MATH250 NOTES

Contents

1 Unit 1

1.1 Lecture 1: Systems of Linear Equations [1.1-1.2]

A system of linear equations is when you have more than 1 equation with more than 1 unknown.

Example 1.1.1

Solve the following system:

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 &= 4 \end{cases}$$

Solution:

$$\begin{aligned} x_1 + 2x_2 = 5 &\Rightarrow x_1 = 5 - 2x_2 \\ 2x_1 + x_2 = 4 &\Rightarrow 2(5 - 2x_2) + x_2 = 4 \Rightarrow 10 - 3x_2 = 4 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Note that the solution to the previous system $(x_1, x_2) = (1, 2)$ also corresponds to the point of intersection of the lines that each equation represents. This then implies that non-parallel lines have a single solution, paralell lines have no solutions, and scalar multiples of the same line have infinitely many solutions.

Example 1.1.2: Systems with Different Number of Solutions

1. One Solution:

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 &= 4 \end{cases}$$

2. No solution:

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + 4x_2 &= 6 \end{cases}$$

3. Infinite Solutions

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + 4x_2 &= 10 \end{cases}$$

Elementary Row Operations

Consider the system

$$\begin{cases} E_1 \\ E_2 \\ \vdots \\ E_n \end{cases}$$

Where E_i is the i-th equation. Then the following operations on the system **do not change the solution**

1. Changing the order of equations: $E_i \leftrightarrow E_j$ where $i \neq j$ for the i-th and j-th equation
2. Scaling equations: $E_i \to \lambda E_i$ where $\lambda \in \mathbb{R}\,(\lambda \neq 0)$
3. Combining equations: $E_i \to E_i + \lambda E_j$ where $i \neq j$ and $\lambda \in \mathbb{R}\,(\lambda \neq 0)$

Example 1.1.3

Solve the following system:

$$\begin{cases} E_1 & = x_1 + 2x_2 = 5 \\ E_2 & = 2x_1 + x_2 = 4 \end{cases}$$

Solution:

$$\begin{aligned} -2E_1 &= -2x_1 - 4x_2 = -10 \\ E_2 - 2E_1 &= (2x_1 - x_2) - 2x_1 - 4x_2 = 4 - 10 \\ &\Rightarrow -3x_2 = -6 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Matricies

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix A has size $m \times n$ if it has m **rows** and n **columns**. A matrix is written as

$$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Where each a_{ij} is called an element of the matrix. i denotes the row and n denotes the column of the element.

With this in mind, note that we can represent the previous system as a matrix:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix}$$

where each row represents an equation and each element corresponds to either the coefficent of a variable or the solution. With this system in matrix form we can manipulate the rows as follows:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -3 & -6 \end{pmatrix}$$

Since each row corresponds to an equation, if we take a look at the bottom row we can see

$$-3x_2 = -6 \Rightarrow x_2 = 2 \Rightarrow x_1 = 1$$

Thus allowing us to reach the same final answer with a lot less hassle.

General Form of a System of Linear Equations

Consider any system of linear equations in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases}$$

Now, notice we can rewrite this as a matrix:

$$A = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Everything to the **left** of the solid line is refered to as the *coefficent matrix.* A itself is referred to as the **augmented matrix** for the system.

Row Echelon Form

Given a matrix A, A_{REF} is an equivalent matrix that satisfies the following properites:

1. All zero rows are below non-zero rows
2. each next leading element is in the column to the right of the previous leading element (called pivots)

Note that the **leading element** of a row is simply the first non-zero element in that row.

A matrix can also be put in RREF (reduced row echelon form) if it is alreay in REF, each pivot is 1, and the only non-zero element in the pivot column is the pivot. This would be A_{RREF}. Now, lets try to combine all we've done by applying basic ERO to a simple matrix to put it in REF.

Example 1.1.4: Basic REF

Consider the system

$$\begin{cases} x_1 + 2x_2 & = 5 \\ 2x_1 + x_2 = 4 \end{cases}$$

Find $\left(A \mid B\right)$

Solution: It's augmented matrix can be put into REF as follows

$$\left(A \mid B\right) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 4 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) = \left(A \mid B\right)_{\text{REF}}$$

Example 1.1.5: Basic RREF

Consider again the system from ex (1.1.4) and find the RREF of the augmented matrix.

Solution: Notice we can put it into RREF as follows:

$$(A \mid B)_{\text{REF}} = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{3}R_2]{} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow[R_1 \to R_1 - 2R_2]{} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) = (A \mid B)_{\text{RREF}}$$

It's important to note that the RREF for any given matrix is *unique*.

Example 1.1.6: Solve the System

Solve the system

$$\begin{cases} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= -1 \end{cases}$$

Solution:

$$\begin{cases} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= -1 \end{cases} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now because there is no pivot in the third column of our matrix, x_3 is what's known as a 'free variable'. All this means is that x_3 can take on any value, we'll notate this by $x_3 = t$ with $t \in \mathbb{R}$.

$$\Rightarrow \begin{cases} x_3 &= t \\ -3x_2 + 3x_3 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \end{cases}$$

$$\Rightarrow X_{\text{gen}} = \begin{pmatrix} -t \\ 1 + t \\ t \end{pmatrix}$$

1.2 Lecture 2: Vector and Matrix Equations [1.3-1.4]

As a reminder from last time, given a system with an augmented matrix $(A \mid B)$ the system will have

1. <u>no</u> solutions if there is a pivot in the last column of $(A \mid B)_{\text{REF}}$
2. <u>one</u> solution if $(A \mid B)_{\text{REF}}$ is $m \times n$ with n pivots (and no pivots in the last column)

3. infinite solutions if $\left(A \mid B\right)_{\text{REF}}$ is $m \times n$ with less than n pivots (and no pivots in the last column)

Vectors

For a vector $\vec{AB}$, point A is the tail and point B is the head. Two vectors, $\vec{AB}$ and $\vec{CD}$ are equal if, and only if, their magnitude and directions are equal. Vectors in $\mathbb{R}^2$ can be notated as $\left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$ where the point (a_1, a_2) is the head and the tail is (usually) assumed to the origin. Generally, for $\mathbb{R}^n$ we have $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ which will always have size $n \times 1$.

Since we can represent vectors as matricies, we can perform the standard matrix operations on them:

Matrix Operations

Let A and B be matricies. Then

1. $A + B = C$ where A and B are the same size. This addition is defined as $c_{ij} = a_{ij} + b_{ij}$
2. $\lambda A = \hat{A}$ where $\hat{a_{ij}} = \lambda a_{ij}$ for $\lambda \in \mathbb{R}$
3. $A + (-A) = \hat{0}$ where $\hat{0}$ is the zero-matrix which has the same size as A with each element being 0
4. Consider matricies $\underset{m\times n}{A}$ and $\underset{n\times 1}{X}$, then AX is defined as

$$\underset{m\times 1}{A} = \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n \\ \vdots & & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \dots & a_{mn}x_n \end{pmatrix} \text{ if } \underset{m\times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } \underset{n\times 1}{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example 1.2.1: Basic Matrix Multiplication

Consider the the matricies

$$\underset{2\times 3}{A} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ and } \underset{3\times 1}{B} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

Find AB.

Solution:

$$\underset{2\times 1}{AB} = \begin{pmatrix} (1*2) + (-1*4) + (2*7) \\ (3*2) + (4*4) + (5*7) \end{pmatrix} = \begin{pmatrix} 12 \\ 57 \end{pmatrix}$$

It's important to note that an alternative form of defining matrix multiplication is as follows:

$$AX = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{pmatrix}$$

This is known as the **linear combination of vectors**.

Linear Combination of Vectors

Let $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ be a set of vectors from $\mathbb{R}^n$. Then,

$$c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k}, \qquad (c_1, c_2, \ldots, c_k \in \mathbb{R})$$

is called the **linear combination of vectors** $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}$.

We can combine the previous two definitions of matrix multiplication and the linear combination of vectors to get this next fact: if we consider a vector

$$(\vec{u_k}) = \begin{pmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{pmatrix}$$

then, $c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k} = AX$ where

$$A = \begin{pmatrix} (\vec{u_1}) & (\vec{u_2}) & \ldots & (\vec{u_k}) \end{pmatrix} \text{ and } X = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

This is especially important when working with systems of equations as we can represent systems as linear combinations of vectors. Given any general system we can rewrite it as a linear combination of vectors as such

$$\left.\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_1 \end{cases}\right] \Leftrightarrow AX = B$$

B is said to be a *linear combination of columns of* A if, and only if, A and B are **compatible**. For A and B to be compatible essentially just means that $\left(A \mid B\right)_{\text{REF}}$ has no pivots in the last column.

Span

$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \operatorname{Span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ where $c_1, c_2, \ldots, c_k$ are <u>all</u> possible numbers

For a single vector $\vec{v}$, Span $\{\vec{v}\}$ is simply the set containing all scaled multiples of $\vec{v}$.

Example 1.2.2: Span of Two Vectors

Notice that Span $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\} = \mathbb{R}^2$. This implies that any vector from $\mathbb{R}^2$ can be written as

$$c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

We can prove this fact by considering the augmented matrix for $AX = B$:

$$\left(\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & 4 & b_2 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & -2 & b_2 - 2b_1 \end{array}\right)$$

Therefore, since there is no pivote in the last column, this system has a single solution for any vector

$B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Example 1.2.3

Given the following vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}$ and $\vec{b}$, determine if $\vec{b}$ is a linear combination of $\vec{a_1}, \vec{a_2}, \vec{a_3}$.

$$\vec{a_1} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \vec{a_2} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \vec{a_3} = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

We can start by noticing that $\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3} \Leftrightarrow A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ where $A = \begin{pmatrix} (\vec{a_1}) & (\vec{a_2}) & (\vec{a_3}) \end{pmatrix}$. Thus,

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow[R_2 \to R_2 + 2R_1]{} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, since there is no pivot in the final column, B is a linear combination of A.

1.3 Lecture 3: Homogenous Systems of Linear Equations [1.5]

Homogenous Systems of Linear Equations

A **homogenous system of linear equations** is any system in the form

$$Ax = \hat{0}$$

Thus, any system in the form

$$Ax = B$$

is called a **non-homogenous system**.

For a non-homogenous system we can follow the steps

$$Ax = B \Rightarrow \begin{pmatrix} A \mid B \end{pmatrix} \xrightarrow[\text{ERO}]{} \begin{pmatrix} A \mid B \end{pmatrix}_{\text{REF}} \Rightarrow \text{backwards substitution into the equations}$$

For each non-homogenous system:

1. if the augmented matrix has a pivote in the last column, it has no solutions
2. if A is $m \times n$ and the augmented matrix has less than n pivots, then $Ax = B$ has an infinite amount of solutions
3. if A is $m \times n$ and the augmented matrix has exactly n pivots then $Ax = B$ has <u>one</u> solution

For a homogenous system we can follow similar steps:

$$Ax = \hat{0} \Rightarrow \begin{pmatrix} A \mid 0 \end{pmatrix} \xrightarrow[\text{ERO}]{} A_{\text{REF}} \Rightarrow \text{backwards substitution to the equations}$$

Note that since the solution matrix is simply just the zero matrix, we dont need to consider it in our augmented matrix - thats why you see A_{REF}. Unlike non-homogenous systems there are only two options for solutions:

1. $x = \hat{0}$ is <u>always</u> a solution, this is called the *trivial solution*. If A_{REF} is $m \times n$ large and has n pivots, then $x = \hat{0}$ is the **only** solution

2. if A_{REF} is $m \times n$ and has less than n pivots, then $Ax = \hat{0}$ has an **infinite** amount of solutions

Example 1.3.1: Solve the following System

Solve the below system

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 & = 0 \\ 2x_2 - x_2 + x_3 - 2x_4 & = 0 \\ 3x_1 + x_2 - x_3 + x_4 & = 0 \end{cases}$$

We can start by putting this in the form $Ax = \hat{0}$ where

$$\underset{3\times 4}{A} = \begin{pmatrix} 1 & 2 & -3 & -1 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & -1 & 1 \end{pmatrix}, \quad \underset{4\times 1}{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Next, we need to put A into REF:

$$\underset{3\times 4}{A} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}]{} \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & -5 & 7 & -4 \\ 0 & -5 & 8 & -2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & -5 & 7 & -4 \\ 0 & 0 & 1 & 2 \end{pmatrix} = A_{\text{REF}}$$

Since there is no pivot in the fourth column, x_4 must be a free variable. We can rename it as $x_4 = t$ for $t \in \mathbb{R}$. Thus

$$\left.\begin{cases} x_3 + 2x_4 & = 0 \Rightarrow x_3 = -2t \\ -5x_2 + 7x_3 - 4x_3 & = 0 \Rightarrow x_2 = -\frac{18}{5}t \\ x_1 + 2x_2 - 3x_3 + x_4 & 0 = 0 \Rightarrow x_1 = \frac{1}{5}t \end{cases}\right] \Rightarrow X = \begin{pmatrix} \frac{1}{5}t \\ -\frac{18}{5}t \\ -2t \\ t \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 \\ -18 \\ -10 \\ 5 \end{pmatrix} t$$

Notice that since t is any arbitary value we can make $t = 5t$ in order to eliminate the fractions. As well, the second matrix with t on the outside is known as the *parametric form* of the solution.

