#3 [6.5]: Let A and $\vec{b}$ be the matricies below for the system $A\vec{x} = \vec{b}$

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

(a) Construct the normal equation for $\hat{x}$

Solution: Begin by noticing that the system is inconsistent and so we must approximate the solution using the least squares method: $A^T A\hat{x} = A^T \vec{b}$ where $\hat{x}$ is the vector such that $\|b - A\hat{x}\| \leq \|b - A\vec{x}\|$ for all other $\vec{x}$. We start by finding $A^T A$ and $A^T \vec{b}$:

$$A^T A = \begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

We can now setup the normal equations for $\hat{x}$:

$$\boxed{\begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}}$$

(b) Solve for $\hat{x}$

Solution: To solve for $\hat{x}$ we can first notice that $\left(A^T A\right)^{-1}$ exists since $\det A^T A = 216 \neq 0$. Thus, using the formula for the inverse of a 2×2 matrix we have

$$\left(A^T A\right)^{-1} = \frac{1}{216} \begin{pmatrix} 42 & -6 \\ -6 & 6 \end{pmatrix}.$$

Now,

$$A^T A\hat{x} = A^T \vec{b} \Rightarrow \hat{x} = \left(A^T A\right)^{-1} \left(A^T \vec{b}\right).$$

Thus, using our calculation of $A^T \vec{b}$ from part (a), we have

$$\hat{x} = \frac{1}{216} \begin{pmatrix} 42 & -6 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 6 \\ -6 \end{pmatrix} = \frac{1}{216} \begin{pmatrix} 288 \\ -72 \end{pmatrix} = \boxed{\begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix}}$$

#7 [7.1]: Determine if the matrix below is orthogonal. Is it is, find the inverse.

$$\begin{pmatrix} .6 & .8 \\ .8 & -.6 \end{pmatrix}$$

Solution: Recall the definition of an orthogonal matrix is a matrix Q such that $Q^T Q = I$. We can check if the above matrix is orthogonal using this definition:

$$\begin{pmatrix} .6 & .8 \\ .8 & -.6 \end{pmatrix} \begin{pmatrix} .6 & .8 \\ .8 & -.6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, by definition, the above matrix **is orthogonal**. Thus, the inverse is simply the transpose, if the above matrix is Q, then

$$Q^T = \boxed{Q^{-1} = \begin{pmatrix} .6 & .8 \\ .8 & -.6 \end{pmatrix}}$$

#17 [7.1]: Orthogonally diagonalize the matrix given below with eigenvalues $\lambda = -4, 4, 7$

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{pmatrix}$$

Solution: We start by finding the eigenvectors. For $\lambda = -4$ we have

$$A + 4I = \begin{pmatrix} 5 & 1 & 5 \\ 1 & 9 & 1 \\ 5 & 1 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 9 & 1 \\ 5 & 1 & 5 \\ 5 & 1 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 9 & 1 \\ 5 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{pmatrix} 1 & 9 & 1 \\ 0 & -44 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow X = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{u}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

Now for $\lambda = 4$:

$$A - 4I = \begin{pmatrix} -3 & 1 & 5 \\ 1 & 1 & 1 \\ 5 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & 1 & 1 \\ -3 & 1 & 5 \\ 5 & 1 & -3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + 3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & -4 & -8 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow X = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

For $\lambda = 7$:

$$A - 7I = \begin{pmatrix} -6 & 1 & 5 \\ 1 & -2 & 1 \\ 5 & 1 & -6 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & -2 & 1 \\ -6 & 1 & 5 \\ 5 & 1 & -6 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + 6R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & -11 & 11 \\ 0 & 11 & -11 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & -11 & 11 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow X = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \vec{u}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

Where $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are unit vectors in the respective eigenspaces. We can now form P and D:

$$\boxed{P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \quad D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}}$$

#5 [7.2]: Find the matrix of the quadratic form. Assume $\vec{x} \in \mathbb{R}^3$.

(a) $3x_1^2 + 2x_2^2 - 5x_3^2 - 6x_1x_2 + 8x_1x_3 - 4x_2x_3$

> **Solution:** Denote the above quadratic form as $Q(\vec{x})$. The coefficients of the quadratic terms correspond to the main diagonal of the matrix. The coefficients of the cross-product terms correspond to twice the (i, j) and (j, i) entries. With this in mind we can form the matrix as follows,
>
> $$\begin{pmatrix} 3 & -3 & 4 \\ -3 & 2 & -2 \\ 4 & -2 & -5 \end{pmatrix}$$

(b) $6x_1x_2 + 4x_1x_3 - 10x_2x_3$

> **Solution:** We can use similar logic to part (a) while noticing that there are no quadratic terms so all the diagonal entries are 0. Thus, the corresponding matrix is
>
> $$\begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & -5 \\ 2 & -5 & 0 \end{pmatrix}$$

#11 [7.2]: Classify the quadratic form $Q(\vec{x}) = 2x_1^2 - 4x_1x_2 - x_2^2$. Then make a change of variable, $\vec{x} = P\vec{y}$, that transforms the quadratic form into one with no cross-product term and write the new quadratic form.

> **Solution:** Using similar logic as in problem (5) we can form the corresponding matrix as
>
> $$A = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}$$
>
> We can now find the characteristic equation of A (using the known characteristic equation for a 2×2 matrix) and use it find the eigenvalues:
>
> $$\lambda^2 - \lambda - 6 \Rightarrow (\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda = 3, -2$$
>
> Since the eigenvalues of A are both positive and negative, we can classify Q as **indefinite**. To find the change of variable we must first find the eigenvectors of A:
>
> For $\lambda = 3$:
>
> $$A - 3I = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix}$$
>
> $$\Rightarrow X = t\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow \vec{u}_1 = \begin{pmatrix} -2/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$
>
> For $\lambda = -2$:
>
> $$A + 2I \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xrightarrow{R_2 \to 2R_2 + R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix}$$
>
> $$\Rightarrow X = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix}$$
>
> Using the unit vectors $\vec{u}_1$ and $\vec{u}_2$, as well as $\lambda_1 = 3$ and $\lambda_2 = -2$ we can form P (since the set $\{\vec{u}_1, \vec{u}_2\}$ is orthogonal) and D:
>
> $$P = \begin{pmatrix} -2/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 2/\sqrt{3} \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$
>
> Performing the change of variables $\vec{x} = P\vec{y}$ gives the new quadratic form of
>
> $$\vec{y}^T D\vec{y} = \boxed{3y_1^2 - 2y_2^2}$$