MATH250 NOTES

Contents

1 Unit 1

1.1 Systems of Linear Equations

A system of linear equations is when you have more than 1 equation with more than 1 unknown.

Example 1.1.1

Solve the following system:

$$\begin{cases} x_1 + 2x_2 & = 5 \\ 2x_1 + x_2 & = 4 \end{cases}$$

$$\begin{aligned} x_1 + 2x_2 = 5 &\Rightarrow x_1 = 5 - 2x_2 \\ 2x_1 + x_2 = 4 &\Rightarrow 2(5 - 2x_2) + x_2 = 4 \Rightarrow 10 - 3x_2 = 4 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Note that the solution to the previous system $(x_1, x_2) = (1, 2)$ also corresponds to the point of intersection of the lines that each equation represents. This then implies that non-parallel lines have a single solution, paralell lines have no solutions, and scalar multiples of the same line have infinitely many solutions.

Example 1.1.2: Systems with Different Number of Solutions

1. One Solution:
$$\begin{cases} x_1 + 2x_2 & = 5 \\ 2x_1 + x_2 & = 4 \end{cases}$$
2. No solution:
$$\begin{cases} x_1 + 2x_2 & = 5 \\ 2x_1 + 4x_2 & = 6 \end{cases}$$
3. Infinite Solutions
$$\begin{cases} x_1 + 2x_2 & = 5 \\ 2x_1 + 4x_2 & = 10 \end{cases}$$

Elementary Row Operations

Consider the system

$$\begin{cases} E_1 \\ E_2 \\ \vdots \\ E_n \end{cases}$$

Where E_i is the i-th equation. Then the following operations on the system **do not change the solution**

1. Changing the order of equations: $E_i \leftrightarrow E_j$ where $i \neq j$ for the i-th and j-th equation
2. Scaling equations: $E_i \to \lambda E_i$ where $\lambda \in \mathbb{R}\,(\lambda \neq 0)$
3. Combining equations: $E_i \to E_i + \lambda E_j$ where $i \neq j$ and $\lambda \in \mathbb{R}\,(\lambda \neq 0)$

Example 1.1.3

Solve the following system:

$$\begin{cases} E_1 & = x_1 + 2x_2 = 5 \\ E_2 & = 2x_1 + x_2 = 4 \end{cases}$$

$$\begin{aligned} -2E_1 &= -2x_1 - 4x_2 = -10 \\ E_2 - 2E_1 &= (2x_1 - x_2) - 2x_1 - 4x_2 = 4 - 10 \\ &\Rightarrow -3x_2 = -6 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Matricies

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix A has size $m \times n$ if it has m **rows** and n **columns**. A matrix is written as

$$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Where each a_{ij} is called an element of the matrix. i denotes the row and n denotes the column of the element.

With this in mind, note that we can represent the previous system as a matrix:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix}$$

where each row represents an equation and each element corresponds to either the coefficent of a variable or the solution. With this system in matrix form we can manipulate the rows as follows:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -3 & -6 \end{pmatrix}$$

Since each row corresponds to an equation, if we take a look at the bottom row we can see

$$-3x_2 = -6 \Rightarrow x_2 = 2 \Rightarrow x_1 = 1$$

Thus allowing us to reach the same final answer with a lot less hassle.

General Form of a System of Linear Equations

Consider any system of linear equations in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = b_m \end{cases}$$

Now, notice we can rewrite this as a matrix:

$$A = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Everything to the **left** of the solid line is refered to as the *coefficent matrix*. A itself is referred to as the **augmented matrix** for the system.

Row Echelon Form

Given a matrix A, A_{REF} is an equivalent matrix that satisfies the following properites:

1. All zero rows are below non-zero rows
2. each next leading element is in the column to the right of the previous leading element (called pivots)

Note that the **leading element** of a row is simply the first non-zero element in that row.

A matrix can also be put in RREF (reduced row echelon form) if it is alreay in REF, each pivot is 1, and the only non-zero element in the pivot column is the pivot. This would be A_{RREF}. Now, lets try to combine all we've done by applying basic ERO to a simple matrix to put it in REF.

Example 1.1.4: Basic REF

Consider the system

$$\begin{cases} x_1 + 2x_2 & = 5 \\ 2x_1 + x_2 = 4 \end{cases}$$

It's augmented matrix can be put into REF as follows

$$(A \mid B) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 4 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) = (A \mid B)_{\text{REF}}$$

Example 1.1.5: Basic RREF

Consider again the system from ex (1.1.4) and notice we can put it into RREF as follows:

$$(A \mid B)_{\text{REF}} = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{3}R_2]{} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow[R_1 \to R_1 - 2R_2]{} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) = (A \mid B)_{\text{RREF}}$$

It's important to note that the RREF for any given matrix is *unique*.

Example 1.1.6

$$\begin{cases} x_1 + 2x_2 - x_3 & = 2 \\ 2x_1 + x_2 + x_3 & = 1 \\ x_1 - x_2 + 2x_3 & = -1 \end{cases} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$