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EVERLASTING learning

FUNDAMENTALS OF COMPUTER GRAPHICS (CSIT304)

B-SPLINE CURVES

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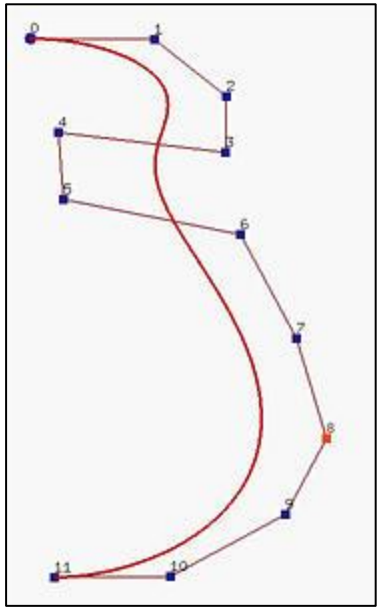
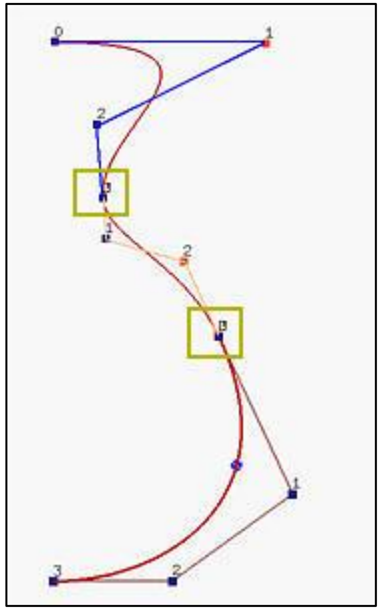
BEZIER CURVES: ISSUES

- No local control

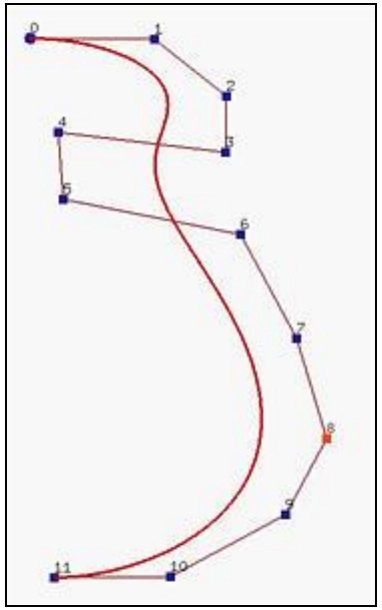
- Degree of curve is fixed by the number of control points

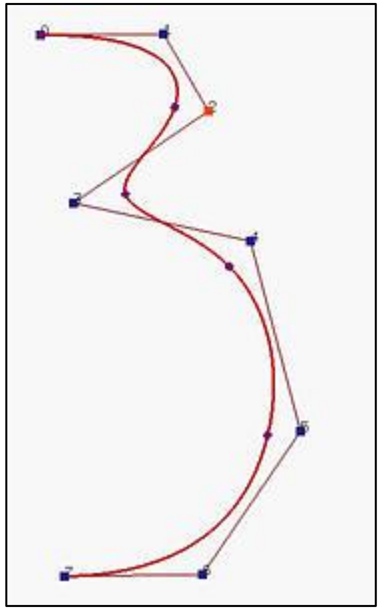




B SPLINE

- Each control point has a unique basis function
- Local control is facilitated
- Is it possible that we still can use lower degree curve segments without worrying about the G1 continuous condition?
- B-spline curves are generalizations of Bézier curves and are developed to answer this question.





B SPLINE (LOCAL CONTROL)

- Bézier basis functions are used as weights.
- B-spline basis functions are much more complex.
- There are two unique properties:
  - The domain is subdivided by knots, and
  - Basis functions are **not non-zero on the entire interval**.
- Each B-spline basis function is non-zero on a few adjacent subintervals
- As a result, B-spline basis functions are quite "local".

