1. Consider a situation where the coordinate axes are oriented along the principal directions. Then the state of stress is given by (you don't have to explain or prove this):

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{p1} & 0 & 0 \\ 0 & \sigma_{p2} & 0 \\ 0 & 0 & \sigma_{p3} \end{bmatrix}$$

The deviatoric part of the above stress matrix is given by: $[\boldsymbol{\sigma}^D] = [\boldsymbol{\sigma}] - \sigma^m[\mathbf{I}]$, where $[\mathbf{I}]$ is the identity matrix and $\sigma^m = \frac{1}{3}(\sigma_{p1} + \sigma_{p2} + \sigma_{p3})$.

A widely used method to represent the stress field in a body is in terms of something called the von Mises stress which is defined as $\sigma_{\text{vonMises}} = \sqrt{\frac{3}{2}\left(\sum_{j=1}^{3}\sum_{i=1}^{3}\left(\sigma_{ij}^D\right)^2\right)}$, where σ_{ij}^D is an element of $[\boldsymbol{\sigma}^D]$ in the i-th row and j-th column.

Show that $\sigma_{\text{vonMises}} = \sqrt{I_1^2 - 3I_2}$, where $I_1 = \text{trace}(\boldsymbol{\sigma})$ and $I_2 = \sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p3}\sigma_{p1}$.

[6 marks]

1

$$[\sigma^D] = [\sigma] - \sigma^m I$$

$$= \begin{bmatrix} \sigma_{p_1} - \sigma^m & 0 & 0 \\ 0 & \sigma_{p_2} - \sigma^m & 0 \\ 0 & 0 & \sigma_{p_3} - \sigma^m \end{bmatrix}$$

$$\sum_{j=1}^{3}\sum_{i=1}^{3} \sigma_{ij}^D \sigma_{ij}^D = (\sigma_{11}^D)^2 + (\sigma_{22}^D)^2 + (\sigma_{33}^D)^2$$

$$= (\sigma_{p_1} - \sigma^m)^2 + (\sigma_{p_2} - \sigma^m)^2 + (\sigma_{p_3} - \sigma^m)^2$$

$$= \sigma_{p_1}^2 + \sigma_{p_2}^2 + \sigma_{p_3}^2 - 2\sigma^m(\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3}) + 3(\sigma^m)^2$$

$$= (\sigma_{p_1} + \sigma_{p_2} + \sigma_{p_3})^2 - 2(\sigma_{p_1}\sigma_{p_2} + \sigma_{p_2}\sigma_{p_3} + \sigma_{p_3}\sigma_{p_1}) - 2\sigma^m \times 3\sigma^m + 3(\sigma^m)^2$$

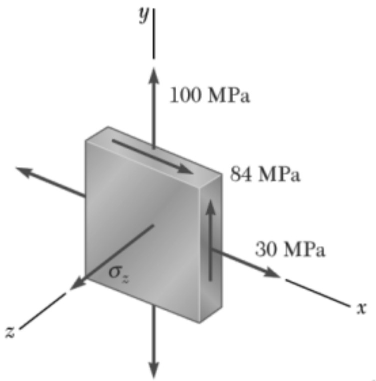
$$= I_1^2 - 2I_2 - 3(\sigma_m)^2$$

$$= I_1^2 - 2I_2 - 3\left(\frac{I_1}{3}\right)^2$$

$$= \frac{2I_1^2}{3} - 2I_2$$

$$\therefore \sqrt{\frac{3}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\sigma_{ij}^{D}\sigma_{ij}^{D}} = \sqrt{I_1^2 - 3I_2}$$

2. The state of stress at a point is depicted on a stress element as shown in the figure: $\sigma_{xx} = 30$ MPa, $\tau_{xy} = 84$ MPa, and $\sigma_{yy} = 100$ MPa. If $\sigma_{zz} = -60$ MPa, determine *all* the principal stresses and the *absolute* maximum shear stress. **[6 marks]**



