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Assignment no 2

Q no: 1 (a)

1. Closure under Addition:
let $u = (r_1, g_1, b_1)$ and $v = (r_2, g_2, b_2)$
$u \oplus v = (r_1+r_2, g_1+g_2, b_1+b_2)$
Since all are real numbers $u \oplus v \in V$.

2. Closure under Scalar Multiplication
let $u = (r, g, b) \in V$ and $k \in \mathbb{R}$
$k \odot u = (kr, kg, kb)$. Since kr, kg, kb are real $k \odot v \in V$

Addition Axioms
3 i) Commutative → $u \oplus v = v \oplus u$ → $\forall u, v \in V$
4 ii) Associative → $u \oplus (v \oplus w) = (u \oplus v) + w$ $\forall u, v, w \in V$
5 (iii) Zero Vector → The zero vector is $0 = (0,0,0)$, for any $u = (r, g, b) \in V$
$u \oplus 0 = \cancel{(r+0)} (r+0, g+0, b+0) = (r, g, b)$ u.

6) Additive Inverse.
For any $u = (r, g, b) \in V$, the additive inverse is $-u = (-r, -g, -b)$
$u \oplus -u = (r + (-r), g + (-g), b + (-b))$
$= (0, 0, 0) = 0$

Scalar Multiplication Axioms.
7) Associativity of S.M → $(k_1 k_2) \odot u = k_1 \odot (k_2 \odot u)$
8) Distributivity over Vector Addition → $k \odot (u \oplus v) = (k \odot u) \oplus (k \odot v)$
9) Distributivity over scalar Addition → $(k_1 + k_2) \odot u = (k_1 \odot u) \oplus (k_2 \odot u)$
10) Multiplication by Scalar Identity $1 \odot u = u$

Since All ten axioms are satisfied. V with standard operation from a vector space.

(b)

i) Existance of zero vector fails.
A vector space's identity should be 0 such that $u \oplus 0 = u$ for all $u \in V$
There is only 1 additive identity $0 = (0,0,0)$.
let $u = (2, 2, 2)$
Acc to given instruction.
$u \oplus 0 = (2, 2, 2) \oplus (0, 0, 0) = (\min(2+0, 1), \min(2+0, 1), \min(2+0, 1))$

$u \oplus 0 = (1,1,1) \neq (2,2,2)$

Conclusion :: it fails.

2) Existance of Additive Inverse

let $u = (-3,-3,-3) \in V$.

The inverse would be $-u = (3,3,3)$

$u \oplus (-u) = (-3,-3,-3) \oplus (3,3,3) = (\min(-3+3,1), \min(-3+3,1), \min(-3+3,1))$

$= (1,1,1)$ which is not equal to $(0,0,0)$

so, it fails.

3) Scalar Multiplication

let $u = (4,4,4)$ and $k = 3$

$k \odot u = 3 \odot (4,4,4) = (\min(3 \cdot 4,1), \min(3 \cdot 4,1), \min(3 \cdot 4,1))$

$= (1,1,1)$ which is not equal $(12,12,12)$

as ku should be equal to (kr, kg, kb).

so, it FAILS

4) Distributivity over vector addition

$k \odot (u+v) = (k \odot u) \oplus (k \odot v)$

let $k = 0.5$, $u = (1,1,1)$ and $(1,1,1)$

L.H.S

$u \oplus v = (\min(1+1,1), \min(1+1,1), \min(1,1,1))$

~~$k \odot u$~~ $= (1,1,1)$

$k \odot (u \oplus v) = (\min(0.5 \cdot 1,1), \min(0.5 \cdot 1,1), \min(0.5 \cdot 1,1))$

$= (0.5, 0.5, 0.5)$

R.H.S

$(k \odot u)$ ~~$\oplus (k \odot v)$~~ $= (\min(0.5 \cdot 1,1), \min(0.5 \cdot 1,1), \min(0.5 \cdot 1,1))$

$= (0.5, 0.5, 0.5)$

similar for the other term

$(k \odot u) \oplus (k \odot v) = (0.5, 0.5, 0.5) \oplus (0.5, 0.5, 0.5)$

$= (\min(0.5+0.5,1), \min(0.5+0.5,1), \min(0.5+0.5,1))$

$= (1,1,1)$

$(1,1,1) \neq (0.5, 0.5, 0.5)$

The axiom does not holds and it fails.

