1/

(a) For non-trivial solutions:

$$\begin{vmatrix} \sigma_{xx}-\sigma_p & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma_p & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz}-\sigma_p \end{vmatrix} = 0$$

$$\Rightarrow (\sigma_{xx}-\sigma_p)\left\{(\sigma_{yy}-\sigma_p)(\sigma_{zz}-\sigma_p) - \tau_{yz}^2\right\} - \tau_{xy}\left\{\tau_{xy}(\sigma_{zz}-\sigma_p) - \tau_{zx}\tau_{yz}\right\} + \tau_{zx}\left\{\tau_{xy}\tau_{yz} - \tau_{zx}(\sigma_{yy}-\sigma_p)\right\} = 0$$

$$\Rightarrow (\sigma_{xx}-\sigma_p)\left\{\sigma_{yy}\sigma_{zz} - \sigma_p(\sigma_{yy}+\sigma_{zz}) + \sigma_p^2 - \tau_{yz}^2\right\} - \tau_{xy}\left\{\tau_{xy}\sigma_{zz} - \tau_{xy}\sigma_p - \tau_{zx}\tau_{yz}\right\} + \tau_{zx}\left\{\tau_{xy}\tau_{yz} - \tau_{zx}\sigma_{yy} + \tau_{zx}\sigma_p\right\} = 0$$

$$\Rightarrow \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_p(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz}) + \sigma_p^2\sigma_{xx} - \sigma_{xx}\tau_{yz}^2 - \sigma_p\sigma_{yy}\sigma_{zz} + \sigma_p^2(\sigma_{yy}+\sigma_{zz}) - \sigma_p^3 + \sigma_p\tau_{yz}^2$$
$$- \sigma_{zz}\tau_{xy}^2 + \sigma_p\tau_{xy}^2 + \tau_{xy}\tau_{yz}\tau_{zx} + \tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{yy}\tau_{zx}^2 + \sigma_p\tau_{zx}^2 = 0$$

$$\Rightarrow \quad -\sigma_p^3 + \sigma_p^2\left(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\right) - \sigma_p\left(\sigma_{xx}\sigma_{yy}+\sigma_{zz}\sigma_{xx}+\sigma_{yy}\sigma_{zz}-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2\right)$$
$$+\left(\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}\right) = 0$$

$$\Rightarrow \quad \sigma_p^3 - \sigma_p^2\left(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\right) + \sigma_p\left(\sigma_{xx}\sigma_{yy}+\sigma_{zz}\sigma_{xx}+\sigma_{yy}\sigma_{zz}-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2\right)$$
$$-\left(\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}\right) = 0$$

$$\therefore \quad I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$
$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$
$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}$$

Some students have directly written down the expressions without going through the steps. Even if those expressions are correct they have received less marks than students who showed the full derivation but somehow, due to some error in the intermediate algebra, did not get all the expressions correctly.

(b) The equation $\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$ is the equation for determining the principal stresses. Now, if we choose different coordinate axes, the various components of stresses, i.e. $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}$ will individually vary but the values of their combinations as they appear in I_1, I_2, and I_3 must not vary. If these values were to vary, it would change the equation and consequently lead to different values of the principal stresses. But we know that principal stresses are a physical fact and they must be independent of the choice of the coordinate system. That is why I_1, I_2, and I_3 are "invariants".

Only a very few students mentioned the fact associated with the principal stresses.

(c) $\vec{T}^{(x)}$ is the traction vector on the plane whose unit outward normal is along x.

$$\therefore \vec{T}^{(x)} = [\underset{=}{\sigma}]^T[\hat{n}] \rightarrow [\hat{n}] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \vec{T}^{(x)} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{zx} \end{bmatrix} \rightarrow \text{So, } \vec{T}^{(x)} = \sigma_{xx}\hat{i} + \tau_{xy}\hat{j} + \tau_{zx}\hat{k}$$

For $\vec{T}^{(y)}$, $[\hat{n}] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. And, we obtain : $\vec{T}^{(y)} = \tau_{xy}\hat{i} + \sigma_{yy}\hat{j} + \tau_{yz}\hat{k}$

For $\vec{T}^{(z)}$, $[\hat{n}] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. And, we obtain : $\vec{T}^{(z)} = \tau_{zx}\hat{i} + \tau_{yz}\hat{j} + \sigma_{zz}\hat{k}$

$$\therefore \left|\vec{T}^{(x)}\right|^2 + \left|\vec{T}^{(y)}\right|^2 + \left|\vec{T}^{(z)}\right|^2 = \left(\sigma_{xx}^2 + \tau_{xy}^2 + \tau_{zx}^2\right) + \left(\tau_{xy}^2 + \sigma_{yy}^2 + \tau_{yz}^2\right) + \left(\tau_{zx}^2 + \tau_{yz}^2 + \sigma_{zz}^2\right)$$

$$= \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + 2\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)$$

$$= \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)^2 - 2\sigma_{xx}\sigma_{yy} - 2\sigma_{yy}\sigma_{zz} - 2\sigma_{zz}\sigma_{xx} + 2\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)$$

$$= \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)^2 - 2\left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right)$$

$$= I_1^2 - 2I_2$$

A few students have directly written down the expressions of $\vec{T}^{(x)}$, $\vec{T}^{(y)}$ and $\vec{T}^{(z)}$ without properly showing the steps starting from $\vec{T} = [\underset{\sim}{\sigma}]^T[\hat{n}]$ and without clearly writing the different forms of $[\hat{n}]$. They have received less marks than students who showed the proper steps clearly.

2/

$\sigma_{yy} = 60\text{ MPa}$, $\sigma_{zz} = 90\text{ MPa}$, $\tau_{yz} = -20\text{ MPa}$

$\sigma_{xx} = 0$, $\tau_{xy} = 0$, $\tau_{xz} = 0$

Since $\tau_{xy} = 0$ & $\tau_{xz} = 0$, the plane $\perp$ to the x-direction is a principal plane and $\sigma_{xx} = 0$ itself is one of the principal stresses.

