Exercise #1: Consider the system of second order differential equations

$$x''(t) = -x(t)$$
$$y''(t) = -y(t).$$

As well, consider the vector function

$$\vec{x}(t) = (x(t), y(t)) = A\vec{u}(t)$$

for

$$A = \begin{bmatrix} x_0 & u_0 \\ y_0 & v_0 \end{bmatrix} \quad \text{and} \quad \vec{u}(t) = (\cos t, \sin t)$$

where

$$x(0) = x_0 \quad \text{and} \quad x'(0) = u_0$$
$$y(0) = y_0 \quad \text{and} \quad y'(0) = v_0.$$

Now, suppose A is invertible, that is suppose that A^{-1} exists. We will show that

$$\|A\vec{x}\,(t)\,\|^2 = 1$$

It follows that

$$\begin{aligned} \|A\vec{x}\,(t)\,\|^2 &= \|A\left(A^{-1}\vec{u}\,(t)\right)\|^2 \\ &= \|AA^{-1}\vec{u}\,(t)\,\|^2 \\ &= \|I\vec{u}\,(t)\,\|^2 \\ &= \|\vec{u}\,(t)\,\|^2 \\ &= \sqrt{\cos^2(t) + \sin^2(t)} \\ &= \boxed{1} \end{aligned}$$

where I is the identity matrix. We now define the matrix M as

$$M = \left(A^{-1}\right)^T \left(A^{-1}\right) \tag{1}$$

where $\left(A^{-1}\right)^T$ denotes the transpose of A^{-1}. Next, we will show that

$$\vec{x}\,(t) \cdot M\vec{x}\,(t) = 1$$

Now, starting with $M\vec{x}(t)$, we can see that

$$\begin{aligned} M\vec{x}\,(t) &= \left(A^{-1}\right)\left(A^{-1}\right)\vec{x}\,(t) \\ &= \left(A^{-1}\right)^T\left(A^{-1}\right)A\vec{u}\,(t) \\ &= \left(A^{-1}\right)^T\vec{u}\,(t) \end{aligned}$$

Taking the dot product of shows,

$$\begin{aligned} \vec{x}\,(t) \cdot M\vec{x}\,(t) &= \vec{x}\,(t) \cdot \left(A^{-1}\right)^T\vec{u}\,(t) \\ &= \vec{x}^T\left(A^{-1}\right)^T\vec{u}\,(t) \\ &= \vec{x}^T\left(A^{-1}\right)^T A^{-1}\vec{x} \\ &= \left(\vec{x}A^{-1}\right)^T\left(A^{-1}\vec{x}\right) \\ &= \boxed{\|A^{-1}\vec{x}\|^2 = 1} \end{aligned} \tag{2}$$

Note that the parenthesis in (2) were dropped for notions sake. Finally, we will show that if

$$M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

then

$$\vec{x}\,(t) \cdot M\vec{x}\,(t) = ax^2(t) + by(t)^2 + 2cx(t)y(t) = 1.$$

We show this through the direct calculation of the multiplication of matricies:

$$\begin{aligned}\vec{x}\,(t) \cdot M\vec{x}\,(t) &= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \cdot \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} ax(t) + cy(t) \\ cx(t) + by(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) & y(t) \end{bmatrix} \begin{bmatrix} ax(t) + cy(t) \\ cx(t) + by(t) \end{bmatrix} \\ &= \boxed{ax(t)^2 + by(t)^2 + 2cx(t)y(t) = 1}\end{aligned}$$

Exercise #2: We now assume the existance of an orthonormal basis of $\mathbb{R}^2$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ such that $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of M corresponding to eigenvalues λ_1 and λ_2 such that

$$M\mathbf{u}_1 = \lambda_1\mathbf{u}_1 \quad \text{and} \quad M\mathbf{u}_1 = \lambda_2\mathbf{u}_2. \tag{3}$$

In order to show that both λ_1 and λ_2 are positive, we will consider the expression $\|A^{-1}\mathbf{u}_1\|^2$. Upon inspection of this expression we can expand it using the definition of norm to get,

$$\|A^{-1}\mathbf{u}_1\|^2 = \left(\mathbf{u}_1 A^{-1}\right)^T \left(A^{-1}\mathbf{u}_1\right) = \mathbf{u}_1^T \left(A^{-1}\right)^T A^{-1}\mathbf{u}_1.$$

But, from (1), we know that $M = \left(A^{-1}\right)^T \left(A^{-1}\right)$. So,

$$\begin{aligned}\|A^{-1}\mathbf{u}_1\|^2 = \mathbf{u}_1^T \left(A^{-1}\right)^T A^{-1}\mathbf{u}_1 &= \mathbf{u}_1^T M\mathbf{u}_1 \\ &= \mathbf{u}_1^T \lambda_1 \mathbf{u}_1 && \text{(from (3))} \\ &= \lambda_1 \mathbf{u}_1^T \mathbf{u}_1 && (\text{since } \lambda_1 \in \mathbb{R}) \\ &= \lambda_1 \|\mathbf{u}_1\|^2.\end{aligned}$$

However, since the left hand side is positive and the norm of $\mathbf{u}_1$ is also positive, λ_1 must also be positive. An identical argument can be followed for λ_2 using the expression $\|A^{-1}\mathbf{u}_2\|^2$ to arrive at the same conclusion that it must also be positive.

