

MATH250 NOTES

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1 Unit 1

1.1 Lecture 1: Systems of Linear Equations [1.1-1.2]

A system of linear equations is when you have more than 1 equation with more than 1 unknown.

Example 1.1.1

Solve the following system:

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + x_2 = 4 \end{cases}$$

Solution:

$$\begin{aligned} x_1 + 2x_2 = 5 &\Rightarrow x_1 = 5 - 2x_2 \\ 2x_1 + x_2 = 4 &\Rightarrow 2(5 - 2x_2) + x_2 = 4 \Rightarrow 10 - 3x_2 = 4 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Note that the solution to the previous system $(x_1, x_2) = (1, 2)$ also corresponds to the point of intersection of the lines that each equation represents. This then implies that non-parallel lines have a single solution, parallel lines have no solutions, and scalar multiples of the same line have infinitely many solutions.

Example 1.1.2: Systems with Different Number of Solutions

1. One Solution:

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + x_2 = 4 \end{cases}$$

2. No solution:

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + 4x_2 = 6 \end{cases}$$

3. Infinite Solutions

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + 4x_2 = 10 \end{cases}$$

Elementary Row Operations

Consider the system

$$\begin{cases} E_1 \\ E_2 \\ \vdots \\ E_n \end{cases}$$

Where E_i is the i -th equation. Then the following operations on the system **do not change the solution**

1. Changing the order of equations: $E_i \leftrightarrow E_j$ where $i \neq j$ for the i -th and j -th equation
2. Scaling equations: $E_i \rightarrow \lambda E_i$ where $\lambda \in \mathbb{R} (\lambda \neq 0)$
3. Combining equations: $E_i \rightarrow E_i + \lambda E_j$ where $i \neq j$ and $\lambda \in \mathbb{R} (\lambda \neq 0)$

Example 1.1.3

Solve the following system:

$$\begin{cases} E_1 &= x_1 + 2x_2 = 5 \\ E_2 &= 2x_1 + x_2 = 4 \end{cases}$$

Solution:

$$\begin{aligned} -2E_1 &= -2x_1 - 4x_2 = -10 \\ E_2 - 2E_1 &= (2x_1 - x_2) - 2x_1 - 4x_2 = 4 - 10 \\ &\Rightarrow -3x_2 = -6 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix A has size $m \times n$ if it has m **rows** and n **columns**. A matrix is written as

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Where each a_{ij} is called an element of the matrix. i denotes the row and n denotes the column of the element.

With this in mind, note that we can represent the previous system as a matrix:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix}$$

where each row represents an equation and each element corresponds to either the coefficient of a variable or the solution. With this system in matrix form we can manipulate the rows as follows:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -3 & -6 \end{pmatrix}$$

Since each row corresponds to an equation, if we take a look at the bottom row we can see

$$-3x_2 = -6 \Rightarrow x_2 = 2 \Rightarrow x_1 = 1$$

Thus allowing us to reach the same final answer with a lot less hassle.

General Form of a System of Linear Equations

Consider any system of linear equations in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}$$

Now, notice we can rewrite this as a matrix:

$$A = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

Everything to the **left** of the solid line is referred to as the *coefficient matrix*. A itself is referred to as the **augmented matrix** for the system.

Row Echelon Form

Given a matrix A , A_{REF} is an equivalent matrix that satisfies the following properties:

1. All zero rows are below non-zero rows
2. each next leading element is in the column to the right of the previous leading element (called pivots)

Note that the **leading element** of a row is simply the first non-zero element in that row.

A matrix can also be put in RREF (reduced row echelon form) if it is already in REF, each pivot is 1, and the only non-zero element in the pivot column is the pivot. This would be A_{RREF} . Now, let's try to combine all we've done by applying basic ERO to a simple matrix to put it in REF.

Example 1.1.4: Basic REF

Consider the system

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 &= 4 \end{cases}$$

Find $(A \mid B)$

Solution: It's augmented matrix can be put into REF as follows

$$(A \mid B) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 4 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array} \right) = (A \mid B)_{\text{REF}}$$

Example 1.1.5: Basic RREF

Consider again the system from ex (1.1.4) and find the RREF of the augmented matrix.

Solution: Notice we can put it into RREF as follows:

$$(A \mid B)_{\text{REF}} = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array} \right) \xrightarrow{R_2 \rightarrow -\frac{1}{3}R_2} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \right) = (A \mid B)_{\text{RREF}}$$

It's important to note that the RREF for any given matrix is *unique*.

Example 1.1.6: Solve the System

Solve the system

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = -1 \end{cases}$$

Solution:

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = -1 \end{cases} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array} \right) \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array} \right) \\ \xrightarrow{R_3 \rightarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Now because there is no pivot in the third column of our matrix, x_3 is what's known as a 'free variable'. All this means is that x_3 can take on any value, we'll notate this by $x_3 = t$ with $t \in \mathbb{R}$.

$$\Rightarrow \begin{cases} x_3 = t \\ -3x_2 + 3x_3 = -3 \\ x_1 + 2x_2 - x_3 = 2 \end{cases} \\ \Rightarrow X_{\text{gen}} = \begin{pmatrix} -t \\ 1+t \\ t \end{pmatrix}$$

1.2 Lecture 2: Vector and Matrix Equations [1.3-1.4]

As a reminder from last time, given a system with an augmented matrix $(A \mid B)$ the system will have

1. no solutions if there is a pivot in the last column of $(A \mid B)_{\text{REF}}$
2. one solution if $(A \mid B)_{\text{REF}}$ is $m \times n$ with n pivots (and no pivots in the last column)

3. infinite solutions if $(A \mid B)_{\text{REF}}$ is $m \times n$ with less than n pivots (and no pivots in the last column)

Vectors

For a vector \vec{AB} , point A is the tail and point B is the head. Two vectors, \vec{AB} and \vec{CD} are equal if, and only if, their magnitude and directions are equal. Vectors in \mathbb{R}^2 can be notated as $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ where the point (a_1, a_2) is the head and the tail is (usually) assumed to the origin. Generally, for \mathbb{R}^n we have $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ which will always have size $n \times 1$.

