

Consider the equation

$$\mathbf{x}''(t) = -K\vec{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1) \quad (1)$$

and

$$\mathbf{f}(t) = \sum_{j=1}^4 \mathbf{f}_j(t) \quad (2)$$

for

$$\begin{aligned}\mathbf{f}_1(t) &= (1, 3) \cos(\omega_1 t) \\ \mathbf{f}_2(t) &= (1, -1) \cos(\omega_2 t) \\ \mathbf{f}_3(t) &= (3, -1) \cos(\omega_3 t) \\ \mathbf{f}_4(t) &= (1, 0) \cos(\omega_4 t)\end{aligned}$$

and

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

1. If  $\mathbf{x}_0(t)$  is a solution to

$$\mathbf{x}''(t) = -K\mathbf{x}(t), \quad \mathbf{x}(0) = (1, 2), \quad \mathbf{x}'(t) = (1, 1)$$

and for  $j = 1, 2, 3, 4$  we have that  $\mathbf{x}_j(t)$  is a solution to

$$\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}_j(t), \quad \mathbf{x}(0) = (0, 0), \quad \mathbf{x}'(t) = (0, 0), \quad (3)$$

then the solution of (1) and (2) is given by

$$\mathbf{x}(t) = \sum_{j=0}^4 \mathbf{x}_j(t).$$

**Solution:**

*Proof.* By the linearity of the derivative operator we know that

$$\mathbf{x}'(t) = \sum_{j=0}^4 \mathbf{x}'_j(t) \quad \text{and} \quad \mathbf{x}''(t) = \sum_{j=0}^4 \mathbf{x}''_j(t).$$

Expanding the sum for the second derivative gives

$$\begin{aligned}\sum_{j=0}^4 \mathbf{x}''_j(t) &= \mathbf{x}''_0(t) + \mathbf{x}''_1(t) + \mathbf{x}''_2(t) + \mathbf{x}''_3(t) + \mathbf{x}''_4(t) \\ &= (-K\mathbf{x}_0) + (-K\mathbf{x}_1 + \mathbf{f}_1) + (-K\mathbf{x}_2 + \mathbf{f}_2) + (-K\mathbf{x}_3 + \mathbf{f}_3) + (-K\mathbf{x}_4 + \mathbf{f}_4) \\ &= -K(\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4) + (\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4) \\ &= -K \left( \sum_{j=0}^4 \mathbf{x}_j \right) + \sum_{j=1}^4 \mathbf{f}_j \\ &= -K\mathbf{x}(t) + \mathbf{f}(t).\end{aligned} \quad (4)$$

But

$$\sum_{j=0}^4 \mathbf{x}_j'' = \mathbf{x}''(t),$$

so

$$\boxed{\mathbf{x}''(t) = -K\mathbf{x}(t) + \mathbf{f}(t)}.$$

In order to confirm that this is a solution, we will check that initial conditions hold true as well:

$$\begin{aligned}\sum_{j=0}^4 \mathbf{x}_j(0) &= \mathbf{x}_0(0) + \sum_{j=1}^4 \mathbf{x}_j(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \sum_{j=0}^4 \mathbf{x}'_j(0) &= \mathbf{x}'_0(0) + \sum_{j=1}^4 \mathbf{x}'_j(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned}$$

which are initial values provided in (1). □

2. Compute  $\mathbf{x}_0(t)$

**Solution:** We will diagonalize  $K$  in order to perform a change of variables as a means of solving for  $\mathbf{x}_0$ . Since  $K$  is a  $2 \times 2$  matrix, we can easily solve for the eigenvalues using the characteristic equation:

$$\lambda^2 - (3+6)\lambda + \det K = 0 \Rightarrow \lambda^2 - 9\lambda + 14 = 0 \Rightarrow \lambda_1 = 7, \lambda_2 = 2.$$

Now we solve for the eigenvalues. Starting with  $\lambda_1 = 7$ :

$$K - 7I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_1 \rightarrow -\frac{1}{2}R_1} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And for  $\lambda_2 = 2$ :

$$K - 2I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

It follows that

$$K = \underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}}_V \underbrace{\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}}_{V^{-1}}. \quad (5)$$

Now, since  $\mathbf{x}'' = -K\mathbf{x}$  (we will use  $\mathbf{x}_0$  and  $\mathbf{x}$  interchangably for the rest of the proof), by (5),

$$\mathbf{x}'' = -VDV^{-1}\mathbf{x} \Rightarrow V^{-1}\mathbf{x}'' = -DV^{-1}\mathbf{x}.$$