Theorem 1.1. *The general solution of a consisten non-homogenous system $Ax = B$ has the form $X_{gen} = X_{genhom} + X_p$ where X_{genhom} is the general solution of the corresponding homogenous system $Ax = \hat{0}$ and X_p is any particular solution of $Ax = B$.*

Proof. Let X_1 and X_2 be any two solutions of $Ax = B$. That is,

$$\left.\begin{cases} AX_1 & = B \\ AX_2 & = B \end{cases}\right] \Rightarrow AX_1 - AX_2 = 0 \Leftrightarrow A(X_1 - X_2) = 0$$

Therefore, $X_1 - X_2$ is a solution of the homogenous system. With some clever 'renaming' we can see:

$$X_1 - X_2 = X_{\text{hom}} \Rightarrow X_1 = X_{\text{hom}} + X_2 \Rightarrow X_{\text{gen}} = X_{\text{genhom}} + X_p$$

□

Example 1.3.2: Solve the System

$$\begin{cases} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + x_2 - 2x_3 &= 2 \\ x_1 + x_2 - x_3 &= 2 \end{cases}$$

As always, we start by forming the augmented matrix and putting it into REF:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 2 & 1 & -2 & 2 \\ 1 & 1 & -1 & 2 \end{array}\right) \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_2}]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -2 \end{array}\right) \xrightarrow[\substack{R_2 \to -\frac{1}{3}R_2 \\ R_3 \to R_3 - \frac{1}{3}R_2}]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 2R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now, since there is no pivot in the third column, x_3 must be a free variable, so we have $x_3 = t$ for $t \in \mathbb{R}$. Using some very basic backwards substitution we can see that $x_2 = 2$ and $x_1 = x_3 = t$. As well, notice that the final form of the augmented matrix is actually in RREF and not REF, because of how easy it was to put the matrix in RREF from REF, there was almost no reason **not** to put it into RREF. We can now write our solution vector,

$$X_{\text{gen}} = \begin{pmatrix} t \\ 2 \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}_{X_p} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{X_{\text{genhom}}}$$

Example 1.3.3: Find the General Homogenous Solution

Just to prove the point, lets find the general homogenous solution for the above system, $Ax = \hat{0}$.

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_2 \\ R_3 \to R_3 - R_1}]{} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_{\text{RREF}}$$

With backwards substitution we can easily see that $x_3 = t$ for $t \in \mathbb{R}$, $x_2 = 0$, and $x_1 = t$. So,

$$X_{\text{genhom}} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Which, as expected, is exactly the same vector we saw in the previous example.

Example 1.3.4: Solve the System

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 3x_1 - x_2 - x_3 &= 0 \end{cases}$$

Once again, we must put A into REF:

$$A \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 5 & -4 & 3 \\ 0 & 5 & -4 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 5 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, because there are no pivots in the third and fourth column, x_3 and x_4 are free variables which makes x_1 and x_2 our basic variables $\therefore x_3 = t, x_4 = s$ for $s, t \in \mathbb{R}$

$$\begin{cases} 5x_2 - 4x_3 + 3x_4 & = 0 \Rightarrow x_2 = \frac{4}{5}x_3 - \frac{3}{5}x_4 = \frac{4}{5}t - \frac{3}{5}s \\ x_1 - 2x_2 + x_3 - x_4 & = 0 \Rightarrow x_1 = 2x_2 - x_3 + x_4 = 2(\frac{4}{5}t - \frac{3}{5}s) - t + s \end{cases}$$

Before continuing, it's smart to make $t = 5t$ and $s = 5s$ in order to eliminate the fractions. Then we have:

$$X_{\text{gen}} = \begin{pmatrix} 3t - s \\ 4t - 3s \\ 5t \\ 5s \end{pmatrix} = t \begin{pmatrix} 3 \\ 4 \\ 5 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$

1.4 Lecture 4: Span of a Set of Vectors [1.3]

Rank of a Matrix

Let $\underset{m\times n}{A}$ be a matrix. Then $\text{rank}(A)$ is the number of pivots in A_{REF}. $\text{rank}(A)$ is equivalent to the number of non-zero rows in A_{REF}.

It follows from the definition of $\text{rank}(A)$ that the number of free variables in a system is $n - \text{rank}(A)$. This quanitity is known as $\text{null}(A)$.

If $\underset{n\times n}{A}$ is a matrix such that $\text{rank}(A) = n$, then $A_{\text{REF}} = I$ where I is the identity matrix.

Identity Matrix

The identity matrix, I, is a square matrix which is automatically in RREF and its only non-zero elements are the pivots. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Another important fact about the rank of matrix is that it can only be as great as the minimimum of n and m for a matrix of size $m \times n$. Formally,

$$\text{rank}\left(\underset{m\times n}{A}\right) \leq \min\{m, n\}$$

where $\min\{x_1, x_2, \ldots, x_n\}$ returns the smallest value out of $x_1, x_2, \ldots, x_n$.

Span of Vectors from $\mathbb{R}^n$

The span of a set of vectors from $\mathbb{R}^n$, $\text{Span}\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ is the set of all possible linear combinations of $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}$:

$$AC = c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k}, \quad (c_1, c_2, \ldots, c_k \in \mathbb{R})$$

where

$$A = \begin{pmatrix} (\vec{u_1}) & (\vec{u_2}) & \dots & (\vec{u_k}) \end{pmatrix} \text{ and } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

The span of the zero vector, is also the zero vector: Span $\left\{\vec{0}\right\} = \vec{0}$

Example 1.4.1: Span of a Single Vector

Consider a vector $u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then Span $\{\vec{u_1}\} = c\vec{u_1}$ for any $c \in \mathbb{R}$. Geometrically, this is the set of vectors in the same, or opposite, direction as $\vec{u_1}$ with a scaled magnitude. When put together, Span $\{\vec{u_1}\}$ gives a line in $\mathbb{R}^2$.

Example 1.4.2: Span of Two Vectors

Consider $\vec{u_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{u_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, Span $\{\vec{u_1}, \vec{u_2}\} = \mathbb{R}^2$ because any vector from $\mathbb{R}^2$ can be writtin in the form

$$c_1\vec{u_1} + c_2\vec{u_2} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Proof. Consider the system

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \underset{2\times 2}{A} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Then, the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 1 & b_2 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - 2b_1 \end{array}\right) = (A \mid B)_{\text{REF}}$$

$\therefore$ since the augmented matrix in REF has no pivots in the final column, the system is compatible and contains a solution for all vectors $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}$. □

Note as well, that we could've written the final justification using the rank of augmented matrix as reasoning: '$\because$ rank$(A) = 2$, so the system is compatible...'. As well, note that since A is a square matrix and rank$(A) = 2$, that also implies that $A_{\text{RREF}} = I_2$ (it should be pretty easy to put A into RREF and see this for yourself).

Example 1.4.3: Span of Three Vectors

Consider three vectors

$$\vec{u_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{u_2} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad \vec{u_3} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

and consider the span of $\vec{u_1}, \vec{u_2}$ and $\vec{u_3}$, Span $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$. Now notice that $u_2 = 2u_1$ and $u_3 = 3u_3$. This implies that Span $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} =$ Span $\{\vec{u_1}\}$. u_2 and u_3 are considered 'redudant' in the span.

Proof. Consider the matrix representing the vectors of the generation set:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ no pivots in the second and third columns implies that the second and third vectors are redudant. $\square$

Example 1.4.4: Span of Three Vectors (again)

ow consider $\vec{u_1}, \vec{u_2}, \vec{u_3}$ such that

$$\vec{u_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{u_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{u_3} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Notice that Span $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ **must** be reducible since the corresponding matrix has dimensions $2 \times 3 \Rightarrow \text{rank}A \leq \min 2, 3 = 2 \therefore$ (at least) one vector is redudant. We can figure out which one as follows

$$\underset{2\times 3}{A} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -6 \end{pmatrix}$$

Thus, no pivot in the third column implies that the third vector is redudant:

$$\therefore \text{Span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \text{Span}\{\vec{u_1}, \vec{u_2}\}$$

Example 1.4.5: Proving a Vector is in a Span

Show that $\begin{pmatrix} 4 \\ 6 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$. We can start with the system

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 1 & 6 \end{array}\right) \xrightarrow[R_2 \to R_2 - 3R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -6 \end{array}\right)$$

This gives

$$\left\{\begin{array}{ll} -5c_2 & = 6 \\ c_1 + 2c_2 & = 4 \end{array}\right] \Rightarrow c_2 = \frac{6}{5} \text{ and } c_2 = \frac{8}{5}$$

Note that we knew that $\begin{pmatrix} 4 \\ 5 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ since the system was compatible, we just found the values of c_1 and c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Example 1.4.6: Geometric Interpretation of $\mathbb{R}^3$ Span

Consider vectors $\vec{u_1}, \vec{u_2}$ and give a geometric interpretation of their span, Span $\{\vec{u_1}, \vec{u_2}\}$

$$\vec{u_1} = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}, \quad \vec{u_2} = \begin{pmatrix} 12 \\ 3 \\ -9 \end{pmatrix}$$

Start by noticing that $\vec{u_2} = \frac{3}{2}\vec{u_1} \therefore u_1 \parallel u_2 \Rightarrow \text{Span}\{\vec{u_1}, \vec{u_2}\}$ must be a line in $\mathbb{R}^3$ along $\vec{u_1}$.

Example 1.4.7: Geometric Interpretation of $\mathbb{R}^3$ Span (again)

onsider vectors $\vec{u_1}, \vec{u_2}$ and give a geometric interpretation of their span, Span $\{\vec{u_1}, \vec{u_2}\}$

$$\vec{u_1} = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}, \quad \vec{u_2} = \begin{pmatrix} 12 \\ 3 \\ 9 \end{pmatrix}$$

Since $\vec{u_1} \nparallel \vec{u_2}$, Span $\{\vec{u_1}, \vec{u_2}\}$ makes a plane in $\mathbb{R}^3$ (spanned by $\vec{u_1}, \vec{u_2}$)

Example 1.4.8

Let A and b be matricies such that

$$\underset{3\times 3}{A} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix} = \begin{pmatrix} (\vec{u_1}) & (\vec{u_2}) & (\vec{u_3}) \end{pmatrix} \text{ and } b = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$$

Now consider $\mathcal{W} = \text{Span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$.

(a) Is $b \in \mathcal{W}$?

Solution: Start by assuming that $c \in \mathcal{W} \therefore c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3} = b \Leftrightarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right) \xrightarrow[R_3 \to R_3 + 2R_1]{} \left(\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right)$$

Thus, $\text{rank}(A_{\text{REF}}) = 3 \therefore$ the system is compatible and $b \in \mathcal{W}$

(b) Is Span $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ reducible?

Solution:

$$\underset{3\times 3}{A} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ 0 & 6 & -5 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

$\text{rank}(A_{\text{REF}}) = 3 \therefore$ the span is irreducible.

