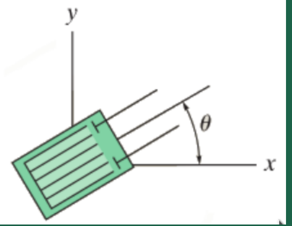
1. A point in a body under plane stress conditions has a state of stress given by non-zero σ_{xx} and σ_{yy} but with $\tau_{xy} = 0$. The material constants are E and ν. A strain gauge placed at the point is orientated in such a way that its reading of the normal strain (along its own axis) is dependent only on σ_{yy} but not on σ_{xx}. Show that the orientation θ is given by $\tan\theta = \dfrac{1}{\sqrt{\nu}}$. **[10 marks]**



Q1

There are different approaches of doing this problem. I am showing just one way.

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \cancelto{0}{\sigma_{zz}})\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \cancelto{0}{\sigma_{zz}})\right]$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{2G} = 0$$

$$\varepsilon_{x'x'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + 2\cancelto{0}{\varepsilon_{xy}}\sin\theta\cos\theta$$

$$= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy})\cos^2\theta + \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})\sin^2\theta$$

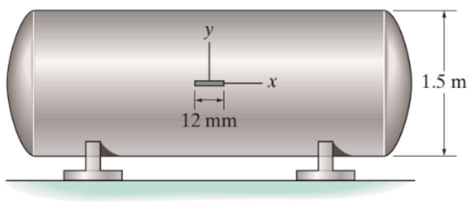
$$= \frac{1}{E}\sigma_{xx}(\cos^2\theta - \nu\sin^2\theta) + \frac{1}{E}\sigma_{yy}(\sin^2\theta - \nu\cos^2\theta)$$

Since $\varepsilon_{x'x'}$ is not dependent on σ_{xx}, we have $\cos^2\theta - \nu\sin^2\theta = 0 \Rightarrow \tan^2\theta = \frac{1}{\nu}$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{\nu}} \quad (\text{+ve due to } \theta < 90^\circ)$$

Q2

2. The strain gauge is placed on the surface of a thin-walled steel boiler (a kind of pressure vessel) as shown. The boiler wall has an inner diameter of 1.5 m and a wall thickness of 12 mm. For the steel, $E = 200$ GPa and $\nu = 0.3$.
   (a) If the strain gauge is 12 mm long, determine the pressure in the boiler when the gauge undergoes an elongation of 5×10^{-3} mm.
   (b) Determine the maximum *in-plane* shear strain in the wall.
   [10 + 2 = 12 marks]



Axial dir^n : 2

Circumferential dir^n : 1

$$\varepsilon_2 = \frac{\Delta l}{l} = \frac{5\times10^{-3}\ \text{mm}}{12\ \text{mm}}$$

$$= 417\times10^{-6}$$

(a) $\sigma_1 = \dfrac{pr}{t}$, $\sigma_2 = \dfrac{pr}{2t}$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \cancel{\sigma_3})\right] \qquad \sigma_3 \simeq 0$$

$$\Rightarrow \varepsilon_2 = \frac{1}{E}\left[\frac{pr}{2t} - \nu\frac{pr}{t}\right]$$

$$\Rightarrow \varepsilon_2 = \frac{pr}{Et}\left(\frac{1}{2} - \nu\right)$$

$$\Rightarrow p = \frac{2Et\varepsilon_2}{(1-2\nu)r}$$

$$= 6.67\ \text{MPa}$$

(b) For max in-plane shear strain, we need ε_1 also.

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\sigma_2\right] = \frac{1}{E}\left[\frac{pr}{t} - \nu\frac{pr}{2t}\right]$$

