AIFA: Fuzzy Relation

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Relations

- AI = {Akash, Ashish}
- MATH = {Sundar, Sekhar}
- AI x MATH = {(Akash,Sundar), (Akash,Sekhar), (Ashish,Sundar), (Ashish, Sekhar)}

- Relation R = Close Friends
- $R \subseteq \{(Akash, Sundar), (Ashish, Sekhar)\}$

- How to go beyond absolute membership?

Fuzzy Relations

- A Fuzzy relation for N sets is defined as an extension of the crisp relation to include membership grade

- $R = \{\mu_R(x_1, \dots, x_N)/(x_1, \dots, x_N) | x_i \in X_i\}$

Fuzzy Relation

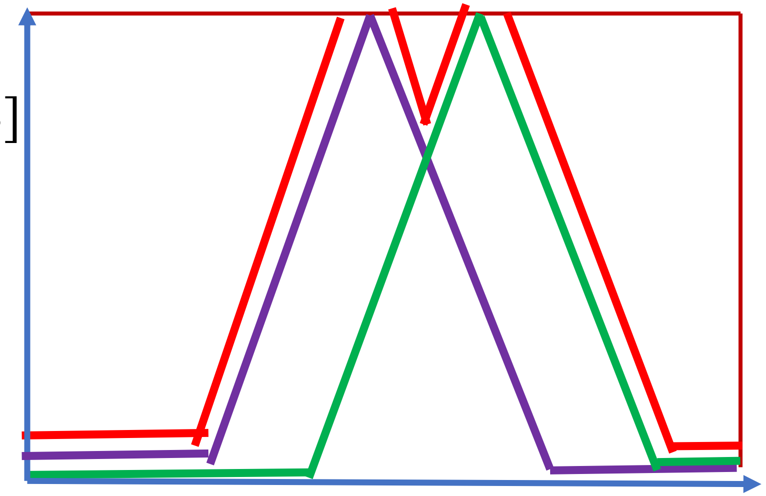
- Fuzzy relation describes interactions between variables

- It defines the mapping of variables from one fuzzy set to another
  - Like crisp relation, we can also define the relation over fuzzy sets

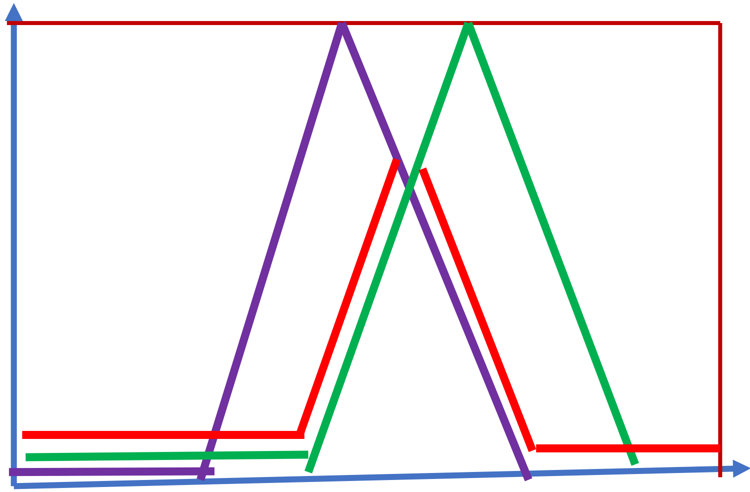
- Main fuzzy operations and compositions are followings:
  - Union
  - Intersection
  - Complement
  - Difference
  - Max-min composition
  - Max-product composition