Q no. 2

a) 8 - Axiom $V = R^4$

Zero Vector θ :- $(0,0,0,0)$

Vector Addition :- $(u_1, u_2, u_3 + u_4) + (v_1, v_2, v_3, v_4)$
$= (u_1+v_1, u_2+v_2, u_3+v_3, u_4+v_3)$

Scalar Multiplication : $k(u_1, u_2, u_3, u_4) =$
(ku_1, ku_2, ku_3, ku_4).

b) Determine whether it is a subspace of R^4

W_2 is not a subspace of R^4

(ii) W_2 fails under Vector Addition and the most fundamental control, Non-Emptiness (Zero-vector).

1) Contains 0 — Is $(0,0,0,0)$ in W_2? Since $0 \neq 1$.
Check constraint: $0+0+0+0=0$ — the zero vector $0 = (0,0,0,0)$ is not in W_2

2) Closed under Addition — let $u, v \in W_2$, Is $u+v \in W_2$? — let $u = (1,0,0,0) \in W_2$, Since $(1+0+0+0) = 1$.
let $v = (0,1,0,0) \in W_2$ (Since $(0+1+0+0=1)$)
Then $u+v = (1,1,0,0)$.

Check constraint $1+1+0+0=2$ Since $2 \neq 1$,
$u+v \notin W_2$.

c) $W_3 = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 = 2x_3 \text{ and } x_2 = x_4\}$

(i) W_3 is a Subspace of R^4

(ii) Since W_3 is a subspace, no condition fails.

1) Contains zero vector?
. Check $0 = (0,0,0,0)$
. Constraint 1 : $x_1 = 2x_3 \to 0 = 2(0)$, which is true.
. Constraint 2 : $x_2 = x_4 \Rightarrow 0 = 0$, which is true
. Thus, $0 \in W_3$

2) Closed under Vector Addition
let $u = (u_1, u_2, u_3, u_4) \in W_3$ and $v = (v_1, v_2, v_3, v_4) \in W_3$. This means: $u_1 = 2u_3$, $u_2 = u_4$,
$v_1 = 2v_3$ and $v_2 = v_4$.

$w = u+v = (u_1+v_1, u_2+v_2, u_3+v_3, u_4+v_4)$.
$w_1 = u_1+v_1 = 2u_3 + 2v_3 = 2(u_3+v_3) = 2w_3$
Check constraint 2 (for w_2 and w_4)
~~Thus~~ $w_2 = u_2 + v_2 = u_4 + v_4 = w_4$.
Thus $u+v \in W_3$

3. Closed under Scalar Multiplication?

Let $u = (u_1, u_2, u_3, u_4) \in W_4$ and $k \in \mathbb{R}$. This $u_1 = 2u_3$ and $u_2 = u_4$.

Consider Constraint I (for z_1 and z_3):

$$z_1 = ku_1 + k(2)(ku_3) = 2z_3$$

This holds.

Check Constraint II (for z_2 and z_4)

$\cancel{z_1 = ku_1 = k(2u_3) = 2(ku_3)}$

$$z_2 = ku_2 = ku_4 = z_4$$

This holds

Thus, $ku \in W_3$

Problem 3 Let M be an arbitrary 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We need to find scalars c_1, c_2, c_3, c_4 such that:

$$M = c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4$$

Substitute the matrices from S:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Scalar Multiplication and matrix add

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_4 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$

Express B as a linear Combination of the Matrices in S

$$B = \begin{pmatrix} 3 & -2 \\ 5 & 4 \end{pmatrix}$$

from the result in part (i), it was established that for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the co-efficients are simply $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$

Comparing B to general form

$a = 3$, $b = -2$, $c = 5$, $d = 4$

Therefore

$c_1 = 3$, $c_2 = -2$, $c_3 = 5$, $c_4 = 4$

That B As linear combination of matrices in S

$$B = 3A_1 + (-2)A_2 + 5A_3 + 4A_4$$

$$\begin{pmatrix} 3 & -2 \\ 5 & 4 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Q no: 4

By using Theorem 4.4.3
It states that if no of vector is greater than dimensions then the vector V is linearly dependent. Here $V_n \Rightarrow n = 4$ dimension $\rightarrow 3$

$4 > 3$ so V is linearly dependent.