Next, for the faces $\perp$ to the y-axis and z-axis

$$\sigma_{avg} = \frac{1}{2}(\sigma_{yy} + \sigma_{zz}) = 75\text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{yy} - \sigma_{zz}}{2}\right)^2 + \tau_{yz}^2} = \sqrt{15^2 + 20^2}\text{ MPa} = 25\text{ MPa}$$

$$\therefore \sigma_p = \sigma_{avg} \pm R = 100\text{ MPa}, 50\text{ MPa}$$

Arranged in decreasing order, the principal stresses are:

$\sigma_1 = 100\text{ MPa}$, $\sigma_2 = 50\text{ MPa}$, $\sigma_3 = 0$

(from σ_{xx})

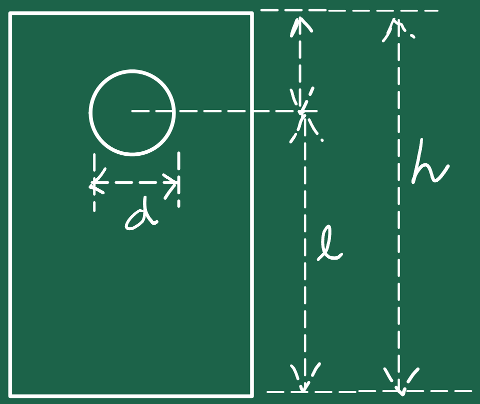
$\therefore$ The abs. max. shear stress, $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 50\,MPa$

The majority of the students have not identified $\sigma_{xx} = 0$ as one of the principal stresses. And, they went on to calculate $\tau_{max,abs}$ as $\frac{1}{2}(100-50)\,MPa = 25\,MPa$. This is a big mistake. And they have received only 2 out of 5 marks.

3 Location of the centroid referred to the bottom edge of the beam c/s :

$$\bar{y} = \frac{(bh)\left\{\frac{h}{2}\right\} - \left(\pi \frac{d^2}{4}\right)\{l\}}{(bh) - \left(\pi \frac{d^2}{4}\right)}$$

$= 24.162$ [Nothing wrong if you have found out this location referred to the top edge or the mid-line.]



$h = 50$ mm
$l = 37.5$ mm
$d = 10$ mm

Moment of inertia about the neutral axis

For beam without hole: $I_1 = \frac{1}{12} bh^3 + bh\left(\frac{h}{2} - \bar{y}\right)^2$

$= 261294 \text{ mm}^4$

For just the hole: $I_2 = \frac{1}{4}\pi\left(\frac{d}{2}\right)^4 + \left(\pi \frac{d^2}{4}\right)\left(l - \bar{y}\right)^2$

$= 14463 \text{ mm}^4$

For beam with hole : $I = I_1 - I_2 = 246831 \text{ mm}^4$

Stress at the top edge of the beam:

$$\sigma = \frac{(PL)(h-\bar{y})}{I} = 25.1 \text{ MPa (tensile)}$$

Stress at the top of the hole:

$$\sigma = \frac{(PL)\left(l + \frac{d}{2} - \bar{y}\right)}{I} = 17.8 \text{ MPa (tensile)}$$

Many students used the formula $\sigma = -\frac{My}{I}$ and ended up with $\sigma < 0$, which is compressive. This is WRONG.

We have to note that $M = -PL$

and so $\sigma > 0$ (tensile).

4 (a) $\varepsilon_{xx} = \varepsilon_a = 0.00250$

$\varepsilon_{yy} = \varepsilon_c = -0.00125$

$$\varepsilon_{x'x'} = \varepsilon_b = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos(2\times45^\circ) + \varepsilon_{xy}\sin(2\times45^\circ)$$

$$\Rightarrow \quad \varepsilon_b = \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy}) + \varepsilon_{xy}$$

$$\Rightarrow \quad \varepsilon_{xy} = \varepsilon_b - \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy}) = 0.000775$$

Since this is a case of PLANE STRESS, we have: $\sigma_{zz} = 0,\ \tau_{zx} = 0,\ \tau_{zy} = 0$

$$\therefore \quad \varepsilon_{zx} = \frac{\tau_{zx}}{2G} = 0 \quad \text{and} \quad \varepsilon_{zy} = \frac{\tau_{zy}}{2G} = 0$$

But $\varepsilon_{zz} \neq 0$

$$\varepsilon_{zz} = \frac{1}{E}\left[\cancelto{0}{\sigma_{zz}} - \nu(\sigma_{xx}+\sigma_{yy})\right]$$

So, we need to find σ_{xx} & σ_{yy}

Many students stopped after finding ε_{xy}!

Some, wrote that $\varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{yz} = 0$ since it is plane stress.

Both of these are big mistakes.

We have :

$$\varepsilon_a \swarrow \varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \cancel{\sigma_{zz}}\right)\right] \quad \sigma_{zz} = 0$$

$$\varepsilon_c \swarrow \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\cancel{\sigma_{zz}} + \sigma_{xx}\right)\right] \quad \sigma_{zz} = 0$$

Only 2 or 3 students obtained ε_{zz} after finding σ_{xx} and σ_{yy}.

Solving the above two equations we obtain

$\sigma_{xx} = 168.67$ MPa and $\sigma_{yy} = -34.34$ MPa

So, $\varepsilon_{zz} = \frac{1}{E}\left[-\nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = -6.157 \times 10^{-4}$

(b) The change in the angle asked is simply $\gamma_{xy} = 2\varepsilon_{xy} = 0.00155$ (in radians)

$= 0.0889^\circ$

Some students have done complicated steps in this very small part to show that the required angle is γ_{xy}!

We need to identify the angle is γ_{xy} from the basic geometrical interpretation of γ_{xy}. → Many students got this correct.

