FLAME UNIVERSITY

EVERLASTING learning

FUNDAMENTALS OF COMPUTER GRAPHICS (CSIT304)

SURFACE REPRESENTATION

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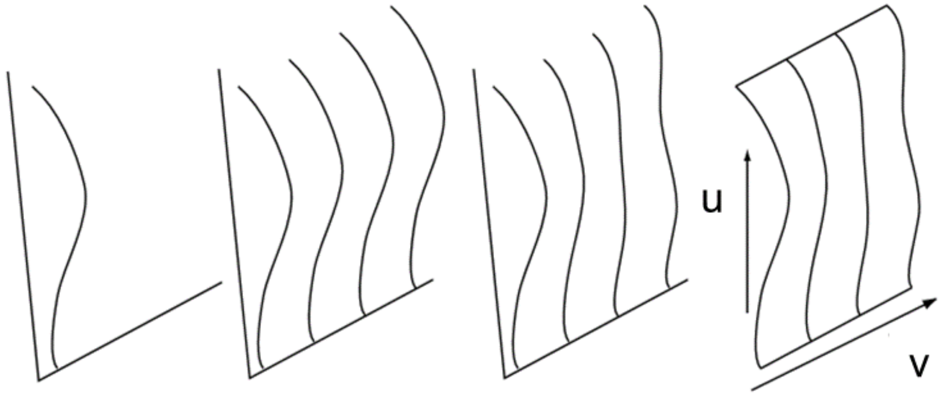
OBJECTIVE

After completing this lecture, students will be able to

- Mathematically represent a surface patches
  - Planar Surface
  - Curved Surface
  - Bi-linear Surface
  - Lofted or Ruled Surface
- Solve related mathematical problems