Since we can represent vectors as matrices, we can perform the standard matrix operations on them:

Matrix Operations

Let A and B be matrices. Then

1. $A + B = C$ where A and B are the same size. This addition is defined as $c_{ij} = a_{ij} + b_{ij}$
2. $\lambda A = \hat{A}$ where $\hat{a}_{ij} = \lambda a_{ij}$ for $\lambda \in \mathbb{R}$
3. $A + (-A) = \hat{0}$ where $\hat{0}$ is the zero-matrix which has the same size as A with each element being 0
4. Consider matrices $A_{m \times n}$ and $X_{n \times 1}$, then AX is defined as

$$A_{m \times 1} = \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n \\ \vdots & & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \dots & a_{mn}x_n \end{pmatrix} \text{ if } A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } X_{n \times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example 1.2.1: Basic Matrix Multiplication

Consider the the matrices

$$A_{2 \times 3} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ and } B_{3 \times 1} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

Find AB .

Solution:

$$AB_{2 \times 1} = \begin{pmatrix} (1 * 2) + (-1 * 4) + (2 * 7) \\ (3 * 2) + (4 * 4) + (5 * 7) \end{pmatrix} = \begin{pmatrix} 12 \\ 57 \end{pmatrix}$$

It's important to note that an alternative form of defining matrix multiplication is as follows:

$$AX = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

This is known as the **linear combination of vectors**.

Linear Combination of Vectors

Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ be a set of vectors from \mathbb{R}^n . Then,

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k, \quad (c_1, c_2, \dots, c_k \in \mathbb{R})$$

is called the **linear combination of vectors** $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$.

We can combine the previous two definitions of matrix multiplication and the linear combination of vectors to get this next fact: if we consider a vector

$$(\vec{u}_k) = \begin{pmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{pmatrix}$$

then, $c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k = AX$ where

$$A = ((\vec{u}_1) \quad (\vec{u}_2) \quad \dots \quad (\vec{u}_k)) \text{ and } X = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

This is especially important when working with systems of equations as we can represent systems as linear combinations of vectors. Given any general system we can rewrite it as a linear combination of vectors as such

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_1 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_1 \end{cases} \Leftrightarrow AX = B$$

B is said to be a *linear combination of columns of A* if, and only if, A and B are **compatible**. For A and B to be compatible essentially just means that $(A \mid B)_{\text{REF}}$ has no pivots in the last column.

Span

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ where c_1, c_2, \dots, c_k are all possible numbers

For a single vector \vec{v} , $\text{Span}\{\vec{v}\}$ is simply the set containing all scaled multiples of \vec{v} .

Example 1.2.2: Span of Two Vectors

Notice that $\text{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\} = \mathbb{R}^2$. This implies that any vector from \mathbb{R}^2 can be written as

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

We can prove this fact by considering the augmented matrix for $AX = B$:

$$\left(\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & 4 & b_2 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & -2 & b_2 - 2b_1 \end{array} \right)$$

Therefore, since there is no pivote in the last column, this system has a single solution for any vector

$$B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Example 1.2.3

Given the following vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and \vec{b} , determine if \vec{b} is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \vec{a}_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

We can start by noticing that $\vec{b} = c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 \Leftrightarrow A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ where

$A = ((\vec{a}_1) \ (\vec{a}_2) \ (\vec{a}_3))$. Thus,

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Therefore, since there is no pivot in the final column, B is a linear combination of A .

1.3 Lecture 3: Homogenous Systems of Linear Equations [1.5]**Homogenous Systems of Linear Equations**

A **homogenous system of linear equations** is any system in the form

$$Ax = \hat{0}$$

Thus, any system in the form

$$Ax = B$$

is called a **non-homogenous system**.

For a non-homogenous system we can follow the steps

$$Ax = B \Rightarrow (A \mid B) \xrightarrow{\text{ERO}} (A \mid B)_{\text{REF}} \Rightarrow \text{backwards substitution into the equations}$$

For each non-homogenous system:

1. if the augmented matrix has a pivot in the last column, it has no solutions
2. if A is $m \times n$ and the augmented matrix has less than n pivots, then $Ax = B$ has an infinite amount of solutions
3. if A is $m \times n$ and the augmented matrix has exactly n pivots then $Ax = B$ has one solution

For a homogenous system we can follow similar steps:

$$Ax = \hat{0} \Rightarrow (A \mid 0) \xrightarrow{\text{ERO}} A_{\text{REF}} \Rightarrow \text{backwards substitution to the equations}$$

Note that since the solution matrix is simply just the zero matrix, we don't need to consider it in our augmented matrix - that's why you see A_{REF} . Unlike non-homogenous systems there are only two options for solutions:

1. $x = \hat{0}$ is always a solution, this is called the *trivial solution*. If A_{REF} is $m \times n$ large and has n pivots, then $x = \hat{0}$ is the **only** solution
2. if A_{REF} is $m \times n$ and has less than n pivots, then $Ax = \hat{0}$ has an **infinite** amount of solutions

Example 1.3.1: Solve the following System

Solve the below system

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 0 \\ 2x_2 - x_2 + x_3 - 2x_4 = 0 \\ 3x_1 + x_2 - x_3 + x_4 = 0 \end{cases}$$

We can start by putting this in the form $Ax = \hat{0}$ where

$$A_{3 \times 4} = \begin{pmatrix} 1 & 2 & -3 & -1 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & -1 & 1 \end{pmatrix}, \quad X_{4 \times 1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Next, we need to put A into REF:

$$A_{3 \times 4} \xrightarrow[\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}]{\quad} \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & -5 & 7 & -4 \\ 0 & -5 & 8 & -2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & -5 & 7 & -4 \\ 0 & 0 & 1 & 2 \end{pmatrix} = A_{\text{REF}}$$

Since there is no pivot in the fourth column, x_4 must be a free variable. We can rename it as $x_4 = t$ for $t \in \mathbb{R}$. Thus

$$\begin{cases} x_3 + 2x_4 = 0 \Rightarrow x_3 = -2t \\ -5x_2 + 7x_3 - 4x_4 = 0 \Rightarrow x_2 = -\frac{18}{5}t \\ x_1 + 2x_2 - 3x_3 + x_4 = 0 \Rightarrow x_1 = \frac{1}{5}t \end{cases} \Rightarrow X = \begin{pmatrix} \frac{1}{5}t \\ -\frac{18}{5}t \\ -2t \\ t \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 \\ -18 \\ -10 \\ 5 \end{pmatrix} t$$

Notice that since t is any arbitrary value we can make $t = 5t$ in order to eliminate the fractions. As well, the second matrix with t on the outside is known as the *parametric form* of the solution.

