Proposition 1. *For all $n \in \mathbb{N}$,*
$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Proof. We will prove this using induction on n. We start by proving the base case, $n = 1$,
$$\sum_{k=0}^{1} k = 1 = \frac{1(1+2)}{2}.$$

Therefore, since the LHS and RHS are equal, the base case is true.
Now we will assume that for some $t \in \mathbb{N}$
$$\sum_{k=0}^{t} k = \frac{t(t+1)}{2},$$
and we must show that this property holds true for $t + 1$, ie.
$$\sum_{k=0}^{t+1} k = \frac{(t+1)(t+2)}{2}.$$

Consider

$$\begin{aligned}
\sum_{k=0}^{t+1} k &= t + 1 + \sum_{k=0}^{t} k \\
&= t + 1 + \frac{t(t+1)}{2} && \text{(by the IH)} \\
&= \frac{2t+2}{2} + \frac{t^2+t}{2} \\
&= \frac{t^2+3t+2}{2} \\
&= \frac{(t+1)(t+2)}{2}
\end{aligned}$$

Which was to be shown. $\qquad\square$

Proposition 2. *For all $n \in \mathbb{N}$, $2n^3 + 3n^2 + n$ is divisible by 6.*

Proof. We will prove this using induction on n. For the base case $n = 0$ we have
$$2(0)^3 + 3(0)^2 + 0 = 0 \Rightarrow 6 \mid 0.$$

Thus, the base case is true.
Now we will assume that for some $k \in \mathbb{N}$,
$$6 \mid 2k^3 + 3k^2 + k$$

We must show that
$$6 \mid 2(k+1)^3 + 3(k+1)^2 + k + 1.$$

But

$$\begin{aligned}
2(k+1)^3 + 3(k+1)^2 + k + 1 &= 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + k + 1 \\
&= 2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3 + k + 1 \\
&= (2k^3 + 3k^2 + k) + 6k^2 + 12k + 6 \\
&= 6t + 6(k^2 + 2k + 1) && \text{(by the IH } 2k^3 + 3k^2 + k = 6t \text{ for } t \in \mathbb{N}) \\
&= 6(t + k^2 + 2k + 1).
\end{aligned}$$

Therefore, since $\mathbb{N}$ is closed under addition and multiplication, $t + k^2 + 2k + 1 \in \mathbb{N}$ and so, by definition $6 \mid 2(k+1)^3 + 3(k+1)^2 + k + 1$. $\qquad\square$

Proposition 3. *For all $n \in \mathbb{Z}, n \geq 4$,*

$$2^n < n!$$

Proof. We will prove this using induction on n. For the base case $n = 4$ we have,

$$2^4 = 16 < 4! = 24.$$

This inequality is true and thus the base case is true.
We will now assume that for some $k \in \mathbb{Z}, k \geq 4$,

$$2^k < k!.$$

We must show that

$$2^{k+1} < (k+1)!.$$

But,

$$2^{k+1} = 2^k 2 < k! \cdot 2 < (k+1)k! = (k+1)!.$$

This inequality holds true because since $k \geq 4$, $k + 1 > 2$ for all k. Thus, $2^{k+1} < (k+1)!$. $\square$

Proposition 4. *For all $n \in \mathbb{N}$*

$$\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$$

Proof. We will prove this using induction on n. For the base case $n = 0$ we have

$$\sum_{i=0}^{0} i \cdot i! = 0 = (0+1)! - 1.$$

Since the LHS and RHS are equivalent, the base case is true.
We will now assume that for some $k \in \mathbb{N}$,

$$\sum_{i=0}^{k} k \cdot k! = (k+1)! - 1.$$

We must show that

$$\sum_{i=0}^{k+1} (k+1) \cdot (k+1)! = (k+2)! - 1.$$

But,

$$
\begin{aligned}
\sum_{i=0}^{k+1} (k+1) \cdot (k+1)! &= (k+1) \cdot (k+1)! + \sum_{i=0}^{k} k \cdot k! \\
&= (k+1) \cdot (k+1)! + (k+1)! - 1 \qquad \text{(by IH)} \\
&= (k+1)!(k+1+1) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1.
\end{aligned}
$$

Which was to be shown. $\square$

Proposition 5. *For all $n \in \mathbb{Z}, n \geq 12$, there exist $k, l \in \mathbb{N}$ such that $n = 4k + 5l$*

Proof. We will prove this using strong induction on n. For the base cases $n = 12, 13, 14, 15$ we have

$$n = 12 = 4(3) + 5(0)$$
$$n = 13 = 4(2) + 5(1)$$
$$n = 14 = 4(1) + 5(2)$$
$$n = 15 = 4(0) + 5(3)$$

We will now assume that for some $t \in \mathbb{Z}, 12 \leq t \leq n$ there exist $k, l \in \mathbb{N}$ such that $t = 4k + 5l$. We must show that $n + 1 = 4k + 5l$. Since $n \geq 15$ we know that $n + 1 \geq 16$ so $(n + 1) - 4 \geq 12$. Then $(n + 1) - 4 = 4k' + 5l' \Rightarrow n + 1 = 4(k' + 1) + 5l'$. Thus $k = k' + 1$ and $l = l'$ so there exist k and l such that $n + 1 = 4k + 5l$ which was to be shown. $\square$