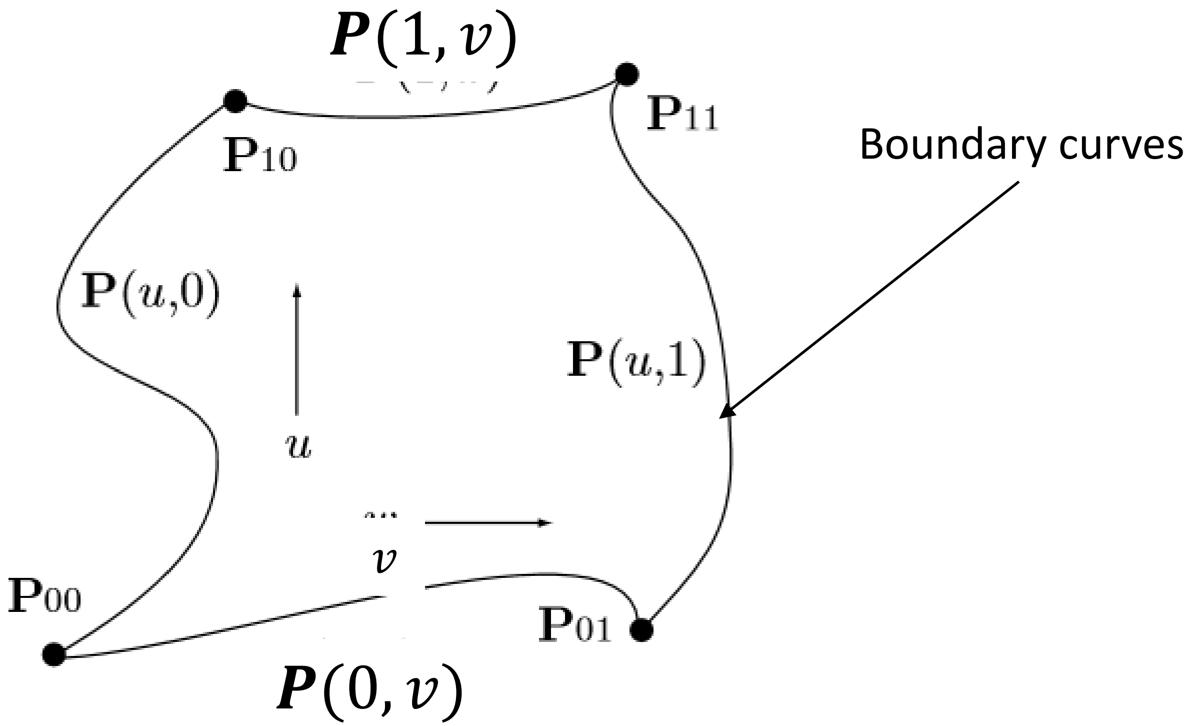
SURFACE

- Representation of surface
  - Explicit
  - Implicit
  - Parametric



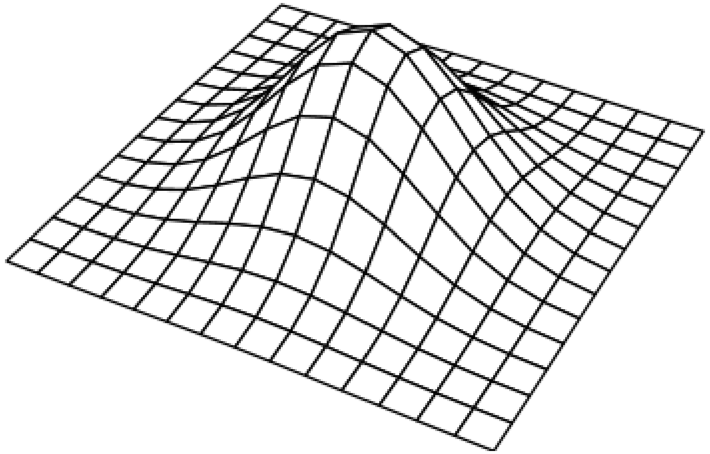
(a) (b) (c) (d)

A SURFACE PATCH



DISPLAYING A SURFACE PATCH

- A surface patch can be displayed either as a wire frame



NON-PARAMETRIC SURFACES

- A quadratic surface would have an equation of the form:

$$ax^2 + ey^2 + hz^2 + 2bxy + 2cxz + 2fyz + 2dx + 2gy + 2iz + 1 = 0$$

- Can we represent this as matrix multiplication?

$$\begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & 1 \end{bmatrix}$$

PARAMETRIC SURFACES

- A point on a surface patch is a function of two parameters

$$P(u,v) = au^2 + 2buv + 2cu + dv^2 + 2ev + f = 0$$

- Can we represent this as matrix multiplication?

$$P(u,v) = [u\ v\ 1] \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} & \mathbf{e} \\ \mathbf{c} & \mathbf{e} & \mathbf{f} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

SURFACE EDGES

- The surface has edges given by the four curves for which one of the parameters is either 0 or 1.

