

TAYLOR POLYNOMIALS REVIEW

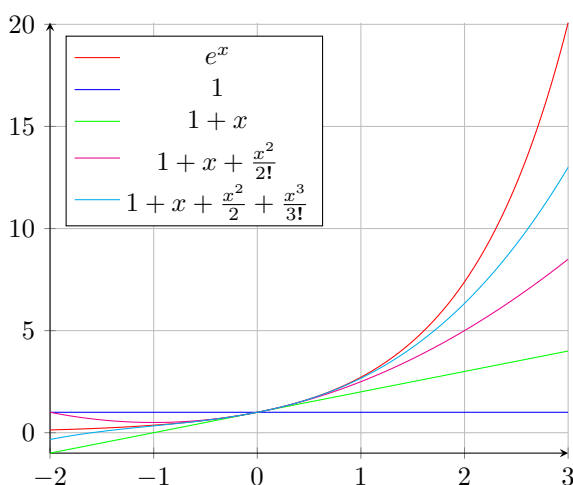
Before we talk about Lagrange Error (also known as Taylor's Theorem) let's do a quick recap of Taylor polynomials. Taylor polynomials are polynomials that we can use to approximate transcendental functions like e^x and $\sin x$. If there is some function f that has n derivatives then,

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is the n th Taylor Polynomial for f centered around c .

* Note that in the special case where $c = 0$ the polynomial is known as the Maclaurin Polynomial

Now as we increase the degree of the Taylor polynomial we can increase the accuracy of our approximation of our function.

Taylor Approximations of e^x 

The graph to the left shows the first 4 terms of the Maclaurin polynomial for e^x . The key thing to note here is that as the polynomial increases in degree (blue \rightarrow green \rightarrow magenta \rightarrow cyan) the polynomial more closely approximates e^x . This is the key idea that we will use to in the calculation of our Lagrange Error Bound^a.

Also notice that because this polynomial was centered around 0, we named it the Maclaurin polynomial for e^x

^a[See this desmos demo for more](#)

TAYLOR'S THEOREM (LAGRANGE ERROR)

Before we talk about how to find the lagrange error let's first think about what the lagrange error bound is *used* for.

Let's assume you have a function $f(x)$ and the n th degree Taylor approximation of f centered around c , $P_n(x)$. Now let's say that you want to approximate the value of f at some value $x = z$, so you find the value of $P_n(z)$. Now remembering that P_n is only an approximation you want to find the error in your approximation. **This is what Taylor's Theorem tells us.**

Just like the alternating series remainder theorem the Lagrange error bound will utilize the next omitted term to find the remainder between your approximation and the real value, in our case, this next neglected term will be the next higher derivative. This makes sense if you recall from the recap section where above we stated that as we increased the number of derivatives, our approximation (Taylor) polynomial got better and more accurate. Now you might be tempted to immediately calculate the $n+1$ th Taylor polynomial, P_{n+1} and evaluate it at z ; however, just like in the alternating series remainder theorem we will be focusing on the next neglected term.

TAYLOR'S THEOREM (LAGRANGE ERROR) (cont.)

With this lets (let's?) start constructing our formula:

$$R_n(x) = \frac{f^{(n+1)}(x-c)}{(n+1)!} (x-c)^{n+1} \quad \text{where } R_n(x) \text{ is our remainder function}$$

Hopefully, you'll recognize this as the $n+1$ th term of our Taylor polynomial P_n . Now there are a few small tweaks we can make to this formula to help us find a better error bound.

First start by considering what this formula tells us in its current state: the remainder (or error), R_n is *exactly* equal to the next neglected term (or next higher derivative). Now while this is nice we want to cover all our bases, so instead of seeking the exact error, lets find the **maximum** error that could possible occur with our approximation. We can start by turning this into an inequality:

$$R_n(x) \leq \frac{f^{(n+1)}(x-c)}{(n+1)!} (x-c)^{n+1} \quad \text{where } R_n(x) \text{ is our remainder function}$$

Then we can focus on making our function as large as possible. We will do this in two steps:

$$R_n(x) \leq \left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| |x-c|^{n+1} \quad \text{where } R_n(x) \text{ is our remainder function} \quad (1)$$

$$R_n(x) \leq \mathbf{max} \left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| |x-c|^{n+1} \quad \text{where } R_n(x) \text{ is our remainder function} \quad (2)$$

To maximize the value of our function we first added absolute value bars in equation (1). Then we physically maximized the function by taking the maximum value of $f^{(n+1)}$ with the **max** function, as seen in equation (2). There are a few things to note here:

1. z is going to be a number between x and c .
2. n is simply the n th derivative of our *original* function. It is very important to note that n comes from the original polynomial and not the one used to find error.
3. c is the center of the taylor polynomial.
4. x is the value we are trying to approximate.
5. **max** $|f^{(n+1)}|$ is the maximum possible value of $f^{(n+1)}$ at z . Often times you won't know what this is so you'll have to guess what this is (we'll talk about this in a minute).

You'll also note that we are now evaluating the $n+1$ th derivative at z instead of $x-c$. This is because we want the maximum value of the derivative at whatever point we are approximating so we choose some number z to represent the function input that gives us the maximum value.

As for actually finding **max** $|f^{(n+1)}|$ there are a few tricks we can use. If our *original* function is a trig function, then we can immediately assume that 1 is our maximum value. This is because $0 \leq \sin, \cos, \tan, \leq 1$. In other scenarios when we are not dealing with trig functions we can use logic to deduce a good maximum value. For example, if $f^n(x) = n^n$ and the original Taylor polynomial is 3 terms long ($n=3$) then using $(n+1)^{(n+1)} = 4^4$ would be a good choice because we know that it *must* be larger than the n th derivative evaluated at that same point. Or, for instance, if our original function was e^x then we know that if $c \leq z \leq x$ that e^x (where x is the number we are approximating the value of) is a good assumption for the max value. The main thing to keep in mind is that the closer your max value is the better your error approximation will be.