Consider the equation

$$\mathbf{x}''(t) = -K\vec{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1) \tag{1}$$

and

$$\mathbf{f}(t) = \sum_{j=1}^{4} \mathbf{f}_j(t) \tag{2}$$

for

$$\begin{aligned} \mathbf{f}_1(t) &= (1, 3)\cos(\omega_1 t) \\ \mathbf{f}_2(t) &= (1, -1)\cos(\omega_2 t) \\ \mathbf{f}_3(t) &= (3, -1)\cos(\omega_3 t) \\ \mathbf{f}_4(t) &= (1, 0)\cos(\omega_4 t) \end{aligned}$$

and

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

1. If $\mathbf{x}_0(t)$ is a solution to
$$\mathbf{x}''(t) = -K\mathbf{x}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1)$$
and for $j = 1, 2, 3, 4$ we have that $\mathbf{x}_j(t)$ is a solution to
$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}_j(t), \quad \mathbf{x}(0) = (0, 0), \quad \mathbf{x}'(t) = (0, 0), \tag{3}$$
then the solution of (1) and (2) is given by
$$\mathbf{x}(t) = \sum_{j=0}^{4} \mathbf{x}_j(t).$$

Solution:

Proof. By the linearity of the derivative operator we know that

$$\mathbf{x}'(t) = \sum_{j=0}^{4} \mathbf{x}_j'(t) \quad \text{and} \quad \mathbf{x}''(t) = \sum_{j=0}^{4} \mathbf{x}_j''(t).$$

Expanding the sum for the second derivative gives

$$\begin{aligned} \sum_{j=0}^{4} \mathbf{x}_j''(t) &= \mathbf{x}_0''(t) + \mathbf{x}_1''(t) + \mathbf{x}_2''(t) + \mathbf{x}_3''(t) + \mathbf{x}_4''(t) \\ &= (-K\mathbf{x}_0) + (-K\mathbf{x}_1 + \mathbf{f}_1) + (-K\mathbf{x}_2 + \mathbf{f}_2) + (-K\mathbf{x}_3 + \mathbf{f}_3) + (-K\mathbf{x}_4 + \mathbf{f}_4) \\ &= -K(\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4) + (\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4) \\ &= -K\left(\sum_{j=0}^{4} \mathbf{x}_j\right) + \sum_{j=1}^{4} \mathbf{f}_j \\ &= -K\mathbf{x}(t) + \mathbf{f}(t). \end{aligned} \tag{4}$$

But
$$\sum_{j=0}^{4} \mathbf{x}_j'' = \mathbf{x}''(t),$$
so
$$\boxed{\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}(t)}.$$
In order to confirm that this is a solution, we will check that initial conditions hold true as well:
$$\sum_{j=0}^{4} \mathbf{x}_j(0) = \mathbf{x}_0(0) + \sum_{j=1}^{4} \mathbf{x}_j(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\sum_{j=0}^{4} \mathbf{x}_j'(0) = \mathbf{x}_0'(0) + \sum_{j=1}^{4} \mathbf{x}_j(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
which are initial values provided in (1). □

2. Compute $\mathbf{x}_0(t)$

Solution: We will diagonalize K in order to perform a change of variables as a means of solving for $\mathbf{x}_0$. Since K is a 2×2 matrix, we can easily solve for the eigenvalues using the characteristic equation:
$$\lambda^2 - (3+6)\lambda + \det K = 0 \Rightarrow \lambda^2 - 9\lambda + 14 = 0 \Rightarrow \lambda_1 = 7, \lambda_2 = 2.$$
Now we solve for the eigenvalues. Starting with $\lambda_1 = 7$:
$$K - 7I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{2}R_1} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$
And for $\lambda_2 = 2$:
$$K - 2I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$
It follows that
$$K = \overbrace{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}^{V} \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \overbrace{\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}}^{V^{-1}}. \tag{5}$$
Now, since $\mathbf{x}'' = -K\mathbf{x}$ (we will use $\mathbf{x}_0$ and $\mathbf{x}$ interchangably for the rest of the proof), by (5),
$$\mathbf{x}'' = -VDV^{-1}\mathbf{x} \Rightarrow V^{-1}\mathbf{x}'' = -DV^{-1}\mathbf{x}.$$
We define $\mathbf{y} = V^{-1}\mathbf{x}$ so
$$\mathbf{y}'' = V^{-1}\mathbf{x}'' = -D\mathbf{y}. \tag{6}$$
From (6) it immediately follows that
$$\mathbf{y}'' = -D\mathbf{y} \Rightarrow \begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = \begin{bmatrix} -\lambda_1 \mathbf{y}_1 \\ -\lambda_2 \mathbf{y}_2 \end{bmatrix} \tag{7}$$
$$\Rightarrow \mathbf{y}(0) = V^{-1}\mathbf{x}(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and,}$$
$$\mathbf{y}'(0) = V^{-1}\mathbf{x}'(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1'(0) \\ \mathbf{y}_2'(0) \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}.$$