We will show that the major axis of the ellipse has endpoints at $\pm\dfrac{1}{\sqrt{\lambda_2}}\mathbf{u}_2$ and the minor axis of the ellipse has endpoints at $\pm\dfrac{1}{\sqrt{\lambda_1}}\mathbf{u}_1$. Now, without the loss of generality we suppose that

$$\lambda_1 \geq \lambda_2 > 0.$$

Since M is a symmetric matrix, M must also be diagonalizable. We consider the diagonalization of M,

$$M = PDP^T, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

It follows that

$$\begin{aligned}\vec{x}(t) \cdot M\vec{x}(t) &= \vec{x}^T Mx \\ &= \vec{x}^T \left(PDP^T\right)\vec{x} \\ &= \left(P^T x\right)^T D \left(P^T \vec{x}\right).\end{aligned} \tag{4}$$

We define

$$\vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = P^T\vec{x}. \tag{5}$$

Substituting (5) into (4),

$$\begin{aligned} 1 = \vec{x}(t)\cdot M\vec{x}(t) &= \left(P^T x\right)^T D\left(P^T\vec{x}\right) \\ &= (\vec{x}')^T D\vec{x}' \\ &= \lambda_1 \left(x'\right)^2 + \lambda_2 \left(y'\right)^2 = 1 \end{aligned} \tag{6}$$

Since the major/minor axis occur when $x' = 0$ or $y' = 0$ we can solve (6) for x' and y' with the other equal to 0, which shows that

$$x' = \frac{1}{\sqrt{\lambda_1}} \quad \text{and} \quad y' = \frac{1}{\sqrt{\lambda_2}}$$

Since $\lambda_1 \geq \lambda_2$ it follows that $\dfrac{1}{\sqrt{\lambda_2}} \geq \dfrac{1}{\sqrt{\lambda_1}}$ so y' corresponds to the major axis and x' the minor. Thus, the endpoints, E_{major} and E_{minor}, are:

$$\boxed{E_{\text{major}} = \pm\frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2 \quad \text{and} \quad E_{\text{minor}} = \pm\frac{1}{\sqrt{\lambda_1}}\mathbf{u}_1}$$

We will now prove that $\|\vec{x}(t)\|$ is maximal if, and only if,

$$\vec{x}(t) = \pm\frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2.$$

To show that this condition is necessary we will assume that $\|\vec{x}\,(t)\,\|$ is maximal and show that

$$\vec{x}\,(t) = \pm\frac{1}{\lambda_2}\mathbf{u}_2.$$

We start with the fact that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$ which means that there exists $x', y' \in \mathbb{R}$ such that

$$\vec{x}(t) = x'\mathbf{u}_1 + y'\mathbf{u}_2. \tag{7}$$

So,

$$\begin{aligned} \|\vec{x}(t)\|^2 &= \vec{x}(t)\cdot\vec{x}(t) \\ &= (x'\mathbf{u}_1 + y'\mathbf{u}_2)\cdot(x'\mathbf{u}_1 + y'\mathbf{u}_2) \\ &= \left(x'\right)^2(\mathbf{u}_1\cdot\mathbf{u}_2) + \left(y'\right)^2(\mathbf{u}_2\cdot\mathbf{u}_2) + 2x'y'(\mathbf{u}_1\cdot\mathbf{u}_2) \\ &= (x')^2 + (y')^2 \end{aligned} \tag{8}$$

with the last equality following directly from the fact that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set. We can now optimize (8) with the constraint of (6). Solving (6) for $(x')^2$ we can easily see that

$$(x')^2 = \frac{1 - \lambda_2(y')^2}{\lambda_1}.$$

Thus,

$$\begin{aligned} \|\vec{x}(t)\|^2 = (x')^2 + (y')^2 &= \frac{1 - \lambda_2(y')^2}{\lambda_1} + (y')^2 \\ &= \frac{1}{\lambda_1} - \frac{\lambda_2}{\lambda_1}(y')^2 + (y')^2 \\ &= \frac{1}{\lambda_1} + (y')^2\left(1 - \frac{\lambda_2}{\lambda_1}\right). \end{aligned} \tag{9}$$

Since $\lambda_1 \geq \lambda_2$, the coefficient of $(y')^2$ will be nonnegative. It follows that the norm is at a maxium when $(y')^2$ is largest. From (6) we can see that this will happen when $x' = 0$, therefore, $y' = \pm\frac{1}{\sqrt{\lambda_2}}$. Using these values of x' and y' in (7) we can see that $\|\vec{x}\,(t)\,\|$ is at a max when

$$\boxed{\vec{x}\,(t) = \pm\frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2.} \tag{10}$$

