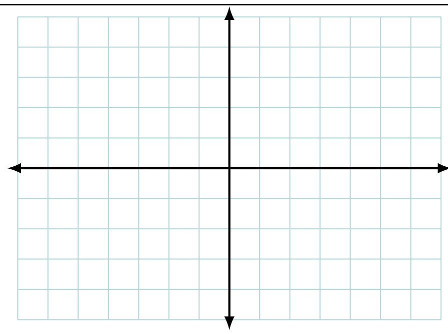
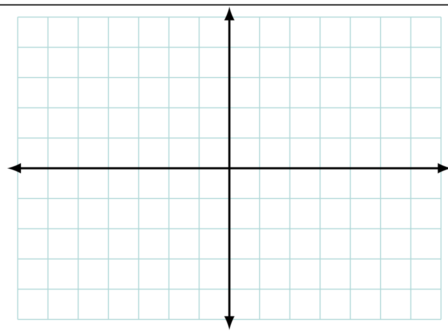
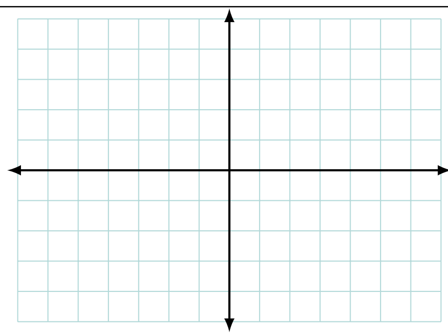
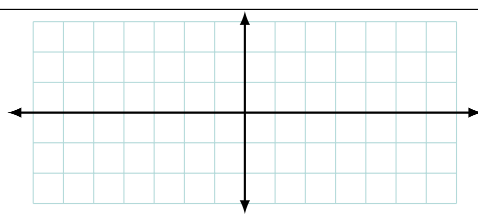
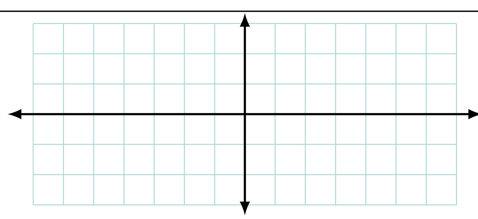
Calculus

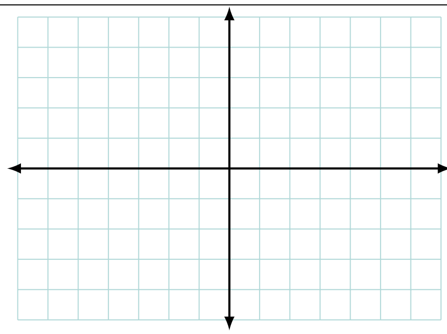
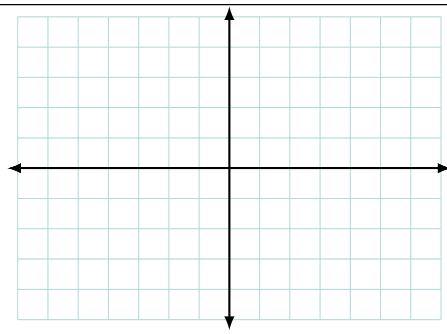
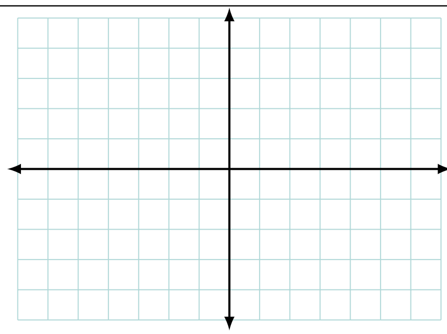
Google Drive Folder

2022	4.84	2027	
2023		2028	
2024		2029	
2025		2030	
2026		2031	

Contents

An Introduction to Limits

Function	Graph	$\lim\limits_{x \to c^-} f(x)$	$\lim\limits_{x \to c^+} f(x)$	$\lim\limits_{x \to c} f(x)$
$\lim\limits_{x \to 2} 2x+1$				
$\lim\limits_{x \to 0} x^2-2x+1$				
$\lim\limits_{x \to 2} \dfrac{x+3}{x+2}$				
$\lim\limits_{x \to \frac{3\pi}{2}} \sin x$				
$\lim\limits_{x \to \pi} -3\cos\theta$				

$\lim_{x\to 5} \dfrac{x^2-25}{x-5}$				
$\lim_{x\to 5} \lvert x+2\rvert - 1$				
$\lim_{x\to 0} \dfrac{1}{x}$				

In short, we can interpret the limit as the expected y-value of a function. It doesn't really matter if the function even attains a the value the limit takes on. The behavior of the graph *around* that point determines everything.

DEFINITION OF THE EXISTENCE OF A LIMIT

A funciton has a limit as x approaches c if and only if the right hand limit and the left hand limit at a point c are the same. That is

$$\text{if } \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L, \text{ then } \lim_{x \to c} = L.$$

There are three cases:

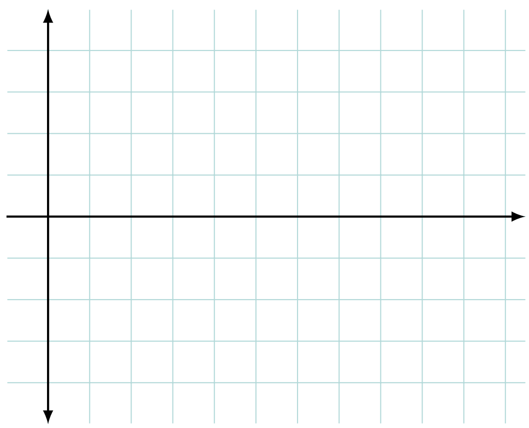
1. A continuous function on the interval $(-\infty, , \infty)$.

2. A non-continuous function where $f(c)$ does not exist.

3. A non-continuous function where $f(c)$ is defined at a different point.

Example: Graph the function.

$$f(x) = \begin{cases} 2 - 2x & [0,\, 1) \\ -(x-3)^2 + 2 & [2,\, 4) \\ -1 & 4 \\ x - 4 & (4,\, 5] \end{cases}$$



$f(0) =$	$\lim_{x \to 0^+} f(x) =$	$\lim_{x \to 0^-} f(x) =$	$\lim_{x \to 0} f(x) =$
$f(1) =$			
$f(2) =$			
$f(3) =$			
$f(4) =$			
$f(5) =$			

A limit is the value of the function we **approach** as we near a particular x from either the left or the right. A limit is **not** necessarily the value of the function at that point.

Strategies for Finding Limits:

1. Substitute in
2. Simplify the equation algebraically, then substitute in
3. Make the graph and determine the limit visually
4. Construct a table of values to conjecture a limit

Examples:

1. $\lim_{x \to 3} \dfrac{x^2 + 1}{2x}$

2. $\lim_{x \to 2} \lfloor x \rfloor$

3. $\lim_{x \to -4.1} \lfloor x \rfloor$

4. $\lim_{x \to 3} \dfrac{x^2 - 9}{x + 3}$

5. $\lim_{t \to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

6. $\lim_{x \to 0} \dfrac{\frac{1}{2+x} - \frac{1}{2}}{x}$

7. $\lim_{x \to 0} \dfrac{(2+x)^3 - 8}{x}$

Properties of Limits

PROPERTIES OF LIMITS
Let f and g be functions such that $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$ for real numbers L, M, c, and k. Then the following properties of limits exist: 1. $\lim\limits_{x \to c} \left(f(x) \pm g(x)\right) = L \pm M$ 2. $\lim\limits_{x \to c} \left(f(x) \cdot g(x)\right) = L \cdot M$ 3. $\lim\limits_{x \to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$ 4. $\lim\limits_{x \to c} \left(kf(x)\right) = kL$ 5. $\lim\limits_{x \to c} \left(f(x)\right)^k = L^k$

Memorize this limit: $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

Examples:

1. $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$

2. $\lim\limits_{x \to 0} \dfrac{x + \sin x}{x}$

Limits Involving Infinity

There are two kinds: one where the answer is infinity and one where you are finding the limit as x approaches infinity.

1. $\lim\limits_{x \to 2^-} \dfrac{1}{x-2}$

2. $\lim\limits_{x \to 2^+} \dfrac{1}{x-2}$

3. $\lim\limits_{x \to 2} \dfrac{1}{x-2}$

4. $\lim\limits_{x \to \infty} \dfrac{1}{x-2}$

For problems involving polynomial and rational functions, we can determine the result of the limit without sketching the graph. Always consider the **dominant term** when evaluating.

1. $\lim\limits_{x \to \infty} x^3 + 2x + 1$

2. $\lim\limits_{x \to \infty} -x^3 + x^2 + 1$

3. $\lim\limits_{x \to \infty} \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

SHORTCUT FOR EVALUATING LIMITS AT INFINITY

Let $\dfrac{f(x)}{g(x)}$ be a rational function.

1. If $\deg f(x) > \deg g(x)$, then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ or $-\infty$.
2. If $\deg f(x) < \deg g(x)$, then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
3. If $\deg f(x) = \deg g(x)$, then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ where a and b are the leading coefficients of $f(x)$ and $g(x)$ respectively.

Examples:

1. $\displaystyle\lim_{x\to\infty} \frac{3x^3 - x + 1}{x + 3}$

2. $\displaystyle\lim_{x\to\infty} \frac{1}{x^2}$

3. $\displaystyle\lim_{x\to\infty} \frac{3x^2 + 5}{2x^2 + x + 3}$

4. $\displaystyle\lim_{x\to\infty} \sqrt{\frac{x+1}{x}}$

LIMITS AND ASYMPTOTES

If $\displaystyle\lim_{x\to a^-} f(x) = \pm\infty$ or if $\displaystyle\lim_{x\to a^+} f(x) = \pm\infty$, then the line $x = a$ is a **vertical asymptote** of the graph $f(x)$.

If $\displaystyle\lim_{x\to\infty} f(x) = b$ or if $\displaystyle\lim_{x\to-\infty} f(x) = b$, then the line $y = b$ is a **horizontal asymptote** of the graph $f(x)$.

Continuity

TEST FOR CONTINUITY
A function $f(x)$ is **continuous** on a given interval if and only if all three of the following are true: 1. $f(c)$ exists 2. $\lim_{x \to c} f(x)$ exists 3. $\lim_{x \to c} f(x) = f(c)$. If the above is not true, then we say c is a **point of discontinuity**.

Examples:

1. $f(x) = x^2 + 2x - 8$
2. $f(x) = \dfrac{x^2 - 9}{x - 3}$
3. $f(x) = \dfrac{1}{x + 5}$
4. $f(x) = \lfloor x \rfloor$

There are 4 kinds of discontinuity: *Match 1-3 with the examples above*

1. Jump Discontinuity
2. Infinite Discontinuity
3. Removable Discontinuity
4. Oscillating Discontinuity*

*Oscillating Discontinuities are rare in the scope of this course, but worth noting nonetheless. To see an example, check out the graph of the **Topologist's Sine Curve**, $y = \sin\left(\frac{1}{x}\right)$.