Theorem 1.1. *The general solution of a consistent non-homogenous system $Ax = B$ has the form $X_{\text{gen}} = X_{\text{genhom}} + X_p$ where X_{genhom} is the general solution of the corresponding homogenous system $Ax = \hat{0}$ and X_p is any particular solution of $Ax = B$.*

Proof. Let X_1 and X_2 be any two solutions of $Ax = B$. That is,

$$\begin{cases} AX_1 = B \\ AX_2 = B \end{cases} \Rightarrow AX_1 - AX_2 = 0 \Leftrightarrow A(X_1 - X_2) = 0$$

Therefore, $X_1 - X_2$ is a solution of the homogenous system. With some clever ‘renaming’ we can see:

$$X_1 - X_2 = X_{\text{hom}} \Rightarrow X_1 = X_{\text{hom}} + X_2 \Rightarrow X_{\text{gen}} = X_{\text{genhom}} + X_p$$

□

Example 1.3.2: Solve the System

$$\begin{cases} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + x_2 - 2x_3 &= 2 \\ x_1 + x_2 - x_3 &= 2 \end{cases}$$

As always, we start by forming the augmented matrix and putting it into REF:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 2 & 1 & -2 & 2 \\ 1 & 1 & -1 & 2 \end{array} \right) &\xrightarrow[R_3 \rightarrow R_3 - R_2]{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -2 \end{array} \right) \xrightarrow[R_3 \rightarrow R_3 - \frac{1}{3}R_2]{R_2 \rightarrow -\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ &\xrightarrow{R_1 \rightarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{aligned}$$

Now, since there is no pivot in the third column, x_3 must be a free variable, so we have $x_3 = t$ for $t \in \mathbb{R}$. Using some very basic backwards substitution we can see that $x_2 = 2$ and $x_1 = x_3 = t$. As well, notice that the final form of the augmented matrix is actually in RREF and not REF, because of how easy it was to put the matrix in RREF from REF, there was almost no reason **not** to put it into RREF. We can now write our solution vector,

$$X_{\text{gen}} = \begin{pmatrix} t \\ 2 \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}_{X_p} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{X_{\text{genhom}}}$$

Example 1.3.3: Find the General Homogenous Solution

Just to prove the point, let's find the general homogenous solution for the above system, $Ax = \hat{0}$.

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_{\text{RREF}}$$

With backwards substitution we can easily see that $x_3 = t$ for $t \in \mathbb{R}$, $x_2 = 0$, and $x_1 = t$. So,

$$X_{\text{genhom}} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Which, as expected, is exactly the same vector we saw in the previous example.

Example 1.3.4: Solve the System

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 3x_1 - x_2 - x_3 &= 0 \end{cases}$$

Once again, we must put A into REF:

$$A \xrightarrow[\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}]{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 5 & -4 & 3 \\ 0 & 5 & -4 & 3 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 5 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, because there are no pivots in the third and fourth column, x_3 and x_4 are free variables which makes x_1 and x_2 our basic variables $\therefore x_3 = t, x_4 = s$ for $s, t \in \mathbb{R}$

$$\begin{cases} 5x_2 - 4x_3 + 3x_4 = 0 \Rightarrow x_2 = \frac{4}{5}x_3 - \frac{3}{5}x_4 = \frac{4}{5}t - \frac{3}{5}s \\ x_1 - 2x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = 2x_2 - x_3 + x_4 = 2(\frac{4}{5}t - \frac{3}{5}s) - t + s \end{cases}$$

Before continuing, it's smart to make $t = 5t$ and $s = 5s$ in order to eliminate the fractions. Then we have:

$$X_{\text{gen}} = \begin{pmatrix} 3t - s \\ 4t - 3s \\ 5t \\ 5s \end{pmatrix} = t \begin{pmatrix} 3 \\ 4 \\ 5 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \end{pmatrix}$$

1.4 Lecture 4: Span of a Set of Vectors [1.3]

Rank of a Matrix

Let A be a matrix. Then $\text{rank}(A)$ is the number of pivots in A_{REF} . $\text{rank}(A)$ is equivalent to the number of non-zero rows in A_{REF} .

It follows from the definition of $\text{rank}(A)$ that the number of free variables in a system is $n - \text{rank}(A)$. This quantity is known as $\text{null}(A)$.

If A is a matrix such that $\text{rank}(A) = n$, then $A_{\text{REF}} = I$ where I is the identity matrix.

Identity Matrix

The identity matrix, I , is a square matrix which is automatically in RREF and its only non-zero elements are the pivots. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Another important fact about the rank of matrix is that it can only be as great as the minimum of n and m for a matrix of size $m \times n$. Formally,

$$\text{rank} \left(\begin{matrix} A \\ m \times n \end{matrix} \right) \leq \min \{m, n\}$$

where $\min \{x_1, x_2, \dots, x_n\}$ returns the smallest value out of x_1, x_2, \dots, x_n .

Span of Vectors from \mathbb{R}^n

The span of a set of vectors from \mathbb{R}^n , $\text{Span} \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ is the set of all possible linear combinations of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$:

$$AC = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k, \quad (c_1, c_2, \dots, c_k \in \mathbb{R})$$

where

$$A = ((\vec{u}_1) \quad (\vec{u}_2) \quad \dots \quad (\vec{u}_k)) \text{ and } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

The span of the zero vector, is also the zero vector: $\text{Span}\{\vec{0}\} = \vec{0}$

Example 1.4.1: Span of a Single Vector

Consider a vector $u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then $\text{Span}\{\vec{u}_1\} = c\vec{u}_1$ for any $c \in \mathbb{R}$. Geometrically, this is the set of vectors in the same, or opposite, direction as \vec{u}_1 with a scaled magnitude. When put together, $\text{Span}\{\vec{u}_1\}$ gives a line in \mathbb{R}^2 .

Example 1.4.2: Span of Two Vectors

Consider $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, $\text{Span}\{\vec{u}_1, \vec{u}_2\} = \mathbb{R}^2$ because any vector from \mathbb{R}^2 can be written in the form

$$c_1\vec{u}_1 + c_2\vec{u}_2 = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Proof. Consider the system

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad {}_{2 \times 2} A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Then, the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 1 & b_2 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - 2b_1 \end{array} \right) = (A \mid B)_{\text{REF}}$$

\therefore since the augmented matrix in REF has no pivots in the final column, the system is compatible and contains a solution for all vectors $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}$. \square

Note as well, that we could've written the final justification using the rank of augmented matrix as reasoning: ' $\therefore \text{rank}(A) = 2$, so the system is compatible...'. As well, note that since A is a square matrix and $\text{rank}(A) = 2$, that also implies that $A_{\text{RREF}} = I_2$ (it should be pretty easy to put A into RREF and see this for yourself).