B-SPLINE (KNOT, DEFINITION)

- Let U be a set of m + 1 non-decreasing numbers
  - u0 <= u2 <= u3 <= ... <= um.
  - The ui's are called knots, the set U the knot vector,
  - The half-open interval [ui, ui+1) the i-th knot span.
- If a knot ui appears k times (i.e., ui = ui+1 = ... = ui+k-1), where k > 1, ui is a multiple knot of multiplicity k, written as ui(k).
- If ui appears only once, it is a simple knot.
- If the knots are equally spaced (i.e., ui+1 - ui is a constant for 0 <= i <= m - 1), the knot vector or the knot sequence is said **uniform**; otherwise, it is **non-uniform**.

B SPLINE CURVES

- The user supplies: the degree p, n+1 **control points**, and m+1 **knot vectors**

- Write the curve as:

$$P(t) = \sum_{i=0}^{n} P_i N_i^p(t)$$

- The functions N_i^p are the *B-Spline basis functions*

B-Spline Animation

B SPLINE BASIS

- The domain is subdivided by knots, and
- Basis functions are **not non-zero** on the entire interval.
- Some knot spans may not exist (Repeat)
  - Simple / Multiple Knots
  - Uniform/ Non-Uniform Knots

B-Spline Basis Plots

- The ***i*-th B-spline basis function** of ***degree p***

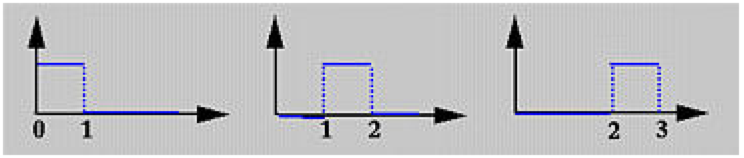
$$N_i^0(t) = \begin{cases} 1, t_i \leq t \leq t_{i+1} \\ Otherwise \end{cases}$$

$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_i} N_{i+1}^{p-1}(t)$$

Cox-de Boor recursion formula

EXPLANATION

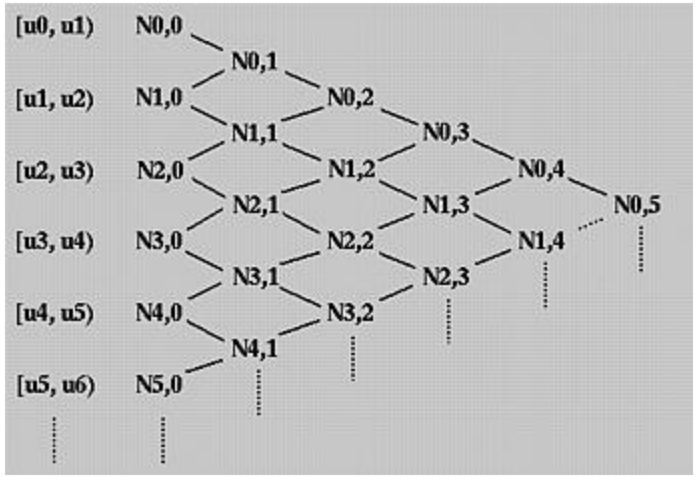
- If the degree is zero (i.e., p = 0)
    - These basis functions are all step functions.
- Basis function $N_{i,0}(u)$ is 1 if u is in the i-th knot span [ui, ui+1).
- For example,
    - If we have four knots u0 = 0, u1 = 1, u2 = 2 and u3 = 3,
    - Knot spans 0, 1 and 2 are [0,1), [1,2), [2,3)
    - The basis functions of degree 0 are $N_{0,0}(u)$ = 1 on [0,1) and 0 elsewhere, $N_{1,0}(u)$ = 1 on [1,2) and 0 elsewhere, and $N_{2,0}(u)$ = 1 on [2,3) and 0 elsewhere.



