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FUNDAMENTALS OF COMPUTER GRAPHICS (CSIT304)

THREE-DIMENSIONAL GEOMETRIC TRANSFORMATIONS

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INTRODUCTION

- Extended from two-dimensional methods by including considerations for the z coordinate.
- In some cases– particularly, rotation–the extension to three dimensions is less obvious.
- A three-dimensional position, expressed in homogeneous coordinates, is represented as a four-element column vector
- Each geometric transformation operator is now a 4×4 matrix
- In 3D, the matrix multiplication order remains same as that of 2D

TYPES OF TRANSFORMATION

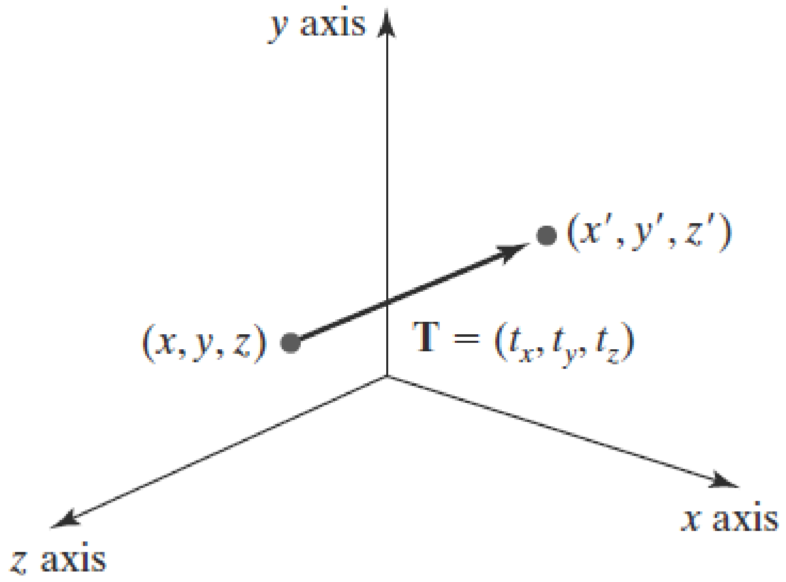
- 3D Transformation
  - Translation
  - Scaling
  - Rotation
    - Around X-Axis
    - Around Y-Axis
    - Around Z-Axis
  - Shear
    - X-Direction
    - Y-Direction
    - Z-Direction
  - Reflection
    - Relative to XY Plane
    - Relative to YZ Plane
    - Relative to ZX Plane

TRANSLATION

- 3D Translation is a process of moving an object from one position to another in a three dimensional plane.

- A position P = (x, y, z) in 3D is translated to a location P′ =(x′, y′, z′) by adding translation distances t_x, t_y, and t_z to the Cartesian coordinates of P

- **How to perform inverse translation?**



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

SCALING

- The matrix expression for the three-dimensional scaling transformation of a position **P** = *(x, y, z)* **relative to the coordinate origin**
- This is a simple extension of two-dimensional scaling.

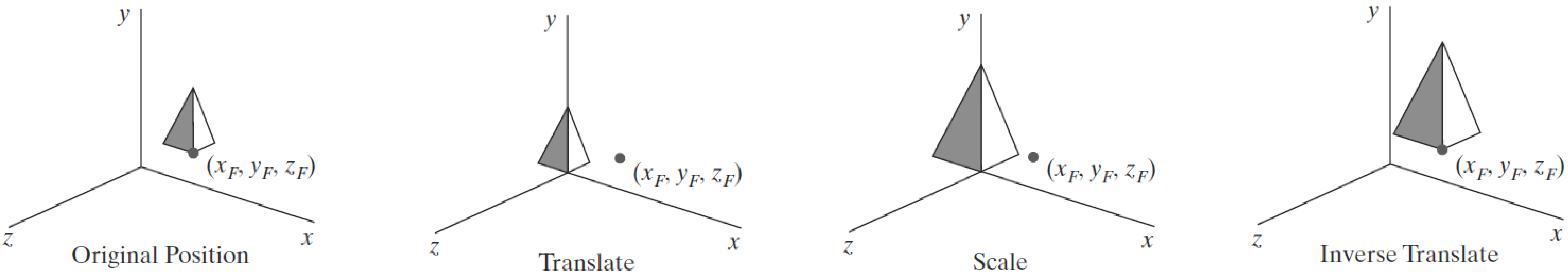
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- **What happens if scale parameter is greater or less than 1?**

SCALING ABOUT A FIXED POINT

- Scaling transformation with respect to any selected fixed position (x_f, y_f, z_f)

1. Translate the fixed point to the origin.
2. Apply the scaling transformation relative to the coordinate
3. Translate the fixed point back to its original position.



SCALING ABOUT A FIXED POINT

- Scaling transformation with respect to any selected fixed position (x_f, y_f , z_f)

1. Translate the fixed point to the origin.
2. Apply the scaling transformation relative to the coordinate
3. Translate the fixed point back to its original position.

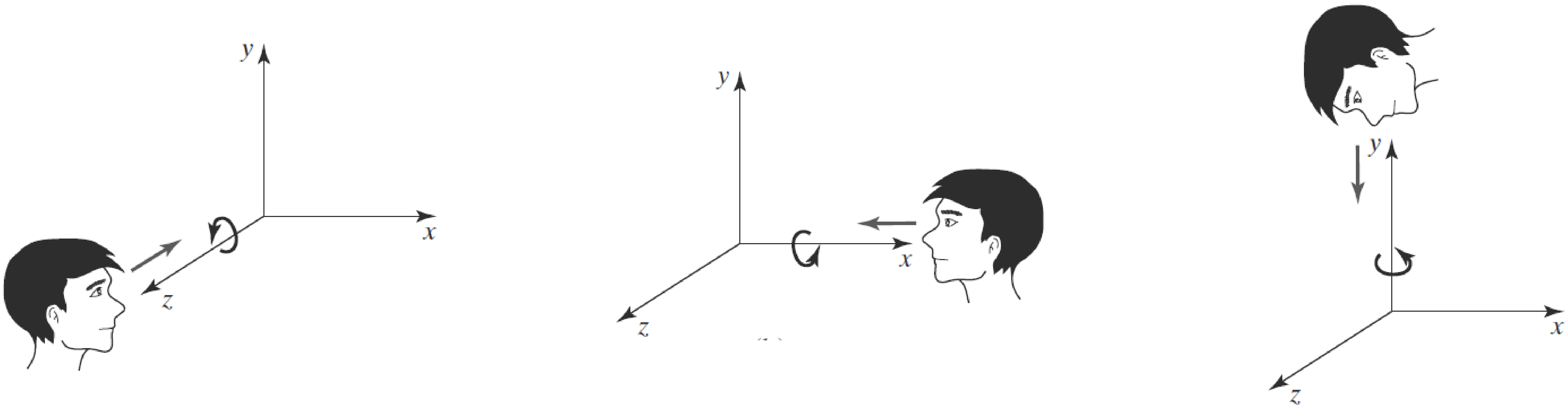
$$\mathbf{T}(x_f, y_f, z_f) \cdot \mathbf{S}(s_x, s_y, s_z) \cdot \mathbf{T}(-x_f, -y_f, -z_f) = \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How to perform inverse scaling?