$$= \frac{pr}{Et}\left(1 - \frac{\nu}{2}\right)$$

$$= 1771 \times 10^{-6}$$

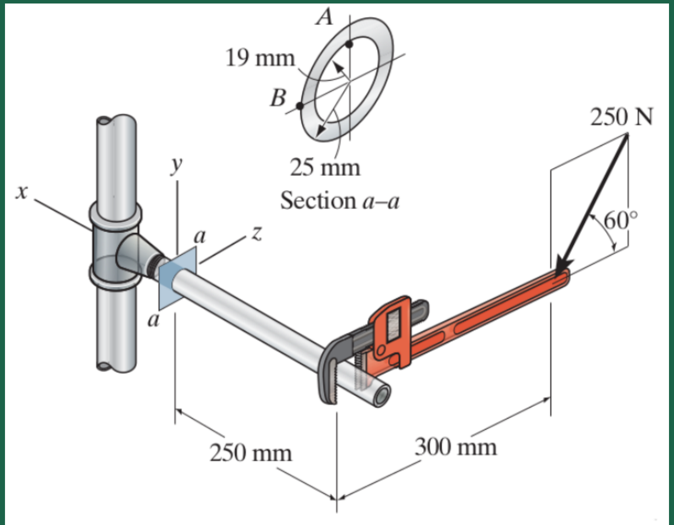
$\therefore$ Max in-plane shear strain : $\frac{1}{2}(\varepsilon_1 - \varepsilon_2)$

$$= 677 \times 10^{-6}$$

Or, $\gamma_{max,\, in\text{-}plane} = 2 \times 677 \times 10^{-6}$

$$= 1354 \times 10^{-6}$$

3. Determine the normal stress and the shear stress at the point A on the cross-section of the pipe at section $a-a$. **[16 marks]**



Q3

Externally applied force:

$$\vec{F}_o = \left(-250\sin 60^\circ \hat{j} - 250\cos 60^\circ \hat{k}\right) \text{N}$$

$$= \left(-216.506\hat{j} - 125\hat{k}\right) \text{N}$$

$\therefore$ At the c/s: $\vec{F} = -\vec{F}_o = \left(216.506\hat{j} + 125\hat{k}\right) \text{N}$

Moment of $\vec{F}_o$ about the origin:

$\vec{M}_o = \vec{r} \times \vec{F}_o$, where $\vec{r}_o = \left(-0.250\hat{i} + 0.300\hat{k}\right)$ m

$$= \left(64.952\hat{i} - 31.25\hat{j} + 54.127\hat{k}\right) \text{N m}$$

$\therefore$ Moment $\vec{M}$ "at" the c/s:

$$\vec{M} = -\vec{M}_o = \left(-64.952\hat{i} + 31.25\hat{j} - 54.127\hat{k}\right) \text{N m}$$

Now, $I_y = I_z = \frac{\pi}{4}\left(R_o^4 - R_i^4\right) = 204442 \text{ mm}^4$

$$J = \frac{\pi}{2}\left(R_o^4 - R_i^4\right) = 408884 \text{ mm}^4$$

Normal stress at A:

M_y does not contribute.

M_z contributes

$$\sigma_{xx}\Big|_A = \left|\frac{M_z\, y}{I_z}\right| \quad (\text{tensile})$$

$$= 5.03 \text{ MPa}$$

No contribution from any axial stress because $F_x = 0$

Shear Stress at A:

First note that $\tau_{xy}\Big|_A = 0$ because $\tau_{yx}\Big|_A = 0$. This is because pt A is part of a free surface (the inner hollow curved surface)

$\tau_{xz}\Big|_A$ will have 2 contributions: torsion + shear (due to bending)

$$\tau_{xz}\Big|_A^{\text{torsion}} = \left|\frac{T\rho_A}{J}\right| = \left|\frac{M_x \rho_A}{J}\right| = \frac{64.95 \times 0.019}{408884 \times 10^{-12}} \ \mathrm{N/m^2} = 3.02 \ \mathrm{MPa}$$

↳ This is along the −ve z-dir^n.

$$\tau_{xz}\Big|_A^{\text{shear}} = \left|\frac{V_z Q}{I t}\right|$$

↳ This is along +ve z-dir^n.