$N_{0,0}(u)$ $N_{1,0}(u)$ $N_{2,0}(u)$

B SPLINE BASIS: OBSERVATIONS 1

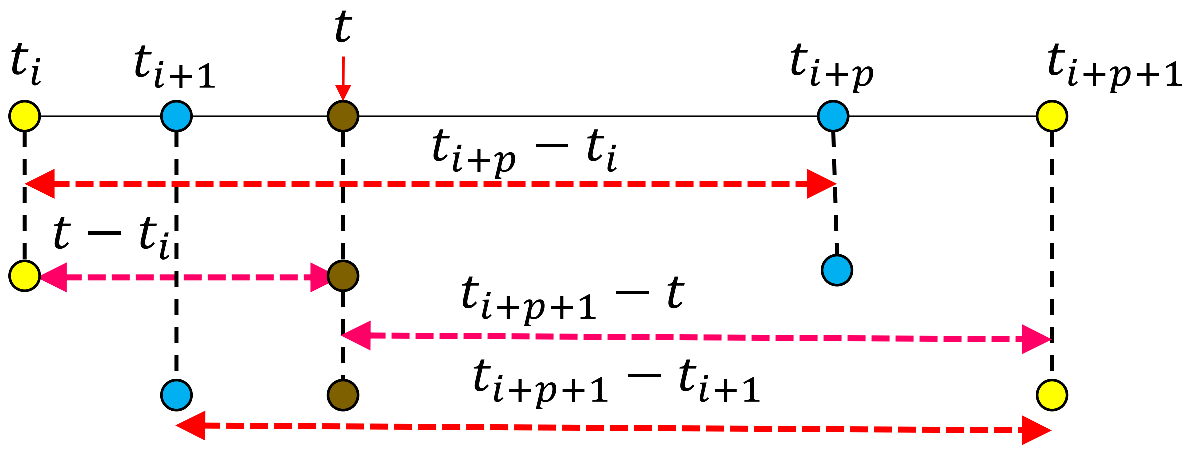
- Non-zero domain of a basis function



Basis function $N_{i,p}(u)$ is non-zero on $[t_i, t_{i+p+1})$

B SPLINE BASIS: OBSERVATIONS 2

- Influence of the basis function coefficients



Linear combination of two intervals, where both are linear in u

EXAMPLE

- Suppose the knot vector is $T = \{\ 0, 0.25, 0.5, 0.75, 1\ \}$.
- Hence, $n = 4$ and $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$ and $t_4 = 1$.

$$P(t) = \sum_{i=0}^{n} P_i N_i^p(t)$$

$$N_i^0(t) = \begin{cases} 1, t_i \leq t \leq t_{i+1} \\ \quad Otherwise \end{cases}$$

$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_i} N_{i+1}^{p-1}(t)$$

Degree	Basis Function	Range	Equation
0	N_0^0		
	N_1^0		
	N_2^0		
	N_3^0		
1	N_0^1		
	N_1^1		
	N_2^1		

EXAMPLE

- Suppose the knot vector is $T = \{\ 0, 0.25, 0.5, 0.75, 1\ \}$.
- Hence, n = 4 and $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$ and $t_4 = 1$.

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$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_i} N_{i+1}^{p-1}(t)$$

Degree	**Basis Function**	**Range**	**Equation**
0	N_0^0	[0,0.25)	=1
	N_1^0	[0.25,0.5)	=1
	N_2^0	[0.5,0.75)	=1
	N_3^0	[0.75,1)	=1
1	N_0^1		
	N_1^1		
	N_2^1		

EXAMPLE

- Suppose the knot vector is $T = \{ 0, 0.25, 0.5, 0.75, 1 \}$.
- Hence, $n = 4$ and $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$ and $t_4 = 1$.

$$P(t) = \sum_{i=0}^{n} P_i N_i^p(t)$$

$$N_i^0(t) = \begin{cases} 1, t_i \le t \le t_{i+1} \\ \quad Otherwise \end{cases}$$

$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_i} N_{i+1}^{p-1}(t)$$

Degree	**Basis Function**	**Range**	**Equation**
0	N_0^0	[0,0.25)	=1
	N_1^0	[0.25,0.5)	=1
	N_2^0	[0.5,0.75)	=1
	N_3^0	[0.75,1)	=1
1	N_0^1		
	N_1^1		
	N_2^1		

EXAMPLE

- Suppose the knot vector is $T = \{ 0, 0.25, 0.5, 0.75, 1 \}$.
- Hence, $n = 4$ and $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$ and $t_4 = 1$.

$$P(t) = \sum_{i=0}^{n} P_i N_i^p(t)$$

$$N_i^0(t) = \begin{cases} 1, t_i \le t \le t_{i+1} \\ \quad Otherwise \end{cases}$$

$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_i} N_{i+1}^{p-1}(t)$$

Degree	Basis Function	Range	Equation
0	N_0^0	[0,0.25)	=1
	N_1^0	[0.25,0.5)	=1
	N_2^0	[0.5,0.75)	=1
	N_3^0	[0.75,1)	=1
1	N_0^1	[0,0.25)	4t
		[0.25,0.5)	2(1-t)
	N_1^1		
	N_2^1		