ROTATION

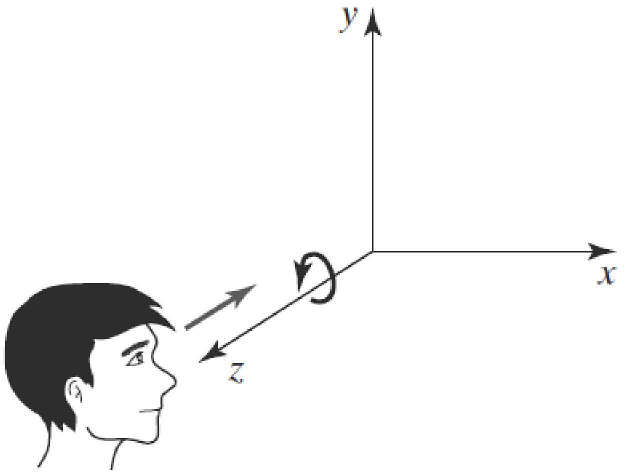
- We can rotate an object about any axis in space
- The easiest rotation axes to handle are those that are parallel to the Cartesian-coordinate axes.
- A combinations of coordinate-axis rotations (along with appropriate translations) to specify a rotation about any other line in space.
- First consider the operations involved in coordinate-axis rotations
- Second, consider the calculations needed for other rotation axes.

ILLUSTRATIONS



- By convention, **positive rotation** angles produce **counterclockwise rotations** about a coordinate axis
- Assuming that we are looking in the **negative direction along that coordinate axis**

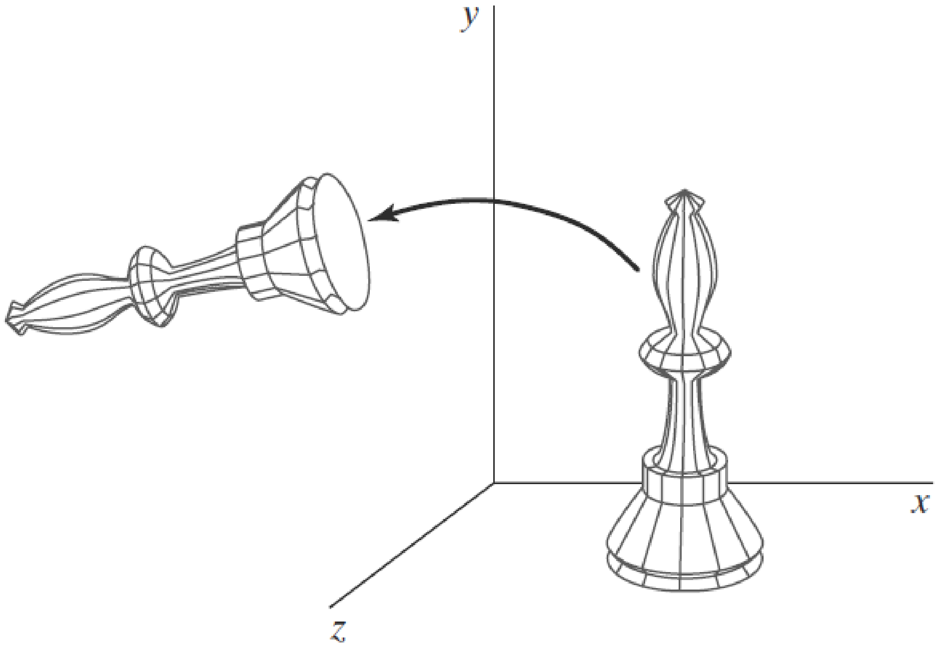
Z-AXIS ROTATION



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

Parameter θ specifies the rotation angle about the z axis

Illustration



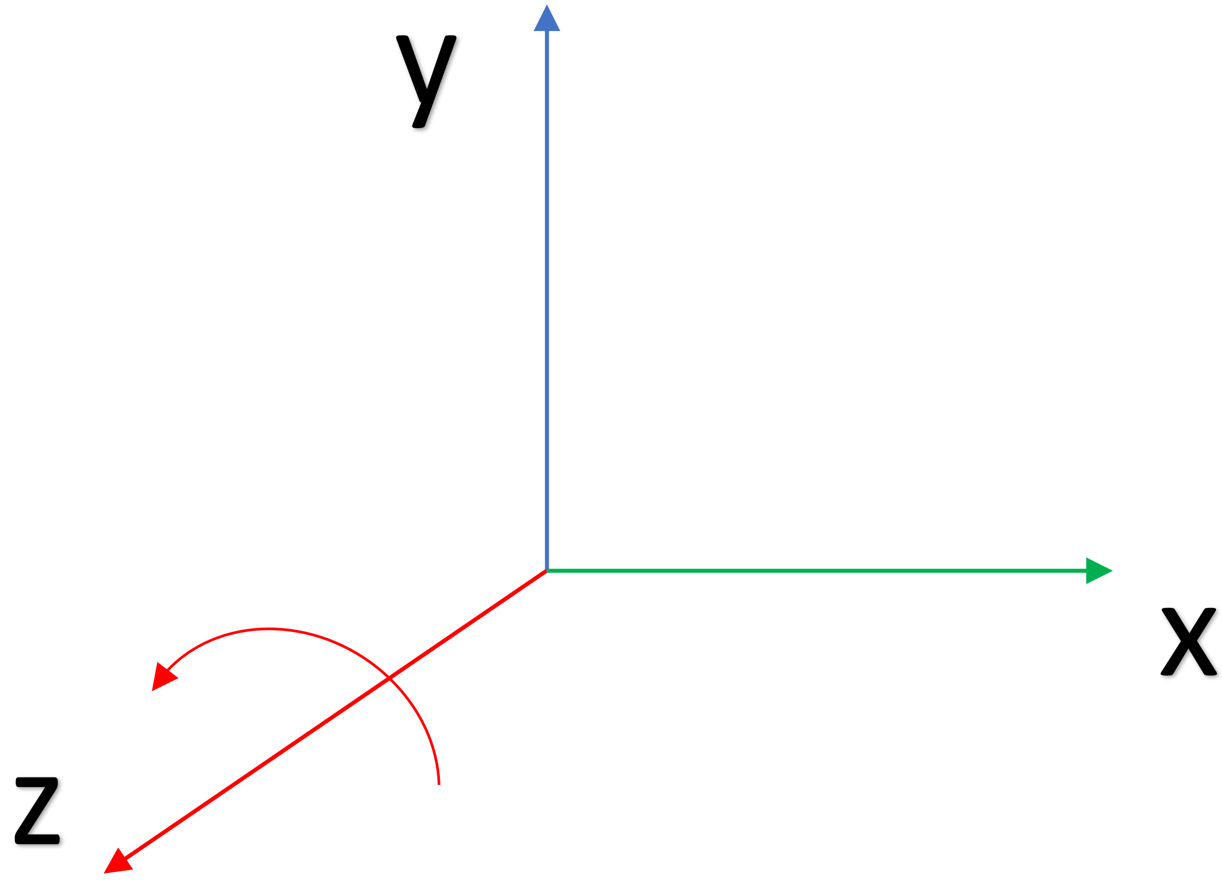
Matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