PROPERTIES OF CONTINUOUS FUNCTIONS
If f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$. 1. $f \pm g$ 2. $f \cdot g$ 3. $k \cdot f$ for any scalar k 4. $\frac{f}{g}$ when $g(c) \neq 0$ 5. $f \circ g$ or $f(g(x))$ if f is continuous at $g(c)$.

1. Explain how the function $y = \left| \dfrac{x \sin x}{x^2 + 1} \right|$ is always continuous.

2. Find a value of a so that the function $f(x) = \begin{cases} 2x + 3 & x \leq 2 \\ ax + 1 & x > 2 \end{cases}$ is continuous.

Intermediate Value Theorem

THE INTERMEDIATE VALUE THEOREM
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if $f(a) \leq y_0 \leq f(b)$, then $y_0 = f(c)$ for some $c \in [a, b]$.

Examples:

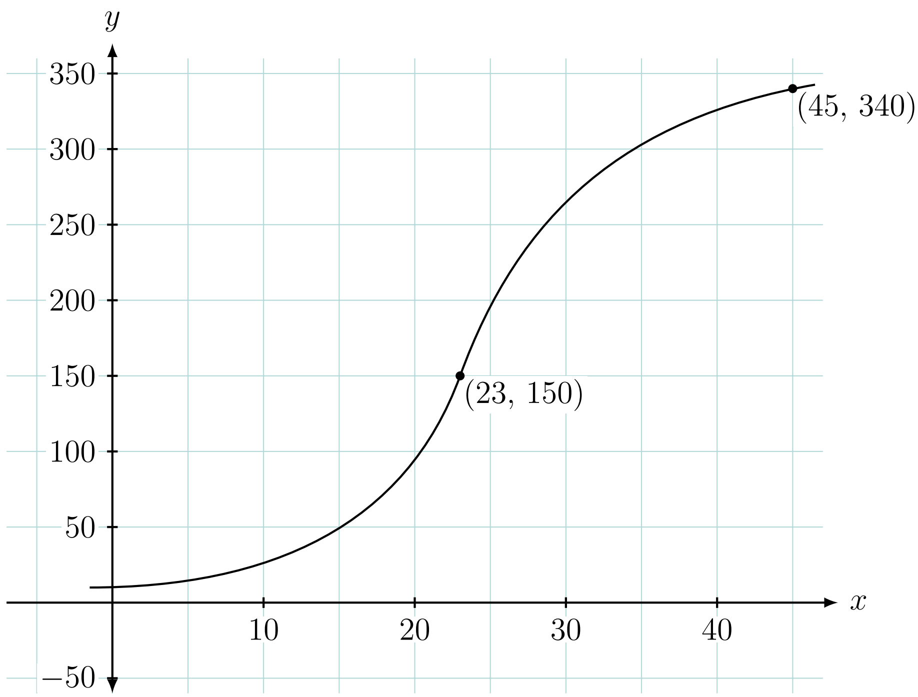
1. Apply the IVT to verify the existence of a point where $f(x) = 0$ on the interval $[0, 5]$ given $f(x) = x^3 - 2x - 2$.

2. Is there a real number that is exactly one more than its cube? Justify your answer.

Rates of Change & Tangent Lines

The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. This is just a fancy way of describing the **slope**.

Example: Find the average rate of change in the population of fruit flies from day 23 to day 45.



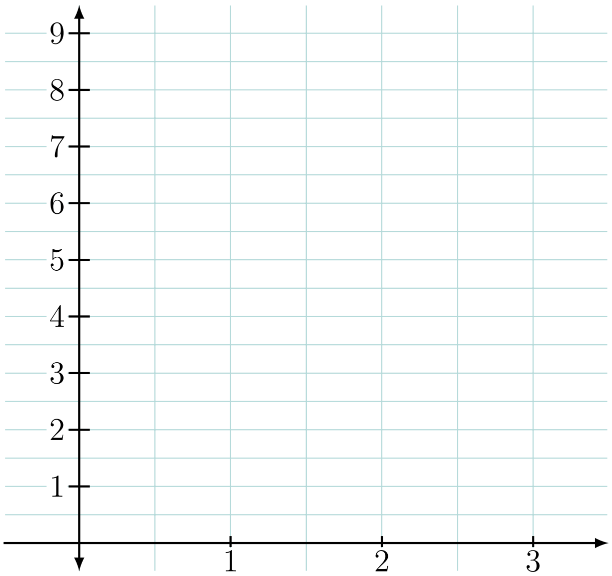
A line through two points on a curve is called a **secant** to the curve. The slope of the secant line is the average rate of change.

Did the number of flies increase by 8.6 each day?

How fast was the fly population growing on day 23?

The population growth on one particular day poses a problem: it would be the slope of the line at *only one point*. We need *two* points to use the slope formula as we know it. How do we circumvent this issue?

Example: Given $y = x^2$, what is the average rate of change on [0, 3]?



What is the instantaneous rate of change at $x = 1$?

DEFINITION OF TANGENT LINE WITH SLOPE m
If f is defined on an open interval containing c, and if the limit $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$ exists, then the line passing through $(c,\ f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c,\ f(c))$.

Examples:

1. Write the equation of the tangent line to the curve $f(x) = 3x^2 + 1$ at $x = -1$.

2. Write a general formula for the slope of the tangent line to $f(x) = 2x^2 - x$ for any point $x = a$.

3. Write an equation for the line **normal** to the curve $f(x) = 4 - x^2$ at $x = 1$.

4. Find the equations of all lines tangent to $y = 9 - x^2$ that pass through $(1,\ 12)$.

Definition of the Derivative

DEFINITION OF THE DERIVATIVE OF A FUNCTION
The **derivative** of f at x is given by $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$ provided the limit exists. For all x for which this limit exists, f' is a function of x.

The derivative of a function is the formula for the slope of the tangent line.

Examples: Compute the derivatives of the following functions. *Note that these require some algebraic tricks to compute.*

1. $f(x) = \sqrt{x - 3}$

2. $f(x) = \dfrac{1}{x - 5}$

Rules for Differentiation A

Let $y = f(x)$ be a function of x. If the limit $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ exists and is finite, we call this limit the derivative of f at x and say that f is *differentiable* at x.

Rules for Differentiation:

1. **Constant Rule:**
   (a) $y = 5$
2. **Power Rule:**
   (a) $y = x^2$
   (b) $f(x) = x^3$
   (c) $\dfrac{d}{dx}\left(x^4\right) =$
   (d) $y = x^{13}$
   (e) $\dfrac{d}{dx}\left(x^{79}\right) =$
3. **Constant Multiple Rule:**
   (a) $\dfrac{d}{dx}\left(7x^5\right) =$
   (b) $y = -3x^4$
   (c) $\dfrac{d}{dx}\left(\dfrac{2}{3}x^3\right) =$
4. **Sum and Difference Rule:**
   (a) $\dfrac{d}{dx}\left(x^3 + 7x^2 - 5x + 4\right) =$
   (b) $y = \dfrac{1}{4}x^4 - \dfrac{x^3}{3} + 4$
5. **Higher Order Derivatives:**
   (a) $y = x^3 - 5x^2 + 2$

   (b) $y = \dfrac{x+1}{x}$

For a derivative to exist, the left-hand and right-hand derivatives must be the same in order for there to be derivative at that point *and* for derivatives, it must be continuous at that point.

1. Determine whether the curve has a tangent at the indicated point. If it does, give the slope at that point. If not, explain why.

   (a) $f(x) = \begin{cases} -x & x < 0 \\ x^2 - x & x \geq 0 \end{cases}$ at $x = 0$

   (b) $f(x) = \begin{cases} \sin x & 0 \leq x < 3\pi/4 \\ \cos x & 3\pi/4 \leq x \leq 2\pi \end{cases}$ at $x = 3\pi/4$

2. True or False: The graph of $f(x) = |x|$ has a tangent line at x=0. Justify your answer.

Rules for Differentiation B

More Rules:

1. **Product Rule:** $\dfrac{d}{dx}(uv) = v \cdot \dfrac{du}{dx} + u \cdot \dfrac{dv}{dx}$
   (a) $y = (3x-1)(2x+5)$

   (b) $f(x) = (1+x^2)(-3x-4)$

2. **Quotient Rule:** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$
   (a) $y = \dfrac{x^2+1}{x^2-1}$

   (b) $\dfrac{d}{dx}\left(\dfrac{x^2}{1-x^3}\right) =$

3. **Extended Power Rule:** $\dfrac{d}{dx}\left(u^n\right) = nu^{n-1} \cdot \dfrac{du}{dx}$
   (a) $y = (x^2-3x+1)^5$

   (b) $f(x) = \left(x^2+1\right)^{-}2$

   (c) $\dfrac{d}{dx}\left[(x^2+1)^3(x-1)^2\right] =$

Examples: Make sure to choose the *correct* strategy!

1. $y = \dfrac{1}{(x^2-1)^5}$

2. $\dfrac{d}{dx}\left(\dfrac{2x-1}{x+7}\right)^3 =$

3. $g(x) = \dfrac{(x-1)(x^2-2x)}{x^4}$

4. Let u and v be functions that are differentiable at $x = 2$ and that $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$. Find the values of the following derivatives at $x = 2$.

   (a) $\dfrac{d}{dx}(uv) =$

   (b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) =$

   (c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) =$

Distance, Velocity, & Acceleration

Distance: The function $s(t)$ gives the position s on a line as a function of time t.
Displacement: The displacement of an object over the time interval from t to $t + \Delta t$ is $\Delta s = f(t + \Delta t) - f(t)$.
Velocity: The derivative of the position equation is velocity because it describes how fast the position is changing. $v(t) = s'(t)$

1. Average Velocity: the average velocity over a period of time. $\dfrac{\Delta s}{\Delta t}$, where Δs represents **displacement** and Δt represents the **time traveled**.
2. Instantaneous Velocity: how fast **and in what direction** an object is moving at a given instant.
   (a) Velocity is **positive** when the object is moving forward or up and the distance is increasing.
   (b) Velocity is **negative** when the object is moving back or down and the distance is decreasing.
3. Speed: how fast an object is moving at a given instant, $|v(t)| = \left|\dfrac{ds}{dt}\right|$. The only difference between velocity and speed is that velocity includes **direction**.

Acceleration: The derivative of the velocity equation is acceleration because it describes **how fast the velocity is changing**. $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

Examples:
For each of the following, s represents the position of a moving body with s in feet and t in seconds. Find the body's velocity and acceleration.

