

**Exercise 4:** Consider the initial data of

$$x_0 = 1, \quad u_0 = 0, \quad y_0 = 1, \quad v_0 = \sqrt{3},$$

and find  $R_2 = \max_{t \in \mathbb{R}} \|\vec{x}(t)\|$  as well as  $R_1 = \min_{t \in \mathbb{R}} \|\vec{x}\|$  and the times that correspond to  $R_1$  and  $R_2$ .

**Solution:** We start by finding

$$A = \begin{pmatrix} 1 & 0 \\ 1 & \sqrt{3} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \Rightarrow (A^{-1})^T = \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix}.$$

It follows that

$$\begin{aligned} M &= (A^{-1})^T (A^{-1}) \\ &= \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{pmatrix}. \end{aligned}$$

To find the eigenvectors and eigenvalues of  $M$  we start by solving the characteristic equation as follows:

$$\begin{aligned} \lambda^2 - \lambda \left( \frac{5}{3} \right) + 1/3 &= 0 \Rightarrow 3\lambda^2 - 5\lambda + 1 = 0 \\ \Rightarrow \lambda &= \frac{5 \pm \sqrt{25 - 4(3)}}{6} \\ \Rightarrow \lambda_1 &= \frac{5 + \sqrt{13}}{6}, \lambda_2 = \frac{5 - \sqrt{13}}{6} \end{aligned}$$

From the first set of challenge problems we know that  $\|\vec{x}(t)\|$  is maximal if, and only if,

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_2}} \vec{u}_2,$$

where  $\vec{u}_2$  is an orthonormal eigenvector of  $M$ . It follows that

$$R_2 = \max \|\vec{x}(t)\| = \left\| \pm \frac{1}{\sqrt{\lambda_2}} \vec{u}_2 \right\| = \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{6}{5 - \sqrt{13}}} = \boxed{\sqrt{\frac{5 + \sqrt{13}}{2}}}.$$

Using the conclusion also from the first set of challenge problems that  $\|\vec{x}(t)\|$  is minimal if, and only if,

$$\vec{x}(t) = \pm \frac{1}{\sqrt{\lambda_1}} \vec{u}_1,$$

we can arrive at the conclusion

$$R_1 = \min \|\vec{x}(t)\| = \left\| \pm \frac{1}{\sqrt{\lambda_1}} \vec{u}_1 \right\| = \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{6}{5 + \sqrt{13}}} = \boxed{\sqrt{\frac{5 - \sqrt{13}}{2}}}.$$

Now, in order to find the values of  $t$  that correspond to  $R_1$  and  $R_2$  we must find the eigenvectors of  $M$ . For  $\lambda_1$ :

$$\begin{aligned} M - \lambda_1 I_2 &= \begin{pmatrix} \frac{3-\sqrt{13}}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{-3-\sqrt{13}}{6} \end{pmatrix} \\ &\xrightarrow[R_1 \rightarrow 6R_1]{R_2 \rightarrow 6R_2} \begin{pmatrix} 3 - \sqrt{13} & -2 \\ -2 & -3 - \sqrt{13} \end{pmatrix} \\ \Rightarrow \vec{u}_1 &= \begin{pmatrix} 2 \\ 3 - \sqrt{13} \end{pmatrix}. \end{aligned}$$

Similarly, for  $\lambda_2$ :

$$\begin{aligned} M - \lambda_2 I_2 &= \begin{pmatrix} \frac{3+\sqrt{13}}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{-3+\sqrt{13}}{6} \end{pmatrix} \\ &\xrightarrow{\substack{R_1 \rightarrow 6R_1 \\ R_2 \rightarrow 6R_2}} \begin{pmatrix} 3 + \sqrt{13} & -2 \\ -2 & -3 + \sqrt{13} \end{pmatrix} \\ &\Rightarrow \vec{u}_2 = \begin{pmatrix} 2 \\ 3 - \sqrt{13} \end{pmatrix}. \end{aligned}$$

Now, from the first set of challenge problems we know that when  $\|\vec{x}(t)\|$  is maximal

$$\vec{u}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \pm \vec{v}_2$$

But,

$$\begin{aligned} \pm \vec{v}_2 &= \pm \frac{1}{\sqrt{\lambda_2}} A^{-1} \vec{u}_2 = \pm \frac{1}{\sqrt{\lambda_2}} \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 \\ 3 + \sqrt{13} \end{pmatrix} \\ &= \pm \frac{1}{\sqrt{\lambda_2}} \begin{pmatrix} 2 \\ \frac{1+\sqrt{13}}{\sqrt{3}} \end{pmatrix}. \end{aligned}$$

