Exam 3: COT 2000

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1 Question 1

(a) The first four terms:

$$a_0 = 5, a_1 = 17, a_2 = 29, a_3 = 41$$

(b) To show convergence: -

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5 + \lim_{n\to\infty} \left(\frac{1}{2}\right)^n$$

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$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

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$$\lim_{n\to\infty} a_n = 5 + 0 = 5$$

ans: $a_n = 5 + \left(\frac{1}{2}\right)^n$ converges to 5 as n approaches infinity.

2 Question 2

(a) $b_n = 2n,$ for $n \geq 1$
(b) $b_n = 3(-1)^{n+1},$ for $n \geq 1$
(c) $b_n = \frac{n}{2n+1},$ for $n \geq 1$

3 Question 3

(a) $\sum_{k=2}^{7}(k+3) = 5 + 6 + 7 + 8 + 9 + 10 = 45$
(b) $\prod_{k=2}^{4}(k+1)^2 = 3^2 \cdot 4^2 \cdot 5^2 = 3600$

4 Question 4

Let $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 4, 6, 8, 10\}$.
(a) Define $f : X \to Y$ by:

$$f(1) = 8, \quad f(3) = 6, \quad f(5) = 4, \quad f(7) = 8, \quad f(9) = 10.$$

Is f one-to-one? Is f onto? Explain your answers.

- f is **not one-to-one** because $f(1) = 8$ and $f(7) = 8$, implying f is not injective.
- f is **not onto** because $2 \in Y$ is not mapped by any element in X, implying f is not surjective.

(b) Define $g : X \to Y$ by:

$$g(1) = 2, \quad g(3) = 4, \quad g(5) = 6, \quad g(7) = 8, \quad g(9) = 10.$$

Is g one-to-one? Is g onto? Explain your answers.

- g is **one-to-one** because all values are distinct, implying g is injective.
- g is **onto** because every element in Y is mapped by some element in X, implying g is surjective.

5 Question 5

(a) F is not one-to-one (e is repeated). F is not onto (g is not in the range).
(b) G is one-to-one. G is onto.
(c) To make F and G one-to-one correspondences: F: a→e, b→f, c→g (remove d) G: a→e, b→f, c→g, d→h (add h to Y)

6 Question 6

(a) h(x) = 2x + 1
i. One-to-one proof:

$$h(x_1) = h(x_2) \implies 2x_1 + 1 = 2x_2 + 1 \implies x_1 = x_2$$

ii. Onto proof: For any $y \in \mathbb{Z}$, $x = \frac{y-1}{2} \in \mathbb{Z}$ satisfies $h(x) = y$
(b) $g(x) = x^2 - 1$
i. Not one-to-one: g(1) = g(-1) = 0
ii. Not onto: g(x) $\geq$ -1 for all x, so y ¡ -1 has no preimage