(c) Is Span $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solution: Yes, they're both $\mathbb{R}^3$

1.5 Lecture 5: Linear Dependence and Independence [1.7]

Linear Independence and Dependence

A set of vectors from $\mathbb{R}^n$, $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$, is **linearly independent** if

$$c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_k\vec{u_k} = 0$$

is satisfied if, and only if, $c_1 = c_2 = \cdots = c_k = 0$. Otherwise the set is called **linearly dependent**.

Now, notice that the above equation is actually equivalent to the matrix equation $AC = 0$ where

$$\underset{n\times k}{A} = \begin{pmatrix} (\vec{u_1}) & (\vec{u_2}) & \ldots & (\vec{u_k}) \end{pmatrix} \text{ and } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

It should be evident that we can now view this as a homogenous system of linear equations. Thus, if $\operatorname{rank}(A_{\text{REF}}) = k$ the only solution to the homogenous system is the trivial solution, $\vec{0}$. We can also define linear dependence this way: if $\operatorname{rank}(A_{\text{REF}}) < k$, the system must have non trivial solutions and the set of vectors must be linearly dependent.

Example 1.5.1

Consider the set of vectors below and determine if it is linearly independent or dependent:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Solution: This problem is actually fairly easy and requires zero calculations. Consider the matrix formed from these vectors, A. Notice that A must have dimensions 3×4 and recall that for any $m \times n$ matrix, the rank of said matrix is always less than or equal to $\min m, n$. Applying this to A, we can see that $\operatorname{rank}(A_{\text{REF}}) \leq \min 3, 4 = 3$. Therefore, since there can be only a max of 3 pivots, at least one column will not have a pivot, so the system will have non-trivial solutions. This implies that the set of vectors is linearly **dependent**.

A set of vectors that is linearly dependent contains at least one vector that can be represented as a linear combination of the other vectors in the set. This should also remind you of sets of vectors that have reducible spans.

Example 1.5.2

Consider the set of vectors in the above example, and determine which vector(s) are redudant.

Solution:

$$\underset{3\times 4}{A} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 4 & 1 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 + 2R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -11 & -5 & -2 \end{pmatrix} \xrightarrow[R_3 \to \frac{9}{7}R_3]{R_3 \to R_3 + \frac{11}{7}R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 1 & * \end{pmatrix}$$

$\therefore$ no pivot in the fourth column implies that the fourth vector is redudant.

In the above example, notice that we really didn't even need to calculate the element a_{34} since it's value has no effect on whether or not the fourth column will have a pivot or not.

Example 1.5.3

Consider the set of vectors below and find a value of h such that the set is linearly dependent

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 10 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ -7 \\ h \end{pmatrix} \right\}$$

Solution:

$$\underset{3\times 3}{A} = \begin{pmatrix} 1 & 3 & 2 \\ -3 & 10 & -6 \\ 2 & -6 & h \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 3R_1} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & h-4 \end{pmatrix}$$

Thus, in order for the set to be linearly dependent, one column must *not* have a pivot. This will happen if $h - 4 = 0 \Rightarrow h = 4$.

Example 1.5.4

Determine the column vectors of A are linearly independent or depdendent where $A =$

$$\begin{pmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 5 \\ 5 & 4 & 6 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{pmatrix} \xrightarrow[R_4 \to R_4 - 5R_1]{R_3 \to R_3 + 4R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_2]{\substack{R_4 \to R_4 + 4R_2 \\ R_3 \leftrightarrow R-4}} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ if $\vec{u_1}, \vec{u_2}, \vec{u_3}$ are the column vectors, then they are linearly independent because $\operatorname{rank}(A_{REF}) = 3$

Example 1.5.5

Determine the value of h such that the vectors are linearly dependent

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ h \\ -9 \end{pmatrix} \right\} \Rightarrow \underset{3\times 3}{A} = \begin{pmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 5R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & h-15 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ since there is no pivot in the third column, the vectors will be linearly independent $\forall h \in \mathbb{R}$.

1.6 Lecture 5: Linear Transformations [1.8]

Transformations

A linear transformation $\mathcal{T} : \mathbb{R}^m \to \mathbb{R}^n$ given by the map $\underset{m\times n}{A} \cdot \underset{n\times 1}{\vec{x}} \mapsto \vec{b}$ is a function that can be thought of as finding all vectors $\vec{x} \in \mathbb{R}^n$ that turn into $\vec{b} \in \mathbb{R}^m$ when multiplied by A. $\mathbb{R}^n$ is called the **domain** and $\mathbb{R}^m$ is called the **co-domain**.

Example 1.6.1

Consider a linear map $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $A\vec{x} \mapsto \vec{b}$. For the given matrix A and vectors $\vec{u}$ and $\vec{v}$, find $\mathcal{T}(\vec{u})$ and $\mathcal{T}(\vec{v})$.

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Since $\mathcal{T}(\vec{x}) = A\vec{x}$, we have

$$\mathcal{T}(\vec{u}) = A\vec{u} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} \text{ and, } \mathcal{T}(\vec{v}) = A\vec{v} = \begin{pmatrix} 5a \\ \frac{1}{2}a \\ \frac{1}{2}c \end{pmatrix}$$

Linear Transformations

A transformation, $\mathcal{T}(\vec{x})$, is **linear**, if

$$\mathcal{T}(a\vec{u} + b\vec{v}) = a\mathcal{T}(\vec{u}) + b\mathcal{T}(\vec{v}) \Leftrightarrow A\left(a\vec{X} + b\vec{Y}\right) = aA\vec{X} + bB\vec{Y}$$

Note that not all transformations are linear transformations, there is a distinct difference!

Basic Linear Transformations

- Dilation Transformation: $\mathcal{T}(\vec{u}) = r\vec{u}$ for $r \in \mathbb{R}$
- 90° Transformation: $\mathcal{T}(\vec{u}) = A\vec{u} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u}$
- Shear Transformation: $\mathcal{T}(\vec{u}) = A\vec{u} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \vec{u}$

Note that for the **shear** transformation, the y-coordinate ($\hat{\jmath}$) of $\vec{u}$ stays the same, only the x-coordinate ($\hat{\imath}$) is stretched.

Example 1.6.2: Mapping to the Zero Vector

Consider the following matrix A and find all vectors $\vec{x}$ such that they get mapped to the zero vector when multiplied by A:

$$\underset{3\times 4}{A} = \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{pmatrix}$$

We can start by noticing that if we define $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^3$ such that $\mathcal{T}(\vec{x}) = A\vec{x}$, then we just need to solve $\mathcal{T}(\vec{x}) = \vec{0} \Rightarrow A\vec{x} = \vec{0}$:

$$A \xrightarrow[R_3 \to R_3 - 2R_1]{} = \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -5 & 6 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} = \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, since there is no pivot in the third and forth columns, $x_3 = s$ and $x_3 = t$ are free variables (for $s, t, \in \mathbb{R}$). Backwards substitution into the equations gives:

$$\left.\begin{cases} x_2 - 4x_3 + 3x_4 & = 0 \Rightarrow x_2 = 4s - 3t \\ x_1 - 4x_2 + 7x_3 - 5x_4 & = 0 \Rightarrow x_1 = 9s - 7t \end{cases}\right] \Rightarrow X = s \begin{pmatrix} 9 \\ 4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Example 1.6.3

Consider $\mathcal{T}$ given by $A\vec{X} \mapsto \vec{B}$ and determine if $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in \mathcal{T}^{\text{img}}$, where A is the same matrix from the above example. Notice that all this questions is really asking is if the system $A\vec{x} = \vec{b}$ is consistent. We can figure this out fairly easily:

$$\left(A \mid b\right) = \left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{array}\right) \xrightarrow{\text{ERO}} \left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\therefore$ the system is consistent so $b \in \mathcal{T}^{\text{img}}$

1.7 Lecture 7: The Matrix of a Linear Transformation [1.9]

Recall that a function $\mathcal{T}(X) : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear tranformation if

1. $\mathcal{T}(X+Y) = \mathcal{T}(X) + \mathcal{T}(Y)$, and
2. $\mathcal{T}(kX) = k\mathcal{T}(X)$ for $k \in \mathbb{R}$

It's important to notice the similarities to the properities of matrix multiplication:

1. $\underset{m\times n}{A}\left(\underset{n\times 1}{X} + \underset{n\times 1}{Y}\right) = AX + AY$
2. $\underset{m\times n}{A}\left(k\underset{n\times 1}{X}\right) = kAX$

Theorem 1.2. *Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a matrix, $\underset{m\times n}{A}$ such that*

$$\mathcal{T}(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$

Proof. For any matrix X,

$$X = I_n X = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

Then,

$$\begin{aligned}
\mathcal{T}(\vec{x}) &= \mathcal{T}(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\
&= \mathcal{T}(x_1 e_1) + \mathcal{T}(x_2 e_2) + \cdots + \mathcal{T}(x_n e_n) \\
&= x_1 \mathcal{T}(e_1) + x_2 \mathcal{T}(e_2) + \cdots + x_n \mathcal{T}(e_n) \\
&= \left([\mathcal{T}(e_1)] \quad [\mathcal{T}(e_2)] \cdots \quad [\mathcal{T}(e_n)]\right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
&= A\vec{x}
\end{aligned}$$

□

Note that I_n is the identity matrix which is an $n \times n$ matrix in which every element is a 0 except for elements, $a_{11}, a_{22}, \ldots, a_{nn}$. Note as well, that e_i is the i-th column of the identity matrix. For example, consider I_2 and I_3:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then e_1 for I_2 would be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Similarly, e_3 for I_3 would be $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

One-to-one and Onto

Consider a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ given by the map $\vec{x} \mapsto \left(\underset{m\times n}{A}\right)\vec{x}$. Then, $\mathcal{T}$ is called **onto** when there exists at least one vector $\vec{x} \in \mathbb{R}^n$ for *any* vector $\vec{B} \in \mathbb{R}^m$. $\mathcal{T}$ is called **one-to-one** when $A\vec{x} = \vec{B}$ has only one solution.

One way to think about it is that $\mathcal{T}$ is **onto** if any vector in $\mathbb{R}^m$ can be 'made' by applying the transformation to a vector $\vec{x}$. $\mathcal{T}$ is **one-to-one** if for any solution to the equation $A\vec{x} = \vec{B}$, the matrix $\vec{x}$ is the sole, unique, solution to that equation. Note as well, that a transformation can be <u>both</u> one-to-one and onto.

Determining if a tranformation is one-to-one or onto

Consider a linear transformation $\mathcal{T}(\vec{x}) = A\vec{x}$ for a matrix $\underset{m\times n}{A}$.

- If $\operatorname{rank}(A) = m$ (implying that $m \leq n$), then each row in A contains a pivot which ensures that the augmented matrix $\left(A \mid B\right)_{\text{REF}}$ will have no pivots in the final column. In other words, the system $A\vec{x} = \vec{B}$ will be consisten for all $\vec{B} \in \mathbb{R}^m$ and $\mathcal{T}$ is **onto**.
- If $\operatorname{rank}(A) = n$ (implying $n \leq m$), then each column in A contains a pivot which ensures that there are no free variables in the system $A\vec{x} = \vec{B}$. In other words, any solution to $A\vec{x} = \vec{B}$ will be unique and $\mathcal{T}$ is **one-to-one**.

Example 1.7.1: Determine if a Transformation is One-to-one or Onto

Consider the transformation $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 2x_1 + x_2 - x_3 + x_4 \\ x_1 + 3x_2 + 5x_3 - 2x_4 \\ 3x_1 - x_2 + 2x_3 + 3x_4 \end{pmatrix}$$

determine if $\mathcal{T}$ is one-to-one, onto, both, or neither.