$$= 0.298 \ \mathrm{MPa}$$

$$Q = \frac{4R_o}{3\pi} \times \frac{1}{2}\pi R_o^2 - \frac{4R_i}{3\pi} \times \frac{1}{2}\pi R_i^2$$

$$= \frac{2}{3}\left(R_o^3 - R_i^3\right) = 5844 \ \mathrm{mm}^3$$

$$t = 2 \times (R_o - R_i)$$

$$= 12 \ \mathrm{mm}$$

$$\therefore \ \tau_{xz}\Big|_A = (3.02 - 0.298) \ \mathrm{MPa}$$

$$= 2.72 \ \mathrm{MPa}$$

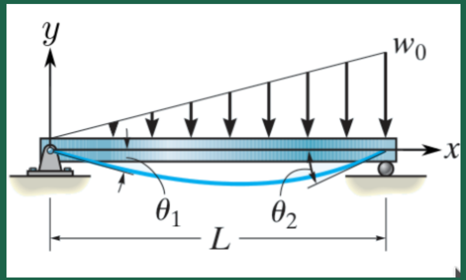
Q4

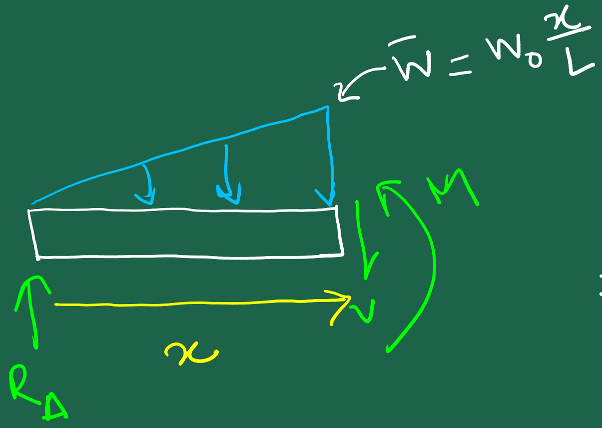
$\sum M_B = 0$

$\Rightarrow \left(\frac{1}{2} w_0 L\right) \times \frac{L}{3} = R_A L$

$\Rightarrow R_A = \frac{w_0 L}{6}$

4. A simply-supported beam of constant flexural rigidity (EI) is subjected to a linearly varying distributed load, $w(x) = \frac{w_0 x}{L}$. Determine the expression of the beam deflection, $y(x)$, by solving the 2nd-order differential equation for deflection of beams together with the use of appropriate boundary conditions. Your answer MUST be expressed in the form: $y = -\frac{w_0 x}{KEIL}\left(ax^4 + bx^3L + cx^2L^2 + dL^3x + eL^4\right)$, where the values of K, a, b, c, d, and e are to be determined. K must be positive. **[14 marks]**





$\frac{1}{2} \bar{w} x \times \frac{x}{3} + M - R_A x = 0$

$\Rightarrow M = R_A x - \frac{\bar{w} x^2}{6}$

$\Rightarrow M = \frac{w_0 x L}{6} - \frac{w_0 x^3}{6L}$

Now, $EI \frac{d^2 y}{dx^2} = M = \frac{w_0 x L}{6} - \frac{w_0 x^3}{6L}$

$\Rightarrow \frac{d^2 y}{dx^2} = \frac{w_0}{6EI}\left(xL - \frac{x^3}{L}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{w_0}{6EI}\left(\frac{x^2 L}{2} - \frac{x^4}{4L}\right) + c_1$$

$$\Rightarrow \quad y = \frac{w_0}{6EI}\left(\frac{x^3 L}{6} - \frac{x^5}{20L}\right) + c_1 x + c_2$$

@ $x = 0,\ y = 0 \Rightarrow c_2 = 0$

@ $x = L,\ y = 0 \Rightarrow \frac{w_0}{6EI}\left(\frac{L^4}{6} - \frac{L^4}{20}\right) + c_1 L = 0$

$$\Rightarrow \quad c_1 = \frac{-w_0}{6EI}\left(\frac{L^3}{6} - \frac{L^3}{20}\right)$$

$$\Rightarrow \quad c_1 = \frac{-w_0}{6EI}\left(\frac{10-3}{60}\right)L^3$$

$$\Rightarrow \quad c_1 = \frac{-7w_0 L^3}{360EI}$$

$$\therefore\ y = \frac{w_0}{6EI}\left(\frac{x^3L}{6} - \frac{x^5}{20L}\right) - \frac{7w_0L^3x}{360EI}$$

$$= \frac{w_0}{36EI}\left(x^3L - \frac{6x^5}{20L}\right) - \frac{7w_0L^3x}{360EI}$$

