MATH292 NOTES

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1.1   Lecture 1

1.1.1   What is a Differential Equation?

A **differential equation** relates a function to its derivatives. An **ordinary differential equation** (ODE) is a differential equation in which the function only has one independent variable. Differential equations often arise from physical laws such as Newtons second law:

$$x''(t) = \frac{1}{m} F(x(t))$$

1.1.2   First Order ODEs

> **Order**
>
> The order of a differential equation refers to highest derivative found in the differential equation.

It follows that a first order ODE refer to an ODE that only contains first derivatives. The general form for a first order ODE is

$$G(x, y, y') = 0 \tag{1}$$

for some real-valued function G. We can apply the *Implicit Function Theorem* on G if G is continuously differentiable and

$$G(x_0, y_0, z_0) = 0 \quad \text{and} \quad \frac{\partial}{\partial z} G(x_0, y_0, z_0) \neq 0$$

to give a much 'nicer' form of

$$y' = g(x, y)$$

The solution to a first order ODE is the function $y(x)$ that satisfies (1). The **standard form** of a first order ODE is an equation that gives the derivative y' as a function of y. There are two main types of standard forms.

> **Standard Forms**
>
> An **autonomous** first order ODE is one in which the derivative only depends on y. The standard form of an autonomous first order ODE is
>
> $$y' = f(y).$$
>
> A **non-autonomous** first order ODE is one in which the derivative depends on x as well as the value of the function y itself. The standard form of a non-autonomous first order ODE is
>
> $$y' = g(x, y).$$

> **Seperable ODEs**
>
> Seperable first order ODEs are ODEs in the form
>
> $$f(y)y' = g(x).$$
>
> These can be solved with (relative) ease byt integrating both sides and then solving algebraically.

Example 1.1.1: A Seperable First Order ODE

Solve the differential equation
$$y' = \mathbf{e}^{x+y}$$

Solution:

$$y' = \mathbf{e}^{x+y} \Rightarrow \mathbf{e}^{-y} y' = e^x$$
$$\Rightarrow \int \mathbf{e}^{-y} dy = \int e^x dx$$
$$\Rightarrow -\mathbf{e}^{-y} = \mathbf{e}^x + C$$
$$\Rightarrow \mathbf{e}^{-y} = -\mathbf{e}^x + C$$
$$\Rightarrow y = -\ln(\mathbf{e}^x + C)$$

1.1.3   Vector Fields and Integral Curves

Consider a function $\vec{x}'(t) = v(\vec{x}(t))$ that is continuous on some open subset $U \subseteq \mathbb{R}^n$ and has values in $\mathbb{R}^n$. Then, $v(x)$ is a **vector field**. To solve this equation means to find all continuously differentiable functions $\vec{x}(t)$ defined on some time interval (a, b) with values in U for which the equation is satisfied $\forall t \in (a, b)$.

Vector Function and Integral Curves

An **autonomous first order system of ODEs** in standard form is an equation in the form

$$\vec{x}'(t) = v(\vec{x}(t)).$$

A **non-autonomous first order system of ODEs** in standard form is an equation in the form

$$\vec{x}'(t) = v(t, \vec{x}(t)).$$

The solutions to either an autonomous or non-autonomous system, $\vec{x}(t)$ are called **integral curves** of the vector field.

As a basic example consider the vector field on $\mathbb{R}^2$ given by $v(x, y) = (-y, x)$. Then, the differential equation $\vec{x}'(t) = v(\vec{x}(t))$ is simply a more efficient way to write

$$x'(t) = -y(t)$$
$$y'(t) = x(t)$$

We might solve this by considering the function $f(\vec{x}) = \|\vec{x}\| = x^2 + y^2$ (don't worry about where this function came from for now!) and apply the multivariable chain rule,

$$\frac{d}{dt}[f(\vec{x}(t))] = \nabla f(\vec{x}(t)) \cdot \vec{x}'(t) = (2x, 2y) \cdot (-y, x) = 0.$$

Recall that $\vec{x}'(t) = v(\vec{x}(t)) = (-y, x)$. Since

$$\frac{df(\vec{x}(t))}{dt} = 0$$

we know that $f(\vec{x}(t))$ must be a constant function. Thus, $f(\vec{x}(t)) = f(x, y) = C = x^2 + y^2$ for some constant $C \in \mathbb{R}$. It should be clear to see that $f(\vec{x}(t))$ simply represents a circle centered at the origin with radius $\sqrt{C}$.

