1. (a) Find the general solution of

$$tx'(t) = x(t) + 3t^2$$

Solution: First, notice that,

$$tx'(t) = x(t) + 3t^2 \Rightarrow x'(t) - \frac{1}{t}x(t) = 3t.$$

This is a first order linear differential equation. Let

$$\mu(t) = e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = \frac{1}{t}$$

be the integrating factor. Then,

$$\begin{aligned} \frac{1}{t}x' - \frac{1}{t^2}x = 3 &\Rightarrow \int (tx)'\,dt = \int 3\,dt \\ &\Rightarrow \frac{1}{t}x = 3t + C \\ &\Rightarrow \boxed{x(t) = 3t^2 + Ct} \end{aligned}$$

(b) Find the flow transformation $\Phi_{t,t_0}(x)$ specified by this equation

Solution: Let us choose C such that $x(t_0) = x$,

$$x(t_0) = 3t_0^2 + Ct_0 \Rightarrow C = \frac{x - 3t_0^2}{t_0}.$$

Thus, our flow transformation will be

$$x(t) = \boxed{\Phi_{t,t_0}(x) = 3t^2 + \frac{x - 3t_0^2}{t_0}t}$$

(c) Find the particular solution $x(t)$ that satisfies $x(1) = 2$

Solution:

$$\begin{aligned} x(t) = 3t^2 + Ct &\Rightarrow x(1) = 2 = 3(1)^2 + C(1) \\ &\Rightarrow C = -1 \\ &\Rightarrow \boxed{x(t) = 3t^2 - 1} \end{aligned}$$

2. (a) Find the general solution of

$$x'(t) + \frac{1}{3}x(t) = e^t x^4(t)$$

Solution: Start by dividing the entire equation by $x^4(t)$ to get

$$x^{-4}x' + \frac{1}{3}x^{-3} = e^t.$$

Now, this is clearly a Bernoulli differential equation, so consider the change of variable

$$v = x^{-3} \Rightarrow v' = -3x^{-4}x' \Rightarrow -\frac{1}{3}v' = x^{-4}x'.$$

Applying our change of variables we can see that

$$-\frac{1}{3}v' + \frac{1}{3}v = e^t \Rightarrow v' - v = -3e^t.$$

This is now a linear differential equation, so we will let our integrating factor be

$$\mu(t) = e^{\int -1\,dt} = e^{-t}.$$

Multiplying through by the integrating factor and solving shows

$$\begin{aligned} e^{-t}v' - e^{-t}v = -3 &\Rightarrow \int (e^{-t}v)'\,dt = \int -3\,dt \\ &\Rightarrow e^t v = -3t + C \\ &\Rightarrow v = -3te^t + Ce^t \\ &\Rightarrow x^{-3} = -3te^t + Ce^t \\ &\Rightarrow \boxed{x(t) = \left(-3te^t + Ce^t\right)^{-1/3}} \end{aligned}$$

(b) Find the particular solution $x(t)$ that satisfies $x(1) = 2$. Over what time interval $t \in (a, b)$ is the solution a continuously differentiable function?

Solution: First we will solve for C,

$$\begin{aligned} x(1) = 2 &\Rightarrow 2 = (-3(1)e^1 + Ce^1)^{-1/3} \\ &\Rightarrow 2^{-3} = e(C-3) \\ &\Rightarrow \frac{1}{8e} = C - 3 \\ &\Rightarrow \frac{1}{8e} + 3 = C \end{aligned}$$

Thus, our particular solution is

$$\boxed{x(t) = \left(-3te^t + \left(\frac{1}{8e} + 3\right)e^t\right)^{-1/3} = \frac{1}{\sqrt[3]{e^t\left(3(1-t) + \frac{1}{8e}\right)}}}$$

Since this is a rational function we must check when the denominator equals 0,

$$3(1-t)+\frac{1}{8e}=0 \Rightarrow 3+\frac{1}{8e}=3t$$
$$\Rightarrow t=1+\frac{1}{24e}.$$

Note that we were able to drop e^t from the the denominator since it will never be 0. Thus, since our interval must contain the initial condition,

$$\boxed{x(t) \text{ is continuously differentiable for all } t \in \left(-\infty, 1+\frac{1}{24e}\right).}$$

3. (a) Find the general solution of

$$tx'(t)=tx^2(t)-x(t)-\frac{1}{t}$$

Solution: First, rewrite the equation as

$$x'=x^2-\frac{1}{t}x-\frac{1}{t^2}.$$

This is a Riccati equation. Since all the coefficents are powers of t we will consider a particular solution in the form $x_1=ct^\alpha$ for $c, \alpha \in \mathbb{R}$,

$$\alpha ct^{\alpha-1}=c^2t^{2\alpha}-ct^{\alpha-1}-t^{-2}$$

Since all the powers of t must be equal we can solve for α:

$$\alpha-1=2\alpha=-2 \Rightarrow \alpha=-1.$$

This allows us to cancel all the t's from the equation leaving us with

$$c=c^2-c-1 \Rightarrow c^2-1=0$$
$$\Rightarrow (c-1)(c+1)=0$$
$$\Rightarrow c=-1,1.$$