DERIVATIVES OF B SPLINE CURVE

$$P(t) = \sum_{i=0}^{n} P_i N_i^p(t)$$

The derivative of **each of these basis functions** can be computed as follows:

$$\frac{d}{dt}\left(P(t)\right) = \sum_{i=0}^{n} P_i N_i^p(t)' = \frac{p}{t_{i+p} - t_i} N_i^{p-1}(t) - \frac{p}{t_{i+p+1} - t_{i+1}} N_{i+1}^{p-1}(t)$$

Plugging these derivatives back

$$\frac{d}{dt}\left(P(t)\right) = \sum_{i=0}^{n-1} N_{i+1}^{p-1}(t) Q_i \qquad where, Q_i = \frac{p}{t_{i+p+1} - t_{i+1}}(P_{i+1} - P_i)$$

Derivative of a B-spline curve **is another B-spline curve of degree $p - 1$** on the **original knot vector** with a **new set of n control points, $Q_0, Q_1, \dots, Q_{n-1}$**

PROPERTIES OF B-SPLINE

- $\mathrm{N}_{\mathrm{i}}^{\mathrm{p}}(t)$ is a degree p polynomial in t
- **Non-negativity:** For all i, p and t, $N_i^p(t)$ is non-negative
- **Local Support:** $N_i^p(t)$ is a non-zero polynomial on $[t_i, t_{i+p+1})$
- On any span $[t_i, ti_{+p+1})$, **at most $p + 1$ degree p basis functions are non-zero**
  - $N_{i-p}^p(t)$, $N_{i-p+1}^p(t)$, …, $N_i^p(t)$
- **Partition of Unity**
  - The sum of all non-zero degree p basis functions on span $[t_i, t_{i+p+1})$ is unity, i.e. $\sum_{k=0}^{p} N_{i-k}^p = 1$

PROPERTIES OF B-SPLINE

- m = n+p+1

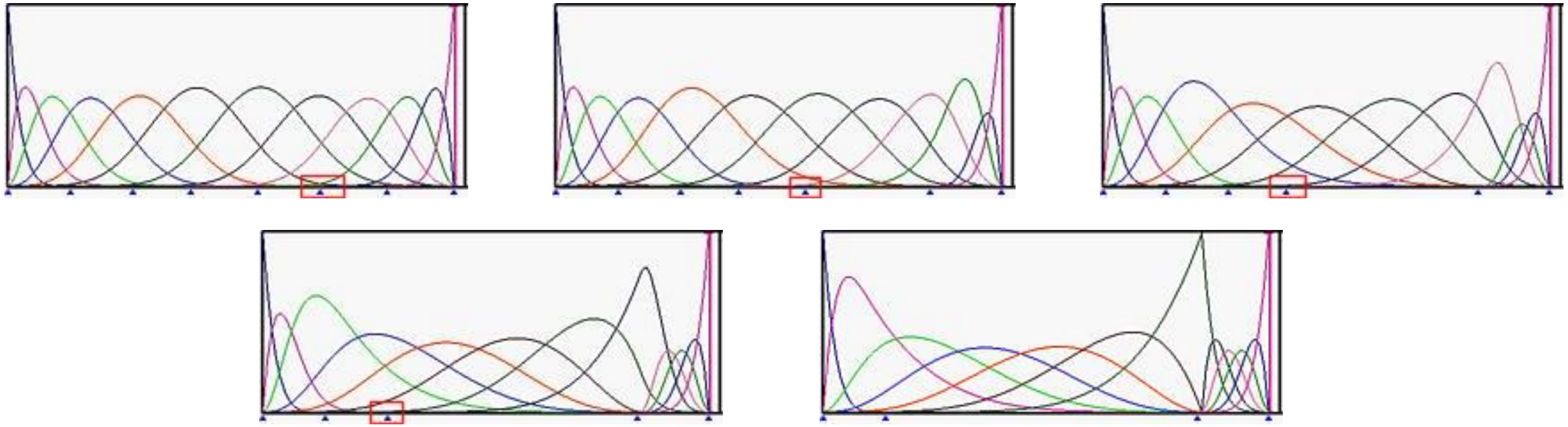
- Basis function $N_i^p(t)$ is a composite curve of degree p polynomials with joining points at knots in $[t_i, t_{i+p+1})$

- At a knot of multiplicity k, basis function $N_i^p(t)$ is $\mathrm{C}^{\mathrm{p}-k}$ continuous.