> **Slope Function**
>
> let $v(x, y) = (f(x, y), g(x, y))$ be any continuous vector field on an open set U of $\mathbb{R}^2$. On the subset open V on which $f(x, y) \neq 0$, define the slope function fo the vector field to be
>
> $$s(x, y) = \frac{g(x, y)}{f(x, y)}.$$

We define the slope function because aan integral curve of V is the function $y(x)$ that satisfies the differential equation $y' = s(x, y)$. This is important because there can be occasions in which you may be able to solve the slope equation $y' = s(x, y)$ despite not being able to explicitly solve for the integral curves of a vector field.

1.1.4 First Order Linear Equations

Differential equations in the form

$$x'(t) = p(t)x(t) + q(t) \tag{2}$$

are called **first order linear equations**. By letting p and q be continuous on some interval (a, b) (with $a = -\infty$ and $b = \infty$ allowed) we consider the primative (or anti-derivative) of p, $P(t) = \int p(t)\, dt$. Now, notice that

$$\frac{d}{dt}\left[x(t)\mathbf{e}^{-P(t)}\right] = [x'(t) - p(t)x(t)]\, \mathbf{e}^{-P(t)}.$$

But,

$$x'(t) - p(t)x(t) = q(t).$$

So,

$$\frac{d}{dt}\left[x(t)\mathbf{e}^{-P(t)}\right] = [x'(t) - p(t)x(t)]\, \mathbf{e}^{-P(t)}$$

$$\Rightarrow \frac{d}{dt}\left[x(t)\mathbf{e}^{-P(t)}\right] = q(t)\mathbf{e}^{-P(t)}$$

$$\Rightarrow \int \frac{d}{dt}\left[x(t)\mathbf{e}^{-P(t)}\right]\, dt = \int q(t)\mathbf{e}^{-P(t)}\, dt$$

$$\Rightarrow x(t)\mathbf{e}^{-P(t)} + C = \int q(t)\mathbf{e}^{-P(t)}\, dt$$

$$\Rightarrow x(t) = \mathbf{e}^{P(t)}\left(\int q(t)\mathbf{e}^{-P(t)}\, dt + C\right). \tag{3}$$

And thus (3) is the general solution to a first order linear differential equation.

You may be asked to solve an **initial value problem** (IVP) which is simply when an initial condition is provided (i.e. $x(t_0) = x_0$). In order to solve an IVP for a first order linear ODE all you must do is solve for the general solution and then plug in the initial condition and solve for the constant of integration. Usually when solving an IVP we have two main questions:

1. Is there a solution? (existence)

2. If there is a solution, is it unique? (uniqueness)

Example 1.1.2: First Order Linear Differential Equation

Solve the differential equation
$$x'(t) = tx(t) + 1$$

Solution: We'll start by noticing that $p(t) = t$ so $P(t) = \frac{1}{2}t^2$. Thus,

$$x'(t) = tx(t) + 1 \Rightarrow \frac{d}{dt}\left[x(t)\mathbf{e}^{-\frac{1}{2}t^2}\right] = 1\mathbf{e}^{-\frac{1}{2}t^2}$$

$$\Rightarrow \int \frac{d}{dt}\left[x(t)\mathbf{e}^{-\frac{1}{2}t^2}\right] dt = \int \mathbf{e}^{-\frac{1}{2}t^2} dt$$

$$\Rightarrow x(t)\mathbf{e}^{-\frac{1}{2}t^2} + C = \int \mathbf{e}^{-\frac{1}{2}t^2} dt$$

$$\Rightarrow x(t) = \boxed{e^{\frac{1}{2}t^2}\left(\int \mathbf{e}^{-\frac{1}{2}t^2} dt + C\right)}$$

For those curious, the integral on the right handside is not an easy integral to evaluate (it requires the power series for $\mathbf{e}^x$) so for simplicity sake we leave it as is and thus the boxed answer is our general solution in simplest form.

1.1.5 Extras

Flow Transformation

For a first order linear equation in the form $x'(t) = p(t)x(t) + q(t)$ with p and q continuous on (a, b), define
$$\Phi_{t_1, t_0}(x) = x(t_1)$$
for all $t_0 \neq t_1 \in (a, b)$ and where $x(t)$ is the unique solution to the IVP $x(t_0) = x_0$.