Operations on Fuzzy Set

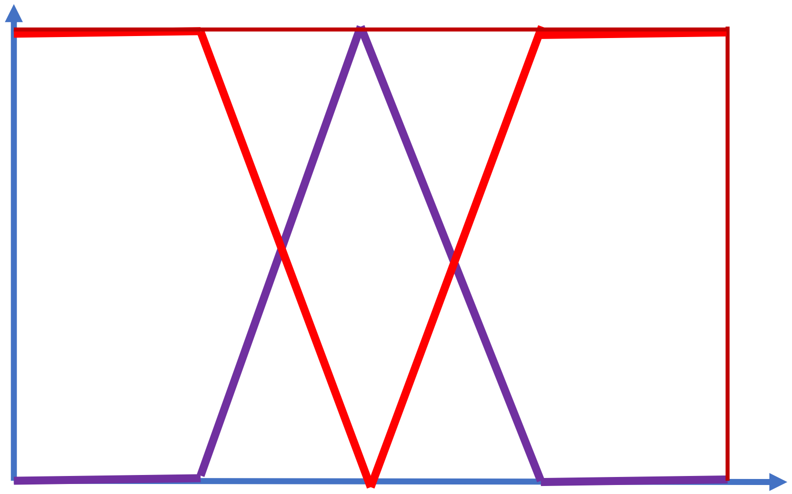
- Union:
  - $A \cup B$
  - $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X$ [Every member of A and B]
  - *Example:*
    - *The First Fuzzy Set is : {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}*
    - *The Second Fuzzy Set is : {'a': 0.9, 'b': 0.9, 'c': 0.4, 'd': 0.5}*
    - *Fuzzy Set Union is : {'a': 0.9, 'b': 0.9, 'c': 0.6, 'd': 0.6}*



- Intersection:
  - $A \cap B$
  - $\mu_{A \cup B}(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X$ [Every member of A and B]
  - *Example:*
    - *The First Fuzzy Set is : {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}*
    - *The Second Fuzzy Set is : {'a': 0.9, 'b': 0.9, 'c': 0.4, 'd': 0.5}*
    - *Fuzzy Set Intersection is : {'a': 0.2, 'b': 0.3, 'c': 0.4, 'd': 0.5}*



Operations on Fuzzy Set



- Complement
  - A'
  - $\mu_{A'}(x) = 1 - \mu_A(x)$
  - *Example:*
    - *The Fuzzy Set is : {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}*
    - *Fuzzy Set Complement is : {'a': 0.8, 'b': 0.7, 'c': 0.4, 'd': 0.4}*

- Difference:
  - $A - B$
  - $\mu_{A-B}(x) = \min(\mu_A(x), 1 - \mu_B(x)), \forall x \in X$ [Every member of A and B]
  - *Example:*
    - *The First Fuzzy Set is : {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}*
    - *The Second Fuzzy Set is : {'a': 0.9, 'b': 0.9, 'c': 0.4, 'd': 0.5}*
    - *Fuzzy Set Intersection is : {'a': 0.1, 'b': 0.1, 'c': 0.6, 'd': 0.5}*

Operations on Fuzzy Set

- Containment
  - $A \subseteq B$
  - $\mu_A(x) \leq \mu_B(x)\ ), \forall x \in X$

- Equality:
  - $A = B$
  - $\mu_A(x) = \mu_B(x), \forall x \in X$

Compositions of two relations

- Max-Min Composition

- Given two relation matrices R and S, the max-min composition is defined as $T = R \circ S$
  - $T(x,z) = \max\{\min\{R(x,y), S(y,z) and\ \forall_y \in Y\}\}$

Max-Min Composition

- $X = \{1,3,5\}; Y = \{1,3,5\}$
- $R = \{(x, y)|y = x + 2\}; S = \{(x, y)|x < y\}$
- $R\ and\ S\ is\ on\ X \times Y$

- $R = \{(1,3), (3,5)\}$
- $S = \{(1,3), (1,5), (3,5)\}$

- $R =$

	1	3	5
1	0	1	0
3	0	0	1
5	0	0	0

- $S =$

	1	3	5
1	0	1	1
3	0	0	1
5	0	0	0

$R \circ S$

	1	3	5
1	0	0	1
3	0	0	0
5	0	0	0

Fuzzy Cartesian Product

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$
- $R = A \times B \subset X \times Y$
- $\mu_R(X, Y) = \mu_{A\times B}(X, Y) = \min\{\mu_A(x), \mu_B(y)\}$
- $A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$, B $= \{(b_1, 0.5), (b_2, 0.6)\}$

