1)

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

a) Calc eigenvalues λ_1, λ_2.

$$(A - \lambda I) = 0$$

$$\begin{pmatrix} (1-\lambda) & 0 \\ 3 & (2-\lambda) \end{pmatrix} = 0 \qquad \text{(as } I \text{ is a diag matrix)}$$

$$2 - 3\lambda + \lambda^2 = 0 \qquad vA = \lambda v$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\boxed{\lambda_1 = 2, \lambda_2 = 1}$$ (values should be sorted $\lambda_1 > \lambda_2$)

b) Determine Eigenvector $\vec{v_i}$ for λ_i

$\lambda_1 = 2$,

$$\vec{v}A = \lambda \vec{v}$$

$$\Rightarrow \vec{v}(A - \lambda I) = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 0 \\ 3 & 0 \end{pmatrix} \cdot \vec{v} = \vec{0}$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ choose v based on above Eq^n

Can be any value?

$\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \vec{v} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ as $(0,0)$ is no direction or trivial soln

c) $$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|A| = ad - bc.$

$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(Inv exist if $|A| \neq 0$)

d) $$\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Show. $A = P \lambda P^{-1}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = \underline{A}$$

e) $A^{k} = (P \lambda P^{-1})^{k}$

$$A^k = (P\lambda P^{-1})^k$$

$$= P\lambda \underbrace{P^{-1} P}_{I} \lambda \underbrace{P^{-1} P}_{I} \lambda P^{-1} \cdots \quad P\lambda P^{-1}$$

$$= P\lambda^k P^{-1} \quad (\because \lambda \text{ is diagonal matrix})$$

$$= P \begin{pmatrix} 2^k & 0 \\ 0 & 1^k \end{pmatrix} \cdot P^{-1}$$

$$= P \begin{pmatrix} 2^k & 0 \\ 0 & 1^k \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3.2^k & 2^k \\ -1^k & 0 \end{pmatrix}$$

$$= \begin{pmatrix} +1^k & 0 \\ 3(2^k - 1^k) & 2^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(2^k - 1) & 2^k \end{pmatrix}$$

$$\therefore A^{15} = \begin{pmatrix} 1 & 0 \\ 3(2^{15} - 1) & 2^{15} \end{pmatrix}$$

P b 2

Prob 2

Initial distribution at time t

$$\vec{v}_0 = (T, 0, 0, 0)^T$$

1st pt is heated to

Adjacency matrix A :
(1/3 stay at current node.
1/3 moves left, 1/3 moves right)

$$\vec{v}_t = A \cdot \vec{v}_{t-1} = A^t \vec{v}_0$$

a)

$$\lim_{t\to\infty} \vec{v}_t = \lim_{t\to\infty} A^t \vec{v}_0 = \lim_{t\to\infty} P \Lambda^t P^{-1} \cdot \vec{v}_0$$

(λ is already given)

$$= \lim_{t\to\infty} P \begin{pmatrix} 1^t & 0 & 0 & 0 \\ 0 & \frac{1}{3}^t & 0 & 0 \\ 0 & 0 & \frac{1}{3}^t & 0 \\ 0 & 0 & 0 & \left(-\frac{1}{3}\right)^t \end{pmatrix} P^{-1} \vec{v}_0$$

$$= \lim_{t\to\infty} P(*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} \vec{v}_0$$

