Swarm Intelligence
Exercise 4

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Problem 13: PageRank Algorithm

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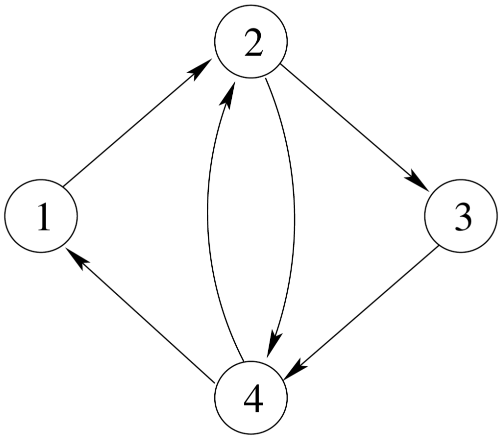
Idea and Goal of the PageRank Algorithm

- Random-Walk across the internet
- The more often a page is reached, the higher its PageRank
- Determination of the relevance of websites

Problem 13 (a)

Start at website 1 and run two steps.
Where will one end up with which probability?



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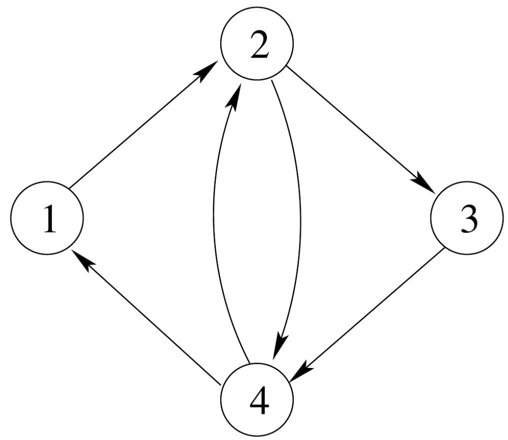

Two possibilities:

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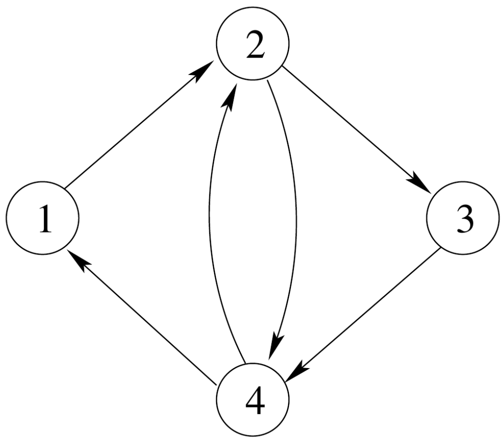

Two possibilities:

- $1 \to 2 \to 3$ with 50% chance

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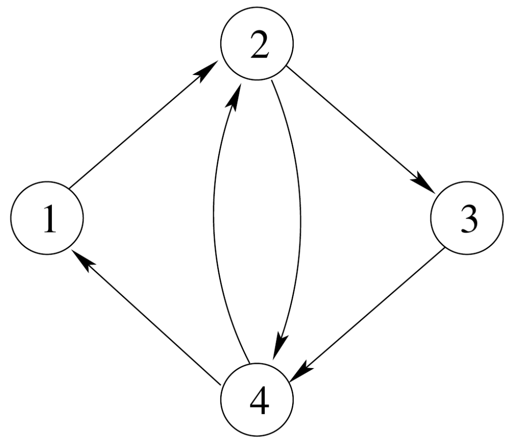


Two possibilities:

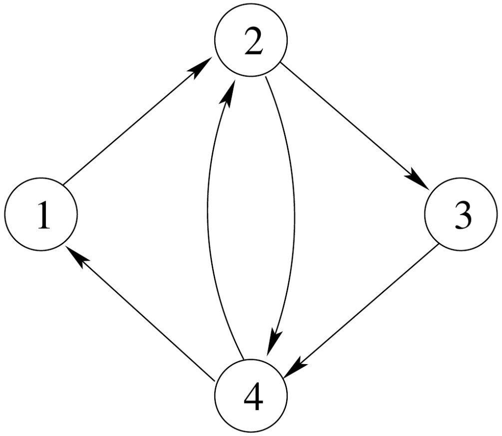
- $1 \to 2 \to 3$ with 50% chance
- $1 \to 2 \to 4$ with 50% chance

Problem 13 (b)

Determine the transition matrix A, such that

$$\pi_{t+1} = \pi_t \cdot A$$

holds.



