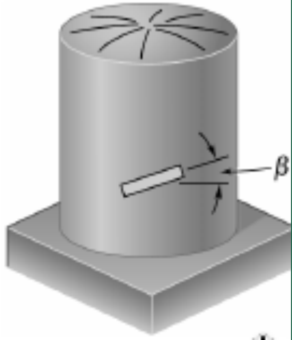
1. A strain gauge forming an angle $\beta = 18°$ with a horizontal plane is used to determine the gauge pressure in the cylindrical steel tank shown (a kind of pressure vessel). The cylindrical wall of the tank is 6 mm thick, has a 600-mm inside diameter, and is made of a steel with $E = 200$ GPa and $\nu = 0.3$. Determine the pressure in the tank indicated by a strain gauge reading of 280×10^{-6}. **[12 marks]**



$$\sigma_{xx} = \frac{pr}{t}$$

$$\sigma_{yy} = \frac{pr}{2t}$$

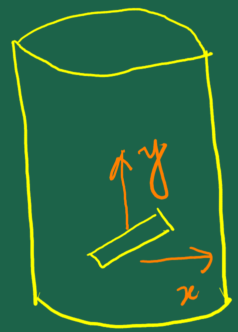
$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right)$$

$$= \frac{pr}{Et}\left(1 - \frac{\nu}{2}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right)$$

$$= \frac{pr}{Et}\left(\frac{1}{2} - \nu\right)$$

$$\varepsilon_{xy} = 0 \quad (\because \tau_{xy} = 0)$$



$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos 2\beta + \cancelto{0}{\varepsilon_{xy}} \sin 2\beta$$

$$= \frac{1}{2}\frac{pr}{Et}\left(1 - \frac{\nu}{2} + \frac{1}{2} - \nu\right) + \frac{1}{2}\frac{pr}{Et}\left(1 - \frac{\nu}{2} - \frac{1}{2} + \nu\right)\cos 2\beta$$

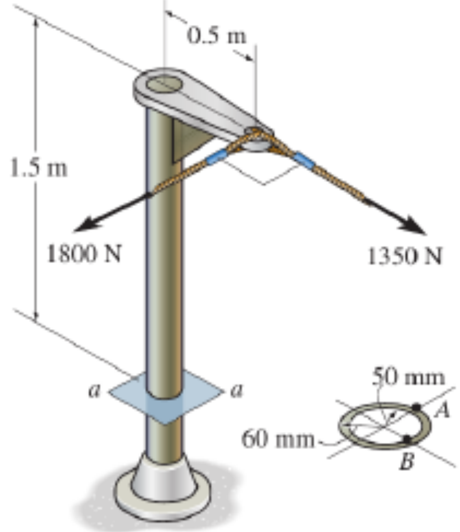
$$= \frac{1}{2}\frac{pr}{Et}\left[\frac{3}{2}(1-\nu) + \frac{1}{2}(1+\nu)\cos 2\beta\right]$$

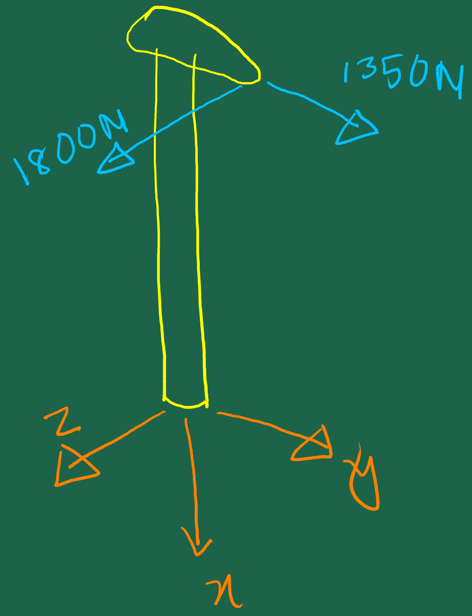
$$\therefore \quad p = \frac{2Et\,\varepsilon_{x'x'}}{r\left[\frac{3}{2}(1-\nu) + \frac{1}{2}(1+\nu)\cos 2\beta\right]}$$

$$= 1.42 \text{ MPa}$$

This was a "gift".
Those who did the major step correctly received 10.
A few students received less because they did not even use stress/strain transformation.