When we raise fractions to the power infinity, they tend to zero

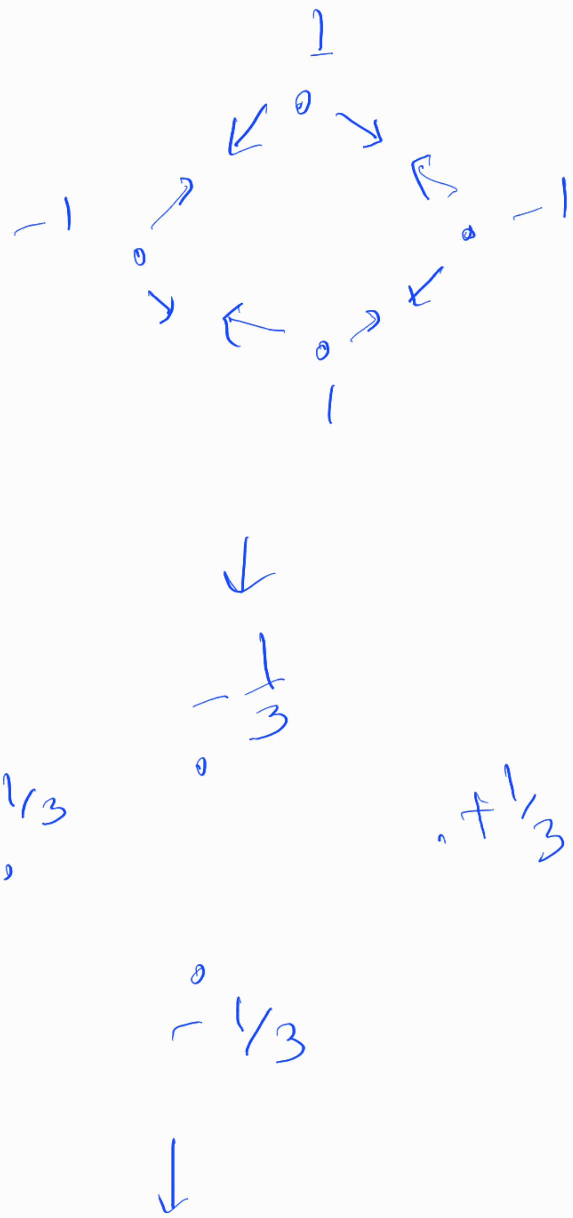
$$= \lim_{t\to\infty} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{pmatrix} P^{-1} \vec{v}_0$$

$$= \lim_{t \to \infty} \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{T}{4} (1, 1, 1, 1)^T$$

b)

Decay with Oscillations



3.

$$\begin{pmatrix} f_{i+1} \\ f_i \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_{i-1} \end{pmatrix} \quad \left(\text{in general, } f_{i+1} = f_i + f_{i-1} \right.$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2 \times 2)$$

$$f_{i+1} = a f_i + b f_{i-1}$$
$$(a = 1,\ b = 1)$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f_i = c f_i + d f_{i-1}$$
$$(c = 1,\ d = 0)$$

(1) find λ values

$$A - \lambda I = 0$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \Rightarrow -\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0 \qquad \Rightarrow \lambda = \frac{+1 \pm \sqrt{1+4}}{} = \frac{1 \pm \sqrt{5}}{2}$$

Gaussian Elimination Process

② Find Eigenvectors of A.

choose $v_1 = \begin{pmatrix} -1 \\ \lambda_2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ \lambda_1 \end{pmatrix}$

Proof:

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v_i} = \lambda_i \vec{v_i}$

let $v_i = \begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow x + y = \lambda x$
$\Rightarrow x = \lambda y$

$\Rightarrow y(1+\lambda) = \lambda^2 y$
$\Rightarrow y(\lambda^2 - \lambda - 1) = 0$

say $x = 1$, $y = \frac{1}{\lambda_1}$ for v_1

$x = 1$ $y = \frac{1}{\lambda_2}$ for v_2

$$\begin{pmatrix} 1-\lambda_i & 1 \\ 1 & -\lambda_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1-\lambda_i)x + y = 0$

$x - y\lambda_i = 0$

3) Decomposition:

$$A = \begin{pmatrix} -1 & -1 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 \\ -\lambda_2 & -1 \end{pmatrix}$$

$$\lambda_2 - \lambda_1 = \left(\frac{1-\sqrt{5}}{2}\right) - \left(\frac{1+\sqrt{5}}{2}\right) = -\sqrt{5}$$

$$\lambda_1 \lambda_2 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = -1$$

Result:

$$\begin{pmatrix} f_{i+1} \\ f_i \end{pmatrix} = A^i \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = P \Lambda^i P^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$