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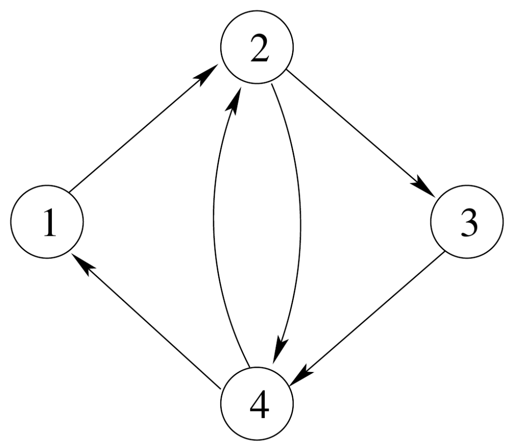

$$A = \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{pmatrix}$$

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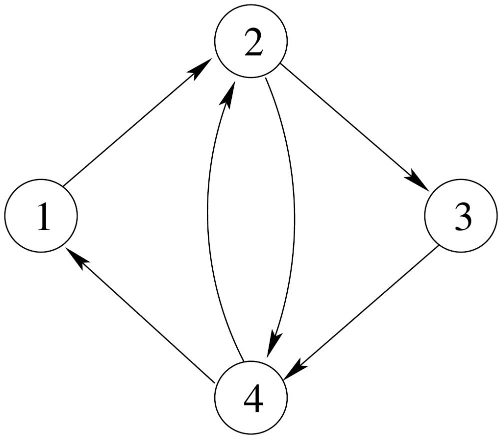
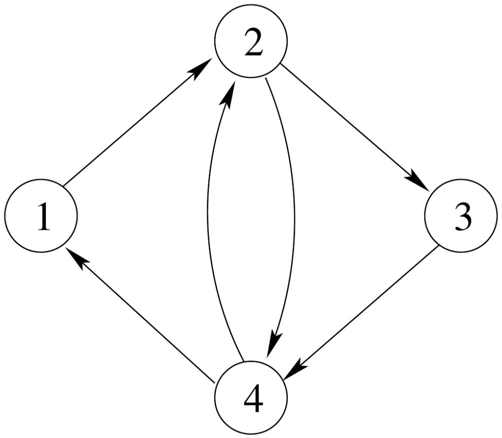


$$A = \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{pmatrix} \quad \rightarrow \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Problem 13 (c)

Determine the limit π to which this Markov chain converges.



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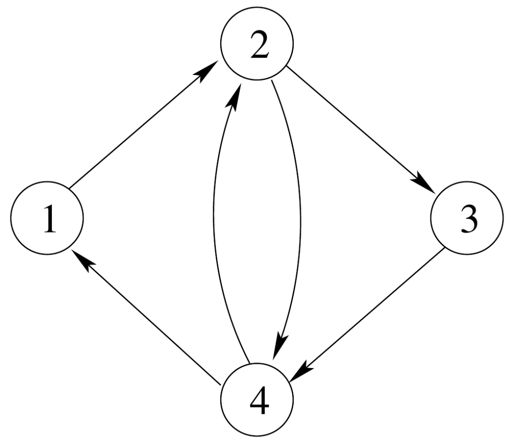

Proof of convergence:

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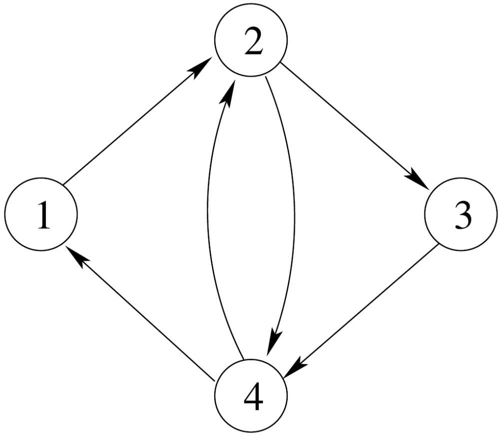

Proof of convergence:

- A is **irreducible**, since every node is reachable from every other node

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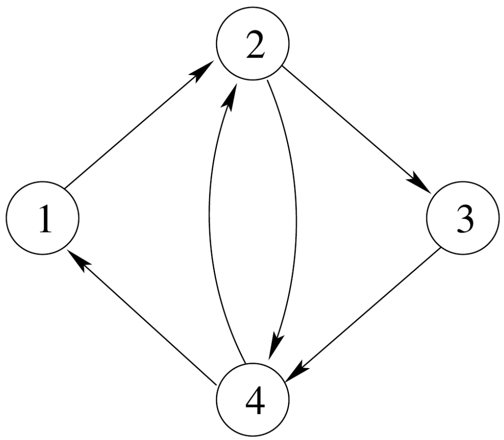

Proof of convergence:

- A is **irreducible**, since every node is reachable from every other node
- A is **aperiodic**, since node 2 can be revisited after two as well as three steps:

$$2 \to 4 \to 2$$
$$2 \to 3 \to 4 \to 2$$

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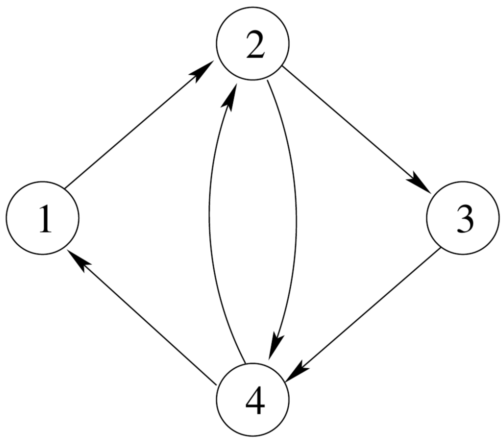

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- A is **aperiodic**, since node 2 can be revisited after two as well as three steps:

$$2 \to 4 \to 2$$
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Thus the Markov chain does converge.

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Compute (left) eigenvector π corresponding to the largest eigenvalue 1:

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$$
\begin{aligned}
& \pi \cdot (A - \mathbb{1}) = (0, \ldots, 0) \\
\Leftrightarrow \quad & (A - \mathbb{1})^\top \cdot \pi^\top = (0, \ldots, 0)^\top
\end{aligned}
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Kernel of $(A - \mathbb{1})^\top$:

$$(A - \mathbb{1})^\top = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} \\ 1 & -1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & \frac{1}{2} & 1 & -1 \end{pmatrix}$$

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Choose normalized eigenvector $\pi = \frac{1}{6} \cdot (1, 2, 1, 2)$

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Thus the following PageRanks come up:

$$(2, 4, 1, 3)$$

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Remove link from 3 to 4. What is the problem now?

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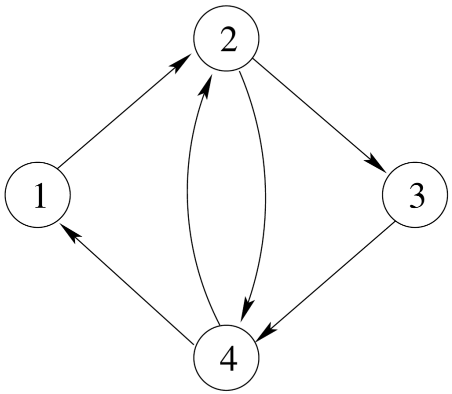
Previously:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$