~~Sol~~
Sol:

eq: $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$$c_1\begin{bmatrix}1\\2\\-1\end{bmatrix} + c_2\begin{bmatrix}3\\1\\4\end{bmatrix} + c_3\begin{bmatrix}2\\-1\\5\end{bmatrix} + v_4\begin{bmatrix}0\\5\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented M = $\left[\begin{array}{cccc|c}1 & 3 & 2 & 0 & 0\\2 & 1 & -1 & 5 & 0\\-1 & 4 & 5 & 2 & 0\end{array}\right]$

$$\sim \begin{bmatrix}1 & 3 & 2 & 0\\0 & -5 & -5 & 5\\0 & 7 & 7 & 2\end{bmatrix} \quad \begin{array}{l}R_2 - 2R_1\\R_3 + R_1\end{array}$$

$$\sim \begin{bmatrix}1 & 3 & 2 & 0\\0 & 1 & 1 & -1\\0 & 7 & 7 & 2\end{bmatrix} \quad -1/5 R_2$$

$$\sim \begin{bmatrix}1 & 0 & -1 & 3\\0 & 1 & 1 & -1\\0 & 0 & 0 & 9\end{bmatrix} \quad \begin{array}{l}R_3 - 7R_2\\R_1 - 3R_2\end{array}$$

$$\sim \begin{bmatrix}1 & 0 & -1 & 3\\0 & 1 & 1 & -1\\0 & 0 & 0 & 1\end{bmatrix} \quad 1/9 R_3$$

$$\sim \begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \quad \begin{array}{l}R_1 - 3R_3\\R_2 + R_3\end{array}$$

eq formed

$c_1 - c_3 = 0$

$c_2 + c_3 = 0$

$c_4 = 0$

$c_1 = c_3$

$c_2 = -c_3$

let $c_3 = t$ be the free variable where $t \in \mathbb{R}$ and $t \neq 0$ for non-trivial sol

then $c_1 = t$, $c_3 = t$, Therefore, the system is Dependent linearly.

$c_2 = -t$, $c_4 = 0$

b) Dependency Relationship:

from above part.

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

if we let $c_3 = 1$, the values would be:

$c_1 = 1$, $c_2 = -1$, $c_3 = 1$, $c_4 = 0$

then

$v_1 - v_2 + v_3 = 0$

$v_1 = v_2 - v_3$

and v_4 is independent.

3) By using determinant method as the matrix would be square

let $W = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix}$

$|W| = 1(c-0) - 0 + a(0)$

$|W| = c$

$c \neq 0$

Therefore, the system is dependent.

problem 5:

$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$B' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

i) $A_B = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$

$[A_B] = (3, 1, 2, 4)$.

ii) $\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_4 - R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & +\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \begin{array}{l} \\ -\frac{1}{2} R_2 \\ -\frac{1}{2} R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \begin{array}{l} \\ R_1 - R_2 \\ R_3 - R_4 \\ \\ \end{array}$$

$$P_{B \to B'} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

3) Using results from part ii) and iii)

$$[A]_{B'} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3+1 \\ 3-1 \\ 2+4 \\ 2-4 \end{bmatrix}$$

$$[A]_{B'} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

This means $A = 2S_1 + 1S_2 + 3S_3 - 1S_4$

Problem 6

(i)

Augmented Matrix $= \begin{bmatrix} 1 & 2 & -1 & 3 & -2 \\ 2 & 4 & -2 & 7 & -3 \\ 3 & 6 & -3 & 10 & -5 \end{bmatrix}$

$\sim \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $R_2 - 2R_1$, $R_3 - 3R_1$

$\sim \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 - 3R_2$

$x_1 + 2x_2 - x_3 - 5x_5 = 0$

$x_4 + x_5 = 0$

$x_4 = -x_5 \rightarrow$ eq (i)

