AIFA
Inference through Resolution in FOPL

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Koustav Rudra

Generalized Modus Ponens

- For atomic sentences $P_i, P_i', and\ q$, where there is a substitution θ such that
  - SUBST(θ, P_i') = SUBST(θ, P_i), for all i
  - $\frac{P_1', P_2', \ldots, P_n', (P_1 P_2 \ldots P_n \rightarrow q)}{SUBST(\theta, q)}$
  - Unification of P_1 with P_1', P_2 with P_2', …

Horn sentences

- Atomic sentences
  - perfect_sq(36)

- Implication with a conjunction of atomic sentences on the left and a single atom on the right
  - $\forall_{x,y}$ perfect_sq(x)∧prime(y)∧divides(x,y)→divides(x,square(y))

- No Existential quantifier

Modus Ponens - completeness

- Reasoning with Modus Ponens is incomplete

- Consider the example –
  - $\forall_x$ P(x) ➔ Q(x)
  - $\forall_x$ Q(x) ➔ S(x)

  $\forall_x$ ~P(x) ➔ R(x)

  $\forall_x$ R(x) ➔ S(x)

  - We should be able to conclude S(A)
  - The problem is that $\forall_x$ ~P(x) ➔ R(x) cannot be converted to Horn form and thus cannot be used by Modus Ponens

Godel Completeness Theorem

- For first-order logic, any sentence that is entailed by another set of sentences can be proved from that set
  - Godel did not suggest a proof procedure
  - In 1965 Robinson published his resolution algorithm

- Entailment in first-order logic is semi-decidable, that is,
  - we can show that sentences follow from premises if they do,
  - but we cannot always show if they do not.

Generalized Resolution Rule

- Generalized Resolution Rule:
- For atoms pi, qi, ri, si, where unify(p_j,q_k)= θ, we have

- $p_1 \wedge p_2 \wedge p_3 \ldots \wedge p_{n1} \rightarrow r_1 \vee r_2 \vee \ldots r_{n2}$
- $s_1 \wedge s_2 \wedge s_3 \ldots \wedge s_{n3} \rightarrow q_1 \vee q_2 \vee \ldots q_{n4}$

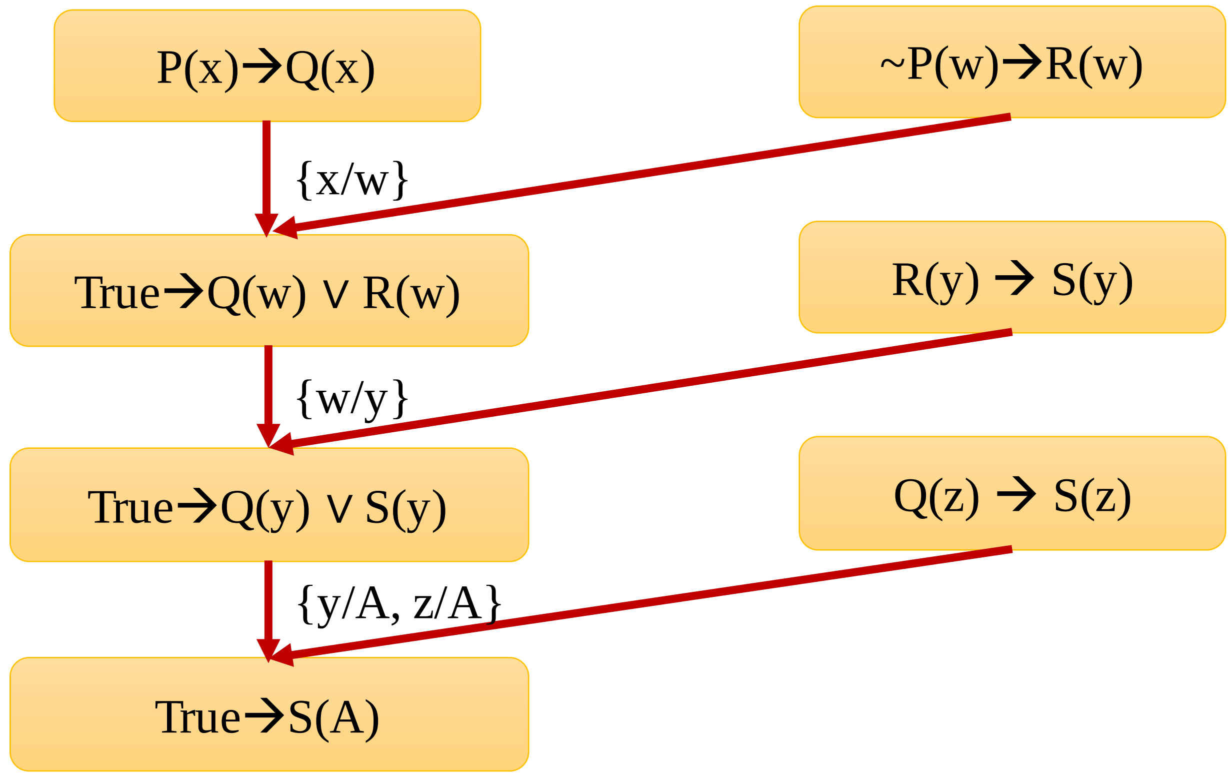
- SUBST(θ,
- $p_1 \wedge \ldots p_{j-1} \wedge p_{j+1} \ldots \wedge p_{n1} \wedge s_1 \wedge \ldots \wedge s_{n3} \rightarrow r_1 \vee \ldots r_{n2} \vee q_1 \vee \ldots q_{k-1} \vee q_{k+1} \ldots \vee q_{n4}$