Example 1.4.3: Span of Three Vectors

Consider three vectors

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

and consider the span of \vec{u}_1, \vec{u}_2 and \vec{u}_3 , $\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Now notice that $u_2 = 2u_1$ and $u_3 = 3u_1$. This implies that $\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \text{Span}\{\vec{u}_1\}$. u_2 and u_3 are considered 'redundant' in the span.

Proof. Consider the matrix representing the vectors of the generation set:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

\therefore no pivots in the second and third columns implies that the second and third vectors are redundant. \square

Example 1.4.4: Span of Three Vectors (again)

Now consider $\vec{u}_1, \vec{u}_2, \vec{u}_3$ such that

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Notice that $\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ **must** be reducible since the corresponding matrix has dimensions $2 \times 3 \Rightarrow \text{rank} A \leq \min 2, 3 = 2$. \therefore (at least) one vector is redundant. We can figure out which one as follows

$$A_{2 \times 3} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -6 \end{pmatrix}$$

Thus, no pivot in the third column implies that the third vector is redundant:

$$\therefore \text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \text{Span}\{\vec{u}_1, \vec{u}_2\}$$

Example 1.4.5: Proving a Vector is in a Span

Show that $\begin{pmatrix} 4 \\ 6 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$. We can start with the system

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -6 \end{pmatrix}$$

This gives

$$\begin{cases} -5c_2 = 6 \\ c_1 + 2c_2 = 4 \end{cases} \Rightarrow c_2 = -\frac{6}{5} \text{ and } c_1 = \frac{8}{5}$$

Note that we knew that $\begin{pmatrix} 4 \\ 6 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ since the system was compatible, we just found the values of c_1 and c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Example 1.4.6: Geometric Interpretation of \mathbb{R}^3 Span

Consider vectors \vec{u}_1, \vec{u}_2 and give a geometric interpretation of their span, $\text{Span}\{\vec{u}_1, \vec{u}_2\}$

$$\vec{u}_1 = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 12 \\ 3 \\ -9 \end{pmatrix}$$

Start by noticing that $\vec{u}_2 = \frac{3}{2}\vec{u}_1 \therefore u_1 \parallel u_2 \Rightarrow \text{Span}\{\vec{u}_1, \vec{u}_2\}$ must be a line in \mathbb{R}^3 along \vec{u}_1 .

Example 1.4.7: Geometric Interpretation of \mathbb{R}^3 Span (again)

Consider vectors \vec{u}_1, \vec{u}_2 and give a geometric interpretation of their span, $\text{Span}\{\vec{u}_1, \vec{u}_2\}$

$$\vec{u}_1 = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 12 \\ 3 \\ 9 \end{pmatrix}$$

Since $\vec{u}_1 \nparallel \vec{u}_2$, $\text{Span}\{\vec{u}_1, \vec{u}_2\}$ makes a plane in \mathbb{R}^3 (spanned by \vec{u}_1, \vec{u}_2)

Example 1.4.8

Let A and b be matrices such that

$$A_{3 \times 3} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix} = ((\vec{u}_1) \quad (\vec{u}_2) \quad (\vec{u}_3)) \text{ and } b = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$$

Now consider $\mathcal{W} = \text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

(a) Is $b \in \mathcal{W}$?

Solution: Start by assuming that $c \in \mathcal{W} \therefore c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 = b \Leftrightarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 + 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array} \right)$$

Thus, $\text{rank}(A_{\text{REF}}) = 3 \therefore$ the system is compatible and $b \in \mathcal{W}$

(b) Is $\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ reducible?

Solution:

$$A_{3 \times 3} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_1} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ 0 & 6 & -5 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

$\text{rank}(A_{\text{REF}}) = 3 \therefore$ the span is irreducible.

(c) Is $\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solution: Yes, they're both \mathbb{R}^3

1.5 Lecture 5: Linear Dependence and Independence [1.7]

Linear Independence and Dependence

A set of vectors from \mathbb{R}^n , $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$, is **linearly independent** if

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k = 0$$

is satisfied if, and only if, $c_1 = c_2 = \dots = c_k = 0$. Otherwise the set is called **linearly dependent**.

Now, notice that the above equation is actually equivalent to the matrix equation $AC = 0$ where

$$A_{n \times k} = ((\vec{u}_1) \ (\vec{u}_2) \ \dots \ (\vec{u}_k)) \text{ and } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

It should be evident that we can now view this as a homogenous system of linear equations. Thus, if $\text{rank}(A_{\text{REF}}) = k$ the only solution to the homogenous system is the trivial solution, $\vec{0}$. We can also define linear dependence this way: if $\text{rank}(A_{\text{REF}}) < k$, the system must have non trivial solutions and the set of vectors must be linearly dependent.

Example 1.5.1

Consider the set of vectors below and determine if it is linearly independent or dependent:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Solution: This problem is actually fairly easy and requires zero calculations. Consider the matrix formed from these vectors, A . Notice that A must have dimensions 3×4 and recall that for any $m \times n$ matrix, the rank of said matrix is always less than or equal to $\min m, n$. Applying this to A , we can see that $\text{rank}(A_{\text{REF}}) \leq \min 3, 4 = 3$. Therefore, since there can be only a max of 3 pivots, at least one column will not have a pivot, so the system will have non-trivial solutions. This implies that the set of vectors is linearly **dependent**.

A set of vectors that is linearly dependent contains at least one vector that can be represented as a linear combination of the other vectors in the set. This should also remind you of sets of vectors that have reducible spans.

Example 1.5.2

Consider the set of vectors in the above example, and determine which vector(s) are redundant.

Solution:

$$A_{3 \times 4} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 4 & 1 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 4R_1]{R_2 \rightarrow R_2 + 2R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -11 & -5 & -2 \end{pmatrix} \xrightarrow[R_3 \rightarrow \frac{9}{7}R_3]{R_3 \rightarrow R_3 + \frac{11}{7}R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 1 & * \end{pmatrix}$$

\therefore no pivot in the fourth column implies that the fourth vector is redundant.

In the above example, notice that we really didn't even need to calculate the element a_{34} since it's value has no effect on whether or not the fourth column will have a pivot or not.

Example 1.5.3

Consider the set of vectors below and find a value of h such that the set is linearly dependent

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 10 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ -7 \\ h \end{pmatrix} \right\}$$

Solution:

$$A_{3 \times 3} = \begin{pmatrix} 1 & 3 & 2 \\ -3 & 10 & -6 \\ 2 & -6 & h \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 + 3R_1} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & h - 4 \end{pmatrix}$$

Thus, in order for the set to be linearly dependent, one column must *not* have a pivot. This will happen if $h - 4 = 0 \Rightarrow h = 4$.