$$= \frac{w_0}{360EI}\left(10x^3L - 3\frac{x^5}{L} - 7L^3x\right)$$

$$= \frac{-w_0x}{360EIL}\left(-10x^2L^2 + 3x^4 + 7L^4\right)$$

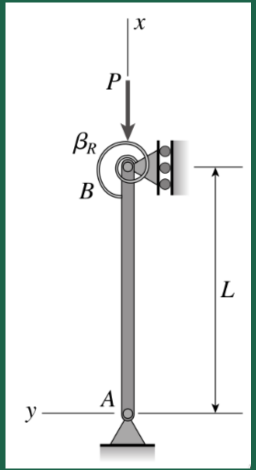
$\therefore$ $K = 360$, $c = -10$,

$a = 3$, $d = 0$

$b = 0$, $e = 7$

5. A column with constant flexural rigidity (EI) is pinned at both its ends. At the top end ($x = L$), there is a rotational spring of stiffness β_R. Showing all the steps involving the solution of the 4th-order differential equation together with the use of the proper boundary conditions, show that the equation for finding the critical buckling load will be: $\frac{\beta_R L}{EI}(kL\cot(kL) - 1) = k^2L^2$, where $k^2 = \frac{P}{EI}$. **[14 marks]**

NOTE: You don't have to solve this equation for the critical load.



Q5

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + k^2\frac{d^2y}{dx^2} = 0$$

Let $y = e^{mx}$. Then,

$$m^4 + k^2 m^2 = 0$$

$$m = 0, 0, \pm ik$$

We obtain: $y = A\sin kx + B\cos kx + Cx + D$

Boundary Conditions:

@ $x = 0$, $y = 0$

@ $x = L$, $y = 0$

@ $x = 0$, $M = EI\frac{d^2y}{dx^2} = 0$

@ $x = L$, $M = EI\frac{d^2y}{dx^2} = -\beta\frac{dy}{dx}$ $\Rightarrow$ i.e. β times the slope or $\beta\theta$

We have: $\frac{dy}{dx} = Ak\cos kx - Bk\sin kx + C$

$$\frac{d^2y}{dx^2} = -Ak^2\sin kx - Bk^2\cos kx$$

Now, @ $x=0,\ y=0$

$$\Rightarrow B+D=0 \quad -(1)$$

@ $x=0,\ M=0$

$$\Rightarrow \quad B=0$$

$$\therefore \quad D=0 \quad (\text{Using } (1))$$

@ $x=L,\ y=0$

$$\Rightarrow A\sin KL + B\cos KL + CL + D = 0$$

$$\Rightarrow A\sin KL + CL = 0$$

$$\Rightarrow \quad C = -\frac{A\sin KL}{L}$$

$@x = L, \quad EI\frac{d^2y}{dx^2} = -\beta\frac{dy}{dx}$

$$EI\left(-AK^2\sin KL - BK^2\cos KL\right) - \beta\left(AK\cos KL - BK\sin KL + C\right) = 0$$

$$\Rightarrow -EI\,AK^2\sin KL - \beta AK\cos KL - \beta\left(\frac{-A\sin KL}{L}\right)$$

$$\Rightarrow \beta L\,AK\cot KL - \beta A - EI\,AK^2L = 0$$

$$\Rightarrow \beta A\left(KL\cot(KL) - 1\right) - EI\,AK^2L = 0$$

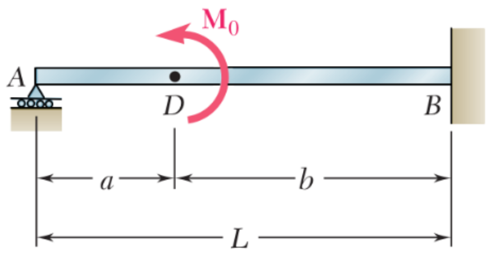
$$\Rightarrow A\left[\frac{\beta L}{EI}\left(KL\cot(KL) - 1\right) - K^2L^2\right] = 0$$