- Convex hull property

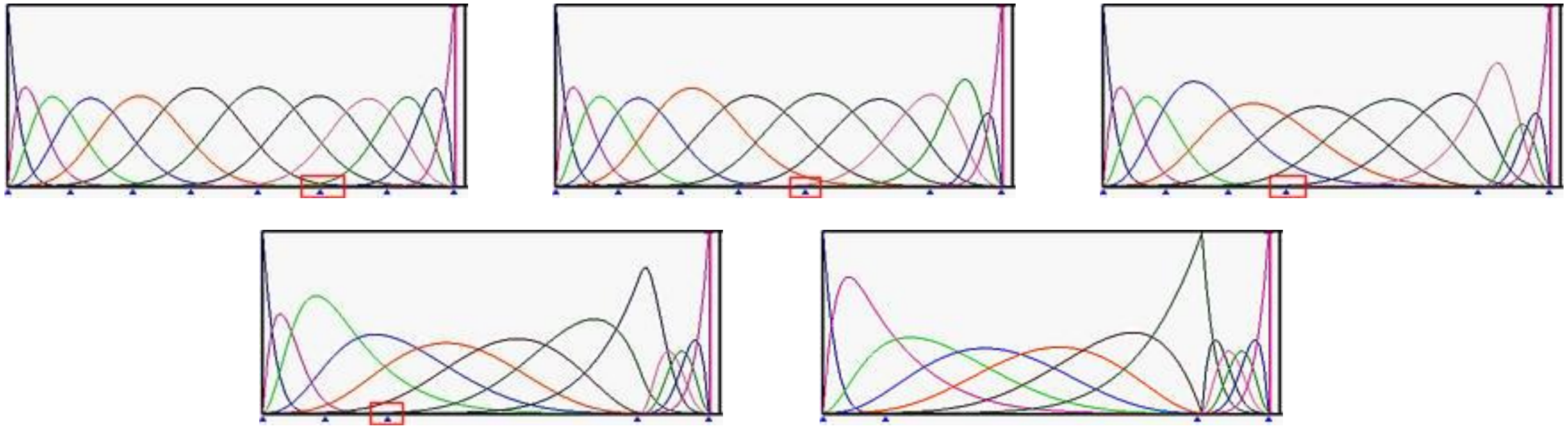
IMPACT OF MULTIPLE KNOTS

- Significant impact on the computation of basis functions
- Counting properties
- **Each knot of multiplicity k reduces at most $k-1$ basis functions' non-zero domain.**



