#23 [2.3]: Can a square matrix with identical columns be invertible? Why or why not?

Solution: Any square matrix with at least two identical columns **cannot** be invertible. This is because if two columns are identical then there exist non-trivial solutions to the system $A\vec{x} = \vec{0}$: Let

$$A = \begin{pmatrix} (\vec{a_1}) & (a_2) & \cdots & (a_n) \end{pmatrix}$$

be an $n \times n$ matrix. Now assume that two columns are identical or, $\vec{a_i} = \vec{a_j}$ for $i \neq j$. Then for the linear combination

$$c_1\vec{a_1} + c_2\vec{a_2} + \cdots + c_n\vec{a_n} = \vec{0}$$

we have the solution $c_i = 1, c_j = -1$ and $c_k = 0\,(\forall k \neq i, j)$. Thus, there exists a non-trivial solution, and so the columns of A are linearly dependent and thus A is **not** invertible.

#41 [2.3]: Let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation as defined below. Show that $\mathcal{T}$ is invertible and find a formula for $\mathcal{T}^{-1}$.

$$\mathcal{T}(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$$

Solution: We can begin by finding the matrix associated with $\mathcal{T}$,

$$A = \begin{pmatrix} (\mathcal{T}(E_1)) & (\mathcal{T}(E_2)) \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}$$

Then, $\det(A) = -1 \neq 0$ so A is invertible and thus $\mathcal{T}$ is invertible. Let $\mathcal{S}(\vec{x}) = \mathcal{T}^{-1}(\vec{x})$. Then,

$$\mathcal{S}(\vec{x}) = A^{-1}\vec{x} = \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix} \vec{x}$$

Note that we were able to quickly find A^{-1} using the formula for 2×2 matricies:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

#3 [3.1]: Compute the determinant for the given matrix below using a cofactor expansion across the first row. Then, compute the determinant by a cofactor expansion down the second column.

$$\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix}$$

Solution: Along the first row we have:

$$\Delta = 2\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 2(-1-6) + 2(-3-2) + 3(9-1) = \boxed{0}$$

Along the second column we have:

$$\Delta = (-1)^{1+2}(-2)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + (-1)^{2+2}(1)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + (-1)^{3+2}(3)\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 2(-3-2) + 1(-2-3) - 3(4-9) = \boxed{0}$$

#13 [3.1]: Compute the determinant for the given matrix below by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

Solution:

$$\Delta = (-1)^{3+2}(3) \begin{vmatrix} 4 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & -1 & 2 \end{vmatrix} = -3(-1)^{2+2}(2) \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} = -6\left(4(1) - 3(10) + (-5)(-5)\right) = \boxed{6}$$