SEQUENCES REVIEW

Review and Definition

A sequence is an ordered list of numbers that follows a specific rule or pattern. Sequence terms are standardly notated in the following manner:

$$a_1, a_2, a_3, \ldots, a_n$$

where a_1 is the first term, a_2 the second term, and a_n represents the n-th term. The actual sequence may be denoted in multiple different ways:

$$\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

In the second and third representations, a_n will be given by a formula, either explicitly inside the curly braces or in another way. In the third notation, the $n = 1$ represents the starting index of the sequence (note that not all sequences start at 1) and the ∞ is the ending index. Note that not all sequences are infinite, for example, $\{a_n\}_{n=3}^{10}$ is also a valid sequence.

1. Write the first five terms of the sequence $\{2^n\}_{n=0}^{10}$
2. Write the general formula for the sequence $\{5, 10, 15, 20, \ldots\}$

When it comes to defining sequences there are two main ways to do so: the **explicit** formula or the **recursive** formula. The explicit formula defines a rule for any a_n given any n meanwhile the recursive formula defines a formula for a_n based on a_{n-1}. To see the difference look at the following examples:

Explicit	Recursive
$a_n = 5n$	$a_{n+1} = 5 + a_n;\ \ a_1 = 5$

A few key things are important to note here when looking at the recursive formula. First, notice that the term that is defined is a_{n+1} and not a_n since the a_n term is being used in the formula. With this, you should be aware that the recursive formula could've also been written in the following manner:

$$a_n = 5 + a_{n-1};\ \ a_1 = 5$$

Also notice how a 'base term' is defined, a_1. This tells us our starting term and using that we can begin to find other terms in the sequence.

SERIES

Review and Definition

While sequences may be ordered lists of numbers, series are sums of the elemnets of a sequence. Given some sequence, $\{a_n\}_{n=1}^{\infty}$ with elements $\{a_1, a_2, \ldots, a_n\}$ we can define the following:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{n=1}^{\infty} a_n \end{aligned}$$

The s_n are call **partial sums** and will form the *sequence* $\{s_n\}_{n=1}^{\infty}$.

Summation Notation

As you saw another way to represent the sum s_n was with a capital sigma, $\sum$. Given a summation, $\sum\limits_{n=1}^{\infty} a_n$ the n is called the **index of summation** or index for short. It's important to note that the index is simply a dummy variable that takes on value only because of its context and enviornment. Because of that,

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_k = \sum_{j=1}^{\infty} a_j$$

All three of the above series are the same series, save for the fact that we've choosen a different letter to represent the index. As with sequences it's also important to know that the starting and ending values can be anything, we don't have to start at $n = 0$ or $n = 1$ and we don't have to end at ∞. In fact, you might often see the notation $\sum a_n$ to represent a generic series in which the starting and ending values are irrelevant to the presented fact.

PROPERTIES OF SERIES

If $\sum a_n$ and $\sum b_n$ are two series[a],

1. If c is any number ($c \in \mathbb{R}$) then

$$\sum ca_n = c \sum a_n$$

2. Series can be combined by addition and subtraction

$$\sum a_n \pm \sum b_n = \sum a_n \pm b_n$$

[a] Assume $\sum a_n$ and $\sum b_n$ are convergent

Index Shifts

The next topic to discuss is the **index shift**. The index shift is a technique used to manipulate a given series so that it starts/ends at a different value. First start with the following series:

$$\sum_{n=1}^{k} a_n$$

and now say that we want the series to start at $n = 0$ instead of $n = 1$. Lets start by defining $i = n-1$ since plugging in $n = 1$ gives $i = 1 - 1 = 0$. Before we can change our series we first have to figure out what i will equal at the the start and end of our summation. Like previously mentioned, when $n = 1$, $i = 0$. As for the end we can apply the same logic, when $n = k$, $i = k - 1$. Finally we need to change the inside of our summation, to do that let's rewrite $i = n - 1$ as $n = i + 1$ and then plug that in to a_n to give a_{i+1}. Putting this all together gives,

$$\sum_{n=1}^{k} a_n = \sum_{i=0}^{k-1} a_{i+1}$$

And again, since the index letter doesn't matter, we could change it back to n if we'd like. Similar to the above work, we can also shift the index to increase by following similar steps. Define $i = n + 1$, solve for the limits and change the term:

$$\sum_{n=1}^{k} a_n = \sum_{i=2}^{k+1} a_{i-1}$$

Although not an index shift, we can also rewrite series by adding and removing terms. First, consider the sequence $\{a_n\}_{n=1}^{k} = \{a_1, a_2, a_3, \ldots, a_k\}$ and the series $\sum\limits_{n=1}^{k} a_n$. Now, notice that,

$$\sum_{n=1}^{k} a_n = a_1 + \sum_{n=1}^{k} a_n$$

and that

$$\sum_{n=1}^{k} a_n = a_k + \sum_{n=1}^{k-1} a_n$$

Now, we can apply this multiple times over to strip out the first or last n terms from a series without it changing the value.

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1. Rewrite the series $\sum\limits_{z=3}^{32} \frac{1}{2^z}$ to start at $z = 0$

2. Strip out the first and last term from the series $\sum\limits_{q=0}^{10} \frac{2}{5}z$