$$P(u,v) = au^2 + 2buv + 2cu + dv^2 + 2ev + f = 0$$

$$P(0,v) = ?$$

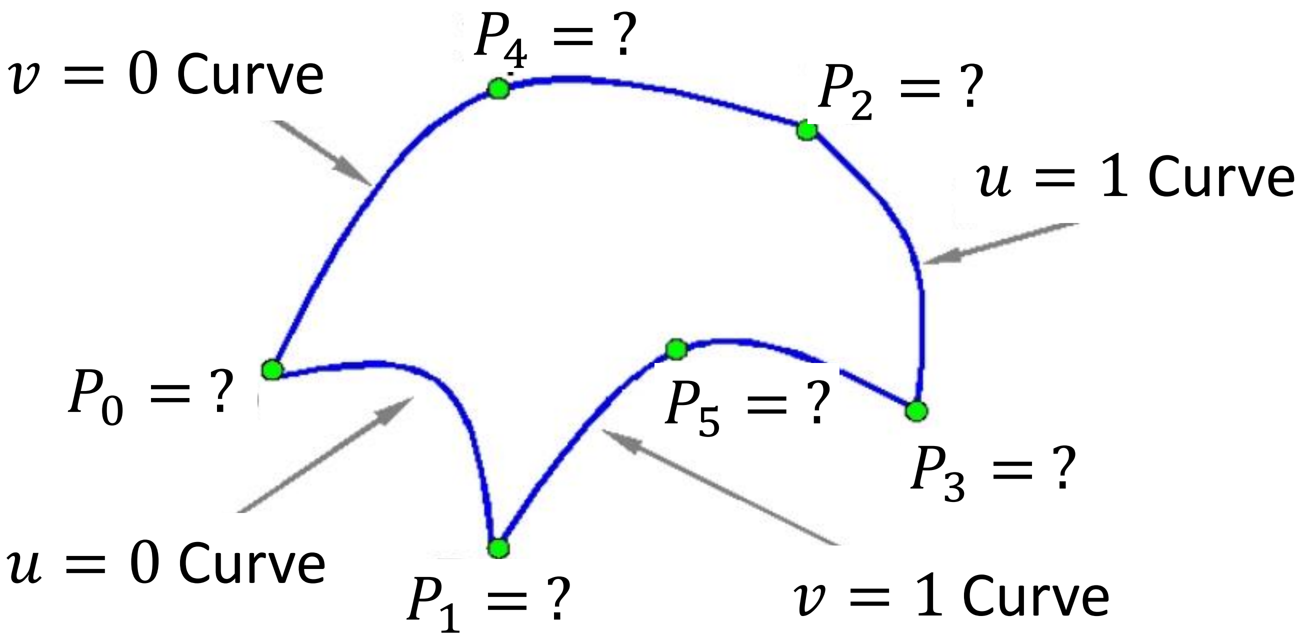
$$P(1,v) = ?$$

$$P(u,0) = ?$$

$$P(u,1) = ?$$

VISUALIZATION

$$P(u, v) = au^2 + 2buv + 2cu + dv^2 + 2ev + f = 0$$



	u	v
P_0	0	0
P_1	0	1
P_2	1	0
P_3	1	1
P_4	½	0
P_5	½	1

CURVED SURFACE REPRESENTATION

- A curve is represented as a vector valued function of a single variable

- A surface is represented by a bivariate vector-valued function

- How to represent a curve on the surface?

- How a point on the surface may be represented?

LINEAR COONS SURFACE

- If the four boundary curves $P(u, 0), P(u, 1), P(0, v)$, and $P(1, v)$ are known, then?

