1 & 0 \\ 1 & 1 & 1 & 1 & .5 & .3 & 0 & 0 \\ .8 & .6 & .4 & .2 & 0 & 0 & 0 & 0 \\ .6 & .4 & .2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Once we define the petite fuzzy relation, we can answer two kinds of questions:

- What is the degree that a female with a specific height and a specific weight is considered to be petite?
- What is the possibility that a petite person has a specific pair of height and weight measures? (fuzzy relation becomes a possibility distribution)

Application: Fuzzy Relation Petite

Given a two-dimensional fuzzy relation and the possible values of one variable, infer the possible values of the other variable using similar fuzzy composition as described earlier.

Definition: Let X and Y be the universes of discourse for variables x and y, respectively, and x_i and y_j be elements of X and Y. Let R be a fuzzy relation that maps X x Y to [0,1] and the possibility distribution of X is known to be $\Pi_x(x_i)$. The compositional rule of inference infers the possibility distribution of Y as follows:

max-min composition: $$\Pi_Y(y_j) = \max_{x_i}(\min(\Pi_X(x_i), \Pi_R(x_i, y_j)))$$

max-product composition: $$\Pi_Y(y_j) = \max_{x_i}(\Pi_X(x_i) \times \Pi_R(x_i, y_j))$$

Application: Fuzzy Relation Petite

Problem: We may wish to know the possible weight of a petite female who is about 5’4”.

Assume About 5’4” is defined as

About-5’4” = {0/5’, 0/5’1”, 0.4/5’2”, 0.8/5’3”, 1/5’4”, 0.8/5’5”, 0.4/5’6”}

Using max-min compositional, we can find the weight possibility distribution of a petite person about 5’4” tall:

$$\tilde{P} = \begin{array}{c|cccccccc} & 90 & 95 & 100 & 105 & 110 & 115 & 120 & 125 \\ \hline 5' & 1 & 1 & 1 & 1 & 1 & 1 & .5 & .2 \\ 5'1'' & 1 & 1 & 1 & 1 & 1 & .9 & .3 & .1 \\ 5'2'' & 1 & 1 & 1 & 1 & 1 & .7 & .1 & 0 \\ 5'3'' & 1 & 1 & 1 & 1 & .5 & .3 & 0 & 0 \\ 5'4'' & .8 & .6 & .4 & .2 & 0 & 0 & 0 & 0 \\ 5'5'' & .6 & .4 & .2 & 0 & 0 & 0 & 0 & 0 \\ 5'6'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

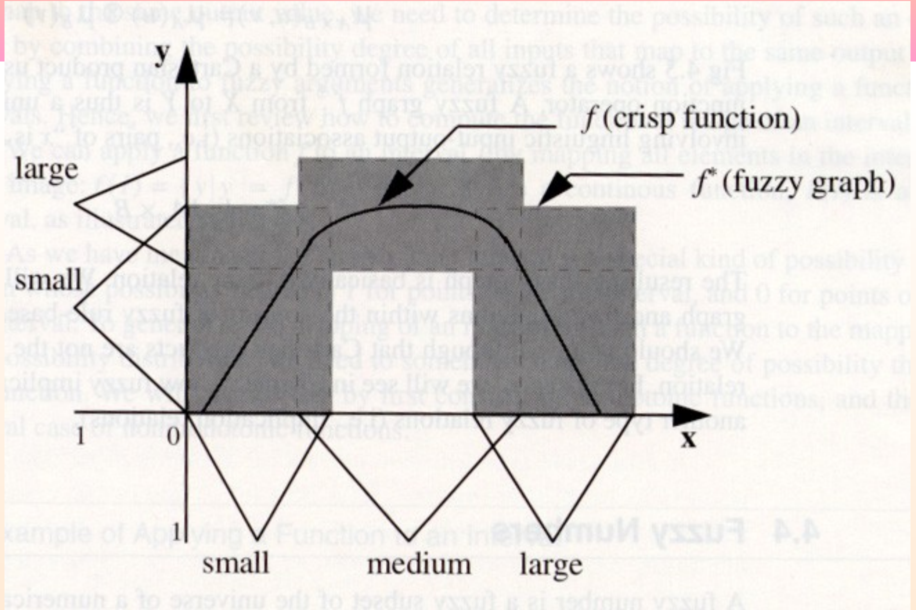
$$\Pi_{weight}(90) = (0 \wedge 1) \vee (0 \wedge 1) \vee (.4 \wedge 1) \vee (.8 \wedge 1) \vee (1 \wedge .8) \vee (.8 \wedge .6) \vee (.4 \wedge 0)$$

$$= 0.8$$

Similarly, we can compute the possibility degree for other weights. The final result is

$$\Pi_{weight} = \{0.8/90, 0.8/95, 0.8/100, 0.8/105, 0.5/110, 0.4/115, 0.1/120, 0/125\}$$

Fuzzy Graphs



- A fuzzy relation may not have a meaningful linguistic label.
- Most fuzzy relations used in real-world applications do not represent a concept, rather they represent a functional mapping from a set of input variables to one or more output variables.
- Fuzzy rules can be used to describe a fuzzy relation from the observed state variables to a control decision (using fuzzy graphs)
- A fuzzy graph describes a functional mapping between a set of input linguistic variables and an output linguistic variable.

