1. Consider a situation where the coordinate axes are oriented along the principal directions. Then the state of stress is given by (you don't have to explain or prove this):

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{p1} & 0 & 0 \\ 0 & \sigma_{p2} & 0 \\ 0 & 0 & \sigma_{p3} \end{bmatrix}$$

A plane which is equally inclined to these coordinate axes is referred to as an octahedral plane, and the unit outward normal to this plane $\hat{\mathbf{n}} = [n_1 \; n_2 \; n_3]^\mathsf{T}$ is such that $|n_1| = |n_2| = |n_3| = 1/\sqrt{3}$ (again, you don't have to prove this).

(a) Show that the expression for the normal stress component on this plane is $\sigma_{nn} = \dfrac{1}{3} I_1$.

(b) Show that the expression for the shear stress component (i.e. the traction component lying in the same plane that contains the traction vector and the unit outward normal) is $\sigma_{ns} = \dfrac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2}$.

Here, $I_1 = \text{trace}(\boldsymbol{\sigma})$ and $I_2 = \sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p3}\sigma_{p1}$.

[3 + 4 = 7 marks]

1(a)

$$\sigma_{nn} = [\hat{n}]^T [\underset{\sim}{\sigma}] [\hat{n}]$$

$$= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{p_1} & 0 & 0 \\ 0 & \sigma_{p_2} & 0 \\ 0 & 0 & \sigma_{p_3} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{p_1} n_1 \\ \sigma_{p_2} n_2 \\ \sigma_{p_3} n_3 \end{bmatrix}$$

$$= \sigma_{p_1} n_1^2 + \sigma_{p_2} n_2^2 + \sigma_{p_3} n_3^2 = \frac{1}{3}(\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3}) = \frac{1}{3} I_1$$

1(b) $\tau_{ns}^2 = |\vec{T}|^2 - \sigma_{nn}^2$

Now, $\vec{T} = [\underset{\approx}{\sigma}]^T [\hat{n}]$

$$= \begin{bmatrix} \sigma_{p_1} n_1 \\ \sigma_{p_2} n_2 \\ \sigma_{p_3} n_3 \end{bmatrix}$$

$$|\vec{T}|^2 = (\sigma_{p_1} n_1)^2 + (\sigma_{p_2} n_2)^2 + (\sigma_{p_3} n_3)^2$$

$$= \frac{1}{3}(\sigma_{p_1}^2 + \sigma_{p_2}^2 + \sigma_{p_3}^2)$$

So, $\tau_{ns}^2 = \frac{1}{3}(\sigma_{p_1}^2 + \sigma_{p_2}^2 + \sigma_{p_3}^2) - \frac{1}{9}(\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3})^2$

$$= \frac{1}{3}\left[(\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3})^2 - 2(\sigma_{p_1}\sigma_{p_2} + \sigma_{p_2}\sigma_{p_3} + \sigma_{p_3}\sigma_{p_1})\right] - \frac{1}{9}(\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3})^2$$

$$= \frac{2}{9}\left(\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3}\right)^2 - \frac{2}{3}\left(\sigma_{p_1}\sigma_{p_2} + \sigma_{p_2}\sigma_{p_3} + \sigma_{p_3}\sigma_{p_1}\right)$$

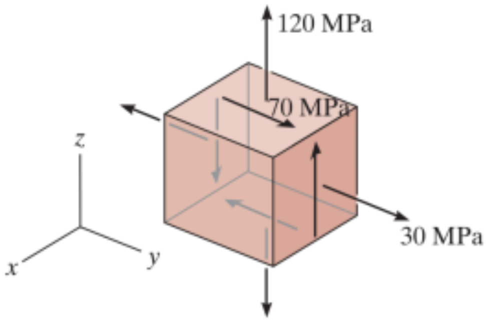
$$= \frac{2}{9}\left[I_1^2 - 3 I_2 \right]$$

$$\therefore \quad \tau_{ns} = \frac{\sqrt{2}}{3}\left(I_1^2 - 3 I_2\right)^{1/2}$$

2.

2. The state of stress at a point is depicted on a stress element as shown in the figure. Determine the principal stresses and the absolute maximum shear stress. **[5 marks]**

NOTE: The direction of the arrows represent the *actual physical* direction of the stress components. It is *your* responsibility to take the correct signs following the positivity convention in your calculations.



$\sigma_{yy} = 30 \text{ MPa}$

$\sigma_{zz} = 120 \text{ MPa}$

$\tau_{yz} = 70 \text{ MPa}$ (not -70 MPa!)

$\sigma_{xx} = \tau_{xy} = \tau_{xz} = 0 \rightarrow$ Indicates plane stress.

Additionally, $\tau_{xy} = \tau_{xz} = 0$ implies σ_{xx} itself is a principal stress. $\rightarrow$ Very Important!

