#29 [1.7]: Describe the possible echelon forms of the matrix A such that A is a 3×3 matrix with linearly independent columns

Solution:

$$\boxed{\begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{pmatrix}}$$

Where $\blacksquare \in \mathbb{R} \setminus 0$, $* \in \mathbb{R}$ and $0 = 0$. This is the only possible echelon form for A since in order for the columns to be linearly independent the system $A\vec{x} = \vec{0}$ must only have the trivial solution. This will occur when each column in A contains a pivot.

#37 [1.7]: Given the matrix A below, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\vec{x} = \vec{0}$ **without using row operations**

$$A = \begin{pmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{pmatrix}$$

Solution: Start by noticing that

$$A\vec{x} = \vec{0} \Leftrightarrow x_1 \begin{pmatrix} 2 \\ -5 \\ -3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -4 \\ -4 \\ 1 \end{pmatrix} = \vec{0}$$

Now, for any given row in A, the third column is the sum of the first two columns. Thus

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} \Rightarrow A\vec{x} = \vec{0}$$

#11 [1.8]: Let $\vec{b}$ and A be the matricies below. Is $\vec{b}$ in the range of the linear transformation $\vec{x} \mapsto A\vec{x}$? Why or why not?

$$A = \begin{pmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Solution: Consider the augmented matrix $(A|B)$:

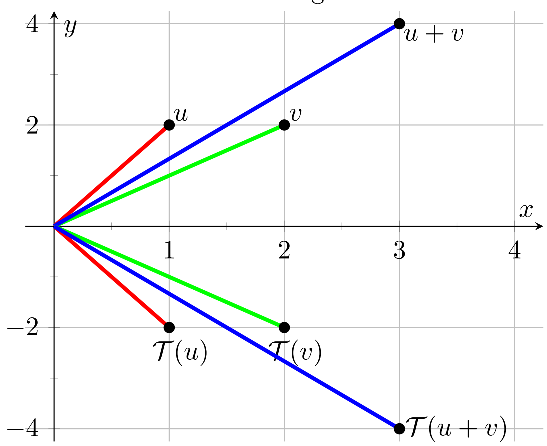
$$\left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & 8 & 6 & 2 \end{array}\right)$$

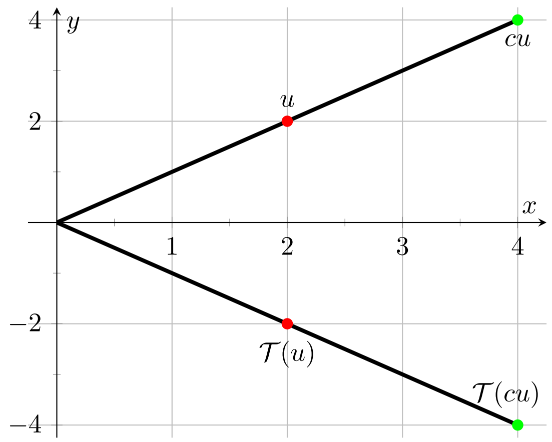
$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{cccc|c} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ no pivots in the final column implies that the system $A\vec{x} = \vec{b}$ is consistent so $\vec{b} \in \mathcal{T}^{\text{img}}$ where $\mathcal{T}$ is the linear transformation given by the map $\vec{x} \mapsto A\vec{x}$

#31 [1.8]: Let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each point through the x_1-axis. Make two sketches that illustrate the two main properties of linear transformations.

Solution: See the below figures:





#15 [1.9]: Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$\begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - 3x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{pmatrix}$$

Solution: Recall that any linear transformation can be put in the from $\mathcal{T}(\vec{x}) = A\vec{x}$ where A is the matrix with column vectors $\mathcal{T}(\mathbf{e}_1) \dots \mathcal{T}(\mathbf{e}_n)$. Thus,

$$A = \big((\mathcal{T}(\mathbf{e}_1)) \quad (\mathcal{T}(\mathbf{e}_2)) \quad (\mathcal{T}(\mathbf{e}_3)) \big) = \boxed{\begin{pmatrix} 2 & 0 & -3 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}}$$

#33 [1.9]: Determine if the transfromation $\mathcal{T}$ given by the map $\vec{x} \mapsto A\vec{x}$, where A is given below, is one-to-one, onto, both, or neither.