1. $s(t) = 16t^2 + 3$

2. $s(t) = 832t - 16t^2$

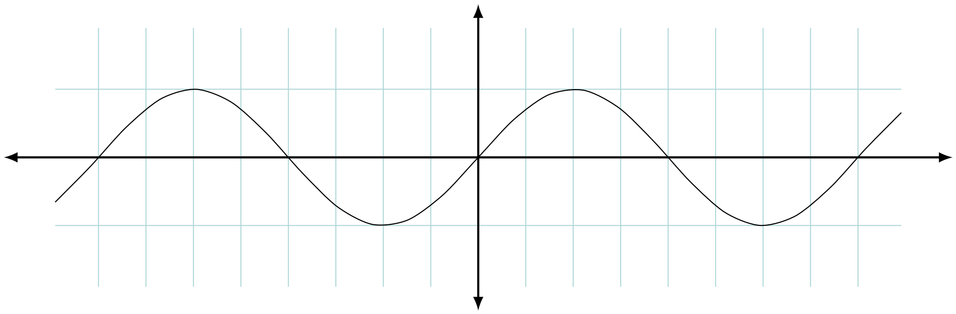
3. The position of a body at time t is $s = t^3 - 4t^2 - 3t$. Find the acceleration each time the velocity is zero.

4. A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where s is measured in meters and t is measured in seconds.
   (a) Find the displacement in the first 2 seconds.
   (b) Find the average velocity in the first 4 seconds.
   (c) Find the instantaneious velocity when $t = 4$.
   (d) Find the acceleration when $t = 4$. Describe the motion of the particle. At what values of t does the particle change direction?
   (e) Where is the particle when s is a minimum?

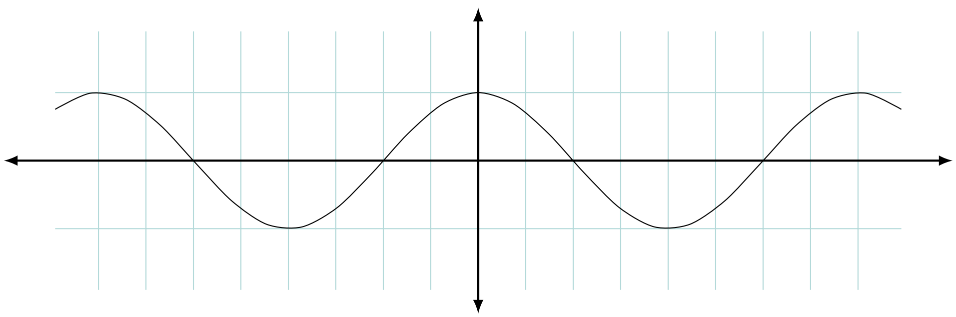
5. A ball is thrown vertically upward at a speed of 30 m/s. Its height in meters t seconds later is given by $h(t) = 30t - 5t^2$.
   (a) Find the maximum height of the ball.
   (b) Find the acceleration of the ball.

Derivatives of Sine and Cosine

Below is the graph of $y = \sin x$. Using the fact that y' is the slope of the tangent line, make the graph of y' for $y = \sin x$.



Similarly, attempt with the graph of $y = \cos x$.



DERIVATIVES OF SINE AND COSINE	
$\dfrac{d}{dx}(\sin x) =$	$\dfrac{d}{dx}(\cos x) =$

Examples: Find the derivative of each of the following.

1. $y = 2\sin x$
2. $f(x) = \dfrac{-\sin x}{10}$
3. $u = x + \cos x$

The Chain Rule

In your head, consider the derivative of the function $y = (3x + 4)^2$. To do it, we have to "take the derivative of the inside". More formally, this is another differentiation rule that we call the **Chain Rule**.

THE CHAIN RULE
If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ or, equivalently $$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$$

Examples:

1. Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	-3
3	3	-4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x

(a) $2f(x)$, $x = 2$

(b) $f(x) \cdot g(x)$, $x = 3$

(c) $\dfrac{f(x)}{g(x)}$, $x = 2$

(d) $f(g(x))$, $x = 2$

(e) $\dfrac{1}{g^2(x)}$, $x = 3$

(f) $\sqrt{f^2(x) + g^2(x)}$, $x = 2$

2. Find $\dfrac{dy}{dx}$ for the two similar functions.

   (a) $\dfrac{d}{dx}\left(\sin^2 x\right) =$

   (b) $\dfrac{d}{dx}\left(\sin x^2\right) =$

3. $f(u) = \left(\dfrac{u-1}{u+1}\right)^2$, $g(x) = \dfrac{1}{x^2} - 1$ and find $(f \circ g)'(-1)$.

4. Let $y = x^2 + 7x - 5$. Evaluate $\dfrac{dy}{dt}$ when $x = 1$ and $\dfrac{dx}{dt} = \dfrac{1}{3}$.

Derivatives of Trigonometric Functions

DERIVATIVE RULES FOR ALL TRIG FUNCTIONS
$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ $\quad$ $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$
$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$ $\quad$ $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$
$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$ $\quad$ $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$

Try out the following derivatives. Remember, **all other derivative rules still apply**!!

1. $\dfrac{d}{dx}\left(\cos^2(3x)\right) =$

2. $\dfrac{d}{dx}\left(\sec^2(5x)\right) =$

3. $\dfrac{d}{dx}\left(\sqrt{\tan(3x)}\right) =$

4. $\dfrac{d}{dx}\left(x^2 \sin x\right) =$

5. $\dfrac{d^2}{dx^2}\left(\dfrac{1}{\cos x}\right) =$

WARNING: You are *going* to need to memorize multiple trig identities for later units. You will start practicing that for tonight's homework as well.

Implicit Differentiation

What we have been doing so far is explicit differentiation, as y has always been an explicit function of x. Today, we will continue our journey of taking derivatives but now with equations that are not functions!

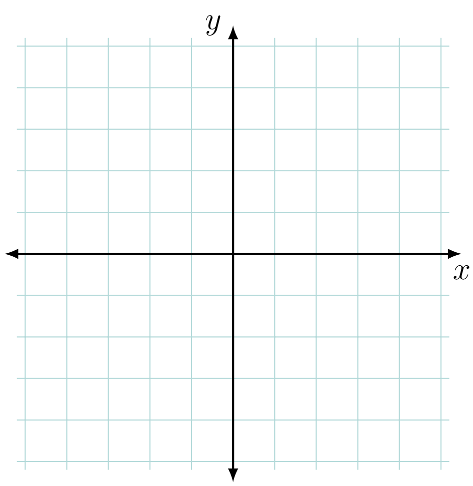
STEPS FOR IMPLICIT DIFFERENTIATION
1. Differentiate both sides of the equation with respect to x, doing x as you normally would. The derivative of y becomes dy/dx or y'. 2. Collect terms with dy/dx on one side of the equation. 3. Factor out the dy/dx. 4. Solve for dy/dx.

Examples: Find $\dfrac{dy}{dx}$.

1. $y^2 = x$

2. $x^2 + y^2 = 25$

3. The Folium of Descartes is $x^3 + y^3 = 9xy$. Find and graph the tangent lines at the points (4, 2) and (2, 4).



4. The line that is normal to the curve $y = x^2 + 2x - 3$ at $(1, 0)$ intersects the curve at what other point?

Example: Find $\dfrac{d^2y}{dx^2}$ for $2x^3 - 3y^2 = 8$.

Second derivatives of Implicit Functions is a skill that is always exercised on the AP Exam. The level of simplification might vary, but the overall procedure and *form* of the final answer is important.

Two more examples of second derivatives (time-permitting):

1. $5y^2 + 2 = 5x^2$

2. $y + \cos y = x + 1$

Derivatives of Transcendental Functions

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

$$\frac{d}{dx}(\arcsin u) = \frac{du}{\sqrt{1-u^2}} \qquad \frac{d}{dx}(\arccos u) = \frac{-du}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}(\arctan u) = \frac{du}{1+u^2} \qquad \frac{d}{dx}(\text{arccot } u) = \frac{-du}{1+u^2}$$

$$\frac{d}{dx}(\text{arcsec } u) = \frac{du}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}(\text{arccsc } u) = \frac{-du}{|u|\sqrt{u^2-1}}$$

Examples: Find $\dfrac{dy}{dx}$.

1. $y = \sin^{-1} x^2$

2. $y = \cos^{-1}\left(\dfrac{1}{x}\right)$

3. $y = \cot^{-1} \sqrt{x}$

4. $y = x\sqrt{1-x^2} - \cos^{-1} x$

5. $y = x \arcsin x + \sqrt{1 - x^2}$

6. A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \arctan \sqrt{t}$. What is the velocity of the particle when $t = 16$?

DERIVATIVES OF THE EXPONENTIAL FUNCTION
Let u be a differentiable function of x. $\frac{d}{dx}\left(e^u\right) = u' \cdot e^u$ $\qquad$ $\frac{d}{dx}\left(a^u\right) = u' \cdot a^u \cdot \ln a$

Examples:

1. $y = e^{\sin x}$

2. $y = e^{(x+1)}$

3. $y = \cos\left(e^x\right)$

4. $y = (1 + 2x)3^{-2x}$

5. $y = e^{3^x}$

6. $y = e^{x^2} e^{-x}$

DERIVATIVES OF THE LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

$$\frac{d}{dx}(\ln u) = \frac{u'}{u} \qquad \frac{d}{dx}(\log_a u) = \frac{u'}{u \cdot \ln a}$$

Examples:

1. $y = \ln(5x)$
2. $y = \ln(\cos x)$
3. $y = \ln(\ln x)$
4. $y = (\ln x)^2$
5. $y = \log_3(1 + x\ln 3)$
6. $y = \dfrac{1}{\log_2 x}$

Use properties of logarithms before differentiating to **make your life easier!**

1. $\dfrac{d}{dx}\left[\ln\left(x\sqrt{x^2+1}\right)\right] =$

2. $y = \ln\left(\tan\left(\dfrac{x-1}{x+1}\right)\right)$

Logarithmic Differentiation

For the two previous problems, we could use properties of logs to make life much easier. Wouldn't it be great to apply those same rules to other hard problems such as $y = \sqrt[3]{\dfrac{x+1}{x-1}}$? We can, just **take the** ln **of both sides!**

1. $y = x^x$, $x > 0$

2. $y = x^{\tan x}$

3. $y^5 = \sqrt{\dfrac{(x+1)^5}{(x+2)^{10}}}$

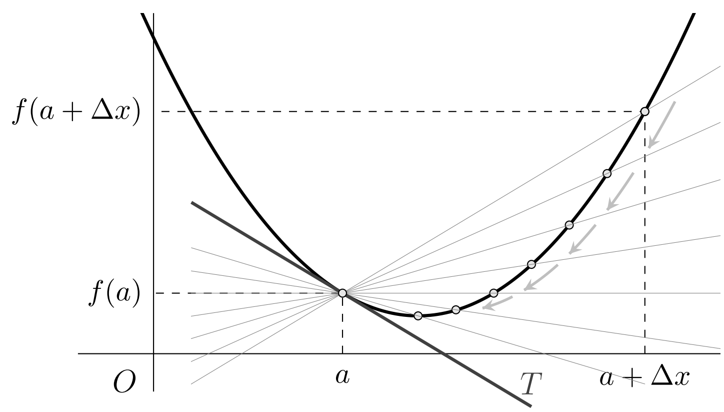
4. $y^2 = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

Differentiability & Graphing Derivatives

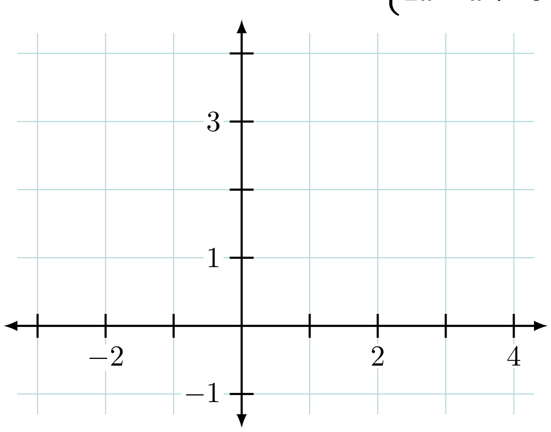
Recall...