ROTATION ON OTHER AXIS

- Transformation equations for rotations about the other two coordinate axes can be obtained with a cyclic permutation of the coordinate parameters x, y, and z
- $x \rightarrow y \rightarrow z \rightarrow x$
- Thus, to obtain the x-axis and y-axis rotation transformations, we cyclically replace x with y, y with z, and z with x

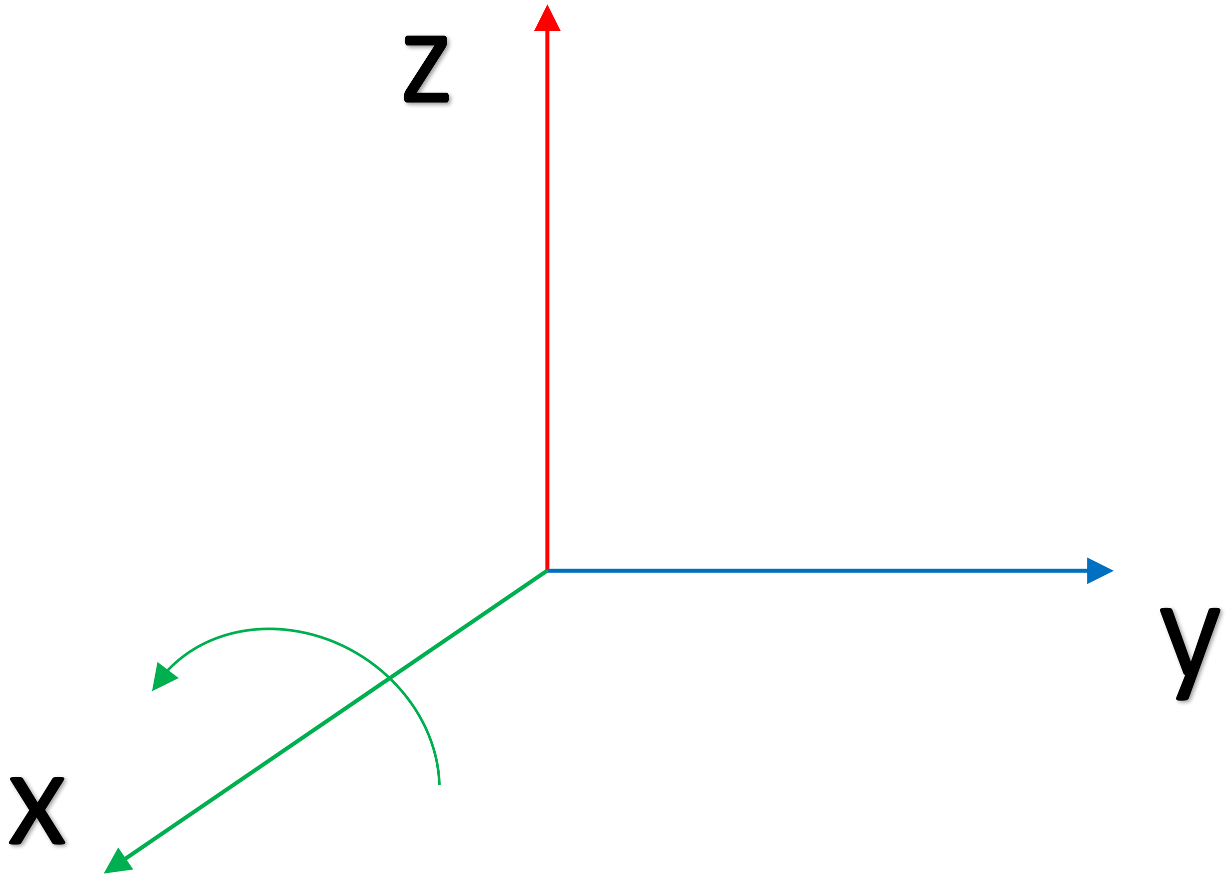
ROTATION ON OTHER AXIS (Z)