Rewriting (7) gives

$$\begin{bmatrix}\mathbf{y}_1''\\ \mathbf{y}_2''\end{bmatrix} = -\begin{bmatrix}7 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}.$$

We know from the first set of challenge problems that the solution to a system of differential equations in the form of (7) is

$$\mathbf{y}(t) = \mathbf{s}_0\cos(\omega t) + \mathbf{t}_0\sin(\omega t)$$

where $\mathbf{y}(0) = \mathbf{s}_0$ and $\mathbf{y}'(0) = \mathbf{t}_0$. Substituting in our known values we can find

$$\mathbf{y}(t) = \begin{bmatrix}\cos(\sqrt{7}t) + \frac{3}{5}\sin(\sqrt{7}t)\\ -\frac{1}{5}\sin(\sqrt{2}t)\end{bmatrix}. \tag{8}$$

Using our change of variable we can solve for $\mathbf{x} = V\mathbf{y}$. Substituting (8) we can see

$$\begin{aligned}\mathbf{x}(t) &= V\mathbf{y}(t)\\ &= \begin{bmatrix}7 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\cos(\sqrt{7}t) + \frac{3}{5}\sin(\sqrt{7}t)\\ -\frac{1}{5}\sin(\sqrt{2}t)\end{bmatrix}\\ &= \boxed{\begin{bmatrix}7\cos(\sqrt{7}t) + \frac{21}{5}\sin(\sqrt{7}t)\\ -\frac{2}{5}\sin(\sqrt{2}t)\end{bmatrix} = \mathbf{x}(t)}\end{aligned}$$

Now, for any $\mathbf{x}_j(t)$ that is a solution to (3) we know that for $\mathbf{v}_1, \mathbf{v}_2$ eigenvectors of K

$$\mathbf{x}(t) = y_1(t)\mathbf{v}_1 + y_2(t)\mathbf{v}_2 \tag{9}$$

since K is a symmetric matrix. Then,

$$\begin{aligned}\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f} &\Rightarrow \mathbf{x}''(t) + K\mathbf{x}(t) = \mathbf{f}\\ &\Rightarrow (y_1(t)\mathbf{v}_1 + y_2(t)\mathbf{v}_2)'' + K(y_1(t)\mathbf{v}_1 + y_2\mathbf{v}_2) = \mathbf{f}\\ &\Rightarrow y_1''(t)\mathbf{v}_1 + y_2''(t)\mathbf{v}_2 + 7y_1(t)\mathbf{v}_1 + 2y_2(t)\mathbf{v}_2 = \mathbf{f} \qquad \text{(Since } K\mathbf{v}_i = \lambda_i\mathbf{v}_i\text{)}\\ &\Rightarrow \mathbf{v}_1(y_1''(t) + 7y_1(t)) + \mathbf{v}_2(y_2''(t) + 2y_2(t)) = \mathbf{f}.\end{aligned}$$

Thus, we have the equations

$$y_1''(t) + 7y_1(t) = f_1 \quad \text{and} \quad y_2''(t) + 2y_2(t) = f_2.$$

For an equation of the form $y'' + \lambda y = f$, we may guess that the solution is of the form

$$\begin{aligned}y(t) &= A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t) + C\cos(\omega t) \quad \text{or}\\ y(t) &= A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t) + C\sin(\omega t).\end{aligned} \tag{10}$$

Consider the first case in which the final term is $\cos(\omega t)$ and $f = \alpha\cos(\omega t)$ term. Then,

$$y''(t) = -\lambda(A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)) - \omega^2 C\cos(\omega t).$$

Adding $\lambda y(t)$ to both sides shows that

$$y''(t) + \lambda y(t) = \lambda C\cos(\omega t) - \omega^2 C\cos(\omega t) = (\lambda - \omega^2)C\cos(\omega t).$$

It follows that

$$(\lambda - \omega^2)C\cos(\omega t) = \alpha\cos(\omega t) \Rightarrow C = \frac{\alpha}{\lambda - \omega^2}.$$

Applying the initial conditions, we can see that

$$\begin{aligned} y(t) &= A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t) + \frac{\alpha}{\lambda-\omega^2}\cos(\omega t) \\ \Rightarrow y(0) &= 0 = A + \frac{\alpha}{\lambda-\omega^2} \Rightarrow A = -\frac{\alpha}{\lambda-\omega^2} \\ \Rightarrow y'(0) &= 0 = B. \end{aligned}$$