5

We know $$\tan 2\theta_p^{\varepsilon} = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}$$

and $$\tan 2\theta_p^{\sigma} = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

If we can show that one of the above can be derived from the other, the proof is done.

$$\tan 2\theta_p^{\varepsilon} = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} = \frac{2\,\tau_{xy}/2G}{\frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] - \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right]}$$

$$= \frac{2\,\tau_{xy} \times (E/2G)}{(\sigma_{xx} - \sigma_{yy})(1+\nu)}$$

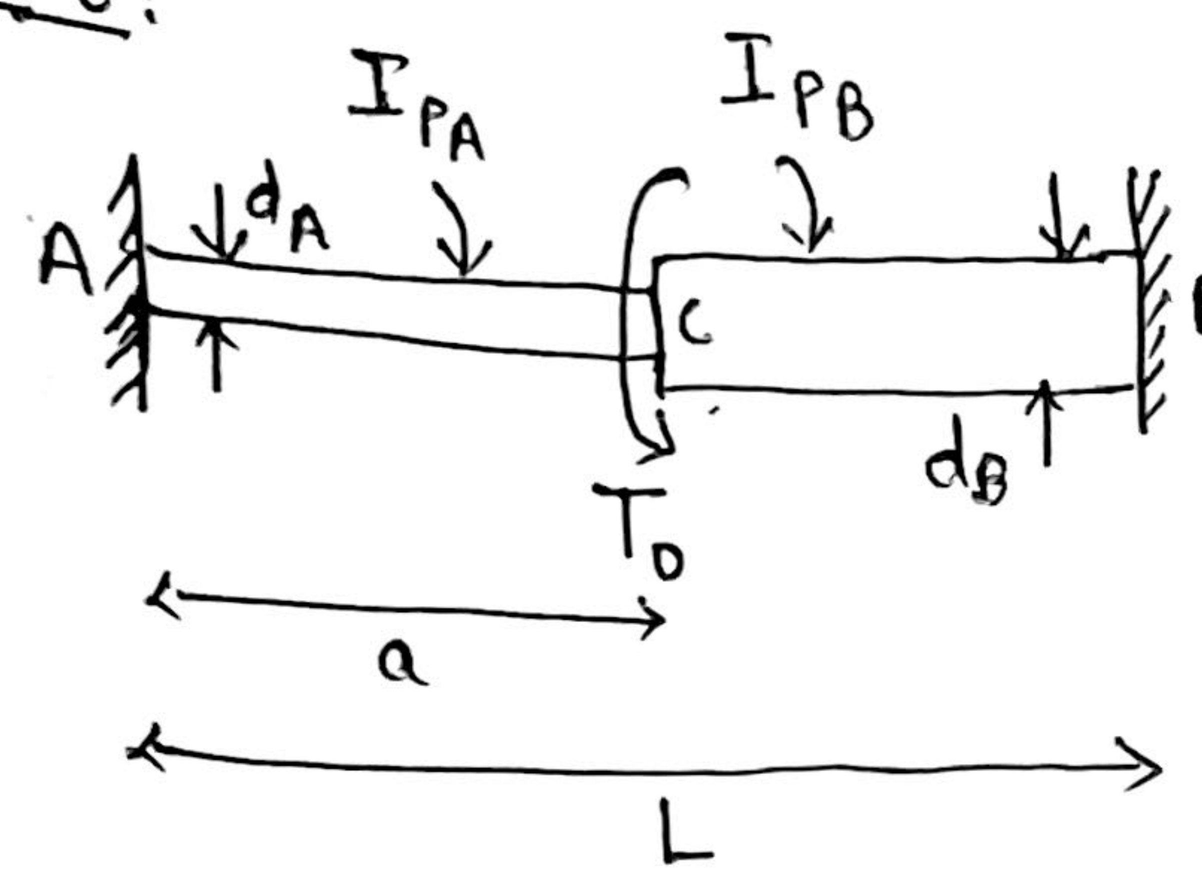
$$= \frac{2\,\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad \left[\text{Since } G = \frac{E}{2(1+\nu)}\right]$$

$$= \tan 2\theta_p^{\sigma}$$

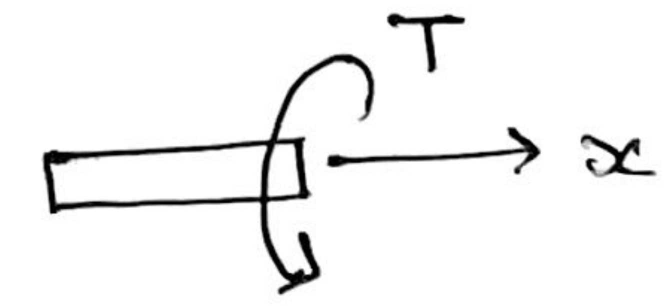
Hence, proved

It can be done similarly for the xz and the yz planes

Prob 6:

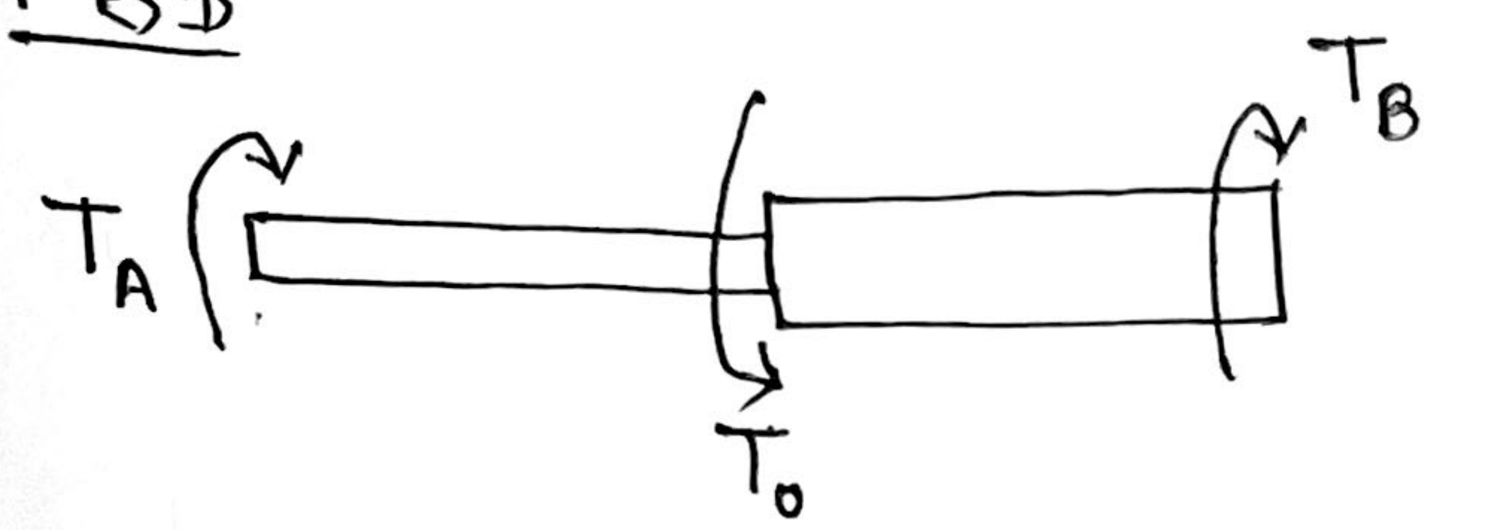


Sign convention for torsion (internal)