2. Determine the normal stress and the shear stress at the point B on the cross-section of the post at section $a-a$. Note that B is a point on the inner boundary of the annulus. **[16 marks]**





$$V_y = -1350\ \text{N}$$
$$V_z = -1800\ \text{N}$$
$$T = -900\ \text{Nm}$$
$$M_y = -2700\ \text{Nm}$$
$$M_z = 2025\ \text{Nm}$$

No marks for these

$$I_y = I_z = \frac{\pi}{4}\left(60^4 - 50^4\right) = 5.27 \times 10^6\ \text{mm}^4$$
$$J = \frac{\pi}{2}\left(60^4 - 50^4\right) = 10.54 \times 10^6\ \text{mm}^4$$

1 mark

$$(Q_y)_B = 0$$

$$(Q_z)_B = \frac{4}{3\pi} r_o \left(\frac{\pi}{2} r_o^2\right) - \frac{4}{3\pi} r_i \left(\frac{\pi}{2} r_i^2\right) = 6.067 \times 10^4 \text{ mm}^3$$ (1 mark)

Normal Stress:

$$\sigma_B = -\frac{M_z y}{I_z} = -19.21 \text{ MPa}$$

(1 mark for the -ve sign or writing compressive stress)

(5 marks for σ_B magnitude)

Shear Stress:

Due to torsion: $\tau_{xz}\big|_T = \frac{T r_i}{J} = 4.269$ MPa (3 marks)

Due to transverse load $\tau_{xz}\big|_V = \frac{V_z Q_z}{I_y t} = 1.036$ MPa (3 marks)

Both these shear stresses are in the same dir^n.

$$\therefore \quad \tau_{xz}\big|_{total} = \tau_{xz}\big|_T + \tau_{xz}\big|_V = 5.305 \text{ MPa}$$

2 marks for the addition

4. A thin annular disk of inner radius r_i and outer radius r_o is fixed at its outer boundary to a rigid support. However axial displacement is not constrained. At the inner boundary, there is a force (per unit area) f acting radially inward. Starting from the governing differential equation for radial displacement in the form: $\dfrac{d^2u}{dr^2} + \dfrac{1}{r}\dfrac{du}{dr} - \dfrac{u}{r^2} = 0$, and using the appropriate boundary conditions, determine the radial stress at the outer boundary. **[12 marks]**

Let $u = r^m$. Substituting in the eqn and solving gives us:

$$u = C_1 r + C_2 \frac{1}{r}$$ — (1 mark)

@ $r = r_0$, $u = 0 \Rightarrow C_1 r_0 + C_2 \frac{1}{r_0} \Rightarrow C_2 = -C_1 r_0^2$

Now, this is is a plane stress case. So,

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}\right]$$

$$= \frac{E}{1-\nu^2}\left[\frac{du}{dr} + \nu\frac{u}{r}\right]$$

$$= \frac{E}{1-\nu^2}\left[C_1 - \frac{C_2}{r^2} + \nu\left(C_1 + \frac{C_2}{r^2}\right)\right]$$

(6 marks deducted for those who wrote the formula for plane strain. I had said on the last day of class that I would give 0 to those who make this mistake. But I changed my mind.)

@ $r = r_i$, $\sigma_{rr} = f$

$$\Rightarrow \frac{E}{1-\nu^2}\left[c_1(1+\nu) - \frac{c_2}{r_i^2}(1-\nu)\right] = f$$

$$\Rightarrow \frac{E}{1-\nu^2}\left[c_1(1+\nu) + \frac{c_1 r_o^2}{r_i^2}(1-\nu)\right] = f$$

$$\Rightarrow c_1 = \frac{f}{\frac{E}{1-\nu^2}\left[1+\nu + \frac{r_o^2}{r_i^2}(1-\nu)\right]}$$

$$\therefore \sigma_{rr}\Big|_{r=r_o} = \frac{E}{1-\nu^2}\left[c_1(1+\nu) - \frac{c_2}{r_o^2}(1-\nu)\right]$$