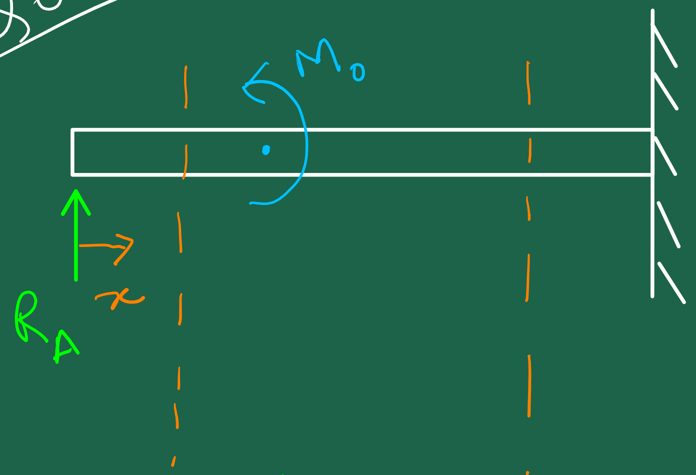
If $A = 0$, then $C = 0 \rightarrow$ trivial sol^n

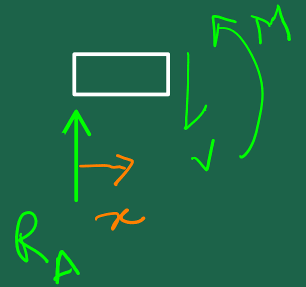
$$\therefore \frac{\beta L}{EI}\left(KL\cot(KL) - 1\right) = K^2L^2$$

Q6

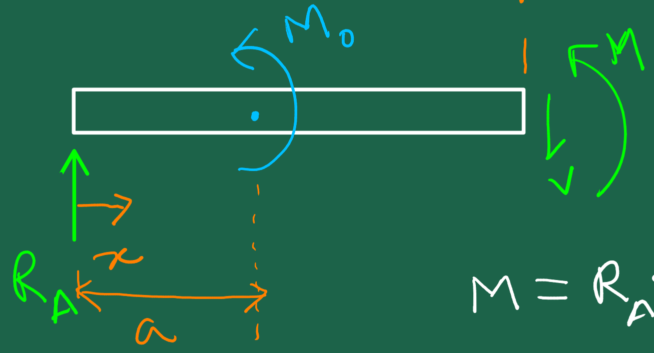
6. A beam AB is supported by rollers at end A, is clamped to the wall at B, and is subjected to a moment M_0 as shown in the figure. The flexural rigidity is EI, a constant. Using Castigliano's theorem, determine the reaction force at the the end A. **[12 marks]**







$$M = R_A x$$



$$M = R_A x - M_0$$

$$\Delta_A = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_A}\,dx$$

$$= \int_0^a \frac{R_A x}{EI}(x)\,dx + \int_a^L \frac{(R_A x - M_0)}{EI}(x)\,dx$$

$$= \frac{R_A a^3}{3EI} + \frac{R_A}{EI}\left(\frac{L^3 - a^3}{3}\right) - \frac{M_0}{EI}\left(\frac{L^2 - a^2}{2}\right)$$

$$= \frac{R_A}{EI}\left(\frac{a^3}{3} + \frac{L^3}{3} - \frac{a^3}{3}\right) - \frac{M_0}{EI}\left(\frac{L^2 - a^2}{2}\right)$$

$$= \frac{R_A L^3}{3EI} - \frac{M_0(L^2-a^2)}{2EI}$$

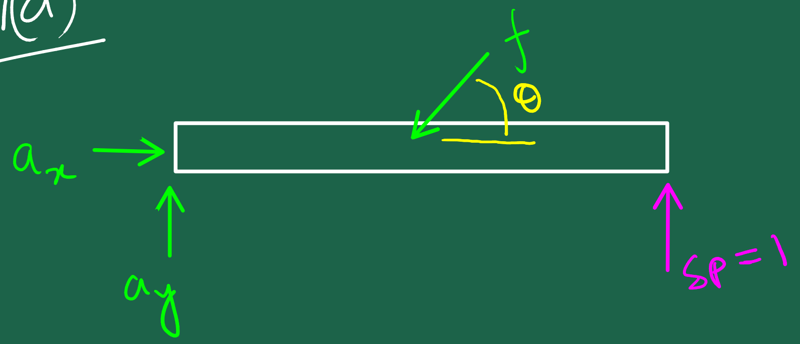
But $\Delta_A = 0$.

