Exercise 4: Consder the intial data of

$$x_0 = 1, \quad u_0 = 0, \quad y_0 = 1, \quad v_0 = \sqrt{3},$$

and find $R_2 = \max_{t\in\mathbb{R}} \|\vec{x}(t)\|$ as well was $R_1 = \min_{t\in\mathbb{R}} \|\vec{x}\|$ and the times that correspond to R_1 and R_2.

Solution: We start by finding

$$A = \begin{pmatrix} 1 & 0 \\ 1 & \sqrt{3} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \Rightarrow (A^{-1})^T = \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix}.$$

It follows that

$$\begin{aligned} M &= (A^{-1})^T (A^{-1}) \\ &= \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{pmatrix}. \end{aligned}$$

To find the eigenvectors and eigenvalues of M we start by solving the characteristic equation as follows:

$$\begin{aligned} \lambda^2 - \lambda\left(\frac{5}{3}\right) + 1/3 = 0 &\Rightarrow 3\lambda^2 - 5\lambda + 1 = 0 \\ &\Rightarrow \lambda = \frac{5 \pm \sqrt{25 - 4(3)}}{6} \\ &\Rightarrow \lambda_1 = \frac{5+\sqrt{13}}{6}, \lambda_2 = \frac{5-\sqrt{13}}{6} \end{aligned}$$

From the first set of challenge problems we know that $\|\vec{x}(t)\|$ is maximal if, and only if,

$$\vec{x}(t) = \pm\frac{1}{\sqrt{\lambda_2}}\vec{u_2},$$

where $\vec{u}_2$ is an orthonormal eigenvector of M. It follows that

$$R_2 = \max\|\vec{x}(t)\| = \left\|\pm\frac{1}{\sqrt{\lambda_2}}\vec{u}_2\right\| = \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{6}{5-\sqrt{13}}} = \boxed{\sqrt{\frac{5+\sqrt{13}}{2}}.}$$

Using the conclusion also from the first set of challenge problems that $\|\vec{x}(t)]\|$ is minimal if, and only if,

$$\vec{x}(t) = \pm\frac{1}{\sqrt{\lambda_1}}\vec{u}_1,$$

we can arrive at the conclusion

$$R_1 = \min\|\vec{x}(t)\| = \left\|\pm\frac{1}{\sqrt{\lambda_1}}\vec{u}_1\right\| = \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{6}{5+\sqrt{13}}} = \boxed{\sqrt{\frac{5-\sqrt{13}}{2}}.}$$

Now, in order to find the values of t that correspond to R_1 and R_2 we must find the eigenvectors of M. For λ_1:

$$\begin{aligned} M - \lambda_1 I_2 &= \begin{pmatrix} \frac{3-\sqrt{13}}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{-3-\sqrt{13}}{6} \end{pmatrix} \\ &\xrightarrow[R_2\to 6R_2]{R_1\to 6R_1} \begin{pmatrix} 3-\sqrt{13} & -2 \\ -2 & -3-\sqrt{13} \end{pmatrix} \\ &\Rightarrow \vec{u}_1 = \begin{pmatrix} 2 \\ 3-\sqrt{13} \end{pmatrix}. \end{aligned}$$

Similarly, for λ_2:

$$\begin{aligned} M - \lambda_2 I_2 &= \begin{pmatrix} \frac{3+\sqrt{13}}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{-3+\sqrt{13}}{6} \end{pmatrix} \\ &\xrightarrow[R_2 \to 6R_2]{R_1 \to 6R_1} \begin{pmatrix} 3+\sqrt{13} & -2 \\ -2 & -3+\sqrt{13} \end{pmatrix} \\ &\Rightarrow \vec{u}_2 = \begin{pmatrix} 2 \\ 3-\sqrt{13} \end{pmatrix}. \end{aligned}$$

