

#5 [5.3]: The matrix  $A$  below is factored in the form  $PDP^{-1}$ . Use the Diagonalization Theorem to find the eigenvalues of  $A$  and a basis for each eigenspace.

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}$$

**Solution:** By the diagonalization theorem we know that the entries across the main diagonal of  $D$  are the eigenvalues of  $A$ . We also know that columns of  $P$  form the corresponding eigenvectors to the eigenvalues in  $D$ . Thus,

Eigenvalues: 5, 1, 1

and,

$\text{Basis for } \lambda = 5 : \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ Basis for } \lambda = 1 : \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

#19 [5.3]: Diagonalize the matrix below

$$\begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

**Solution:** Let  $A$  be the matrix above. Since  $A$  is an upper triangular matrix, we know that the eigenvalues of  $A$  are the entries along the main diagonal. Thus,  $\lambda = 5, 3, 2, 2$ . We must find the eigenvectors for each eigenvalue. For  $\lambda = 5$ ,

$$A - 5I = \begin{pmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \Rightarrow \begin{cases} x_4 = 0 \\ x_3 = 0 \\ x_2 = 0 \\ x_1 = t \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For  $\lambda = 3$ ,

$$\begin{pmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} x_4 = 0 \\ x_3 = 0 \\ x_2 = 2t \\ x_1 = 3t \end{cases} \Rightarrow \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

For  $\lambda = 2$ ,

$$\begin{pmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_4 = s \\ x_3 = t \\ x_2 = 2s - t \\ x_1 = -s - t \end{cases} \Rightarrow \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Thus,

$P = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

#11 [6.1]: Find a unit vector in the direction of the given vector below

$$\begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix}$$

**Solution:** Let  $\vec{x}$  be the vector above. Then

$$\|\vec{x}\| = \sqrt{\frac{49}{16} + \frac{1}{4} + 1} = \frac{\sqrt{69}}{4} \Rightarrow \frac{\vec{x}}{\|\vec{x}\|} = \boxed{\begin{pmatrix} 7/\sqrt{69} \\ 2/\sqrt{69} \\ 4/\sqrt{69} \end{pmatrix}}$$

#35 [6.1]: Suppose that a vector  $\vec{y}$  is orthogonal to vectors  $\vec{u}$  and  $\vec{v}$ . Show that  $\vec{y}$  is orthogonal to the vector  $\vec{u} + \vec{v}$ .

**Solution:**

*Proof.* Suppose the above. Then,

$$\vec{y} \cdot (\vec{u} + \vec{v}) = \vec{y} \cdot \vec{u} + \vec{y} \cdot \vec{v} = 0$$

Since  $\vec{y}$  is orthogonal to  $\vec{v}$  and  $\vec{u}$ . Thus,  $\vec{y}$  is orthogonal to  $\vec{u} + \vec{v}$ . □

#17 [6.2]: Determine if the two vectors below form an orthogonal set. If they do, turn the set into an orthonormal set.

$$\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

**Solution:** Let  $\vec{u}$  and  $\vec{v}$  be the vectors above. We can check if  $\{\vec{u}, \vec{v}\}$  is an orthogonal set by finding  $\vec{u} \cdot \vec{v}$ :

$$\vec{u} \cdot \vec{v} = \frac{1}{3} \left( -\frac{1}{2} \right) + \frac{1}{3} \left( \frac{1}{2} \right) = 0 \therefore \vec{u} \perp \vec{v}$$

We can now normalize the vectors as follows:

$$\begin{aligned} \|\vec{u}\| &= \frac{1}{\sqrt{3}} \Rightarrow \frac{\vec{u}}{\|\vec{u}\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \\ \|\vec{v}\| &= \frac{1}{\sqrt{2}} \Rightarrow \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

Thus, the orthonormal set is

$$\boxed{\left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}}$$

#35 [6.2]: Let  $U$  be a square matrix with orthonormal columns. Explain why  $U$  is invertible.

**Solution:** Since  $U$  has orthonormal columns, the columns of  $U$  form a basis for  $\mathbb{R}^n$  (if  $U$  is  $n \times n$ ). Since the columns of  $U$  form a basis, they must be linearly independent. We know that any square matrix with linearly independent columns must be invertible since there only exist trivial solutions to the homogenous equation.

#9 [6.3]: Let  $W$  be the subspace spanned by the  $\vec{u}$ 's, and write  $\vec{y}$  as the sum of a vector in  $W$  and a vector orthogonal to  $W$ .

$$\vec{y} = \begin{pmatrix} 4 \\ 3 \\ 3 \\ -1 \end{pmatrix}, \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \vec{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

**Solution:** We will perform orthogonal decomposition on  $\vec{y}$  using  $\vec{u}_1, \vec{u}_2$ , and  $\vec{u}_3$ .

$$\begin{aligned} \hat{y} &= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} \vec{u}_3 \\ &= 2\vec{u}_1 + \frac{2}{3}\vec{u}_2 - \frac{2}{3}\vec{u}_3 = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow \vec{y} - \hat{y} &= \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \end{pmatrix} \\ \Rightarrow \vec{y} &= \boxed{\begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \end{pmatrix}} \end{aligned}$$

#15 [6.3]: Let  $\vec{y}$ ,  $\vec{u}_1$ , and  $\vec{u}_2$  be given below. Find the distance from  $\vec{y}$  to the plane  $\mathbb{R}^3$  spanned by  $\vec{u}_1$  and  $\vec{u}_2$ .

$$\vec{y} = \begin{pmatrix} 5 \\ -9 \\ 5 \end{pmatrix}, \vec{u}_1 = \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

**Solution:** We will start by finding  $\vec{y}$  in  $\text{Span}\{\vec{u}_1, \vec{u}_2\}$ :

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{35}{35} \vec{u}_1 + \frac{-28}{14} \vec{u}_2 = \begin{pmatrix} 3 \\ -9 \\ -1 \end{pmatrix}$$

Now we can calculate the distance:

$$\vec{y} - \hat{y} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} \Rightarrow \|\vec{y} - \hat{y}\| = \boxed{\sqrt{40}}$$

#3 [6.4]: The given set below is a basis for a subspace  $W$ . Use the Gram-Schmidt process to produce an orthogonal basis for  $W$ .

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

**Solution:** Let  $\vec{x}_1$  and  $\vec{x}_2$  be the vectors above. We will use the Gram-Schmidt process with  $\vec{x}_1$  and  $\vec{x}_2$ :

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{x}_2 - \frac{1}{2} \vec{v}_1 = \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix} \end{aligned}$$

Thus, our orthogonal basis is

$$\left\{ \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix} \right\}$$

#9 [6.4]: Find an orthogonal basis for the column space of the matrix below

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$$

**Solution:** We can start by noticing that

$$\text{Col}(A) = \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} \right\}.$$

Now, let  $\vec{x}_1$ ,  $\vec{x}_2$ , and  $\vec{x}_3$  be the vectors in  $\text{Col}(A)$ , respectively. We will now perform the Gram-Schmidt process using  $\vec{x}_1$ ,  $\vec{x}_2$ , and  $\vec{x}_3$ .

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{x}_2 - \frac{-40}{20} \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \vec{x}_3 - \frac{30}{20} \vec{v}_1 - \frac{-10}{20} \vec{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

Therefore, our orthogonal basis will be:

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$$