Example 1.5.4

Determine the column vectors of A are linearly independent or dependent where $A =$

$$\begin{pmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 5 \\ 5 & 4 & 6 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{pmatrix} \xrightarrow[R_4 \rightarrow R_4 - 5R_1]{R_3 \rightarrow R_3 + 4R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 4 \\ 0 & -3 & 20 \\ 0 & 4 & -9 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 3R_2]{R_4 \rightarrow R_4 + 4R_2, R_3 \leftrightarrow R - 4} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

\therefore if $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are the column vectors, then they are linearly independent because $\text{rank}(A_{REF}) = 3$

Example 1.5.5

Determine the value of h such that the vectors are linearly dependent

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ h \\ -9 \end{pmatrix} \right\} \Rightarrow A_{3 \times 3} = \begin{pmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + 3R_1]{R_2 \rightarrow R_2 - 5R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & h - 15 \\ 0 & 0 & 0 \end{pmatrix}$$

\therefore since there is no pivot in the third column, the vectors will be linearly independent $\forall h \in \mathbb{R}$.

1.6 Lecture 5: Linear Transformations [1.8]

Transformations

A linear transformation $\mathcal{T} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by the map $A \cdot \vec{x} \mapsto \vec{b}$ is a function that can be thought of as finding all vectors $\vec{x} \in \mathbb{R}^n$ that turn into $\vec{b} \in \mathbb{R}^m$ when multiplied by A . \mathbb{R}^n is called the **domain** and \mathbb{R}^m is called the **co-domain**.

Example 1.6.1

Consider a linear map $\mathcal{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $A\vec{x} \mapsto \vec{b}$. For the given matrix A and vectors \vec{u} and \vec{v} , find $\mathcal{T}(\vec{u})$ and $\mathcal{T}(\vec{v})$.

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Since $\mathcal{T}(\vec{x}) = A\vec{x}$, we have

$$\mathcal{T}(\vec{u}) = A\vec{u} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathcal{T}(\vec{v}) = A\vec{v} = \begin{pmatrix} 5a \\ \frac{1}{2}a \\ \frac{1}{2}c \end{pmatrix}$$

Linear Transformations

A transformation, $\mathcal{T}(\vec{x})$, is **linear**, if

$$\mathcal{T}(a\vec{u} + b\vec{v}) = a\mathcal{T}(\vec{u}) + b\mathcal{T}(\vec{v}) \Leftrightarrow A(a\vec{X} + b\vec{Y}) = aA\vec{X} + bA\vec{Y}$$

Note that not all transformations are linear transformations, there is a distinct difference!

Basic Linear Transformations

- Dilation Transformation: $\mathcal{T}(\vec{u}) = r\vec{u}$ for $r \in \mathbb{R}$
- 90° Transformation: $\mathcal{T}(\vec{u}) = A\vec{u} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u}$
- Shear Transformation: $\mathcal{T}(\vec{u}) = A\vec{u} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \vec{u}$

Note that for the **shear** transformation, the y -coordinate (j) of \vec{u} stays the same, only the x -coordinate (i) is stretched.

Example 1.6.2: Mapping to the Zero Vector

Consider the following matrix A and find all vectors \vec{x} such that they get mapped to the zero vector when multiplied by A :

$$A = \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{pmatrix}_{3 \times 4}$$

We can start by noticing that if we define $\mathcal{T} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ such that $\mathcal{T}(\vec{x}) = A\vec{x}$, then we just need to

solve $\mathcal{T}(\vec{x}) = \vec{0} \Rightarrow A\vec{x} = \vec{0}$:

$$A \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -5 & 6 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, since there is no pivot in the third and forth columns, $x_3 = s$ and $x_4 = t$ are free variables (for $s, t, \in \mathbb{R}$). Backwards substitution into the equations gives:

$$\begin{cases} x_2 - 4x_3 + 3x_4 = 0 \Rightarrow x_2 = 4s - 3t \\ x_1 - 4x_2 + 7x_3 - 5x_4 = 0 \Rightarrow x_1 = 9s - 7t \end{cases} \Rightarrow X = s \begin{pmatrix} 9 \\ 4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Example 1.6.3

Consider \mathcal{T} given by $A\vec{X} \mapsto \vec{B}$ and determine if $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in \mathcal{T}^{\text{img}}$, where A is the same matrix from the above example. Notice that all this questions is really asking is if the system $A\vec{x} = \vec{b}$ is consistent. We can figure this out fairly easily:

$$(A \mid b) = \left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{array} \right) \xrightarrow{\text{ERO}} \left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

\therefore the system is consistent so $b \in \mathcal{T}^{\text{img}}$

1.7 Lecture 7: The Matrix of a Linear Transformation [1.9]

Recall that a function $\mathcal{T}(X) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a linear transformation if

1. $\mathcal{T}(X + Y) = \mathcal{T}(X) + \mathcal{T}(Y)$, and
2. $\mathcal{T}(kX) = k\mathcal{T}(X)$ for $k \in \mathbb{R}$

It's important to notice the similarities to the properties of matrix multiplication:

1. $A_{m \times n} \begin{pmatrix} X \\ Y \end{pmatrix}_{\substack{n \times 1 \\ n \times 1}} = AX + AY$
2. $A_{m \times n} \begin{pmatrix} kX \\ \end{pmatrix}_{n \times 1} = kAX$

Theorem 1.2. Let $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then, there exists a matrix, $A_{m \times n}$ such that

$$\mathcal{T}(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$

Proof. For any matrix X ,

$$X = I_n X = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

Then,

$$\begin{aligned} \mathcal{T}(\vec{x}) &= \mathcal{T}(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= \mathcal{T}(x_1 e_1) + \mathcal{T}(x_2 e_2) + \cdots + \mathcal{T}(x_n e_n) \\ &= x_1 \mathcal{T}(e_1) + x_2 \mathcal{T}(e_2) + \cdots + x_n \mathcal{T}(e_n) \\ &= ([\mathcal{T}(e_1)] \quad [\mathcal{T}(e_2)] \cdots [\mathcal{T}(e_n)]) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= A\vec{x} \end{aligned}$$

□

Note that I_n is the identity matrix which is an $n \times n$ matrix in which every element is a 0 except for elements, $a_{11}, a_{22}, \dots, a_{nn}$. Note as well, that e_i is the i -th column of the identity matrix. For example, consider I_2 and I_3 :

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then e_1 for I_2 would be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Similarly, e_3 for I_3 would be $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

One-to-one and Onto

Consider a linear transformation $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by the map $\vec{x} \mapsto \begin{pmatrix} A \\ m \times n \end{pmatrix} \vec{x}$. Then, \mathcal{T} is called **onto** when there exists at least one vector $\vec{x} \in \mathbb{R}^n$ for *any* vector $\vec{B} \in \mathbb{R}^m$. \mathcal{T} is called **one-to-one** when $A\vec{x} = \vec{B}$ has only one solution.