$R = A \times B$

	b_1	b_2
a_1	0.2	0.2
a_2	0.5	0.6
a_3	0.4	0.4

Operations on Fuzzy Relations: Max-Min

	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.9
x_3	0.8	0.6

	z_1	z_2	z_3
y_1	0.6	0.4	0.7
y_2	0.5	0.8	0.9

	z_1	z_2	z_3
x_1	0.5	0.4	0.5
x_2	0.5	0.8	0.9
x_3	0.6	0.6	0.7

Compositions of two relations

- Max-Product Composition

- Given two relation matrices R and S, the max-min composition is defined as $T = R \circ S$
  - $T(x,z) = \max\{\{R(x,y) * S(y,z) and\ \forall_y \in Y\}\}$

Operations on Fuzzy Relations: Max-Product

	y_1	y_2
x_1	0.6	0.3
x_2	0.2	0.9

	z_1	z_2	z_3
y_1	1	0.5	0.3
y_2	0.8	0.4	0.7

	z_1	z_2	z_3
x_1	0.6	0.3	0.21
x_2	0.72	0.36	0.63

Fuzzy Rule

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Fuzzy Rule

- A Fuzzy implication of the form:
  - If X is A then Y is B
  - A and B are two linguistic variables defined by Fuzzy sets A and B
  - On the universe of discourses X and Y, respectively

- Antecedent: X is A
- Consequence: Y is B

If-Then Rules

- Use Fuzzy sets and Fuzzy operators as the subjects and verbs of fuzzy logic to form rules
  - If x is A then Y is B
  - Where A and B are linguistic terms defined by fuzzy sets on the sets X and Y respectively

  - If velocity is small, then current needed is small

  - If temp is high, put cooler as moderate

Fuzzy Implication: Example 1

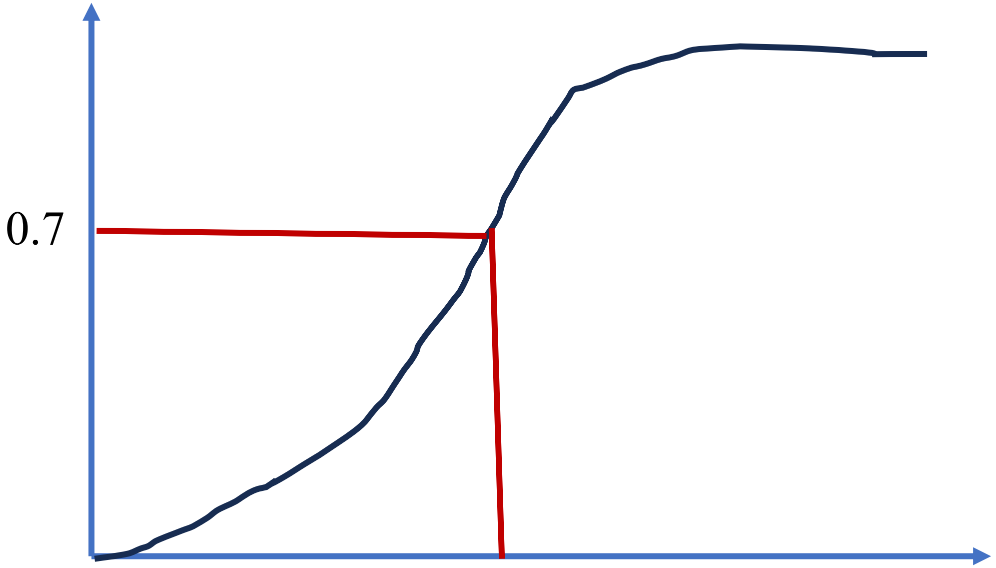
- If pressure is high then temperature is low