$$\therefore \frac{R_A L^3}{3EI} = \frac{M_0}{EI}\left(\frac{L^2-a^2}{2}\right)$$

$$\Rightarrow R_A = \frac{3M_0(L+a)(L-a)}{2L^3}$$

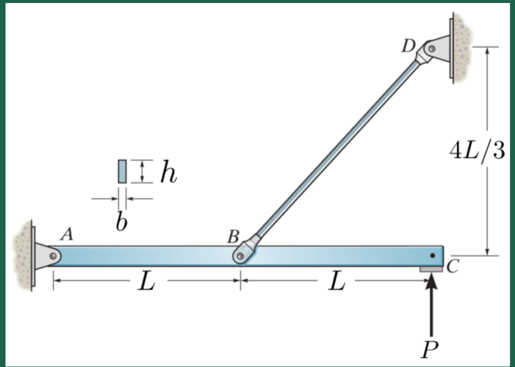
$$= \frac{3M_0(L+a)\,b}{2L^3}$$

Q 7(a)



7. The bar ABC has a rectangular cross-section $(b \times h)$ while the attached rod BD has a circular cross-section of diameter d. Both members are made of the same material (Young's modulus, E). A point load P is acting at C in the vertically upward direction.

   (a) Determine the vertically upward deflection of the end C, using the principle of virtual work. Neglect contribution due to shear. **[12 marks]**



$$BD = \sqrt{\left(\frac{4L}{3}\right)^2 + L^2} = \frac{5L}{3}$$

$$\sin\theta = \frac{4}{5}$$

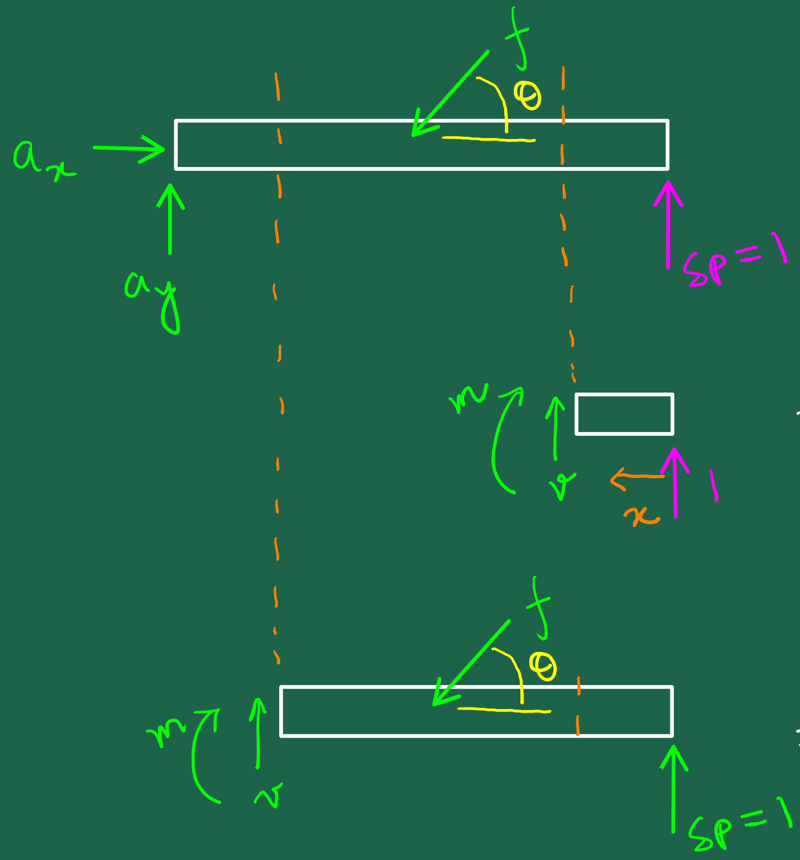
$$\cos\theta = \frac{3}{5}$$

$$\sum M_A = 0$$

$$\Rightarrow f \sin\theta\,(L) = \delta P\,(2L)$$

$$\Rightarrow f = \frac{2}{\sin\theta} = \frac{2}{4/5} = \frac{5}{2}$$

The forces and moments corresponding to P will be P times the forces and moments corresponding to $\delta P = 1$.