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

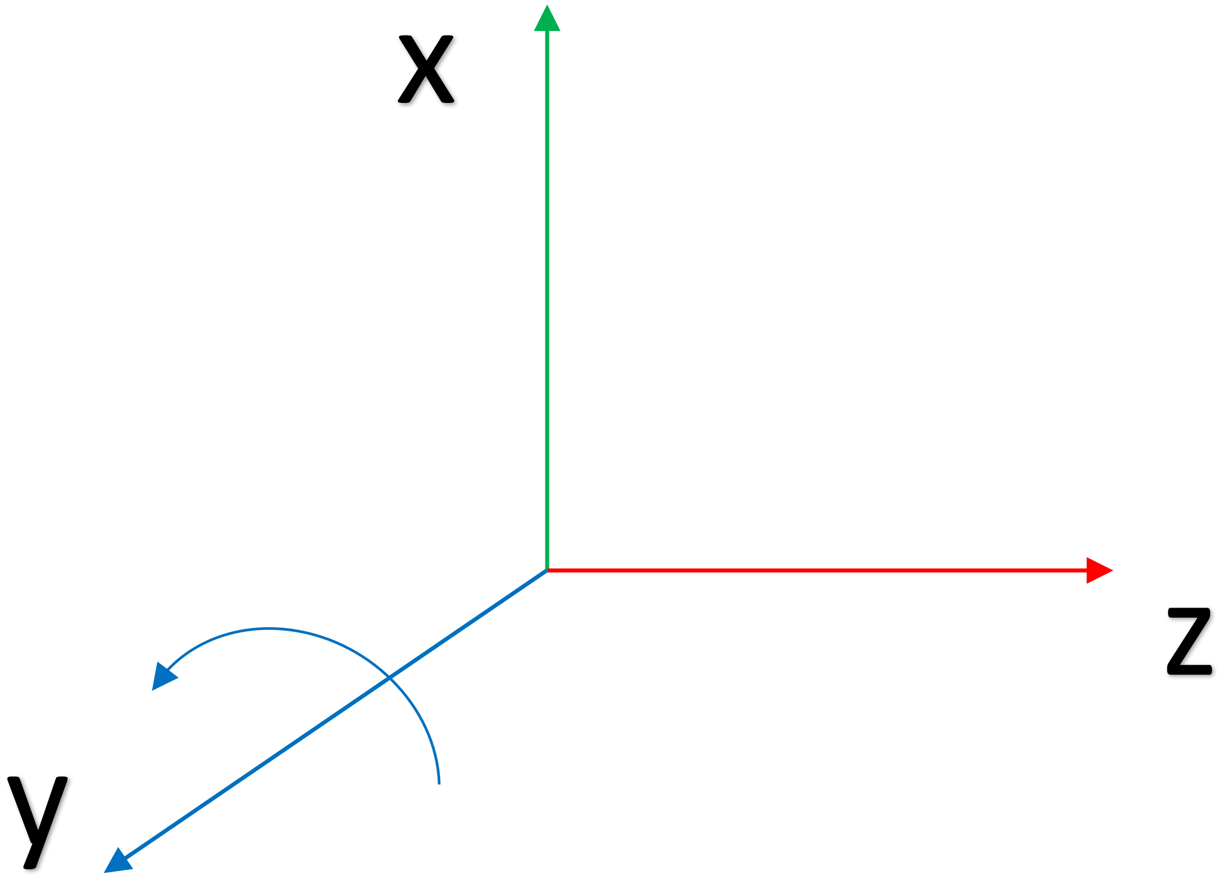
ROTATION ON OTHER AXIS (X)



$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

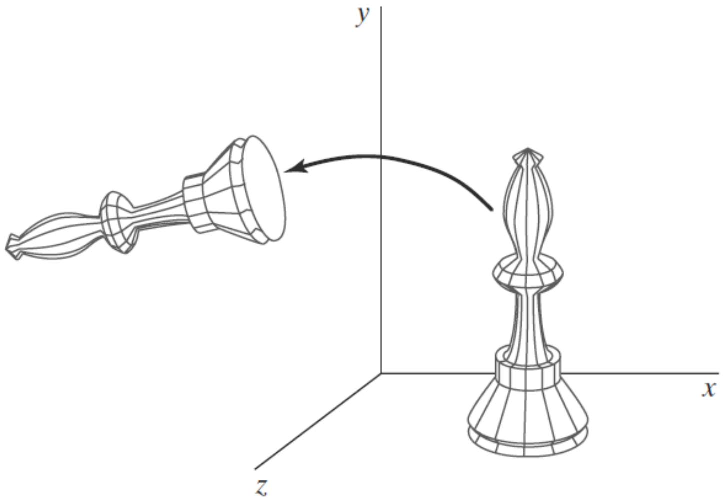
ROTATION ON OTHER AXIS (Y)



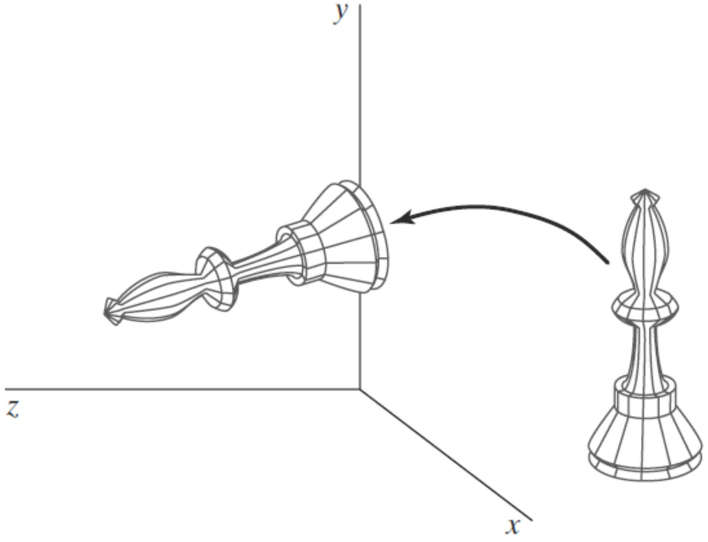
$$z' = z\cos\theta - x\sin\theta$$
$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