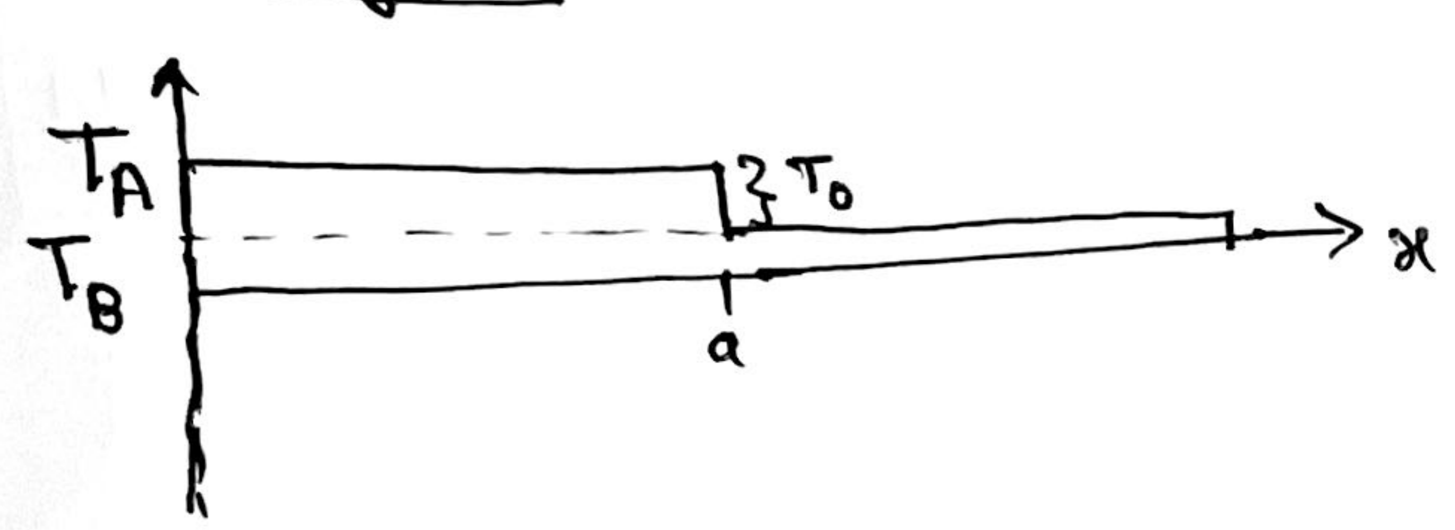


T is +ve.

FBD



Torsion diagram.



$$T_B = (T_A - T_0) \quad — (1)$$

here T_A & T_B are unknowns and we need to use rotational constraint condition to dertermined them as the problem is indeterminate.

we have, $\theta_A = 0$ and $\theta_B = 0$ as the shaft is completely fixed.

also we know, $\dfrac{d\theta}{dx} = \dfrac{T}{GI_p}$

integrating,

$$\Rightarrow \int_{\theta_A}^{\theta_B} d\theta = \int_0^a \frac{T_A}{GI_{PA}}\,dx + \int_a^L \frac{(T_A - T_0)}{GI_{PB}}\,dx$$

$$\Rightarrow 0 = \frac{T_A a}{GI_{PA}} + \frac{(T_A - T_0)(L-a)}{GI_{PB}}$$

from (1)

$$\Rightarrow T_A = \frac{T_0 I_{PA}(L-a)}{aI_{PB} + (L-a)I_{PA}} \quad \& \quad T_B = \frac{T_0 I_{PB}\, a}{aI_{PB} + (L-a)I_{PA}} \quad — (2)$$

Here we note that Both T_A & T_B are +ve. & $T_A > T_B$ (from ①)

(a)

$$\tau_{max}^{A} = \frac{T_A d_A}{2 I_{PA}} \quad \text{for part AC}$$

$$\& \quad \tau_{max}^{B} = \frac{T_B d_B}{2 I_{PB}} \quad \text{for Part BC}$$

max shear stress.

for $\tau_{max}^{A} = \tau_{max}^{B}$

$$\Rightarrow \frac{\cancel{T_0} \cancel{I_{PA}} (L-a) d_A}{2K \cancel{I_{PA}}} = \frac{\cancel{T_0} \cancel{I_{PB}} a \, d_B}{2K \cancel{I_{PB}}}$$

where $K = aI_{PB} + (L-a) I_{PA}$

$$\Rightarrow \frac{(L-a)}{a} = \frac{d_B}{d_A}$$

$$\Rightarrow \frac{L}{a} - 1 = \frac{d_B}{d_A} \Rightarrow \boxed{\frac{a}{L} = \frac{d_A}{d_A + d_B}}$$ (Ans)

(b) for internal torque to be same,

$$T_A = T_B$$

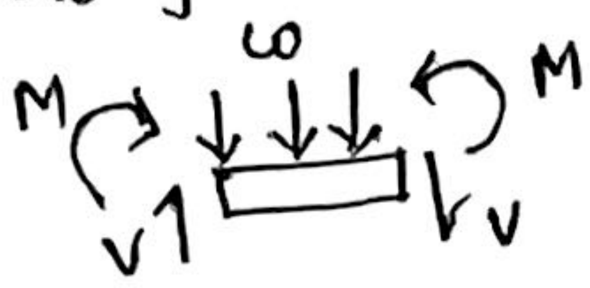
$$\Rightarrow \frac{\cancel{T_0} I_{PA} (L-a)}{\cancel{K}} = \frac{\cancel{T_0} I_{PB} \, a}{\cancel{K}}$$

$$\Rightarrow \frac{L-a}{a} = \frac{I_{PB}}{I_{PA}}$$

$$\Rightarrow \boxed{\frac{a}{L} = \frac{I_{PA}}{I_{PA} + I_{PB}}}$$ (Ans.)

Prob. 7.

[Note: you need to use the following relations for B.M & S.F:—



Sign convention.