2/

$$\sigma_{avg} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

$$= \frac{1}{2}(30 + 100) \text{ MPa}$$

$$= 65 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{30 - 100}{2}\right)^2 + 84^2} \text{ MPa}$$

$$= 91 \text{ MPa}$$

$$\sigma_{p_1, p_2} = \sigma_{avg} \pm R = 156 \text{ MPa}, -26 \text{ MPa}$$

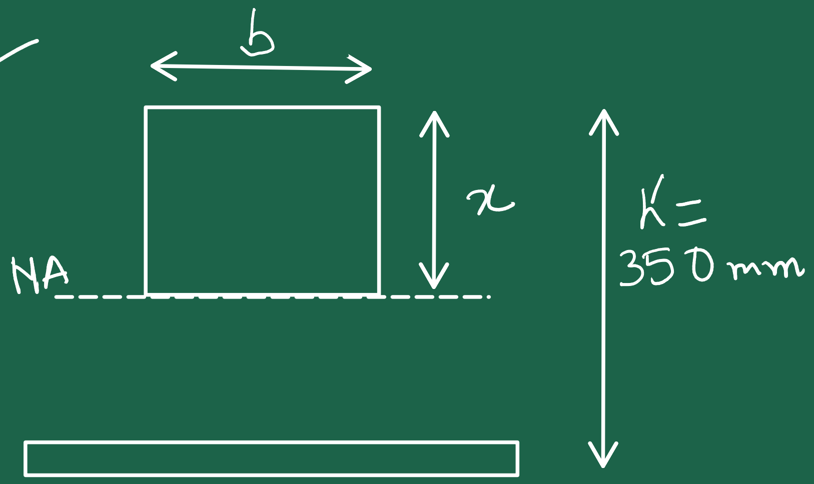
σ_{p_3} is $\sigma_{zz} = -60$ MPa itself. Since $\tau_{xz} = 0$ and $\tau_{yz} = 0$, σ_{zz} is a principal stress.

Principal stresses arranged in decreasing order are: 156 MPa, −26 MPa, −60 MPa.

$$\therefore \tau_{max,abs} = \frac{|\sigma_{max} - \sigma_{min}|}{2} = 108 \text{ MPa}$$

Many students have not identified $\sigma_{zz} = -60$ MPa as one of the principal stresses. Directly 3 marks have been deducted.

3



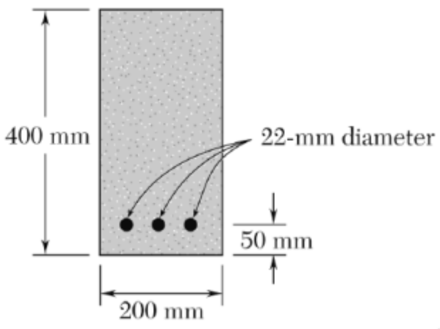
3. A concrete beam is reinforced by three steel rods placed as shown in the figure. The Young's modulus for concrete is 20 GPa and that for steel is 200 GPa. The allowable stress for concrete is 9 MPa and that for steel is 140 MPa. The beam is subjected to a bending moment such that the top part of the cross-section is in compression while the bottom part is in tension.

(a) What is the location of the neutral axis (in mm) with respect to the top of the cross-section?

(b) What is the area moment of inertia (or, second moment of area) (in mm^4) of the cross-section about the neutral axis?

(c) Determine the largest bending moment (in N·m) that can be sustained by the beam.

[2 + 3 + 5 = 10 marks]



$$n = \frac{200\ \text{GPa}}{20\ \text{GPa}} = 10$$

$$A_s = 3\frac{\pi}{4}d^2 = 1140\ \text{mm}^2$$

(a) Location of neutral axis (NA) wrt top:

$$b x \frac{x}{2} - n A_s (h' - x) = 0$$

Solve for x: only valid sol^n is 150.7 mm

Students have been given a wide variety of marks depending on how they have proceeded and written the sol^n.