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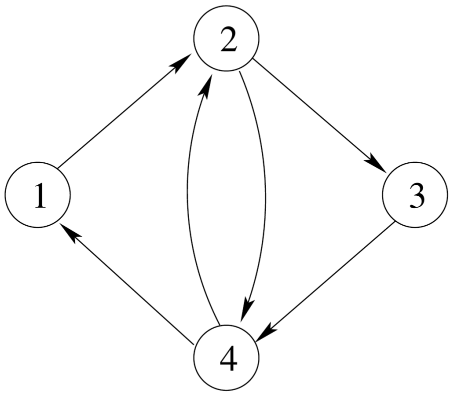
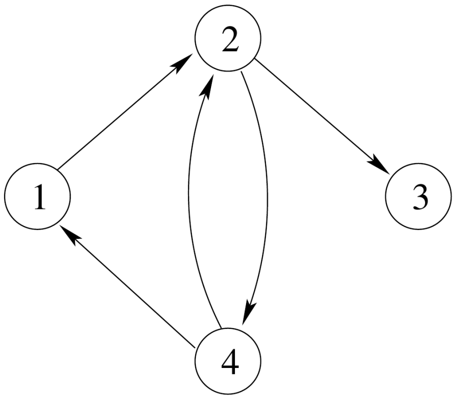


Without edge $(3 \to 4)$:

$$B =$$



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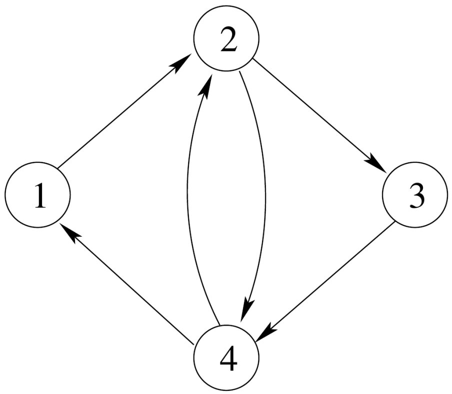

Without edge $(3 \rightarrow 4)$:

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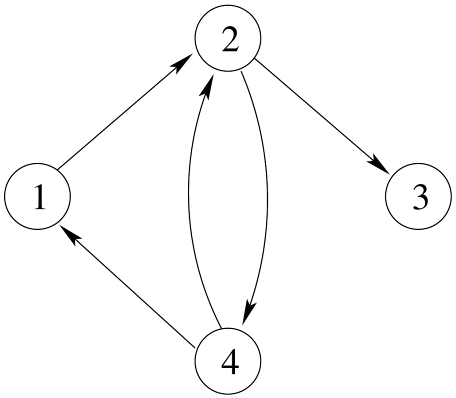

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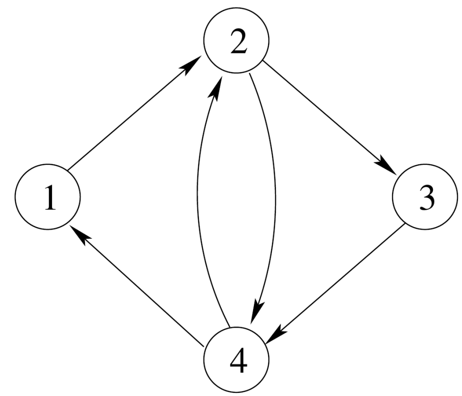

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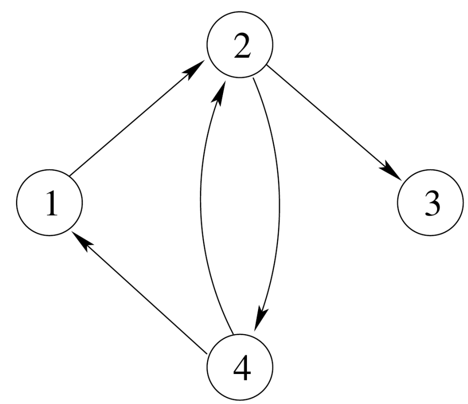

$p \cdot B^t$ converges to $(0, 0, 1, 0)$ for $t \rightarrow \infty$, although page 3 seems to be rather irrelevant ($\rightarrow$ absorption).