Solution: We can start by rewriting $\mathcal{T}$ in the form $\mathcal{T}(\vec{x}) = A\vec{x}$:

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A\vec{x}$$

Since A is a 3×4 matrix, $\operatorname{rank}(A) \leq 3 \therefore \mathcal{T}$ <u>cannot be one-to-one</u>. Now, as per usual, in order to determine the rank of A we must find A_{REF}. We can start with $A \xrightarrow{R_1 \to R_2}$

$$\begin{pmatrix} 1 & 3 & 5 & -2 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 2 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & -5 & -11 & 5 \\ 0 & -10 & -13 & 9 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & -5 & -11 & 5 \\ 0 & 0 & 9 & -1 \end{pmatrix}$$

Thus, $\operatorname{rank}(A) = 3 = n$ so $\mathcal{T}$ is **onto**

Rotation Transformations

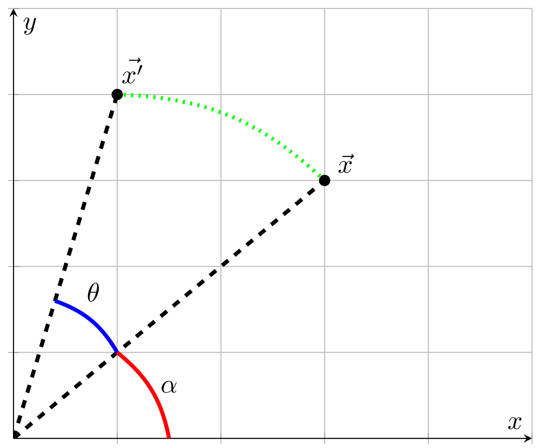
Consider vectors $\vec{x}$ which has an angle α from the x-axis and $\vec{x'}$ which has an angle $\alpha + \theta$ from the x-axis. There exists a transformation $\mathcal{T}$ such that $\mathcal{T}(\vec{x}) = \vec{x'}$. Just like all linear transformations, $\mathcal{T}$ can be written in the form $\mathcal{T}(\vec{x}) = A\vec{x}$ where

$$A = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{Rotation Matrix}}$$

Thus, if $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{x'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ then,

$$\mathcal{T}\left(\vec{x}\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

If you're curious where the definition of the rotation matrix comes from, consider the two vectors $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{x'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$:



We can represent $\vec{x}$ in polar coordinates as

$$x = r\cos\alpha \text{ and } y = r\sin\alpha$$

Then,

$$x' = r\cos\left(\alpha + \theta\right) \text{ and } y' = r\sin\left(\alpha + \theta\right)$$

This gives,

$$\begin{aligned} x' &= r\cos\left(\alpha+\theta\right) = \overbrace{r\cos\alpha}^{x}\cos\theta - \overbrace{r\sin\alpha}^{y}\sin\theta = x\cos\theta - y\sin\theta \\ y' &= \underbrace{r\sin\alpha}_{y}\cos\theta + \underbrace{r\cos\alpha}_{x}\sin\theta = x\sin\theta + y\sin\theta \\ &\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Example 1.7.2: Prove $\mathcal{T}$ is a Linear Transformation

Consider a transformation $\mathcal{T}$ such that

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix}$$

determine if $\mathcal{T}$ is a linear transformation.

Solution: Notice that $\mathcal{T}(\vec{x}) = A\vec{x}$ for

$$\underset{4\times 4}{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } \underset{4\times 1}{\vec{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Because $\mathcal{T}$ can be put in the form $\mathcal{T}(\vec{x}) = A\vec{x}$, it must be a linear transformation.

Example 1.7.3: Determine if $\mathcal{T}$ is One-to-one or Onto

Consider the linear transformation $\mathcal{T}$ from the above example. Determine if $\mathcal{T}$ is one-to-one, onto, both, or neither.

Solution:

$$\underset{4\times 4}{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $\text{rank}(A) \neq 4 = m = n$ so $\mathcal{T}$ is neither one-to-one nor onto.

Example 1.7.4: Find the Matrix A

Given that $\mathcal{T}$ is a linear transformation such that

$$\mathcal{T}(\vec{x}) = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - 3x_2 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{pmatrix}$$

determine the value of the matrix A.

Solution: We can find A as the coefficent matrix of $\mathcal{T}$:

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Example 1.7.5: Determine if $\mathcal{T}$ is One-to-one or Onto

Consider the linear transformation $\mathcal{T}(\vec{x}) = A\vec{x}$ such that A is the matrix in the above example. Determine if $\mathcal{T}$ is one-to-one, onto, both, or neither.

Solution: We can easily figure this out by looking at the rank of A_{REF}.

$$A \xrightarrow[R_2 \leftrightarrow R_1]{R_2 \to \frac{1}{4}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[R_2 \to -\frac{1}{3}R_3]{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $\text{rank}(A_{\text{RREF}}) = 3 = m = n$ so $\mathcal{T}$ is both one-to-one and onto.

2 Unit 2

2.1 Lecture 8: Matrix Operations [2.1]

Before I start, I must note that I was not able to physically make it to lecture 8, so these notes are simply just a brief overview of concepts covered in lecture.

So far, we've very briefly been introduced to the idea of matrix multiplication through our studies of systems of equation in the form $A\vec{x} = \vec{b}$ which we know can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

in order to represent the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_m & = b_n \end{cases}$$

You'll notice that in order to go from the matrix to the system we simply take each row of the matrix (or each coefficent) and multiply it by the corresponding variable. This is exactly how matrix multiplication is defined.

Matrix Multiplication

Consider two matricies $\underset{m\times n}{A}$ and $\underset{n\times k}{B}$ such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ \vdots & & \ddots & \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}$$

Label the columns in B as $\mathbf{b_1}, \mathbf{b2}, \ldots \mathbf{b_k}$. Then their product, AB, is defined as follows:

$$\underset{m\times k}{AB} = \begin{pmatrix} A\mathbf{b_1} & A\mathbf{b_2} & \cdots & A\mathbf{b_k} \end{pmatrix}$$

It's extremely important to note that matrix multiplication is only defined two matricies, A and B, <u>if, and only if, the number of columns in A is equal to the number of rows in B.</u>

One other way to think of matrix multiplication that if A and B are defined as above then,

$$AB = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \sum_{i=1}^{n} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{mi}b_{ik} \\ \sum_{i=1}^{n} a_{2i}b_{i1} & \sum_{i=1}^{n} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{2i}b_{ik} \\ \vdots & & \ddots & \\ \sum_{i=1}^{n} a_{mi}b_{i1} & \sum_{i=1}^{n} a_{mi}b_{i2} & \cdots & \sum_{i=1}^{n} a_{mi}b_{ik} \end{pmatrix}$$

Now, granted, this looks very complicated and extremely hard to use. So, why did I bother including this here? Well, if you look closely at each summation, what you'll notice is that each sumation is simply the sum of the product of a corresponding *row* from A with a column from B. So, for example, the first element in AB is the sum of each element in the **first** *row* in A multiplied by its corresponding element in the **first** *column* in B. The second element in AB is the **first** *row* in A multiplied by the **second** *column* in B. This is probably best explained with a few examples:

Example 2.1.1: Basic Matrix Multiplication

Let A and B be defined as below. Determine if AB is defined, and if it is find it.

$$\underset{2\times 2}{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } \underset{2\times 2}{B} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

Solution: First, since A and B are both 2×2 matricies, we know that AB must exists since A and B, obviously, have the same number of rows and columns as each other. We can then find AB:

$$AB = \begin{pmatrix} (1\cdot 2)+(2\cdot 1) & (1\cdot 4)+(2\cdot 3) \\ (3\cdot 2)+(4\cdot 1) & (3\cdot 4)+(4\cdot 3) \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 10 & 24 \end{pmatrix}$$

Keep in mind that we also could've found the product AB using the other definition of matrix multiplication:

$$AB = \begin{pmatrix} A\mathbf{b_1} & A\mathbf{b_2} \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \right) = \left(\begin{pmatrix} 4 \\ 10 \end{pmatrix} \begin{pmatrix} 10 \\ 24 \end{pmatrix} \right) = \begin{pmatrix} 4 & 10 \\ 10 & 24 \end{pmatrix}$$

Example 2.1.2: Basic Matrix Multiplication

Let A and B be defined as below. Determine if AB is defined, and if it is find it.

$$\underset{3\times 2}{A} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{pmatrix} \text{ and } \underset{2\times 2}{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Solution: Since A has 2 columns and B has 2 rows, we know that AB exists. Thus,

$$AB = \begin{pmatrix} (1\cdot 1)+(3\cdot 0) & (1\cdot 1)+(3\cdot 1) \\ (5\cdot 1)+(7\cdot 0) & (5\cdot 1)+(7\cdot 1) \\ (9\cdot 1)+(11\cdot 0) & (9\cdot 1)+(11\cdot 1) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 5 & 12 \\ 9 & 20 \end{pmatrix}$$

Theorem 2.1 (Properties of Matrix Multiplication)**.** *Consider a matrix* $\underset{m\times n}{A}$ *and let* B *and* C *be matricies such that the necessary products are defined.*

1. *Associative Law:* $A(BC) = (AB)C$
2. *Left Distributive Law:* $A(B+C) = AB + AC$
3. *Right Distributive Law:* $(B+C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ *for any* $r \in \mathbb{R}$
5. *Identity for Matrix Multiplication:* $I_m A = A = A I_n$

Recall that I_n is the $n \times n$ matrix such that every element is zero except for $a_{11}, a_{22}, \ldots, a_{nn}$, referred to as the **Identity Matrix**.

Its important to note a few things about the above theorem.

1. There is NO communative law, thus, you cannot assume $AB = BA$
2. There is NO cancelation law: $AB = BC$ does NOT imply that $A = C$
3. If AB is the zero matrix, you CANNOT conclude that either $A = 0$ or $B = 0$

The Power of a Matrix

Let A be any $n \times n$ matrix. Then A^k is defined as

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

If $k = 0$ then $A^k = I_n$ where I_n is the identity matrix in $\mathbb{R}^n$.

The definition of the power of a matrix should feel extremely similar to the definition of a real number to any power: repeated multiplication.

Transpose of a Matrix

Let A be any $m \times n$ matrix. Then A^{T} is the **transpose** of A, which is a $n \times m$ matrix who's <u>rows are made from the columns of A and whose columns are made from the rows</u>.

For example, let A and B be defined as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then,

$$A^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ and } B^{\mathrm{T}} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Theorem 2.2 (Properties of the Transpose)**.** *Let A and B be matricies such that necessary sums and products are defined. Then,*

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ *for any* $r \in \mathbb{R}$
4. $(AB)^T = B^T A^T$

2.2 Lecture 9: Inverse of a Matrix and Characterizations of Inveritble Matricies [2.2-2.3]

Inveritble Matricies

An $n \times n$, A, matrix is called <u>invertible</u> if there exists an $n \times n$ matrix, B, such that $AB = BA = I_n$. Such B is called the inverse of A, denoted A^{-1}. A^{-1} is always a unique matrix for any A.

Theorem 2.3. *An $n \times n$ matrix is inveritble if, and only if, rank$(A) = n$*

Proof. Consider a matrix $\underset{n\times n}{A}$ such that $\operatorname{rank}(A_{\text{RREF}}) = n$. Then $A_{\text{RREF}} = I_n$. Assume it takes k iterations of elementary row operations (ERO) to obtain A_{RREF} from A. Then, $A_{\text{RREF}} = I_n = E_k \cdots E_2 E_1$. So, $I = BA \Leftrightarrow A^{-1} = B$ and $B^{-1} = A$ and $B^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.
If A^{-1} exists then consider $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$. Thus, the system has only one solution and so $\operatorname{rank}(A) = n$. □

If you're confused on why $\vec{x} = A^{-1}\vec{b}$ implies that the system only has a single unique solution, consider that if $\vec{x} = A^{-1}\vec{b}$ then we have

$$A\vec{x} = A\left(A^{-1}\vec{b}\right) = \left(AA^{-1}\right)\vec{b} = I\vec{b} = \vec{b}$$

so, $A^{-1}\vec{b}$ must be a solution. To show that it is unique, consider any solution $\vec{u}$. Then,

$$A\vec{u} = \vec{b} \Rightarrow A^{-1}A\vec{u} = A^{-1}\vec{b} \Rightarrow I\vec{u} = A^{-1}\vec{b} \Rightarrow \vec{u} = A^{-1}\vec{b}$$

Then, we know from 1.3, that if a system has a single solution, its augmented matrix (or A_{REF}) has a rank equal to n. It should follow from this that if A^{-1} exists the homogenous system, $A\vec{x} = \vec{0}$ has a single solution, the trivial solution: $\vec{x} = \vec{0}$.