DEFINITION OF THE DERIVATIVE OF A FUNCTION
The **derivative** of f at x is given by $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ provided the limit exists. For all x for which this limit exists, f' is a function of x. **Alternate Form:** The derivative of a function f at the point $x = a$ is the limit $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ provided the limit exists.

Graphical Representation of the above definitions:



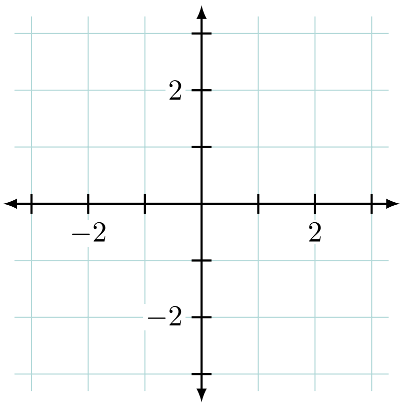
Consider the function $y = \begin{cases} x^2 & x \le 0 \\ 2x & x > 0 \end{cases}$. What is the derivative of the function at $x = 0$?



Four Instances where a Derivative will Fail to Exist:

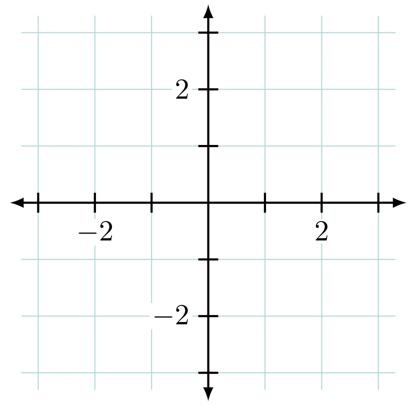
1. At a **corner**, where the one-sided derivatives differ.

   Example: $f(x) = |x|$



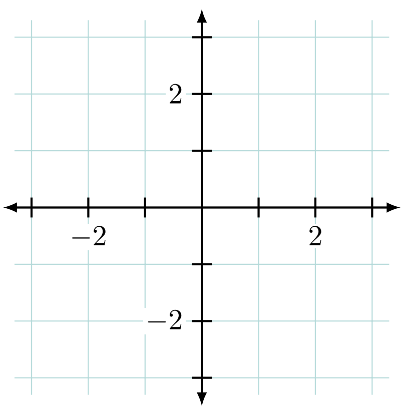
2. At a **cusp**, where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other.

   Example: $f(x) = x^{2/3}$



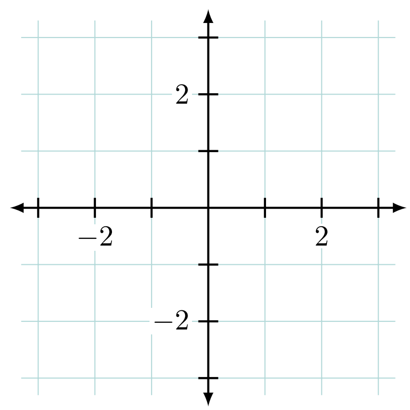
3. At a **vertical tangent**, where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides.

   Example: $f(x) = \sqrt[3]{x}$



4. At a **point of discontinuity**.

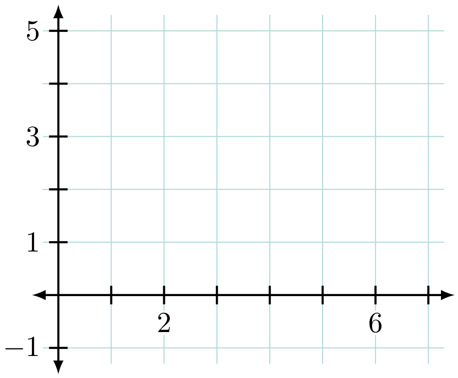
   Example: $f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$

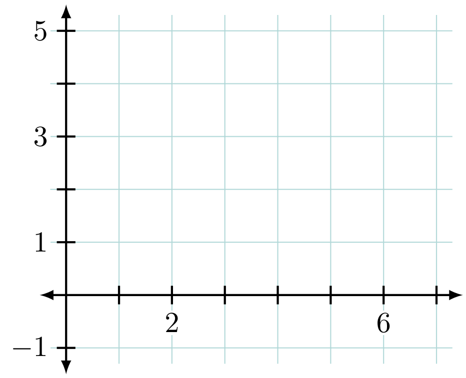


- **A function must be continuous at a to be differentiable at a.**
- **If a function is continuous at a, that does not mean that it is differentiable.**
- **If a function is differentiable at a, then the function is continuous at a.**

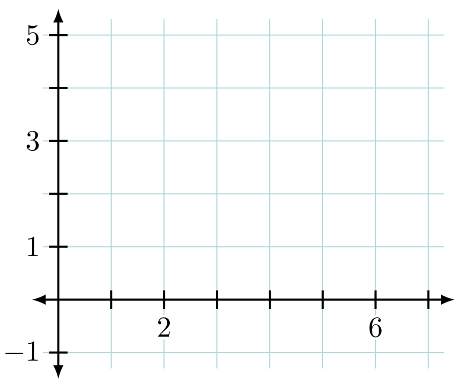
The derivative at a point is the slope of the line tangent to the curve at that point. We use this fact to graph derivatives of functions.

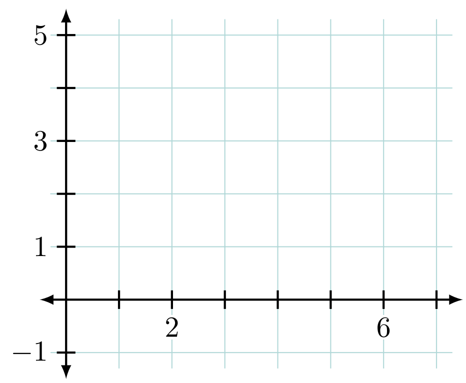
Examples:
Graph $f(x) = (x-2)^2 + 1$, then graph the derivative of the function.



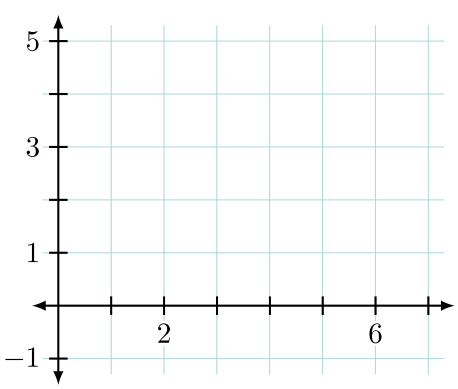


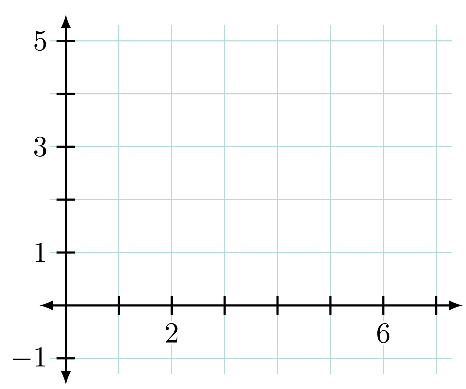
Graph a positive cubic function passing through the point (0, 1) and having a maximum at (2, 4) and a minimum at (6, 1). Graph the derivative of the function.





Now let's try it backwards!!! Graph $f'(x) = \begin{cases} 2 & x > 1 \\ -1 & x < 1 \end{cases}$ below. Given that $f(0) = 0$ and $f(x)$ is continuous, graph $f(x)$.





Extreme Values of Functions

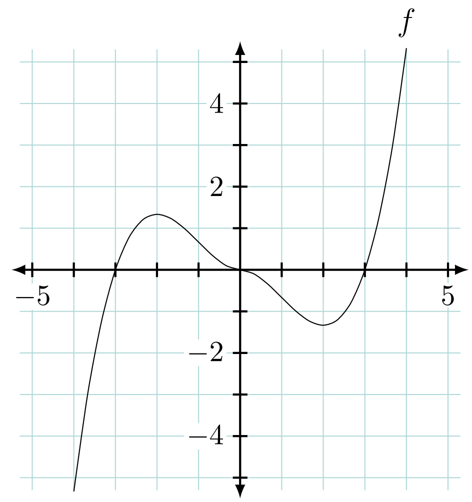
Extreme values help us answer questions like, *What is the most effective dose of medicine?*, *What is the least expensive way to pipe oil from an offshore refinery down the coast?*

Absolute (or global) maximum and minimums means there is no greater/lesser value for $f(x)$ anywhere.

Local maximum and minimum means there is no greater/lesser value for $f(x)$ nearby.

Domain Matters! Given that $f(x) = \frac{1}{3}x^3 - x^2$, find the absolute extrema and the local extrema on the following intervals:

	local max	abs max	local min	abs min
1. $(-\infty, \infty)$				
2. $[-3, 6]$				
3. $(0, 6]$				
4. $[-3, -1]$				



THE EXTREME VALUE THEOREM
If $f(x)$ is continuous on the closed interval $[a, b]$, then $f(x)$ has both a maximum value and a minimum value on the interval.

Naturally the question becomes: **How can we find the extreme values?** ______________!!!!

DEFINITION OF LOCAL EXTREME VALUES
If a function $f(x)$ has a local maximum or minimum value at an interior point c of its domain, and if $f'(x)$ exists at c, then $f'(c) = 0$.

But sometimes a maximum or a minimum may occur when $f'(c) \neq 0$. What might cause this to happen?

____

Instead, lets introduce a definition that encompasses ALL these points:

DEFINITION OF A CRITICAL POINT
Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical point** (or **critical number**) of f. As a consequence, **relative extrema only occur at critical points!**

Example:
Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval [1, 2].

More Examples:

1. Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

2. Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4-x^2}}$.

3. Find the extreme values of $f(x) = \begin{cases} 5-2x^2 & x \leq 1 \\ x+2 & x > 1 \end{cases}$.

4. Find the extreme values of $f(x) = \ln\left|\dfrac{x}{1+x^2}\right|$.

5. Let $f(x) = |x^3 - 9x|$
   (a) Does $f'(0)$ exist?
   (b) Does $f'(3)$ exist?
   (c) Does $f'(-3)$ exist?
   (d) Determine all extrema of $f(x)$.