~~$x_1 = x_3 + x_5 - 2x_2$~~ x_3

$x_1 = -2x_2 + x_3 + 5x_5$

Let free vars be

$x_2 = r$

$x_3 = s$

$x_5 = t$

The general solution vector is:

$x = \begin{bmatrix} -2r+s+5t \\ r \\ s \\ -t \\ t \end{bmatrix}$

Basis:-

$x = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Let $B = \{b_1, b_2, b_3\}$ be the basis

$b_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

b) The Dimension of Vector Space is 3.

c) Adding $x_1+x_2+x_3=0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2r+s+5t \\ r \\ s \\ -t \\ t \end{bmatrix}$$

$x_1+x_2+x_3=0$

$-2r+s+5t+r+s=0$

$-r+2s+5t=0$

Solving

$r = 2s+5t \Rightarrow \because r = x_2$

Now,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2(2s+5t)+s+5t \\ 2s+5t \\ s \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -3s-5t \\ 2s+5t \\ s \\ -t \\ t \end{bmatrix}$$

Therefor

$$x = s\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore,

$$B' = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \rightarrow \text{New basis.}$$

The Dimension changed from 3 to 2.

Part B

1) let Matrix be $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Axiom 1:

$A+B \in V$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix}$$

$$\begin{bmatrix} a_{11}+a_{11}' & a_{12}+a_{12}' & a_{13}+a_{13}' \\ a_{21}+a_{21}' & a_{22}+a_{22}' & a_{23}+a_{23}' \\ a_{31}+a_{31}' & a_{32}+a_{32}' & a_{33}+a_{33}' \end{bmatrix} \in V$$

Axiom-5 $(A+(-A)=0)$

$\rightarrow$ Inverse.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} -a_{11}' & -a_{12}' & -a_{13}' \\ -a_{21}' & -a_{22}' & -a_{23}' \\ -a_{31}' & -a_{32}' & -a_{33}' \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverse exists

Axiom-6 $k(U) = kU$

$$k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

Axiom-10 $(IU = U)$

$$I\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let $W = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

2) Show W is subspace of P_4

1) Contains zero vector $(0 \in W)$?

Check Constraints $p_0(1) = 0$ and $p_0(-1) = 0$

Both are true.

2) Closed under Addition

$p_1(1) = 0$, $p_1(-1) = 0$, $p_2(1) = 0$, $p_2(-1) = 0$

$q(1) = p_1(1) + p_2(1) = 0 + 0 = 0$

$q(-1) = p_1(-1) + p_2(-1) = 0 + 0 = 0$

3) Closed under Scaler Multiplication

$p(1) = 0$ and $p(-1) = 0$.

let $r(x) = kp(x)$,

$r(1) = kp(1) = k \cdot 0 = 0$

$r(-1) = kp(-1) = k \cdot 0 = 0$

The set W is subspace of P_4

Finding Dimension and an Explicit Base

let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

Given $p(1) = 0$ and $p(-1) = 0$

1. $p(1) = 0$: $a_0 + a_1 + a_2 + a_3 + a_4 = 0 \rightarrow$ eq (1)
2. $p(-1) = 0$: $a_0 - a_1 + a_2 - a_3 + a_4 = 0 \rightarrow$ eq (ii)

Subtract eq (ii) from eq (i)

$\cancel{a_0} + a_1 + \cancel{a_2} + a_3 + \cancel{a_4} - \cancel{a_0} + a_1 - \cancel{a_2} + a_3 - \cancel{a_4} = 0$

$2a_1 + 2a_3 = 0$

$a_1 = -a_3$

Adding both eq

$a_0 + a_1 + a_2 + a_3 + a_4 + a_0 - a_1 + a_2 - a_3 + a_4 = 0$

$2a_0 + 2a_2 + 2a_4 = 0$

$a_0 = -a_2 - a_4$

put in the ~~eq~~ (A) polynomial

$p(x) = (-a_2 - a_4) + (-a_3)x + a_2x^2 + a_3x^3 + a_4x^4$

$= a_2(x^2-1) + a_3(x^3-x) + a_4(x^4-1)$.