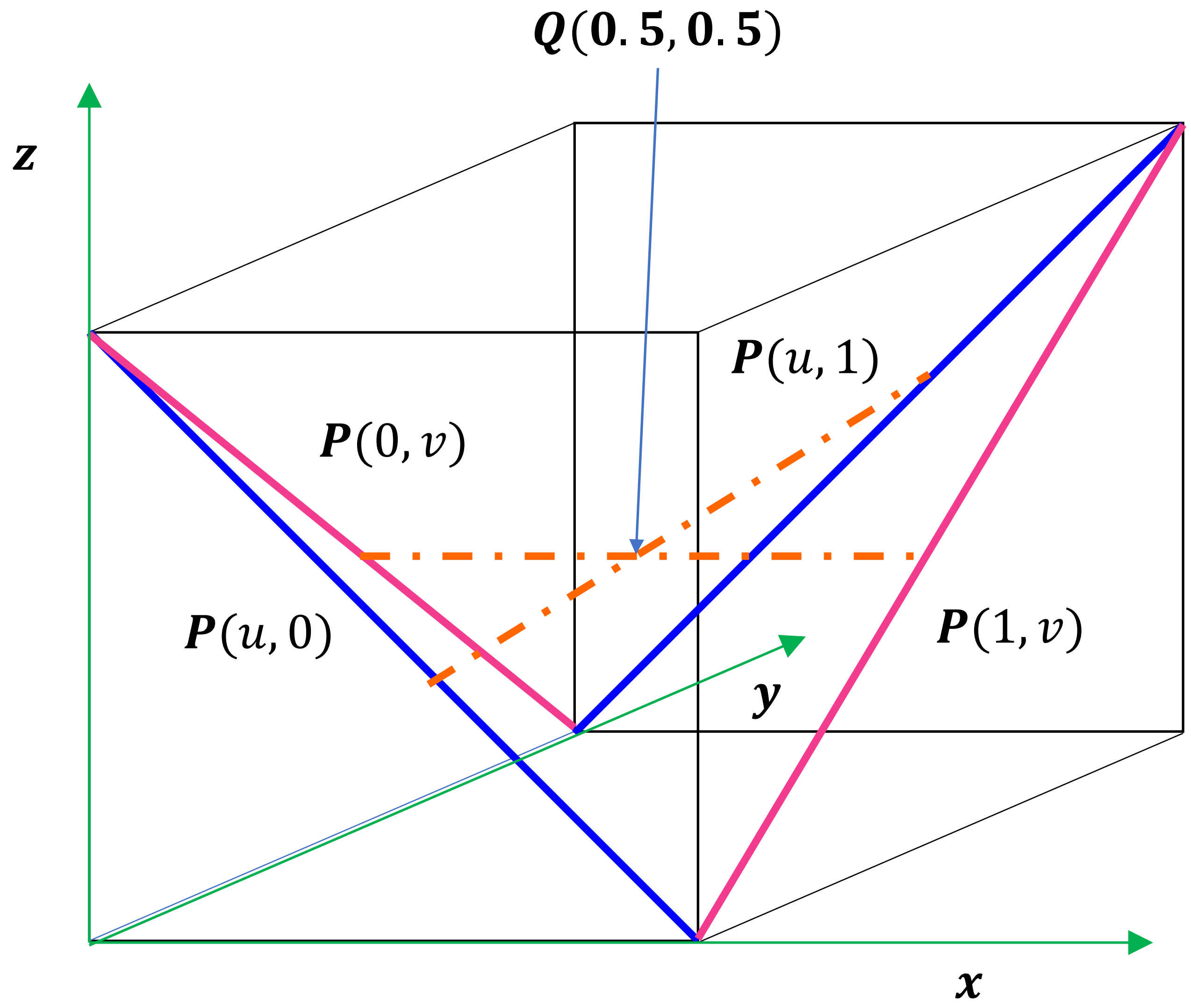
- Intuitive solution

- What about the corner points and at the edges?

- Solution?

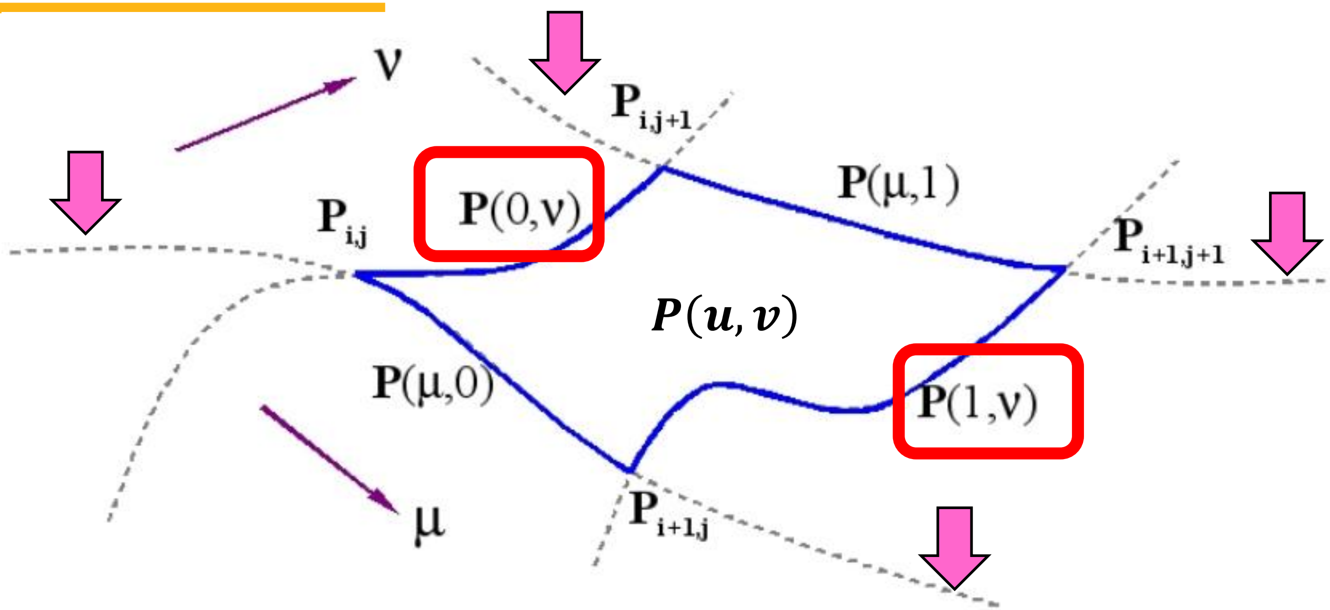
- Matrix representation of the linear Coons surface

EXAMPLE



$\boldsymbol{Q(0.5, 0.5)}$
z
$\boldsymbol{P}(u, 1)$
$\boldsymbol{P}(0, v)$
$\boldsymbol{P}(u, 0)$
$\boldsymbol{P}(1, v)$
y
x