Extension Principle

- Provides a general procedure for extending crisp domains of mathematical expressions to fuzzy domains.
- Generalizes a common point-to-point mapping of a function *f*(.) to a mapping between fuzzy sets.

Suppose that *f* is a function from X to Y and A is a fuzzy set on X defined as

$$A = \mu_A(x_1)/(x_1) + \mu_A(x_2)/(x_2) + ... + \mu_A(x_n)/(x_n)$$

Then the extension principle states that the image of fuzzy set A under the mapping *f*(.) can be expressed as a fuzzy set B,

$$B = f(A) = \mu_A(x_1)/(y_1) + \mu_A(x_2)/(y_2) + ... + \mu_A(x_n)/(y_n)$$

Where $y_i = f(x_i)$, i=1,…,n. If f(.) is a many-to-one mapping then

$$\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$$

Extension Principle: Example

Let A=0.1/-2+0.4/-1+0.8/0+0.9/1+0.3/2

and

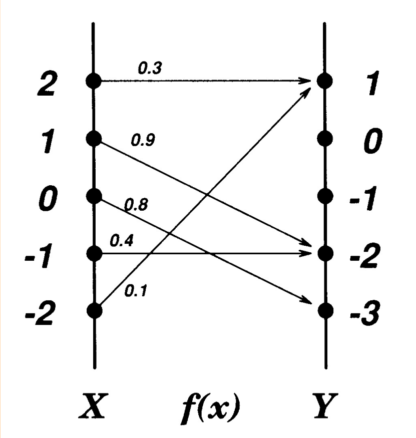
$f(x) = x^2-3$

Upon applying the extension principle, we have

B = 0.1/1+0.4/-2+0.8/-3+0.9/-2+0.3/1

= 0.8/-3+max(0.4, 0.9)/-2+max(0.1, 0.3)/1

= 0.8/-3+0.9/-2+0.3/1

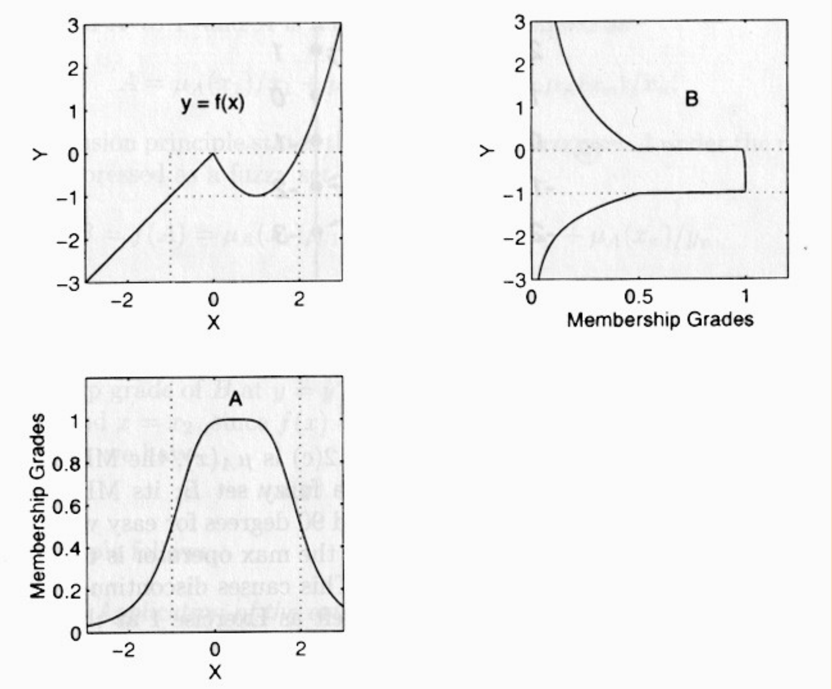


Extension Principle: Example

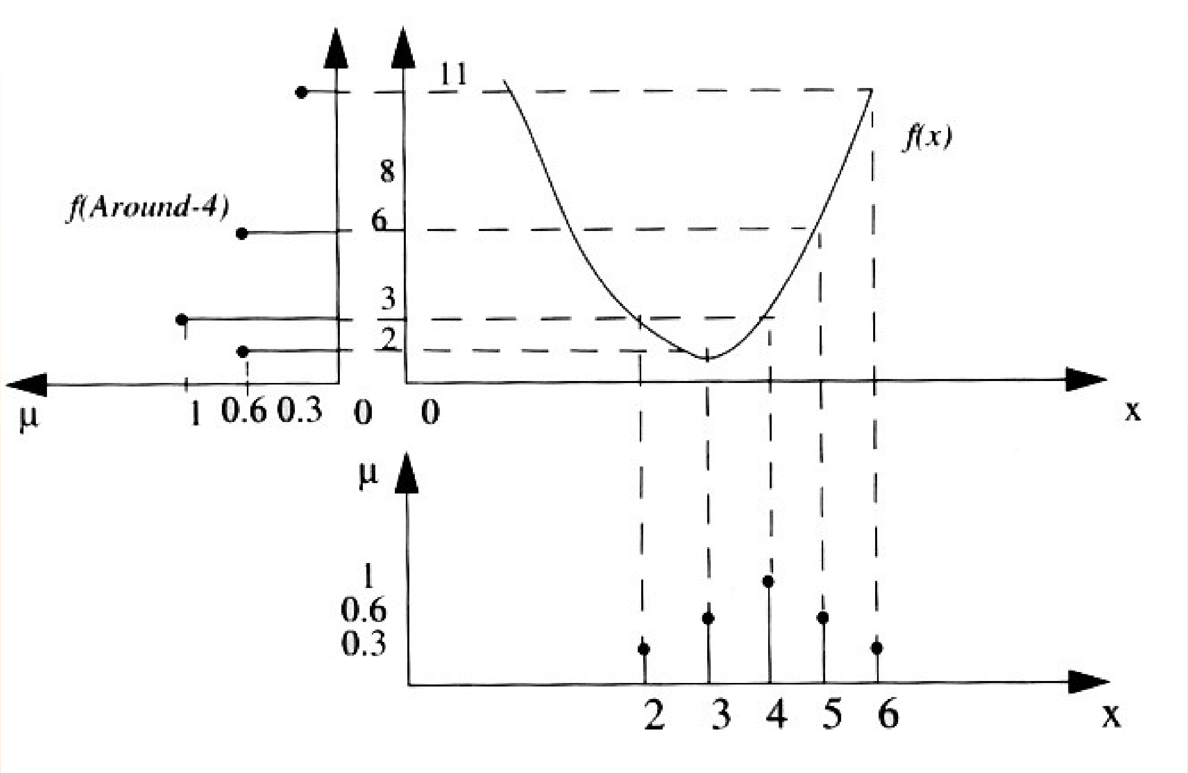
Let $\mu_A(x) = \text{bell}(x;1.5,2,0.5)$

and

$$f(x) = \begin{cases} (x-1)^2-1, & \text{if } x >=0 \\ x, & \text{if } x <=0 \end{cases}$$



Extension Principle: Example



Around-4 = 0.3/2 + 0.6/3 + 1/4 + 0.6/5 + 0.3/6

and

$Y = f(x) = x^2 - 6x + 11$

