1.2.5

(a) If $x \in (A \cap B)^c$ then $x \notin A \cap B$. Thus, either $x \notin A \Rightarrow x \in A^c$ or $x \notin B \Rightarrow x \in B^c$. It follows that $x \in A^c \cup B^c$ so, $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$ then either $x \in A^c$ or $x \in B^c$. Without loss of generality assume that $x \in A^c$, which implies that $x \notin A \Rightarrow x \notin A \cap B \Rightarrow x \in (A \cap B)^c$, so $A^c \cup B^c \subseteq (A \cap B)^c$. Since both sets are subsets of each other, we conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) *Proof.* In order to show that $(A \cup B)^c \subseteq A^c \cap B^c$ consider $x \in (A \cup B)^c$. Then

$$\begin{aligned} x \in (A \cup B)^c &\Rightarrow x \notin A \cup B \\ &\Rightarrow x \notin A \wedge x \notin B \\ &\Rightarrow x \in A^c \wedge x \in B^c \\ &\Rightarrow x \in A^c \cap B^c. \end{aligned}$$

And so $(A \cup B)^c \subseteq A^c \cap B^c$.
Now, consider $x \in A^c \cap B^c$. Then

$$\begin{aligned} x \in A^c \cap B^c &\Rightarrow x \in A^c \wedge x \in B^c \\ &\Rightarrow x \notin A \wedge x \notin B \\ &\Rightarrow x \notin A \cup B \\ &\Rightarrow x \in (A \cup B)^c. \end{aligned}$$

And so $A^c \cap B^c \subseteq (A \cup B)^c$ thus $(A \cup B)^c = A^c \cap A^C$. □

1.2.9

(a) If we have $A = [0, 4]$ and $B = [-1, 1]$ then,

$$\begin{aligned} f^{-1}(A) &= [0, 2] \\ f^{-1}(B) &= [0, 1] \\ f^{-1}(A \cup B) &= [0, 2] = f^{-1}(A) \cup f^{-1}(B) \\ f^{-1}(A \cap B) &= [0, 1] = f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

So, for f, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) *Proof.* Consider $x \in g^{-1}(A \cap B)$. Then $x \in \{a \mid g(a) \in A \cap B\}$. So,

$$g(x) \in A \quad \text{and} \quad g(x) \in B$$

which implies that

$$x \in g^{-1}(A) \quad \text{and} \quad x \in g^{-1}(B).$$

Thus, $x \in g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$.
Now, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in \{a \mid g(a) \in A \wedge g(a) \in B\}$, so

$$g(x) \in A \quad \text{and} \quad g(x) \in B.$$

It follows that $g(x) \in A \cap B$, so $x \in g^{-1}(A \cap B)$. Thus, $g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$, and so, $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.
Now, we consider $x \in g^{-1}(A \cup B)$. By definition,

$$x \in \{a \mid g(a) \in A \cup B\},$$

therefore, either $g(x) \in A$ or $g(x) \in B$. It follows that $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$, so $x \in g^{-1}(A) \cup g^{-1}(B)$. Thus, $g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B)$.
For $x \in g^{-1}(A) \cup g^{-1}(B)$, we can see that

$$x \in \{a \mid g(a) \in A \vee g(a) \in B\}.$$

Regardless if $g(x) \in A$ or $g(x) \in B$, $g(x) \in A \cup B$ since it must be in at least one of A or B. It follows that $x \in g^{-1}(A \cup B)$, so $g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B)$ □

1.2.10

(a) Consider the reverse implication that if $a < b + \epsilon$ for all $\epsilon > 0$ then $a < b$. Take $a = b$ and notice that $a < b + \epsilon$ but $a \not< b$. Thus because the backwards direction doesn't work, the entire statement is false.

(b) Apply the same reasoning as above with $a = b$. Then $a < b + \epsilon$ for all $\epsilon > 0$ but $a \not< b$. Thus, ths statement is false.

(c) This statement is true. The forwards direction is trivial a if $a \leq b$ then $a < b + \epsilon$ for all $\epsilon > 0$. We show the backwards direction by contradiction. That is, suppose that $a < b + \epsilon$ for all $\epsilon > 0$ and that $a > b$. Now choose $\epsilon = \frac{a-b}{2}$ which is greater than 0 since $a > b$. But,

$$b + \epsilon = b + \frac{a-b}{2} = \frac{a+b}{2} < a.$$

This is a contraidction since $a < b + \epsilon$ for all $\epsilon > 0$. Thus our supposition must be false and $a \leq b$.

