(b) PROBLEMS ON DIRECTIONAL DERIVATIVE

1. Find the direction derivative of $\phi = x^2yz + 4xz^2$ @ $(1,-2,-1)$ along $(2i-j-2k)$

$\Rightarrow \quad \nabla\phi = (2xyz + 4z^2)i + (x^2z)j + (x^2y + 8xz)k$

$= (4+4)i + (-1)j + (-2-8)k$

$= 8i - j - 10k$

$$\hat{n} = \frac{2i - j - 2k}{\sqrt{4+(-1)^2+(-2)^2}}$$

$$= \frac{2i - j - 2k}{\sqrt{9}}$$

$$= \frac{2i - j - 2k}{3}$$

$$\nabla\phi \cdot \hat{n} = \frac{(8i - j - 10k) \cdot (2i - j - 2k)}{3}$$

$$= \frac{1(16i^2 + j^2 + 20k^2)}{3}$$

$$= \frac{37}{3}$$

② $\phi = 4xz^3 - 3x^2y^2z$ @ $(2, -1, 2)$
along $(2i - 3j + 6k)$

$-3z^2y^2$

$$\nabla\phi = (4z^3 - 6xy^2z)i + (-6x^2yz)j + (12xz^2)k$$
$$= (32 - 24)i + (-6(4)(-1)2)j + (12(2)(2^2))k$$
$$= 8i + 48j + 84k$$

$$\hat{n} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{7}$$

$$\nabla\phi \cdot \hat{n} = \frac{(8i + 48j + 84k)(2i - 3j + 6k)}{7}$$

$$= \frac{16 - 144 + 504}{7}$$

$$= \frac{376}{7}$$

★ ANGLE B/W TWO SURFACES

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

① Find the angle b/w the surfaces $x^2+y^2+z^2=9$ & $x^2+y^2-z=3$ @ $(2,-1,2)$

$\Rightarrow$ $\nabla\phi_1 = (2x)i + (2y)j + (2z)k$

$\nabla\phi_2 = (2x)i + (2y)j - k$

$\nabla\phi_1 = 4i - 2j + 4k$

$\nabla\phi_2 = 4i - 2j - k$

$$\cos\theta = \frac{(4i-2j+4k)(4i-2j-k)}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$= \frac{(4i-2j+4k)(4i-2j-k)}{6\sqrt{21}}$$

6

$$\theta = \cos^{-1}\left(\frac{16+4-4}{6\sqrt{21}}\right)$$

$$= \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$$

$$= 54.4^\circ$$

(4) find div $\vec{F}$ & curl $\vec{F}$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

$\Rightarrow \nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k$

Let $x^3+y^3+z^3-3xyz = \phi$

$\vec{F} = \nabla\phi$

$= (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$

$\nabla\cdot\vec{F} = \left(\frac{\delta}{\delta x} i + \frac{\delta}{\delta y} j + \frac{\delta}{\delta z} \hat{k}\right)\cdot\left[(3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}\right]$

$= \cancel{6x-3z-3y-3z}$

$= 6x + 6y + 6z$

$= 6(x+y+z)$

$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$

$= \hat{i}(-3x+3x) - \hat{j}(-3y+3y) + \hat{k}(-3z+3z)$

$= 0\hat{i} - 0\hat{j} + 0\hat{k}$

$= \vec{0}$

$= 0$

* Solenoidal & Irrotational Vectors.

- A vector $\vec{F}$ is said to be solenoial div $\vec{F}=0$
- A $\vec{F}$ is said to irrotational if curl $\vec{F}=0$
- Irrotational vector feild is also called conservative feild or potential feild
- If $\vec{F}$ is irrotational there exists a scalar point functh ϕ such that $\vec{F}=\nabla\phi$. (grad ϕ) the ϕ is called scalar potential of $\vec{F}$.

① Show that $\vec{F}=\frac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is both solenoidal & irrotational

$$\text{div}\,\vec{F}=\nabla\cdot\vec{F}$$

$$=\left(\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}\right)\left(\frac{x\hat{i}+y\hat{j}}{x^2+y^2}\right)$$

$$\cancel{=\frac{-1}{x^2}-\frac{1}{y^2}+0}$$

$$=\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2}+\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}$$

$$=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$=\frac{x^2+y^2-x^2-y^2}{(x^2+y^2)^2}=0$$

div. $\vec{F}=0$ so the $\vec{F}$ is solenoidal

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x/x^2+y^2 & y/x^2+y^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}\left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right)$$

$$= \vec{0}$$

$\vec{F}$ is irrotation

(4) Find the value of constant a such that vector feild, $\vec{F} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational find the scalar functⁿ ϕ such that $\vec{F} = \nabla\phi$.