PARAMETRIC SURFACE PATCH



$P(u,v) = ?$ [INTERPOLATION]

$$P(u,v) = P(u,0)(1-v) + P(u,1)v + P(0,v)(1-u) + P(1,v)u$$ [INCORRECT]

$$P(u,v) = P(u,0)(1-v) + P(u,1)v + P(0,v)(1-u) + P(1,v)u - P(0,0)(1-u)(1-v) - P(0,1)(1-u)v + P(1,0)u(1-v) - P(1,1)uv$$

EXAMPLE OF CONSTRUCTING A COON'S PATCH

IDENTIFY THE CORNER POINTS

$(\boldsymbol{y}, \boldsymbol{v})$

$(\boldsymbol{x}, \boldsymbol{u})$

$P(0,0) = ?$

$P(0,1) = ?$

	2	3	4	5	6	7
7	.	.	.	.	.	.
8	.	.	10	9		
9	.	14	12	11	10	.
10	.	15	13	14	10	.
11	.	.	10	11		

$P(1,0) = ?$

$P(1,1) = ?$

EXAMPLE OF CONSTRUCTING A COON'S PATCH

(y, v)

(x, u)

	2	3	4	5	6	7
7	.	.	.	.	.	.
8	.	.	10	9		
9	.	14	12	11	10	.
10	.	15	13	14	10	.
11	.	.	10	11		

- Gradients at the corners of the patch

$$P(0,0) = (9,4,12)$$

$$\frac{\partial(P(0,0)}{\partial u} = ((10,4,13) - (8,4,10))/2 = (1,0,1.5)$$

$$\frac{\partial(P(0,1)}{\partial u} =$$

$$\frac{\partial(P(1,0)}{\partial u} =$$

$$\frac{\partial(P(1,1)}{\partial u} =$$

EXAMPLE OF CONSTRUCTING A COON'S PATCH

(y, v)

(x, u)

	2	3	4	5	6	7
7	.	.	.	.	.	.
8	.	.	10	9		
9	.	14	12	11	10	.
10	.	15	13	14	10	.
11	.	.	10	11		

- Gradients at the corners of the patch

$$P(0,0) = (9,4,12)$$

$$\frac{\partial(P(0,0)}{\partial v} = ((9,5,11) - (9,3,14))/2 = (0,1,-1.5)$$

$$\frac{\partial(P(0,1)}{\partial v} =$$

$$\frac{\partial(P(1,0)}{\partial v} =$$

$$\frac{\partial(P(1,1)}{\partial v} =$$

EXAMPLE OF CONSTRUCTING A COON'S PATCH

- To find the bounding contours we use the cubic spline patch equation.

$$P(u,0) = \boldsymbol{a_3}u^3 + \boldsymbol{a_2}u^2 + \boldsymbol{a_1}u + \boldsymbol{a_0}$$

$$\boldsymbol{a_0} = \boldsymbol{P_0} = (\boldsymbol{9, 4, 12})$$

$$\boldsymbol{a_1} = \boldsymbol{P'_0} = (\boldsymbol{1, 0, 1.5})$$

$$\boldsymbol{a_2} = -\boldsymbol{3P_0} - \boldsymbol{2P'_0} - \boldsymbol{3P_1} - \boldsymbol{P'_1}$$

$$\boldsymbol{a_3} = \boldsymbol{2P_0} + \boldsymbol{P'_0} - \boldsymbol{2P_1} + \boldsymbol{P'_1}$$

SURFACE REPRESENTATION

OBJECTIVE

- After completing this lecture, students will be able to
  - Derive mathematical expressions for
    - Bi-cubic surface
  - Surface sub-division
  - Geometric properties of surface

BICUBIC SURFACE PATCH

- Discussed so far simple surface patches from a conceptual point of view.
- Useful patch descriptions
- Boundary points are given
- Need to calculate the basis functions
- Steps:
  - Calculate for a fixed u, calculated for a fixed v
  - Add the two set of curves to generate the surface

BICUBIC SURFACE REPRESENTATION

1. Start with the cubic spline expression and derive for boundary conditions
2. Represent the curve as a function of the extreme points and the blending functions
3. For a fixed u, represent the curve
4. For a fixed v, represent the curve
5. Add them
6. Do the correction
7. Final expression

SUBDIVISION OF SURFACE PATCH

- To construct a surface patch with an interactive algorithm
- Changing the boundary (global) points doesn’t change the shape
- Local control is required
- Add more points
- How many points we should add?
- Constraints ?