$$\underset{4\times 4}{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Solution: Start by moving row 1 to the bottom and shifting the other rows up:

$$A_{\text{REF}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that $\text{rank}\left(A_{\text{REF}}\right) \neq 4 \therefore$ $\boxed{\mathcal{T} \text{ is neither one-to-one nor onto.}}$

#7 [2.1]: If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

Solution: The number of columns in A must match the number of rows in B so B must have 3 rows. The number of columns of B must match the number of columns in AB so B must have 7 columns. $\boxed{\text{Thus, } B \text{ is a } 3 \times 7 \text{ matrix.}}$

#25 [2.1]: If A and AB are the matricies given below, determine the first and second columns of B

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \text{ and } AB = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{pmatrix}$$

Solution: Let since A has size 2×2 and AB has size 2×3, B must have size 2×3 as well. Let

$$B = \begin{pmatrix} x_1 & x_3 & * \\ x_2 & x_4 & * \end{pmatrix}$$

Where $* \in \mathbb{R}$. Then,

$$AB = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 & x_3 & * \\ x_2 & x_4 & * \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{pmatrix}$$

By definition of matrix multiplication we can find the two systems:

$$\left.\begin{cases} x_1 - 2x_2 & = -1 \\ -2x_1 + 5x_2 & = 6 \end{cases}\right] \Rightarrow x_2 = 4 \text{ and } x_1 = 7 \qquad \left.\begin{cases} x_3 - 2x_4 & = 2 \\ -2x_3 + 5x_4 & = -9 \end{cases}\right] \Rightarrow x_4 = -5 \text{ and } x_3 = -8$$

Thus,

$$\boxed{B = \begin{pmatrix} 7 & -8 & * \\ 4 & -5 & * \end{pmatrix}}$$

#7 [2.2]: Use the inverse of a matrix to solve the system below

$$\begin{cases} 8x_1 + 3x_2 & = 2 \\ 5x_1 + 2x_2 & = 3 \end{cases}$$

Solution: Consider the system in the from $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$$

Then, $\vec{x} = A^{-1}\vec{b}$ where

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix}$$

Therefore

$$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 7 \\ -18 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

#43 [2.2]: Find the inverses of

$$\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{pmatrix}$$

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B, and then prove that $AB = I$ and $BA = I$.

Solution: Let the first matrix above be M_1 and the second M_2. We can find the inverse of M_1 by augmenting it with I_3

$$(M_1|I_3) = \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 1 & 1 & -1 & 0 & 1\end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 0 & 1 & 0 & -1 & 1\end{pmatrix}$$

Thus,

$$M_1^{-1} = \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}$$

We can follow a similar process for M_2 with I_4:

$$(M_2|I_4) = \begin{pmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1\end{pmatrix} \xrightarrow{\text{ERO}} \begin{pmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1\end{pmatrix}$$

Thus,

$$M_2^{-1} = \begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{pmatrix}$$

We can now reasonable guess that for an $n \times n$ matrix A in the corresponding form will have an inverse

$$A^{-1} = B = \begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ -1 & 1 & 0 & \cdots & 0\\ 0 & -1 & 1 & \cdots & 0\\ \vdots & \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & -1 & 1\end{pmatrix}$$

Thus, $AB = BA = I_n$.

Proof. We can start with the proof that $AB = I_n$. Begin by noticing that for $1 \leq j \leq n-1$,

$$a_j - a_{j+1} = \mathbf{e}_j \text{ and } b_j = \mathbf{e}_j - \mathbf{e}_{j+1}$$

As well as the fact that $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$. Now, consider AB in the form

$$AB = \begin{pmatrix}A\mathbf{b}_1 & A\mathbf{b}_1 & \cdots & A\mathbf{b}_n\end{pmatrix}$$

For any j we can see that

$$A\mathbf{b}_j = A\left(\mathbf{e}_j - \mathbf{e}_{j+1}\right) = A\mathbf{e}_j - A\mathbf{e}_{j+1} = \mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$$

Thus,

$$AB = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} = I_n$$

It follows that by the uniqueness of inverse matricies that $B = A^{-1}$ therefore by definition of an inverse matrix

$$AA^{-1} = A^{-1}A = BA = AB = I_n$$

$\square$

[0] LaTeX code for this document can be found on github here