Partial marks were given only when the steps were clear.

(b) $I = \frac{1}{12} bx^3 + bx\left(\frac{x}{2}\right)^2 + nA_s(h' - x)^2$

$= 681 \times 10^6 \ mm^4$

(c) Using allowable stress of concrete (at top edge):

$\sigma^c_{all} = \frac{Mx}{I} \Rightarrow M_1 = 40.7 \ kN.m$

σ^c_{all}: 9MPa

Using allowable stress of steel (at the steel rods location):

$\sigma^s_{all} = n\frac{M(h'-x)}{I} \Rightarrow M_2 = 47.8 \ kN.m$

σ^s_{all}: 140MPa

So, max M possible is: $\min(M_1, M_2) = 40.7 \ kN.m$

For this part, many students have done only M_1 or M_2. They have been given 0. In contrast, those who understood the need to calculate both were given partial marks even if answers were wrong.

4. Consider the displacement field given by: $u_x = kxy$, $u_y = kxy$, $u_z = 2k(x+y)z$, where k is a constant.

(a) Determine the strain matrix, using linear strain-displacement relations.

(b) What is the normal strain in the direction, $N_x = N_y = N_z = 1/\sqrt{3}$?

(c) Let the material in which the above strain field exists follow the generalized Hooke's law (E, ν). It is known that ν cannot exceed 1/2. Now, a student claims that on the plane $z = 0$, a state of plane stress exists, i.e. $\sigma_{zz} = 0$, $\sigma_{zx} = 0$, $\sigma_{yz} = 0$. Assumming these 0 values, compare the expression of ε_{zz} from the stress-strain relations with that from the strain matrix, and check if the student's claim leads to a contradiction or not.

[3 + 2 + 3 = 8 marks]

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(a) $$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = ky$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = kx$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 2k(x+y)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}k(x+y)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = kz$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = kz$$

$$\therefore [\underset{\approx}{\varepsilon}] = \begin{bmatrix} ky & \frac{1}{2}k(x+y) & kz \\ \frac{1}{2}k(x+y) & kx & kz \\ kz & kz & 2k(x+y) \end{bmatrix}$$

Those who did not write the strain matrix have been penalized.

(b) $\varepsilon_N = [\hat{N}]^T [\underset{\approx}{\varepsilon}] [\hat{N}]$

$$= \begin{bmatrix} N_x & N_y & N_z \end{bmatrix} [\underset{\approx}{\varepsilon}] \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

Show the steps

$$= \frac{1}{3} 4K (x+y+z)$$

(c) $\left.\begin{array}{l} \sigma_{zx} = 0 \Rightarrow \varepsilon_{zx} = \frac{1}{2G} \sigma_{zx} = 0 \\ \sigma_{yz} = 0 \Rightarrow \varepsilon_{yz} = \frac{1}{2G} \sigma_{yz} = 0 \end{array}\right\}$ (consistent with $z=0$)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \cancel{\sigma_{zz}}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \cancel{\sigma_{zz}}\right)\right]$$

$$\longrightarrow \varepsilon_{xx} + \varepsilon_{yy} = \frac{1-\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\Rightarrow \sigma_{xx} + \sigma_{yy} = \frac{E}{1-\nu}\left(\varepsilon_{xx} + \varepsilon_{yy}\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\cancel{\sigma_{zz}} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\Rightarrow \varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) = -\frac{\nu}{E}\,\frac{E}{1-\nu}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) = -\frac{\nu}{1-\nu}\,k(y+x)$$

But from part (a) $\varepsilon_{zz} = 2k(x+y)$

$$\therefore \quad 2k(x+y) = -\frac{\nu}{1-\nu}\,k(y+x)$$

$$\Rightarrow \quad 2 = -\frac{\nu}{1-\nu}$$

$$\Rightarrow \quad 2-2\nu = -\nu \Rightarrow \nu = 2 \rightarrow$$

But this is a contradiction because $\nu \leq \frac{1}{2}$.