CONSTRAINT ON SURFACE SUBDIVISION

- The algorithm
  - Takes a surface defined by n points
  - Partitions it into several patches, so that
    - Together they represent the same surface
    - Each patch is now represented with n points
    - They are computed from the original set of points

BEZIER AND B-SPLINE SURFACE

OBJECTIVE

- After completing this lecture, students will be able to
  - Derive the mathematical expressions for
    - Bezier surface
    - B-spline surface
  - Construct Bezier surface
- Calculate expressions for
  - Surface subdivision
  - Degree elevation

BÉZIER CURVE

- Bézier equation

$$P(t) = \sum_{i=0}^{n} P_i B_i^n(t)$$

$B_i^n(t)$= Bernstein Basis = $\binom{n}{i} t^i (1-t)^{n-i}$

P_i= Control points

The corresponding point of t on the new curve is obtained by ***translating*** *the corresponding point of t on the original curve in the* ***direction of $\boldsymbol{v}$ with a distance $\boldsymbol{B_i^n(t)}$***

BEZIER SURFACE

- $m+1 \times n+1$ point grid

- A rectangular Bezier patch is defined as:

$$Q(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) P_{i,j} B_j^n(v)$$

- In Matrix form this can be written as: $Q(u,v) = UNPN^T V^T$

$$U = [u^m, u^{m-1}, \dots, u, 1] \qquad V = [v^n, v^{n-1}, \dots, v, 1] \qquad N = ?$$

EXAMPLE

- Given the six three-dimensional points

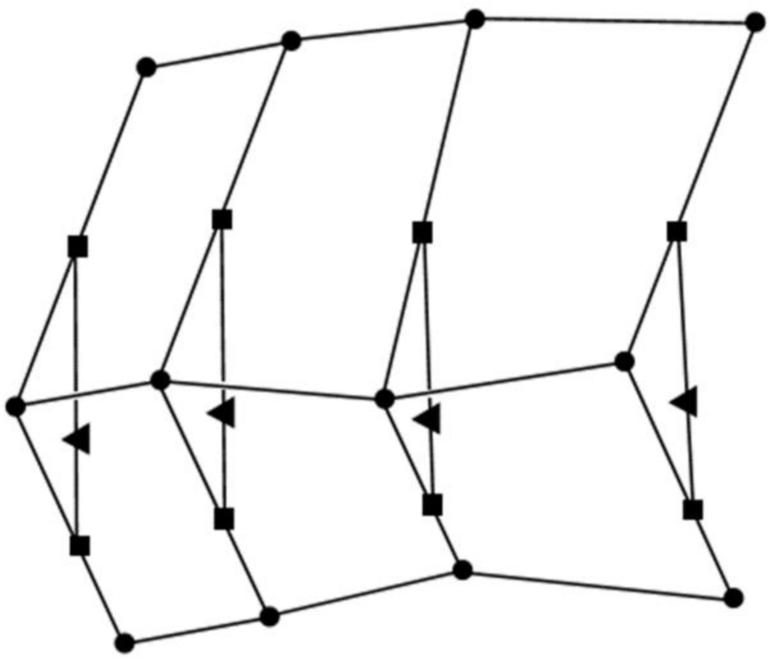
$$\begin{matrix} \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \end{matrix}$$

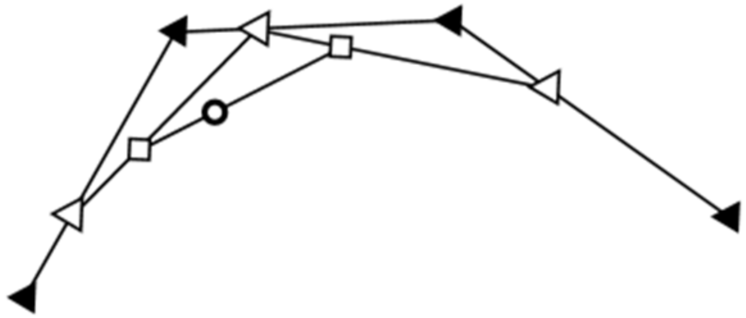
- **Problem:** Generate the corresponding Bezier Surface patch
- Step 1: Find the orders m and n of the surface
- Step 2: Calculate the weight functions $B_i^1(v), B_j^2(u)$
  - $i = 0,1$
  - $j = 0,1,2$
  - $m = 1, n = 2$