Thus,  $\|\vec{x}(t)\|$  is maximal when

$$\tan t = \frac{\frac{1+\sqrt{13}}{\sqrt{3}}}{2} \Rightarrow \boxed{t = \arctan \left( \frac{1+\sqrt{13}}{2\sqrt{3}} \right) + \pi n, \quad n \in \mathbb{N}.}$$

Using the fact that  $\|\vec{x}(t)\|$  is minimal when

$$\vec{u}(t) = \pm \vec{v}_2 = \pm \frac{1}{\sqrt{\lambda_1}} A^{-1} \vec{u}_2,$$

we can follow a similar manner of calculation to see that

$$\tan t = \frac{\frac{1-\sqrt{13}}{\sqrt{3}}}{2} \Rightarrow \boxed{t = \arctan \left( \frac{1-\sqrt{13}}{2\sqrt{3}} \right) + \pi n, \quad n \in \mathbb{N}.}$$

**Exercise 5:** Given

$$V = (\vec{v}_1 \quad \vec{v}_1), \quad S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad U = (\vec{u}_1 \quad \vec{u}_2),$$

we must show that

$$A\vec{x} = \sigma(\vec{x} \cdot \vec{v}_1)\vec{u}_1 + \sigma_2(\vec{x} \cdot \vec{v}_2)\vec{u}_2 = USV^T\vec{x}$$

**Solution:** Consider

$$\begin{aligned} USV^T\vec{x} &= (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} \vec{x} \\ &= (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \vec{x} \cdot \vec{v}_1 \\ \vec{x} \cdot \vec{v}_2 \end{pmatrix} \\ &= (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} \sigma_1(\vec{x} \cdot \vec{v}_1) \\ \sigma_2(\vec{x} \cdot \vec{v}_2) \end{pmatrix} \\ &= [\sigma_1(\vec{x} \cdot \vec{v}_1)\vec{u}_1 + \sigma_2(\vec{x} \cdot \vec{v}_2)\vec{u}_2] \end{aligned}$$

**Exercise 6:** Let

$$A = \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix}.$$

Compute a singular value decomposition of  $A$ .

**Solution:** We start by finding

$$\begin{aligned} A^T &= \begin{pmatrix} 11 & 2 \\ -5 & -10 \end{pmatrix} \\ \Rightarrow A^T A &= \begin{pmatrix} 11 & 2 \\ -5 & -10 \end{pmatrix} \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix} = \begin{pmatrix} 125 & -75 \\ -75 & 125 \end{pmatrix}. \end{aligned}$$

Now, we solve for the eigenvalues using the characteristic equation:

$$0 = \lambda^2 - \lambda(250) + (10000) \Rightarrow (\lambda - 50)(\lambda - 200) = 0 \Rightarrow \lambda_1 = 50, \lambda_2 = 200.$$

For  $\lambda_1 = 50$ :

$$A^T A - 50I = \begin{pmatrix} 75 & -75 \\ -75 & 75 \end{pmatrix} \rightarrow \begin{pmatrix} 75 & -75 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 75 \\ 75 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For  $\lambda_2 = 200$ :

$$A^T A - 200I = \begin{pmatrix} -75 & -75 \\ -75 & -75 \end{pmatrix} \rightarrow \begin{pmatrix} -75 & -75 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 75 \\ -75 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can normalize to get the following eigenvectors:

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

Since  $\sigma_i = \sqrt{\lambda_i}$ , we know that

$$\sigma_1 = \sqrt{50} \quad \text{and} \quad \sigma_2 = \sqrt{200},$$

so,

$$\begin{aligned} A\vec{v}_1 = \sigma_1 \vec{u}_1 &\Rightarrow \frac{1}{\sigma_1} A\vec{v}_1 = \vec{u}_1 & A\vec{v}_2 = \sigma_2 \vec{u}_2 &\Rightarrow \frac{1}{\sigma_2} A\vec{v}_2 = \vec{u}_2 \\ &\Rightarrow \frac{1}{\sqrt{50}} \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} && \Rightarrow \frac{1}{\sqrt{200}} \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \quad \text{Thus,} \\ &= \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix} = \vec{u}_1 && = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} = \vec{u}_2 \end{aligned}$$

$A = USV^T$  for

$$U = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}, \quad S = \begin{pmatrix} \sqrt{50} & 0 \\ 0 & \sqrt{200} \end{pmatrix}, \quad V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

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<sup>0</sup>LATEX code for this document can be found on github [here](#)