Now, from the first set of challenge problems we know that when $\|\vec{x}(t)\|$ is maximal

$$\vec{u}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \pm \vec{v}_2$$

But,

$$\begin{aligned} \pm\vec{v}_2 = \pm\frac{1}{\sqrt{\lambda_2}} A^{-1}\vec{u}_2 &= \pm\frac{1}{\sqrt{\lambda_2}} \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 \\ 3+\sqrt{13} \end{pmatrix} \\ &= \pm\frac{1}{\sqrt{\lambda_2}} \begin{pmatrix} 2 \\ \frac{1+\sqrt{13}}{\sqrt{3}} \end{pmatrix}. \end{aligned}$$

Thus, $\|\vec{x}(t)\|$ is maximal when

$$\tan t = \frac{\frac{1+\sqrt{13}}{\sqrt{3}}}{2} \Rightarrow \boxed{t = \arctan\left(\frac{1+\sqrt{13}}{2\sqrt{3}}\right) + \pi n, \quad n \in \mathbb{N}.}$$

Using the fact that $\|\vec{x}(t)\|$ is minimal when

$$\vec{u}(t) = \pm\vec{v}_2 = \pm\frac{1}{\sqrt{\lambda_1}} A^{-1}\vec{u}_2,$$

we can follow a similar manner of calculation to see that

$$\tan t = \frac{\frac{1-\sqrt{13}}{\sqrt{3}}}{2} \Rightarrow \boxed{t = \arctan\left(\frac{1-\sqrt{13}}{2\sqrt{3}}\right) + \pi n, \quad n \in \mathbb{N}.}$$

See (1) for a possible sketch of the ellipse

Exercise 5: Given

$$V = \begin{pmatrix} \vec{v_1} & \vec{v_1} \end{pmatrix}, \quad S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad U = \begin{pmatrix} \vec{u_1} & \vec{u_2} \end{pmatrix},$$

we must show that

$$A\vec{x} = \sigma(\vec{x} \cdot \vec{v_1})\vec{u_1} + \sigma_2(\vec{x} \cdot \vec{v_2})\vec{u_2} = USV^T\vec{x}$$

Solution: Consider

$$\begin{aligned} USV^T x &= \begin{pmatrix} \vec{u_1} & \vec{u_2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \vec{v_1} \\ \vec{v_2} \end{pmatrix} \vec{x} \\ &= \begin{pmatrix} \vec{u_1} & \vec{u_2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \vec{x} \cdot \vec{v_1} \\ \vec{x} \cdot \vec{v_2} \end{pmatrix} \\ &= \begin{pmatrix} \vec{u_1} & \vec{u_2} \end{pmatrix} \begin{pmatrix} \sigma_1(\vec{x} \cdot \vec{v_1}) \\ \sigma_2(\vec{x} \cdot \vec{v_2}) \end{pmatrix} \\ &= \boxed{\sigma_1(\vec{x} \cdot \vec{v_1})\vec{u_1} + \sigma_2(\vec{x} \cdot \vec{v_2})\vec{u_2}} \end{aligned}$$

Exercise 6: Let

$$A = \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix}.$$

Compute a singular value decomposition of A.

Solution: We start by finding

$$\begin{aligned} A^T &= \begin{pmatrix} 11 & 2 \\ -5 & -10 \end{pmatrix} \\ \Rightarrow A^T A &= \begin{pmatrix} 11 & 2 \\ -5 & -10 \end{pmatrix} \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix} = \begin{pmatrix} 125 & -75 \\ -75 & 125 \end{pmatrix}. \end{aligned}$$

Now, we solve for the eigenvalues using the characteristic equation:

$$0 = \lambda^2 - \lambda(250) + (10000) \Rightarrow (\lambda - 50)(\lambda - 200) = 0 \Rightarrow \lambda_1 = 50, \lambda_2 = 200.$$