THE FIRST DERIVATIVE TEST

The following applies to a continuous function $f(x)$.

1. If $f'(x)$ changes sign from positive to negative at a critical point c, then $f(x)$ has a local maximum value at c.
2. If $f'(x)$ changes sign from negative to positive at a critical point c, then $f(x)$ has a local minimum value at c.
3. If $f'(x)$ does not change sign at a critical point c, then $f(x)$ has no local extreme value at c.

Examples: Identify any local extrema using the first derivative test. Then identify any absolute extrema.

1. $f(x) = x^2 - 12x - 5$
2. $g(x) = \left(x^2 - 3\right) e^x$

There are times when a curve can hold water or spill water. We call this ability **concavity**.

THE CONCAVITY TEST

The graph of a twice differentiable function $f(x)$ is

1. concave up if $f''(x) > 0$.
2. concave down if $f''(x) < 0$.

The point where the graph has a tangent line and when the concavity changes is a **point of inflection**. When $f''(x) = 0$ or $f''(x)$ DNE are both candidates for POI's.

THE SECOND DERIVATIVE TEST
1. If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$. 2. If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$. The test fails if $f''(c) = 0$ or if $f''(c)$ does not exist.

Examples: Find the local extreme values of the following.

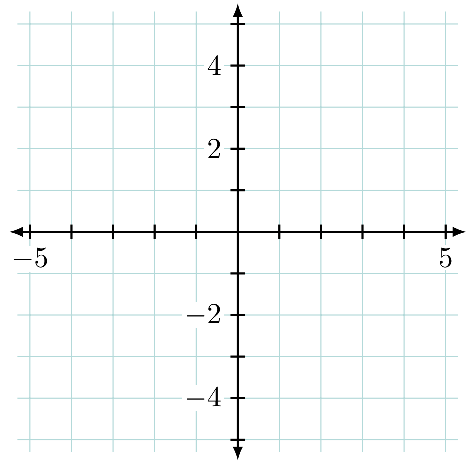
1. $y = x^4$
2. $f(x) = \sqrt[3]{x}$

Curve Sketching

Summary for Graphing Functions

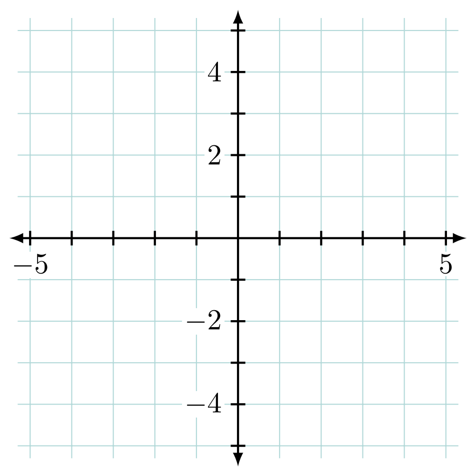
1. Use pre-calculus and algebra
   (a) know the general shape
   (b) find the y-intercept ($x = 0$)
   (c) find the roots/zeros ($y = 0$)
   (d) determine the vertical asymptotes (denom=0) and limits
   (e) determine the horizontal or slant asymptotes
   (f) use limits $x \to \pm\infty$ to determine end behavior
2. Use the first derivative
   (a) determine the critical points
   (b) make a first derivative sign chart to see where function is increasing or decreasing
   (c) apply first derivative test to determine extrema
3. Use the second derivative
   (a) determine possible points of inflection
   (b) apply the second derivative test to determine extrema
   (c) make a sign chart to determine concavity

Example 1: $y = x^3 - 3x^2 + 4$



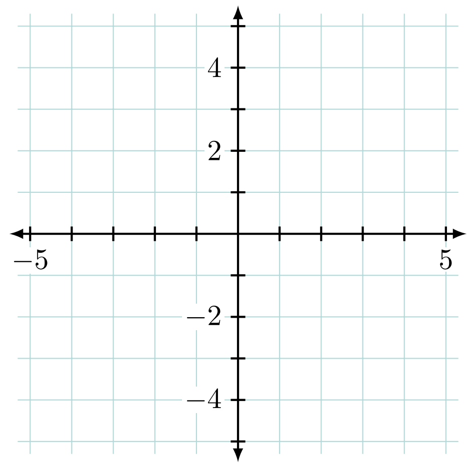
1. The first derivative is
2. The second derivative is
3. The function is increasing on
4. The function is decreasing on
5. The function is concave up on
6. The function is concave down on
7. The absolute maximum is
8. The absolute minimum is
9. The local maximum(s) is(are)
10. The local minimums(s) is(are)
11. The point(s) of inflection is(are)

Example 2: $y = \dfrac{5}{2x - 4}$



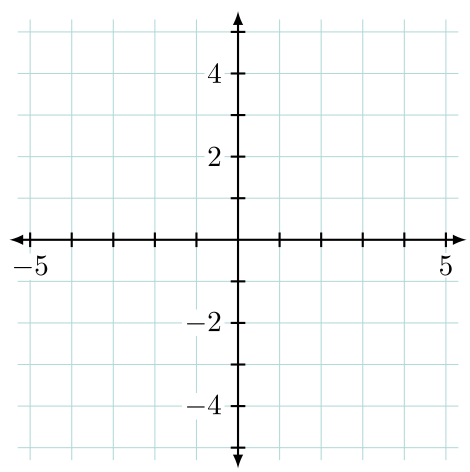
1. The first derivative is
2. The second derivative is
3. The function is increasing on
4. The function is decreasing on
5. The function is concave up on
6. The function is concave down on
7. The absolute maximum is
8. The absolute minimum is
9. The local maximum(s) is(are)
10. The local minimums(s) is(are)
11. The point(s) of inflection is(are)

Example 3: $y = \dfrac{x^2}{x^2 - 1}$



1. The first derivative is
2. The second derivative is
3. The function is increasing on
4. The function is decreasing on
5. The function is concave up on
6. The function is concave down on
7. The absolute maximum is
8. The absolute minimum is
9. The local maximum(s) is(are)
10. The local minimums(s) is(are)
11. The point(s) of inflection is(are)

Example 4: $y = \dfrac{x^2 + x - 2}{x - 2}$



1. The first derivative is
2. The second derivative is
3. The function is increasing on
4. The function is decreasing on
5. The function is concave up on
6. The function is concave down on
7. The absolute maximum is
8. The absolute minimum is
9. The local maximum(s) is(are)
10. The local minimums(s) is(are)
11. The point(s) of inflection is(are)

Mean Value Theorem *(for derivatives)*

THE MEAN VALUE THEOREM
If f is continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b), then there exists a number c in (a, b) such that $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Think about it graphically:

Another example: if a car accelerating from 0 takes 8 seconds to go 352 ft, its average velocity for the 8 second interval is 44 ft/sec (30 mph). At some point during the acceleration, the speedometer *must* read exactly 30 mph!

Examples: For each of the following functions on the indicated intervals, determine whether or not the MVT applies. If it does, find the value(s) where the mean value is attained.

1. $f(x) = x^3 + 1$ on $[1,\, 2]$

2. $f(x) = \sqrt{x^2} + 1$ on $[-1,\, 1]$

3. $f(x) = \dfrac{3}{x+2}$ on $[-3,\, 1]$

4. $f(x) = \begin{cases} x^3 + 3 & x < 1 \\ x^2 + 1 & x \geq 1 \end{cases}$ on $[-2,\, 2]$

Modeling & Optimization

1. An open-top box is to be made by cutting congruent squares of side length x from the corners of a 20-by-25-inch sheet of tin and bending up the sides. Find how large the cut-out squares should be so that the box can hold as much as possible. Find the resulting maximum volume. [1, 7]

2. Find the largest possible value of $2x + y$ if x and y are the lengths of the sides of a right triangle whose hypotenuse is $\sqrt{5}$ units long. [2]

3. Find two positive numbers whose sum is 36 and whose product is as large as possible. [3, 6]

4. The city recreation department plans to build a rectangular playground having an area of 3600 square meters and surround it by a fence. Find the dimensions of the rectangle that uses the least amount of fencing. [4, 8, 11, 16]

5. Find the radius and height of a right circular cylinder that minimizes total surface area for a volume of 36π cubic inches. [5, 9]

6. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have. [10, 19]

7. A carpenter has been asked to build an open box with a square base. The sides of the box will cost \$3 per square meter, and the base will cost \$4 per square meter. Find the dimensions of the box of greatest volume that can be constructed for \$48. [12,13,14,17]

8. Find the dimensions of the right circular cylinder of greatest volume inscribed in a right circular cone of radius 10 inches and a height of 24 inches. [15, 18]

Related Rates

1. A snowball is melting at a rate of 2 cubic feet per hour. If it remains spherical, at what rate is the radius changing when the radius of the snowball is 20 inches? At what rate is the surface area changing when the radius is 20 inches? [1, 2, 3]

2. A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second. The height of the funnel is 20 centimeters and the radius of the top is 4 centimeters. How fast is the fluid level dropping when the level stands 5 centimeters above the vertex? [4, 5]

3. Sand is poured on a conical pile at a rate of 20 cubic meters per minute. The height of the pile is always equal to the radius of the base of the pile. When the pile is 3 meters high, how fast is the height of the pile increasing? [6]

You can now complete [14, 16, 18, 20, 22, 24, and 25]

4. A ladder leans against a vertical wall with the bottom of the ladder 8 feet from the wall on a horizontal floor. At that time, the bottom end of the ladder is being pulled away at the rate of 3 ft/sec and the top of the ladder slips down the wall at the rate of 4 ft/sec. How long is the ladder? [7, 8, 9, 10, 15, 17, 19, 21, 23]

5. An object moves on a parabola $3y = x^2$. When $x = 3$, the x-coordinate of the object is increasing at the rate of 1 foot per minute. How fast is the y-coordinate increasing at that moment? [11, 12]

6. A balloon rises vertically at the rate of 5 ft/sec. A person on the ground 200 feet away from the spot below the rising balloon watches the balloon ascend. At what rate is the distance between the balloon and the observer changing when the balloon is 50 feet above the ground? **Don't confuse this problem with the following one!**

7. A balloon rises vertically at the rate of 10 feet per second. A person watches the balloon ascend from a point on the ground that is 100 feet away from the spot below the rising balloon. At what rate (radian/second) is the observer's head rotating upward to follow the balloon when the balloon is 50 feet above the level of the observer's head? [13]

8. A man 6 feet tall walks at a rate of 5 ft/sec toward a street light that is 16 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light? [26, 27]

L'Hopital's Rule

Remember when we computed $\lim_{x\to 0}\dfrac{\sin x}{x}$? We got $\frac{0}{0}$, which is called an **indeterminate form**. While we could use Squeeze Theorem to show that this limit becomes 1, it would be easier to show using **L'Hopital's Rule**.

L'HOPITAL'S RULE
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(x) \neq 0$. Then $$\lim_{x\to a}\frac{f(a)}{g(a)} = \frac{f'(a)}{g'(a)}$$

DO NOT USE QUOTIENT RULE!