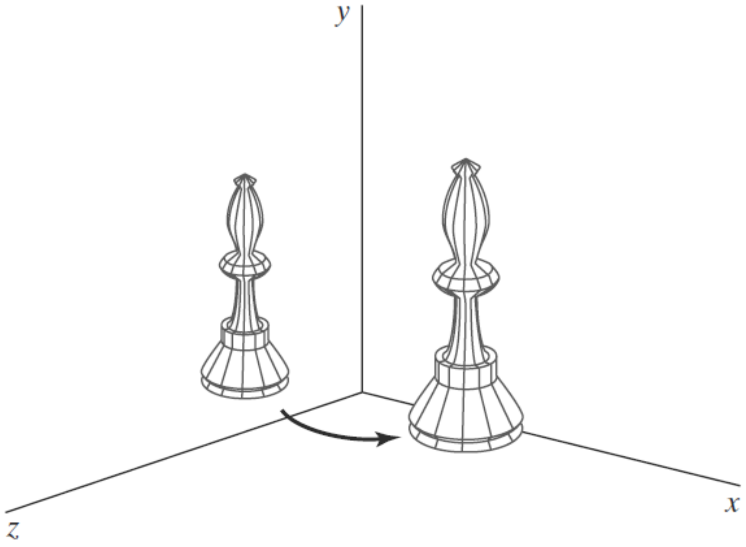
ILLUSTRATION



About z-axis



About x-axis



About y-axis

GENERAL 3D ROTATIONS

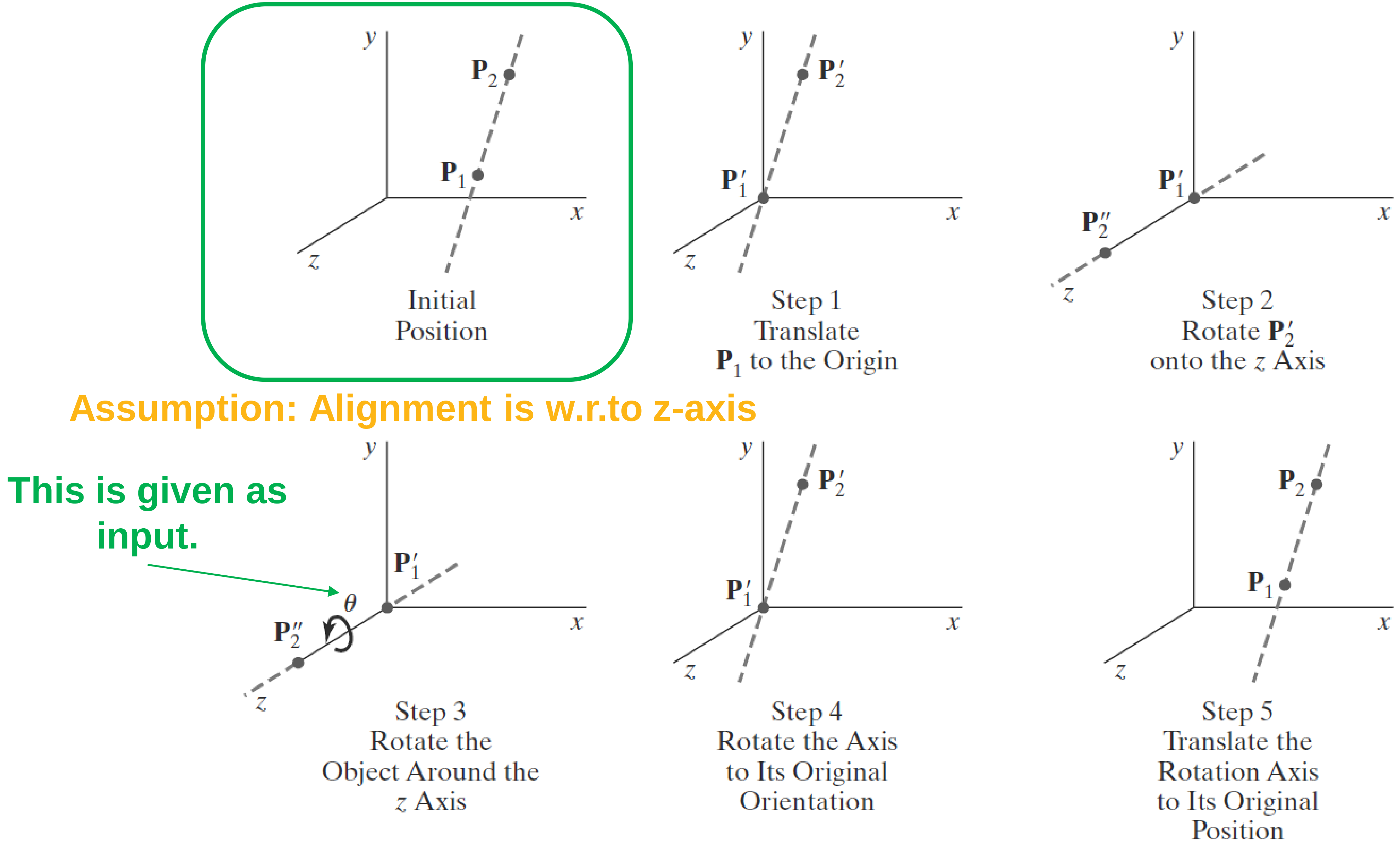
- Rotation matrix for any axis
  - Does not coincide with a coordinate axis
- Composite transformation
  - Involving combinations of translations and the coordinate-axis rotations.
- Given
  - Rotation axis and the rotation angle
- The required rotation needs 5 steps.

1. Translate the object so that the rotation axis passes through the coordinate origin.
2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.
3. Perform the specified rotation about the selected coordinate axis.
4. Apply inverse rotations to bring the rotation axis back to its original orientation.
5. Apply the inverse translation to bring the rotation axis back to its original spatial position.

ILLUSTRATION

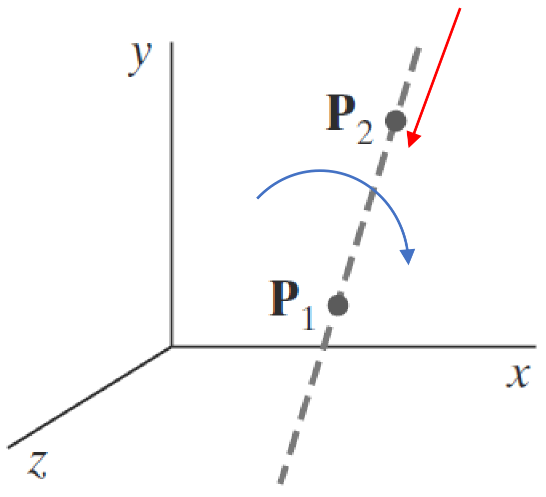
At what angle to rotate?

This is to be computed



HOW TO CALCULATE THE ANGLE?