One way to think about it is that \mathcal{T} is **onto** if any vector in \mathbb{R}^m can be ‘made’ by applying the transformation to a vector \vec{x} . \mathcal{T} is **one-to-one** if for any solution to the equation $A\vec{x} = \vec{B}$, the matrix \vec{x} is the sole, unique, solution to that equation. Note as well, that a transformation can be both one-to-one and onto.

Determining if a transformation is one-to-one or onto

Consider a linear transformation $\mathcal{T}(\vec{x}) = A\vec{x}$ for a matrix $A_{m \times n}$.

- If $\text{rank}(A) = m$ (implying that $m \leq n$), then each row in A contains a pivot which ensures that the augmented matrix $(A \mid B)_{\text{REF}}$ will have no pivots in the final column. In other words, the system $A\vec{x} = \vec{B}$ will be consistent for all $\vec{B} \in \mathbb{R}^m$ and \mathcal{T} is **onto**.
- If $\text{rank}(A) = n$ (implying that $n \leq m$), then each column in A contains a pivot which ensures that there are no free variables in the system $A\vec{x} = \vec{B}$. In other words, any solution to $A\vec{x} = \vec{B}$ will be unique and \mathcal{T} is **one-to-one**.

Example 1.7.1: Determine if a Transformation is One-to-one or Onto

Consider the transformation $\mathcal{T} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 2x_1 + x_2 - x_3 + x_4 \\ x_1 + 3x_2 + 5x_3 - 2x_4 \\ 3x_1 - x_2 + 2x_3 + 3x_4 \end{pmatrix}$$

determine if \mathcal{T} is one-to-one, onto, both, or neither.

Solution: We can start by rewriting \mathcal{T} in the form $\mathcal{T}(\vec{x}) = A\vec{x}$:

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A\vec{x}$$

Since A is a 3×4 matrix, $\text{rank}(A) \leq 3 \therefore \mathcal{T}$ cannot be one-to-one. Now, as per usual, in order to determine the rank of A we must find A_{REF} . We can start with $A \xrightarrow{R_1 \leftrightarrow R_2}$

$$\begin{pmatrix} 1 & 3 & 5 & -2 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 2 & 3 \end{pmatrix} \xrightarrow[\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & -5 & -11 & 5 \\ 0 & -10 & -13 & 9 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & -5 & -11 & 5 \\ 0 & 0 & 9 & -1 \end{pmatrix}$$

Thus, $\text{rank}(A) = 3 = n$ so \mathcal{T} is **onto**

Rotation Transformations

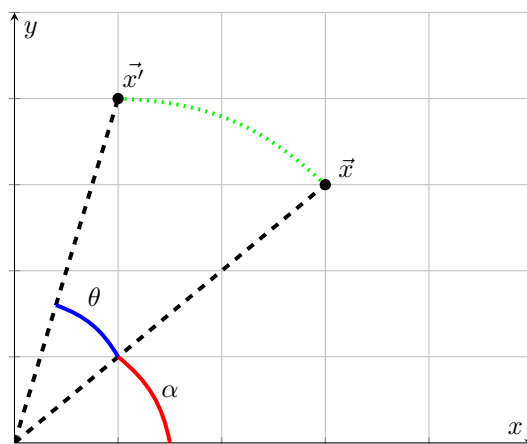
Consider vectors \vec{x} which has an angle α from the x -axis and \vec{x}' which has an angle $\alpha + \theta$ from the x -axis. There exists a transformation \mathcal{T} such that $\mathcal{T}(\vec{x}) = \vec{x}'$. Just like all linear transformations, \mathcal{T} can be written in the form $\mathcal{T}(\vec{x}) = A\vec{x}$ where

$$A = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{\text{Rotation Matrix}}$$

Thus, if $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ then,

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

If you're curious where the definition of the rotation matrix comes from, consider the two vectors $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$:



We can represent \vec{x} in polar coordinates as

$$x = r \cos \alpha \text{ and } y = r \sin \alpha$$

Then,

$$x' = r \cos(\alpha + \theta) \text{ and } y' = r \sin(\alpha + \theta)$$

This gives,

$$\begin{aligned} x' &= r \cos(\alpha + \theta) = \overbrace{r \cos \alpha}^x \cos \theta - \overbrace{r \sin \alpha}^y \sin \theta = x \cos \theta - y \sin \theta \\ y' &= \overbrace{r \sin \alpha}^y \cos \theta + \overbrace{r \cos \alpha}^x \sin \theta = x \sin \theta + y \cos \theta \\ \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Example 1.7.2: Prove \mathcal{T} is a Linear Transformation

Consider a transformation \mathcal{T} such that

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix}$$

determine if \mathcal{T} is a linear transformation.

Solution: Notice that $\mathcal{T}(\vec{x}) = A\vec{x}$ for

$$A_{4 \times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } \vec{x}_{4 \times 1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Because \mathcal{T} can be put in the form $\mathcal{T}(\vec{x}) = A\vec{x}$, it must be a linear transformation.

Example 1.7.3: Determine if \mathcal{T} is One-to-one or Onto

Consider the linear transformation \mathcal{T} from the above example. Determine if \mathcal{T} is one-to-one, onto, both, or neither.

Solution:

$$A_{4 \times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $\text{rank}(A) \neq 4 = m = n$ so \mathcal{T} is neither one-to-one nor onto.

Example 1.7.4: Find the Matrix A

Given that \mathcal{T} is a linear transformation such that

$$\mathcal{T}(\vec{x}) = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - 3x_2 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{pmatrix}$$

determine the value of the matrix A .

Solution: We can find A as the coefficient matrix of \mathcal{T} :

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Example 1.7.5: Determine if \mathcal{T} is One-to-one or Onto

Consider the linear transformation $\mathcal{T}(\vec{x}) = A\vec{x}$ such that A is the matrix in the above example. Determine if \mathcal{T} is one-to-one, onto, both, or neither.