IMPACT OF MULTIPLE KNOTS

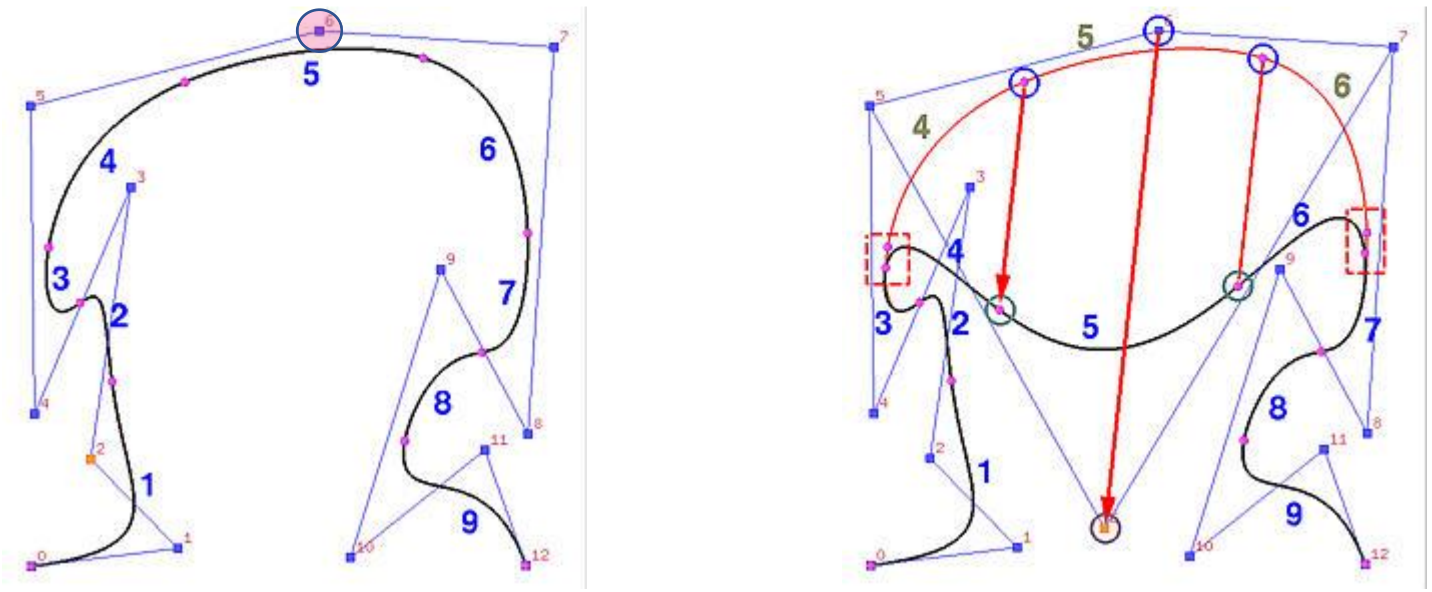
- At each internal knot of multiplicity *k*, the number of non-zero basis functions is at most $p - k + 1$, where *p* is the degree of the basis functions.



B SPLINE MOVING CONTROL POINTS

- Local control scheme

p=4,
n=12,
m=17



Span	$[t_4, t_5)$	$[t_5, t_6)$	$[t_6, t_7)$	$[t_7, t_8)$	$[t_8, t_9)$	$[t_9, t_{10})$	$[t_{10}, t_{11})$	$[t_{11}, t_{12})$	$[t_{12}, t_{13})$
Segment	1	2	3	4	5	6	7	8	9

B-SPLINE CURVES: KNOT INSERTION

- Adding a new knot into the existing knot vector
  - Without changing the shape of the curve.

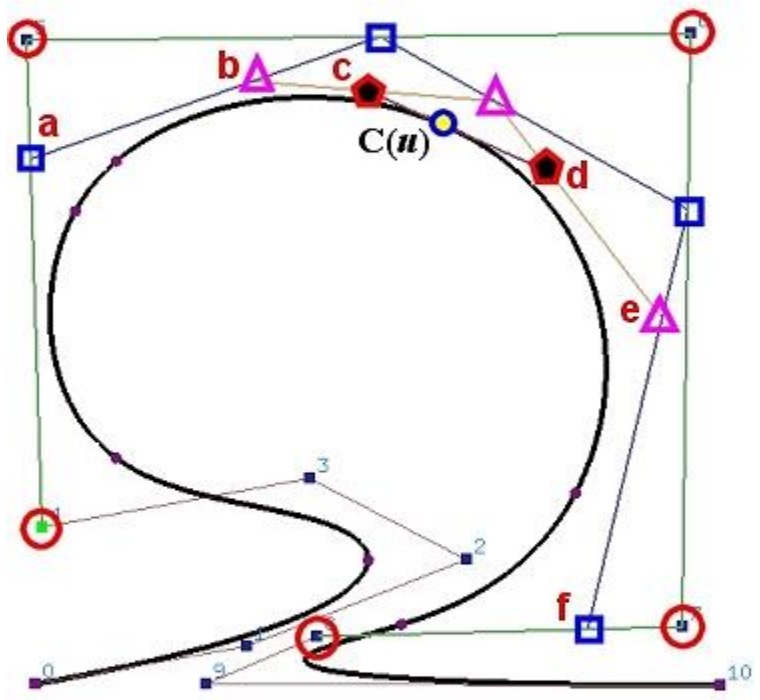
- $m = n + p + 1$

- Inserting a new knot causes a new control point to be added

- Some existing control points are removed and replaced with new ones by **corner cutting**.

SUB-DIVISION

- Follows exactly the same procedure for subdividing a Bézier curve.



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