$$m = x$$

$$m + f\sin\theta\,(x - L) - x = 0$$

$$\Rightarrow m + \frac{5}{2} \times \frac{4}{5}(x - L) - x = 0$$

$$\Rightarrow m = 2L - x$$

Using the Principle of Virtual Work:

$$1 \times \Delta = \underbrace{\int_0^{2L} m \frac{M}{EI_1} dx}_{\text{Bending in AC}} + \underbrace{\int_0^{5L/3} n \frac{N}{A_2 E} ds}_{\text{Axial loading in BD}} + \underbrace{\int_L^{2L} a_x \frac{A_x}{A_1 E} dx}_{\text{Axia loading in AB}}$$

$$= \int_0^L x \frac{Px}{EI_1} dx + \int_L^{2L} (2L - x) \frac{P(2L-x)}{EI_1} dx + \int_0^{5L/3} \left(\frac{5}{2}\right) \frac{\left(\frac{5P}{2}\right)}{A_2 E} ds + \int_L^{2L} \left(\frac{3}{2}\right) \frac{\left(\frac{3}{2}P\right)}{A_1 E} dx$$

$$= \frac{PL^3}{3EI_1} + \frac{P}{EI_1} \int_L^{2L} (x - 2L)^2 dx + \frac{25P}{4}\left(\frac{5L}{3A_2 E}\right) + \frac{9P}{4}\left(\frac{L}{A_1 E}\right)$$

$$= \frac{PL^3}{3EI_1} + \frac{P}{EI_1} \int_{-L}^{0} \bar{x}^2 d\bar{x} + \frac{125PL}{12A_2 E} + \frac{9PL}{4A_1 E}$$

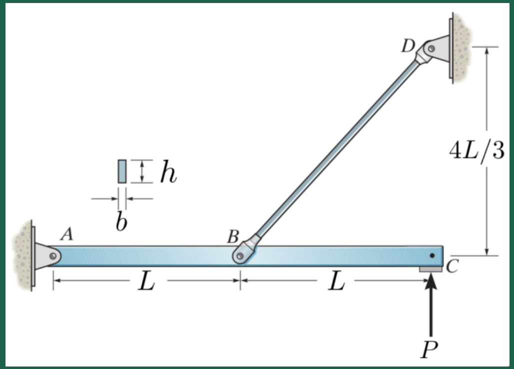
$$= \frac{PL^3}{3EI_1} + \frac{PL^3}{3EI_1} + \frac{125PL}{12A_2 E} + \frac{9PL}{4A_1 E}$$

$$= \frac{2PL^3}{3EI_1} + \frac{125PL}{12A_2E} + \frac{9PL}{4A_1E}$$

$$= \frac{2PL^3}{3E\left(\frac{1}{12}bh^3\right)} + \frac{125PL}{12\left(\pi d^2/4\right)E} + \frac{9PL}{4bhE}$$

$$= \frac{8PL^3}{Ebh^3} + \frac{125PL}{3\pi d^2E} + \frac{9PL}{4bhE}$$

(b) Taking $h = 3b$, determine the relation between b and d such that for a given P, both the part AB and the member BD will have a tendency to buckle together *out of the plane*. **[10 marks]**



Q7(b)

For BD: $F = \frac{5}{2}P = \frac{\pi^2 E\left(\pi d^4/64\right)}{\left(0.5\,\frac{5L}{3}\right)^2}$

For AB: $A_x = \frac{3}{2}P = \frac{\pi^2 E\left(\frac{1}{12}hb^3\right)}{(0.5L)^2}$

For out-of-plane buckling of AB:

$I = \frac{1}{12}hb^3$

$$\therefore \frac{\cancel{2}}{5}\,\frac{\cancel{\pi^2}\,\cancel{E}\left(\pi d^4/64\right)}{\left(\cancel{0.5}\,\frac{5L}{3}\right)^2} = \frac{\cancel{2}}{3}\,\frac{\cancel{\pi^2}\,\cancel{E}\left(\frac{1}{12}hb^3\right)}{(\cancel{0.5}L)^2}$$

$$\Rightarrow \frac{9\pi d^4}{25\times 64}\times\frac{3}{5}\times 12 = hb^3 = (3b)\,b^3$$

$$\Rightarrow \frac{27\pi d^4}{125\times 16} = b^4 \Rightarrow b = 0.4538\,d$$

or $d = 2.2036\,b$