To show that this is a sufficient condition we will assume that

$$\vec{x}\,(t) = \pm\frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2$$

and will show that $\|\vec{x}(t)\|$ is at a max. Consider any other vector satisfying (6) such that $x' \neq 0$. Then,

$$y' = \pm\sqrt{\frac{1-\lambda_1(x')^2}{\lambda_2}}$$

However, since λ_1 and $(x')^2$ are both positive quantities,

$$\left|\pm\sqrt{\frac{1-\lambda_1(x')^2}{\lambda_2}}\right| < \left|\pm\frac{1}{\sqrt{\lambda_2}}\right|.$$

And so, no bigger norm is possible. An extremely similar argument can be made (by solving (6) for (y')) to show the same conclusions for the minima.

Exercise #3: We define

$$\mathbf{v}_1 = \frac{1}{\sqrt{\lambda_1}}A^{-1}\mathbf{u}_1 \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{\sqrt{\lambda_2}}A^{-1}\mathbf{u}_2$$

with $\mathbf{u}_1, \mathbf{u}_2, \lambda_1$, and λ_2 defined as in exercise #2. In order to show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for $\mathbb{R}^2$ we must show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors and are orthogonal to each other. First we will show that the two vectors are unit vectors. Consider

$$\begin{aligned}
\|\mathbf{v_i}\|^2 &= \mathbf{v}_i \cdot \mathbf{v}_i \\
&= \frac{1}{\lambda_i}(A^{-1}\mathbf{u}_i)\cdot(A^{-1}\mathbf{u}_i) \\
&= \frac{1}{\lambda_i}\left(A^{-1}\mathbf{u}_i\right)^T\left(A^{-1}\mathbf{u}_i\right) \\
&= \frac{1}{\lambda_i}\mathbf{u}_i^T(A^{-1})^T A^{-1}\mathbf{u}_i \\
&= \frac{1}{\lambda_i}\mathbf{u}_i^T M\mathbf{u}_i && \text{(from (1))} \\
&= \frac{1}{\lambda_i}\mathbf{u}_i^T\left(\lambda_i\mathbf{u}_i\right) = \mathbf{u}_i^T\mathbf{u}_i \\
&= 1 \Rightarrow \|\mathbf{v}_i\| = 1
\end{aligned}$$

for $i = 1, 2$. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors. Next to show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal consider,

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \left(A^{-1}\mathbf{u}_1\right) \cdot \left(A^{-1}\mathbf{u}_2\right) \\
&= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \left(A^{-1}\mathbf{u}_1\right)^T \left(A^{-1}\mathbf{u}_2\right) \\
&= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T (A^{-1})^T A^{-1} \mathbf{u}_2 \\
&= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T M \mathbf{u}_2 && \text{(from (6))} \\
&= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T \lambda_2 \mathbf{u}_2 \\
&= \frac{\lambda_1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T \mathbf{u}_2 = 0.
\end{aligned}$$

Thus $\mathbf{v}_1$ and $\mathbf{v}_2$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set, it also forms a basis for $\mathbb{R}^2$.
Next, we will show that $\|\vec{x}(t)\|$ is maximal if and only if t satisfies

$$(\cos t, \sin t) = \pm \mathbf{v}_2.$$

Earlier we showed that $\|\vec{x}(t)\|$ is maximal if and only if $x(t) = \pm \frac{1}{\sqrt{\lambda_2}} \mathbf{u}_2$ which means that the max norm is

$$\|x(t)\| = \left\| \pm \frac{1}{\sqrt{\lambda_2}} \mathbf{u}_2 \right\| = \left\| \frac{1}{\sqrt{\lambda_2}} \right\| = \frac{1}{\sqrt{\lambda_2}}$$

since $\mathbf{u}_2$ is a unit vector. To show that $\|\vec{x}(t)\|$ is maximal if and only if t satisfies

$$(\cos t, \sin t) = \pm \mathbf{v}_2$$

recall from (10) that if $\|\vec{x}(t)\|$ is maximal then

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}} \mathbf{u}_2.$$

Multiplying both sides by A^{-1}

$$A^{-1} \vec{x}\,(t) = \pm \frac{1}{\sqrt{\lambda_2}} A^{-1} \mathbf{u}_2.$$

But the left hand side is equal to $\vec{u}\,(t)$ and the right hand side is equal to $\mathbf{v}_2$. Thus,

$$\boxed{\vec{v}(t) = (\cos t, \sin t) = \mathbf{v}_2}$$

Following the same logic backwards provides the proof for the backwards direction. An identical argument follows for when $\|\vec{x}(t)\|$ is at a mimima.

[0] LaTeX code for this document can be found on github here