For $\lambda_1 = 50$:

$$A^T A - 50I = \begin{pmatrix} 75 & -75 \\ -75 & 75 \end{pmatrix} \to \begin{pmatrix} 75 & -75 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v_1} = \begin{pmatrix} 75 \\ 75 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 200$:

$$A^T A - 200I = \begin{pmatrix} -75 & -75 \\ -75 & -75 \end{pmatrix} \to \begin{pmatrix} -75 & -75 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v_2} = \begin{pmatrix} 75 \\ -75 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can normalize to get the following eigenvectors:

$$\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

Since $\sigma_i = \sqrt{\lambda_i}$, we know that

$$\sigma_1 = \sqrt{50} \quad \text{and} \quad \sigma_2 = \sqrt{200},$$

so,

$$\begin{aligned} A\vec{v_1} = \sigma_1 \vec{u_1} &\Rightarrow \frac{1}{\sigma_1} A\vec{v_1} = \vec{u_1} \\ &\Rightarrow \frac{1}{\sqrt{50}} \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix} = \vec{u}_1 \end{aligned}$$

$$\begin{aligned} A\vec{v_2} = \sigma_2 \vec{u_2} &\Rightarrow \frac{1}{\sigma_2} A\vec{v_2} = \vec{u_2} \\ &\Rightarrow \frac{1}{\sqrt{200}} \begin{pmatrix} 11 & -5 \\ 2 & -10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} = \vec{u}_2 \end{aligned}$$

Thus, $A = USV^T$ for

$$\boxed{U = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}, \quad S = \begin{pmatrix} \sqrt{50} & 0 \\ 0 & \sqrt{200} \end{pmatrix}, \quad V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}$$

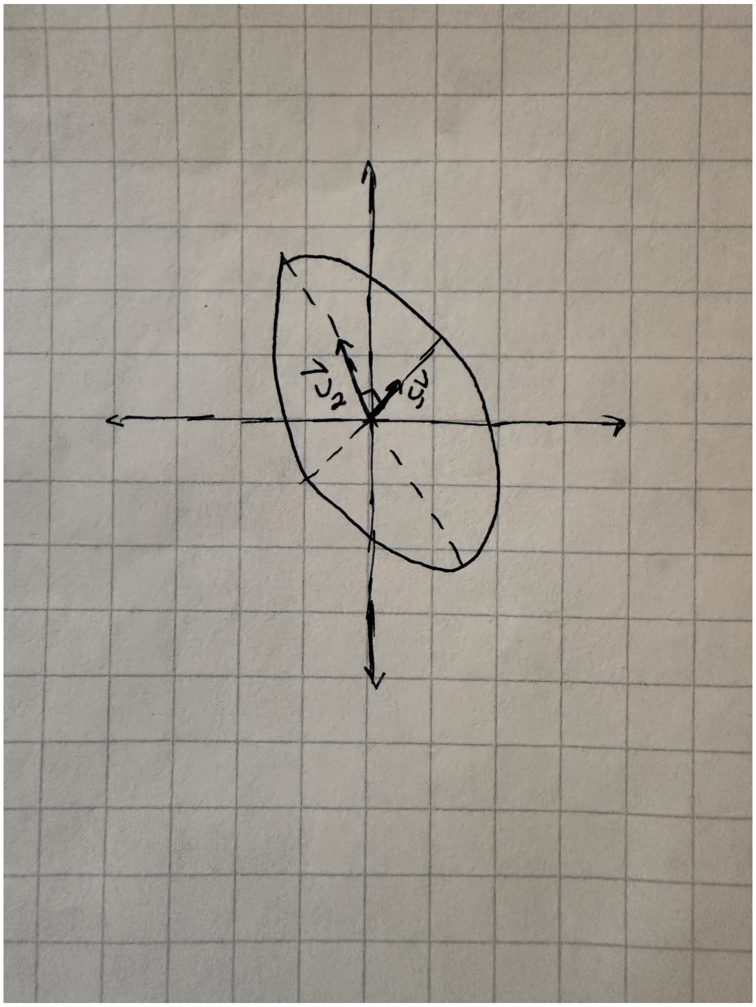

Figure 1: Sketch of Ellipse

[0]LaTeX code for this document can be found on github [here]