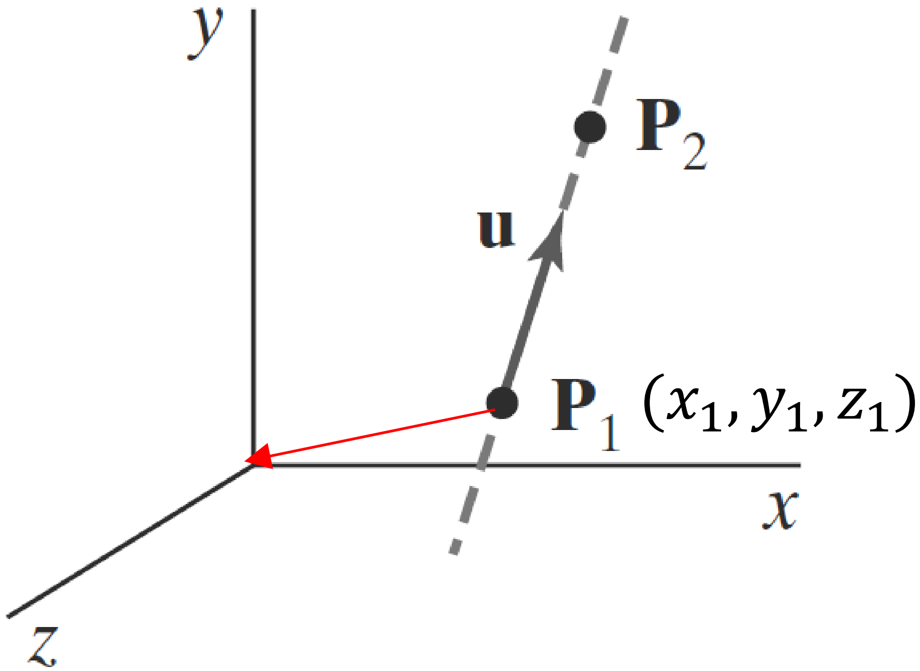
- A rotation axis can be defined with
  - Two coordinate positions
  - Other forms of representations are not considered
- The direction of rotation is
  - **Counterclockwise** when looking along the axis from **P**2 to **P**1
- The components of the rotation-axis vector are then computed as



$$\boldsymbol{V} = \boldsymbol{P_2} - \boldsymbol{P_1} = (x2 - x1, y2 - y1, z2 - z1)$$

Unit Rotation axis vector $\boldsymbol{u} = \frac{\boldsymbol{V}}{|\boldsymbol{V}|} = \left(\frac{x2-x1}{|\boldsymbol{V}|}, \frac{y2-y1}{|\boldsymbol{V}|}, \frac{z2-z1}{|\boldsymbol{V}|}\right)$

STEP 1: TRANSLATION



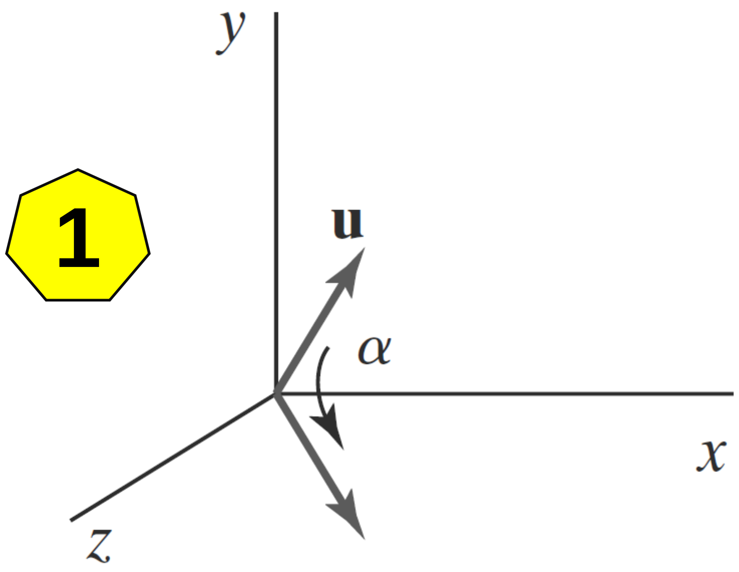
- Counterclockwise rotation when viewing along the axis from P2 to P1

- Move the point P1 to the origin

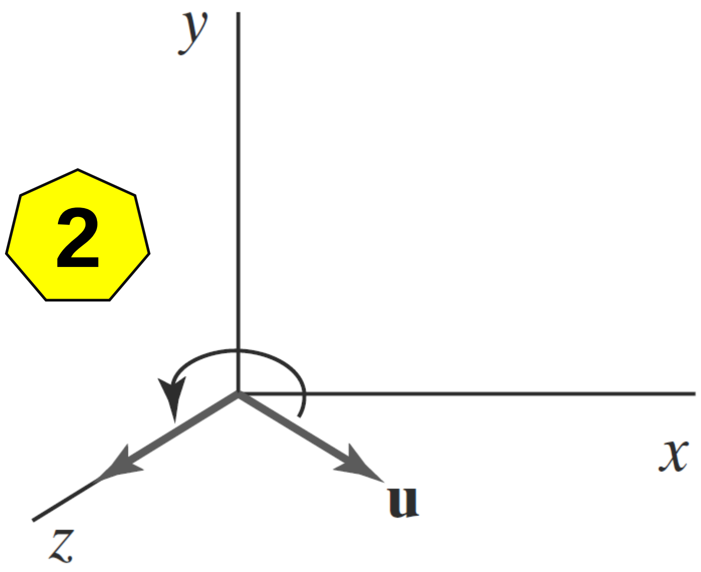
$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