Examples:

1. $\lim_{x\to 0}\dfrac{3x - \sin x}{x}$

2. $\lim_{x\to 0}\dfrac{\sqrt{x+1} - 1}{x}$

3. $\lim_{x\to 0}\dfrac{\sqrt{x+1} - 1 - x/2}{x - 2}$

4. $\lim_{x\to \pi/2}\dfrac{1 + \sec x}{\tan x}$

Differentials

DEFINITION OF A DIFFERENTIAL
Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is $dy = f'(x)\,dx$.

Examples: Find the differential dy and evaluate dy for the given values of x and dx.

1. $y = x^5 + 37x$, $x = 1$, $dx = 0.01$

2. $y = \sin 3x$, $x = \pi$, $dx = -0.02$

3. $x + y = xy$, $x = 2$, $dx = 0.05$

We can use differentials to estimate the change in the value of a function! Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of $f(x)$ when x changes from a to $a + dx$ is $df = f'(a)\,dx$.

4. The radius of a circle is to be increased from $r = 10$ to $r = 10.1$. Use differentials to estimate the change in area. Calculate the estimated percent change.

Linearization & Newton's Method

This section has been replaced with a small take-home assignment. Expect to see this material on a quiz of test. It is not dense enough to merit the use of in-class instruction. Come find me outside of class if you have questions.

Link to File:

Basic Integration Rules

Given dy/dx, find y.

1. $\dfrac{dy}{dx} = 6x$

2. $\dfrac{dy}{dx} = 4x^3 + 2$

Are you *sure*?

DEFINITION OF THE INDEFINITE INTEGRAL
The indefinite integral of f with respect to x is the set of all antiderivatives of a function $f(x)$. $$\int f(x)\,dx = F(x) + C$$

We use the following general formula for polynomials:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq 1$$

Examples:

1. $\displaystyle\int 6x\,dx$

2. $\displaystyle\int (4x^3 + 2)\,dx$

3. $\displaystyle\int -dx$

4. $\displaystyle\int \frac{x^5 + 3x - 2}{x^3}\,dx$

5. $\int \left(7\sqrt[3]{x^4} + x\right)\, dx$

How would you approach integrating $\int 4x^3\left(x^4 - 1\right)\, dx$?

You would distribute and then do each term individually. However, there are going to be integrals where that method is not going to be preferred. Therefore, I will softly introduce a new method called **Integration by Substitution**.

Examples: Evaluate the following integrals.

1. $\int 15x^2(5x^3 - 2)^4\, dx$

2. $\int x^2\sqrt{5 + 2x^3}\, dx$

Solve the Differential Equation

3. $\dfrac{dy}{dx} = 3x^2\sqrt{1 + x^3}$, given that $y = 1$ when $x = 0$.

Integration of Trigonometric Functions

Before we begin, let's check out the list of basic trig integrals that you should have memorized (these are just derivatives in reverse, so nothing new here).

INTEGRALS OF BASIC TRIG FUNCTIONS	
$\int \sin x\,dx = -\cos x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \cos x\,dx = \sin x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Also... you are going to need to know a lot of trig identities.

Examples: Where possible, try rewriting using trig identities before integrating.

1. $\displaystyle\int \frac{1}{\cos^2 x}\,dx$

2. $\displaystyle\int \frac{\sin x}{\tan x}\,dx$

The following examples require **Integration by Substitution**. The key to doing these problems is learning when the angle should be chosen as u or when to let the entire trig function be u.

3. $\displaystyle\int \cos(2x)\,dx$

4. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx$

5. $\displaystyle\int 16x\sin^3\left(2x^2+1\right)\cos\left(2x^2+1\right)\,dx$

INTEGRAL OF SECANT AND COSECANT	
$\displaystyle\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert + C$	$\displaystyle\int \csc u\,du = -\ln\lvert\csc u + \cot u\rvert + C$

You **have** to memorize these. The algebraic trick to do the integration by hand is more annoying to remember than just having this formula ingrained in your memory.

1. $\displaystyle\int 7x\sec(x^2)\,dx$

2. $\displaystyle\int \frac{\cos x + \sin x}{\tan x}\,dx$

Integration of Transcendental Functions

INTEGRAL OF U^{-1}
$$\int \frac{du}{u} = \ln\lvert u\rvert + C$$

Examples:

1. $\displaystyle\int \frac{2}{x}\,dx$

2. $\displaystyle\int \frac{\sec^2(2x)}{1+\tan(2x)}\,dx$

3. $\displaystyle\int \frac{2}{2-3x}\,dx$

4. $\displaystyle\int \tan x\,dx$

INTEGRAL OF EXPONENTIAL FUNCTIONS
$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

Examples:

1. $\displaystyle\int xe^{x^2}\,dx$

2. $\displaystyle\int \frac{1}{e^x}\,dx$

3. $\displaystyle\int 5^x\,dx$

4. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

INTEGRALS OF INVERSE TRIG FUNCTIONS

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\text{arcsec}\frac{|u|}{a} + C$$

Examples:

1. $\displaystyle\int \frac{9}{2x^2+1}\,dx$

2. $\displaystyle\int \frac{-1}{3x\sqrt{9x^2-25}}\,dx$

3. $\displaystyle\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$

Finding the Constant of Integration

1. Given $f'(x) = 4x^5$ and $f(1) = 1$, find $f(x)$.

2. Given $f'(x) = (x^2 - 3x + 2)(x - 2)$ and $f(-2) = 0$, find $f(x)$.

Solve the differential equations subject to the given initial conditions.

3. $\dfrac{dy}{dx} = 4(x-7)^3$, $y = 10$ when $x = 8$.

4. The graph of y passes through the point (4, 13) with slope 5 and $\dfrac{d^2y}{dx^2} = \dfrac{3x}{8}$. Find y.

Applications of Integration

5. A particle moving on a straight line has acceleration $a(t) = 5 - 3t$ in ft/sec^2, and its velocity is 7 ft/sec at 2 seconds. If $s(t)$ is the distance from the origin, find $s(2) - s(1)$.

6. A motorist applies the brakes on a car moving at 45 miles per hour on a straight road, and the brakes cause a constant deceleration of 22 ft/sec^2. In how many seconds will the car stop, and how many feet will the car have traveled after the time the brakes were applied?

7. Find the equation of a curve such that y'' is always 2, and at the point (2, 6) the slope of the tangent line is 10.

8. A stone is thrown straight up from a building ledge that is 120 feet above the ground with an initial velocity of 96 ft/sec. When will the stone reach its maximum height? What will the maximum height be? When will the stone hit the ground? With what speed will it hit the ground?

Definite Integration

THE DEFINITE INTEGRAL
$$\int_a^b f(x\,dx) = F(x)\|_a^b = F(b) - F(a)$$

Examples: Compute the following.

1. $\displaystyle\int_0^{\pi} \sin x\,dx$

2. $\displaystyle\int_{-2}^{1} \left(x^2+1\right)^2\,dx$

3. $\displaystyle\int_1^2 \left(3 - \frac{6}{x^2}\right)\,dx$

4. $\displaystyle\int_0^{\pi} \sin^2\theta\,d\theta$

5. $\displaystyle\int_{-1}^{0} x\sqrt{1-x^2}\,dx$

Changing the Limits of Integration

Be careful that you do not write down the limits of integration when your integral is in terms of u unless you change the limits of integration.

Examples:

1. $\displaystyle\int_{-1}^{0} \frac{x^3}{\sqrt{x^4+9}}\,dx$

2. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

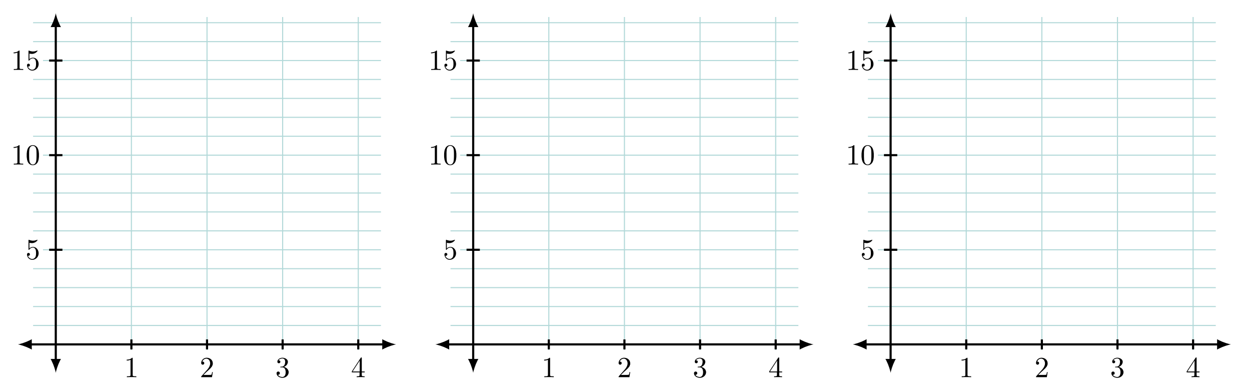
3. $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos 3x)\sin 3x\,dx$

Area Under the Curve with RAM

A train moves along a track at a steady rate of 75 miles per hour from 7 am to 9 am. What is the total distance traveled by the train? What if the train doesn't travel at a steady rate? Then how do you find the distance?

Our goal is to calculate the area under a curve. We do that by breaking the area (which we cannot calculate) into rectangles (whose area we can calculate) and adding them together.

1. Given $y = x^2$, what is the area under the curve from $x = 0$ to $x = 4$?



RAM is the Rectangular Approximation Method

LRAM is the left hand values of y

RRAM is the right hand values of y

MRAM is the midpoint of L and R

2. Find the area between the x-axis and the curve $y = -x^2 + 2x + 24$ using MRAM and 5 intervals (5 rectangles). MAKE A SKETCH along with your approximation.

3. The table to the right shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance traveled by the engine, using MRAM over 5 subintervals. Again, make a sketch!

Time (sec)	Velocity (in/sec)
0	0
1	12
2	22
3	10
4	5
5	13
6	11
7	6
8	2
9	6
10	0

4. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the table below.

   (a) Give an upper and lower estimate of the total quantity of oil that has escaped over the 8 hours.

   (b) The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gallons of oil, approximately how many more hours will elapse in the worst case before all the oil has leaked? In the best case?

Time (h)	Leakage (gal/h)
0	50
1	70
2	97
3	136
4	190
5	265
6	369
7	516
8	720

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

5. Use a Midpoint Riemann sum with two subintervals of equal length and values (directly above) from the table to approximate $\int_1^{1.4} f'(x)\,dx$.

Trapezoidal Rule

THE TRAPEZOIDAL RULE
To approximate $\int_a^b f(x)\,dx$, use $T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n)$, where $[a, b]$ is partitioned into n subintervals of equal length $h = \frac{b-a}{n}$.

Fun Fact: $T = \frac{\mathrm{LRAM}_n + \mathrm{RRAM}_n}{2}$. However, this is not the same as MRAM!