Matrix Exponentials

Consider $\mathbf{e}^A$ where A is an $n \times n$ matrix over $\mathbb{C}$ defined as
$$\mathbf{e}^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

If $A = PDP^{-1}$ (i.e. if A is diagonalized) then,
$$\mathbf{e}^A = P\mathbf{e}^D P^{-1}$$

where
$$\mathbf{e}^D = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \text{ if } D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

1.2 Lecture 2

1.2.1 Solution Curves

> **Solution Curves**
>
> Consider the first order ODE given by
>
> $$x'(t) = p(t)x(t) + q(t)$$
>
> defined on an open interval (a, b). A **solution curve** is the graph of any specific solution.

Note that the general solution of any first order ODE is of the form

$$x(t) = cf(t) + g(t)$$

with $c \in \mathbb{R}$ and $f(t) \neq 0$.

Any two different solutions curves of a given first order ODE will never intersect. This follows from the fact that any two different solutions of the ODE correspond to different values of the constant, c, and since $f(t) \neq 0$, the values of the two solutions at any t must be different. Hence, through each point in the (t, x) plane given by $(a, b) \times \mathbb{R}$, there is exactly on solution curve. Conversely, every family of functions of the form

$$x(t) = cf(t) + g(t)$$

with g and f continuously differentiable and such that $f(t) \neq 0$, $\forall t \in (a, b)$ is the solution set of a first order linear equation on (a, b). To see this combine

$$x(t) = cf(t) + g(t)$$

and

$$c = \frac{1}{f(t)}(x(t) - g(t))$$

to deduce

$$x'(t) = \frac{f'(t)}{f(t)}(x(t) - g(t)) + g'(t)$$
$$= \frac{f'(t)}{f(t)}x(t) + \left(g'(t) - \frac{f'(t)}{f(t)}g(t)\right)$$

1.2.2 Flow Transformations

Consider the ODE

$$x'(t) = \frac{-x(t)}{t} + t$$

on the interval $t > 0$. The field of solution curves extends over the entire right half of the (t, x) plane. If we think of the equation as describing the motion of a point on the line, such that the point is at x_0 at time t_0, then by following the unique solution curves through (t_0, x_0), we can see where the particle is at every other $t > 0$. Through each point in the right half of the plane there is exactly one solution curve. We define a function $\Phi_{t_1, t_0} : \mathbb{R} \to \mathbb{R}$ as follows: $\Phi_{t_1, t_0}(x_0)$ is the intersection of the solution curve passing through (t_0, x_0) with the line $t = t_1$. In our graph, locate the point at height x_0 in the vertical line $t = t_0$. Follow the solution curve $x(t)$ through this point until it intersects the line $t = t_1$. The height of the intersection, which is $x(t_1)$, is $\Phi_{t_1, t_0}(x_0)$.

Flow Transformations

Consider the IVP

$$x'(t) = p(t)x(t) + q(t), \quad x(t_0) = x_0$$

with p, q continuous on (a, b). For every $t_0 \neq t_1 \in (a, b)$, define

$$\Phi_{t_1, t_0}(x_0) = x(t_1)$$

where x is the unique solution to the IVP.

One important fact about flow transformations is that for any distinct $t_0, t_1, t_2 \in (a, b)$ and $x_0 \in \mathbb{R}$, we have

$$\Phi_{t_2, t_1}(\Phi_{t_1, t_0}(x_0)) = \Phi_{t_2, t_0}(x_0).$$

Proof. The LHS is obtained by starting from the point (t_0, x_0) and moving along the solution curve of the ODE to meet the vertical line $t = t_1$, then moving along the solution curve passing $(t_1, x(t_1))$ until it intersects the vertical line $t = t_2$ where x is the unique solution to the IVP $x(t_0) = x_0$. The RHS is obtained by starting from the point (t_0, x_0) and moving along the solution curve of the ODE to meet the vertical line $t = t_2$. Both sides are now the point $(t_2, x(t_2))$. $\qquad \square$

Example 1.2.1

Compute the flow transformation for the IVP

$$x'(t) = \frac{-x(t)}{t} + t, \quad x(t_0) = x_0.$$

Solution: This is a basic linear differential equation and be solved using an integrating factor of $\mu(t) = e^{\int \frac{1}{t} dt} = t$. Upon solving you should arrive at a final solution of

$$x(t) = \frac{t^2}{3} + \frac{c}{t}$$

and so, solving for c,

$$c = t_0 \left(x_0 - \frac{t_0^2}{t} \right) \Rightarrow x(t) = \frac{t^2}{3} + \frac{t_0 x_0 - \frac{t_0^3}{3}}{t}.$$

Thus,

$$\Phi_{t_1, t_0}(x_0) = x(t_1) = \frac{t_1^2}{3} + \frac{3 t_0 x_0 - t_0^3}{3 t_1}$$

1.2.3 Bernoulli Equations

Bernoulli Equations

Any ODE of the form

$$x'(t) = p(t)x(t) + q(t)x^n(t), \quad n \neq 0, 1 \in \mathbb{R}$$

is called a **Bernoulli Equation**.