Thus, we have $x_1=t^{-1}$ or $x_1=-t^{-1}$. Choosing $x_1=t^{-1}$ we can solve for the general solution with the formula

$$x(t)=x_1(t)+u(t)$$

where $u(t)$ is a solution to the equation

$$u'=(q+2rx_1)u+ru^2$$

for $q=-t^{-1}$ (coefficent of the linear term) and $r=1$ (coefficent of the quadratic term). We can solve this as follows:

$$\begin{aligned} u' = (-t^{-1} + 2(1)t^{-1})u + u^2 &\Rightarrow u'u^{-2} - t^{-1}u^{-1} = 1 \\ &\Rightarrow \text{Let } v = u^{-1} \Rightarrow v' = -u^{-2}u' \\ &\Rightarrow v' + t^{-1}v = -1 \\ &\Rightarrow \mu(t) = e^{\int t^{-1}\,dt} = e^{\ln t} = t \\ &\Rightarrow tv' + v = -t \\ &\Rightarrow \int (tv)'\,dt = \int -t\,dt \\ &\Rightarrow tv = -\frac{1}{2}t^2 + C \\ &\Rightarrow v = -\frac{1}{2}t + \frac{C}{t} \\ &\Rightarrow u^{-1} = -\frac{1}{2} + \frac{C}{t} \\ &\Rightarrow u = \left(-\frac{1}{2}t + \frac{C}{t}\right)^{-1}. \end{aligned}$$

Thus, our general solution is

$$x(t) = x_1(t) + u(t) = \frac{1}{t} + \frac{1}{-\frac{1}{2}t + \frac{C}{t}} = \boxed{\frac{t^2 + C}{t(C - t^2)}}$$

(b) For any (x_0, t_0) with $t_0 > 0$, find the solution $x(t)$ of this equation that satisfies $x(t_0) = x_0$

Solution: We can solve for C using the initial conditions as follows,

$$\begin{aligned} x(t_0) = x_0 \Rightarrow x_0 &= \frac{t_0^2 + C}{t_0\,(C - t_0^2)} \\ &\Rightarrow x_0 t_0 (C - t_0^2) = t_0^2 + C \\ &\Rightarrow C x_0 t_0 - x_0 t_0^3 = t_0^2 + C \\ &\Rightarrow C(x_0 t_0 - 1) = t_0^2 + x_0 t_0^3 \\ &\Rightarrow C = \frac{t_0^2 + x_0 t_0^3}{x_0 t_0 - 1}. \end{aligned}$$

Thus, our particular solution is

$$x(t) = \frac{t^2 + \frac{t_0^2 + x_0 t_0^3}{x_0 t_0 - 1}}{t\left(\frac{t_0^2 + x_0 t_0^3}{x_0 t_0 - 1} - t^2\right)} = \boxed{\frac{x_0 t_0 (t^2 + t_0^2) - (t^2 - t_0^2)}{t[x_0 t_0 (t_0^2 - t^2) + (t^2 + t_0^2)]}}$$

(c) Write down a formula for the flow transformation Φ_{t,t_0} generated by this equation. Verify explicity that $\Phi_{3,2}(\Phi_{2,1}(x)) = \Phi_{3,1}(x)$ for all x

Solution: We choose C such that $x(t_0) = x$. By substituing x_0 for x in our work in part (b) we can see that

$$C = \frac{xt_0^3 + t_0^2}{xt_0 - 1} \Rightarrow x(t) = \frac{xt_0(t^2 + t_0^2) - (t^2 - t_0^2)}{t[xt_0(t_0^2 - t^2) + (t^2 + t_0^2)]}.$$

It follows that the flow transformation is

$$\Phi_{t_1,t_0}(x) = x(t_1) = \boxed{\frac{xt_0(t_1^2 + t_0^2) - (t_1^2 - t_0^2)}{t_1[xt_0(t_0^2 - t_1^2) + (t_1^2 + t_0^2)]}}$$

In order to verify that $\Phi_{3,2}(\Phi_{2,2}(x)) = \Phi_{3,1}(x)$ we will need the following:

$$\begin{aligned}
\Phi_{2,1}(x) &= \frac{x(1+4) - (4-1)}{2(x(1-4) + (4+1))} = \frac{5x-3}{10-6x} \\
\Phi_{3,2}(x) &= \frac{x(2)(9+4) - (9-4)}{3(x(2)(4-9) + (9+4))} = \frac{26x-5}{39-30x} \\
\Phi_{3,1}(x) &= \frac{x(9+1) - (9-1)}{3(x(1-9) + (9+1))} = \frac{10x-8}{30-24x}.
\end{aligned}$$