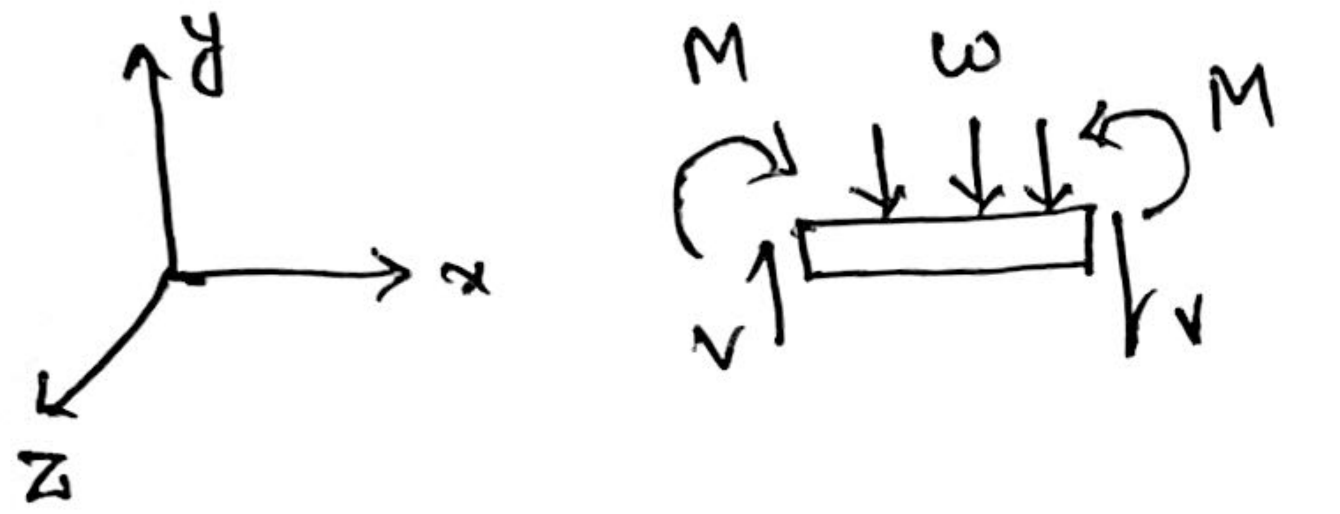
1. $\frac{dV}{dx} = -\omega$

2. $\frac{dM}{dx} = V$ & $\frac{d^2M}{dx^2} = -\omega$

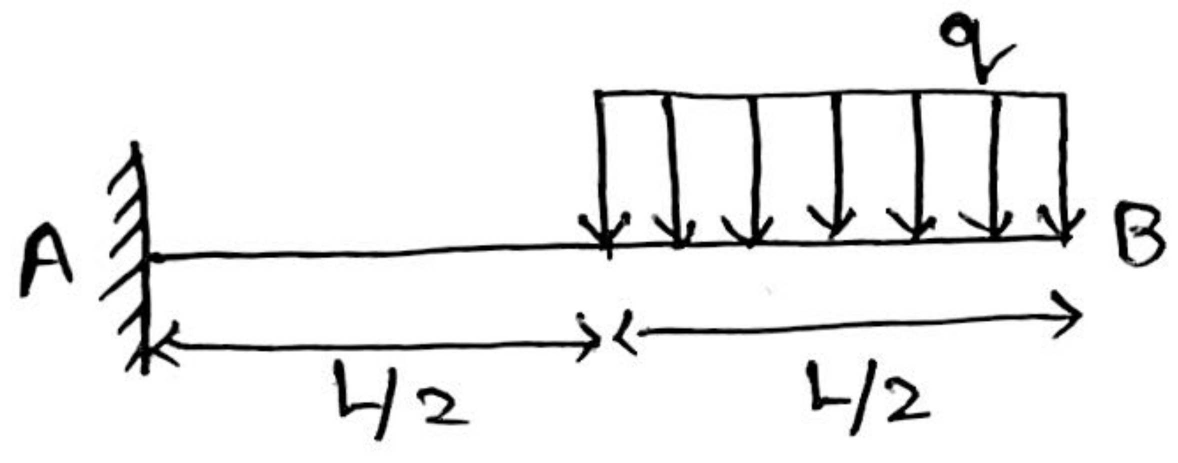
along with the jump conditions discussed in the class.]

Sol^n.

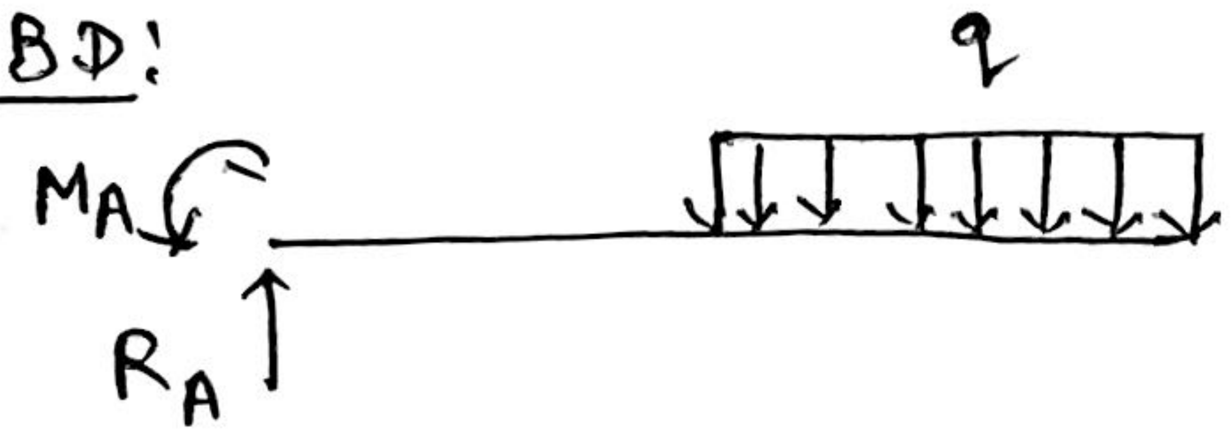
Sign convention:— (for S.F & B.M)



(a)



FBD:

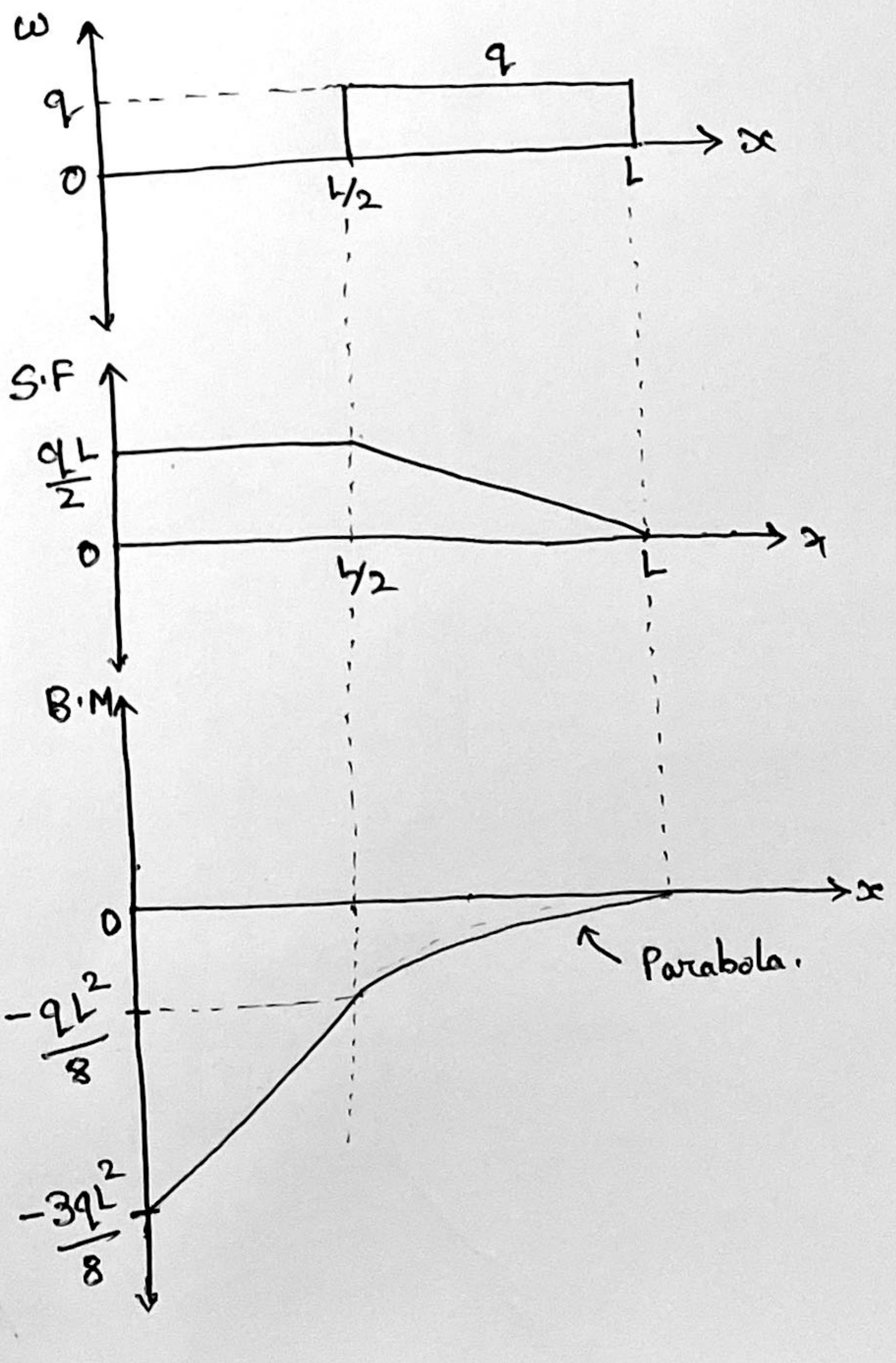


$\Sigma F_y = 0 \Rightarrow R_A = qL/2$

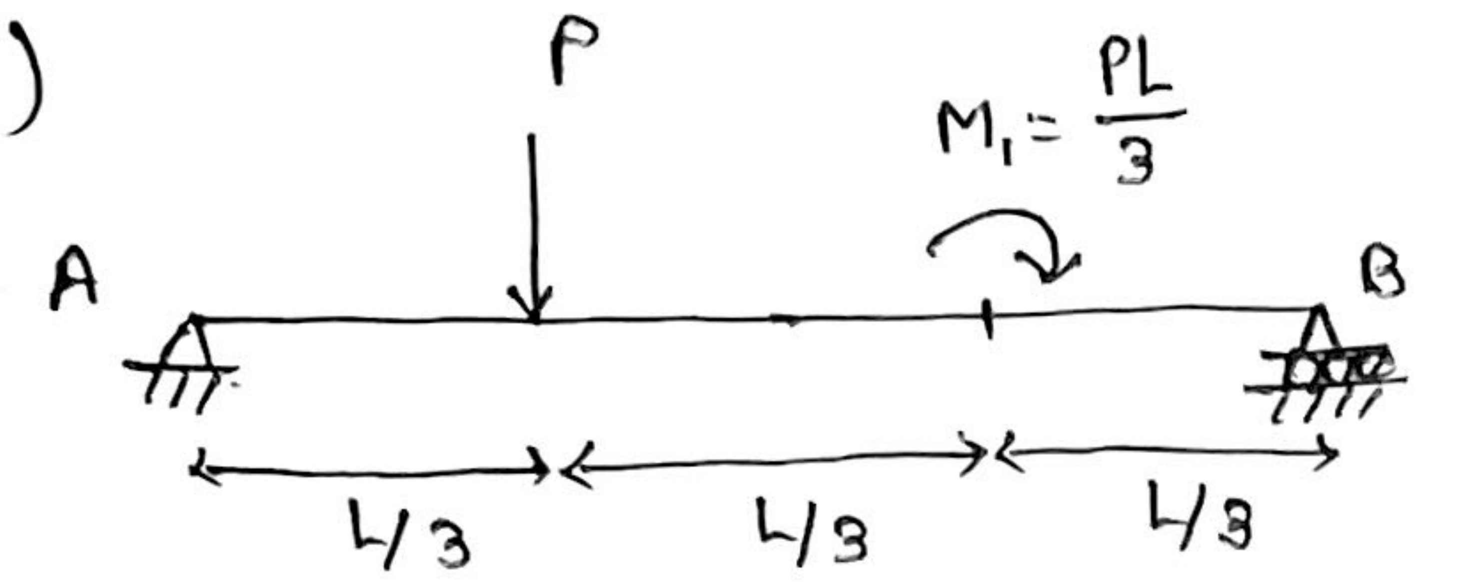
$\Sigma M_z = 0 \Rightarrow M_A = qL/2 \cdot \frac{3L}{4}$

