1.4.4: Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup T = b$

Solution:

Proof. By definition of T, b must be an upper bound for T. Now, for any other upper bound of T, call it c, we must have $c \geq b$. This is because if $c < b$ then c cannot be an upper bound for if $b \in T$ (i.e $b \in \mathbb{Q}$) then c is not greater than or equal to every element of T, and if $b \notin T$ (i.e b is irrational), then by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a number $r \in \mathbb{Q}$ such that $c < r < b$ so $r \in T$, which again means that c cannot be an upper bound. Thus b is least upper bound so $\sup T = b$. □

1.4.5: Prove that for any two real numbers $a < b$, there exists an irration number, t, such that $a < t < b$ by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution:

Proof. By the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists $r \in \mathbb{Q}$ such that

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

So,

$$a < r + \sqrt{2} < b,$$

but $r + \sqrt{2}$ is irrational (since a rational number plus an irrational number will always be irrational - by assumption of exercise 1.4.1), completing the proof. □

1.4.8: Give an example of each or state that the request is impossible. When a request is impossible proivde a compelling arugment for why this is the case.

(a) Two sets A and B with $A \cap B = \varnothing, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

Solution: Consider $A = \{x \in \mathbb{Q} \mid 0 < x^2 < 2\}$ and $B = \{x \in \mathbb{R} \setminus \mathbb{Q} \mid 0 < x^2 < 2\}$. Then $A \cap B = \varnothing$, $\sup A = \sup B = \sqrt{2}$ and $\sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open interval $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

Solution: Let $J_n = (1 - \frac{1}{n}, 1 + \frac{1}{n})$ such that each J_n is nested and open. Then

$$\bigcup_{n=1}^{\infty} J_n = \{1\}$$

which is nonempty and only contains a single element.

(c) A sequence of nested unbounded closed interval $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \varnothing$.

Solution: Consider $L_n = [n, \infty)$. Then

$$\bigcap_{n=1}^{\infty} L_n = \varnothing$$

since eventually $n > x$ for all $x \in \mathbb{R}$.

(d) A sequence of closed bounded (not necesssarily nested) interval $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \varnothing$ for all $N \in \mathbb{N}$, but $\bigcap n = 1^{\infty} I_n = \varnothing$.

Solution: This is impossible. We define $K_n = I_1 \cap I_2 \cap \cdots \cap I_n$. Then each K_n is closed and bounded since each I_n is also closed and bounded. We cannow apply the nested interval theorem since $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$,

$$\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

1.5.9:

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2} + \sqrt{3}$ are algebraic

Solution: For the polynomial $x^2 - 2 = 0$, we know that $\sqrt{2}$ is a root, so $\sqrt{2} \in \mathbb{A}$. For the polynomial $x^3 - 2 = 0$, we know that $\sqrt[3]{2}$ is a root, so $\sqrt{2} \in \mathbb{A}$. For the polynomial $x^4 - 10x^2 + 1 = 0$, we know that $\sqrt{2} + \sqrt{3}$ is a root, so $\sqrt{2} + \sqrt{3} \in \mathbb{A}$. Note that we found this polynomial by considering $x - (\sqrt{2} + \sqrt{3}) = 0$ and then solving for x before squaring the entire polynomial twice over to find integer coefficients.

(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

Solution:

Proof. We start by representing each polynomial with integer coefficients as a tuple of coefficients:

$$(a_n, a_{n-1}, \ldots, a_1, a_0)$$

and consider the set of all polynomials with integer coefficients in this manner. However, this is the same as the set $\mathbb{Z}^{n+1}$ and since the product of a finite number of countable sets is also countable, the set of all polynomials with integer coefficients must also be countable. Since this set is countable and each polynomial of degree n can have at most n roots, it follows that A_n must also be countable (by being the union of a countable amount of sets with finite elemnts). □

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution: Since the set of algebraic numbers, $\mathbb{A}$ is countable, the set of transcendentals, $\mathbb{R} \setminus \mathbb{A}$, must be uncountable. This is because every real number is either algebraic or transcendental, so $A \cup (\mathbb{R} \setminus \mathbb{A}) = \mathbb{R}$. Since $\mathbb{R}$ is uncountable, $\mathbb{R} \setminus \mathbb{A}$ must be uncountable as well since the union of two countable sets cannot be uncountable.

1.5.11:

(a) Explain how partitioning X and Y in such a way that f maps A onto B and g maps B' onto A' would lead to a proof that $X \sim Y$.

Solution: Since g is both onto and $1-1$ we know that $g^{-1} : A' \to B'$ exists. We define $h : X \to Y$ as follows:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A' \end{cases}.$$

Since f and g^{-1} are bijections, h must be one as well. Thus $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y)$ and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n \mid n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) \mid n \in \mathbb{N}\}$ is a similar collection in Y.