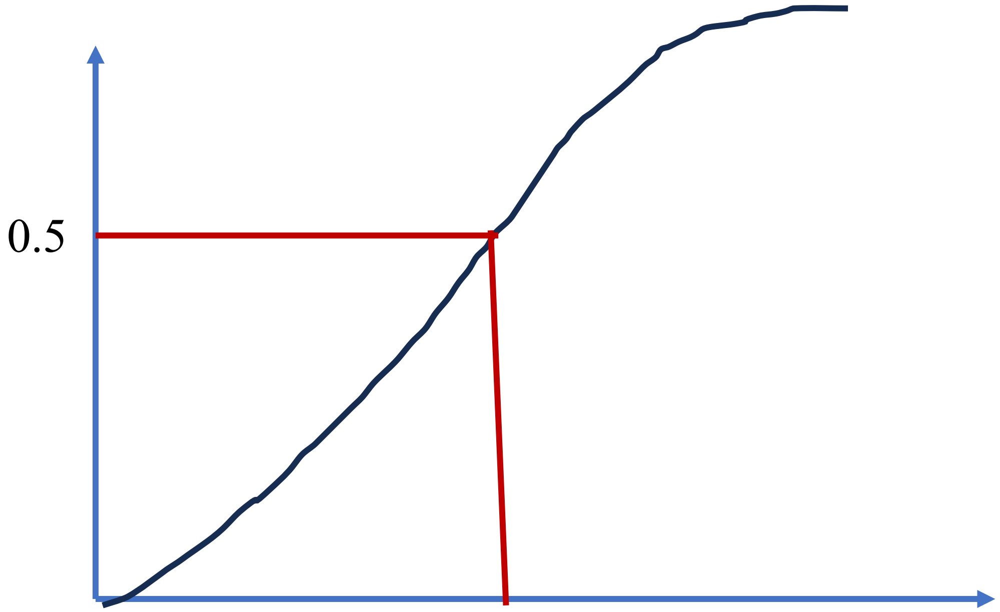
- If mango is yellow then mango is sweet else mango is sour

- The Fuzzy Implication is denoted as $R: A \rightarrow B$

If-Then Rules

- If x is A is true then …
- If 70 is high is true then …





If-Then Rules

- If x is A is and y is B then …
- If 70 is high and 5 is medium then …

0.7 0.8

0.7

Evaluation of Fuzzy Rules

- In Boolean logic: p $\Longrightarrow q$
  - If p is true then q is true

- In Fuzzy logic: p $\Longrightarrow q$
  - If p is true to some degree then q is true to some degree

- 0.5p $\Longrightarrow 0.5q$
  - Partial premise implies partially

- How?

Max-min rule of composition

- Given N observations E_i over X and hypothesis H_i over Y we have N rules:
- If E_1 $then$ H_1
- If E_2 $then$ H_2

- If E_N $then$ H_N

- $\mu_H = \max\{\min\left(\mu_{E_1}\right), \min\left(\mu_{E_2}\right), \ldots, \min(\mu_{E_N})\}$

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$R = A \times B$

	b_1	b_2
a_1	0.2	0.2
a_2	0.5	0.6
a_3	0.4	0.4

Fuzzy Implication: Example 2

- Suppose P and T represent Pressure and Temperature
- P = {1,2,3,4}
- T = {10, 15, 20, 25, 30, 35, 40, 45, 50}
- $T_{HIGH} = \{(20,0.2), (25,0.4), (30,0.6), (35,0.6), (40,0.7), (45,0.8), (50,0.8)\}$
- $P_{LOW} = \{(1,0.8), (2,0.8), (3,0.6), (4,0.4)\}$
- if temperature is HIGH, then pressure is LOW, $R: T_{HIGH} \rightarrow P_{LOW}$

	1	2	3	4
20	0.2	0.2	0.2	0.2
25	0.4	0.4	0.4	0.4
30	0.6	0.6	0.6	0.4
35	0.6	0.6	0.6	0.4
40	0.7	0.7	0.6	0.4
45	0.8	0.8	0.6	0.4
50	0.8	0.8	0.6	0.4

