# MATH250 NOTES

# Contents

# 1 Unit 1

## 1.1 Lecture 1 (1.1-1.2)

A system of linear equations is when you have more than 1 equation with more than 1 unknown.

**Example 1.1.1**

Solve the following system:

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 &= 4 \end{cases}$$

$$\begin{aligned} x_1 + 2x_2 = 5 &\Rightarrow x_1 = 5 - 2x_2 \\ 2x_1 + x_2 = 4 &\Rightarrow 2(5 - 2x_2) + x_2 = 4 \Rightarrow 10 - 3x_2 = 4 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Note that the solution to the previous system $(x_1, x_2) = (1, 2)$ also corresponds to the point of intersection of the lines that each equation represents. This then implies that non-parallel lines have a single solution, paralell lines have no solutions, and scalar multiples of the same line have infinitely many solutions.

**Example 1.1.2: Systems with Different Number of Solutions**

1. One Solution:
$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 &= 4 \end{cases}$$
2. No solution:
$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + 4x_2 &= 6 \end{cases}$$
3. Infinite Solutions
$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + 4x_2 &= 10 \end{cases}$$

Elementary Row Operations

Consider the system

$$\begin{cases} E_1 \\ E_2 \\ \vdots \\ E_n \end{cases}$$

Where $E_i$ is the $i$-th equation. Then the following operations on the system **do not change the solution**

1. Changing the order of equations: $E_i \leftrightarrow E_j$ where $i \neq j$ for the $i$-th and $j$-th equation
2. Scaling equations: $E_i \to \lambda E_i$ where $\lambda \in \mathbb{R}\,(\lambda \neq 0)$
3. Combining equations: $E_i \to E_i + \lambda E_j$ where $i \neq j$ and $\lambda \in \mathbb{R}\,(\lambda \neq 0)$

**Example 1.1.3**

Solve the following system:

$$\begin{cases} E_1 & = x_1 + 2x_2 = 5 \\ E_2 & = 2x_1 + x_2 = 4 \end{cases}$$

$$\begin{aligned} -2E_1 &= -2x_1 - 4x_2 = -10 \\ E_2 - 2E_1 &= (2x_1 - x_2) - 2x_1 - 4x_2 = 4 - 10 \\ &\Rightarrow -3x_2 = -6 \\ &\Rightarrow x_2 = 2 \\ &\Rightarrow x_1 = 1 \end{aligned}$$

Matricies

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix $A$ has size $m \times n$ if it has $m$ **rows** and $n$ **columns**. A matrix is written as

$$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Where each $a_{ij}$ is called an element of the matrix. $i$ denotes the row and $n$ denotes the column of the element.

With this in mind, note that we can represent the previous system as a matrix:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix}$$

where each row represents an equation and each element corresponds to either the coefficent of a variable or the solution. With this system in matrix form we can manipulate the rows as follows:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -3 & -6 \end{pmatrix}$$

Since each row corresponds to an equation, if we take a look at the bottom row we can see

$$-3x_2 = -6 \Rightarrow x_2 = 2 \Rightarrow x_1 = 1$$

Thus allowing us to reach the same final answer with a lot less hassle.

General Form of a System of Linear Equations

Consider any system of linear equations in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = b_m \end{cases}$$

Now, notice we can rewrite this as a matrix:

$$A = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Everything to the **left** of the solid line is refered to as the *coefficent matrix*. $A$ itself is referred to as the **augmented matrix** for the system.

Row Echelon Form

Given a matrix $A$, $A_{\text{REF}}$ is an equivalent matrix that satisfies the following properites:

1. All zero rows are below non-zero rows
2. each next leading element is in the column to the right of the previous leading element (called pivots)

Note that the **leading element** of a row is simply the first non-zero element in that row.

A matrix can also be put in RREF (reduced row echelon form) if it is alreay in REF, each pivot is 1, and the only non-zero element in the pivot column is the pivot. This would be $A_{\text{RREF}}$. Now, lets try to combine all we've done by applying basic ERO to a simple matrix to put it in REF.

**Example 1.1.4: Basic REF**

Consider the system

$$\begin{cases} x_1 + 2x_2 &= 5 \\ 2x_1 + x_2 = 4 \end{cases}$$

It's augmented matrix can be put into REF as follows

$$(A \mid B) = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 4 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) = (A \mid B)_{\text{REF}}$$

**Example 1.1.5: Basic RREF**

Consider again the system from ex (1.1.4) and notice we can put it into RREF as follows:

$$(A \mid B)_{\text{REF}} = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -6 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{3}R_2]{} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow[R_1 \to R_1 - 2R_2]{} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) = (A \mid B)_{\text{RREF}}$$

It's important to note that the RREF for any given matrix is *unique*.

**Example 1.1.6: Solve the System**

$$\begin{cases} x_1 + 2x_2 - x_3 & = 2 \\ 2x_1 + x_2 + x_3 & = 1 \\ x_1 - x_2 + 2x_3 & = -1 \end{cases} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now because there is no pivot in the third column of our matrix, $x_3$ is what's known as a 'free variable'. All this means is that $x_3$ can take on any value, we'll notate this by $x_3 = t$ with $t \in \mathbb{R}$.

$$\Rightarrow \begin{cases} x_3 & = t \\ -3x_2 + 3x_3 & = -3 \\ x_1 + 2x_2 - x_3 & = 2 \end{cases}$$

$$\Rightarrow X_{\text{gen}} = \begin{pmatrix} -t \\ 1 + t \\ t \end{pmatrix}$$

## 1.2 Lecture 2 (1.3-1.4)

As a reminder from last time, given a system with an augmented matrix $\left(A \mid B\right)$ the system will have

