2.5.2:

Solution: If $v(x) = \tan(x)$ then $x'(t) = v(x) = \tan(x)$. By Barrows formula we have that

$$\begin{aligned} t(x) &= 0 + \int_{x_0}^{x} \frac{1}{\tan z}\, dz \\ &= \int_{x_0}^{x} \cot z \, dz \\ &= \ln|\sin x|\Big|_{x_0}^{x} = \ln\left|\frac{\sin x}{\sin x_0}\right|. \end{aligned}$$

Thus,

$$\begin{aligned} t(x) &= \ln\left|\frac{\sin x}{\sin x_0}\right| \\ &= e^t \sin x_0 = \sin x \\ &\Rightarrow \boxed{\arcsin(e^t \sin x_0) = x(t)} \end{aligned}$$

Since $\arcsin(x)$ is continunous for $x \in (1,1)$ we must have

$$\begin{aligned} -1 \le e^t \sin x_0 \le 1 &\Rightarrow \frac{-1}{\sin x_0} \le e^t \le \frac{1}{\sin x_0} \\ &\Rightarrow \ln\left(\frac{-1}{\sin x_0}\right) \le t \le \ln\left(\frac{1}{\sin x_0}\right). \end{aligned}$$

Which is the interval for which the solution is defined.

2.5.3:

Solution:

(a) If $v(x) = (1-x^4)^{1/2}$ then $x'(t) = (1-x^4)^{1/2}$. By Barrows Theorem, a solution exists and is defined for all $t \in \mathbb{R}$ if, and only if,

$$\int_{-1}^{0} \frac{1}{|\sqrt{1-x^4}|}\, dx = \int_{0}^{1} \frac{1}{|\sqrt{1-x^4}|}\, dx = \infty$$

since $(-1,1)$ is a maximal interval for v. But, for $x \in (-1,0]$

$$T_a \int_{-1}^{0} \frac{1}{\sqrt{1-x^4}}\, dx \le \int_{-1}^{0} \frac{1}{2\sqrt{1+x}}\, dx = T,$$

and for $x \in [0,1)$

$$T_b = \int_{0}^{1} \frac{1}{\sqrt{1-x^4}}\, dx \le \int_{0}^{1} \frac{1}{2\sqrt{1-x}}\, dx = T.$$

For some $T, T_a, T_b \in \mathbb{R}$. Thus, the solutions only exists within $(-1,1)$ for $t \in (T_a, T_b)$.

(b) If $v(x) = (1-x^4)^2$ then $x'(t) = (1-x^4)^2$. By Barrows Theorem, a solution exists within $(-1,1)$ if, and only if

$$\int_{-1}^{0} \frac{1}{|(1-x^4)^2|}\, dx = \int_{0}^{1} \frac{1}{|(1-x^4)^2|}\, dx = \infty.$$

It is clear that these integrals diverge since $\lim_{x\to\infty} \frac{1}{|(1-x^4)^2|} = \infty$. Thus, the solution does exist and remains within $(-1,1)$ for all $t \in \mathbb{R}$.

2.5.10:

Solution:

(a) Consider the derivative of $H(x(t), x'(t))$

$$\begin{aligned}\frac{d}{dt}H(x(t),x'(t)) &= \frac{d}{dt}\left(\frac{1}{2}(x'(t))^2 + v(x(t))\right)\\ &= x'(t)x''(t) + v'(x(t))x'(t)\\ &= -x'(t)v'(x(t)) + v'(x(t))x'(t)\\ &= 0.\end{aligned}$$

So, $H(x,y)$ must be a constant function. It follows that

$$\boxed{H(x(t),x'(t)) = H(x(t_0),x'(t_0)).}$$

If $H(v_0, x_0)$ is constant then

$$\begin{aligned}H(v_0,x_0) &= \frac{1}{2}(x')^2 + v(x)\\ \Rightarrow 2(H(v_0,x_0) - v(x)) &= (x')^2\\ \Rightarrow &\boxed{\sqrt{2(H(v_0,x_0) - v(x))} = x'}\end{aligned}$$

which was to be shown.

