Algorithm for $|\langle\psi|\phi\rangle|^2$

- Assume $|\psi\rangle$ is stabilized by $G = \langle g_1, \cdots g_n\rangle$ and $|\phi\rangle$ by $H = \langle h_1, \cdots h_n\rangle$.
- We can express $|\phi\rangle\langle\phi| = \prod_{j=1}^{n} \frac{1}{2}(\mathbb{I} + h_j)$, So

  $|\langle\psi|\phi\rangle|^2 = \langle\psi| \prod_{j=1}^{n} \frac{1}{2}(\mathbb{I}+h_j)|\psi\rangle.$

- Recall the algorithm for "measurement" on a stabilizer state $|\psi\rangle$.
  - To measure a Pauli-string P:

    1. If $P \in G \Rightarrow$ One has $P(+) = \langle\psi|\frac{1}{2}(\mathbb{I}+P)|\psi\rangle = 1$, $|\psi_+\rangle = |\psi\rangle$

       and $P(-) = \langle\psi|\frac{1}{2}(\mathbb{I}-P)|\psi\rangle = 0$
    2. If $-P \in G \Rightarrow P(+) = \langle\psi|\frac{1}{2}(\mathbb{I}+P)|\psi\rangle = 0.$ $P(-) = 1.$
    3. If $\pm P \notin G \Rightarrow P(+) = \langle\psi|\frac{1}{2}(\mathbb{I}+P)|\psi\rangle = \frac{1}{2} = P(-).$

       The after measurement state $|\psi_\pm\rangle = \frac{1}{\sqrt{P_\pm}}\frac{1}{2}(\mathbb{I}\pm P)|\psi\rangle.$

    The stab. group of $|\psi_\pm\rangle$ can be expressed as the following:
    1. Find a new basis of generator g_j' such that only one g_j (say g_1) anticommute with P.
    2. Replace g_1 with $\pm P$. So $|\psi_\pm\rangle$ is stabized by $\langle P, g_2', \cdots g_n'\rangle$

- We can use this "measurement" process to calculate $|\langle\psi|\phi\rangle|^2$.

Algo: pval $= 1$.
$|\psi_m\rangle = |\psi\rangle$ (stabilizer-wise).
For $i = 1 \dots j$,
- (measure h_i):

  if $h_i \in G$: continue
  elseif $-h_i \in G$: return $|\langle\psi|\phi\rangle|^2 = 0$.
  elseif $h_i \notin G$,
  pval = pval $\times \frac{1}{2}$
  $|\psi_m\rangle = \frac{1}{\sqrt{1/2}} \cdot \frac{1}{2}(\mathbb{I}+h_i)|\psi_m\rangle$. (Forcing the new state to be the after-meas. state with outcome $= +$).

end.
return pval.