Arithmetic Operations on Fuzzy Numbers

Applying the extension principle to arithmetic operations, we have

Fuzzy Addition: $\mu_{A+B}(z) = \bigoplus_{\substack{x,y \\ x+y=z}} \mu_A(x) \otimes \mu_B(y)$

Fuzzy Subtraction: $\mu_{A-B}(z) = \bigoplus_{\substack{x,y \\ x-y=z}} \mu_A(x) \otimes \mu_B(y)$

Fuzzy Multiplication: $\mu_{A\times B}(z) = \bigoplus_{\substack{x,y \\ x\times y=z}} \mu_A(x) \otimes \mu_B(y)$

Fuzzy Division: $\mu_{A/B}(z) = \bigoplus_{\substack{x,y \\ x/y=z}} \mu_A(x) \otimes \mu_B(y)$

Arithmetic Operations on Fuzzy Numbers

Let A and B be two fuzzy integers defined as

A = 0.3/1 + 0.6/2 + 1/3 + 0.7/4 + 0.2/5

B = 0.5/10 + 1/11 + 0.5/12

Then

F(A+B) = 0.3/11+ 0.5/12 + 0.5/13 + 0.5/14 +0.2/15 +
0.3/12 + 0.6/13 + 1/14 + 0.7/15 + 0.2/16 +
0.3/13 + 0.5/14 + 0.5/15 + 0.5/16 +0.2/17

Get max of the duplicates,

F(A+B) =0.3/11 + 0.5/12 + 0.6/13 + 1/14 + 0.7/15
+0.5/16 + 0.2/17

Summary

- A fuzzy relation is a multidimensional fuzzy set
- A composition of two fuzzy relations is an important technique
- A fuzzy graph is a fuzzy relation formed by pairs of Cartesian products of fuzzy sets
- A fuzzy graph is the foundation of fuzzy mapping rules
- The extension principle allows a fuzzy set to be mapped through a function
- Addition, subtraction, multiplication, and division of fuzzy numbers are all defined based on the extension principle