$= \frac{3qL^2}{8}$

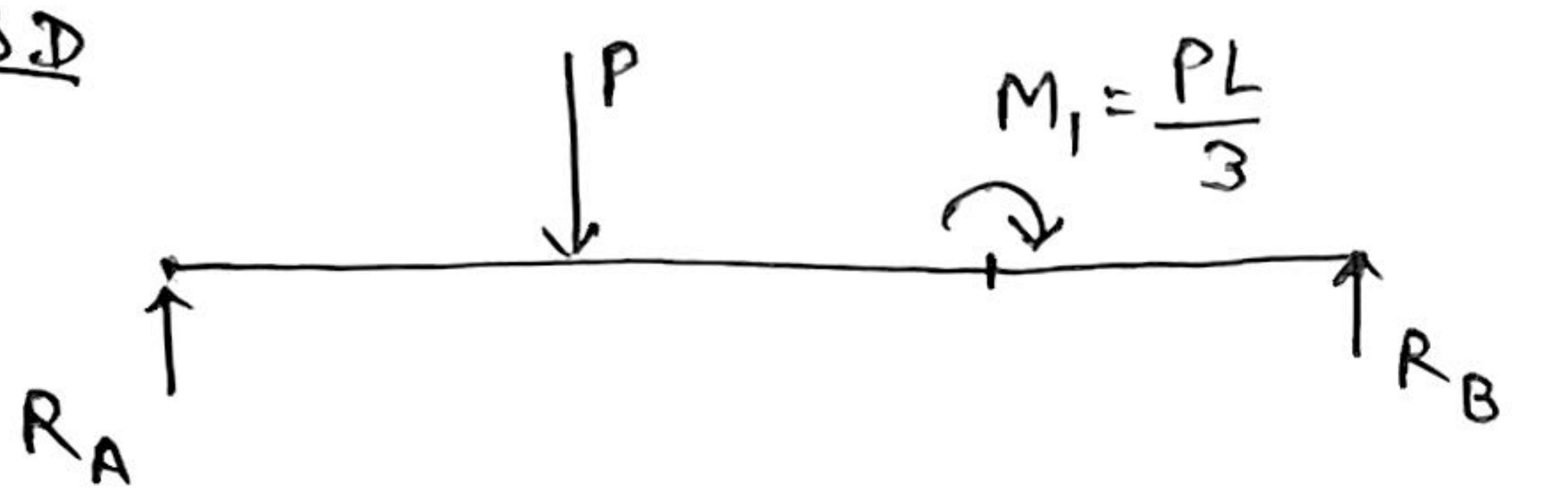
S.F & B.M diagram:—



(b)



FBD



$$\sum F_y = 0 \Rightarrow R_A + R_B = P \quad — (1)$$

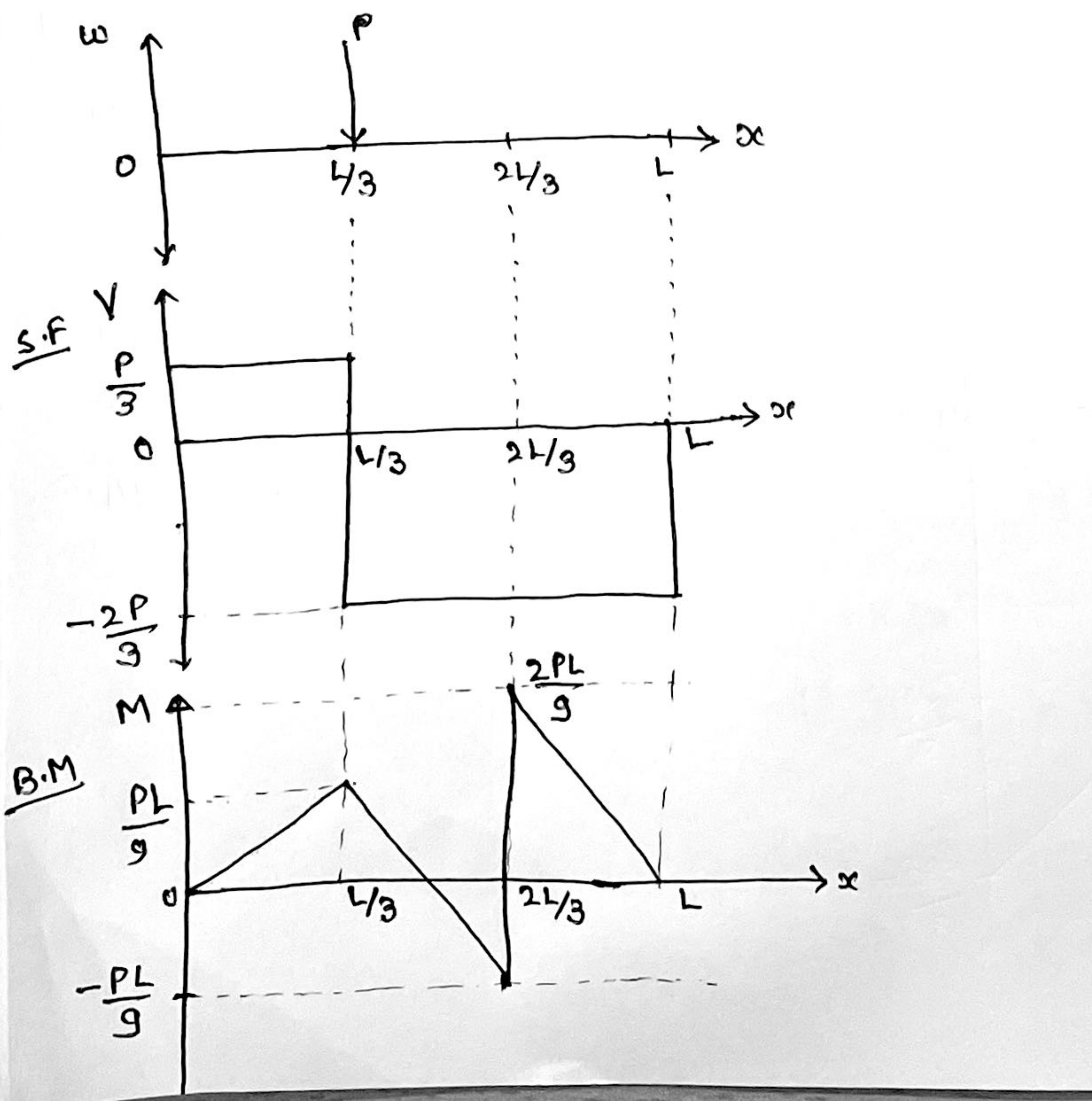
$$\sum M_A = 0 \Rightarrow R_B L = \frac{LP}{3} + \frac{PL}{3}$$