How to Find the Inverse of a 2×2 Matrix

Consider 2×2 matricies A, B with rank 2 such that $AB = I_2$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Since $\operatorname{rank}(A) = 2$, A is invertible. Then,

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives,

$$\left.\begin{cases} a\alpha + b\gamma & = 1 \\ c\alpha + d\gamma & = 0 \end{cases}\right] \Rightarrow \left(\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right) \xrightarrow{\text{ERO}} \left(\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & \gamma \end{array}\right)_{\text{RREF}}$$

$$\left.\begin{cases} a\beta + b\delta & = 0 \\ c\beta + d\delta & = 1 \end{cases}\right] \Rightarrow \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right) \xrightarrow{\text{ERO}} \left(\begin{array}{cc|c} 1 & 0 & \beta \\ 0 & 1 & \delta \end{array}\right)_{\text{RREF}}$$

We can then combine to form one large matrix

$$(A \mid I) \xrightarrow{\text{ERO}} (I \mid A^{-1}) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \alpha & \beta \\ 0 & 1 & \gamma & \delta \end{array}\right)$$

Note that this method can actually be generalized such that for any matrix $\underset{n\times n}{A}$, augmenting it with the identity matrix and using ERO to find $(A \,|\, I)_{\text{RREF}}$ will always result in $(I \,|\, A^{-1})$

Example 2.2.1

Find the inverse of A and verify that $AA^{-1} = I$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$$

Then,

$$AA^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Now, you might have noticed that in the previous example, A^{-1} actaully looked extremely similar to A. In fact, A and A^{-1} are made up elements of the exact same magnitude. This phenomenon will occur for the inverses of every 2×2 matrix with rank 2 and, in fact, we have a formula that explains the similarity.

Formula for Finding the Inverse of a 2×2 matrix

Let A be a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then,

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where the quanity $ad - bc$ is called the **determinant of** A, known as $\det A$.

It should follow from the last line that $\det A \neq 0 \Leftrightarrow A^{-1}$ exists.

Example 2.2.2

For A below, does A^{-1} exist? If so, find A^{-1}

$$A = \begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix}$$

Solution: Start by finding $\det A$:

$$\det A = (-4 \cdot -9) - (4 \cdot 9) = 36 - 36 = 0$$

Thus, by the fact above, A^{-1} must not exist. Or, in other words, A is <u>not</u> invertible.

Example 2.2.3

For A below, does A^{-1} exist? If so, find A^{-1}.

$$A = \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{pmatrix}$$

Solution: Start by finding the rank of A:

$$A \xrightarrow{R_3 \to R_3 + 3R_1} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $\operatorname{rank}(A) < 3$ so A is <u>not</u> invertible.

Example 2.2.4

For A below, does A^{-1} exist? If so, find A^{-1}.

$$A = \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 5 \\ -3 & 6 & 0 \end{pmatrix}$$

Solution: Start by finding the rank of A:

$$A \xrightarrow{R_3 \to R_3 + 3R_1} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 5 \\ 0 & -9 & -12 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

Therefore, $\text{rank}\,(A) = 3$ so A <u>is</u> invertible. Now, consider

$$\left(A \mid I_3\right) = \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_1} \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 0 & -9 & -12 & 3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 0 & 0 & 3 & 3 & 3 & 1 \end{array}\right) \xrightarrow[R_3 \to \frac{1}{3}R_3]{R_2 \to \frac{1}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + 4R_3]{R_2 \to R_2 - \frac{5}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 5 & 4 & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{4}{3} & -\frac{5}{9} \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array}\right) \xrightarrow{R_1 \to R_1 + 5R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{10}{3} & -\frac{8}{3} & -\frac{13}{9} \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{4}{3} & -\frac{5}{9} \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array}\right)$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} -\frac{10}{3} & -\frac{8}{3} & -\frac{13}{3} \\ -\frac{5}{3} & -\frac{4}{3} & -\frac{5}{9} \\ 1 & 1 & 3 \end{pmatrix}}$$

Example 2.2.5: Is A invertible?

Let A be the matrix below. Determine if A is inveritble.

$$A = \begin{pmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{pmatrix}$$

Solution: Since A has a 0 column, $\text{rank}\,(A) < 3 \therefore A$ is <u>not</u> invertible

Example 2.2.6: Is A invertible?

Let A be the matrix below. Determine if A is invertible.

$$A = \begin{pmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

Solution:

$$A \xrightarrow[\substack{R_3\to R_3-2R_1\\ R_2\to -\frac{1}{4}R_2}]{R_2\to R_2+3R_1} \begin{pmatrix} -1 & -3 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{R_4\to R_4+R_2} \begin{pmatrix} -1 & -3 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, $\operatorname{rank}(A) = 4 \therefore A$ $\underline{\text{is}}$ invertible.

2.3 Lecture 10: Review for Midterm 1

Systems of Equations

Homogenous systems of equations are systems in the form $A\vec{x} = \vec{0}$. These systems always contain at least one solution, $\vec{x} = \vec{0}$ (known as the trivial solution). *Non-Homogenous* systems of equations are system in the form $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$.

Example 2.3.1: Solve the System

$$\begin{cases} 2x_1 + x_2 - x_3 & = 4 \\ x_1 - 2x_2 + x_3 & = 2 \end{cases}$$

Solution: We can form an augmented matrix and use elementary row operations to find a solution:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -2 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 2 & 1 & -1 & 4 \end{array}\right) \xrightarrow{R_2\to R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 5 & -3 & 0 \end{array}\right)$$

Now, because there is no pivot in the third column, x_3 is a free variable: let $x_3 = 5t$ for $t \in \mathbb{R}$. Then, we can back substitute from the matrix to the equations to get:

$$\left.\begin{cases} 5x_2 - 3x_3 & = 0 \\ x_1 - 2x_2 + x_3 & = 2 \end{cases}\right] \Rightarrow x_2 = 3t \text{ and } x_1 = 2 + t$$

We can formally write our solution as follows:

$$X_{\text{gen}} = \begin{pmatrix} 2+t \\ 3t \\ 5t \end{pmatrix} = t \underbrace{\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}}_{X_{\text{genhom}}} + \overbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}^{X_p}$$

Where X_{genhom} is the general homogenous solution and X_p is a particular solution.

A system is **consistent** if there exists at least one solution. $A\vec{x} = \vec{b}$ is consisten if the augmented matrix, $(A \mid B)_{\text{REF}}$ has no pivots in the final column. Equivalently: if $\operatorname{rank}\left(\underset{m\times n}{A}\right) = m$ then $A\vec{x} = \vec{b}$ is always consistent. We can note a few things from this fact. If $A\vec{x} = \vec{b}$ is consistent, then the span of the columns of A is $\mathbb{R}^m$. As well, consider a linear transformation $\mathcal{T}:\mathbb{R}^n \to \mathbb{R}^m$ given by the map $\vec{x} \mapsto A\vec{x}$, then $\mathcal{T}$ will be onto. If the system $A\vec{x}=\vec{b}$ has only a single solution, then $\operatorname{rank}\left(\underset{m\times n}{A}\right)=n$. Again, consider a linear transformation, $\mathcal{T}:\mathbb{R}^m \to \mathbb{R}^n$ given by the mapping $\vec{x} \mapsto A\vec{x}$, then $\mathcal{T}$ will be one-to-one. If a linear transformation is one-to-one **and** onto, then it is called invertible.

Recall that $\operatorname{rank}\left(\underset{m\times n}{A}\right) \leq \min\{m,n\}$, so in order for a linear transformation to be both one-to-one and onto, m must equal n (in other words: A must be a square matrix).

Span/Linear Dependence

Consider a set of vectors, $\{\vec{u_1},\vec{u_2}\ldots,\vec{u_k}\}$. Then, the **span** of that set of vectors, denoted $\operatorname{Span}\{\vec{u_1},\vec{u_2},\ldots,\vec{u_k}\}$, is the set of all possible linear combinations of $\vec{u_1},\vec{u_2},\ldots,\vec{u_k}$:

$$c_1\vec{u_1}+c_2\vec{u_2}+\cdots+c_k\vec{c_k}=AC,\quad A=\begin{pmatrix}(\vec{u_1}) & (\vec{u_2}) & \cdots & (\vec{u_k})\end{pmatrix} \text{ and } C=\begin{pmatrix}c_1\\c_2\\\vdots\\c_k\end{pmatrix}$$

The set $\{\vec{u_1},\vec{u_2},\ldots,\vec{u_k}\}$ is called the **generating set**.

Consider $AC=\vec{0}$. If the only solution to $AC=\vec{0}$ is the trivial solution, then the vectors $\vec{u_1},\vec{u_2},\ldots,\vec{u_k}$ are called **linearly independent**. If there exists a non-trivial solution to $AC=\vec{0}$, then the vectors are called **linearly dependent**. If the vectors are linearly dependent, then at least on vector in the generating set is redudant. We can determine which vector is redudant by finding A_{REF} and noting that any columns without a pivot correspond to redudant vectors. Notice that we can easily determine if vectors are linearly independent or dependent simply by just finding A_{REF}. If $\operatorname{rank}(A_{\text{REF}})<n$ then the columns will always be linearly dependent. If $\operatorname{rank}(A_{\text{REF}})=n$, then the columns will always be linearly independent.

Example 2.3.2

Determine a value of r such that the set below is linearly dependent.

$$\left\{\begin{pmatrix}1\\-1\\3\end{pmatrix},\begin{pmatrix}2\\4\\-2\end{pmatrix},\begin{pmatrix}r\\1\\2\end{pmatrix}\right\}$$

Solution: Consider

$$A=\begin{pmatrix}1 & 2 & r\\-1 & 4 & 1\\3 & -2 & 2\end{pmatrix}\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2+R1}\begin{pmatrix}1 & 2 & r\\0 & 6 & 1+r\\0 & -8 & 2-3r\end{pmatrix}\xrightarrow{R_3\to R_3+\frac{8}{6}R_2}\begin{pmatrix}1 & 2 & r\\0 & 6 & 1+r\\0 & 0 & \frac{10}{3}-\frac{5}{3}r\end{pmatrix}=A_{\text{REF}}$$

Therefore, in order for $\operatorname{rank}(A_{\text{REF}})$ to be less then $n=3$, r must be equal to 2. So, $\boxed{r=2}$

Inverses

A $n\times n$ matrix is invertible if, and only if, it has rank n.

Example 2.3.3

Let A be the matrix below. Find A^{-1}

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$$

Solution: We can find A^{-1} by augmenting A with I and finding $(A\,|\,I)_{\text{RREF}} = \big(I\,|\,A^{-1}\big)$.

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 5 & -5 & -2 & 1 & 0 \\ 0 & -2 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & -2 & 5 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 3 & \frac{1}{5} & \frac{2}{5} & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{3}R_3]{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{4}{5} & -\frac{2}{5} & -1 \\ 0 & 1 & -1 & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{15} & \frac{2}{15} & \frac{1}{3} \end{array}\right) \xrightarrow{R_2 \to R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{4}{5} & -\frac{2}{5} & -1 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{15} & \frac{2}{15} & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{15} & \frac{4}{15} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{15} & \frac{2}{15} & \frac{1}{3} \end{array}\right) \Rightarrow \boxed{A^{-1} = \begin{pmatrix} \frac{2}{15} & \frac{4}{15} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{15} & \frac{2}{15} & \frac{1}{3} \end{pmatrix}}$$

3 Unit 3

3.1 Lecture 12: Introduction to Determinants [3.1]

Before I start, I must note that I was not able to physically make it to lecture 12, so these notes are simply just a brief overview of concepts covered in lecture.

Back in section (2.2) we mentioned that a 2×2 matrix is invertible if, and only if, its **determinant** was non-zero and we defined the determinant of a two by two matrix as

$$\det A = ad - bc, \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The idea of the determinant is one that we will extend to any $n \times n$ matrix. Before we introduce a formal definition of and method for finding the determinant, we must first learn some new notation.

Submatrix Notation

Let A be any $n \times n$ matrix. Then let A_{ij} denote the submatrix obtained by removing the i-th row and j-th column from A.