STEP 2: ALIGNMENT

- Put the rotation axis onto the z axis
- This is going to take 2 rotations

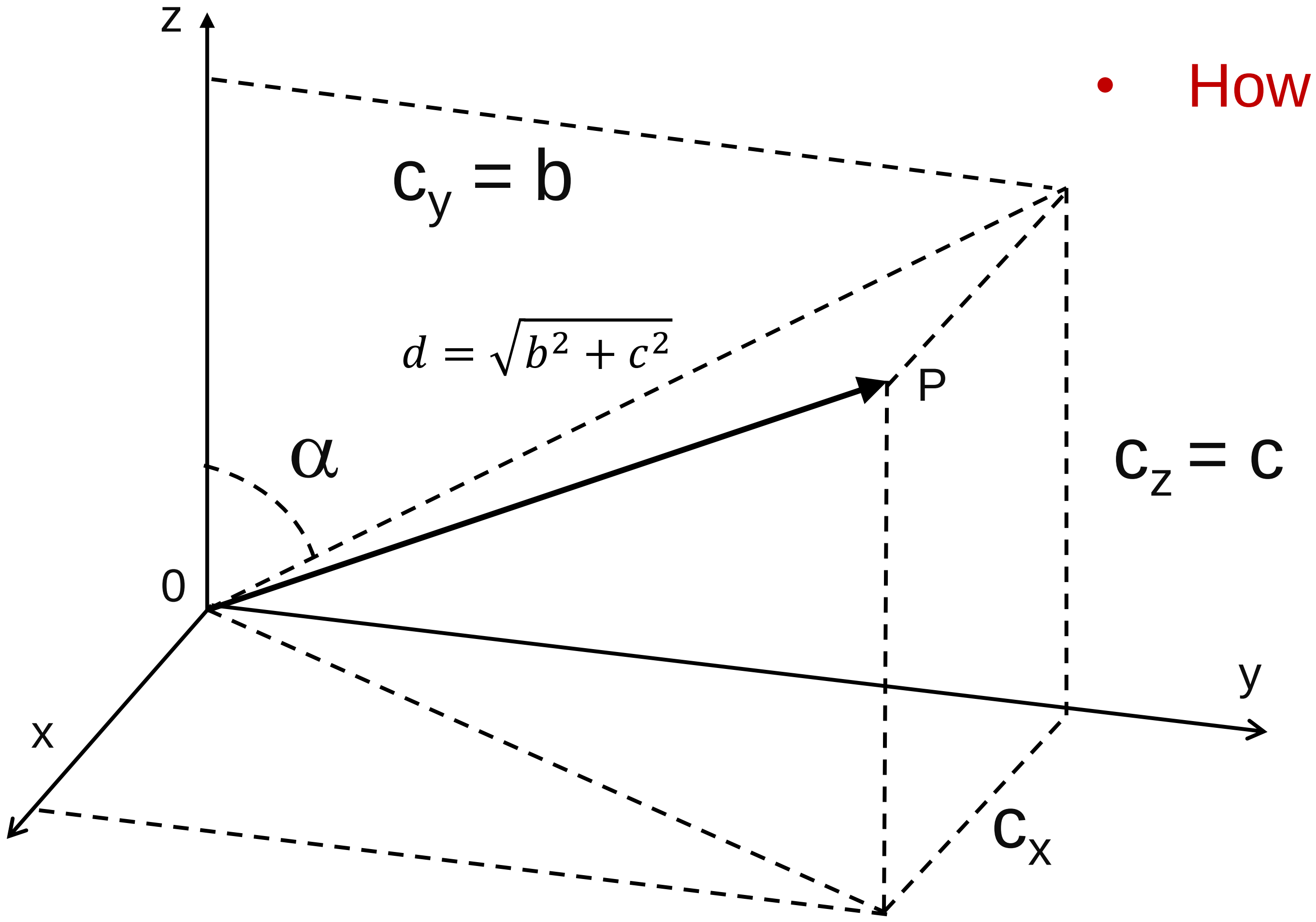


About x-axis
(to place the axis in the xz plane)



About y-axis
(to place the result coincident with the z-axis).

STEP 2.1: ROTATION ABOUT X-AXIS



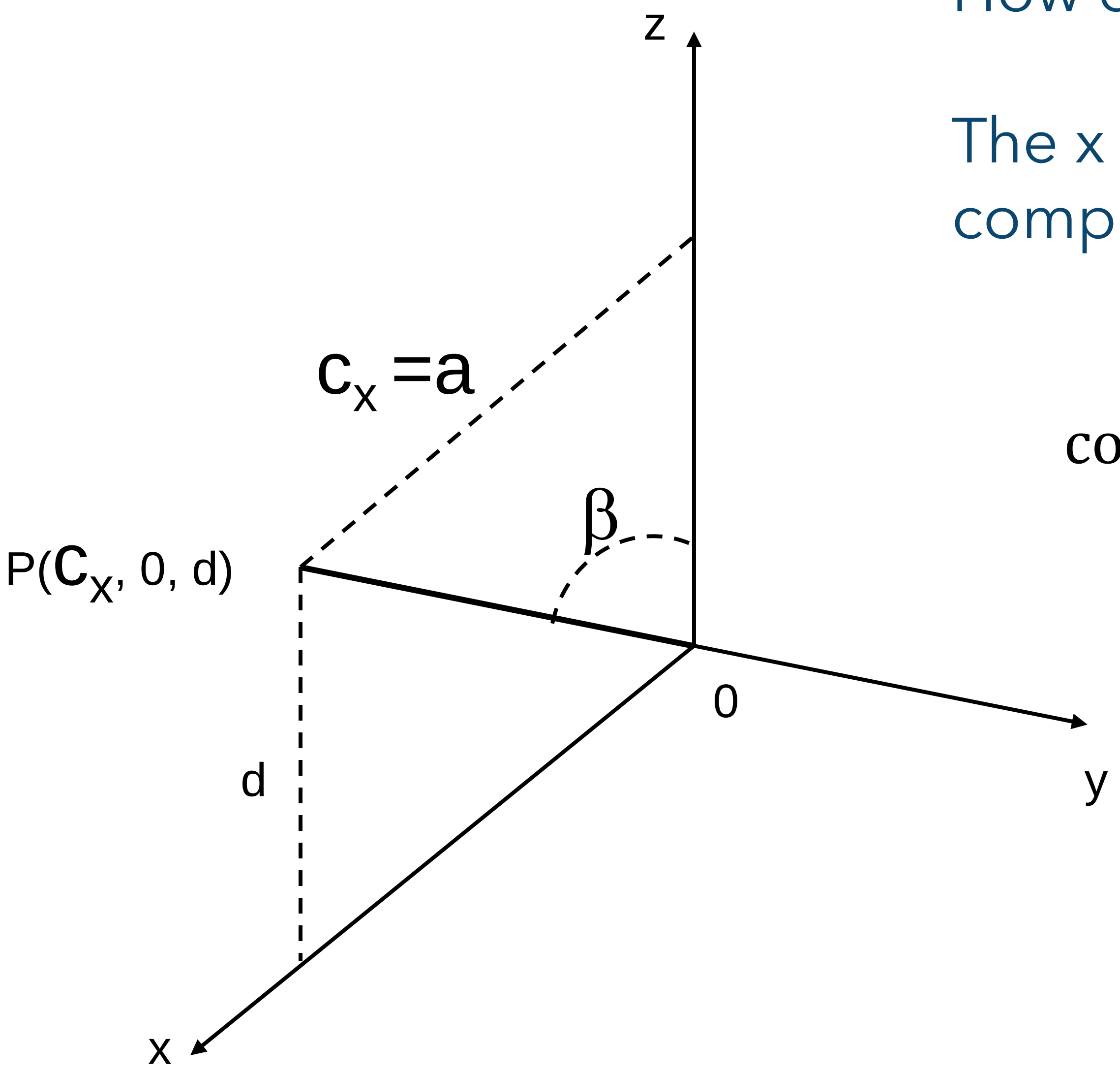
- Rotation about x by α
- α is the angle between the projection of u in the yz plane and the positive z axis
- How do we determine α?

$$\cos\alpha = \frac{c}{d} \qquad \sin\alpha = \frac{b}{d}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

STEP 2.1: ROTATION ABOUT Y-AXIS



How do we determine β?