(b) If $x_0 = 1$ and $v_0 = 0$, then $H(0,1) = \frac{1}{2}$. So,

$$\begin{aligned}x' &= \pm\sqrt{1-x^2} && (1)\\ \Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx &= \int dt\\ \Rightarrow \arcsin(x) &= t + C && (2)\\ \Rightarrow x &= \sin(t+C). && (3)\end{aligned}$$

Without the loss of generality, take $t_0 = 0$. Then, if $x(0) = x_0 = 1$ we have that $C = \frac{\pi}{2}$. But

$$x = \sin\left(t + \frac{\pi}{2}\right) = \cos t.$$

From (1) we know that

$$(x')^2 + x^2 = 1$$

which has equilibrium points at $x = \pm 1$ since

$$(x')^2 + (\pm 1)^2 = 1 \Rightarrow x' = 0.$$

Thus, $x(t) = 1$ for $t \in [0, T_1]$ since it is at rest for some arbitrary time $T_1 \geq 0$. For $t \in (T_1, T_2)$, $x(t)$ will follow a cosine wave until reaching $x = -1$ at which point it will stay at $x = -1$ for $x \in [T_2, T_3]$ with $T_3 \geq 0$ before following the cosine wave again to $x = 1$.

The only solution to be differentiable is $x(t) = \cos t$ for if $x'' = -v'(x) = -x$, then $x'' = -1$ when $x = 1$. But if the particle is at rest at $x = 1$, then $x'' = 0$. Since $0 \neq 1$ this is a contradiction and the only way for x'' to exist is if the particle is never at rest.

(c) The equilibrium points of $(x')^2 + x^4 = 1$ are $x = \pm 1$. Similar to (b) at $x = 1$, $x' = 0$, so we may stay at $x = 1$ for any choosen time T meaning there are infinitely many solutions. For a solution to be twice differntiable (when $x' \neq 0$),

$$2x'x'' - 4x^3x' = 0 \Rightarrow 2x'(x'' + 2x^3) = 0 \Rightarrow x'' = -2x^3.$$

Once again, without the loss of generality take $t_0 = 1$ so that x is at rest. Then $x'' = 0$ since x is at rest but $x'' = -2$ which is a contradiction. Thus, there is only a single solution that is twice differentiable, when x is never at rest.

3.5.2:

Solution:

(a) If $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$, we can find $\mathbf{w}$ such that $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$. Since $\mathbf{w}(\mathbf{u}(t)) = \mathbf{u}'(t)$ we know that

$$\mathbf{w}(\mathbf{u}(t)) = \begin{pmatrix} u' \\ v' \end{pmatrix}$$

and thus it will suffice to find u' and v'. For $u'(\mathbf{x}(t))$ we have,

$$\begin{aligned} u'(x,y) &= \frac{\partial u}{\partial x}x' + \frac{\partial u}{\partial y}y' \\ &= \frac{1}{2}\left(\frac{(x-y)-(x+y)}{(x-y)^2}\right)x' + \frac{1}{2}\left(\frac{(x-y)-(-1)(x+y)}{(x-y)^2}y'\right) \\ &= \frac{-y}{(x-y)^2} + \frac{x}{(x-y)^2}. \end{aligned}$$

From $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$,

$$\mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x - y\sqrt{x^2-y^2} \\ y - x\sqrt{x^2-y^2} \end{pmatrix} = \mathbf{v}(\mathbf{x}(t)), \tag{4}$$

so,

$$\begin{aligned} v'(x,y) &= \frac{-y}{(x-y)^2}(x - y\sqrt{x^2-y^2}) + \frac{x}{(x-y)^2}(y - x\sqrt{x^2-y^2}) \\ &= \frac{-xy + y^2\sqrt{x^2-y^2} + xy - x^2\sqrt{x^2-y^2}}{(x-y)^2} \\ &= \frac{\sqrt{x^2-y^2}(y^2-x^2)}{(x-y)^2} \\ &= \frac{-\sqrt{x^2-y^2}(x^2-y^2)}{(x-y)^2} \\ &= \frac{-\sqrt{x^2-y^2}(x-y)(x+y)}{(x-y)^2} \\ &= \frac{-\sqrt{x^2-y^2})(x+y)}{x-y} \\ &= -2u. \end{aligned}$$

