

Consider the equation

$$\mathbf{x}''(t) = -K\vec{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1) \quad (1)$$

and

$$\mathbf{f}(t) = \sum_{j=1}^4 \mathbf{f}_j(t) \quad (2)$$

for

$$\begin{aligned}\mathbf{f}_1(t) &= (1, 3) \cos(\omega_1 t) \\ \mathbf{f}_2(t) &= (1, -1) \cos(\omega_2 t) \\ \mathbf{f}_3(t) &= (3, -1) \cos(\omega_3 t) \\ \mathbf{f}_4(t) &= (1, 0) \cos(\omega_4 t)\end{aligned}$$

and

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

1. If  $\mathbf{x}_0(t)$  is a solution to

$$\mathbf{x}''(t) = -K\mathbf{x}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1)$$

and for  $j = 1, 2, 3, 4$  we have that  $\mathbf{x}_j(t)$  is a solution to

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}_j(t), \quad \mathbf{x}(0) = (0, 0), \quad \mathbf{x}'(t) = (0, 0),$$

then the solution of (1) and (2) is given by

$$\mathbf{x}(t) = \sum_{j=0}^4 \mathbf{x}_j(t).$$

**Solution:**

*Proof.* By the linearity of the derivative operator we know that

$$\mathbf{x}'(t) = \sum_{j=0}^4 \mathbf{x}'_j(t) \quad \text{and} \quad \mathbf{x}''(t) = \sum_{j=0}^4 \mathbf{x}''_j(t).$$

Expanding the sum for the second derivative gives

$$\begin{aligned}\sum_{j=0}^4 \mathbf{x}''_j(t) &= \mathbf{x}''_0(t) + \mathbf{x}''_1(t) + \mathbf{x}''_2(t) + \mathbf{x}''_3(t) + \mathbf{x}''_4(t) \\ &= (-K\mathbf{x}_0) + (-K\mathbf{x}_1 + \mathbf{f}_1) + (-K\mathbf{x}_2 + \mathbf{f}_2) + (-K\mathbf{x}_3 + \mathbf{f}_3) + (-K\mathbf{x}_4 + \mathbf{f}_4) \\ &= -K(\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4) + (\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4) \\ &= -K \left( \sum_{j=0}^4 \mathbf{x}_j \right) + \sum_{j=1}^4 \mathbf{f}_j \\ &= -K\mathbf{x}(t) + \mathbf{f}(t).\end{aligned} \quad (3)$$

But

$$\sum_{j=0}^4 \mathbf{x}_j'' = \mathbf{x}''(t),$$

so

$$\boxed{\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}(t)}.$$

In order to confirm that this is a solution, we will check that initial conditions hold true as well:

$$\begin{aligned}\sum_{j=0}^4 \mathbf{x}_j(0) &= \mathbf{x}_0(0) + \sum_{j=1}^4 \mathbf{x}_j(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \sum_{j=0}^4 \mathbf{x}'_j(0) &= \mathbf{x}'_0(0) + \sum_{j=1}^4 \mathbf{x}'_j(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned}$$

which are initial values provided in (1). □

2. Compute  $\mathbf{x}_0(t)$

**Solution:** We will diagonalize  $K$  in order to perform a change of variables as a means of solving for  $\mathbf{x}_0$ . Since  $K$  is a  $2 \times 2$  matrix, we can easily solve for the eigenvalues using the characteristic equation:

$$\lambda^2 - (3+6)\lambda + \det K = 0 \Rightarrow \lambda^2 - 9\lambda + 14 = 0 \Rightarrow \lambda_1 = 7, \lambda_2 = 2.$$

Now we solve for the eigenvalues. Starting with  $\lambda_1 = 7$ :

$$K - 7I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_1 \rightarrow -\frac{1}{2}R_1} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And for  $\lambda_2 = 2$ :

$$K - 2I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

It follows that

$$K = \underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}}_V \underbrace{\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}}_{V^{-1}}. \quad (4)$$

Now, since  $\mathbf{x}'' = -K\mathbf{x}$  (we will use  $\mathbf{x}_0$  and  $\mathbf{x}$  interchangably for the rest of the proof), by (4),

$$\mathbf{x}'' = -VDV^{-1}\mathbf{x} \Rightarrow V^{-1}\mathbf{x}'' = -DV^{-1}\mathbf{x}.$$

We define  $\mathbf{y} = V^{-1}\mathbf{x}$  so

$$\mathbf{y}'' = V^{-1}\mathbf{x}'' = -D\mathbf{y}. \quad (5)$$

From (5) it immediately follows that

$$\begin{aligned}\mathbf{y}'' &= -D\mathbf{y} \Rightarrow \begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = \begin{bmatrix} -\lambda_1 \mathbf{y}_1 \\ -\lambda_2 \mathbf{y}_2 \end{bmatrix} \\ \Rightarrow \mathbf{y}(0) &= V^{-1}\mathbf{x}(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and,} \\ \mathbf{y}'(0) &= V^{-1}\mathbf{x}'(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1'(0) \\ \mathbf{y}_2'(0) \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}.\end{aligned} \quad (6)$$

Rewriting (6) gives

$$\begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = - \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

We know from the first set of challenge problems that the solution to a system of differential equations in the form of (6) is

$$\mathbf{y}(t) = \mathbf{s}_0 \cos(\omega t) + \mathbf{t}_0 \sin(\omega t)$$

where  $\mathbf{y}(0) = \mathbf{s}_0$  and  $\mathbf{y}'(0) = \mathbf{t}_0$ . Substituting in our known values we can find

$$\mathbf{y}(t) = \begin{bmatrix} \cos(\sqrt{7}t) + \frac{3}{5} \sin(\sqrt{7}t) \\ -\frac{1}{5} \sin(\sqrt{2}t) \end{bmatrix}. \quad (7)$$

Using our change of variable we can solve for  $\mathbf{x} = V\mathbf{y}$ . Substituting (7) we can see

$$\begin{aligned} \mathbf{x}(t) &= V\mathbf{y}(t) \\ &= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(\sqrt{7}t) + \frac{3}{5} \sin(\sqrt{7}t) \\ -\frac{1}{5} \sin(\sqrt{2}t) \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 7 \cos(\sqrt{7}t) + \frac{21}{5} \sin(\sqrt{7}t) \\ -\frac{2}{5} \sin(\sqrt{2}t) \end{bmatrix}} = \mathbf{x}(t) \end{aligned}$$

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<sup>0</sup>LATEX code for this document can be found on github [here](#)