The x component is cx=a and the z component is d.

$$\cos\beta = \frac{d}{1} \qquad \sin\alpha = \frac{-a}{1}$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

STEP 3: ROTATION ABOUT Z AXIS

- The rotation axis is with the positive z axis.
- The specified rotation angle θ can now be applied as a rotation about the z axis

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

STEP 4 & 5: REPOSITION

- To complete the required rotation about the given axis
  - Transform the rotation axis back to its original position.

- This is done by applying the inverse of transformations

- Therefore, the composite transformation

$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x^{-1}(\alpha) \cdot \mathbf{R}_y^{-1}(\beta) \cdot \mathbf{R}_z(\theta) \cdot \mathbf{R}_y(\beta) \cdot \mathbf{R}_x(\alpha) \cdot \mathbf{T}$$

THREE-DIMENSIONAL REFLECTIONS

- A reflection in a 3-D space can be performed relative to
  - a selected reflection axis or
  - with respect to a reflection plane

- Reflection matrices are set up similarly to those for two dimensions.

- Reflections relative to a given axis
  - Equivalent to 180◦ rotations about that axis.

3D REFLECTION MATRICES

$$T_{XY} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

XY plane

$$T_{YZ} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

YZ plane

$$T_{ZX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ZX plane

THREE-DIMENSIONAL SHEARS

- These transformations can be used to modify object shapes, just as in two dimensional applications.

- They are also applied in three-dimensional viewing transformations for perspective projections.

- For three-dimensional we can also generate shears relative to the z axis.

3D SHEAR ALONG X-AXIS

- Shearing in X axis is achieved by using the following shearing equations-
- $X_{new} = X_{old}$
- $Y_{new} = Y_{old} + Sh_y \; X_{old}$
- $Z_{new} = Z_{old} + Sh_z \; X_{old}$

3D SHEAR ALONG Y-AXIS

- Shearing in Y axis is achieved by using the following shearing equations-
- $X_{new} = X_{old} + Sh_x \times Y_{old}$
- $Y_{new} = Y_{old}$
- $Z_{new} = Z_{old} + Sh_z \times Y_{old}$

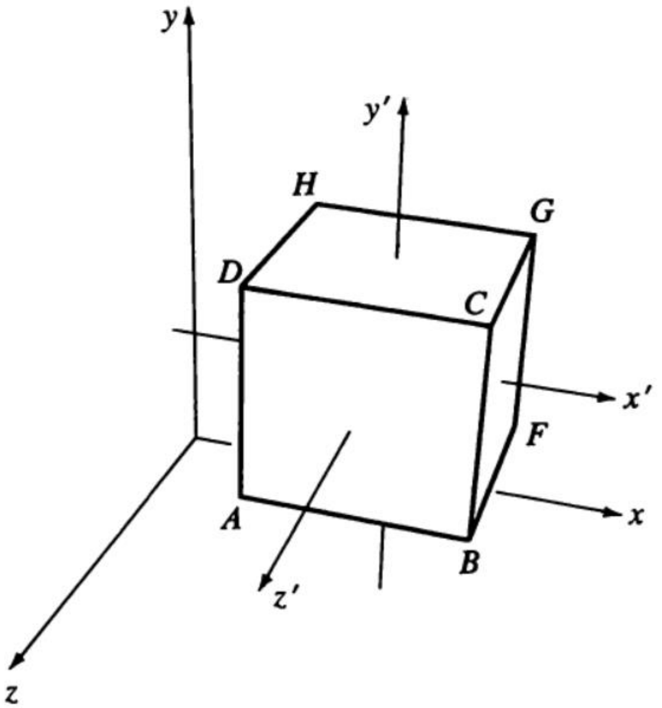
3D SHEAR ALONG Y-AXIS

- Shearing in Z axis is achieved by using the following shearing equations-
- $X_{new} = X_{old} + Sh_x \times Z_{old}$
- $Y_{new} = Y_{old} + Sh_y \times Z_{old}$
- $Z_{new} = Z_{old}$

NUMERICAL EXAMPLES

PROBLEM 1

Consider the block in the figure below is a defined by the position vectors [X]

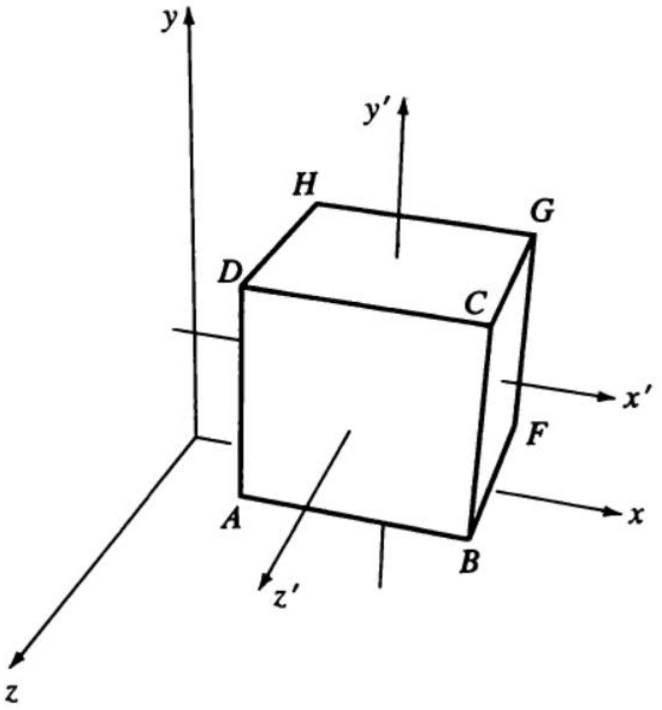


$$[X] = \begin{array}{c} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}$$

relative to the global xyz-axis system. Let's rotate the block θ = +30° about the local x'-axis passing through the centroid of the block. The origin of the local axis system is assumed to be the centroid of the block.

PROBLEM 1

Consider the block in the figure below is a defined by the position vectors [X]



$$[X] = \begin{array}{c} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}$$

To rotate the block ϕ= -45° about the y'-axis, followed by a rotation of θ = +30° about the x'-axis.

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THANK YOU