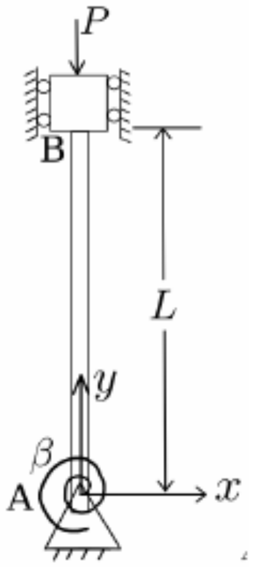
$$= \frac{E}{1-\nu^2}\left[c_1(1+\nu) + \frac{c_1 \cancel{r_o^2}}{\cancel{r_o^2}}(1-\nu)\right]$$

$$= \frac{E}{1-\nu^2}\left[2c_1\right] = \frac{2f}{1+\nu + \frac{r_o^2}{r_i^2}(1-\nu)}$$

Serious penalty for those who wrote wrong BCs, especially who wrote something like

$\sigma_{rr} = -P_o$ @ $r = r_o$.

5. Consider a column with constant flexural rigidity (EI) as shown in the figure. The bottom end A ($x = 0$) is pinned and attached with a rotational spring of stiffness β. The top end B ($x = L$) is fixed to the vertical guide. Showing all the steps involving the solution of the 4th-order differential equation together with the use of the proper boundary conditions, determine the critical condition(s) for buckling. **[12 marks]**



[In the figure the labels of the x and y axes were interchanged by mistake.]

The general solution of the 4th order differential eqn, $\frac{d^4 y}{dx^4} + k^2 \frac{d^2 y}{dx^2} = 0$ is

$$y = A \sin kx + B \cos kx + Cx + D \quad \text{— (1 mark)}$$

$$\frac{dy}{dx} = Ak \cos kx - Bk \sin kx + C$$

$$\frac{d^2 y}{dx^2} = -Ak^2 \sin kx - Bk^2 \cos kx$$

(There are different ways of proceeding with the simplifications after writing the BCs. But the BCs are most important.)

$@\ x=0,\quad y=0$ — (2 marks)

$\Rightarrow B+D=0$ — (1)

$@\ x=0,\quad EI\dfrac{d^2y}{dx^2}=-\beta\dfrac{dy}{dx}$ — (2 marks)

$\Rightarrow EI(-BK^2)=-\beta(AK+C) \Rightarrow C=\dfrac{EIK^2}{\beta}B-AK$ — (2)

$@\ x=L,\quad y=0$ — (2 marks)

$\Rightarrow A\sin KL+B\cos KL+CL+D=0$ — (3)

$@\ x=L,\quad \dfrac{dy}{dx}=0$ — (2 marks)

$\Rightarrow AK\cos KL-BK\sin KL+C=0$

$\Rightarrow AK\cos KL-BK\sin KL+\dfrac{EIK^2}{\beta}B-AK=0$ (Using (2))

$\Rightarrow AK(\cos KL-1)=BK\left(\sin KL-\dfrac{EIK}{\beta}\right)$ — (4)

Using ① and ② in ③, we have:

$$A\sin KL + B\cos KL + L\left(\frac{EIK^2}{\beta}B - AK\right) - B = 0$$

$$\Rightarrow A(\sin KL - KL) = B\left(1 - \cos KL - \frac{EIK^2 L}{\beta}\right) \quad \text{— ⑤}$$

Dividing ④ by ⑤, we obtain:

$$\frac{\cos KL - 1}{\sin KL - KL} = \frac{K\left(\sin KL - \frac{EIK}{\beta}\right)}{1 - \cos KL - \frac{EIK^2 L}{\beta}}$$

(Some students may have obtained other forms of the above critical condition. Those are fine as long as they have proceeded from the correct BCs.)

I kept 3 marks for the simplification. Not many students proceeded well. But those who tried to proceed somewhat, I gave ½, 1 or 2 marks depending on their effort.