

#37 [3.2]: Show that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$

Solution:

Proof. Consider an $n \times n$ matrix A such that it is invertible, and call its inverse A^{-1} . By definition of an inverse, $AA^{-1} = I_n$, so $\det(AA^{-1}) = \det(I_n) = 1$. But, neither determinant is 0 (since A is invertible, $\det A \neq 0$). So,

$$\det(AA^{-1}) = \det(A) \cdot \det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det A}$$

Thus, we have shown that the determinant of A^{-1} is $\frac{1}{\det A}$. □

#41 [3.2]: Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$

Solution:

Proof. Consider a matrix U such that $U^T U = I$. Then,

$$\det(U^T U) = \det(U^T) \cdot \det(U) = \det I = 1$$

But, for all matrices $\det A = \det A^T$. Thus,

$$\det(U^T) \cdot \det(U) = \det(U) \cdot \det(U) = (\det U)^2 = 1 \Rightarrow \det U = \pm 1$$

Hence we have shown that $\det U = \pm 1$ □

#17 [3.3]: Show that if A is 2×2 , then Theorem 8 gives the same formula for A^{-1} as that given by Theorem 4 in section 2.2.

Solution:

Proof. Consider any 2×2 matrix A such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By the formula in section 2.2 we have

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1}$$

We will show this is equivalent to the inverse found by applying Theorem 8, B^{-1} . We can start by finding the cofactors of A :

$$C_{11} = (-1)^{1+1} (d) = d$$

$$C_{12} = (-1)^{1+2} (c) = -c$$

$$C_{21} = (-1)^{2+1} (b) = -b$$

$$C_{22} = (-1)^{2+2} (a) = a$$

We can now find C^T :

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus,

$$B^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So, $A^{-1} = B^{-1}$ and the two methods for finding the inverse of a 2×2 matrix are equivalent. \square

#21 [3.3]: Find the area of the parallelogram whose vertices are

$$(-2, 0), (0, 3), (1, 3), (-1, 0)$$

Solution: We start by shifting the parallelogram to have a vertex at the origin. Our new list of vertices is

$$(0, 0), (2, 3), (3, 3), (1, 0).$$

We can form a matrix A of the vertices

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$$

Taking the absolute value of the determinant of A gives $|\det A| = |-3| = 3 \text{un}^2$ for our area.

#5 [4.1]: Determine if the set of all polynomials of the form $\mathbf{p}(t) = at^2$ for $a \in \mathbb{R}$ is a subspace of \mathbb{P}_n for an appropriate value of n .

Solution: First of all, since \mathbf{p} is a polynomial of degree 2, the appropriate value of n is $n = 2$.

We can start by checking scalar multiplication. Consider $\lambda \mathbf{p}(t)$ for $\lambda \in \mathbb{R}$. Then

$$\mathbf{p}(t) = \lambda at^2 = \alpha t^2, \quad \alpha = \lambda a.$$

But $\alpha \in \mathbb{R}$ since \mathbb{R} is closed under multiplication. Thus $\lambda \mathbf{p}(t)$ is in our subspace, and so our subspace is closed under scalar multiplication.

We will now check vector addition. Consider $\mathbf{p}_1(t) = at^2$ and $\mathbf{p}_2(t) = bt^2$. Then

$$\mathbf{p}_1(t) + \mathbf{p}_2(t) = at^2 + bt^2 = (a + b)t^2$$

but $a + b \in \mathbb{R}$ since \mathbb{R} is closed under addition. Thus, $\mathbf{p}_1(t) + \mathbf{p}_2(t)$ is in our subspace and our subspace is closed under vector addition.

It is also worth noticing that $\vec{0}$ is in our subspace since $\mathbf{p}(0) = 0$.

Thus, since our subspace is closed under scalar multiplication, vector addition, and contains the zero vector, it is a subspace of \mathbb{P}_2 .

#19 [4.1]: Consider the set of functions

$$\mathcal{W} = \{y(t) \mid y(t) = c_1 \cos \omega t + c_2 \sin \omega t\}$$

for a fixed ω and arbitrary c_1, c_2 . Show that \mathcal{W} is vector space.

Solution:

Proof. We will start by showing that \mathcal{W} is closed under scalar multiplication. Consider $\lambda y(t)$ for $\lambda \in \mathbb{R}$,

$$\lambda y(t) = \lambda (c_1 \cos \omega t + c_2 \sin \omega t) = \lambda c_1 \cos \omega t + \lambda c_2 \sin \omega t.$$

But $\lambda c_1, \lambda c_2 \in \mathbb{R}$ so $\lambda y(t) \in \mathcal{W}$ and thus, \mathcal{W} is closed under scalar multiplication.

We will now show that \mathcal{W} is closed under vector addition. Consider $y_1(t), y_2(t) \in \mathcal{W}$. Then,

$$y_1(t) + y_2(t) = (c_1 \cos \omega t + c_2 \sin \omega t) + (c_3 \cos \omega t + c_4 \sin \omega t) = (c_1 + c_3) \cos \omega t + (c_2 + c_4) \sin \omega t.$$

But, $(c_1 + c_3), (c_2 + c_4) \in \mathbb{R}$ since \mathbb{R} is closed under addition. Thus $y_1(t) + y_2(t) \in \mathcal{W}$ and \mathcal{W} is closed under vector addition.