For example, consider a 4×4 matrix

$$\underset{4\times 4}{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

Then,

$$A_{11} = \begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix} \qquad A_{34} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$$

The Determinant of a $n \times n$ Matrix

For $n \geq 2$ the **determinant** of a $n \times n$ matrix A, is

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} \det A_{1j}$$

Note that the terms being added are occasionally written in a slightly different form:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}, \text{ where } C_{ij} = (-1)^{i+j} \det A_{ij}$$

The variable C_{ij} is also known as that (i, j)**-cofactor** of A. With this in mind, we introduce the next theorem

Theorem 3.1. *The determinant of a $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across any row is*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and the expansion down any column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

We also have the following theorm

Theorem 3.2. *For any triangular matrix A, $\det A$ is the product of all the entries on the main diagonal of A.*

3.2 Lecture 13: Properties of Determinents [3.1]

Cofactors

Consider a matrix $\underset{n\times n}{A}$. Then

$$\det A = |A| = \underbrace{\sum_{j=1}^{n} a_{ij} C_{ij}}_{i\text{-th row expansion}} = \underbrace{\sum_{i=i}^{n} a_{ij} C_{ij}}_{j\text{-th column expansion}}$$

where $C_{ij} = (-1)^{i+j} M_{ij}$. M_{ij} is the matrix obtained by removing the i-th row and j-th column from A. C_{ij} is called the **cofactor**.

Properties of Determinents

Consider an $n \times n$ matrix A, then

1. $\det A = \det A^T$
2. $A \xrightarrow{R_i \leftrightarrow R_j} \tilde{A} \Rightarrow \det \tilde{A} = -\det A \quad (i \neq j)$
3. $A \xrightarrow{R_i \to \lambda R_i} \tilde{A} \Rightarrow \det \tilde{A} = \lambda \det A \quad (\lambda \neq 0)$
4. $A \xrightarrow{\text{ERO}} \tilde{A} = \lambda A \Rightarrow \det \tilde{A} = \lambda^n \det A$
5. $A \xrightarrow{R_i \to R_i + \lambda R_j} \tilde{A} \Rightarrow \det \tilde{A} = \det A \quad (i \neq j)(\lambda \neq 0)$
6. $\det (A_{\text{RREF}}) = a_{11} \times a_{22} \times \cdots \times a_{nn}$
7. $\det A \neq 0 \Leftrightarrow \text{rank}\,(A) = n$
8. $\det A \neq 0 \Leftrightarrow A^{-1}$ exists
9. $\det (AB) = \det (A) \det (B)$
10. $\det \left(A^{-1}\right) = \frac{1}{\det A}$
11. If $\det A \neq 0$ the, $A^{-1} = \frac{C^T}{\det A}$ where C is the matrix of cofactors

One should note that it follows from property (6) that the determinant of any triangular matrix is simply the product of the elements on the main diagonal. You should also note that properties (7) and (8) are equivalent statements since an $n \times n$ matrix is only invertible if, and only if, it has rank n.

Cramer's Rule

Consider an $n \times n$ matrix A and an $n \times 1$ matrix b such that $A\vec{x} = b$ and $\det A \neq 0$. Then, the system has solutions

$$x_1 = \frac{\Delta_1}{\det A}, x_2 = \frac{\Delta_2}{\det A}, \cdots, x_n = \frac{\Delta_n}{\det A}$$

where each Δ_k is the determinent of the matrix obtained from A by replacing the k-th column in A with b.

Example 3.2.1: Solve the system using Cramer's Rule

$$\begin{cases} x_1 + 2X_2 - x_3 &= 4 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_3 &= 6 \end{cases}$$

Solution: Start wi th the following matricies:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$$

Then we can easily calculate $\det A$ as follows:

$$\det A = \begin{vmatrix} 5 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} = -6 \neq 0$$

Not only have we found that $\det A = -6$ but since it is non-zero we also know we can apply Cramer's rule. Now we can find $\Delta_1, \Delta_2, \Delta_3$:

$$\Delta_1 = \begin{vmatrix} 4 & 2 & -1 \\ 3 & -1 & 1 \\ 6 & 0 & 1 \end{vmatrix} = -4 \Rightarrow x_1 = \frac{-4}{-6} = \frac{2}{3}$$

$$\Delta_2 = \begin{vmatrix} 1 & 4 & -1 \\ 2 & 3 & 1 \\ -1 & 6 & 1 \end{vmatrix} = -30 \Rightarrow x_2 = \frac{-30}{-6} = 5$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ -1 & 0 & 6 \end{vmatrix} = -40 \Rightarrow x_3 = \frac{-40}{-6} = \frac{20}{3}$$

Thus,

$$X = \begin{pmatrix} \frac{2}{3} \\ 5 \\ \frac{20}{3} \end{pmatrix}$$

Example 3.2.2: Find A^{-1} if it exists

$$A = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$$

Solution: Start with $\det A = 5 - 6 = -1 \neq 0$ to verify that A^{-1} does exist. Then we can find the cofactors as follows

$$C_{11} = 1 \quad C_{12} = -2 \quad C_{21} = -3 \quad C_{22} = 5$$

Since A is 2×2 each cofactor will simply be the only element remaning after removing the corresponding row and column, and then changing the sign based on $(-1)^{i-j}$.
We can now construct A^{-1}:

$$C = \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} \Rightarrow A^{-1} = -\begin{pmatrix} 1 & -3 \\ -2^5 & \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}$$

We can easily verify this is correct using the formula for the inverse of a 2×2 matrix given in section (2.2)

We will now note down a few of the important applications of determinants in math.
Consider two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$. Then their cross product will be

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now, consider a third vector $\vec{c} = \langle c_1, c_2, c_3 \rangle$ and define the triple product of vectors to be $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$. This can also be represented as the determinant of the matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 3.2.3: Determine if the Given Set is Linearly Independent

$$\left\{ \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ -2 \end{pmatrix} \right\}$$

Solution: Recalling property (7), we know that we only have to find the determinant of the matrix with columns made up of vectors from the set and check that the determinant is non-zero.

$$\begin{vmatrix} 4 & -6 & -3 \\ 6 & 0 & -5 \\ 2 & 6 & -2 \end{vmatrix} = 4\,(30) - (-6)\,(-2) + (-3)\,(-36) = 120 - 12 - 108 = 0$$

$\det A = 0 \therefore$ the set of vectors is **not** linearly independent.

4 Unit 4

4.1 Lecture 14: Vector Spaces and Subspaces [2.8/4.1]

Consider $\mathbb{R}^n$ and vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and notice that the following properties always hold true:

1. $\forall\, \vec{v}, \vec{u} \in \mathbb{R}^n, \vec{u} + \vec{v} \in \mathbb{R}^n$ (Closure under addition)
2. $\forall\, \vec{v} \in \mathbb{R}^n\,, \forall\, \lambda \in \mathbb{R}, \lambda\vec{v} \in \mathbb{R}^n$ (Closure under scalar multiplication)
3. $\vec{0} \in \mathbb{R}^n$ (Inclusion of the zero vector)

If you are unaware, the symbol $\forall$ means 'for all' and the symbold $\in$ means 'contained in'. So, property (1) would be read 'for all vectors v and u in $\mathbb{R}^n$, their sum is also in $\mathbb{R}^n$'.
Now these properties that $\mathbb{R}^n$ has make it special. In fact because of these three properties we call $\mathbb{R}^n$ a **vector space**.

Subspaces of $\mathbb{R}^n$

Consider a subset of $\mathbb{R}^n$, W. W is a subspace of $\mathbb{R}^n$ if

1. $\forall\, \vec{v}, \vec{u} \in W, \vec{u} + \vec{v} \in W$ (Closure under addition)
2. $\forall\, \vec{v} \in W\,, \forall\, \lambda \in \mathbb{R}, \lambda\vec{v} \in W$ (Closure under scalar multiplication)
3. $\vec{0} \in W$ (Inclusion of the zero vector)

It's important to note that the *trivial* subspaces of $\mathbb{R}^n$ are $\mathbb{R}^n$ itself and the zero subspace: $\left\{\vec{0}\right\}$

Example 4.1.1

If $V \subseteq \mathbb{R}^2 = \left\{\vec{v} \in \mathbb{R}^2 \mid \vec{v} \text{ is a unit vector}\right\}$ (read 'V is a subset of $\mathbb{R}^2$ such that V is the set of all vectors in $\mathbb{R}^2$ that are unit vectors'), is V a subspace of $\mathbb{R}^2$?

Solution: If we were to write out V we would have

$$V = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

as these are the only unit vectors in $\mathbb{R}^2$. From this it should be very clear that $\vec{0} \notin V \therefore V$ is **not** a subspace of $\mathbb{R}^2$.

Example 4.1.2

If $V \subseteq \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\}$, is V a subspace of $\mathbb{R}^2$?

Solution: Consider the parameterization $x_2 = t$ and $x_1 = 1 - t$. Then

$$X = \begin{pmatrix} 1 - t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Once again, from this parameterization it should be easy to see that $\vec{0} \notin V \therefore V$ is **not** a subspace of $\mathbb{R}^2$.

Example 4.1.3

Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \right\}$. Is V a subspace of $\mathbb{R}^3$?

Solution: Just like before consider the parameterization $x_2 = s, x_3 = t, x_1 = -2s+t$. This gives

$$X = \begin{pmatrix} -2s+t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

From this we can see that all three properties are satisfied thus V **is** a subspace of $\mathbb{R}^2$.

You may notice that the above example invovles an equality to the Span of a set of vectors, despite said span not being mentioned once. Well, it is included because of the following theorem:

Theorem 4.1. *The span of a set of vectors from $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.*

Proof. Consider two vectors $\vec{v}, \vec{u} \in \mathbb{R}^n$ and the span, $\text{Span}\{x_1, x_2, \cdots, x_k\}$. Then $\vec{v}$ and $\vec{u}$ can be expressed as a linear combination as follows:

$$\begin{aligned} \vec{u} &= c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_k\vec{x_k} \\ \vec{v} &= \tilde{c_1}\vec{x_1} + \tilde{c_2}\vec{x_2} + \cdots + \tilde{c_k}\vec{x_k} \\ \Rightarrow \vec{u} + \vec{v} &= (c_1 + \tilde{c_1})\,\vec{x_1} + (c_2 + \tilde{c_2})\,\vec{x_2} + \cdots + (c_k + \tilde{c_k})\,\vec{x_k} \end{aligned}$$

Then, $\vec{u}, \vec{v}, u \vec{+} v \in \text{Span}\{x_1, x_2, \cdots x_k\}$ by the definition of span. $\square$

Example 4.1.4

Consider the set of all polynomials of degree $3, P_3$. Is P_3 a vector space?

Solution: Let P be any polynomial of degree 3. Then

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

We can have $P(x) = 0$ if $\forall x\, a_0 = a_1 = a_2 = a_3 = 0$, therefore $\vec{0} \in P_3$.
Now consider another polynomial Q of degree three:

$$Q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$$

Then,

$$R(x) = P(x) + Q(x) = (a_0 + b_0) + (a_1 + b_1)\,x + (a_2 + b_2)\,x^2 + (a_3 + b_3)\,x^3$$

But, R is still a degree 3 polynomial, so $R \in P_3$, and thus P_3 is closed under addition. We can also consider any scalar $\lambda \in \mathbb{R}$. Then

$$Q(x) = \lambda\, P(x) = \lambda\, a_0 + \lambda\, a_1x + \lambda\, a_2x^2 + \lambda\, a_3x^3$$

Again, $Q \in P_3$ because Q is still a degree 3 polynomial. Thus P_3 **is** a vector space.

