Problems:

1. Find the MLE of mean and variance in the case of a normal population.

Sol:

Let $X \sim N(\mu, \sigma^2)$ then,

$$L = \prod_{i=1}^{n} \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} \right]$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}} \right)^{n} e^{-\sum_{i=1}^{n} \frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\log L = n \log \left(\frac{1}{\sigma\sqrt{2\pi}} \right) - \sum_{i=1}^{n} \frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2.$$

$$= n\log(\sigma^2 2\pi)^{-1/2} - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$= -\frac{n}{2}\left\{\log\sigma^2 - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\cancel{\sigma}}\right)^2\right.$$

$$\log L = -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \rightarrow (1)$$

The likelihood equn.. of μ & σ^2 are,

$$\frac{\partial}{\partial\mu}\log L = 0 \qquad\qquad \frac{\partial}{\partial\sigma^2}\log L = 0$$

using (1)

$$\frac{\partial}{\partial\mu}\log L = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}2(x_i-\mu)^{\cancel{2}}(-1)$$

$$\frac{\partial}{\partial\mu}\log L = 0$$

$$\Rightarrow \frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^{\cancel{2}} = 0$$

$$\sum_{i=1}^{n}(x_i-\mu)^{\cancel{2}} = 0$$

$$\sum_{i=1}^{n}x_i - n\mu = 0$$

$$\Rightarrow n\mu = \sum_{i=1}^{n}x_i$$

$$\hat{\mu} = \frac{1}{n}x_i = \bar{x}$$

$$\frac{\partial^2}{\partial\mu^2}\log L = \frac{1}{\sigma^2}\sum_{i=1}^{n}(-1)$$

$$= -\frac{n}{\sigma^2} < 0$$

$\therefore$ $\bar{x}$ is MLE for μ.

using ①

$$\frac{\partial^2}{\partial \sigma^2} \log L = -\frac{n}{2}\frac{1}{\sigma^2} - \frac{1}{2}\left(-\sigma^{-4}\right)\sum (x_i-\mu)^2 \; 710$$

$$x^{-2} = -2x^{-2-1} = -2x^{-3}$$

$$= \frac{1}{2\sigma^2}\left[-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$\frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{1}{\sigma^2} = -\frac{1}{(\sigma^2)^2}$$

$$\frac{\partial}{\partial \sigma^2} \log L = 0$$

$$\Rightarrow \frac{1}{2\sigma^2}\left[-n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] = 0$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = n$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = s^2,$$ the sample variance.

Hense MLE for σ^2 is s^2

2. Find the MLE for the Parameter λ of a Poisson distr. on the basis of a sample of size n.

Sol: w.k.t, the probl. func. of the Poisson distribution with parameter λ is given by,

$$P(X=x) = f(x,\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \; x = 0,1,2,\ldots$$

Likelihood func. of random sample $x_1, x_2, \ldots, x_n$ of n observations from this population is,

$$L = \prod_{i=1}^{n} f(x_i,\lambda) = \prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$\log L = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} (x_i!)}$$

$$\log L = \log(e^{-n\lambda}) + \log(\lambda^{\sum_{i=1}^{n} x_i}) - \sum_{i=1}^{n} \log(x!)$$

$$= -n\lambda + n\bar{x}\log\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{\partial \log L}{\partial \lambda} = -n + \frac{n\bar{x}}{\lambda}$$

Likelihood equ of the parameter λ is given by,

$$\frac{\partial}{\partial L} \log L = 0$$

$$\Rightarrow -n + \frac{n\bar{x}}{\lambda} = 0$$

$$\Rightarrow -1 + \frac{\bar{x}}{\lambda} = 0$$

$$\Rightarrow \frac{\bar{x}}{\lambda} = 1$$

$$\Rightarrow \boxed{\lambda = \bar{x}}$$

$\therefore$ MLE for λ is the Sample mean $\bar{x}$.