SCAFFOLDING CONSTRUCTION

- The scaffolding construction (or **de Casteljau algorithm**)

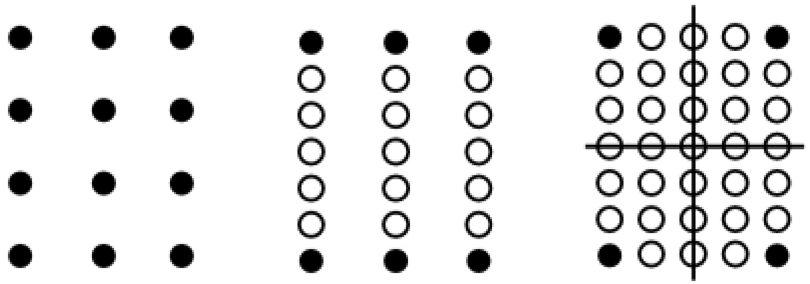
- Extension of the idea discussed in Bezier curves





SUBDIVIDING RECTANGULAR PATCHES

- To change the surface shape, if there are not enough points
- Just adding points is not a good solution because
  - this changes the shape of the surface
- A better solution is to subdivide the patch into four connected surface patches



Step 1: Select values for u and w

Step 2: Apply de Casteljau Algo (column wise)

Step 3: Apply de Casteljau Algo (row wise)

DEGREE ELEVATION

- In an application it is desired that all involved curves to have the same degree
- Increase the degree of a Bézier curve *without* changing its shape.
- Degree elevation
- Existing degree = n; elevated degree = n+1
- Existing control points = $\{P_0, \dots, P_n\}$; New control points = $\{Q_0, \dots, Q_{n+1}\}$, such that $P_0 = Q_0$ & $P_n = Q_{n+1}$

$$\boxed{\boldsymbol{Q_i = \frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i; 1 \leq i \leq n}}$$

DEGREE ELEVATION FOR SURFACE

- Extension of the idea discussed in case of curves

- Bezier patch of degree $m \times n$

- Definition: $$Q(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) P_{i,j} B_j^n(v)$$

- Degree elevated patch is defined by (m+2), (n+2) control points

$$\mathbf{Q}_{ij} = \left(\frac{i}{m+1}, 1 - \frac{i}{m+1}\right) \begin{bmatrix} \mathbf{P}_{i-1,j-1} & \mathbf{P}_{i-1,j} \\ \mathbf{P}_{i,j-1} & \mathbf{P}_{i,j} \end{bmatrix} \begin{bmatrix} \frac{j}{n+1} \\ 1 - \frac{j}{n+1} \end{bmatrix},$$
$$\text{for} \quad i = 0, 1, \ldots, m+1, \quad \text{and} \quad j = 0, 1, \ldots, n+1.$$

EXAMPLE

- Starting with the 2×3 control points

$$\begin{matrix} \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \end{matrix}$$

- Elevation: 1×2 to 2×3

$$\begin{matrix} \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \end{matrix} \Rightarrow \begin{matrix} \mathbf{R}_{20} & \mathbf{R}_{21} & \mathbf{R}_{22} \\ \mathbf{R}_{10} & \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{00} & \mathbf{R}_{01} & \mathbf{R}_{02} \end{matrix} \Rightarrow \begin{matrix} \mathbf{Q}_{20} & \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{10} & \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{00} & \mathbf{Q}_{01} & \mathbf{Q}_{02} & \mathbf{Q}_{03} \end{matrix}$$

B-SPLINE SURFACE

- B-Spline surface can be represented as:

$$Q(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{j+1}^{i+1} N_k^i(u) M_L^j(v)$$

$$N_1^i(u) = \begin{cases} 1 \; if \; x_i \leq u \leq x_i + 1 \\ 0 \; otherwise \end{cases}$$

$$N_k^i(u) = \frac{(u - x_i)N_{k-1}^i(u)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - u)N_{k-1}^{i+1}(u)}{x_{i+k} - x_{i+1}}$$

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THANK YOU