Within the yz-plane,

$$\sigma_{avg} = \frac{\sigma_{yy} + \sigma_{zz}}{2} = 75 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{yy} - \sigma_{zz}}{2}\right)^2 + \tau_{yz}^2} = 83.22 \text{ MPa}$$

$$\sigma_{P_1} = \sigma_{avg} + R = 158.22 \text{ MPa}$$

$$\sigma_{P_2} = \sigma_{avg} - R = -8.22 \text{ MPa}$$

From before, $\sigma_{xx} \equiv \sigma_{P_3} = 0$ MPa

Note: $\sigma_{P_1} > \sigma_{P_3} > \sigma_{P_2}$

$$\therefore \tau_{abs,max} = \frac{\sigma_{P_1} - \sigma_{P_2}}{2}$$

$$= 83.22 \text{ MPa}$$

Some students have obtained this <u>but without</u> mentioning anything regarding the $\sigma_{xx} = 0$ being also a principal stress. I have cut their marks.

In contrast, some students correctly recognized σ_{xx} as a principal stress but due to some calculation error got wrong values. I have cut less marks for these cases.

4. Consider a case of plane stress (in the xy-plane) for a material with Young's modulus, E, and Poisson's ratio, ν, and which follows the generalised Hooke's law.

(a) Show that: $\sigma_{xx} = \dfrac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy})$ and $\sigma_{yy} = \dfrac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx})$.

(b) Derive the strain transformation equations from the stress transformation equations.

[3 + 5 = 8 marks]

4(a)

Plane stress in the xy-plane implies:

$$\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \cancelto{0}{\sigma_{zz}}\right)\right] \Rightarrow E\varepsilon_{xx} = \sigma_{xx} - \nu\sigma_{yy} \quad \text{— (1)}$$

$$\varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \nu\left(\cancelto{0}{\sigma_{zz}} + \sigma_{xx}\right)\Big] \Rightarrow E\varepsilon_{yy} = \sigma_{yy} - \nu\sigma_{xx} \quad \text{— (2)}$$

(1) $+ \nu \times$ (2)

$$\Rightarrow E(\varepsilon_{xx} + \nu\varepsilon_{yy}) = \sigma_{xx}(1-\nu^2) \Rightarrow \sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy})$$

Similarly, (2) $+ \nu \times$ (1) $\Rightarrow \sigma_{yy} = \dfrac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx})$

4(b) From the list of useful formulae at the end of the question paper:

- Stress transformation:

$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Obtain $\sigma_{y'y'}$ from $\sigma_{x'x'}$ using $\theta \to \theta + \pi/2$:

$$\sigma_{y'y'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos(2\theta+\pi) + \tau_{xy}\sin(2\theta+\pi)$$

$$= \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

Now, $\varepsilon_{x'x'} = \frac{1}{E}\left(\sigma_{x'x'} - \nu\,\sigma_{y'y'}\right)$

$$= \frac{1}{E}\left[\frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta - \nu\frac{\sigma_{xx}+\sigma_{yy}}{2} + \nu\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \nu\,\tau_{xy}\sin 2\theta\right]$$

$$\therefore\ E\,\varepsilon_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2}(1-\nu) + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta\,(1+\nu) + \tau_{xy}\sin 2\theta\,(1+\nu) \quad \text{— (3)}$$

From part (a) : $\sigma_{xx}+\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx}+\nu\varepsilon_{yy}+\varepsilon_{yy}+\nu\varepsilon_{xx}\right) = \frac{E}{1-\nu^2}(\varepsilon_{xx}+\varepsilon_{yy})(1+\nu)$

$$= \frac{E}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy}) \quad \text{— (4)}$$

$$\sigma_{xx}-\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx}+\nu\varepsilon_{yy}-\varepsilon_{yy}-\nu\varepsilon_{xx}\right) = \frac{E}{1-\nu^2}(\varepsilon_{xx}-\varepsilon_{yy})(1-\nu)$$

$$= \frac{E}{1+\nu}(\varepsilon_{xx}-\varepsilon_{yy}) \quad \text{— (5)}$$

Additionally, $\tau_{xy} = 2G\,\varepsilon_{xy} = 2\frac{E}{2(1+\nu)}\varepsilon_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$ — (6)

Substituting (4), (5), and (6) in (3) :

$$E\,\varepsilon_{x'x'} = \frac{E}{\cancel{1-\nu}}\frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\cancel{(1-\nu)} + \frac{E}{\cancel{1+\nu}}\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta\,\cancel{(1+\nu)} + \frac{E}{\cancel{1+\nu}}\varepsilon_{xy}\sin 2\theta\,\cancel{(1+\nu)}$$

$$\Rightarrow \varepsilon_{x'x'} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta + \varepsilon_{xy}\sin 2\theta$$

Similarly, it can be done for $\varepsilon_{y'y'}$. (no need to actually do it!)

$$\varepsilon_{x'y'} = \frac{\tau_{x'y'}}{2G} = \frac{1+\nu}{E}\tau_{x'y'}$$

$$= \frac{1+\nu}{E}\left[-\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]$$

$$= \frac{1+\nu}{E}\left[-\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\,\frac{E}{1+\nu}\cos 2\theta + \frac{E}{1+\nu}\varepsilon_{xy}\sin 2\theta\right]$$

$$= -\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta + \varepsilon_{xy}\sin 2\theta$$

Many students forgot to derive the $\varepsilon_{x'y'}$.

