Theorem 1 (Divisibility by a Prime)**.** *Any integer $n > 1$ is divisible by a prime number.*

Theorem 2. *For call integers n, if $n > 2$ then there is a prime number p such that $n < p < n\boldsymbol{!}$*

Proof. Assume that n is an integer greater than 2 and that p is a prime number such that $p \mid (n\mathbf{!} - 1)$ *[by the prime divisibility theorem]* and that $p \leq n$. Then, by definition of factorial, $p \mid n\mathbf{!}$. So,

$$n\mathbf{!} \equiv 0 \pmod{p} \tag{1}$$

However, by assumption since $p \mid (n\mathbf{!} - 1)$ we can say,

$$n\mathbf{!} - 1 \equiv 0 \pmod{p} \tag{2}$$

Manipulating the left hand side of (1) to match (2) gives,

$$\begin{aligned} n\mathbf{!} &\equiv 0 \pmod{p} \\ n\mathbf{!} - 1 &\equiv 0 - 1 \pmod{p} \\ n\mathbf{!} - 1 &\equiv -1 \pmod{p} \\ \implies 0 &\equiv -1 \pmod{p} \end{aligned} \tag{3}$$

But, in order for equation 3 to be true p must be equal to 1 *[for the sake of modular arithmatic assume our domain to be $\mathbb{Z}^+$].* Thus we have reached a contradiction since $p \leq n$ and $n > 2$ so $p \neq 1$. So, $p > n$ and $p \mid (n\mathbf{!} - 1) \implies p \leq (n\mathbf{!} - 1)$. Because $(n\mathbf{!} - 1) < n\mathbf{!}$, we know that $p \leq (n\mathbf{!} - 1) < n\mathbf{!}$ so, $n < p < n\mathbf{!}$ *[which was to be proved.]* ■