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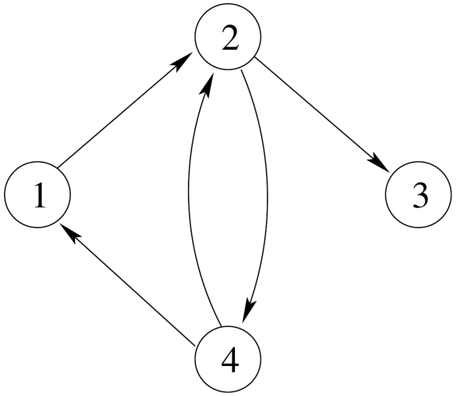

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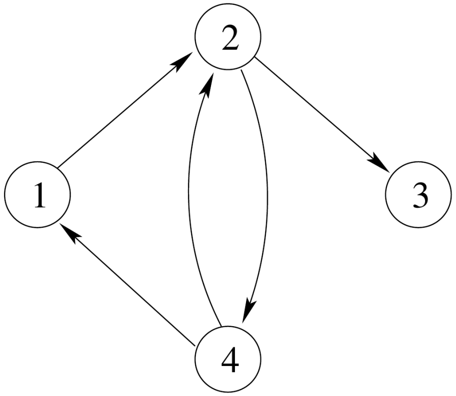

Solution 1: Remove node 3

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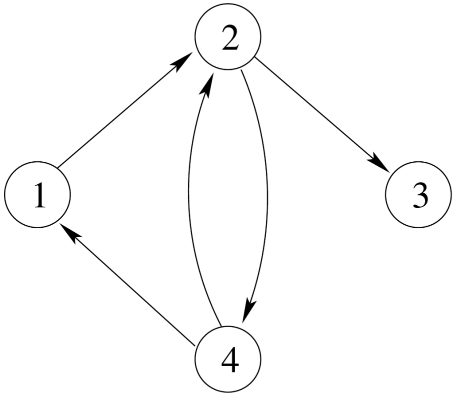

Solution 1: Remove node 3

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \rightarrow \quad \pi_{B_1} = \frac{1}{5} \cdot (1, 2, 2)$$

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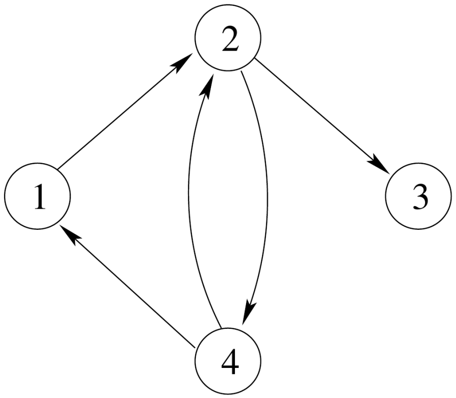

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Solution 2: Add edges from 3 to every other node

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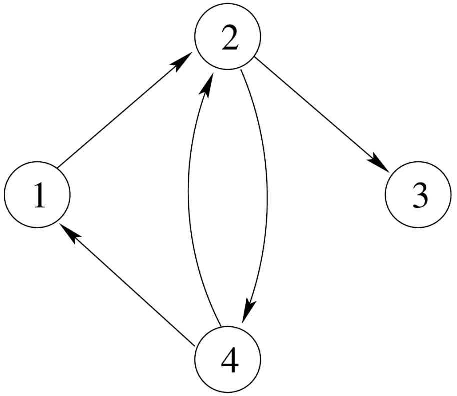

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Solution 2: Add edges from 3 to every other node

$$B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \pi_{B_2} = \frac{1}{17} \cdot (3, 6, 4, 4)$$

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Additional trick: Ensure **irreducibility** and **aperiodicity**

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Remove link from 3 to 4. What is the problem now?

Additional trick: Ensure **irreducibility** and **aperiodicity**

$$\tilde{A} = \alpha \cdot A + (1 - \alpha) \cdot \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

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“Good” compromise for $\alpha = 0.85$

Thank you for your attention! Any Questions?

design

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