Consider the equation

$$\mathbf{x}''(t) = -K\vec{x}(t) + \mathbf{f}(t), \quad \vec{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1) \tag{1}$$

and

$$\mathbf{f}(t) = \sum_{j=1}^{4} \mathbf{f}_j(t) \tag{2}$$

for

$$\mathbf{f}_1(t) = (1, 3)\cos(\omega_1 t)$$
$$\mathbf{f}_2(t) = (1, -1)\cos(\omega_2 t)$$
$$\mathbf{f}_3(t) = (3, -1)\cos(\omega_3 t)$$
$$\mathbf{f}_4(t) = (1, 0)\cos(\omega_4 t)$$

and

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

1. If $\mathbf{x}_0(t)$ is a solution to
$$\mathbf{x}''(t) = -K\mathbf{x}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1)$$

and for $j = 1, 2, 3, 4$ we have that $\mathbf{x}_j(t)$ is a solution to
$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}_j(t), \quad \mathbf{x}(0) = (0, 0), \quad \mathbf{x}'(t) = (0, 0),$$

then the solution of (1) and (2) is given by

$$\mathbf{x}(t) = \sum_{j=0}^{4} \mathbf{x}_j(t).$$

Solution:

Proof. By the linearity of the derivative operator we know that

$$\mathbf{x}'(t) = \sum_{j=0}^{4} \mathbf{x}'_j(t) \quad \text{and} \quad \mathbf{x}''(t) = \sum_{j=0}^{4} \mathbf{x}''_j(t).$$

Expanding the sum for the second derivative gives

$$\sum_{j=0}^{4} \mathbf{x}''_j(t) = \mathbf{x}''_0(t) + \mathbf{x}''_1(t) + \mathbf{x}''_2(t) + \mathbf{x}''_3(t) + \mathbf{x}''_4(t)$$

$$= (-K\mathbf{x}_0) + (-K\mathbf{x}_1 + \mathbf{f}_1) + (-K\mathbf{x}_2 + \mathbf{f}_2) + (-K\mathbf{x}_3 + \mathbf{f}_3) + (-K\mathbf{x}_4 + \mathbf{f}_4)$$
$$= -K(\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4) + (\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4)$$

$$= -K\left(\sum_{j=0}^{4} \mathbf{x}_j\right) + \sum_{j=1}^{4} \mathbf{f}_j$$

$$= -K\mathbf{x}(t) + \mathbf{f}(t).$$

But
$$\sum_{j=0}^{4} \mathbf{x}_j'' = \mathbf{x}''(t),$$
so
$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}(t),$$
which was to be shown. □

2. Compute $\mathbf{x}_0(t)$

Solution: We will diagonalize K in order to perform a change of variables as a means of solving for $\mathbf{x}_0$. Since K is a 2×2 matrix, we can easily solve for the eigenvalues using the characteristic equation:
$$\lambda^2 - (3+6)\lambda + \det K = 0 \Rightarrow \lambda^2 - 9\lambda + 14 = 0 \Rightarrow \lambda_1 = 7, \lambda_2 = 2.$$

Now we solve for the eigenvalues. Starting with $\lambda_1 = 7$:
$$K - 7I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{2}R_1} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

And for $\lambda_2 = 2$:
$$K - 2I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

It follows that
$$K = \overbrace{\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}}^{V} \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \overbrace{\begin{bmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{bmatrix}}^{V^{-1}}. \tag{3}$$

Now, since $\mathbf{x}'' = -K\mathbf{x}$ (we will use $\mathbf{x}_0$ and $\mathbf{x}$ interchangably for the rest of the proof), by (3),
$$\mathbf{x}'' = -VDV^{-1}\mathbf{x} \Rightarrow V^{-1}\mathbf{x}'' = -DV^{-1}\mathbf{x}.$$

We define $\mathbf{y} = V^{-1}\mathbf{x}$ so
$$\mathbf{y}'' = V^{-1}\mathbf{x}'' = -D\mathbf{y}. \tag{4}$$

From (4) it immediately follows that
$$\mathbf{y}'' = -D\mathbf{y} \Rightarrow \begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = \begin{bmatrix} -\lambda_1 \mathbf{y}_1 \\ -\lambda_2 \mathbf{y}_2 \end{bmatrix} \tag{5}$$
$$\Rightarrow \mathbf{y}(0) = V^{-1}\mathbf{x}(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{bmatrix} = \begin{bmatrix} -5/3 \\ -4/3 \end{bmatrix} \quad \text{and,}$$
$$\mathbf{y}'(0) = V^{-1}\mathbf{x}'(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1'(0) \\ \mathbf{y}_2'(0) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Rewriting (5) gives
$$\begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = -\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

We know from the first set of challenge problems that the solution to a system of differential equations in the form of (5) is
$$\mathbf{y}(t) = \mathbf{s}_0 \cos(\omega t) + \mathbf{t}_0 \sin(\omega t)$$

where $\mathbf{y}(0) = \mathbf{s}_0$ and $\mathbf{y}'(0) = \mathbf{t}_0$. Substituting in our known values we can find

$$\mathbf{y}(t) = \begin{bmatrix} -\frac{5}{3}\cos(\sqrt{7}t) - \sin(\sqrt{7}t) \\ -\frac{4}{3}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{bmatrix}. \tag{6}$$

Using our change of variable we can solve for $\mathbf{x} = V\mathbf{y}$. Substituting (6) we can see

$$\mathbf{x}(t) = V\mathbf{y}(t)$$
$$= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{5}{3}\cos(\sqrt{7}t) - \sin(\sqrt{7}t) \\ -\frac{4}{3}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} -\frac{35}{3}\cos(\sqrt{7}t) - 7\sin(\sqrt{7}t) \\ -\frac{8}{3}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t) \end{bmatrix} = \mathbf{x}(t)}$$

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[0]LaTeX code for this document can be found on github [here](#)