$$= \frac{2PL}{3} \quad — (2)$$

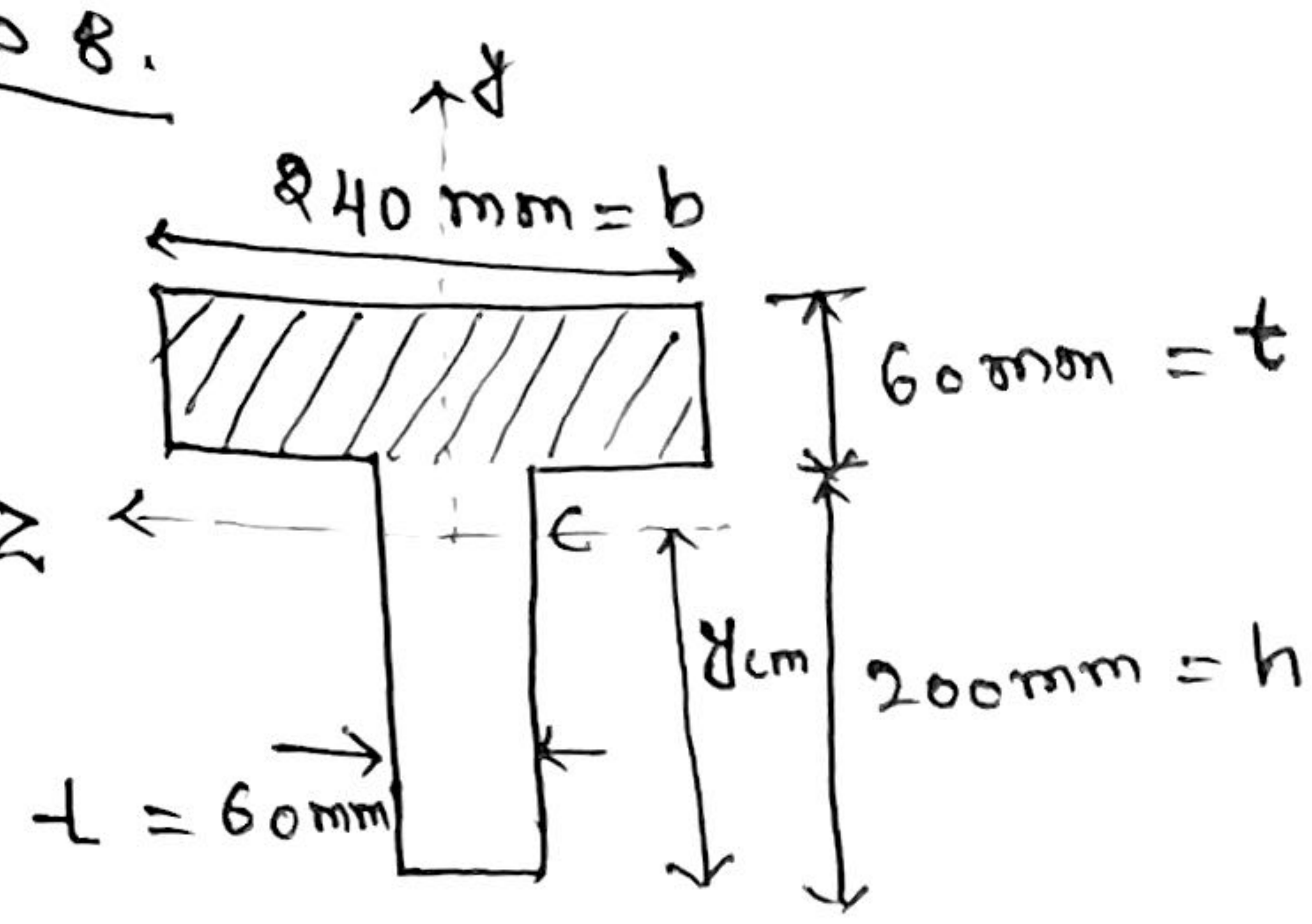
from (1) & (2),

$$R_B = \frac{2P}{3}, \quad R_A = \frac{P}{3}$$

Shear force & Bending moment diagram:-



Prob 8.



Let y_{cm} be the distance of center of the area of the cross-section from the bottom.

$$y_{cm} = \frac{(ht)\,h/2 + bt\left(h + \frac{t}{2}\right)}{ht + bt} = 170.91 \text{ mm.}$$

Second moment of area about centroidal axis :-

$$I_{cm} = \frac{1}{12} b t^3 + bt\left(h - y_{cm} + \frac{t}{2}\right)^2 + \frac{1}{12} t h^3 + ht\left(y_{cm} - h/2\right)^2$$

$$= 1.5494 \times 10^8 \text{ mm}^4.$$

To get the shear ~~force at~~ stress at the joint of the two boards, we need to calculate Q with respect to the centroidal axis of the shaded area.

$$Q = (bt)\left(h - y_{cm} + \frac{t}{2}\right) = 8.5091 \times 10^5 \text{ mm}^3$$

Now the shear flow through the joint can be calculated as,

$$q = \frac{VQ}{I} = \frac{1500 \times 8.5091 \times 10^{-4}}{1.5494 \times 10^{-4}} \text{ N/m} = 8.2378 \times 10^3 \text{ N/m}$$

If the allowable shear in nail is $\tau_{all} = 760$ N and have the spacing s,

Then, $$\frac{\tau_{all}}{s} = q \Rightarrow s = \frac{\tau_{all}}{q} = \frac{760}{8.2378 \times 10^3} = 92.2576 \times 10^{-3} \text{ m.}$$

$$\boxed{s = 92.26 \text{ mm}}\ \text{Ans.}$$