1. no solutions if there is a pivot in the last column of $\left(A \mid B\right)_{\text{REF}}$
2. one solution if $\left(A \mid B\right)_{\text{REF}}$ is $m \times n$ with $n$ pivots (and no pivots in the last column)
3. infinite solutions if $\left(A \mid B\right)_{\text{REF}}$ is $m \times n$ with less than $n$ pivots (and no pivots in the last column)

**Vectors**

For a vector $\vec{AB}$, point $A$ is the tail and point $B$ is the head. Two vectors, $\vec{AB}$ and $\vec{CD}$ are equal if, and only if, their magnitude and directions are equal. Vectors in $\mathbb{R}^2$ can be notated as $\left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$ where the point $(a_1, a_2)$ is the head and the tail is (usually) assumed to the origin. Generally, for $\mathbb{R}^n$ we have $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ which will always have size $n \times 1$.

Since we can represent vectors as matricies, we can perform the standard matrix operations on them:

**Matrix Operations**

Let $A$ and $B$ be matricies. Then

1. $A + B = C$ where $A$ and $B$ are the same size. This addition is defined as $c_{ij} = a_{ij} + b_{ij}$
2. $\lambda A = \hat{A}$ where $\hat{a_{ij}} = \lambda a_{ij}$ for $\lambda \in \mathbb{R}$
3. $A + (-A) = \hat{0}$ where $\hat{0}$ is the zero-matrix which has the same size as $A$ with each element being 0

4. Consider matricies $\underset{m\times n}{A}$ and $\underset{n\times 1}{X}$, then $AX$ is defined as

$$\underset{m\times 1}{A} = \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n \\ \vdots & & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \dots & a_{mn}x_n \end{pmatrix} \text{ if } \underset{m\times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } \underset{n\times 1}{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

**Example 1.2.1: Basic Matrix Multiplication**

Consider the the matricies

$$\underset{2\times 3}{A} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ and } \underset{3\times 1}{B} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

Find $AB$.

$$\underset{2\times 1}{AB} = \begin{pmatrix} (1*2)+(-1*4)+(2*7) \\ (3*2)+(4*4)+(5*7) \end{pmatrix} = \begin{pmatrix} 12 \\ 57 \end{pmatrix}$$

It's important to note that an alternative form of defining matrix multiplication is as follows:

$$AX = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{pmatrix}$$

This is known as the **linear combination of vectors**.

**Linear Combination of Vectors**

Let $\{\vec{u_1}, \vec{u_2}, \dots, \vec{u_k}\}$ be a set of vectors from $\mathbb{R}^n$. Then,

$$c_1\vec{u_1} + c_2\vec{u_2} + \dots + c_k\vec{u_k}, \qquad (c_1, c_2, \dots, c_k \in \mathbb{R})$$

is called the **linear combination of vectors** $\vec{u_1}, \vec{u_2}, \dots, \vec{u_k}$.

We can combine the previous two definitions of matrix multiplication and the linear combination of vectors to get this next fact: if we consider a vector

$$(\vec{u_k}) = \begin{pmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{pmatrix}$$

then, $c_1\vec{u_1} + c_2\vec{u_2} + \dots + c_k\vec{u_k} = AX$ where

$$A = \begin{pmatrix} (\vec{u_1}) & (\vec{u_2}) & \dots & (\vec{u_k}) \end{pmatrix} \text{ and } X = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

This is especially important when working with systems of equations as we can represent systems as linear combinations of vectors. Given any general system we can rewrite it as a linear combination of vectors as such

$$\left.\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_1 \end{cases}\right] \Leftrightarrow AX = B$$

$B$ is said to be a *linear combination of columns of* $A$ if, and only if, $A$ and $B$ are **compatible**. For $A$ and $B$ to be compatible essentially just means that $\left(A \mid B\right)_{\text{REF}}$ has no pivots in the last column.

**Span**

$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \text{Span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ where $c_1, c_2, \ldots, c_k$ are <u>all</u> possible numbers

For a single vector $\vec{v}$, Span $\{\vec{v}\}$ is simply the set containing all scaled multiples of $\vec{v}$.

**Example 1.2.2: Span of Two Vectors**

Notice that Span $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\} = \mathbb{R}^2$. This implies that any vector from $\mathbb{R}^2$ can be written as

$$c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}3\\4\end{pmatrix}$$

We can prove this fact by considering the augmented matrix for $AX = B$:

$$\left(\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & 4 & b_2 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & -2 & b_2 - 2b_1 \end{array}\right)$$

Therefore, since there is no pivote in the last column, this system has a single solution for any vector $B = \begin{pmatrix}b_1\\b_2\end{pmatrix}$

**Example 1.2.3**

Given the following vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}$ and $\vec{b}$, determine if $\vec{b}$ is a linear combination of $\vec{a_1}, \vec{a_2}, \vec{a_3}$.

$$\vec{a_1} = \begin{pmatrix}1\\-2\\0\end{pmatrix} \quad \vec{a_2} = \begin{pmatrix}0\\1\\2\end{pmatrix} \quad \vec{a_3} = \begin{pmatrix}5\\-6\\8\end{pmatrix} \quad \vec{b} = \begin{pmatrix}2\\-1\\6\end{pmatrix}$$

We can start by noticing that $\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3} \Leftrightarrow A\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ where $A = \begin{pmatrix}(\vec{a_1}) & (\vec{a_2}) & (\vec{a_3})\end{pmatrix}$. Thus,

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow[R_2 \to R_2 + 2R_1]{} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, since there is no pivot in the final column, $B$ is a linear combination of $A$.

## 1.3 Lecture 3 (1.5)