One interesting thing comes when we consider vectors in $\mathbb{R}^4$ and P_3. Notice that the general form for a vector in $\mathbb{R}^4$ is

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Now, consider the general (vector) form of any polynomial in P_3:

$$\begin{pmatrix}a_0\\a_1x\\a_2x^2\\a_3x^3\end{pmatrix} = a_0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + a_1\begin{pmatrix}0\\x\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\0\\x^2\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\0\\x^3\end{pmatrix}$$

Now, hopefully, you notice the extremely similarities in the structures of these two vector spaces. In fact, the structure of each of these vector spaces is considered identical in mathematics and $\mathbb{R}^4$ is call **isomorphic** to P_3. We can show this by defining a bijective map $\phi : \mathbb{R}^4 \to P_3$. Obviously, in this case, ϕ is simply the direct mapping of each corresponding vector.
Now, if we consider these vectors *without* their coefficients, what we get is actually called the **standard basis vectors** since

$$\mathbb{R}^4 = \text{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

$$P_3 = \text{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\x\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\x^2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\x^3\end{pmatrix}\right\}$$

Example 4.1.5: C

nsider W such that $W = \text{Span}\{\vec{u}, \vec{v}\}$. Find $\vec{u}$ and $\vec{v}$.

$$W = \begin{pmatrix}5b+2c\\b\\b\end{pmatrix} = \text{Span}\{\vec{u}, \vec{v}\}$$

Solution:

$$W = b\begin{pmatrix}5\\1\\0\end{pmatrix} + c\begin{pmatrix}2\\0\\1\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}5\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\} \Rightarrow \vec{u} = \begin{pmatrix}5\\1\\0\end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix}2\\0\\1\end{pmatrix}$$

We can as well notice that the above span was a subspace of $\mathbb{R}^3$ since it's span was of a set of vectors contained in $\mathbb{R}^3$.

Example 4.1.6

Consider v_1, v_2, v_3 and W as below.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \quad W = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

(a) Is $W \in \operatorname{Span}\{v_1, v_2, v_3\}$?

Solution: Consider $(A \mid W)$ where $A = \left((\vec{v_1}) \quad (\vec{v_2}) \quad (\vec{v_3})\right)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right) \xrightarrow{R_3 \to R_3 + 5R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, the system is consistent so $W \in \operatorname{Span}\{v_1, v_2, v_3\}$

(b) Is $\{v_1, v_2, v_3\}$ linearly independent?

Solution: From the above part we know that $v_3 = 2v_2$ $\therefore$ the set is linearly **depdendent**.

Basis of a Subspace

The basis of a subspace W of $\mathbb{R}^n$ is any set of linearly independent vectors such that its span is W.

4.2 Lecture 15: Column, Row, and Null Spaces [2.9/4.2]

Theorem 4.2. *The span of any non-empty set of vectors from $\mathbb{R}^n$ is a suhspace of $\mathbb{R}^n$*

The Basis of a Subspace

Let $\mathcal{W}$ be a subspace of any given vector space. The **basis** of $\mathcal{W}$ is the smallest generating set and the biggest linear independent set who's span is $\mathcal{W}$. The **dimension** of $\mathcal{W}$ is the number of vectors in a basis.

Rank

Let A be a matrix. Then $\operatorname{rank}(A) = n$ if, and only if, A_{REF} has n pivots and there are n nonzero rows in A_{REF}

Nullity

The **nullity** of A is $\operatorname{nul}(A) = n - \operatorname{rank}(A)$

With these definitions in mind, we introduce the following difference subspaces that are associated with a matrix.

Consider an $m \times n$ matrix A. Then we have the following subspaces

- **Column Space**: The column space of A is defined as

$$\mathrm{Col}\,(A) = \mathrm{Span}\,\{\text{columns of } A\}$$

  and is a subspace of $\mathbb{R}^m$

- **Row Space**: The row space of A is defined as

$$\mathrm{Row}\,(A) = \mathrm{Span}\,\{\text{rows of } A\}$$

  and is a subspace of $\mathbb{R}^n$

- **Null Space**: The null space of A is defined as

$$\mathrm{Nul}\,(A) = \mathrm{Span}\,\{\text{set of all solution of } A\vec{x} = 0\}$$

  and is a subspace of $\mathbb{R}^n$

- **Null Space of A^T**: The null space of A^T is defined as

$$\mathrm{Nul}\left(A^T\right) = \mathrm{Span}\left\{\text{set of all solutions of } A^T\vec{y} = 0\right\}$$

  and is a subspace of $\mathbb{R}^m$

It's important to notice the following equalities

$$\mathrm{Col}\,(A) = \mathrm{Row}\left(A^T\right) \text{ and } \mathrm{Row}\,(A) = \mathrm{Col}\left(A^T\right)$$

as these will lead directly into the fundamental equations of linear algebra we will define at the end of this section.

In order to find the basis of a column space of A, we'll start by finding A_{REF}. Once we have A_{REF}, any column without a pivot corresponds to a redudant vector in the basis.

Example 4.2.1

Consider the matrix A below. Find $\mathcal{W} = \mathrm{Col}\,(A)$ and a basis of $\mathcal{W}$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 1 & 3 & 2 \\ -1 & 5 & -2 \end{pmatrix}$$

Solution: We can easily find

$$\mathrm{Col}\,(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\-2\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\3\\5\end{pmatrix}, \begin{pmatrix}2\\-4\\2\\-2\end{pmatrix}\right\}$$

In order to find the basis of $\mathcal{W}$ we must find A_{REF}.

$$A \xrightarrow[\substack{R_3\to R_3-R_1\\R_4\to R_4+R_1}]{R_2\to R_2+2R_1} \begin{pmatrix}-1 & -1 & 2\\0 & 0 & 0\\0 & 4 & 0\\0 & 4 & 0\end{pmatrix} \xrightarrow[R_3\to R_3-R_4]{R_2\leftrightarrow R_4} \begin{pmatrix}1 & -1 & 2\\0 & 4 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} = A_{\text{REF}}$$

A_{REF} has pivots in the first and second column, but no pivot in the third column. Thus, the third vector is redudant and we can choose a basis to be

$$\left\{\begin{pmatrix}1\\-2\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\3\\5\end{pmatrix}\right\}$$

In order to find a basis of a row space of A, we'll start by finding A_{REF}. Each nonzero column in $(A_{\text{REF}})^T$ is a vector in the basis.

Example 4.2.2

Consider the matrix A below. Find $\mathcal{W} = \mathrm{Row}\,(A)$ and a basis of $\mathcal{W}$.

$$A = \begin{pmatrix}1 & -1 & 2\\-2 & 2 & -4\\1 & 3 & 2\\-1 & 5 & -2\end{pmatrix}$$

Solution: We can easily find

$$\text{Row}\,(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix} \right\} = \mathcal{W}$$

Now, in order to detmine a basis of $\mathcal{W}$ we have to find A_{REF}. But recall that $\text{Row}\,(A) = \text{Col}\left(A^T\right)$, so we can simply take A_{REF} from (4.2) and transpose it:

$$(A_{\text{REF}})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

we can easily see the nonzero columns and find a basis to be

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \right\}$$

Example 4.2.3

Consider the matrix A below. Find $\mathcal{W} = \text{Nul}\,(A)$ and a basis of $\mathcal{W}$.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 1 & 3 & 2 \\ -1 & 5 & -2 \end{pmatrix}$$

Solution: We know that we can find solutions to $A\vec{x} = 0$ by analyzing A_{REF}. Recall that

$$A_{\text{REF}} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

No pivot in the third column tells us that $x_3 = t$ is a free variable. We can also see that $x_2 = 0$ and that $x_1 = -2x_3 = -2t$. Thus,

$$X = \begin{pmatrix} -2t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Thus,

$$\text{Nul}\,(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Because the generating set is only a single vector, it is also the basis for $\text{Nul}\,(A)$.

Example 4.2.4

Consider the matrix A below. Find $\mathcal{W} = \text{Nul}\left(A^T\right)$ and a basis of $\mathcal{W}$.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 1 & 3 & 2 \\ -1 & 5 & -2 \end{pmatrix}$$

Solution: We must find all the solutions to $A^T\vec{y} = 0$ which means we need to find $\left(A^T\right)_{\text{REF}}$

$$A^T = \begin{pmatrix} 1 & -2 & 1 & -1 \\ -1 & 2 & 3 & 5 \\ 2 & -4 & 2 & -2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No pivot in the second and fourth columns tells us that $y_2 = t$ and $y_4 = s$ are free variables. Solving the system we get

$$Y = \begin{pmatrix} 2s + 2t \\ t \\ -s \\ s \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Thus,

$$\text{Nul}\left(A^T\right) = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

One thing you should notice that is the number of vectors in a basis of a subspace also corresponds to the number of nonzero rows in A_{REF}. Connecting this with the definitions of dimension and rank we can see that

$$\dim\left[\text{Row}\left(A\right)\right] = \text{rank}\left(A\right) = \dim\left[\text{Col}\left(A^T\right)\right] \tag{1}$$

Now if we recall from earlier that

$$\text{Col}\left(A\right) = \text{Row}\left(A^T\right)$$

and

$$\dim\left[\text{Col}\left(A\right)\right] = \text{rank}\left(A\right) = \dim\left[\text{Row}\left(A\right)\right]$$

we get

$$\text{rank}\left(A\right) = \text{rank}\left(A^T\right) \tag{2}$$

which is the *first* fundamental equation of linear alegbra.

We now have the following two facts

$$\dim\left[\text{Nul}\left(A\right)\right] = n - \text{rank}\left(A\right) \text{ and } \dim\left[\text{nul}\left(A^T\right)\right] = m - \text{rank}\left(A\right)$$

Combining these with equation (1) we get

$$\dim\left[\text{Row}\left(A\right)\right] + \dim\left[\text{Nul}\left(A\right)\right] = n \tag{3}$$

$$\dim\left[\text{Col}\left(A\right)\right] + \dim\left[\text{Nul}\left(A^T\right)\right] = m \tag{4}$$

Combining all of these we get

Fundamental Equations of Linear Algebra

Below are the three fundamental equations of linear algebra

$$\operatorname{rank}(A) = \operatorname{rank}\left(A^T\right) \tag{5}$$
$$\dim[\operatorname{Row}(A)] + \dim[\operatorname{Nul}(A)] = n \tag{6}$$
$$\dim[\operatorname{Col}(A)] + \dim\left[\operatorname{Nul}\left(A^T\right)\right] = m \tag{7}$$

Example 4.2.5

Consider the matrix A below. Find $\operatorname{Nul}(A)$.

$$\underset{3\times 5}{A} = \begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Notice that $A = A_{\text{REF}}$. With this observation we can get that $\operatorname{rank}(A) = 2$, $\operatorname{nul}(A) = m - \operatorname{rank}(A) = 5 - 2 = 3$, $\operatorname{Nul}(A) \subset \mathbb{R}^5$, and $\dim[\operatorname{Nul}(A)] = 3$. Now from A_{REF} we can see that $x_3 = r, x_4 = s, x_5 = t$ are free variables. Thus,

$$X = r\begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \operatorname{Nul}(A) = \operatorname{Span}\left\{\begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

And, once again, because the generating set is composed of the solutions to $A\vec{x} = 0$, we know that it is linearly independent and thus forms a basis.

4.3 Lecture 16: Linearly Independent Sets, Basis [4.3]

Linear Independence

A set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is **linearly independent** if the equation

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_u\vec{v_k}, \quad c_i \in \mathbb{R}$$

is satisfied if, and only if, $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, the set is **linearly depdendent**.

We can determine if a set is linearly independent by considering the equation $AC = 0$ where

$$\underset{n\times k}{A} = \begin{pmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_k} \\ | & | & & | \end{pmatrix} \text{ and } \underset{k\times 1}{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Then,

1. if $\operatorname{rank}(A) = k$, then $AC = 0$ has only the trivial solution $C = 0$, so the set is linearly *independent*.
2. if $\operatorname{rank}(A) < k$, then $AC = 0$ has an infite amount of solutions so the set is linearly *depdendent*.

Span $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ is a subspace of $\mathbb{R}^n$ if $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k} \in \mathbb{R}^n$

Basis of a Subspace

The basis of a subspace is the smallest linearly independent generating set who's span is the subspace.