Solution: We can easily figure this out by looking at the rank of A_{REF} .

$$A \xrightarrow[R_2 \leftrightarrow R_1]{R_2 \rightarrow \frac{1}{4}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[R_2 \rightarrow -\frac{1}{3}R_2]{R_3 \rightarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $\text{rank}(A_{\text{REF}}) = 3 = m = n$ so \mathcal{T} is both one-to-one and onto.

2 Unit 2

2.1 Lecture 8: Matrix Operations [2.1]

Before I start, I must note that I was not able to physically make it to lecture 8, so these notes are simply just a brief overview of concepts covered in lecture.

So far, we've very briefly been introduced to the idea of matrix multiplication through our studies of systems of equation in the form $A\vec{x} = \vec{b}$ which we know can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

in order to represent the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_n \end{cases}$$

You'll notice that in order to go from the matrix to the system we simply take each row of the matrix (or each coefficient) and multiply it by the corresponding variable. This is exactly how matrix multiplication is defined.

Matrix Multiplication

Consider two matrices $A_{m \times n}$ and $B_{n \times k}$ such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ \vdots & & \ddots & \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}$$

Label the columns in B as $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$. Then their product, AB , is defined as follows:

$$AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{pmatrix}_{m \times k}$$

It's extremely important to note that matrix multiplication is only defined two matrices, A and B , if, and only if, the number of columns in A is equal to the number of rows in B .

One other way to think of matrix multiplication that if A and B are defined as above then,

$$AB = \begin{pmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \cdots & \sum_{i=1}^n a_{1i}b_{ik} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{i2} & \cdots & \sum_{i=1}^n a_{2i}b_{ik} \\ \vdots & & \ddots & \\ \sum_{i=1}^n a_{mi}b_{i1} & \sum_{i=1}^n a_{mi}b_{i2} & \cdots & \sum_{i=1}^n a_{mi}b_{ik} \end{pmatrix}$$

Now, granted, this looks very complicated and extremely hard to use. So, why did I bother including this here? Well, if you look closely at each summation, what you'll notice is that each summation is simply the sum of the product of a corresponding **row** from A with a column from B . So, for example, the first element in AB is the sum of each element in the **first row** in A multiplied by its corresponding element in the **first column** in B . The second element in AB is the **first row** in A multiplied by the **second column** in B . This is probably best explained with a few examples:

Example 2.1.1: Basic Matrix Multiplication

Let A and B be defined as below. Determine if AB is defined, and if it is find it.

$$A_{2 \times 2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B_{2 \times 2} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

Solution: First, since A and B are both 2×2 matrices, we know that AB must exist since A and B , obviously, have the same number of rows and columns as each other. We can then find AB :

$$AB = \begin{pmatrix} (1 \cdot 2) + (2 \cdot 1) & (1 \cdot 4) + (2 \cdot 3) \\ (3 \cdot 2) + (4 \cdot 1) & (3 \cdot 4) + (4 \cdot 3) \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 10 & 24 \end{pmatrix}$$

Keep in mind that we also could've found the product AB using the other definition of matrix multiplication:

$$AB = (A\mathbf{b}_1 \quad A\mathbf{b}_2) = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right) = \left(\begin{pmatrix} 4 \\ 10 \end{pmatrix} \quad \begin{pmatrix} 10 \\ 24 \end{pmatrix} \right) = \begin{pmatrix} 4 & 10 \\ 10 & 24 \end{pmatrix}$$

Example 2.1.2: Basic Matrix Multiplication

Let A and B be defined as below. Determine if AB is defined, and if it is find it.

$$A_{3 \times 2} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{pmatrix} \text{ and } B_{2 \times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Solution: Since A has 2 columns and B has 2 rows, we know that AB exists. Thus,

$$AB = \begin{pmatrix} (1 \cdot 1) + (3 \cdot 0) & (1 \cdot 1) + (3 \cdot 1) \\ (5 \cdot 1) + (7 \cdot 0) & (5 \cdot 1) + (7 \cdot 1) \\ (9 \cdot 1) + (11 \cdot 0) & (9 \cdot 1) + (11 \cdot 1) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 5 & 12 \\ 9 & 20 \end{pmatrix}$$

Theorem 2.1 (Properties of Matrix Multiplication). Consider a matrix $A_{m \times n}$ and let B and C be matrices such that the necessary products are defined.

1. *Associative Law:* $A(BC) = (AB)C$
2. *Left Distributive Law:* $A(B + C) = AB + AC$
3. *Right Distributive Law:* $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ for any $r \in \mathbb{R}$
5. *Identity for Matrix Multiplication:* $I_m A = A = A I_n$

Recall that I_n is the $n \times n$ matrix such that every element is zero except for $a_{11}, a_{22}, \dots, a_{nn}$, referred to as the **Identity Matrix**.

Its important to note a few things about the above theorem.

1. There is NO commutative law, thus, you cannot assume $AB = BA$
2. There is NO cancelation law: $AB = BC$ does NOT imply that $A = C$
3. If AB is the zero matrix, you CANNOT conclude that either $A = 0$ or $B = 0$

The Power of a Matrix

Let A be any $n \times n$ matrix. Then A^k is defined as

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

If $k = 0$ then $A^k = I_n$ where I_n is the identity matrix in \mathbb{R}^n .

The definition of the power of a matrix should feel extremely similar to the definition of a real number to any power: repeated multiplication.

Transpose of a Matrix

Let A be any $m \times n$ matrix. Then A^T is the **transpose** of A , which is a $n \times m$ matrix who's rows are made from the columns of A and whose columns are made from the rows.

For example, let A and B be defined as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then,

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Theorem 2.2 (Properties of the Transpose). *Let A and B be matrices such that necessary sums and products are defined. Then,*

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ for any $r \in \mathbb{R}$
4. $(AB)^T = B^T A^T$

2.2 Lecture 9: Inverse of a Matrix and Characterizations of Invertible Matrices [2.2-2.3]

Invertible Matrices

An $n \times n$, A , matrix is called invertible if there exists an $n \times n$ matrix, B , such that $AB = BA = I_n$. Such B is called the inverse of A , denoted A^{-1} . A^{-1} is always a unique matrix for any A .