We can now verify that

$$\begin{aligned}
\Phi_{3,2}(\Phi_{2,1}(x)) = \frac{26\left(\frac{5x-3}{10-6x}\right) - 5}{39 - 30\left(\frac{5x-3}{10-6x}\right)} &= \frac{26(5x-3) - 5(10-6x)}{39(10-6x) - 30(5x-3)} \\
&= \frac{130x - 78 - 50 + 30x}{390 - 234x - 150x + 90} \\
&= \frac{160x - 128}{480 - 384x} \\
&= \boxed{\frac{10x-8}{30-24x} = \Phi_{3,1}(x)}
\end{aligned}$$

4. Consider the equation

$$x'(t) = 2t\frac{t^2 + x(t)}{t^2 - x(t)} \tag{1}$$

with the change of variable

$$y(t) = \frac{x(t)}{t^2} \tag{2}$$

for $t > 0$.

(a) Show that $x(t)$ solves (1) for $t > 0$ if, and only if, $y(t)$ solves a seperable equation for $t > 0$.

Solution: We start by assuming that $x(t)$ solves (1) for $t > 0$ in order to show that $y(t)$ solves a seperable equation for $t > 0$. If $x(t)$ solves (1) then,

$$y(t) = \frac{x(t)}{t^2} \Rightarrow x = t^2 y \Rightarrow x' = 2ty + t^2 y'.$$

Thus,

$$
\begin{aligned}
2t\left(\frac{t^2+t^2y}{t^2-t^2y}\right) &= 2ty+t^2y \\
\Rightarrow 2t\left(\frac{1+y}{1-y}-y\right) &= t^2y' \\
\Rightarrow 2t\left(\frac{1+y^2}{1-y}\right) &= t^2y' \\
\Rightarrow \frac{2}{t}\left(\frac{1+y^2}{1-y}\right) &= \frac{dy}{dt} \\
\Rightarrow \frac{2}{t}dt &= \frac{1-y}{1+y^2}dy. \qquad (3)
\end{aligned}
$$

This is a seperable equation. We can prove that assuming $y(t)$ solves a seperable equation is a sufficient condition for $x(t)$ solving (1) by assuming that $y(t)$ solves (3) and following the steps above in reverse.

(b) Find the solution of (2) with $y(1) = y_0, y_0 \neq 1$

Solution: Starting with (3) we have

$$
\begin{aligned}
\int \frac{2}{t}\,dt = \int \frac{1-y}{1+y^2}\,dt &= \int \frac{1}{1+y^2} - \frac{y}{1+y^2}\,dy \\
\Rightarrow 2\ln t + C &= \arctan(y) - \frac{1}{2}\ln(1+y^2).
\end{aligned}
$$

Applying the initial condition we can see that

$$y(1) = y_0 \Rightarrow C = \arctan(y_0) - \frac{1}{2}\ln(1+y_0^2).$$

Thus, the general solution is

$$\boxed{2\ln t + \arctan(y_0) - \frac{1}{2}\ln(1+y_0^2) = \arctan(y) - \frac{1}{2}\ln(1+y^2)}$$

(c) Find the solution of (1) with $x(1) = x_0, x_0 \neq 1$

Solution: We start by noticing that

$$x(t) = t^2y(t) \Rightarrow x(1) = y(1) \Rightarrow x_0 = y_0.$$

Thus, the general solution for $x(t)$ is

$$\boxed{2\ln t + \arctan(x_0) - \frac{1}{2}\ln(1+x_0^2) = \arctan(t^{-2}x) - \frac{1}{2}\ln(1+t^{-4}x^2)}$$

5. Find the general solution of the equation

$$tx''(t) = 1 + (x'(t))^2$$

Solution: Consider the change of variables $y = x'$ and substitute:

$$\begin{aligned} ty' = 1 + y^2 \Rightarrow t\frac{dy}{dt} &= 1 + y^2 \\ &\Rightarrow \int \frac{1}{1+y^2}\,dy = \int \frac{1}{t}\,dt \\ &\Rightarrow \arctan(y) = \ln(t) + C \\ &\Rightarrow y = \tan(\ln(t) + C). \end{aligned}$$

It follows that

$$x(t) = \int y(t)\,dt = \boxed{\int \tan(\ln t + C)\,dt}$$

for $C \in \mathbb{R}$. Note that I was unable to figure out how to integrate the explicit function for y.

[0] $\LaTeX$ code for this document can be found on github here