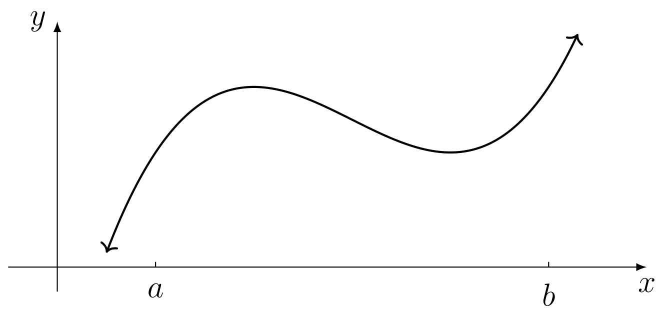
1. An observer measures the outside temperature sporadically from noon until midnight, recording the temperatures in the following table. What is the average temperature for the 12-hour period?

Time	Noon	1	2	4	5	8	9	10	Mdnt
Temp	63	65	66	70	69	65	64	62	55

Riemann Approximations

Basically, all the rectangle approximations *are* Riemann approximations (or "Riemann Sums"). While this is a separate section in most textbooks, let this be another word that equates to a process you already know.

The Definite Integral as Area Under the Curve



The area under the curve can be estimated using $S_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$, a Riemann sum for f on the interval $[a, b]$.

LIMIT DEFINITION OF THE DEFINITE INTEGRAL
Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$. Then the definite integral of f over $[a, b]$ is given by $$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\, \Delta x,$$ where each c_k is chosen arbitrarily in the k^{th} subinterval.

Think! Interpret the above definition and translate into less ”mathy” talk:

Using the Notation

The interval $[-1, 3]$ is partitioned into n subintervals of equal length $\Delta x = 4/n$. Let m_k denote the midpoint of the k^{th} subinterval. Express the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3\,(m_l)^2 - 2m_k + 5\right) \Delta x$ as an integral.

Limit to Integral and Integral to Limit

This section is to be completed after finishing Unit 5: Sequences and Series. The skills required to be fluent in this section will be strengthened in that unit immensely. In my personal opinion, it would be too many unrelated skills to be able to master amongst all the other content this unit holds.

There will be a standalone assessment on this topic at some point after the conclusion of Unit 5. The module involving this topic will be unlocked but not expected to be completed for quite some time.

Link to Notes

DEFINITION OF THE DEFINITE INTEGRAL

Formal: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve of $y = f(x)$ from a to b is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx$$

Informal: The integral is the NET area between the curve and the x-axis. Parts of the area above the curve are positive and parts below the curve are negative.

1. Evaluate $\displaystyle\int_0^{5\pi/2} \sin x\,dx$.

For the following, evaluate the definite integral and find the TOTAL area between the curve and the x-axis.

2. $\displaystyle\int_2^{\pi+2} \sin(x-2)\,dx$

3. Evaluate the integral $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$ using your calculator and confirm graphically that this is the area between the curve and the x-axis.

This even works for constant functions! Remember the train problem from the other day? A train moves along a track at a steady rate of 75 miles per hour from 7 am to 9 am. Express the total distance traveled as an integral. Evaluate the integral.

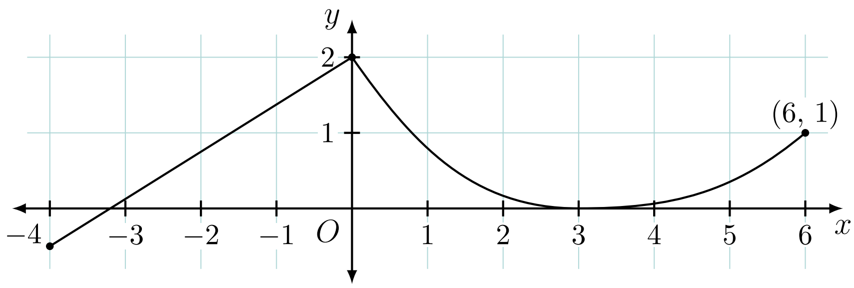
AVT & MVT

AVERAGE VALUE OF A FUNCTION
If f is integrable on $[a,\, b]$, its average (mean) value on $[a,\, b]$ is $$\text{ave}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

1. Find the average value of $f(x) = 3x^2 - 2x$ on $[1,\, 4]$.

THE MEAN VALUE THEOREM FOR DEFINITE INTEGRALS
If f is continuous on $[a,\, b]$, then at some point c in $[a,\, b]$, $$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

2. Suppose we have a circle of radius r centered at the origin. What is the average length of the chords perpendicular to the diameter $[-r,\, r]$ on the x-axis?



Graph of f

3. A continuous function f is defined on the closed interval $[-4, 6]$. The graph of f consists of a line segments and a curve that is tangent to the $x-$axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

   Is there a value of a, with $-4 \le a < 6$, for which the Mean Value Theorem, applied on the interval $[a, 6]$, guarantees a value c, with $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

The Fundamental Theorem of calculus

The **indefinite integral** is a family of functions. $F(x)$ is the antiderivative of $f(x)$.

$$\int f(x)\,dx = F(x) + C$$

The **definite integral** is a number. The number represents the net area between the graph of a function and the x-axis. *Note: areas below the x-axis are negative.*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The **definite integral as a function of $\boldsymbol{x}$** is a function. It is called the **accumulation function**. You can think of F as accumulating the area under the graph of f as the values of t increase from a to x. *Imagine a paint roller!*

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

Examples:

1. Write the antiderivative F for the given function, f.

   (a) $f(x) = -\sin x$

   (b) $f(x) = -\sin\left(x^2\right)$ so that $F(0) = 3$.

2. Construct a function of the form $y = C + \int_a^x f(t)\,dt$ with derivative $\dfrac{dy}{dx} = \tan x$ that satisfies the condition $F(3) = 5$.

DERIVATIVE OF THE INTEGRAL

If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

.

IMPORTANT NOTE: Chain Rules does apply to this process.

Examples:

1. $\dfrac{d}{dx}\displaystyle\int_1^x (2t+1)\,dt$

2. $\dfrac{d}{dx}\displaystyle\int_{-2}^x \sqrt{1+e^{5t}}\,dt$

3. $\dfrac{d}{dx}\displaystyle\int_x^5 3t\sin t\,dt$

4. Find $\dfrac{dy}{dx}$ for $y = \displaystyle\int_1^{x^2} \cos t\,dt$

5. Find $\dfrac{dy}{dx}$ for $y = \displaystyle\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt$

Rewrite the above definition using a u-substitution notation:

6. Let $g(x) = \int_1^x \left(5 - 8\sqrt{\ln t}\right)\, dt$ for $x > 1$. Let $h(x) = \int_1^{x^2} \left(5 - 8\sqrt{\ln t}\right)\, dt$ for $x > 1$.

   (a) Write an equation of the line tangent to g at $x = 3$.

   (b) What is $h'(x)$?

   (c) On which open interval(s) is g decreasing? Justify your answer. `(Calculator allowed)`

   (d) Find all values of x for which h has relative extrema. Label them as maximum of minimum and justify your answer.

Separation of Variables

Use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $y = 3$ when $x = 4$.

2. $\dfrac{dy}{dx} = x\sqrt{y}$, $y = 1$ when $x = 0$.

3. $\dfrac{dy}{dx} = 2xy^2$ and passes through the point $(1,\ 1)$.

4. $\dfrac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$.

5. $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ and contains $(e,\ 1)$.

Exponential Growth and Decay

If y changes at a rate proportional to the amount present (that is, if $\frac{dy}{dt} = ky$), and if $y = y_0$ at $t = 0$, then we have

$$y = y_0 e^{kt}$$

The constant k is the growth constant if $k > 0$ or decay if $k < 0$.

Continually Compounded Interest:

Discretely Compounded Interest:

Half-Life:

Newton's Law of Cooling:

Examples:

1. Suppose you deposit \$800 in an account that pays %6.3 annual interest. How much will you have 8 years later if the interest is compounded a) continuously and b) quarterly?

2. A hard-boiled egg at 98°C is put in a pan under running 18°C water to cool. After 5 minutes, the egg's temperature is found to be 38°C. How much longer will it take the egg to reach 20°C?

3. Scientists who use Carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Differential Equations

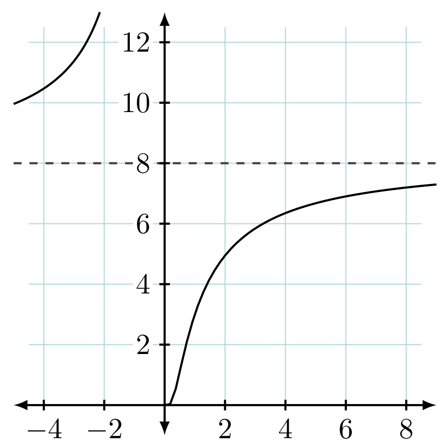
Often times, physical phenomena can be described by differential equations. We say this in previous examples outlined in the section before this. We will explore some other types of differential equations here today as well.

Slope Fields

Slope Fields has been replaced with a take-home packet and quiz. This content will still be on the unit test but will not reviewed in class!! Link to Packet can be found below (clickable if using PDF version as well). Solutions can be found in the folder at start of book.

This is the same link that will appear on Canvas.

1. Find the equation of the curve that passes through the point (1, 3) and has a slope of $\dfrac{y}{x^2}$ at the point (x, y), as shown in the graph below.



Logistic Differential Equations

Before we talk about the LDE, first let's consider the following scenario:

1. Let $N(t)$ be the population of wild coyotes in observed in a study of a wildlife reserve, with t in years. The growth of the population is proportional to $650-N$. Initially, there are 300 coyotes. After two years, there is observed to be 500 coyotes. Find $N(3)$.

Correct answer! However, there is something *inaccurate* about our work. What is it? Why?

A 'Normal' Differential Equation

Differential Equation: $\dfrac{dy}{dt}=ky$ General Solution: $y=Ce^{kt}$

This type of differential equation allows for **unlimited growth**. Obviously, we do not live in a world of infinites, so there cannot be an infinite number of coyotes as time goes on. We need a **carrying capacity**, L, to limit our growth.

The Logistic Differential Equation

Differential Equation:
$\dfrac{dy}{dt}=ky\left(1-\dfrac{y}{L}\right)$ or $\dfrac{dy}{dt}=\dfrac{ky}{L}(L-y)$ General Solution: $y=\dfrac{L}{Ce^{-kt}+1}$

Memorize both forms of the differential equation and its general solution for the BC Exam!

2. 40 elk are released into a game refuge. After 5 years, the population of elk is observed to be 104. The game warden says that the refuge can support no more than 4000 elk at any given time.

   (a) Write a differential equation for the population of elk.