Example

- $\forall_x$ P(x) ➔ Q(x)
- $\forall_x$ Q(x) ➔ S(x)

$\forall_x$ ~P(x) ➔ R(x)

$\forall_x$ R(x) ➔ S(x)



How to automate the things?

Resolution in Predicate Logic

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Conversion to Normal Form

- A formula is said to be in clause form if it is of the form:
  - $\forall_{x1}\forall_{x2}\ldots\ \forall_{xn}$[C1∧C2 ∧… ∧Ck]
  - Each of these clauses could be disjunction of some predicates

- All first-order logic formulas could be converted to clausal form

- $\forall_{x}\{\mathrm{p(x)} \rightarrow \exists_{z}\{\sim\forall_{y}\left[q(x,y)\rightarrow p\left(f(x_{1})\right)\right] \wedge \forall_{y}[q(x,y)\rightarrow p(x)]\}\}$

Conversion to Normal Form

- Step1: Take the existential closure and eliminate redundant quantifiers. This introduces $\exists_{x1}$ and eliminates $\exists_z$, so:

- $\forall_x\{p(x) \rightarrow \exists_z\{\sim\forall_y[q(x,y) \rightarrow p(f(x_1))] \wedge \forall_y[q(x,y) \rightarrow p(x)]\}\}$

- $\exists_{x1}\ \forall_x\{p(x) \rightarrow \{\sim\forall_y[q(x,y) \rightarrow p(f(x_1))] \wedge \forall_y[q(x,y) \rightarrow p(x)]\}\}$

Conversion to Normal Form

- Step2: Rename any variable that is quantified more than once
  - y has been quantified twice, so:

- $\exists_{x1} \forall_x \{p(x) \rightarrow \{\sim\forall_y [q(x,y) \rightarrow p(f(x_1))] \wedge \forall_y [q(x,y) \rightarrow p(x)]\}\}$

- $\exists_{x1} \forall_x \{p(x) \rightarrow \{\sim\forall_y [q(x,y) \rightarrow p(f(x_1))] \wedge \forall_z [q(x,z) \rightarrow p(x)]\}\}$

Conversion to Normal Form

- Step3: Eliminate implication

- $\exists_{x1}\ \forall_x\{p(x) \rightarrow \{\sim\forall_y[q(x,y) \rightarrow p(f(x_1))] \land \forall_z[q(x,z) \rightarrow p(x)]\}\}$

- $\exists_{x1}\ \forall_x\{\sim p(x) \lor \{\sim\forall_y[\sim q(x,y) \lor p(f(x_1))] \land \forall_z[\sim q(x,z) \lor p(x)]\}\}$