To solve a Bernoulli equation we use the change of variables

$$v = x^{1-n} \Rightarrow v' = (1-n)x^{-n}x'$$

under the assumption that $x(t) \neq 0$ for all t. We can then manipulate our Bernoulli Equation in the following manner:

$$\begin{aligned}
x'(t) = p(t)x(t) + q(t)x^n(t) &\Rightarrow x'(t) - p(t)x(t) = q(t)x^n(t) \\
&\Rightarrow x^{-n}(t)x'(t) - p(t)x^{1-n}(t) = q(t) \\
&\Rightarrow \frac{1}{1-n}v' - vp(t) = q(t).
\end{aligned}$$

Thus resulting in a first order linear differential equation that we know how to solve. You will then solve for v, followed by using your solution for v to solve for $x(t)$.

Example 1.2.2

Solve the following ODE:
$$x'(t) = x^2(t)$$

Solution:

$$\begin{aligned}
x'(t) = x^2(t) &\Rightarrow x^{-2}x' = 1 \\
&\Rightarrow v = x^{-1} \Rightarrow v' = -x^{-2}x' \\
&\Rightarrow -v' = 1 \\
&\Rightarrow v = -t + c \\
&\Rightarrow x^{-1} = -t + c \\
&\Rightarrow x(t) = \frac{1}{-t+c}
\end{aligned}$$

for $c \in \mathbb{R}$.

1.2.4   Riccati Equations

Riccati Equations

Any ODE of the form
$$x'(t) = p(t) + q(t)x(t) + r(t)x^2(t)$$
is called a **Riccati Equation**.

There is no general method to solve a Riccati equation unless you're given a particular solution, $x_1(t)$, in which case the general solution will be of the form

$$x(t) = x_1(t) + u(t)$$

where $u(t)$ is the general solution of the Bernoulli equation

$$u'(t) = [q(t) + 2r(t)x_1(t)]\, u(t) + r(t)u^2(t). \tag{4}$$

Now, substituting in for x into the Riccati equation we see that

$$(x_1 + u)' - p - q(x_1 + u) - r(x_1 + u)^2 = 0$$
$$\Rightarrow (x_1' - p - qx_1 - rx_1^2) + (u' - (q + 2rx_1)u - ru^2) = 0$$
$$\Rightarrow u' - (q + 2rx_1)u - ru^2 = 0.$$

With the last implication following from the fact that the first term is simply just the Riccati equation rewritten.

Example 1.2.3

Solve the following ODE:
$$x'(t) = -\frac{1}{t}x(t) + \frac{1}{t^3}x^2(t) + 2t$$

Solution: Since all the coefficients are powers of gt, it is natural to see if there is a solution of the form
$$x_1(t) = ct^\alpha$$
for $c, \alpha \in \mathbb{R}$. Substituting this in the ODE, we get

$$c\alpha t^{\alpha-1} = -ct^{\alpha-1} + c^2 t^{2\alpha-3} + 2t.$$

Since this is a functional equivalence, all the powers of t on the LHS must be equal to all the powers of t on the RHS. This leads us to the fact that

$$\alpha - 1 = \alpha - 1 = 2\alpha - 3 = 1 \Rightarrow \alpha = 2.$$

Now, since all the powers of t are equal, we can cancel them which leaves us with

$$2c = -c + c^2 + 2 \Rightarrow c^2 + c + 2 = 0 \Rightarrow (c-1)(c-2) = 0.$$

Thus, either $x_1 = t^2$ or $x_1 = 2t^2$. With this in mind, we can now solve for u using (4) and choosing $x_1 = t^2$.

