

Challenge Problems 1

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Exercise #1: Consider the system of second order differential equations

$$\begin{aligned}x''(t) &= -x(t) \\ y''(t) &= -y(t).\end{aligned}$$

As well, consider the vector function

$$\vec{x}(t) = (x(t), y(t)) = A\vec{u}(t)$$

for

$$A = \begin{bmatrix} x_0 & u_0 \\ y_0 & v_0 \end{bmatrix} \quad \text{and} \quad \vec{u}(t) = (\cos t, \sin t)$$

where

$$\begin{aligned}x(0) &= x_0 & \text{and} & & x'(0) &= u_0 \\ y(0) &= y_0 & \text{and} & & y'(0) &= v_0.\end{aligned}$$

Now, suppose A is invertible, that is suppose that A^{-1} exists. We will show that

$$\|A\vec{x}(t)\|^2 = 1$$

It follows that

$$\begin{aligned}\|A\vec{x}(t)\|^2 &= \|A(A^{-1}\vec{u}(t))\|^2 \\ &= \|AA^{-1}\vec{u}(t)\|^2 \\ &= \|I\vec{u}(t)\|^2 \\ &= \|\vec{u}(t)\|^2 \\ &= \sqrt{\cos^2(t) + \sin^2(t)} \\ &= \boxed{1}\end{aligned}$$

where I is the identity matrix. We now define the matrix M as

$$M = (A^{-1})^T (A^{-1}) \tag{1}$$

where $(A^{-1})^T$ denotes the transpose of A^{-1} . Next, we will show that

$$\vec{x}(t) \cdot M\vec{x}(t) = 1$$

Now, starting with $M\vec{x}(t)$, we can see that

$$\begin{aligned} M\vec{x}(t) &= (A^{-1}) (A^{-1})^T \vec{x}(t) \\ &= (A^{-1})^T (A^{-1}) A\vec{u}(t) \\ &= (A^{-1})^T \vec{u}(t) \end{aligned}$$

Taking the dot product of shows,

$$\begin{aligned} \vec{x}(t) \cdot M\vec{x}(t) &= \vec{x}(t) \cdot (A^{-1})^T \vec{u}(t) \\ &= \vec{x}^T (A^{-1})^T \vec{u}(t) \\ &= \vec{x}^T (A^{-1})^T A^{-1} \vec{x} \\ &= (\vec{x} A^{-1})^T (A^{-1} \vec{x}) \\ &= \boxed{\|A^{-1} \vec{x}\|^2 = 1} \end{aligned} \tag{2}$$

Note that the parenthesis in (2) were dropped for notions sake. Finally, we will show that if

$$M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

then

$$\vec{x}(t) \cdot M\vec{x}(t) = ax^2(t) + by(t)^2 + 2cx(t)y(t) = 1.$$

We show this through the direct calculation of the multiplication of matrices:

$$\begin{aligned} \vec{x}(t) \cdot M\vec{x}(t) &= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \cdot \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} ax(t) + cy(t) \\ cx(t) + by(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) & y(t) \end{bmatrix} \begin{bmatrix} ax(t) + cy(t) \\ cx(t) + by(t) \end{bmatrix} \\ &= \boxed{ax(t)^2 + by(t)^2 + 2cx(t)y(t) = 1} \end{aligned}$$

Exercise #2: We now assume the existence of an orthonormal basis of \mathbb{R}^2 , $\{\mathbf{u}_1, \mathbf{u}_2\}$ such that \mathbf{u}_1 and \mathbf{u}_2 are eigenvectors of M corresponding to eigenvalues λ_1 and λ_2 such that

$$M\mathbf{u}_1 = \lambda_1 \mathbf{u}_1 \quad \text{and} \quad M\mathbf{u}_2 = \lambda_2 \mathbf{u}_2. \tag{3}$$

In order to show that both λ_1 and λ_2 are positive, we will consider the expression $\|A^{-1}\mathbf{u}_1\|^2$. Upon inspection of this expression we can expand it using the definition of norm to get,

$$\|A^{-1}\mathbf{u}_1\|^2 = (\mathbf{u}_1 A^{-1})^T (A^{-1} \mathbf{u}_1) = \mathbf{u}_1^T (A^{-1})^T A^{-1} \mathbf{u}_1.$$

But, from (1), we know that $M = (A^{-1})^T (A^{-1})$. So,

$$\begin{aligned} \|A^{-1}\mathbf{u}_1\|^2 &= \mathbf{u}_1^T (A^{-1})^T A^{-1} \mathbf{u}_1 = \mathbf{u}_1^T M \mathbf{u}_1 \\ &= \mathbf{u}_1^T \lambda_1 \mathbf{u}_1 && \text{(from (3))} \\ &= \lambda_1 \mathbf{u}_1^T \mathbf{u}_1 && \text{(since } \lambda_1 \in \mathbb{R}) \\ &= \lambda_1 \|\mathbf{u}_1\|^2. \end{aligned}$$

However, since the left hand side is positive and the norm of \mathbf{u}_1 is also positive, λ_1 must also be positive. An identical argument can be followed for λ_2 using the expression $\|A^{-1}\mathbf{u}_2\|^2$ to arrive at the same conclusion that it must also be positive.