Set of Support

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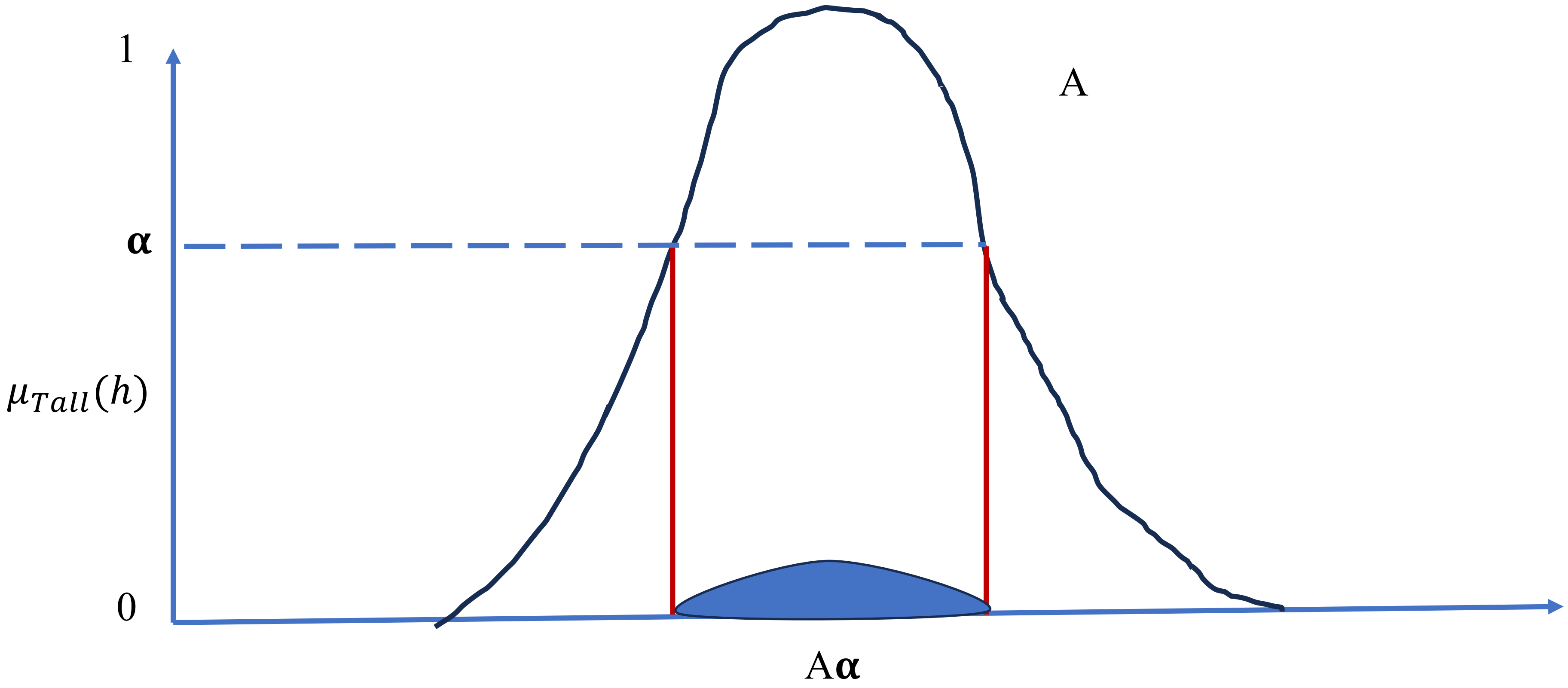
Set of Support

- It is a Crisp set
- Consist of elements whose membership values in the corresponding Fuzzy Set is greater than zero

α Cut / Horizontal Cut

- It is a crisp set
- Consist of elements whose membership values in the corresponding Fuzzy Set is greater than **α**



Thank You