We define  $\mathbf{y} = V^{-1}\mathbf{x}$  so

$$\mathbf{y}'' = V^{-1}\mathbf{x}'' = -D\mathbf{y}. \quad (6)$$

From (6) it immediately follows that

$$\begin{aligned}\mathbf{y}'' &= -D\mathbf{y} \Rightarrow \begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = \begin{bmatrix} -\lambda_1 \mathbf{y}_1 \\ -\lambda_2 \mathbf{y}_2 \end{bmatrix} \\ \Rightarrow \mathbf{y}(0) &= V^{-1}\mathbf{x}(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1(0) \\ \mathbf{y}_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and,} \\ \mathbf{y}'(0) &= V^{-1}\mathbf{x}'(0) \Rightarrow \begin{bmatrix} \mathbf{y}_1'(0) \\ \mathbf{y}_2'(0) \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}.\end{aligned} \quad (7)$$

Rewriting (7) gives

$$\begin{bmatrix} \mathbf{y}_1'' \\ \mathbf{y}_2'' \end{bmatrix} = - \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

We know from the first set of challenge problems that the solution to a system of differential equations in the form of (7) is

$$\mathbf{y}(t) = \mathbf{s}_0 \cos(\omega t) + \mathbf{t}_0 \sin(\omega t)$$

where  $\mathbf{y}(0) = \mathbf{s}_0$  and  $\mathbf{y}'(0) = \mathbf{t}_0$ . Substituting in our known values we can find

$$\mathbf{y}(t) = \begin{bmatrix} \cos(\sqrt{7}t) + \frac{3}{5} \sin(\sqrt{7}t) \\ -\frac{1}{5} \sin(\sqrt{2}t) \end{bmatrix}. \quad (8)$$

Using our change of variable we can solve for  $\mathbf{x} = V\mathbf{y}$ . Substituting (8) we can see

$$\begin{aligned} \mathbf{x}(t) &= V\mathbf{y}(t) \\ &= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(\sqrt{7}t) + \frac{3}{5} \sin(\sqrt{7}t) \\ -\frac{1}{5} \sin(\sqrt{2}t) \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 7 \cos(\sqrt{7}t) + \frac{21}{5} \sin(\sqrt{7}t) \\ -\frac{2}{5} \sin(\sqrt{2}t) \end{bmatrix}} = \mathbf{x}(t) \end{aligned}$$

Now, for any  $\mathbf{x}_j(t)$  that is a solution to (3) we know that for  $\mathbf{v}_1, \mathbf{v}_2$  eigenvectors of  $K$

$$\mathbf{x}(t) = y_1(t)\mathbf{v}_1 + y_2(t)\mathbf{v}_2 \quad (9)$$

since  $K$  is a symmetric matrix. Then,

$$\begin{aligned} \mathbf{x}''(t) &= -K\mathbf{x}(t) + \mathbf{f} \Rightarrow \mathbf{x}''(t) + K\mathbf{x}(t) = \mathbf{f} \\ &\Rightarrow (y_1(t)\mathbf{v}_1 + y_2(t)\mathbf{v}_2)'' + K(y_1(t)\mathbf{v}_1 + y_2\mathbf{v}_2) = \mathbf{f} \\ &\Rightarrow y_1''(t)\mathbf{v}_1 + y_2''(t)\mathbf{v}_2 + 7y_1(t)\mathbf{v}_1 + 2y_2(t)\mathbf{v}_2 = \mathbf{f} \quad (\text{Since } K\mathbf{v}_i = \lambda_i \mathbf{v}_i) \\ &\Rightarrow \mathbf{v}_1(y_1''(t) + 7y_1(t)) + \mathbf{v}_2(y_2''(t) + 2y_2(t)) = \mathbf{f}. \end{aligned}$$

Thus, we have the equations

$$y_1''(t) + 7y_1(t) = f_1 \quad \text{and} \quad y_2''(t) + 2y_2(t) = f_2.$$

For an equation of the form  $y'' + \lambda y = f$ , we may guess that the solution is of the form

$$\begin{aligned} y(t) &= A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t) + C \cos(\omega t) \quad \text{or} \\ y(t) &= A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t) + C \sin(\omega t). \end{aligned} \quad (10)$$

Consider the first case in which the final term is  $\cos(\omega t)$  and  $f = \alpha \cos(\omega t)$  term. Then,

$$y''(t) = -\lambda(A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t)) - \omega^2 C \cos(\omega t).$$

Adding  $\lambda y(t)$  to both sides shows that

$$y''(t) + \lambda y(t) = \lambda C \cos(\omega t) - \omega^2 C \cos(\omega t) = (\lambda - \omega^2)C \cos(\omega t).$$