$B_w = \{(x^3-1), (x^3-x), (x^4-1)\}$.

~~Pr~~ Problem:-

Problem:- 7

$B = \begin{bmatrix} 2 & 4 \\ 2 & -1 \end{bmatrix}$ $\quad B' = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$

$P_{B \to B'} = \left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$

$= \left[\begin{array}{cc|cc} 2 & 4 & 1 & 1 \\ 0 & -5 & 2 & -2 \end{array}\right] R_2 - R_1$

$B' | B = \left[\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 2 & 4 \\ 0 & 2 & -4 & -13 \end{array}\right] R_2 - 3R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 2 & 4 \\ 0 & 1 & -2 & -13/2 \end{array}\right] \frac{1}{2}R_2$

$\sim \left[\begin{array}{cccc} 1 & 0 & 0 & -5/2 \\ 0 & 1 & -2 & -13/2 \end{array}\right] R_1 + R_2$

$P_{B \to B'} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}$

2) Finding $[w]_{B'}$

$$[w]_{B'} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$[w]_{B'} = \begin{bmatrix} 0 & -5 \\ +2 & -13 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -11 \end{bmatrix} \text{ Ans.}$$

Problem: 8

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1) Basis of row space are:

Row(A) = {(1,-3,0,0,3,0), (0,0,1,0,-2,0), (0,0,0,1,1,0), (0,0,0,0,0,1)}

2) Col of row space are:

~~Col(A)~~

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad a_5 = \begin{bmatrix} -2 \\ -1 \\ -1 \\ 2 \end{bmatrix}, \quad a_6 = \begin{bmatrix} 4 \\ 2 \\ 7 \\ -4 \end{bmatrix}$$

3) The RREF eq's are using $(x_1 \cdots x_6)$:

$x_1 - 3x_2 - 3x_5 = 0$

$x_3 - 2x_5 = 0$

$x_4 + x_5 = 0$

$x_6 = 0$

P. vars = x_1, x_3, x_4, x_6 , Free var: x_2, x_5

$x_1 = 3x_2 + 3x_5$

$x_3 = 2x_5$

$x_4 = -x_5$

let $x_2 = s$ and $x_5 = t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3s-3t \\ s \\ 2t \\ -t \\ t \\ 0 \end{bmatrix}$$

$$x = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Basis for Null A = $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

4) As. The rank of matrix A is 4 and Dimension is also 4 of A and whole space.

Yes, the vector b is in the column space of A. The col space col(A) is subspace of $\mathbb{R}^4$. Since Rank of A is 4, col(A) spans all of $\mathbb{R}^4$. Therefore, any vector in $\mathbb{R}^4$, including b is in the column space of A.

Q no: 9

Largest Possible Rank:-

1) The rank of a matrix is the dimension of its col or row space. It cannot exceed the number of rows or columns.

rank(A) ≤ min(rows, cols).

Since A is 5x7 matrix its Rank is 5.

2) Smallest possible nullity

using nullity theorem

rank(A) + nullity(A) = n

nullity (A) = n − rank(A)

n = 7, rank(A) = 5

nullity(A) = 7 − 5 = 2.

The matrix must have two free vars in solution AX = 0.

3) if rank is 4 then

a) nullity (A) = 7 − 4 = 3

There must be 3 free variables.

b) The solution is not unique. A

consistent linear system has a unique solution iff there are zero free vars.
Since nullity is 3, the sol has 3 independent parameters and can be written as sum of particular sol and a non-trivial nullspace : $x = x_p + c_1v_1 + c_2v_2 + c_3v_3$.

3. 5×7 matrix A with nullity(A) = 2

a) What is rank ?

$\text{Rank}(A) + \text{nullity} = n$

$\text{Rank}(A) = n - \text{nullity}(A)$

$n = 7$, $\text{nullity}(A) = 2$

$\text{Rank}(A) = 7 - 2$

$= 5$

b) Yes. the column vectors of A do span $\mathbb{R}^5$
As the rank of the matrix is 5
and $\dim(\text{col}(A)) = \text{rank}(5)$
$\dim(\text{col}(A)) = 5$.

Sign