For $\mathbf{v}(\mathbf{x}(t))$ we have

$$\begin{aligned} v'(x,y) &= \frac{\partial v}{\partial x}x' + \frac{\partial v}{\partial y}y' \\ &= \frac{x}{\sqrt{x^2-y^2}}x' + \frac{-y}{\sqrt{x^2-y^2}}y' \\ &= \frac{1}{\sqrt{x^2-y^2}}(x(x-y\sqrt{x^2-y^2}) - y(y-x\sqrt{x^2-y^2})) && \text{(by 4)} \\ &= \frac{1}{\sqrt{x^2-y^2}}(x^2-y^2) \\ &= \frac{1}{v}v^2 = v. \end{aligned}$$

Thus,

$$\boxed{\mathbf{w}(\mathbf{u}(t)) = \begin{pmatrix} u'(\mathbf{x}(t)) \\ v'(\mathbf{x}(t)) \end{pmatrix} = \begin{pmatrix} -2uv \\ v \end{pmatrix}}$$

(b) We first notice that $\mathbf{w}(\mathbf{u}(t))$ represents a coupled system. Since $\mathbf{u}(0) \in V$, let $\mathbf{u}(0) = (u_0, v_0)$. Then,

$$v' = v \Rightarrow v(t) = v_0 e^t.$$

Substituting this into the equation for u and solving,

$$\begin{aligned} u'(t) = -2uv &\Rightarrow u' = -2v_0 u e^t \\ &\Rightarrow \int_{u_0}^{u} \frac{1}{z}\,dz = \int_0^{t_0} -2v_0 e^t\,dt \\ &\Rightarrow u(t) = u_0 e^{-2v_0 e^t}. \end{aligned}$$

Thus, the general solution is

$$\boxed{\mathbf{u}(t) = \begin{pmatrix} u_0 e^{-2v_0 e^t} \\ v_0 e^t \end{pmatrix}}$$

(c) In order to solve $\mathbf{x}'(t) = \mathbf{w}(\mathbf{u}(t))$, we will convert the solutions from (b) back into the xy-coordinate system. Define

$$\alpha = 2u_0 e^{-2v_0 e^t}$$

such that

$$\begin{aligned} 2u = \alpha &\Rightarrow \frac{x+y}{x-y} = \alpha \\ &\Rightarrow x + y = \alpha x - \alpha y \\ &\Rightarrow x(\alpha - 1) = y(\alpha + 1) \\ &\Rightarrow x = \frac{y(\alpha+1)}{(\alpha-1)}. \end{aligned}$$

Thus,

$$\begin{aligned}
v^2 = x^2 - y^2 \Rightarrow v_0^2 e^{t^2} &= \frac{y^2(\alpha+1)^2}{(\alpha-1)^2} - y^2 \\
\Rightarrow v_0^2 e^{t^2} &= y^2\left(\frac{(\alpha+1)^2}{(\alpha-1)^2} - 1\right) \\
\Rightarrow v_0^2 e^{t^2} &= y^2\left(\frac{(\alpha+1)^2 - (\alpha-1)^2}{(\alpha-1)^2}\right) \\
\Rightarrow v_0^2 e^{t^2} &= y^2\left(\frac{4\alpha}{(\alpha-1)^2}\right) \\
\Rightarrow y^2 &= \frac{(\alpha-1)^2 v_0^2 e^{t^2}}{4\alpha} \\
\Rightarrow y &= \frac{(\alpha-1)v_0 e^t}{2\sqrt{\alpha}}.
\end{aligned}$$

So,

$$\begin{aligned}
x = \frac{y(\alpha+1)}{(\alpha-1)} &= \left(\frac{\alpha+1}{\alpha-1}\right)\left(\frac{(\alpha-1)v_0 e^t}{2\sqrt{\alpha}}\right) \\
&= \frac{(\alpha+1)v_0 e^t}{2\sqrt{\alpha}(\alpha-1)}.
\end{aligned}$$