So, the solution in the case in which $f_i = \alpha\cos(\omega t)$ is

$$y(t) = \frac{\alpha}{\lambda-\omega^2}\left(\cos(\omega t) - \cos(\sqrt{\lambda}t)\right). \tag{11}$$

Returning to (10) and considering the other case in which the final term is $\sin(\omega t)$ and $f = \beta\sin(\omega t)$ term, we have

$$y''(t) = -\lambda(A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)) - \omega^2 C\cos(\omega t).$$

Then,

$$y''(t) + \lambda y(t) = \lambda C\sin(\omega t) - \omega^2 C\sin(\omega t) = (\lambda-\omega^2)C\sin(\omega t).$$

Just like in the first case we can see that

$$(\lambda-\omega^2)C\sin(\omega t) = \beta\sin(\omega t) \Rightarrow C = \frac{\beta}{\lambda-\omega^2}.$$

Applying initial conditions shows

$$\begin{aligned} y(t) &= A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t) + \frac{\beta}{\lambda-\omega^2}\sin(\omega t) \\ \Rightarrow y(0) &= 0 = A \\ \Rightarrow y'(0) &= 0 = \sqrt{\lambda}B + \frac{\beta\omega}{\lambda-\omega^2} \Rightarrow B = -\frac{\beta\omega}{\sqrt{\lambda}(\lambda-\omega^2)}. \end{aligned}$$

So, the solution in the case in which $f = \beta\sin(\omega t)$ is

$$y(t) = \frac{\beta}{\lambda-\omega^2}\left(\sin(\omega t) - \frac{\omega}{\sqrt{\lambda}}\sin(\sqrt{\lambda}t)\right). \tag{12}$$

3. Compute $\mathbf{x}_1(t)$

Solution: For $\mathbf{x}_1$ we have that $\mathbf{f_1} = (1,3)\cos(\omega_1 t)$. Once again, consider the change of variables $\mathbf{y} = V^{-1}\mathbf{x}$. Then,

$$\begin{aligned} \mathbf{y}'' = V^{-1}\mathbf{x}'' &= V^{-1}(-K\mathbf{x} + \mathbf{f}) \\ &= V^{-1}(-VDV^{-1})\mathbf{x} + V^{-1}\mathbf{f} \\ &= -D\mathbf{y} + V^{-1}\mathbf{f} \\ &= -\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{y} + \frac{1}{5}\begin{bmatrix} 7\cos(\omega_1 t) \\ \cos(\omega_1 t). \end{bmatrix} \end{aligned}$$

It follows that $\alpha = \frac{7}{5}$ and $\beta = \frac{1}{5}$, so for

$$\mathbf{x}_1(t) = y_1\mathbf{v}_1 + y_2\mathbf{v_2}$$

we have

$$y_1(t) = \frac{7}{5(7-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t)), \text{ and}$$
$$y_2(t) = \frac{1}{5(2-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{2}t)).$$

Thus,

$$\boxed{\mathbf{x}_1(t) = \frac{7}{5(7-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t))\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{5(2-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{2}t))\begin{bmatrix}2\\-1\end{bmatrix}}$$

4. Compute $\mathbf{x}_2(t)$.

Solution: We follow a similar process for $\mathbf{x}_2$. For $\mathbf{f}_2 = (1,-1)\sin(\omega_2 t)$,

$$V^{-1}\mathbf{f}_2 = \frac{1}{5}\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}\sin(\omega_2 t)\\-\sin(\omega_2 t)\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-\sin(\omega_2 t)\\-3\sin(\omega_2 t)\end{bmatrix},$$

thus $\alpha = -1$ and $\beta = -3$. It follows that

$$\boxed{\mathbf{x}_2(t) = \frac{-1}{5(7-\omega_2^2)}\left(\sin(\omega_2 t) - \frac{\omega_2}{\sqrt{7}}\sin(\sqrt{7})\right)\begin{bmatrix}1\\2\end{bmatrix} - \frac{3}{5(2-\omega_2^2)}\left(\sin(\omega_2 t) - \frac{\omega_2}{\sqrt{2}}\begin{bmatrix}2\\-1\end{bmatrix}\right)}$$

