Name: ______________________________

Answer the questions in the spaces provided on the following pages. If you run out of room for an answer, continue on the back of the page. Show **all** your work!

1. Prove the following using proof by induction

$$\text{For all positive integers, } n,\ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

Proof. Let $P(n)$ be the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Show that $P(1)$ is true: To show that $P(1)$ is true we will show that LHS and the RHS are equivalent. The LHS is simply $1^2 = 1$ and the RHS is $\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$ and so the LHS and RHS equal the same quantity and thus must be equivalent, proving the truthfullness of $P(1)$

Show that if $P(k)$ is true, so is $P(k+1)$: First, we will assume that $P(k)$ is true or that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Next, we will need to show that $P(k+1)$ is true or that

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)\left[(k+1)+1\right]\left[2\left(k+1\right)+1\right]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

We can rewrite the LHS in the following way

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k^2+2k+1)}{6} \\
&= \frac{(k^2+k)(2k+1)}{6} + \frac{6k^2+12k+6}{6} \\
&= \frac{2k^3+2k^2+k^2+k}{6} + \frac{6k^2+12k+6}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}$$

The RHS can be expanded to give

$$\begin{aligned}
\frac{(k+1)(k+2)(2k+3)}{6} &= \frac{(k^2+3k+2)(2k+3)}{6} \\
&= \frac{2k^3+3k^2+6k^2+9k+4k+6}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}$$

Now our LHS and RHS equal the same quantity and must be equivalent, thus our statement holds true through proof by induction. ■

2. Prove the following using proof by induction

$$\text{For all integers } n \geq 1,\ 1+3+5+\cdots+(2n-1)=n^2$$

Solution:

Proof. Let $P(n)$ be the formula

$$1+3+5+\cdots+(2n-1)=n^2$$

Show that $P(1)$ is true: To show that $P(1)$ is true, we will need to show that the LHS and RHS are equivalent. Well, the LHS is $2(1)-1=1$ and the RHS is $1^2=1$ and so both sides equal the same quantity and thus are equivalent, proving the truthfullness of $P(1)$

Show that if $P(k)$ is true, so is $P(k+1)$: We will assume that $P(k)$ is true or that

$$1+3+5+\cdots+(2k-1)=k^2$$

and we must show that $P(k+1)$ is true, or that

$$1+3+5+\cdots+(2\,(k+1)-1)=(k+1)^2$$

We can begin with the LHS as follows,

$$\begin{aligned}1+3+5+\cdots+(2k+1) &= 1+3+5+\cdots+(2k-1)+(2k+1)\\ &= k^2+(2k+1)\\ &= (k+1)(k+1)\\ &= (k+1)^2\end{aligned}$$

And so, our LHS and RHS equal the same quantity and thus are equivalent, proving our statement through proof by induction. ■

3. Prove the following using proof by induction

$$\text{For all integers } n \geq 1,\ 2+4+6+\cdots+2n=n^2+n$$

Solution:

Proof. Let $P(n)$ be the formula

$$2+4+6+\cdots+2n=n^2+n$$

Show that $P(1)$ is true: To show that $P(1)$ is true we will show that the LHS and the RHS are equivalent. The LHS is $2(1)=2$ and the RHS is $1^2+1=2$ and so, since the LHS and RHS are equivalent, $P(1)$ holds true.

Show that if $P(k)$ is true, so is $P(k+1)$: We will assume that $P(k)$ is true or that

$$2+4+6+\cdots+2k=k^2+k$$

and we will show that $P(k+1)$ is true or that

$$2+4+6+\cdots+2(k+1)=(k+1)^2+k=k^2+2k+1+k=k^2+3k+1$$

We can begin with the LHS to show

$$\begin{aligned} 2+4+6+\cdots 2(k+1) &= 2+4+6+\cdots+2k+2(k+1) \\ &= (k^2+k)+2(k+1) \\ &= k^2+k+2k+1 \\ &= k^2+3k+1 \end{aligned}$$

