Assignment 4

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Question 1

(a) Sequence: $a_k = \frac{2k}{5+k}$ for all integers $k \geq 1$
First four terms:

$$a_1 = \frac{2 \cdot 1}{5+1} = \frac{2}{6} = \frac{1}{3}$$
$$a_2 = \frac{2 \cdot 2}{5+2} = \frac{4}{7}$$
$$a_3 = \frac{2 \cdot 3}{5+3} = \frac{6}{8} = \frac{3}{4}$$
$$a_4 = \frac{2 \cdot 4}{5+4} = \frac{8}{9}$$

(b) Sequence: $b_j = \frac{4-j}{4+j}$ for all integers $j \geq 1$
First four terms:

$$b_1 = \frac{4-1}{4+1} = \frac{3}{5}$$
$$b_2 = \frac{4-2}{4+2} = \frac{2}{6} = \frac{1}{3}$$
$$b_3 = \frac{4-3}{4+3} = \frac{1}{7}$$
$$b_4 = \frac{4-4}{4+4} = \frac{0}{8} = 0$$

Question 2

(a) Sequence: $e_m = 2 + \left(\frac{1}{3}\right)^m$ for all integers $m \geq 0$

First four terms:

$$e_0 = 2 + \left(\frac{1}{3}\right)^0 = 2 + 1 = 3$$
$$e_1 = 2 + \left(\frac{1}{3}\right)^1 = 2 + \frac{1}{3} = \frac{7}{3}$$
$$e_2 = 2 + \left(\frac{1}{3}\right)^2 = 2 + \frac{1}{9} = \frac{19}{9}$$
$$e_3 = 2 + \left(\frac{1}{3}\right)^3 = 2 + \frac{1}{27} = \frac{55}{27}$$

(b) Showing convergence:
As $m \to \infty$, $\left(\frac{1}{3}\right)^m \to 0$. Therefore,

$$\lim_{m\to\infty} e_m = \lim_{m\to\infty} \left(2 + \left(\frac{1}{3}\right)^m\right) = 2 + 0 = 2$$

Thus, the sequence e_m converges to 2 as m approaches infinity.

Question 3

(a) Sequence: $a_n = (-1)^{n+1}$ for all integers $n \geq 1$
Explicit formula:

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$

(b) Sequence: $b_n = (-1)^{n+1} \cdot \left\lfloor \frac{n}{2} \right\rfloor$ for all integers $n \geq 1$
Explicit formula:

$$b_n = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2 \\ -n & \text{if } n \equiv 1 \pmod 4 \\ n-1 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

(c) Sequence: $c_n = \frac{n+1}{2}$ for all integers $n \geq 1$
Explicit formula:

$$c_n = \frac{2n-1}{2}$$

Question 4

Summation:

$$\sum_{k=1}^{6}(k+2) = (1+2) + (2+2) + (3+2) + (4+2) + (5+2) + (6+2)$$

Calculating each term:

$$= 3 + 4 + 5 + 6 + 7 + 8 = 33$$

Product:

$$\prod_{k=3}^{5} k^2 = 3^2 \cdot 4^2 \cdot 5^2$$

Calculating each square:

$$= 9 \cdot 16 \cdot 25 = 3600$$

Question 5

Proposition: Prove using mathematical induction that for all integers $n \geq 0$,

$$P(n): \quad 1 + 2 + 4 + 8 + \cdots + 2n = 2^{n+1} - 1$$

(a) Base Case: $P(0)$
For $n = 0$,

$$1 = 2^{0+1} - 1 = 2 - 1 = 1$$

So, $P(0)$ holds true.
(b) Inductive Step: Assume $P(k)$ is true for some integer $k \geq 0$, i.e.,

$$1 + 2 + 4 + 8 + \cdots + 2k = 2^{k+1} - 1$$

Now, prove $P(k + 1)$:

$$1 + 2 + 4 + 8 + \cdots + 2k + 2^{k+1} = 2^{(k+1)+1} - 1$$

Starting from the left-hand side,

$$\begin{aligned} 1 + 2 + 4 + 8 + \cdots + 2k + 2^{k+1} &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

$P(n)$ is true for all integers $n \geq 0$.