Rolf Wanka
Matthias Kergaßner

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Exercises to
Swarm Intelligence
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Sheet 1

Dynamical systems often work "memoryless," i.e., the next state $\vec{x}_{t+1}$ of the system depends only on the current state $\vec{x}_t$ of the system and not additionally on how one got into this state.
In addition, it is often true that one can calculate the new state by multiplying $\vec{x}_t$ by a matrix A, i.e., $\vec{x}_{t+1} = A \cdot \vec{x}_t$. For a concrete example, see Problem 2. Therefore, over the runtime of the system we get $\vec{x}_t = A^t \cdot \vec{x}_0$, where $\vec{x}_0$ denotes the initial state. It is therefore necessary to quickly calculate powers of A, and this is what we want to do in Problem 1.

Problem 1:
Let the matrix $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ be given. Armed only with an inexpensive pocket calculator, our goal is to calculate A^{15} and A^{22}.

(a) Calculate the two eigenvalues λ_1 and λ_2, $\lambda_1 \geq \lambda_2$, of A.

(b) Determine for each eigenvalue λ_i a corresponding eigenvector $\vec{v}_i$.

(c) Let P be the matrix whose first column consists of $\vec{v}_1$ and whose second column consists of $\vec{v}_2$. Calculate P^{-1}.

(d) Show: $A = P \cdot \Lambda \cdot P^{-1}$. Here, Λ is the diagonal matrix $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

(e) Calculate the four *entries* of A^k for any $k \in \mathbb{N}$, and from this general expression the matrices A^{15} and A^{22}.

Problem 2:
Let a ring made of metal be given. On this ring, four uniformly distributed points $1, 2, 3, 4$ are marked, and the point 1 is heated by a flame to the temperature T. Then the flame is removed, and we consider, how (in a little bit idealized manner) the temperature changes on the ring. To do this, we model the temperature changes as following: Each point transfers between two measurements $\frac{1}{3}$ of its temperature to its two neighbors, the residual third remains at this point. No energy is released to the outside. This process of temperature exchance is called *(discrete) diffusion*.

Let $\vec{\vartheta}_t$ denote the temperature distribution on the four points. Thus, at the beginning $\vec{\vartheta}_0 = \begin{pmatrix} T \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

At the next measurement point in time, we have

$$\vec{\vartheta}_1 = \overbrace{\frac{1}{3} \cdot \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}}^{=:A} \cdot \vec{\vartheta}_0$$

and in general $\vec{\vartheta}_t = A^t \cdot \vec{\vartheta}_0$. (Please explain the meaning/origin of matrix A!)

Similar to Problem 1, A can be decomposed:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix}}_{=P} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \end{pmatrix}}_{=\Lambda} \cdot \underbrace{\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}}_{=P^{-1}}$$

(a) Determine $\lim_{t\to\infty} \vec{\vartheta}_t$.

(b) Discuss the influence of $\Lambda_{44} = -\frac{1}{3}$ on the temperatures approaching limit.

Problem 3:
The Fibonacci Numbers $(f_i)_{i=0,1,\ldots}$ are a sequence of numbers used in computer science in many regards, and they occur quite often in nature. They are defined recursively as follows:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_{i+1} &= f_i + f_{i-1} \end{aligned}$$

Determine a 2×2 matrix A such that

$$\begin{pmatrix} f_{i+1} \\ f_i \end{pmatrix} = A \cdot \begin{pmatrix} f_i \\ f_{i-1} \end{pmatrix}$$

and then proceed analogously to Problem 1 in order to represent f_i as an explicit formula.