Summary of Convergence and Divergence Tests

1. Divergence Test: Given $\sum_{n=0}^{\infty} a_n$ does $\lim_{n\to\infty} a_n = 0$? If it does not, then the series must diverge.
   Note that the Divergence Test does not state anything about convergence, it simply tells whether a series diverges or not.

2. Direct Comparison Test: Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with $0 \le a_n \le b_n$:

   (a) If $\sum_{n=0}^{\infty} a_n$ is divergent then so is $\sum_{n=0}^{\infty} b_n$

   (b) If $\sum_{n=0}^{\infty} b_n$ is convergent then so is $\sum_{n=0}^{\infty} a_n$

3. Limit comparison test: Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with $a_n, b_n \ge 0$, let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If L is finite and positive (ie $0 < L < \infty$) then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ have the same convergence
   Note as well that the limit can also be $\lim_{n\to\infty} \frac{b_n}{a_n}$

4. Integral Test: Given $\sum_{n=0}^{\infty} a_n$ and:

   (a) There is a function $f(x)$ such that $f(n) = a_n$
   (b) $f(x)$ is continuous
   (c) $f(x)$ is positive
   (d) $f(x)$ is decreasing

   Then the convergence of $\sum_{n=0}^{\infty} a_n$ will be the same as the convergence of $\int_0^{\infty} f(x)\,dx$
   Note that the conditions of positive and decreasing must only be true ***eventually***

5. Alternating Series Test: Given $\sum_{n=0}^{\infty} (-1)^n a_n$ or $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$ then if:

   (a) a_n is decreasing (ie. $a_n \ge a_{n+1)}$)
   (b) $\lim_{n\to\infty} a_n = 0$

   $\sum_{n=0}^{\infty} (-1)^n a_n$ or $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$ converges

Summary of Convergence and Divergence Tests (cont.)

6. Ratio Test: Given $\sum_{n=0}^{\infty} a_n$, let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
   (a) $L < 1$: $\sum_{n=0}^{\infty} a_n$ converges
   (b) $L > 1$: $\sum_{n=0}^{\infty} a_n$ diverges
   (c) $L = 1$: Ratio test is inconclusive (series could converge or diverge)

   Note that the ratio test will ***always*** *be inconclusive for polynomials, so do not waste your time trying it if a_n is a polynomial*

7. Root Test: Given $\sum_{n=0}^{\infty} a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$
   (a) $L < 1$: $\sum_{n=0}^{\infty} a_n$ converges
   (b) $L > 1$: $\sum_{n=0}^{\infty} a_n$ diverges
   (c) $L = 1$: Root test is inconclusive (series could converge or diverge)

   Note that the root test will ***always*** *be inconclusive for polynomials, so do not waste your time trying it if a_n is a polynomial*

Alternating Series Remainder Theorem

Given $\sum_{n=0}^{\infty}(-1)^n a_n$ or $\sum_{n=0}^{\infty}(-1)^{n+1} a_n$, and an approximation for the sum, S, at $n = N$, S_N, the error between the actual sum and the approximation can by found by using:

$$|S - S_N| = |R_N| \leq a_{N+1}$$

meaning that the error between the true sum and the approximation must be less than or equal to the next neglected term (the next term not included in the approximation)

Special Series

1. P-series: Given $\sum_{n=0}^{\infty} \frac{1}{n^p}$

   (a) If $p > 1$ $\sum_{n=0}^{\infty} \frac{1}{n^p}$ will converge

   (b) If $p \leq 1$ $\sum_{n=0}^{\infty} \frac{1}{n^p}$ will diverge

   Note that if $p = 1$ then the series is known as the harmonic series

2. Geometric Seres: Given $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$ then,

   (a) If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$ converges and its sum can be found with $S = \frac{a}{1-r}$, where a is the first term of the series

   (b) If $|r| \geq 1$ then $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$ diverges and, therefore, its sum *cannot* be found

3. Telescoping Series: Given a series, $\sum_{n=0}^{\infty} a_n$, that has partial sums in the form $(s_1 - s_2) + (s_2 - s_3) + (s_3 - s_4) + \ldots + (s_n - s_{n+1})$ its sum can be found with, $\lim_{n \to \infty} s_1 - s_{n+1}$ however, this is only true if, and only if, $s_n \to$ a finite number as $n \to \infty$

ABSOLUTE/CONDITIONAL CONVERGENCE

- Definition: Let $\sum_{n=0}^{\infty} a_n$ be a **convergent series**. If $\sum_{n=0}^{\infty} |a_n|$ is *convergent*, then $\sum_{n=0}^{\infty}$ is called **absolutely convergent**. If $\sum_{n=0}^{\infty} |a_n|$ is *divergent*, then $\sum_{n=0}^{\infty} a_n$ is called **conditionally convergent**

- Tip: Due to the fact that if $\sum_{n=0}^{\infty} |a_n|$ is convergent then so is $\sum_{n=0}^{\infty} a_n$, **always** test $\sum_{n=0}^{\infty} |a_n|$ first

WHEN TO USE EACH TEST

The short answer: ratio/root test is usually the best test to use **unless:** 1. your a_n is a polynomial or 2. you have one of the special series (p-series, geometric, alternating, or telescoping).
The long answer: most questions won't really ask you to use a specific test so which test you use is up to you. *However*, with that being said, oftentimes a certain test is easier than another. As stated above, ratio/root test *tends* to be easiest, however, do not under estimate the usefulness of the other tests. For example, a series that is extremely similar to a p-series or a geometric series can easily have its convergence be found with a quick comparison test. For a more in depth list of examples see this link.

PRACTICE

As you've heard in many prior units, the best way to learn which series to use when is through lots of practice. So, below are quick links to Pauls Online Notes practice problems for the Sequence's and Series Unit.

- Integral Test Practice
- Comparison Test
- Alternating Series Test
- Absolute Convergence