Conversion to Normal Form

- Step4: Move ~ all the way inwards

- $\exists_{x1}\ \forall_x\{\sim p(x) \vee \{\sim\forall_y[\sim q(x,y) \vee p(f(x_1))] \wedge \forall_z[\sim q(x,z) \vee p(x)]\}\}$

- $\exists_{x1}\ \forall_x\{\sim p(x) \vee \{\exists_y[q(x,y) \wedge \sim p(f(x_1))] \wedge \forall_z[\sim q(x,z) \vee p(x)]\}\}$

Conversion to Normal Form

- Step5: Push the quantifiers to the right

- $\exists_{x1}\ \forall_x\{\sim p(x) \vee \{\exists_y[q(x,y) \wedge \sim p(f(x_1))] \wedge \forall_z[\sim q(x,z) \vee p(x)]\}\}$

- $\exists_{x1}\ \forall_x\{\sim p(x) \vee \{[\exists_y q(x,y) \wedge \sim p(f(x_1))] \wedge [\forall_z \sim q(x,z) \vee p(x)]\}\}$

Conversion to Normal Form

- Step6: Eliminate existential quantifier (Skolemization)

- $\exists_{x1}\ \forall_x\{\sim p(x) \lor \{[\exists_y q(x,y) \land \sim p(f(x_1))] \land [\forall_z \sim q(x,z) \lor p(x)]\}\}$
- Pick out leftmost $\exists_y B(y)$ and replace it by $B(F(x_{i1}, x_{i2,\ldots,} x_{in}))$, where:
  - $x_{i1}, x_{i2,\ldots,} x_{in}$ are all the distinct free variables of $\exists_y B(y)$ that are universally quantified to the left of $\exists_y B(y)$, and
  - F is any n-ary function constant which does not occur already

- $\forall_{x1}\forall_{x2}\forall_{x3}\exists_y B(y, x1, x2, x3)$
  - $\forall_{x1}\forall_{x2}\forall_{x3} B(F(x1, x2, x3), x1, x2, x3)$
- $\forall_{x1}\forall_{x2}\forall_{x3}\exists_y [B1(y, x1) \land B2(x1, x2) \land B3(x2, x3)]$
  - $\forall_{x1}\forall_{x2}\forall_{x3} [B1(F(x1, x2, x3), x1) \land B2(x1, x2) \land B3(x2, x3)]$

Conversion to Normal Form

- Step6: Eliminate existential quantifier (Skolemization)

- $\exists_{x1}\ \forall_x\{\sim p(x) \lor \{[\exists_y q(x,y) \land \sim p(f(x_1))] \land [\forall_z \sim q(x,z) \lor p(x)]\}\}$

- $\forall_x\{\sim p(x) \lor \{[q(x,g(x)) \land \sim p(f(a))] \land [\forall_z \sim q(x,z) \lor p(x)]\}\}$

Conversion to Normal Form

- Step7: Move all universal quantifiers to the left
- $\forall_x\{\sim p(x) \vee \{[q(x, g(x)) \wedge \sim p(f(a))] \wedge [\forall_z \sim q(x, z) \vee p(x)]\}\}$
- $\forall_x \forall_z\{\sim p(x) \vee \{[q(x, g(x)) \wedge \sim p(f(a))] \wedge [\sim q(x, z) \vee p(x)]\}\}$
- Right side we have a set of predicates that are quantified from outside

Conversion to Normal Form

- Step8: Distribute $\wedge$ over $\vee$
- $\forall_x \forall_z \{\sim p(x) \vee \{[q(x, g(x)) \wedge \sim p(f(a))] \wedge [\sim q(x,z) \vee p(x)]\}\}$
- $\forall_x \forall_z \{[\sim p(x) \vee [q(x, g(x)) \wedge \sim p(f(a))]] \wedge [\sim p(x) \vee [\sim q(x,z) \vee p(x)]]\}$
- $\forall_x \forall_z \{[\sim p(x) \vee q(x, g(x))] \wedge [\sim p(x) \vee \sim p(f(a))] \wedge [\sim p(x) \vee \sim q(x,z) \vee p(x)]\}$
- Right side we have a set of predicates that are quantified from outside
- Use Boolean algebra to get CNF

Thank You