

Definition 1. Given a second-order linear homogenous recurrence relation with constant coefficients:

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \forall k \in \mathbb{Z} (k \geq 2)$$

the **characteristic equation of the relation** is

$$t^2 - At - B = 0$$

Lemma 1. Let A and B be real numbers. A recurrence relation of the form

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \forall k \in \mathbb{Z} (k \geq 2)$$

is satisfied by the sequence

$$1, t, t^2, t^3, \dots, t^n, \dots$$

where t is a nonzero real number, if, and only if, t satisfies the equation

$$t^2 - At - B = 0$$

Lemma 2. If r_0, r_1, r_2, \dots and s_0, s_1, s_2, \dots are sequences that satisfy the same second-order linear homogenous recurrence relation with constant coefficients, and if C and D are any numbers, then the sequence a_0, a_1, a_2, \dots defined by the formula

$$a_n = Cr_n + Ds_n \quad \forall n \in \mathbb{Z}^{nonneg}$$

also satisfies the same recurrence relation

Lemma 3. Let A and B be real numbers and suppose that the characteristic equation

$$t^2 - At - B = 0$$

has a single root r . Then the sequences $1, r^1, r^2, r^3, \dots, r^n, \dots$ and $0, r, 2r^2, 3r^3, \dots, nr^n, \dots$ both satisfy the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \forall n \in \mathbb{Z} (n \geq 2)$$

Theorem 1 (Single-Root Theorem). Suppose a sequence a_0, a_1, a_2, \dots satisfies a recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for some real numbers A and B with $B \neq 0$ and $\forall k \in \mathbb{Z} (k \geq 2)$. If the characteristic equation $t^2 - At - B = 0$ has a single (real) root r , then a_0, a_1, a_2, \dots is given by the explicit formula

$$a_n = Cr^n + Dnr^n$$

where C and D are the real numbers whose values are determined by the values of a_0 and any other known value of the sequence

Proof. Suppose for some real numbers A and B , a sequence a_0, a_1, a_2, \dots satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2} \quad \forall k \in \mathbb{Z} (k \geq 2)$, and suppose the characteristic equation $t^2 - At - B = 0$ has one distinct root r . We will show that

$$\forall n \in \mathbb{Z}^{nonneg}, \quad a_n = Cr^n + Dnr^n$$

where C and D are numbers such that

$$a_0 = Cr^0 + D(0)r^0 = C \quad \text{and} \quad a_1 = Cr^1 + D(1)r^1 = Cr + Dr$$

Let $P(n)$ be the equation

$$a_n = Cr^n + Dnr^n$$

We use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 0$. In the basis step, we prove that $P(0)$ and $P(1)$ are true. We do this because in the inductive step we need the equation to hold for $n = 0$ and $n = 1$ in order to prove that it holds for $n = 1$

Show that $P(0)$ and $P(1)$ are true: The truth of $P(0)$ and $P(1)$ is automatic because C and D are exactly those numbers that make the following equations true:

$$a_0 = Cr^0 + D(0)r^0 = C \quad \text{and} \quad a_1 = Cr^1 + D(1)r^1 = Cr + Dr$$

Show that for all integers $k \geq 1$, if $P(i)$ is true for all integers i from 0 through k , then $P(k+1)$ is also true: Suppose that $k \geq 1$ and for all integers i from 0 through k ,

$$a_i = Cr^i + Dir^i$$

We must show that

$$a_{k+1} = Cr^{k+1} + D(k+1)r^{k+1}$$

Now by the inductive hypothesis,

$$a_k = Cr^k + Dkr^k \quad \text{and} \quad a_{k-1} = Cr^{k-1} + D(k-1)r^{k-1}$$

so

$$\begin{aligned} a_{k+1} &= Aa_k + Ba_{k-1} && \text{(by definition of } a_0, a_1, a_2, \dots) \\ &= A(Cr^k + Dkr^k) + B(Cr^{k-1} + D(k-1)r^{k-1}) \\ & && \text{(by inductive hypothesis)} \\ &= C(Ar^k + Bkr^k) + D(Akr^k + B(k-1)r^{k-1}) \\ & && \text{(by combining like terms)} \\ &= Cr^{k+1} + D(k+1)r^{k+1} && \text{(by Lemma 1)} \end{aligned}$$

This was what was to be shown. ■

Remark. The reason the last equality follows from Lemma 2 is that since r satisfies the characteristic equation and is the only root of it, the sequences $1, r^1, r^2, r^3, \dots, r^n, \dots$ and $0, r, 2r^2, 3r^3, \dots, nr^n, \dots$ satisfy the recurrence relation