Substituting in α,

$$\begin{aligned}
x &= \frac{(2u_0 e^{-2v_0 e^t} + 1)v_0 e^t}{2\sqrt{2u_0 e^{-2v_0 e^t}}(2u_0 e^{-2v_0 e^t} - 1)} \\
y &= \frac{(2u_0 e^{-2v_0 e^t} - 1)v_0 e^t}{2\sqrt{2u_0 e^{-2v_0 e^t}}}.
\end{aligned}$$

So,

$$\boxed{\Psi_t(\mathbf{x}) = \left(\frac{(2u_0 e^{-2v_0 e^t} + 1)v_0 e^t}{2\sqrt{2u_0 e^{-2v_0 e^t}}(2u_0 e^{-2v_0 e^t} - 1)}, \frac{(2u_0 e^{-2v_0 e^t} - 1)v_0 e^t}{2\sqrt{2u_0 e^{-2v_0 e^t}}}\right)}$$

3.5.4:

Solution:

(a) We start by finding the eigenvalues of A using the characteristic equation for 2×2 matrix:

$$\lambda^2 + 5\lambda - 6 = 0 \Rightarrow \lambda_1 = -6, \lambda_2 = 1.$$

Then, the eigenvalues are

$$A + 6I = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix} \Rightarrow \vec{u}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$A - I = \begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Thus, the general solution has the form

$$\mathbf{x}(t) = a_1 e^{-6t}(-1, 1) + a_2 e^t(2, 5) \tag{5}$$

for $a_1, a_2 \in \mathbb{R}$. We can solve for a_1 and a_2 using the initial condition $\mathbf{x}(t) = (x_0, y_0)$. If

$$V = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 5 \end{pmatrix} \Rightarrow V^{-1} = \frac{1}{7}\begin{pmatrix} -5 & 2 \\ 1 & 1 \end{pmatrix}.$$

So,

$$(a_1, a_2) = \frac{1}{7}\begin{pmatrix} -5 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{7}(-5x_0 + 2y_0, x_0 + y_0).$$

Thus,

$$\begin{aligned}
\mathbf{x}(t) &= \frac{1}{7}\left((-5x_0 + 2y_0)e^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + (x_0 + y_0)e^{t}\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right) \\
&= \frac{1}{7}\left((5x_0 - 2y_0)e^{6t} + 2(x_0 + y_0)e^{t}, (-5x_0 + 2y_0)e^{-6t} + 5(x_0 + y_0)e^{t}\right) \\
&= \frac{1}{7}\left(x_0(5e^{-6t} + 2e^{t}) + y_0(-2e^{-6t} + 2e^{t}), x_0(-5e^{-6t} + 5e^{t}) + y_0(2e^{-6t} + 5e^{t})\right) \\
&= \boxed{\frac{1}{7}\begin{pmatrix} 5e^{-6t} + 2e^{t} & -2e^{-6t} + 2e^{t} \\ -5e^{-6t} + 5e^{t} & 2e^{-6t} + 5e^{t} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = e^{tA}\mathbf{x_0}}
\end{aligned}$$

From (5) we can clearly see that

$$\lim_{t\to\infty} e^{-6t} = 0 \quad \text{and} \quad \lim_{to\to\infty} e^{t} = \infty$$

so, for $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ we must have $\mathbf{x_0} = k(-1, 1)$ for $k \in \mathbb{R}$.

3.5.5:

Solution:

(a) We start by finding the eigenvalues of A using the characteristic equation:

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3.$$

Then the associated eigenvector is

$$A - 3I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Because there is only a single eigenvector, we follow the process used in example 31 to preform a change of variables. Let

$$V = \begin{pmatrix} \vec{u} & \vec{u}^{\perp} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Then

$$V^{-1}AV = \begin{pmatrix} 3 & -5 \\ 0 & 3 \end{pmatrix}$$

which represents a recursively coupled system

$$\mathbf{y}'(t) = (V^{-1}AV)\mathbf{y}(t).$$

Take $\mathbf{y}(t) = (u(t), v(t))$ so $v'(t) = 3v(t)$ thus

$$v(t) = v_0e^{3t}.$$

It follows that

$$\begin{aligned}
&u'(t) = 3u(t) - 5v(t)\\
\Rightarrow\, &u'(t) = 3u(t) - 5v_0e^{3t}\\
\Rightarrow\, &u'(t)e^{-3t} - 3u(t)e^{-3t} = -5v_0\\
\Rightarrow\, &(u(t)e^{-3t})' = -5v_0\\
\Rightarrow\, &u(t) = (-5v_0t + u_0)e^{3t}.
\end{aligned}$$