1.2.13

(a) *Proof.* Consider the proposition

$$P(n) := (A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for $n \in \mathbb{N}$. We will prove $P(n)$ for all $n \in \mathbb{N}$ using induction. The base case $P(1)$ is trivial as $(A_1)^c = A_1^c$. We now assume $P(k)$ for some $n = k$ and must show that $P(k+1)$ is true. That is, we must show that

$$(A_1 \cup A_2 \cup \cdots A_k \cup A_{k+1})^c = A_1^c \cap A_2^c \cap \cdots \cap A_k^c \cap A_{k+1}^c.$$

But,

$$\begin{aligned}(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})^c &= ((A_1 \cup A_2 \cup \cdots A_k) \cup A_{k+1})^c \\ &= (A_1 \cup A_2 \cup \cdots A_k)^c \cap A_{k+1}^c \\ &= (A_1^c \cap A_2^c \cap \cdots \cap A_k^c) \cap A_{k+1}^c \\ &= A_1^c \cap A_2^c \cap \cdots \cap A_k^c \cap A_{k+1}^c\end{aligned}$$

Which was to be shown. □

(b) Consider the collection of sets B_i such that

$$B_i = \left(0, \frac{1}{i}\right).$$

Then for $n \in \mathbb{N}$,

$$\bigcap_{i=1}^{n} B_i \neq \varnothing$$

since, for example, $\frac{1}{2n} \in \bigcap_{i=1}^{n} B_i$. However,

$$\bigcap_{i=1}^{\infty} B_i = \varnothing$$

since as $i \to \infty$, $\frac{1}{i} \to 0$.

(c) *Proof.* Consider $x \in (\bigcup_{i=1}^{\infty} A_i)^c$. Then $x \notin \bigcup_{i=1}^{\infty} A_i$. The only way this is possible is if $x \notin A_i$ for all i. Then, $x \in A_i^c$, so

$$x \in \bigcap_{i=1}^{\infty} A_i \Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

Taking $x \in \cap_{i=1}^{\infty} A_i^c$, it follows that $x \in A_i^c$ so $x \notin A_i$ for all i. Thus

$$x \notin \bigcup_{i=1}^{\infty} A_i \Rightarrow x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c \Rightarrow \bigcap_{i=1}^{\infty} A_i \subseteq \left(\bigcup_{i=1}^{\infty} A_i\right)^c.$$

Thus, $(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c$ □

1.3.9

(a) *Proof.* Since $\sup A < \sup B$ there exists $x \in \mathbb{R}$ such that $\sup A < x < \sup B$. Since $x < \sup B$, x is not an upper bound for B. This implies the existence of a $b \in B$ such that $x < b$. Combining inequalities we see that $\sup A < x < b$ and therefore b is an upper bound for A since it is greater than $\sup A$. □

(b) Consider the sets $A = B = (0,1)$. Then $\sup A \leq \sup B = 1$, but there does not exist $b \in B$ such that $b > a$ for any $a \in A$.

1.3.10

(a) *Proof.* Consider a fixed $b \in B$. Then $b > a$ for all $a \in A$, by definition of A and B, so A is bounded above by b. It follows from the Axiom of Completeness that $\sup A = s$ must exist since A is nonempty and bounded. If $s \in A$ then $s = \sup A = \max A$ since

$$s \geq a \quad \forall a \in A$$

by definition of supremum. As well, $s \in A$ implies $s \leq b$ for all $b \in B$ and thus the Cut Property is satisfied. If $s \in B$ then $s \geq a$ for all $a \in A$ by definition of A and B, but $s \leq b$ for any $b \in B$ since any $b \in B$ is an upper bound of A is must be greater than or equal to the supremum. Therefore, the Cut Property is also satisfied if $s \in B$. □

(b) *Proof.* Let $A = \{x \mid x \leq e \text{ for some } e \in E\}$. Then $E \subset A$ since every element $e \in E$ is less than or equal to some $e \in E$, namely itself. We now define $B = \mathbb{R} \setminus A$. Thus $A \cap B = \varnothing$ and $A \cup B = \mathbb{R}$. By the Cut Property, there exists $s \in \mathbb{R}$ such that $s \geq a$ for all $a \in A$ and $s \leq b$ for all $b \in B$. Then $s = \sup E$ since $s \geq a$ for all $a \in A$, which includes every element of E and if some $x < s$ then $x \notin B$ so $x \leq e$ which means that x is not a upper bound of E. So, $s = \sup E$. □

(c) Consider the two sets

$$A = (-\infty, 0) \cup \left\{x \in \mathbb{Q} \mid 0 \leq x^2 \leq 2\right\}$$
$$B = \left\{x \in \mathbb{Q} \mid x > 0 \wedge x^2 > 2\right\}.$$

Then, $A \cup B = \mathbb{Q}$ and $A \cap B = \varnothing$ but there does not exist $s \in \mathbb{Q}$ such that $a \leq s$ for all $a \in A$ and $s \leq b$ for all $b \in B$ since $\sup A = \sqrt{2} \notin \mathbb{Q}$. Thus, the Cut Property does not hold for $\mathbb{Q}$.

[0] LaTeX code for this document can be found on github here