It follows that

$$(\lambda - \omega^2)C \cos(\omega t) = \alpha \cos(\omega t) \Rightarrow C = \frac{\alpha}{\lambda - \omega^2}.$$

Applying the initial conditions, we can see that

$$\begin{aligned} y(t) &= A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t) + \frac{\alpha}{\lambda - \omega^2} \cos(\omega t) \\ \Rightarrow y(0) = 0 &= A + \frac{\alpha}{\lambda - \omega^2} \Rightarrow A = -\frac{\alpha}{\lambda - \omega^2} \\ \Rightarrow y'(0) = 0 &= B. \end{aligned}$$

So, the solution in the case in which  $f_i = \alpha \cos(\omega t)$  is

$$y(t) = \frac{\alpha}{\lambda - \omega^2} \left( \cos(\omega t) - \cos(\sqrt{\lambda}t) \right). \quad (11)$$

Returning to (10) and considering the other case in which the final term is  $\sin(\omega t)$  and  $f = \beta \sin(\omega t)$  term, we have

$$y''(t) = -\lambda(A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t)) - \omega^2 C \cos(\omega t).$$

Then,

$$y''(t) + \lambda y(t) = \lambda C \sin(\omega t) - \omega^2 C \sin(\omega t) = (\lambda - \omega^2)C \sin(\omega t).$$

Just like in the first case we can see that

$$(\lambda - \omega^2)C \sin(\omega t) = \beta \sin(\omega t) \Rightarrow C = \frac{\beta}{\lambda - \omega^2}.$$

Applying initial conditions shows

$$\begin{aligned} y(t) &= A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t) + \frac{\beta}{\lambda - \omega^2} \sin(\omega t) \\ \Rightarrow y(0) = 0 &= A \\ \Rightarrow y'(0) = 0 &= \sqrt{\lambda}B + \frac{\beta \omega}{\lambda - \omega^2} \Rightarrow B = -\frac{\beta \omega}{\sqrt{\lambda}(\lambda - \omega^2)}. \end{aligned}$$

So, the solution in the case in which  $f = \beta \sin(\omega t)$  is

$$y(t) = \frac{\beta}{\lambda - \omega^2} \left( \sin(\omega t) - \frac{\omega}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) \right). \quad (12)$$

### 3. Compute $\mathbf{x}_1(t)$

**Solution:** For  $\mathbf{x}_1$  we have that  $\mathbf{f}_1 = (1, 3) \cos(\omega_1 t)$ . Once again, consider the change of variables  $\mathbf{y} = V^{-1}\mathbf{x}$ . Then,

$$\begin{aligned} \mathbf{y}'' &= V^{-1}\mathbf{x}'' = V^{-1}(-K\mathbf{x} + \mathbf{f}) \\ &= V^{-1}(-VDV^{-1})\mathbf{x} + V^{-1}\mathbf{f} \\ &= -D\mathbf{y} + V^{-1}\mathbf{f} \\ &= -\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{y} + \frac{1}{5} \begin{bmatrix} 7 \cos(\omega_1 t) \\ \cos(\omega_1 t) \end{bmatrix} \end{aligned}$$

It follows that  $\alpha = \frac{7}{5}$  and  $\beta = \frac{1}{5}$ , so for

$$\mathbf{x}_1(t) = y_1 \mathbf{v}_1 + y_2 \mathbf{v}_2$$

we have

$$y_1(t) = \frac{7}{5(7 - \omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t)), \text{ and}$$

$$y_2(t) = \frac{1}{5(2 - \omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{2}t)).$$

Thus,

$$\mathbf{x}_1(t) = \frac{7}{5(7 - \omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{7}t)) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5(2 - \omega_1^2)}(\cos(\omega_1 t) - \cos(\sqrt{2}t)) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

4. Compute  $\mathbf{x}_2(t)$ .

**Solution:** We follow a similar process for  $\mathbf{x}_2$ . For  $\mathbf{f}_2 = (1, -1) \sin(\omega_2 t)$ ,

$$V^{-1}\mathbf{f}_2 = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \sin(\omega_2 t) \\ -\sin(\omega_2 t) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -\sin(\omega_2 t) \\ -3 \sin(\omega_2 t) \end{bmatrix},$$

thus  $\alpha = -1$  and  $\beta = -3$ . It follows that

$$\mathbf{x}_2(t) = \frac{-1}{5(7 - \omega_2^2)} \left( \sin(\omega_2 t) - \frac{\omega_2}{\sqrt{7}} \sin(\sqrt{7}t) \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{5(2 - \omega_2^2)} \left( \sin(\omega_2 t) - \frac{\omega_2}{\sqrt{2}} \sin(\sqrt{2}t) \right) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