So,

$$\mathbf{y}(t) = e^{3t}(-5v_0 + u_0, v_0) = e^{3t}\begin{pmatrix}1 & -5t\\ 0 & 1\end{pmatrix}\mathbf{y_0}.$$

Let $\mathbf{y_0} = \mathbf{x_0}$ and $\mathbf{x}(t) = V^{-1}\mathbf{y}$. Then,

$$\begin{aligned}
\mathbf{x}(t) &= V^{-1}e^{3t}\begin{pmatrix}1 & -5t\\ 0 & 1\end{pmatrix}V\mathbf{x_0}\\
&= \frac{e^{3t}}{5}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & -5t\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\mathbf{x_0}\\
&= \boxed{e^{3t}\begin{pmatrix}1-2t & -t\\ 4t & 2t+1\end{pmatrix}\mathbf{x_0} = e^{tA}\mathbf{x_0}}
\end{aligned}$$

(b) Similar to part (b) before, we can see that $\lim_{t\to\infty} e^{3t} = \infty$ for all t, so $\lim_{t\to\infty}\mathbf{x}(t) = \mathbf{0}$ if, and only if $\mathbf{x_0} = (0,0)$.

3.5.6:

Solution:

(a) We start by finding the eigenvectors of A using the characteristic equation:

$$\lambda^2 + 5\lambda - 14 = 0 \Rightarrow \lambda_1 = -7, \lambda_2 = 2.$$

The eigenvalues are

$$\begin{aligned}
A + 7I &= \begin{pmatrix}3 & 2\\ 9 & 6\end{pmatrix}\\
&\to \begin{pmatrix}3 & 2\\ 0 & 0\end{pmatrix}\\
&\Rightarrow \vec{u}_1 = \begin{pmatrix}-2\\ 3\end{pmatrix}
\end{aligned}
\quad\text{and}\quad
\begin{aligned}
A - 2I = \begin{pmatrix}-6 & 2\\ 9 & -3\end{pmatrix} &\to \begin{pmatrix}-3 & 1\\ 0 & 0\end{pmatrix}\\
\Rightarrow \vec{u}_2 = \begin{pmatrix}1\\ 3\end{pmatrix}&.
\end{aligned}$$

Take

$$M(t) = \left(e^{-7t}(-2,3) \quad e^{2t}(1,3)\right) = \begin{pmatrix}-2e^{-7t} & e^{2t}\\ 3e^{-7t} & 3e^{2t}\end{pmatrix}$$

so that

$$M(0) = \begin{pmatrix}-2 & 1\\ 3 & 3\end{pmatrix} \Rightarrow M(0)^{-1} = \frac{1}{9}\begin{pmatrix}-3 & 1\\ 3 & 2\end{pmatrix}.$$

Thus,

$$\begin{aligned}\mathbf{x}(t) = M(t)M(0)^{-1}\mathbf{x_0} &= \begin{pmatrix} -2e^{-7t} & e^{2t} \\ 3e^{-7t} & 3e^{2t} \end{pmatrix} \frac{1}{9} \begin{pmatrix} -3 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x_0} \\ &= \boxed{\frac{1}{9} \begin{pmatrix} 6e^{-7t} + 3e^{2t} & -2e^{-7t} + 2e^{2t} \\ -9e^{-7t} + 9e^{2t} & 3e^{-7t} + 6e^{2t} \end{pmatrix} = \frac{1}{9} e^{tA} \mathbf{x_0}}\end{aligned}$$

(b) Similar to before it is clear that $\lim_{t\to\infty} e^{7t} = 0$ and $\lim_{t\to\infty} e^{2t} = \infty$ so for $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ we must have $\mathbf{x_0} = k(-2, 3)$