

#5 [5.3]: The matrix A below is factored in the form PDP^{-1} . Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}$$

Solution: By the diagonalization theorem we know that the entries across the main diagonal of D are the eigenvalues of A . We also know that columns of P form the corresponding eigenvectors to the eigenvalues in D . Thus,

Eigenvalues: 5, 1, 1

and,

Basis for $\lambda = 5$: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, Basis for $\lambda = 1$: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

#19 [5.3]: Diagonalize the matrix below

$$\begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution: Let A be the matrix above. Since A is an upper triangular matrix, we know that the eigenvalues of A are the entries along the main diagonal. Thus, $\lambda = 5, 3, 2, 2$. We must find the eigenvectors for each eigenvalue. For $\lambda = 5$,

$$A - 5I = \begin{pmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \Rightarrow \begin{cases} x_4 = 0 \\ x_3 = 0 \\ x_2 = 0 \\ x_1 = t \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 3$,

$$\begin{pmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} x_4 = 0 \\ x_3 = 0 \\ x_2 = 2t \\ x_1 = 3t \end{cases} \Rightarrow \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 2$,

$$\begin{pmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_4 = s \\ x_3 = t \\ x_2 = 2s - t \\ x_1 = -s - t \end{cases} \Rightarrow \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Thus,

$P = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

#11 [6.1]: Find a unit vector in the direction of the given vector below

$$\begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix}$$

Solution: Let \vec{x} be the vector above. Then

$$\|\vec{x}\| = \sqrt{\frac{49}{16} + \frac{1}{4} + 1} = \frac{\sqrt{69}}{4} \Rightarrow \frac{\vec{x}}{\|\vec{x}\|} = \begin{pmatrix} 7/\sqrt{69} \\ 2/\sqrt{69} \\ 4/\sqrt{69} \end{pmatrix}$$

#35 [6.1]: Suppose that a vector \vec{y} is orthogonal to vectors \vec{u} and \vec{v} . Show that \vec{y} is orthogonal to the vector $\vec{u} + \vec{v}$.

Solution:

Proof. Suppose the above. Then,

$$\vec{y} \cdot (\vec{u} + \vec{v}) = \vec{y} \cdot \vec{u} + \vec{y} \cdot \vec{v} = 0$$

Since \vec{y} is orthogonal to \vec{v} and \vec{u} . Thus, \vec{y} is orthogonal to $\vec{u} + \vec{v}$. □

#17 [6.2]: Determine if the two vectors below form an orthogonal set. If they do, turn the set into an orthonormal set.

$$\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

Solution: Let \vec{u} and \vec{v} be the vectors above. We can check if $\{\vec{u}, \vec{v}\}$ is an orthogonal set by finding $\vec{u} \cdot \vec{v}$:

$$\vec{u} \cdot \vec{v} = \frac{1}{3} \left(-\frac{1}{2} \right) + \frac{1}{3} \left(\frac{1}{2} \right) = 0 \therefore \vec{u} \perp \vec{v}$$

We can now normalize the vectors as follows:

$$\begin{aligned} \|\vec{u}\| &= \frac{1}{\sqrt{3}} \Rightarrow \frac{\vec{u}}{\|\vec{u}\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \\ \|\vec{v}\| &= \frac{1}{\sqrt{2}} \Rightarrow \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

Thus, the orthonormal set is

$$\left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}$$

#35 [6.2]: Let U be a square matrix with orthonormal columns. Explain why U is invertible.

Solution: Since U has orthonormal columns, the columns of U form a basis for \mathbb{R}^n (if U is $n \times n$). Since the columns of U form a basis, they must be linearly independent. We know that any square matrix with linearly independent columns must be invertible since there only exist trivial solutions to the homogenous equation.

#9 [6.3]: Let W be the subspace spanned by the \vec{u} 's, and write \vec{y} as the sum of a vector in W and a vector orthogonal to W .

$$\vec{y} = \begin{pmatrix} 4 \\ 3 \\ 3 \\ -1 \end{pmatrix}, \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \vec{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Solution: We will perform orthogonal decomposition on \vec{y} using \vec{u}_1, \vec{u}_2 , and \vec{u}_3 .

$$\begin{aligned} \hat{y} &= \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} + \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} \\ &= 2\vec{u}_1 + \frac{2}{3}\vec{u}_2 - \frac{2}{3}\vec{u}_3 = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow \vec{y} - \hat{y} &= \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \end{pmatrix} \\ \Rightarrow \vec{y} &= \boxed{\begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \end{pmatrix}} \end{aligned}$$

#15 [6.3]: Let \vec{y} , \vec{u}_1 , and \vec{u}_2 be given below. Find the distance from \vec{y} to the plane \mathbb{R}^3 spanned by \vec{u}_1 and \vec{u}_2 .

$$\vec{y} = \begin{pmatrix} 5 \\ -9 \\ 5 \end{pmatrix}, \vec{u}_1 = \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

Solution: We will start by finding \vec{y} in $\text{Span}\{\vec{u}_1, \vec{u}_2\}$:

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \frac{35}{35} \vec{u}_1 + \frac{-28}{14} \vec{u}_2 = \begin{pmatrix} 3 \\ -9 \\ -1 \end{pmatrix}$$

Now we can calculate the distance:

$$\vec{y} - \hat{y} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} \Rightarrow \|\vec{y} - \hat{y}\| = \boxed{\sqrt{40}}$$

#3 [6.4]: The given set below is a basis for a subspace W . Use the Gram-Schmidt process to produce an orthogonal basis for W .

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

Solution: Let \vec{x}_1 and \vec{x}_2 be the vectors above. We will use the Gram-Schmidt process with \vec{x}_1 and \vec{x}_2 :

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{x}_2 - \frac{1}{2} \vec{v}_1 = \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix} \end{aligned}$$

Thus, our orthogonal basis is

$$\left\{ \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix} \right\}$$

#9 [6.4]: Find an orthogonal basis for the column space of the matrix below

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$$

Solution: We can start by noticing that

$$\text{Col}(A) = \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} \right\}.$$

Now, let \vec{x}_1, \vec{x}_2 , and \vec{x}_3 be the vectors in $\text{Col}(A)$, respectively. We will now perform the Gram-Schmidt process using \vec{x}_1, \vec{x}_2 , and \vec{x}_3 .

$$\begin{aligned} \vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \vec{x}_2 - \frac{-40}{20} \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \vec{x}_3 - \frac{30}{20} \vec{v}_1 - \frac{-10}{20} \vec{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

Therefore, our orthogonal basis will be:

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$$