As well, $\vec{0} \in \mathcal{W}$ for $c_1 = c_2 = 0$.

Thus, since \mathcal{W} is closed under scalar multiplication and vector addition, \mathcal{W} is a vector space. \square

#11 [2.8]: Give integers p and q such that $\text{Nul}(A)$ is a subspace of \mathbb{R}^p and $\text{Col}(A)$ is a subspace of \mathbb{R}^q .

$$A = \begin{pmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{pmatrix}$$

Solution: Let $p = 4$ and $q = 3$. $\text{Nul}(A)$ will be a set of vectors in \mathbb{R}^4 and $\text{Col}(A)$ will be a set of vectors in \mathbb{R}^3 .

#31 [2.8]: Consider the matrix A and an echelon form of A below. Find a basis for $\text{Col}(A)$ and a basis for $\text{Nul}(A)$.

$$A = \begin{pmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Since the pivot columns are the first and second we know that the basis for $\text{Col}(A)$ will be

$$\left\{ \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \right\}.$$

It then follows that x_3 and x_4 are free variables, so let $x_3 = s$ and $x_4 = t$. Solving the system we see

$$X = \begin{pmatrix} 4s - 7t \\ -5s + 6t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 4 \\ -5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, one basis for $\text{Nul}(A)$ is

$$\left\{ \begin{pmatrix} 4 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

#5 [4.2]: Find an explicit description of $\text{Nul}(A)$ by listing vectors that span the null space.

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: Since our matrix A is already in REF we can go straight to backwards substitution and solving the system. Since there is no pivot in the second and fourth columns x_2 and x_4 are free variables. Let $x_2 = s$ and $x_4 = t$. Then,

$$X = \begin{pmatrix} -4t + 2s \\ s \\ 9t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So,

$$\text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

#15 [4.2]: If the set below is $\text{Col}(A)$, find A .

$$\left\{ \begin{pmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{pmatrix} \mid r, s, t \in \mathbb{R} \right\}$$

Solution: Using the definition of the set, we can find the general homogenous solution, X ,

$$X = s \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 0 \\ -1 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

We can now use these vectors to form the columns of A ,

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 4 \\ -1 & -1 & 3 \end{pmatrix}$$

#11 [2.9]: Below is a matrix A and an echelon form of A . Find bases for $\text{Col}(A)$ and $\text{Nul}(A)$, and then state the dimensions of these subspaces.

$$\begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Since the pivot columns in the echelon form of A are the first, second, and fourth we have one basis for $\text{Col}(A)$ being

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -9 \\ 10 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -7 \\ 11 \end{pmatrix} \right\}.$$

It follows from the definition of dimension that $\dim[\text{Col}(A)] = 3$. Since the third and fourth columns are non-pivot columns, x_5 and x_3 must be free variables. Let $x_5 = t$ and $x_3 = k$. Solving the rest of the system we see that

$$X = k \begin{pmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Thus, one possible basis for $\text{Nul}(A)$ is

$$\left\{ \begin{pmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Once again, by definition we have $\dim[\text{Nul}(A)] = 2$.

#13 [2.9]: Find a basis for the subspace spanned by the given vectors below. What is the dimension of the subspace?

$$\begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \\ -6 \\ 12 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -3 \\ 7 \end{pmatrix}$$

Solution: Consider

$$A = \begin{pmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{pmatrix} \xrightarrow{\text{ERO}} \begin{pmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, one possible basis is

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -3 \\ 7 \end{pmatrix} \right\}.$$

Then, $\dim[\text{Col}(A)] = 3$.

#13 [4.3]: Assume that A is row equivalent to B . Find bases for $\text{Nul}(A)$, $\text{Col}(A)$, and $\text{Row}(A)$.

$$A = \begin{pmatrix} -2 & -4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Since B has pivots in the first and second column we can find one possible basis for $\text{Col}(A)$ to be

$$\left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ -6 \\ 8 \end{pmatrix} \right\}.$$

It also follow that one basis for $\text{Row}(A)$ is

$$\{(1 \ 0 \ 6 \ 5), (0 \ 2 \ 5 \ 3)\}.$$

Noting that x_3 and x_4 are free variables, let $x_3 = 2s$ and $x_4 = 2t$. Solving the rest of the system gives

$$X = s \begin{pmatrix} -12 \\ -5 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -10 \\ -5 \\ 0 \\ 2 \end{pmatrix}.$$

So, a basis for $\text{Nul}(A)$ could be

$$\left\{ \begin{pmatrix} -12 \\ -5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -10 \\ -5 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

#43 [4.3]: Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in \mathbb{P}_3 ? Why or why not?

Solution: Since neither polynomial is a multiple of the other, the set is linearly independent.

⁰LaTeX code for this document can be found on github [here](#)