#37 [3.2]: Show that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$

Solution:

Proof. Consider an $n \times n$ matrix A such that it is invertible, and call its inverse A^{-1}. By definition of an inverse, $AA^{-1} = I_n$, so $\det\left(AA^{-1}\right) = \det\left(I_n\right) = 1$. But, neither determinant is 0 (since A is invertible, $\det A \neq 0$). So,

$$\det\left(AA^{-1}\right) = \det\left(A\right) \cdot \det\left(A^{-1}\right) = 1 \Rightarrow \det\left(A^{-1}\right) = \frac{1}{\det A}$$

Thus, we have shown that the determinant of A^{-1} is $\frac{1}{\det A}$. □

#41 [3.2]: Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$

Solution:

Proof. Consider a matrix U such that $U^T U = I$. Then,

$$\det\left(U^T U\right) = \det\left(U^T\right) \cdot \det\left(U\right) = \det I = 1$$

But, for all matricies $\det A = \det A^T$. Thus,

$$\det\left(U^T\right) \cdot \det\left(U\right) = \det\left(U\right) \cdot \det\left(U\right) = \left(\det U\right)^2 = 1 \Rightarrow \det U = \pm 1$$

Hence we have shown that $\det U = \pm 1$ □

#17 [3.3]: Show that if A is 2×2, then Theorem 8 gives the same formula for A^{-1} as that given by Theorem 4 in section 2.2.

Solution:

Proof. Consider any 2×2 matrix A such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By the formula in section 2.2 we have

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1}$$

We will show this is equivalent to the inverse found by applying Theorem 8, B^{-1}. We can start by finding the cofactors of A:

$$\begin{aligned} C_{11} &= (-1)^{1+1}(d) = d \\ C_{12} &= (-1)^{1+2}(c) = -c \\ C_{21} &= (-1)^{2+1}(b) = -b \\ C_{22} &= (-1)^{2+2}(a) = a \end{aligned}$$

We can now find C^T:

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus,

$$B^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So, $A^{-1} = B^{-1}$ and the two methods for finding the inverse of a 2×2 matrix are equivalent. $\square$

#21 [3.3]: Find the are of the parallelogram who verticies are

$$(-2, 0), (0, 3), (1, 3), (-1, 0)$$

Solution: We start by shifting the parallelogram to have a vertex at the origin. Our new list of verticies is

$$(0, 0), (2, 3), (3, 3), (1, 0).$$

We can form a matrix A of the verticies

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$$

Taking the absolute value of the determinant of A gives $|\det A| = |-3| = 3\mathrm{un}^2$ for our area.

#5 [4.1]: Determine if the set of all polynomials of the form $\mathbf{p}(t) = at^2$ for $a \in \mathbb{R}$ is a subspace of $\mathbb{P}_n$ for an appropiate value of n.

Solution: First of all, since $\mathbf{p}$ is a polynomial of degree 2, the appropiate value of n is $n = 2$.
We can start by checking scalar multiplication. Consider $\lambda \mathbf{p}\,(t)$ for $\lambda \in \mathbb{R}$. Then

$$\mathbf{p}\,(t) = \lambda\, at^2 = \alpha\, t^2, \quad \alpha = \lambda\, a.$$

But $\alpha \in \mathbb{R}$ since $\mathbb{R}$ is closed under multiplication. Thus $\lambda\mathbf{p}(t)$ is in our subspace, and so our subspace is closed under scalar multiplication.
We will now check vector addition. Consider $\mathbf{p}_1(t) = at^2$ and $\mathbf{p}_2(t) = bt^2$. Then

$$\mathbf{p}_1(t) + \mathbf{p}_2(t) = at^2 + bt^2 = (a+b)\, t^2$$

but $a + b \in \mathbb{R}$ since $\mathbb{R}$ is closed under addition. Thus, $\mathbf{p}_1(t) + \mathbf{p}_2(t)$ is in our subspace and our subspace is closed under vector addition.
It is also worth noticing that $\vec{0}$ is in our subspace since $\mathbf{p}\,(0) = 0$.
Thus, since our subspace is closed under scalar multiplication, vector addition, and contains the zero vector, it *is* a subspace of $\mathbb{P}_2$.

#19 [4.1]: Consider the set of functions

$$\mathcal{W} = \{y(t) \mid y(t) = c_1 \cos\omega\, t + c_2 \sin\omega\, t\}$$

for a fixed ω and arbitary c_1, c_2. Show that $\mathcal{W}$ is vector space.

Solution:

Proof. We will start by showing that $\mathcal{W}$ is closed under scalar multiplication. Consider $\lambda\, y(t)$ for $\lambda \in \mathbb{R}$,

$$\lambda\, y(t) = \lambda\,(c_1 \cos\omega\, t + c_2 \sin\omega\, t) = \lambda\, c_1 \cos\omega\, t + \lambda\, c_2 \sin\omega\, t.$$

But $\lambda c_1, \lambda c_2 \in \mathbb{R}$ so $\lambda y(t) \in \mathcal{W}$ and thus, $\mathcal{W}$ is closed under scalar multiplication.
We will now show that $\mathcal{W}$ is closed under vector addition. Consider $y_1(t), y_2(t) \in \mathcal{W}$. Then,

$$y_1(t) + y_2(t) = (c_1 \cos\omega t + c_2 \sin\omega t) + (c_3 \cos\omega t + c_4 \sin\omega t) = (c_1 + c_3)\cos\omega t + (c_2 + c_4)\sin\omega t.$$