We will show that the major axis of the ellipse has endpoints at $\pm \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2$ and the minor axis of the ellipse has endpoints at $\pm \frac{1}{\sqrt{\lambda_1}}\mathbf{u}_1$. Now, without the loss of generality we suppose that

$$\lambda_1 \geq \lambda_2 > 0.$$

Since M is a symmetric matrix, M must also be diagonalizable. We consider the diagonalization of M ,

$$M = PDP^T, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \vec{x}(t) \cdot M\vec{x}(t) &= \vec{x}^T Mx \\ &= \vec{x}^T (PDP^T) \vec{x} \\ &= (P^T x)^T D (P^T \vec{x}). \end{aligned} \tag{4}$$

We define

$$\vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = P^T \vec{x}. \tag{5}$$

Substituting (5) into (4),

$$\begin{aligned} 1 = \vec{x}(t) \cdot M\vec{x}(t) &= (P^T x)^T D (P^T \vec{x}) \\ &= (\vec{x}')^T D \vec{x}' \\ &= \lambda_1 (x')^2 + \lambda_2 (y')^2 = 1 \end{aligned} \tag{6}$$

Since the major/minor axis occur when $x' = 0$ or $y' = 0$ we can solve (6) for x' and y' with the other equal to 0, which shows that

$$x' = \frac{1}{\sqrt{\lambda_1}} \quad \text{and} \quad y' = \frac{1}{\sqrt{\lambda_2}}$$

Since $\lambda_1 \geq \lambda_2$ it follows that $\frac{1}{\sqrt{\lambda_2}} \geq \frac{1}{\sqrt{\lambda_1}}$ so y' corresponds to the major axis and x' the minor. Thus, the endpoints, E_{major} and E_{minor} , are:

$$\boxed{E_{\text{major}} = \pm \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2 \quad \text{and} \quad E_{\text{minor}} = \pm \frac{1}{\sqrt{\lambda_1}}\mathbf{u}_1}$$

We will now prove that $\|\vec{x}(t)\|$ is maximal if, and only if,

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2.$$

To show that this condition is necessary we will assume that $\|\vec{x}(t)\|$ is maximal and show that

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2.$$

We start with the fact that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for \mathbb{R}^2 which means that there exists $x', y' \in \mathbb{R}$ such that

$$\vec{x}(t) = x'\mathbf{u}_1 + y'\mathbf{u}_2. \tag{7}$$

So,

$$\begin{aligned}
 \|\vec{x}(t)\|^2 &= \vec{x}(t) \cdot \vec{x}(t) \\
 &= (x'\mathbf{u}_1 + y'\mathbf{u}_2) \cdot (x'\mathbf{u}_1 + y'\mathbf{u}_2) \\
 &= (x')^2(\mathbf{u}_1 \cdot \mathbf{u}_1) + (y')^2(\mathbf{u}_2 \cdot \mathbf{u}_2) + 2x'y'(\mathbf{u}_1 \cdot \mathbf{u}_2) \\
 &= (x')^2 + (y')^2
 \end{aligned} \tag{8}$$

with the last equality following directly from the fact that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal set. We can now optimize (8) with the constraint of (6). Solving (6) for $(x')^2$ we can easily see that

$$(x')^2 = \frac{1 - \lambda_2(y')^2}{\lambda_1}.$$

Thus,

$$\begin{aligned}
 \|\vec{x}(t)\|^2 &= (x')^2 + (y')^2 = \frac{1 - \lambda_2(y')^2}{\lambda_1} + (y')^2 \\
 &= \frac{1}{\lambda_1} - \frac{\lambda_2}{\lambda_1}(y')^2 + (y')^2 \\
 &= \frac{1}{\lambda_1} + (y')^2 \left(1 - \frac{\lambda_2}{\lambda_1}\right).
 \end{aligned} \tag{9}$$

Since $\lambda_1 \geq \lambda_2$, the coefficient of $(y')^2$ will be nonnegative. It follows that the norm is at a maximum when $(y')^2$ is largest. From (6) we can see that this will happen when $x' = 0$, therefore, $y' = \pm \frac{1}{\sqrt{\lambda_2}}$. Using these values of x' and y' in (7) we can see that $\|\vec{x}(t)\|$ is at a max when

$$\boxed{\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2.} \tag{10}$$

To show that this is a sufficient condition we will assume that

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}}\mathbf{u}_2$$

and will show that $\|\vec{x}(t)\|$ is at a max. Consider any other vector satisfying (6) such that $x' \neq 0$. Then,

$$y' = \pm \sqrt{\frac{1 - \lambda_1(x')^2}{\lambda_2}}$$

However, since λ_1 and $(x')^2$ are both positive quantities,

$$\left| \pm \sqrt{\frac{1 - \lambda_1(x')^2}{\lambda_2}} \right| < \left| \pm \frac{1}{\sqrt{\lambda_2}} \right|.$$

And so, no bigger norm is possible. An extremely similar argument can be made (by solving (6) for (y')) to show the same conclusions for the minima.

