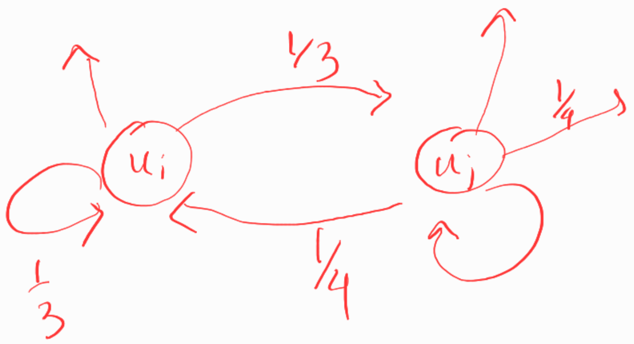
12.




a) Let initial dist, $p = (P_1, \dots, P_n)$

→ Determine the stochastic matrix $M = (m_{ij})_{ij}$

$m_{ij}$ = prob of going from $u_i$ to $u_j$

$$= \underbrace{\frac{1}{d_i + 1}}_{\text{if } i \,\&\, j \text{ are neighbours}} \cdot \mathbb{1} \leftarrow \text{characteristic func.}$$

(iff $v_j \in N(v_i) \cup \{v_i\}$) → else 0

$\{v_i\}$ ↑ self loops

Therefore $p \cdot M^s$ yeilds the distribution after s steps

↑ row vector, so multiply from left.

Remarks:

→ The sum of any row '$i$' of $M$ has to be 1.

Proof:
$$\sum_{j=1}^{N} m_{i,j} = \sum_{j=1}^{N} \frac{1}{d_i + 1} \mathbb{1}_{(v_j \in N(v_i) \cup \{v_i\})}$$

$$= \frac{1}{d_i + 1} \underbrace{\sum_{j=1}^{N} \mathbb{1}_{(v_j \in \dots)}}$$

↳ no. of neighbours of $v_i$ including $v_i$

$$= \frac{1}{d_i+1} \cdot \left| \mathcal{N}(v_i) \cup \{v_i\} \right|$$

↘ cardinality.

( $d_i$ neighbours, 1 for self loop. )

$$= \frac{1}{d_i+1} \cdot (d_i+1) = 1$$

b) $$\pi = \left( \frac{d_1 + 1}{2|E| + |V|} + \frac{d_2 + 1}{2|E| + |V|} + \dots + \frac{d_n + 1}{2|E| + |V|} \right)$$

Prove : $\pi \cdot M = \pi$ holds.

We want to show $j^{th}$ entry of $\pi M$ is same as $j^{th}$ entry of $\pi$.

$$(\pi M)_j = \left( \sum_i \pi_i \cdot m_{ij} \right)$$ ↘ one column

$$= \sum_{i}^{N} \frac{\cancel{(d_i+1)}}{2|E|+|V|} \cdot \frac{1}{\cancel{(d_i+1)}} \cdot \mathbb{1}_{(v_j \in \dots)}$$

$$= \frac{1}{2|E|+|V|} \cdot \underbrace{\sum_i \mathbb{1}\left[v_j \in N(v_i) \cup \{v_i\}\right]}$$

it is the nodes near the node $j$.

$$= \frac{1}{2|E|+|V|} \cdot (d_j+1) = \pi_j$$

Thus Proved

$\Rightarrow \pi$ is a stationary distribution.

(only possible if bidirectional edges)

c) Regular graph $\rightarrow$ any vertex has the degree $d$. From (b), the stationary dist is

$$\pi = \left[ \frac{d+1}{2|E|+|V|} + \cdots + \frac{d+1}{2|E|+|V|} \right] \in \mathbb{R}^n$$

$\left[ |V| = n,\ |E| = \frac{n}{2}(d) \right]$

$\therefore \pi = (\frac{1}{n}, \cdots \frac{1}{n})$

Remarks:

If matrix $M$ (or the Graph) is Irreducible and APeriodic, then there is an unique stationary dist $\pi$. Any other ... dist ... converges to, ie, $\lim p \cdot M^s = \pi$

initial dist $p$ converge ... $s \to \infty$

- M is irreducible → every vertex is reachable from any other vertex. (provided G is connected)
- M is aperiodic → due to self loops.

(M is a Markov chain for our Random walk, → so it has the props of the Graph)

→ Thus existence of and convergence towards $\pi$ is verified.

⇝ Task : Estimate no. of Red nodes in G.

→ Let

$(u_i)_{1 \le i \le T}$ → be the sequence of T visited nodes.

$\hat{r}_T$ → an estimator for rednodes in G.

$$\hat{r}_T = \frac{n}{T} \sum_{t}^{T} \mathbb{1}(u_i \text{ is red})$$

if $T \to \infty$, $\hat{r}_T$ converges to correct no. of red nodes.

d) $P_{t+1} = P_t \cdot M$ (t is timesteps)

$= P_1 \cdot M^t$

- Eigenvalue $\lambda_1 = 1$ exists and corresponds to eigenvector $\pi$. ($\therefore \pi \cdot M = \pi \cdot \underset{\text{eigen value}}{1}$)

- Every other eigenvalue has an absolute value $< 1$.