Theorem 2.3. An $n \times n$ matrix is invertible if, and only if, $\text{rank}(A) = n$

Proof. Consider a matrix $A_{n \times n}$ such that $\text{rank}(A_{\text{RREF}}) = n$. Then $A_{\text{RREF}} = I_n$. Assume it takes k iterations of elementary row operations (ERO) to obtain A_{RREF} from A . Then, $A_{\text{RREF}} = I_n = E_k \cdots E_2 E_1$. So, $I = BA \Leftrightarrow A^{-1} = B$ and $B^{-1} = A$ and $B^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

If A^{-1} exists then consider $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$. Thus, the system has only one solution and so $\text{rank}(A) = n$. \square

If you're confused on why $\vec{x} = A^{-1}\vec{b}$ implies that the system only has a single unique solution, consider that if $\vec{x} = A^{-1}\vec{b}$ then we have

$$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}$$

so, $A^{-1}\vec{b}$ must be a solution. To show that it is unique, consider any solution \vec{u} . Then,

$$A\vec{u} = \vec{b} \Rightarrow A^{-1}A\vec{u} = A^{-1}\vec{b} \Rightarrow I\vec{u} = A^{-1}\vec{b} \Rightarrow \vec{u} = A^{-1}\vec{b}$$

Then, we know from 1.3, that if a system has a single solution, its augmented matrix (or A_{REF}) has a rank equal to n . It should follow from this that if A^{-1} exists the homogenous system, $A\vec{x} = \vec{0}$ has a single solution, the trivial solution: $\vec{x} = \vec{0}$.

How to Find the Inverse of a 2×2 Matrix

Consider 2×2 matrices A, B with rank 2 such that $AB = I_2$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Since $\text{rank}(A) = 2$, A is invertible. Then,

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives,

$$\begin{aligned} \begin{cases} a\alpha + b\gamma = 1 \\ c\alpha + d\gamma = 0 \end{cases} &\Rightarrow \left(\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array} \right) \xrightarrow{\text{ERO}} \left(\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & \gamma \end{array} \right)_{\text{RREF}} \\ \begin{cases} a\beta + b\delta = 0 \\ c\beta + d\delta = 1 \end{cases} &\Rightarrow \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array} \right) \xrightarrow{\text{ERO}} \left(\begin{array}{cc|c} 1 & 0 & \beta \\ 0 & 1 & \delta \end{array} \right)_{\text{RREF}} \end{aligned}$$

We can then combine to form one large matrix

$$(A \mid I) \xrightarrow{\text{ERO}} (I \mid A^{-1}) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \alpha & \beta \\ 0 & 1 & \gamma & \delta \end{array} \right)$$

Note that this method can actually be generalized such that for any matrix $A_{n \times n}$, augmenting it with the identity matrix and using ERO to find $(A \mid I)_{\text{RREF}}$ will always result in $(I \mid A^{-1})$

Example 2.2.1

Find the inverse of A and verify that $AA^{-1} = I$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Solution:

$$\begin{aligned} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array} \right) &\xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array} \right) \xrightarrow{R_2 \rightarrow -\frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array} \right) \\ &\xrightarrow{R_1 \rightarrow R_1 - 3R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array} \right) \\ &\Rightarrow A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

Then,

$$AA^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Now, you might have noticed that in the previous example, A^{-1} actually looked extremely similar to A . In fact, A and A^{-1} are made up elements of the exact same magnitude. This phenomenon will occur for the inverses of every 2×2 matrix with rank 2 and, in fact, we have a formula that explains the similarity.

Formula for Finding the Inverse of a 2×2 matrix

Let A be a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then,

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where the quantity $ad - bc$ is called the **determinant of A** , known as $\det A$.

It should follow from the last line that $\det A \neq 0 \Leftrightarrow A^{-1}$ exists.

Example 2.2.2

For A below, does A^{-1} exist? If so, find A^{-1}

$$A = \begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix}$$

Solution: Start by finding $\det A$:

$$\det A = (-4 \cdot -9) - (4 \cdot 9) = 36 - 36 = 0$$

Thus, by the fact above, A^{-1} must not exist. Or, in other words, A is not invertible.

Example 2.2.3

For A below, does A^{-1} exist? If so, find A^{-1} .

$$A = \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{pmatrix}$$

Solution: Start by finding the rank of A :

$$A \xrightarrow{R_3 \rightarrow R_3 + 3R_1} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 3R_2} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $\text{rank}(A) < 3$ so A is not invertible.

Example 2.2.4

For A below, does A^{-1} exist? If so, find A^{-1} .

$$A = \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 5 \\ -3 & 6 & 0 \end{pmatrix}$$

Solution: Start by finding the rank of A :

$$A \xrightarrow{R_3 \rightarrow R_3 + 3R_1} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 5 \\ 0 & -9 & -12 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 3R_2} \begin{pmatrix} 1 & -5 & -4 \\ 0 & 3 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

Therefore, $\text{rank}(A) = 3$ so A is invertible. Now, consider

$$\begin{aligned} (A \mid I_3) &= \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 + 3R_1} \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 0 & -9 & -12 & 3 & 0 & 1 \end{array} \right) \\ &\xrightarrow{R_3 \rightarrow R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 0 & 0 & 3 & 3 & 3 & 1 \end{array} \right) \xrightarrow{\substack{R_2 \rightarrow \frac{1}{3}R_2 \\ R_3 \rightarrow \frac{1}{3}R_3}} \left(\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array} \right) \\ &\xrightarrow{\substack{R_2 \rightarrow R_2 - \frac{5}{3}R_3 \\ R_1 \rightarrow R_1 + 5R_3}} \left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 5 & 4 & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{4}{3} & -\frac{5}{9} \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 + 5R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{10}{3} & -\frac{8}{3} & -\frac{13}{9} \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{4}{3} & -\frac{5}{9} \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array} \right) \\ &\Rightarrow A^{-1} = \begin{pmatrix} -\frac{10}{3} & -\frac{8}{3} & -\frac{13}{9} \\ -\frac{5}{3} & -\frac{4}{3} & -\frac{5}{9} \\ 1 & 1 & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Example 2.2.5: Is A invertible?

Let A be the matrix below. Determine if A is invertible.

$$A = \begin{pmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{pmatrix}$$

Solution: Since A has a 0 column, $\text{rank}(A) < 3 \therefore A$ is not invertible

Example 2.2.6: Is A invertible?

Let A be the matrix below. Determine if A is invertible.

$$A = \begin{pmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

Solution:

$$A \xrightarrow[\substack{R_3 \rightarrow R_3 - 2R_1 \\ R_2 \rightarrow -\frac{1}{4}R_2}]{R_2 \rightarrow R_2 + 3R_1} \begin{pmatrix} -1 & -3 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{R_4 \rightarrow R_4 + R_2} \begin{pmatrix} -1 & -3 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, $\text{rank}(A) = 4 \therefore A$ is invertible.