$\Rightarrow$ ~~div $\vec{F} = 0$~~

$\nabla \cdot \vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$\hat{i}(0 - 0) + \hat{j}((1-a)z^2 - 3z^2) + \hat{k}((a-2)(2x) - ax)$

$(z^2 - az^2 - 3z)\hat{j} + ((a-2)(2x) - ax)\hat{k} = 0$

$z^2 - az^2 - 3z = 0$

$2ax - 4x - ax = 0$

~~$z^2(1-a) - 3z = 0$~~

$ax = 4x$

$a = 4$

⑤ Find the constants a & b such that $\vec{F} = (axy+z^3)\hat{i} + (3x^2-z)\hat{j} + (bxz^2-y)\hat{k}$ is irrotational also find scalar funct ϕ such that $\vec{F} = \nabla\phi$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (axy+z^3) & (3x^2-z) & (bxz^2-y) \end{vmatrix} = 0$$

$$\hat{i}(-1+1) - \hat{j}(bz^2 - 3z^2) + \hat{k}(6x - ax) = 0$$

$a = 6$
$b = 3$

$$\int \partial\phi = \int (6xy+z^3)\,\partial x$$

$$\phi = 3x^2y + xz^3 + f_1(y,z)$$

$$\int \partial\phi = \int (3x^2 - z)\,\partial y$$

$$\phi = 3x^2y - yz + f_2(x,z)$$

$$\int \partial\phi = \int (3xz^2 - y)\,\partial z$$

$$\phi = xz^3 - yz + f_3(x,y)$$

$f_1(y,z) = -yz$
$f_2(x,z) = xz^3$
$f_3(x,y) = 3x^2y$

$$\phi = 3x^2y + xz^3 - yz$$

★ Gauss Divergence Theorem:

$\oint_S \vec{F}.\hat{n}\,ds$, where $\vec{F} = (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k}$

$$\oint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV$$

$$\nabla.\vec{F} = \frac{\delta}{\delta x}(x^2-yz) + \frac{\delta}{\delta y}(y^2-xz) + \frac{\delta}{\delta z}(z^2-xy)$$

$$= 2x + 2y + 2z$$

$$\int_0^a\int_0^b\int_0^c \nabla.\vec{F} = 2\left\{\iiint x\,dx\,dy\,dz + \iiint y\,dx\,dy\,dz + \iiint z\,dx\,dy\,dz\right.$$

$\cancel{=2}$

$$\int_0^c\int_0^b\int_0^a x\,dx\,dy\,dz = \int_0^c\int_0^b \left[\frac{x^2}{2}\right]_0^a dy\,dz = \int_0^c\int_0^b \frac{a^2}{2}\,dy\,dz = \frac{a^2}{2}\int_0^c\int_0^b 1\,dy\,dz$$

$$= \frac{a^2}{2}\int_0^c [y]_0^b\,dz$$

$$= \frac{a^2bc}{2}$$

$$= \cancel{2}\left\{\frac{a^2bc + ab^2c + abc^2}{\cancel{2}}\right\}$$

$=$

Greens Theorem

$\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the boundary of the region enclosed by $y=\sqrt{x}$ & $y=x^2$

$$\int M dx + N dy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy$$

$M(x,y) = 3x^2-8y^2$
$N(x,y) = 4y-6xy$

$\frac{\delta N}{\delta x} = -6y \qquad \frac{\delta M}{\delta y} = -16y$

$$\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} = -6y - (-16y) = 10y$$

$$\iint_R 10y\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx$$

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}}$$

$$= 5\int_0^1 (x-x^4)$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\cdot\frac{3}{10} = \frac{3}{2}$$

* Stoke's theorem

Verify stoke's theorem for vector $\vec{F} = (x^2+y^2)i - 2xyj$ taken round the rectangle bounded by $x=0, x=a, y=0, y=b$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, dS$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta 0}{\delta y} - \frac{\delta}{\delta z}(-2xy)\right) - \hat{j}\left(\frac{\delta}{\delta x}(0) - \frac{\delta}{\delta z}(x^2+y^2)\right) + \hat{k}\left(\frac{\delta}{\delta x}(-2xy) - \frac{\delta}{\delta y}(x^2+y^2)\right)$$

$$= -2y - 2y = 0$$

$\vec{F} = (x^2+y^2)i - 2xyj$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$

Alog: $y=0, x=0 \to 0$

$dy=0$

$$\int_0^a x^2\,dx = \frac{a^3}{3}$$

Along $x=a,\ y=0 \to b$

$$\int_0^b -2ay\,dy = -ab^2$$

Along $y=b,\ x=a \to 0$

$$\int_a^0 (x^2+b^2)\,dx = -\int_0^a (x^2+b^2)\,dx = -\left(\frac{a^3}{3}+ab^2\right)$$

Along $x=0,\ y=b \to 0$

$$\int_b^0 -2.(0)\,y\,dy = 0$$

$$\frac{a^3}{3} + (-ab^2) + \left[-\left(\frac{a^3}{3}+ab^2\right)\right] + 0 = 0$$