5. Compute $\mathbf{x}_3$.

Solution: For $\mathbf{x}_3$ we have $\mathbf{f}_3 = (3,-1)\sin(\omega_3 t)$ so

$$V^{-1}\mathbf{f}_3 = \frac{1}{5}\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}3\sin(\omega_3 t)\\-\sin(\omega t)\end{bmatrix} = \frac{1}{5}\begin{bmatrix}\sin(\omega_3 t)\\-7\sin(\omega_3 t)\end{bmatrix},$$

thus $\alpha = \frac{1}{5}$ and $\beta = \frac{-7}{5}$. It follows that

$$\boxed{\mathbf{x}_3(t) = \frac{1}{5(7-\omega_2^2)}\left(\sin(\omega_2 t) - \frac{\omega_2}{\sqrt{7}}\sin(\sqrt{7})\right)\begin{bmatrix}1\\2\end{bmatrix} - \frac{7}{5(2-\omega_2^2)}\left(\sin(\omega_2 t) - \frac{\omega_2}{\sqrt{2}}\begin{bmatrix}2\\-1\end{bmatrix}\right)}$$

6. Compute $\mathbf{x}_4$.

Solution: For $\mathbf{x}_4$ we have $\mathbf{f}_4 = (1,0)\cos(\omega_4 t)$ so

$$V^{-1}\mathbf{f}_4 = \frac{1}{5}\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}\cos(\omega_4 t)\\0\end{bmatrix} = \frac{1}{5}\begin{bmatrix}\cos(\omega_4 t)\\-2\cos(\omega_4 t)\end{bmatrix},$$

thus $\alpha = \frac{1}{5}$ and $\beta = \frac{-2}{5}$. It follows that

$$\boxed{\mathbf{x}_4(t) = \frac{1}{5(7-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t))\begin{bmatrix}1\\2\end{bmatrix} - \frac{2}{5(2-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{2}t))\begin{bmatrix}2\\-1\end{bmatrix}}$$

7. Write the solutions to (1) and (2). For which values of $\omega_1, \omega_2, \omega_3$, and ω_4 does the system exhibit resonance.

Solution: From exercise 1, we know that

$$\mathbf{x}(t) = \sum_{j=0}^{4} \mathbf{x}_j(t) = \mathbf{x}_0(t) + \sum_{j=1}^{4} \mathbf{x}_j(t).$$

so

$$\begin{aligned}
x(t) = &\begin{bmatrix}7\cos(\sqrt{7}t) + \frac{21}{5}\sin(\sqrt{7}t)\\ -\frac{2}{5}\sin(\sqrt{2}t)\end{bmatrix}\\
&+ \frac{7}{5(7-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t))\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{5(2-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{2}t))\begin{bmatrix}2\\-1\end{bmatrix}\\
&+ \frac{-1}{5(7-\omega_2^2)}\left(\sin(\omega_2 t) - \frac{\omega_2}{\sqrt{7}}\sin(\sqrt{7}t)\right)\begin{bmatrix}1\\2\end{bmatrix} - \frac{3}{5(2-\omega_2^2)}\left(\sin(\omega_2 t) - \frac{\omega_2}{\sqrt{2}}\sin(\sqrt{2}t)\right)\begin{bmatrix}2\\-1\end{bmatrix}\\
&+ \frac{1}{5(7-\omega_3^2)}\left(\sin(\omega_3 t) - \frac{\omega_3}{\sqrt{7}}\sin(\sqrt{7}t)\right)\begin{bmatrix}1\\2\end{bmatrix} - \frac{7}{5(2-\omega_3^2)}\left(\sin(\omega_3 t) - \frac{\omega_3}{\sqrt{2}}\sin(\sqrt{2}t)\right)\begin{bmatrix}2\\-1\end{bmatrix}\\
&+ \frac{1}{5(7-\omega_4^2)}(\cos(\omega_4 t) - \cos(\sqrt{7}t))\begin{bmatrix}1\\2\end{bmatrix} - \frac{2}{5(2-\omega_4^2)}(\cos(\omega_4 t) - \cos(\sqrt{2}t))\begin{bmatrix}2\\-1\end{bmatrix}.
\end{aligned}$$

The system will exhibit resonance for $\omega_i = \sqrt{7}$ or $\omega_i = \sqrt{2}$ because $y'' + \lambda y = 0$ for either of these values of ω.

8. Of the four given values of ω_1, choose the one that creates a signficantly larger term in the solution, so the solution can be approximated by only the main term.

Solution: For $\omega_1 = 2.65 \approx \sqrt{7}$ we have that

$$\frac{1}{7-\omega_1^2} \approx -44.5$$

which is much larger than any of the other terms. Thus, the main component is

$$\boxed{\frac{7}{5(7-\omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t))\begin{bmatrix}1\\2\end{bmatrix}}$$

[0] LaTeX code for this document can be found on github here