Definition 1. Given a second-order linear homogenous recurrence relation with constant coefficients:

$$a_k = Aa_{k-1} + Ba_{k-1} \quad \forall k \in \mathbb{Z}\,(k \geq 2)$$

the **characteristic equation of the relation** is

$$t^2 - At - B = 0$$

Lemma 1. *Let A and B be real numbers. A recurrence relation of the form*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \forall k \in \mathbb{Z}\,(k \geq 2)$$

is satisfied by the sequence

$$1, t, t^2, t^3, \ldots, t^n, \ldots$$

where t is a nonzero real number, if, and only if, t satisfies the equation

$$t^2 - At - B = 0$$

Lemma 2. *If $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satify the same second-order linear homogenous recurrence relation with constant coefficients, and if C and D are* any *numbers, then the sequence $a_0, a_1, a_2, \ldots$ defined by the formula*

$$a_n = Cr_n + Ds_n \quad \forall n \in \mathbb{Z}^{nonneg}$$

also satisfies the same recurrence relation

Lemma 3. *Let A and B be real numbers and suppose that the characteristic equation*

$$t^2 - At - B = 0$$

has a single root r. Then the sequences $1, r^1, r^2, r^3, \ldots, r^n, \ldots$ and $0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$ both satisfy the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \forall n \in \mathbb{Z}\,(n \geq 2)$$

Theorem 1 (Single-Root Theorem)**.** *Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for some real numbers A and B with $B \neq 0$ and $\forall k \in \mathbb{Z}\,(k \geq 2)$. If the characteristic equation $t^2 - At - B = 0$ has a single (real) root r, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Dnr^n$$

where C and D are the real numbers whose values are determined by the values of a_0 and any other known value of the sequence

Proof. Suppose for some real numbers A and B, a sequence $a_0, a_1, a_2, \ldots$ satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2} \;\; \forall k \in \mathbb{Z}\,(k \geq 2)$, and suppose the characteristic equation $t^2 - At - B = 0$ has one distinct root r. We will show that

$$\forall n \in \mathbb{Z}^{nonneg}, \quad a_n = Cr^n + Dnr^n$$

where C and D are numbers such that

$$a_0 = Cr^0 + D(0)r^0 = C \qquad \text{and} \qquad a_1 = Cr^1 + D(1)r^1 = Cr + Dr$$

Let $P(n)$ be the equation

$$a_n = Cr^n + Dnr^n$$

We use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 0$. In the basis step, we prove that $P(0)$ and $P(1)$ are true. We do this because in the inductive step we need the equation to hold for $n = 0$ and $n = 1$ in order to prove that it holds for $n = 1$

Show that $P(0)$ and $P(1)$ are true: The truth of $P(0)$ and $P(1)$ is automatic because C and D are exactly those numbers that make the follwing equations true:

$$a_0 = Cr^0 + D(0)r^0 = C \qquad \text{and} \qquad a_1 = Cr^1 + D(1)r^1 = Cr + Dr$$

Show that for all integers $k \geq 1$, if $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is also true: Suppose that $k \geq 1$ and for all integers i from 0 through k,

$$a_i = Cr^i + Dir^i$$

We must show that

$$a_{k+1} = Cr^{k+1} + D\,(k+1)\,r^{k+1}$$

Now by the inductive hypothesis,

$$a_k = Cr^k + Dkr^k \qquad \text{and} \qquad a_{k-1} = Cr^{k-1} + D\,(k-1)\,r^{k-1}$$

so

$$\begin{aligned}
a_{k+1} &= Aa_k + Ba_{k-1} && \text{(by definition of } a_0, a_1, a_2, \ldots) \\
&= A\left(Cr^k + Dkr^k\right) + B\left(Cr^{k-1} + D\,(k-1)\,r^{k-1}\right) \\
& && \text{(by inductive hypothesis)} \\
&= C\left(Ar^k + Bkr^k\right) + D\left(Akr^k + B\,(k-1)\,r^{k-1}\right) \\
& && \text{(by combining like terms)} \\
&= Cr^{k+1} + D\,(k+1)\,r^{k+1} && \text{(by Lemma 1)}
\end{aligned}$$

This was what was to be shown.

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Remark. The reason the last equality follows from Lemma 2 is that since r satisfies the characteristic equation and is the only root of it, the sequences $1, r^1, r^2, r^3, \ldots, r^n, \ldots$ and $0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$ satify the recurrence relation