A word of advice:

For these kinds of proof-based problems, please write the steps with academic honesty.

Just writing some steps and then scribbling some arbitrary things followed by "Hence, proved" reflects very poorly on your intellect.

There are some other ways of deriving the above. They have been given full marks. However, in one of the ways, some students arrived at a step which looked like the following:

$$\varepsilon_{x'x'} + \nu\,\varepsilon_{y'y'} = (\quad A \quad) + \nu(\quad B \quad)$$

Then they wrote: "Comparing coefficients of ν", $\varepsilon_{x'x'} = A$

$$\varepsilon_{y'y'} = B$$

Even though the expressions of A and B turn out to be correct, the argument of "comparing coefficients" is not correct. See the following counter-argument:

In the plane strain case: $\cancelto{0}{\varepsilon_{zz}} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]$

$$\Rightarrow \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

Now, if we "compare coefficients of ν", then $\sigma_{zz} = 0$ and $\sigma_{xx} + \sigma_{yy} = 0$

But these are certainly not true!!

I think students came up with this argument "inspired" by what is done in complex numbers ($a + ib = c + id \Rightarrow a = b$ and $c = d$) or in vector algebra ($a\hat{i} + b\hat{j} = c\hat{i} + d\hat{j} \Rightarrow a = c$ and $b = d$).

However, unlike the imaginary unit "i" or unit vectors, the Poisson's ratio ν is just a number.

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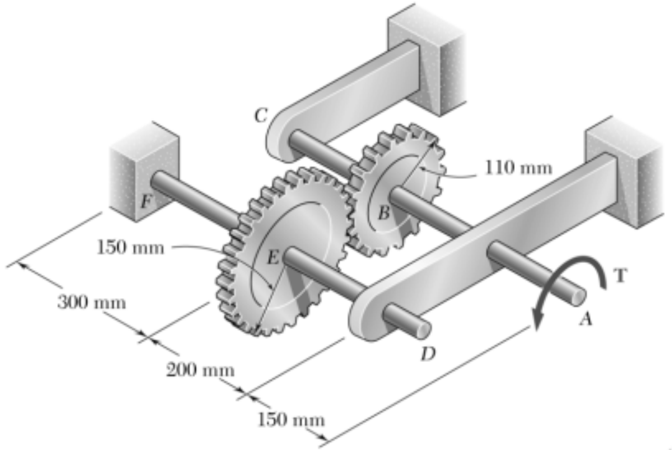
The torque applied at A is fully available at B.

However, what is transmitted to E depends as follows:

5. Two shafts, each of 22 mm diameter and made of the same material are connected by gears as shown in the figure. Given that $G = 77$ GPa and the shaft at F is fixed, determine the angle through which the end A rotates when a torque T = 130 N·m is applied at A. The slender beam-like supports through which the shafts pass do not provide any resistance to their rotation. Hint: When two gears mesh, the contact forces are equal (not the torques!); additionally the arcs of contact are equal. **[10 marks]**



$$\frac{T_{AB}}{r_B} = \frac{T_{EF}}{r_E} \Rightarrow T_{EF} = \frac{r_E}{r_B} T_{AB} = \frac{150}{110} 130\,\text{Nm} = 177.3\ \text{Nm}$$

Many students wrote this as $T_{AB} r_B = T_{EF} r_E$ → Severe penalty for this!

Some students have done some kind of moment balance considering both gears. But that is wrong because they are on two separate axes.

Some students have either used $T_{AB} = T_{EF}$ or written some equations which effectively imply $T_{AB} = T_{EF}$. This is completely wrong!

Next, considering the part EF :

$$\phi_{E/F} = \frac{T_{EF} L_{EF}}{GJ} = 0.03 \text{ rad}$$

$\left[J = \frac{1}{2} \pi r_{shaft}^4 = 23 \times 10^{-9} \text{ m}^4 \right]$

$r_{shaft} = 0.011 \text{ m}$

Taking this length as L_{DF} is conceptually wrong. D does not rotate relative to E.

$$\therefore \phi_E = \phi_{E/F} + \cancel{\phi_F}^{0} = 0.03 \text{ rad}$$

So, $\phi_B = \frac{r_E}{r_B} \phi_E$ (arcs of contact are equal)

$$= \frac{150}{110}(0.03) \text{ rad} = 0.041 \text{ rad}$$

Many students have stopped at this point thinking $\phi_A = \phi_B$, but that is not correct. → Small penalty for this.

$$\phi_A = \phi_{A/B} + \phi_B$$

$$\phi_{A/B} = \frac{T_{AB} L_{AB}}{GJ} = 0.026 \text{ rad}$$

$$\therefore \phi_A = (0.041 + 0.026) \text{ rad}$$
$$= 0.067 \text{ rad} = 3.82^\circ$$

Many students have got all the answers wrong because of some calculation mistake or wrong substitution but the flow of their solution steps indicates that they have conceptually understood the key points of the problem. I have given some marks for such cases.