$$u'(t) = \frac{1}{t}u + \frac{1}{t^3}u^2 \Rightarrow u^{-2}u' - \frac{1}{t}u^{-1} = \frac{1}{t^3}$$
$$\Rightarrow v = u^{-1} \Rightarrow v' = -u^{-2}u'$$
$$\Rightarrow -v' - \frac{1}{t}v = \frac{1}{t^3}$$
$$\Rightarrow v't + v = -\frac{1}{t^2} \qquad\qquad (\mu(t) = t)$$
$$\Rightarrow vt = \int -\frac{1}{t^2}\,dt = \frac{1}{t} + c$$
$$\Rightarrow v = \frac{1}{t^2} + \frac{c}{t} = \frac{1 + ct}{t^2}$$
$$\Rightarrow u = \frac{1}{v} = \frac{t^2}{1 + ct}$$
$$\Rightarrow x(t) = x_1(t) + u(t) = t^2 + \frac{t^2}{1 + ct}$$

1.2.5   Reduction of Order

Some second order ODEs can be reduced to first order equations that we know how to solve. General second order ODEs have the form

$$F(t, x, x', x'') = 0$$

for some real-valued function F defined on a subset of $\mathbb{R}^4$. There are two cases when dealing with reduction of order problems.

Case 1: F does not depend on x. In this case $F(t, x', x'') = 0$ so we let $y = x'$ which turns F into $F(t, y, y')$ which is first order.

Example 1.2.4

Solve the following ODE:

$$tx'' + x' = t^3$$

Solution: Let $y = x'$ and substitute,

$$ty' + y = t^3 \Rightarrow y' + t^{-1}y = t^2$$
$$\Rightarrow \mu(t) = e^{\int t^{-1} dt} = t$$
$$\Rightarrow ty' + y = t^3$$
$$\Rightarrow ty = \int t^3 \, dt = \frac{t^4}{4} + c$$
$$\Rightarrow y = \frac{t^3}{4} + \frac{c}{t}$$
$$\Rightarrow x = \int y(t) \, dt = \int \frac{t^3}{4} + \frac{c}{t} \, dt = \frac{t^4}{16} + c\ln(t) + d$$

for $c, d \in \mathbb{R}$.

Case 2: F does not depend on t. Once again we let $y = x'$, but also define $X(t) = (x(t), y(t))$ and the vector field $v(x, y) = (y, f(x, y))$. Then $x(t)$ solves $x'' = f(x, x')$ if, and only if, $X(t)$ solves $X'(t) = v(X(t))$. Our strategy is to first find a first order ODE for y. Solving this will tell us the curve traced out by $X(T)$ for t near t_0. Once we know the explicity function, $y(x)$, recall that $y = x'$, and so $x'(t) = y(x(t))$ is a first order equation for $x(t)$. To actually find an equation for $y(x)$ note that the solope of the graph of this function, namely $y'(x)$, is given by the slope of the vector $v(x, y) = (y, f(x, y))$ which is

$$y' = \frac{f(x, y)}{y}. \tag{5}$$

And so, (5) is a seperable first order equation we can solve.

Example 1.2.5: Simple Harmonic Motion

Solve the following ODE:

$$x'' + \omega^2 x = 0, \quad \omega \in \mathbb{R}^+$$

Solution: Notice that

$$\frac{dy}{dx} = \frac{f(x,y)}{y} = \frac{-\omega^2 x}{y}$$

$$\Rightarrow \int y \, dy = \int -\omega^2 x \, dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{-\omega^2 x^2}{2} + c$$

$$\Rightarrow \omega^2 x^2 + y^2 = 2c$$

$$\Rightarrow \omega^2 x^2 + (x')^2 = 2c$$

for $c \in \mathbb{R}$. The case $c = 0$ is trivial so let $2c = r^2 \geq 0$. Then $\omega^2 x^2 + (x')^2 = r^2$ so the point $(\omega x(t), x'(t))$ is on the circle centered at the origin with radius r. Using this to convert to polar we can see that

$$\begin{cases} \omega x(t) & = r \cos(\theta(t)) \\ x'(t) & = r \sin(\theta(t)) \end{cases}$$

But,

$$(\omega x(t))' = \omega x'(t)$$

$$\Rightarrow \frac{d}{dt}(r \cos(\theta(t))) = \omega r \sin(\theta(t))$$

$$\Rightarrow -r \sin(\theta(t)) \theta'(t) = \omega r \sin(\theta(t))$$

$$\Rightarrow -x'(t)\theta'(t) = \omega x'(t)$$

$$\Rightarrow -\theta'(t) = \omega$$

$$\Rightarrow \theta(t) = -\omega t + \theta_0$$

for $\theta_0 \in \mathbb{R}$. Hence,

$$x(t) = \frac{r \cos(-\omega t + \theta_0)}{\omega}$$

where $r \geq 0$ and $\theta_0 \in [0, 2\pi)$