Solution: If $A_1 = \varnothing$ then $X = g(Y)$ so g is onto and thus a bijection which implies $X \sim Y$. In the case $A_1 \neq \varnothing$ we may move on to induction. For the base case $A_1 \cap g(Y) = \varnothing$ by definition of A_1, so $A_n \cap A_1 = \varnothing$ for all $n > 1$ since $A_n = g(f(A_{n-1})) \subseteq g(Y)$. We now suppose that $A_m \cap A_n = \varnothing$ for some $n < m$. Then $f(A_n) \cap f(A_m) = \varnothing$ since f is $1-1$. So, $A_{n+1} = g(f(A_n)) \cap g(f(A_m)) = A_{m+1}$ is empty and thus $\{A_n \mid n \in \mathbb{N}\}$ is pairwise disjoint. It follows from this that $\{f(A_n) \mid n \in \mathbb{N}\}$ is also pairwise disjoint in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

Solution: Let $y \in B$. By definition of union, $y \in f(A_n)$ for some n. There exists $x \in A_n$ such that $f(x) = y$ by definition of $f(A_n)$. Since $A_n \subseteq A$, we know that $x \in A$. Thus, we have found $x \in A$ such that $f(x) = y$ for some $y \in B$ which implies that f is onto from A to B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution: We will show that $g(B') = A'$. Suppose that $x \in g(B')$. Then $x = g(y)$ for some $y \in B'$. If $x \in A$ then $x \in A_n$ for some n. If $n = 1$ then $x \notin g(Y)$ which contradicts the assumption that $x \in g(B') \subseteq g(Y)$. If $n > 1$ then $x \in A_n = g(f(A_{n-1}))$. Because g is 1-1, $y = f(A_{n-1}) \subseteq B$ which contradicts the fact that $y \notin B$. So, $x \in A'$.
Now, let $x \in A'$. Then $x \notin A$ so $x \notin A_1 = X \setminus g(Y)$. So $x \in g(Y)$ which implies that $x = g(y)$ for some $y \in Y$. If $y \in B$, then $y \in f(A_n)$ for some n. Then $x = g(y) \in g(f(A_n)) = A_{n+1}$. But then $x \in A_{n+1} \subseteq A$ which implies $x \in A$ which is a contradiction. Thus $y \in B'$.
So, since $g(B') = A'$, it must be that g maps B' onto A'.

1.6.1: Show that $(0, 1)$ is uncountable if, and only if, $\mathbb{R}$ is uncountable.

Solution:

Proof. Forwards: Assume $\mathbb{R}$ is uncountable and consider the function

$$f(x) = \frac{e^x}{e^x + 1}.$$

Then, f is a bijective function with a domain of $\mathbb{R}$ and a range of $(0, 1)$ since $f(x) \neq 0$ and $f(x) \neq 1$ for all $x \in \mathbb{R}$. Thus, since there exists a bijection from $\mathbb{R}$ to $(0, 1)$, we have that $\mathbb{R} \sim (0, 1)$.
Backwards: If $(0, 1)$ is uncountable it immediately follows that $\mathbb{R}$ must be uncountable since if $\mathbb{R}$ was countable, then every (infite) subset must be countable. However, this contradicts our assumption and so $\mathbb{R}$ must be countable. □

1.6.9: Show that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

Solution:

Proof. We will show that there exist injective functions $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathcal{P}(\mathbb{N})$. We define f as follows: for every subset $S \subseteq \mathbb{N}$, take $f(S)$ as the base-10 decimal representation $0.b_1 b_2 b_3 \ldots$ where $b_n = 1$ if, and only if, $n \in S$ and $b_n = 0$ otherwise. Unique subsets will have unique images under f so this function is injective. We define $g : \mathbb{R} \to \mathcal{P}(\mathbb{N})$ as follows: for every number $x \in (0, 1)$ map the binary representation of $x = c_1 c_2 c_3 \ldots$, where $c_n \in \{0, 1\}$, to a subset $S \in \mathcal{P}(\mathbb{N})$ where $n \in S$ if, and only if, $c_n = 1$. This results in g being injective since each binary representation is unique. It follows that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$. □

1.6.10:

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?

Solution: The set of all functions from $\{0, 1\}$ to $\mathbb{N}$ is countable. This is because for any $f : \{0, 1\} \to \mathbb{N}$, we can define a map to $\mathbb{N} \times \mathbb{N}$ given by $(f(0), f(1))$. Since each ordered pair $(a, b) \in \mathbb{N}$ will appear only once (when $f(0) = a$ and $f(1) = b$), this is a bijection and thus the set of all functions is countable.

(b) Is the set of all function from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?

Solution: The set of all functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable. This is because any function $f : \mathbb{N} \to \{0, 1\}$ must assign each $n \in \mathbb{N}$ to either 0 or 1, so for any such function we can associate it with a subset $A \subseteq \mathbb{N}$ such that $n \in A$ if, and only if, $f(n) = 1$. This is a mapping to the power set, $\mathcal{P}(\mathbb{N})$ which is uncountable, so the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ is also uncountable.

(c) Does $\mathcal{P}(\mathbb{N})$ contain an uncountable antichain?

Solution: Yes, $\mathcal{P}(\mathbb{N})$ does contain an uncountable antichain. For each $x \in (0, 1)$ we represent it as binary: $x = 0.b_1 b_2 b_3 \ldots$ for $b_n \in \{0, 1\}$. Now take the binary representation of x and map it to subset of $\mathbb{N} \times \mathbb{N}$ by the map (n, b_n) for each b_n in the binary representation of x (eg. for $x = 0.101\ldots$ we have $\{(1, 1), (2, 0), (3, 1), \ldots\}$). Then, for any $x \neq y$ there will be some index, say $n = k$, at which $b_{x_k} \neq b_{y_k}$. This results in no set being a subset of another. Since there are uncountably many real numbers between 0 and 1, there are uncountable many sequences and this results in an uncountable antichain.

[0] LaTeX code for this document can be found on github here