Exercise #3: We define

$$\mathbf{v}_1 = \frac{1}{\sqrt{\lambda_1}}A^{-1}\mathbf{u}_1 \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{\sqrt{\lambda_2}}A^{-1}\mathbf{u}_2$$

with $\mathbf{u}_1, \mathbf{u}_2, \lambda_1$, and λ_2 defined as in exercise #2. In order to show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for \mathbb{R}^2 we must show that \mathbf{v}_1 and \mathbf{v}_2 are unit vectors and are orthogonal to each other. First we will show that

the two vectors are unit vectors. Consider

$$\begin{aligned}
 \|\mathbf{v}_i\|^2 &= \mathbf{v}_i \cdot \mathbf{v}_i \\
 &= \frac{1}{\lambda_i} (A^{-1}\mathbf{u}_i) \cdot (A^{-1}\mathbf{u}_i) \\
 &= \frac{1}{\lambda_i} (A^{-1}\mathbf{u}_i)^T (A^{-1}\mathbf{u}_i) \\
 &= \frac{1}{\lambda_i} \mathbf{u}_i^T (A^{-1})^T A^{-1} \mathbf{u}_i \\
 &= \frac{1}{\lambda_i} \mathbf{u}_i^T M \mathbf{u}_i && \text{(from (1))} \\
 &= \frac{1}{\lambda_i} \mathbf{u}_i^T (\lambda_i \mathbf{u}_i) = \mathbf{u}_i^T \mathbf{u}_i \\
 &= 1 \Rightarrow \|\mathbf{v}_i\| = 1
 \end{aligned}$$

for $i = 1, 2$. Thus, \mathbf{v}_1 and \mathbf{v}_2 are unit vectors. Next to show that \mathbf{v}_1 and \mathbf{v}_2 are orthogonal consider,

$$\begin{aligned}
 \mathbf{v}_1 \cdot \mathbf{v}_2 &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} (A^{-1}\mathbf{u}_1) \cdot (A^{-1}\mathbf{u}_2) \\
 &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} (A^{-1}\mathbf{u}_1)^T (A^{-1}\mathbf{u}_2) \\
 &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T (A^{-1})^T A^{-1} \mathbf{u}_2 \\
 &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T M \mathbf{u}_2 && \text{(from (6))} \\
 &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T \lambda_2 \mathbf{u}_2 \\
 &= \frac{\lambda_1}{\sqrt{\lambda_1 \lambda_2}} \mathbf{u}_1^T \mathbf{u}_2 = 0.
 \end{aligned}$$

Thus \mathbf{v}_1 and \mathbf{v}_2 . Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set, it also forms a basis for \mathbb{R}^2 . Next, we will show that $\|\vec{x}(t)\|$ is maximal if and only if t satisfies

$$(\cos t, \sin t) = \pm \mathbf{v}_2.$$

Earlier we showed that $\|\vec{x}(t)\|$ is maximal if and only if $x(t) = \pm \frac{1}{\sqrt{\lambda_2}} \mathbf{u}_2$ which means that the max norm is

$$\|x(t)\| = \left\| \pm \frac{1}{\sqrt{\lambda_2}} \mathbf{u}_2 \right\| = \left\| \frac{1}{\sqrt{\lambda_2}} \right\| = \frac{1}{\sqrt{\lambda_2}}$$

since \mathbf{u}_2 is a unit vector. To show that $\|\vec{x}(t)\|$ is maximal if and only if t satisfies

$$(\cos t, \sin t) = \pm \mathbf{v}_2$$

recall from (10) that if $\|\vec{x}(t)\|$ is maximal then

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}} \mathbf{u}_2.$$

Multiplying both sides by A^{-1}

$$A^{-1}\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}} A^{-1} \mathbf{u}_2.$$

But the left hand side is equal to $\vec{u}(t)$ and the right hand side is equal to \mathbf{v}_2 . Thus,

$$\vec{v}(t) = (\cos t, \sin t) = \mathbf{v}_2$$

Following the same logic backwards provides the proof for the backwards direction. An identical argument follows for when $\|\vec{x}(t)\|$ is at a minima.

⁰LaTeX code for this document can be found on github [here](#)