Which is the RHS of $P(k+1)$ and so our LHS and RHS are equivalent, proving the validity of our statement through proof by induction. ■

4. *(Challenge)* Prove the following using proof by induction

$$\text{For all integers } n \geq 1,\ \sum_{i=1}^{n} i\,(i\mathbf{!}) = (n+1)\mathbf{!} - 1$$

Solution:

Proof. Let $P(n)$ be the formula

$$\sum_{i=1}^{n} i\,(i\mathbf{!}) = (n+1)\,\mathbf{!} - 1$$

Show that $P(1)$ **is true:** To show that $P(1)$ is true we will show that the LHS and RHS are equivalent.

LHS	RHS
$\sum_{i=1}^{1} i\,(i\mathbf{!}) = 1(1\mathbf{!}) = 1(1) = 1$	$(1+1)\,\mathbf{!} - 1 = 2\mathbf{!} - 1 = 2 - 1 = 1$

And so, our LHS and RHS are equivalent, thus proving $P(1)$

Show that if $P(k)$ **is true, so is** $P(k+1)$**:** First, we will assume that $P(k)$ is true or that

$$\sum_{i=1}^{k} i\,(i\mathbf{!}) = (k+1)\,\mathbf{!} - 1$$

and we will need to show that $P(k+1)$ is true or that

$$\sum_{i=1}^{k+1} i\,(i\mathbf{!}) = [(k+1)+1]\,\mathbf{!} - 1 = (k+2)\,\mathbf{!} - 1$$

We can begin with the LHS as follows,

$$\begin{aligned}
\sum_{i=1}^{k+1} i\,(i\mathbf{!}) &= (k+1)\,[(k+1)\,\mathbf{!}] + \sum_{i=1}^{k} i\,(i\mathbf{!}) \\
&= (k+1)\,(k+1)\,\mathbf{!} + [(k+1)\,\mathbf{!} - 1] \\
&= [(k+1)\,(k+1)\,\mathbf{!} + (k+1)\,\mathbf{!}] - 1 && \text{(by the associative property of addition)} \\
&= (k+1)\,\mathbf{!}\,[(k+1)+1] - 1 && \text{(by factoring)} \\
&= (k+1)\,\mathbf{!}\,(k+2) - 1 \\
&= (k+2)\,\mathbf{!} - 1 && \text{(by properties of factorials)}
\end{aligned}$$

And so, our LHS is equivalent to the same quantity as the RHS and is thus equal to the RHS, proving our statement through induction. ■

5. *(**Mega** Challenge)* Prove the following using proof by induction

$$\text{For all integers } n \geq 1\text{, 3 is a factor of } 4^n - 1$$

Solution:

Proof. Let $P(n)$ be the statement that 3 is a factor of $4^n - 1$
Show that $P(1)$ **is true:** To show that $P(1)$ is true we will show that 3 is a factor of $4^1 - 1$. $4^1 - 1 = 3$ and $3/3 = 1$ which is an integer, thus 3 is a factor of $4^1 - 1$ and $P(1)$ is valid.
Show that if $P(k)$ **is true, so is** $P(k+1)$**:** We will assume that $P(k)$ is true or that

$$\frac{4^k - 1}{3} \in \mathbb{Z}$$

and we will need to show that

$$\frac{4^{k+1} - 1}{3} \in \mathbb{Z}$$

We can do this by manipulating the LHS

$$\begin{aligned} 4^{k+1} - 1 &= 4^{k+1} - 4^k + 4^k - 1 \\ &= 4^k * 4 - 4^k + 4^k - 1 \\ &= 4^k(4-1) + (4^k - 1) \\ &= 4^k(3) + (4^k - 1) \end{aligned}$$

Since we already know that $4^k - 1$ has a factor of 3 we can conclude that 4^{k+1} must also have a factor of 3 since $\dfrac{4^k(3)}{3} = 4^k \in \mathbb{Z}$ and so adding another factor of 3 will still give a factor of 3. Thus, our statement holds true through proof by induction. ■