Example 4.3.1

Find a basis of $\mathcal{W}$

$$\mathcal{W} = \text{Span}\left\{\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$$

Solution: Consider the matrix who's columns are the vectors in the generating set of $\mathcal{W}$:

$$A = \begin{pmatrix}1 & 1 & 2 & 1\\-2 & 1 & 1 & 0\\3 & -1 & 3 & 2\end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \begin{pmatrix}1 & 1 & 2 & 1\\0 & 3 & 5 & 2\\0 & -4 & -3 & -1\end{pmatrix} \xrightarrow{R_3 \to 3R_3 + 4R_2} \begin{pmatrix}1 & 1 & 2 & 1\\0 & 3 & 5 & 2\\0 & 0 & 11 & 5\end{pmatrix}$$

Now, since there isn't a pivot in the fourth column, the fourth vector in the generating set is redudant. Thus, one possible basis for $\mathcal{W}$ is

$$\left\{\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}\right\}$$

You may also note that $\mathcal{W}$ from the above example is a subspace of $\mathbb{R}^3$ and so Span $\{W\} = \mathbb{R}^3$.

Standard Basis of $\mathbb{R}^n$

The **standard basis** of $\mathbb{R}^n$ is the set

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

where $\mathbf{e}_k$ is the k-th column of I_n (the identity matrix in $\mathbb{R}^n$).

Example 4.3.2

Consider

$$\mathcal{S} = \left\{1 - x^2 + 2x^3, 1 + x, x^2, x^4\right\}$$

(a) Determine if $\mathcal{S}$ is linearly independent.

Solution: By definition of linear independence we must have

$$\begin{aligned} &c_1\left(1-x^2+2x^3\right)+c_2\left(1+x\right)+c_3\left(x^2\right)+c_4\left(x^4\right)=0\\ \Rightarrow &\left(c_1+c_2\right)+\left(c_2\right)x+\left(-c_1+c_3\right)x^2+\left(c_1\right)x^3+\left(c_4\right)x^4=0\\ \Rightarrow &c_1=c_2=c_3=c_4=0 \end{aligned}$$

Thus, $\mathcal{S}$ *is* linearly independent.

(b) Determine if $\text{Span}\left\{\mathcal{S}\right\}=\mathbf{P}_4$

Solution: Since there are only four vectors, $\text{Span}\left\{\mathcal{S}\right\}\neq\mathbb{P}_4$.

Our answer in part b of the above example follows directly from the fact that $\mathbb{P}_n\cong\mathbb{R}^{n+1}$ ($\mathbb{P}_n$ is *isomorphic* to $\mathbb{R}^{n+1}$), and in order for a set of vectors in $\mathbb{R}^n$ to span $\mathbb{R}^n$, there must be exactly n vectors in the generating set. Now, because of this fact, we actually have an easier way to solve part a, rather than splitting up each term by degrees.

We can notice that the standard basis for $\mathbb{P}_4$ is

$$\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\x\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\x^2\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\x^3\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\\x^4\end{pmatrix}\right\}$$

since any polynomial in $\mathbb{P}_4$ can be made up as a linear combination of these vectors. If we modify this basis slightly to get

$$\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},x\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},x^2\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},x^3\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},x^4\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\right\}$$

then we can see how we can represent $\mathcal{S}$ in the following manner:

$$\mathcal{S}=\left\{\begin{pmatrix}1\\0\\-1\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}.$$

Now, consider the matrix who's columns are the vectors from $\mathcal{S}$:

$$A=\begin{pmatrix}1&1&0&0\\0&1&0&0\\-1&0&1&0\\2&0&0&0\\0&0&0&1\end{pmatrix}\xrightarrow[R_4\to R_4-2R_1]{R_3\to R_3+R_1}\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&1&1&0\\0&-2&0&0\\0&0&0&1\end{pmatrix}\xrightarrow[R_4\to R_4+2R_2]{R_3\to R_3-R_2}\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&2&0\\0&0&0&1\end{pmatrix}\xrightarrow[R_4\leftrightarrow R_5]{R_4\to R_4-2R_3}\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}.$$

So, $\text{rank}\left(A\right)=4=k$ so $\mathcal{S}$ is linearly independent.

4.4 Lecture 17: Coordinates in Vector Spaces [4.4]

Theorem 4.3 (Uniqueness Theorem)**.** *Consider a basis* $\left\{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\right\}$ *for* $\mathbb{R}^n$ *and let* $\vec{v} \in \mathbb{R}^n$*. Then if*

$$\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + \cdots + c_n\vec{b_n}, \tag{8}$$

$c_1, c_2, \ldots, c_n$ *is a unique set of numbers for* $\vec{v}$*.*

Proof. Suppose $\vec{v} \in \mathbb{R}^n$ and $\mathbb{R}^n$ has a basis $\left\{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\right\}$. Now, assume that there are two different representations of $\vec{v}$:

$$\begin{aligned}\vec{v} &= c_1\vec{b_1} + c_2\vec{b_2} + \cdots + c_n\vec{b_n}\\ \vec{v} &= \tilde{c_1}\vec{b_1} + \tilde{c_2}\vec{b_2} + \cdots + \tilde{c_n}\vec{b_n}.\end{aligned}$$

Then,

$$0 = (c_1 - \tilde{c_1})\,\vec{b_1} + (c_2 - \tilde{c_2})\,\vec{b_2} + \cdots + (c_n - \tilde{c_n})\,\vec{b_n}.$$

However, this is only possile if, and only if, $c_i = \tilde{c_i}$ for $0 \le i \le n$. Thus, $c_1, c_2, \ldots, c_n$ is unique. $\square$

Consider equation (8) in the form

$$\vec{v} = \hat{B}C \text{ where } \hat{B} = \begin{pmatrix} | & | & & | \\ \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \\ | & | & & | \end{pmatrix} \text{ and } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

We now re-label C to be $[v]_B$ and call it the **vector coordinate of** $\vec{v}$ **in** B. Then, since $\hat{B}$ is invertible,

$$\vec{v} = \hat{B}\,[v]_B \Rightarrow [v]_B = \hat{B}^{-1}\vec{v}$$

Example 4.4.1

Find the vector coordinate of $\vec{v}$ in $\hat{B}$ if

$$\hat{B} = \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right\} \text{ and } \vec{v} = \begin{pmatrix}1\\-4\\4\end{pmatrix}$$

Solution: We can find $\left(\hat{B}\,|\,\vec{v}\right)$ and use it to solve $\hat{B}\,[\vec{v}]_B = \vec{v}$.

$$\left(\hat{B} \mid \vec{v}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -4 \\ 1 & 1 & 2 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right) = \left(\hat{B} \mid \vec{v}\right)_{\text{REF}}.$$

Using this we can solve the system

$$\left\{\begin{array}{ll} c_3 & = 3 \\ -2c_2 + c_3 & = -5 \Rightarrow c_2 = 4 \\ c_1 + c_2 + c_3 & = 1 \Rightarrow c_1 = -6 \end{array}\right] \Rightarrow [\vec{v}]_B = \begin{pmatrix}-6\\4\\3\end{pmatrix}$$

Example 4.4.2

Consider the set $\hat{B}$ below as a basis for $\mathbb{P}_2$. Find teh coordinates of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\hat{B}$.

$$\hat{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Solution: This time around, we will find $\hat{B}^{-1}$ and use it to solve for $[\vec{v}]_B$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \hat{B}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Thus,

$$[\vec{v}]_B = \hat{B}^{-1}\vec{v} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$$

4.5 Lecture 18: The Dimension of a Vector Space and Change of Basis [4.5-4.6]

Theorem 4.4. *If a vector space V has a basis $\mathcal{B} = \left\{ \vec{b_1}, \ldots, \vec{b_n} \right\}$, then any set in V containing more than n vecotrs must be linearly dependent.*

Theorem 4.5. *If a vector V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

If a vector space V is spanned by a finite set set, then V is said to be have <u>finite dimension</u>, and the <u>dimension</u> of V, written as $\dim V$ is the number of vectors in a basis for V. If V is not spanned by a finite set, then V is said to be <u>infinite-dimensional</u>.

The dimension of the zero vector space, $\left\{\vec{0}\right\}$ is defined to be 0. The standard basis for $\mathbb{R}^n$ has n vectors, so $\dim\left[\mathbb{R}^n\right] = n$. The standard basis for $\mathbb{P}_n$ has $n+1$ vectors, so $\dim\left[\mathbb{P}_2\right] = n+1$.

Theorem 4.6. *Let H be a subspace of a finite dimensional vector space V. Any linearly indepedent set in H can be expanded to a basis for H. Also, H is finite dimensional and* $\dim H \leq \dim V$.

Theorem 4.7. *Let V be a p-dimensional vector space with $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.*

The rank of an $m \times n$ matrix A, $\operatorname{rank}(A)$, is the dimension of the column space. The nullity of A, $\operatorname{nul}(A)$, is the dimension of the null space.

The nullity of A is also equal to the number of free variables in the homogenous system $A\vec{x} = 0$.

$\operatorname{rank}(A) + \operatorname{nul}(A) =$ the number of columns in A.

From these above theorems and facts we can extend our understanding of invertible matricies.

Theorem 4.8. *Let A be an $n \times n$ matrix. If A is invertible then,*

1. *$Col(A)$ is a basis of $\mathbb{R}^n$*
2. *$Col(A) = \mathbb{R}^n$*
3. *$rank(A) = n$*
4. *$nul(A) = 0$*
5. *$Nul(A) = \{0\}$*

Change of Coordinates

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ be basis of a vector space V. Then, there exists a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\vec{x}]_{\mathcal{B}}\,.$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}}\,. \end{pmatrix}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Example 4.5.1

Consider the bases $\mathcal{B}$ and $\mathcal{C}$ below. Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

$$\mathcal{B} = \left\{\begin{pmatrix} -9 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ -1 \end{pmatrix}\right\} \text{ and } \mathcal{C} = \left\{\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}\right\}$$

Solution: Consider the vectors in $\mathcal{B}$ to be b_1 and b_2 and similarly for $\mathcal{C}$ to be c_1 and c_2. We must find $[b_1]_{\mathcal{C}}$ and $[b_2]_{\mathcal{C}}$. If we let $[b_1]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $[b_2]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we should have the following relationship:

$$b_1 = c_1x_1 + c_2x_2 = \begin{pmatrix} (c_1) & (c_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$b_2 = c_1y_1 + c_2y_2 = \begin{pmatrix} (c_1) & (c_2) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Now, we can solve both these systems simulatneously by row reducing the matrix

$$\left((c_1) \quad (c_2) \;\middle|\; (b_1) \quad (b_2) \right).$$

We do this as follows

$$\begin{aligned}
\left((c_1) \quad (c_2) \;\middle|\; (b_1) \quad (b_2) \right) &= \left(\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right) \\
&\xrightarrow{R_2 \to R_2 + 4R_1} \left(\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{array}\right) \\
&\xrightarrow{R_2 \to \frac{1}{7}R_2} \left(\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 1 & -5 & -3 \end{array}\right) \\
&\xrightarrow{R_1 \to R_1 - 3R_2} \left(\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right).
\end{aligned}$$

Thus,

$$[b_1]_{\mathcal{C}} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} \text{ and } [b_2]_{\mathcal{C}} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

So,

$$\boxed{\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}}$$

The method used here holds in most cases. In general we have

$$\left((c_1) \quad (c_2) \;\middle|\; (b_1) \quad (b_2) \right) \sim \left(I \;\middle|\; \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \right)$$

Example 4.5.2

Consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ below.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\} \text{ and } \mathcal{C} = \left\{ \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \end{pmatrix} \right\}$$

(a) Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$

Solution: Consider

$$\left((b_1) \quad (b_2) \;\middle|\; (c_1) \quad (c_2)\right).$$

Then,

$$\begin{aligned}
&\left(\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right) \\
&\xrightarrow{R_2 \to R_2 + 3R_1} \left(\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right) \\
&\xrightarrow[R_1 \to R_1 + R_2]{R_2 \to -\frac{1}{2} R_2} \left(\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 0 & 6 & 4 \end{array}\right).
\end{aligned}$$

Thus,

$$\boxed{\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}}$$

(b) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

Solution: Before we start consider the relationship

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [b_1]_{\mathcal{B}} = [b_1]_{\mathcal{C}}$$

and multiply by $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ and notice that we now have

$$[b_1]_{\mathcal{B}} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} [b_1]_{\mathcal{C}}.$$

Therefore, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$! We can apply this property to our problem to see

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$$