

Mean and Variance of F-distribution:

(1)

$$\mu_r(\text{about origin}) = E(F^r) = \int_0^{\infty} F^r f(F) dF$$

$$= \frac{(v_1/v_2)^{v_1/2}}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^{\infty} F^r \frac{F^{(v_1/2)-1}}{\left(1 + \frac{v_1}{v_2} F\right)^{(v_1+v_2)/2}} dF$$

To evaluate the integral, Put $\frac{v_1}{v_2} F = y$, So that

$$dF = \frac{v_2}{v_1} dy, \quad \Rightarrow F = \frac{v_2}{v_1} y, \quad (v_1/2) - 1$$

$$\therefore \mu_r = \frac{(v_1/v_2)^{v_1/2}}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^{\infty} \left(\frac{v_2}{v_1} y\right)^r \cdot \frac{\left(\frac{v_2}{v_1} y\right)^{(v_1/2)-1}}{(1+y)^{(v_1+v_2)/2}} \left(\frac{v_2}{v_1}\right) dy$$

$$= \frac{(v_1/v_2)^{v_1/2}}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^{\infty} \left(\frac{v_2}{v_1}\right)^{r+(v_1/2)-1} \frac{y^{r+(v_1/2)-1}}{(1+y)^{(v_1+v_2)/2}} \left(\frac{v_2}{v_1}\right) dy$$

$$= \frac{\left(\frac{v_2}{v_1}\right)^r}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \int_0^{\infty} \frac{y^{r+(v_1/2)-1}}{(1+y)^{\left(\frac{v_1}{2} + r + \frac{v_2}{2} - r\right)}} dy$$

$$B(x, y) = \int_0^{\infty} t^{x-1} (1+t)^{-x-y} dt$$

$$= \left(\frac{v_2}{v_1}\right)^r \cdot \frac{1}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \cdot B\left(r + \frac{v_1}{2}, \frac{v_2}{2} - r\right), \quad v_2 > 2r$$

$$= \left(\frac{v_2}{v_1}\right)^r \cdot \frac{\sqrt{\frac{v_1}{2} + \frac{v_2}{2}}}{\sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}}} \cdot \frac{\sqrt{\left(r + \frac{v_1}{2}\right)} \sqrt{\left(\frac{v_2}{2} - r\right)}}{\sqrt{\left(r + \frac{v_1}{2} + \frac{v_2}{2} - r\right)}} \quad \left\{ \begin{array}{l} B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\ r < \frac{v_2}{2} \end{array} \right.$$

$$= \left(\frac{v_2}{v_1}\right)^r \frac{\sqrt{\frac{v_1}{2} + \frac{v_2}{2}}}{\sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}}} \cdot \frac{\sqrt{\left(r + \frac{v_1}{2}\right)} \sqrt{\left(\frac{v_2}{2} - r\right)}}{\sqrt{\left(\frac{v_1}{2} + \frac{v_2}{2}\right)}}$$

$$\boxed{\Gamma^r = (r-1) \Gamma^{(r-1)}}$$

$$\therefore \mu_r' = \left(\frac{v_2}{v_1}\right)^r \frac{\sqrt{\left(r + \frac{v_1}{2}\right)} \sqrt{\left(\frac{v_2}{2} - r\right)}}{\sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}}}, \quad r < \frac{v_2}{2} \rightarrow (1)$$

Put $r=1$ we get,

$$\mu_1' = \left(\frac{v_2}{v_1}\right) \frac{\sqrt{\left(1 + \frac{v_1}{2}\right)} \sqrt{\left(\frac{v_2}{2} - 1\right)}}{\sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}}}, \quad r < \frac{v_2}{2}$$

$$\begin{aligned} \sqrt{\left(1 + \frac{v_1}{2}\right)} &= \left(1 + \frac{v_1}{2} - 1\right) \sqrt{\left(1 + \frac{v_1}{2} - 1\right)} \\ &= \frac{v_1}{2} \sqrt{\frac{v_1}{2}} \quad \because \sqrt{(r-1)} = \frac{\Gamma^r}{\Gamma^{(r-1)}} \end{aligned}$$

$$= \left(\frac{v_2}{v_1} \right) \frac{\frac{v_1}{2} \sqrt{\frac{v_1}{2}} \cdot \sqrt{\frac{v_2}{2}}}{\left(\frac{v_2}{2} - 1 \right) \sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}}} \quad (3)$$

$$= \frac{v_2}{v_1} \frac{v_1}{2} \cdot \frac{v_2}{2} \cdot \frac{2}{(v_2 - 2)} \cdot \frac{1}{\left(\frac{v_2}{2} - 1 \right)}$$

$$= \frac{v_2}{v_2 - 2}$$

$$\therefore \mu_1' = \frac{v_2}{v_2 - 2}$$

Put $r = 2$ in (1) we get,

$$\mu_2' = \left(\frac{v_2}{v_1} \right)^2 \frac{\sqrt{\left(2 + \frac{v_1}{2} \right)} \sqrt{\left(\frac{v_2}{2} - 2 \right)}}{\sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}}}, \quad r < \frac{v_2}{2} \rightarrow (2)$$

Consider,

$$\begin{aligned} \sqrt{\left(2 + \frac{v_1}{2} \right)} &= \left(2 + \frac{v_1}{2} - 1 \right) \sqrt{\left(2 + \frac{v_1}{2} - 1 \right)} \\ &= \left(\frac{v_1}{2} + 1 \right) \sqrt{\frac{v_1}{2} + 1} \\ &= \left(\frac{v_1}{2} + 1 \right) \frac{v_1}{2} \sqrt{\frac{v_1}{2}} \end{aligned}$$

$$\sqrt{\left(\frac{v_2}{2} - 2\right)} = \sqrt{\left(\frac{v_2}{2} - 1 - 1\right)} = \frac{\sqrt{\left(\frac{v_2}{2} - 1\right)}}{\left(\frac{v_2}{2} - 2\right)}$$

$$= \frac{\sqrt{\frac{v_2}{2}}}{\left(\frac{v_2}{2} - 2\right) \left(\frac{v_2}{2} - 1\right)}$$

Subl. these in (2) we get,

$$\mu_2' = \left(\frac{v_2}{v_1}\right)^2 \frac{\left(\frac{v_1}{2} + 1\right) \cdot \frac{v_1}{2} \cdot \sqrt{\frac{v_1}{2}} \cdot \sqrt{\frac{v_2}{2}}}{\sqrt{\frac{v_1}{2}} \sqrt{\frac{v_2}{2}} \left(\frac{v_2}{2} - 2\right) \left(\frac{v_2}{2} - 1\right)}$$

$$= \frac{\frac{v_2^2}{v_1} \left(\frac{v_1 + 2}{2}\right) \frac{1}{2}}{\left(\frac{v_2 - 4}{2}\right) \left(\frac{v_2 - 2}{2}\right)}$$

$$\mu_2' = \frac{v_2^2 (v_1 + 2)}{v_1 (v_2 - 4) (v_2 - 2)}, \quad v_2 > 4.$$

$$\therefore \mu_2 = \mu_2' - (\mu_2')^2$$

$$= \frac{v_2^2 (v_1 + 2)}{v_1 (v_2 - 4) (v_2 - 2)} - \left(\frac{v_2}{v_2 - 2}\right)^2$$

$$\therefore \text{mean} = \frac{V_2}{V_2 - 2} \quad V_2 > 2.$$

$$\text{var} = \frac{V_2^2 (V_1 + 2)}{V_1 (V_2 - 4) (V_2 - 2)} - \left(\frac{V_2}{V_2 - 2} \right)^2, \quad V_2 > 4$$