5. Compute  $\mathbf{x}_3$ .

**Solution:** For  $\mathbf{x}_3$  we have  $\mathbf{f}_3 = (3, -1) \sin(\omega_3 t)$  so

$$V^{-1}\mathbf{f}_3 = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \sin(\omega_3 t) \\ -\sin(\omega_3 t) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \sin(\omega_3 t) \\ -7 \sin(\omega_3 t) \end{bmatrix},$$

thus  $\alpha = \frac{1}{5}$  and  $\beta = -\frac{7}{5}$ . It follows that

$$\mathbf{x}_3(t) = \frac{1}{5(7 - \omega_3^2)} \left( \sin(\omega_3 t) - \frac{\omega_3}{\sqrt{7}} \sin(\sqrt{7}t) \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{7}{5(2 - \omega_3^2)} \left( \sin(\omega_3 t) - \frac{\omega_3}{\sqrt{2}} \sin(\sqrt{2}t) \right) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

6. Compute  $\mathbf{x}_4$ .

**Solution:** For  $\mathbf{x}_4$  we have  $\mathbf{f}_4 = (1, 0) \cos(\omega_4 t)$  so

$$V^{-1}\mathbf{f}_4 = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \cos(\omega_4 t) \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \cos(\omega_4 t) \\ -2 \cos(\omega_4 t) \end{bmatrix},$$

thus  $\alpha = \frac{1}{5}$  and  $\beta = -\frac{2}{5}$ . It follows that

$$\mathbf{x}_4(t) = \frac{1}{5(7 - \omega_4^2)} (\cos(\omega_4 t) - \cos(\sqrt{7}t)) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{5(2 - \omega_4^2)} (\cos(\omega_4 t) - \cos(\sqrt{2}t)) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

7. Write the solutions to (1) and (2). For which values of  $\omega_1, \omega_2, \omega_3$ , and  $\omega_4$  does the system exhibit resonance.

**Solution:** From exercise 1, we know that

$$\mathbf{x}(t) = \sum_{j=0}^4 \mathbf{x}_j(t) = \mathbf{x}_0(t) + \sum_{j=1}^4 \mathbf{x}_j(t).$$

so

$$\begin{aligned} x(t) &= \begin{bmatrix} 7 \cos(\sqrt{7}t) + \frac{21}{5} \sin(\sqrt{7}t) \\ -\frac{2}{5} \sin(\sqrt{2}t) \end{bmatrix} \\ &+ \frac{7}{5(7 - \omega_1^2)} (\cos(\omega_1 t) - \cos(\sqrt{7}t)) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5(2 - \omega_1^2)} (\cos(\omega_1 t) - \cos(\sqrt{2}t)) \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &+ \frac{-1}{5(7 - \omega_2^2)} \left( \sin(\omega_2 t) - \frac{\omega_2}{\sqrt{7}} \sin(\sqrt{7}t) \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{5(2 - \omega_2^2)} \left( \sin(\omega_2 t) - \frac{\omega_2}{\sqrt{2}} \sin(\sqrt{2}t) \right) \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &+ \frac{1}{5(7 - \omega_3^2)} \left( \sin(\omega_3 t) - \frac{\omega_3}{\sqrt{7}} \sin(\sqrt{7}t) \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{7}{5(2 - \omega_3^2)} \left( \sin(\omega_3 t) - \frac{\omega_3}{\sqrt{2}} \sin(\sqrt{2}t) \right) \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &+ \frac{1}{5(7 - \omega_4^2)} (\cos(\omega_4 t) - \cos(\sqrt{7}t)) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{5(2 - \omega_4^2)} (\cos(\omega_4 t) - \cos(\sqrt{2}t)) \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \end{aligned}$$

The system will exhibit resonance for  $\omega_i = \sqrt{7}$  or  $\omega_i = \sqrt{2}$  because  $y'' + \lambda y = 0$  for either of these values of  $\omega$ .

8. Of the four given values of  $\omega_1$ , choose the one that creates a significantly larger term in the solution, so the solution can be approximated by only the main term.

**Solution:** For  $\omega_1 = 2.65 \approx \sqrt{7}$  we have that

$$\frac{1}{7 - \omega_1^2} \approx -44.5$$

which is much larger than any of the other terms. Thus, the main component is

$$\frac{7}{5(7 - \omega_1^2)} (\cos(\omega_1 t) - \cos(\sqrt{7}t)) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

<sup>0</sup>LATEX code for this document can be found on github [here](#)