But, $(c_1 + c_3), (c_2 + c_4) \in \mathbb{R}$ since $\mathbb{R}$ is closed under addition. Thus $y_1(t) + y_2(t) \in \mathcal{W}$ and $\mathcal{W}$ is closed under vector addition.
As well, $\vec{0} \in \mathcal{W}$ for $c_1 = c_2 = 0$.
Thus, since $\mathcal{W}$ is closed under scalar multiplication and vector addition, $\mathcal{W}$ is a vector space. □

#11 [2.8]: Give integers p and q such that $\text{Nul}(A)$ is a subspace of $\mathbb{R}^p$ and $\text{Col}(A)$ is a subspace of $\mathbb{R}^q$.

$$A = \begin{pmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{pmatrix}$$

Solution: Let $p = 4$ and $q = 3$. $\text{Nul}(A)$ will be a set of vectors in $\mathbb{R}^4$ and $\text{Col}(A)$ will be a set of vectors in $\mathbb{R}^3$.

#31 [2.8]: Conisder the matrix A and an echelon form of A below. Find a basis for $\text{Col}(A)$ and a basis for $\text{Nul}(A)$.

$$A = \begin{pmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Since the pivot columns are the first and second we know that the basis for $\text{Col}(A)$ will be

$$\left\{ \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \right\}.$$

It then follows that x_3 and x_4 are free variables, so let $x_3 = s$ and $x_4 = t$. Solving the system we see

$$X = \begin{pmatrix} 4s - 7t \\ -5s + 6t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 4 \\ -5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, one basis for $\text{Nul}(A)$ is

$$\left\{ \begin{pmatrix} 4 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

#5 [4.2]: Find an explicit description of $\text{Nul}(A)$ by listing vectors that span the null space.

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: Since our matrix A is already in REF we can go straight to backwards substitution and solving the system. Since there is no pivot in the second and fourth columns x_2 and x_4 are free variables. Let $x_2 = s$ and $x_4 = t$. Then,

$$X = \begin{pmatrix} -4t + 2s \\ s \\ 9t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So,

$$\mathrm{Nul}\,(A) = \mathrm{Span}\left\{ \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

#15 [4.2]: If the set below is $\mathrm{Col}\,(A)$, find A.

$$\left\{ \begin{pmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{pmatrix} \mid r, s, t \in \mathbb{R} \right\}$$

Solution: Using the definition of the set, we can find the general homogenous solution, X,

$$X = s \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 0 \\ -1 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

We can now use these vectors to form the columns of A,

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 4 \\ -1 & -1 & 3 \end{pmatrix}$$

#11 [2.9]: Below is a matrix A and an echelon form of A. Find bases for $\mathrm{Col}\,(A)$ and $\mathrm{Nul}\,(A)$, and then state the dimensions of these subspaces.

$$\begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Since the pivot columns in the echelon form of A are the first, second, and fourth we have one basis for $\text{Col}(A)$ being

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -9 \\ 10 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -7 \\ 11 \end{pmatrix} \right\}.$$

It follows from the definition of dimension that $\dim[\text{Col}(A)] = 3$. Since the third and fourth columns are non-pivot columns, x_5 and x_3 must be free variables. Let $x_5 = t$ and $x_3 = k$. Solving the rest of the system we see that

$$X = k \begin{pmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Thus, one possible basis for $\text{Nul}(A)$ is

$$\left\{ \begin{pmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Once again, by definition we have $\dim[\text{Nul}(A)] = 2$.

#13 [2.9]: Find a basis for the subspace spanned by the given vectors below. What is the dimension of the subspace?

$$\begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \\ -6 \\ 12 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -3 \\ 7 \end{pmatrix}$$

Solution: Consider

$$A = \begin{pmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{pmatrix} \xrightarrow{\text{ERO}} \begin{pmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, one possible basis is

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -3 \\ 7 \end{pmatrix} \right\}.$$

Then, $\dim[\text{Col}(A)] = 3$.

#13 [4.3]: Assume that A is row equivalent to B. Find bases for $\text{Nul}(A)$, $\text{Col}(A)$, and $\text{Row}(A)$.

$$A = \begin{pmatrix} -2 & -4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Since B has pivots in the first and second column we can find one possible basis for $\mathrm{Col}\,(A)$ to be

$$\left\{\begin{pmatrix}-2\\2\\-3\end{pmatrix},\begin{pmatrix}-4\\-6\\8\end{pmatrix}\right\}.$$

It also follow that one basis for $\mathrm{Row}\,(A)$ is

$$\left\{\begin{pmatrix}1 & 0 & 6 & 5\end{pmatrix},\begin{pmatrix}0 & 2 & 5 & 3\end{pmatrix}\right\}.$$

Noting that x_3 and x_4 are free variables, let $x_3 = 2s$ and $x_4 = 2t$. Solving the rest of the system gives

$$X = s\begin{pmatrix}-12\\-5\\2\\0\end{pmatrix} + t\begin{pmatrix}-10\\-5\\0\\2\end{pmatrix}.$$

So, a basis for $\mathrm{Nul}\,(A)$ could be

$$\left\{\begin{pmatrix}-12\\-5\\2\\0\end{pmatrix},\begin{pmatrix}-10\\-5\\0\\2\end{pmatrix}\right\}.$$

#43 [4.3]: Consider the polynomials $\mathbf{p}_1(t) = 1+t^2$ and $\mathbf{p}_2(t) = 1-t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

Solution: Since neither polynomial is a multiple of the other, the set is linearly independent.

[0] $\LaTeX$ code for this document can be found on github here