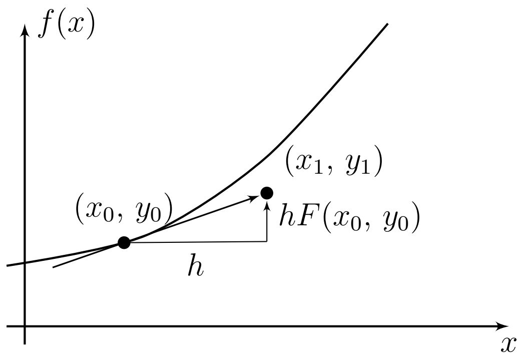
   (b) Write a model for the population, P, in terms of t. (Use the general solution, do not attempt to integrate!)

3. Find $\dfrac{d^2y}{dx^2}$ for the general L.D.E. What can we deduce based on this function?

Solving for the general solution to a LDE requires an integration method not yet covered. We will circle back and derive it as an exercise later.

Euler's Method

Euler's Method: This is an method to approximate the particular solution of a differential equation $y' = F(x, y)$ that passes through (x_0, y_0).



1. Use Euler's Method to approximate the particular solution to the differential equation $y' = x - y$ passing through the point $(0, 1)$. Use a step of $h = 0.2$.

   (a) Perform three iterations.

   (b) approximate $y(0.8)$.

2. Consider the differential equation $\dfrac{dy}{dx} = 3x + 2y + 1$. Let $y = f(x)$ be a particular solution to the differential equation with initial condition $f(0) = 2$. Use Euler's Method, starting at $x = 0$, with step size of $\frac{1}{2}$, to approximate $f(1)$. (No Calc!)

Areas in the Plane

Warm up: Determine the area between the curve and the x-axis for $y = x^2 - x - 6$ on $[-4, 4]$.

AREA IN THE PLANE
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral $$A = \int_a^b [f(x) - g(x)]\, dx$$.

Examples:

1. Determine the area of the region bounded by the graphs of $y = x^2 + 3$, $y = x + 1$, $x = 0$, and $x = 2$.

2. Determine the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$.

3. The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of these regions.

4. Determine the area of the region enclosed by the graphs of $x = 3 - y^2$ and $x = y + 1$.

5. Determine the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$. Use geometry to simplify this computation.

Volumes of Revolution

Problem 1: Sketch the region bounded by the x-axis, the y-axis, the line $x = 4$, and the line $y = 3$. What is the volume of the solid formed when the region is revolved around the x-axis?

Problem 2: What is the volume of the solid formed by revolving around the x-axis the region bounded by the axes and the line $y = -x + 4$?

Problem 3: Consider the shape of the region that is generated if $y = x^2$ on the interval [0, 3] is revolved around the x-axis. What is the volume of this solid?

VOLUME OF A SOLID OF REVOLUTION

The volume of a solid of known integrable cross section can be found by rotating the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis using the integral

$$V = \pi \int_a^b (f(x))^2 \, dx.$$

This is commonly referred to as the **disc method**.

Examples:

1. Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $(0 \le x \le \pi)$ about the x-axis.

2. Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and the x-axis about the x-axis.

3. Find the volume of the solid formed by revolving the region bounded by $y = x^{2/3}$, $x = 0$ and $y = 1$ about the y-axis.

4. Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.

VOLUME OF A REGION USING WASHERS
$$\text{Volume of a region} = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) \, dx.$$ Where $R(x)$ is the radius of the outer washer, $r(x)$ is the inner radius, and the integral represents the height of the *cylinder*.

Examples:

1. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

2. Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2$ and $y = 2 - x^2$ about the x-axis.

3. Find the volume of the solid generated by revolving the region bounded the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

4. The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved around the x axis to form a solid. Find its volume.

5. The area of the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ in the first quadrant is revolved around the line $x = 3$. Find the volume of this solid.

VOLUME OF KNOWN CROSS SECTION
The volume of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b, $$V = \int_a^b A(x)\,dx.$$

Examples:

1. A pyramid 3m high has congruent triangular sides and a square base that is 3m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

2. A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2\sin x$. Each cross section cut perpendicular to the x-axis is a semi-circle whose diameter runs from the x-axis to the curve. Find the volume of the paperweight.

Arc Length

DEFINITION OF ARC LENGTH
Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is $$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$ Similarly, for a function $x = g(y)$, we get $$s = \int_a^b \sqrt{1 + [g'(y)]^2}\, dy.$$

Note: Very few integrals involving arc length can be computed by hand. For the homework, use a calculator after the initial setup.

Examples: Find the length of the arc for the given function on the specified interval

1. $x^2 = (y-1)^3$ on $[0, 8]$.

2. $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$. **(Do by hand)**

3. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on $\left[\dfrac{1}{2}, 2\right]$.

Integration Review

Three Similar Integrals

(a) $\displaystyle\int \frac{4}{x^2+9}\,dx$ (b) $\displaystyle\int \frac{4x}{x^2+9}\,dx$ (c) $\displaystyle\int \frac{4x^2}{x^2+9}\,dx$

Tricky Substitutions:

(d) $\displaystyle\int \frac{x^2}{\sqrt{16 - x^6}}\, dx$

(e) $\displaystyle\int (\cot x)\,[\ln(\sin x)]\; dx$

Integration by Parts

Up to this point, we have *still* been doing only the simple-ish integrals. They can get more complicated, but with a bit of practice, not necessarily harder. Today, we are going to learn how to integrate the product of two different transcendental functions. A.k.a, **undo the product rule**!

INTEGRATION BY PARTS THEOREM
If u and v are functions of x and have continuous derivatives, then $$\int u\,dv = uv - \int v\,du.$$

Hint: Try letting dv be the more complicated part of the integrand that fits a basic integration rule.

Examples: Evaluate the following integrals

1. $\displaystyle\int xe^x\,dx$

2. $\displaystyle\int x^2 \ln x\,dx$

3. $\displaystyle\int_0^1 \arcsin x\,dx$

4. $\displaystyle\int x^2 \sin x\,dx$

5. Using the **Tabular Method**:

   (a) $\displaystyle\int x^2 \sin x \, dx$

   (b) $\displaystyle\int x^2 e^{2x} \, dx$

6. $\displaystyle\int e^{3x} \cos x \, dx$

Trigonometric Combinations

Overarching Goal of this Section: Rewrite an integrand with one factor of cosine and some expression involving sine (or vise versa).

Odd Power Examples:

(a) $\int \sin^5 x \, dx$

(b) $\int \sin^3 x \cos^2 x \, dx$

Even Power Examples:

(a) $\int \sin^2 x \, dx$

(b) $\int \cos^4 x \, dx$

Integrals involving Secant and Tangent:

(a) $\displaystyle\int \tan^6 x \sec^4 x \, dx$

(b) $\displaystyle\int \tan^5 x \sec^7 x \, dx$

Using Other Identities: Evaluate $\displaystyle\int \cos^2 x \tan^3 x \, dx$

Trigonometric Substitution

Integrals containing a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ you can often rewrite using a substitution involving a trigonometric function.

For example, let's consider the form $\sqrt{a^2 - x^2}$ for some $a > 0$. Choose $x = a \sin\theta$ for your substitution.

$$\sqrt{a^2 - x^2} =$$

We will produce the substituions for the rest as we go and complete the table at the end of the section.

Examples: Evaluate each of the following integrals using a trigonometric substitution.

1. $\displaystyle\int \frac{1}{x^2\sqrt{4 - x^2}}\, dx$

2. $\displaystyle\int_0^3 \frac{1}{\sqrt{9 + x^2}}\, dx$

3. $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Expression	**Trig Substitution**	**Interval**	**Identity**
$\sqrt{a^2-x^2}$			
$\sqrt{a^2+x^2}$			
$\sqrt{x^2-a^2}$			

Closing Remark: You do not always *have* to use T.S., however, it does provide a procedure that will always work.

Partial Fraction Decomposition

Let's start off this section with a simple observation:

$$\frac{3}{x+2} - \frac{2}{x-5} \quad = \quad \textit{some math, idk} \quad = \frac{x-19}{x^2-3x-10}$$

Now, let's try to evaluate $\displaystyle\int \frac{x-19}{x^2-3x-10}\,dx$. Unfortunately, there is not going to be a way to integrate this rational function in it's current form. You can try, but the u-substitution does not differ by a constant. RIP, I know.

But, $\displaystyle\int \frac{x-19}{x^2-3x-10}\,dx = \int \left(\frac{3}{x+2} - \frac{2}{x-5}\right)dx = 3\ln|x+2| - 2\ln|x-5| + C$. *Wow!*

Examples: (Warning!! These are long)

1. Use partial fraction decomposition to reduce to distinct linear factors.
   $\displaystyle\int \frac{1}{x^2+x-2}\,dx$

2. Closely examine this one before starting PFD.
$$\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx$$

3. Evaluate.
$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\, dx$$

4. Hmm, there seems to be something different about this one...
$$\int \frac{2x^3 - 2x - 8}{(x^2 - x)(x^2 + 4)}\, dx$$

5. Solve the differential equation: $\dfrac{dy}{dx} = ky\left(1 - \dfrac{y}{L}\right)$

Improper Integrals

Recall the following two conditions from theorems earlier this year:

1. The definition of the definite integral requires that the interval $[a, b]$ is finite.
2. The Fundamental Theorem of Calculus requires that the function is continuous on $[a, b]$.

Integrals that do not possess these properties are called **improper integrals**, and we can still compute them with the addition of limits!

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE BOUNDS

1. If f is continuous on the interval $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
2. If f is continuous on the interval $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. If f is continuous on the interval $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x),\, dx$$
where c is any real number.

If the limit exists, then it is said that improper integral **converges**, otherwise, it **diverges**. In the third case, if either of the integrals on the right diverge, then so does the one on the left.

Examples:

1. Evaluate $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.

2. $\displaystyle\int_{1}^{\infty} e^{-x}\,dx$

3. $\displaystyle\int_{1}^{\infty} (1-x)e^{-x}\,dx$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{x}}{1+e^{2x}}\,dx$

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$
2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$
3. If f is continuous on the interval $[a, b]$ except for some c in (a, b) at which f has an infinite discontinuity, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x),\, dx.$$

If the limit exists, then it is said that improper integral **converges**, otherwise, it **diverges**. In the third case, if either of the integrals on the right diverge, then so does the one on the left.

Examples:

1. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

2. Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$.

3. Evaluate $\displaystyle\int_{-1}^{2} \frac{dx}{x^3}$.

4. Evaluate $\displaystyle\int_{0}^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

A SPECIAL IMPROPER INTEGRAL
$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

Sequences Basics

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers (or, as some might claim, the **natural numbers** $\mathbb{N}$). Although a sequence is a function, it is common to represent sequences by subscript notation.

Example:
$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & ..., & n, & ... \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1, & a_2, & a_3, & a_4, & a_5, & \dots, & a_n, & \dots, \end{array}$$

Each a_n is a **term** of the sequence.

The number a_n is called the **nth term**.

Finding Patterns Describe a pattern for each of the following sequences. Then use your description to write a formula for the nth term of each sequence. As n increases, do the terms appear to be approaching a limit?

(a) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$

(b) $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, ...$

(c) $10, \frac{10}{3}, \frac{10}{6}, \frac{10}{10}, \frac{10}{15}, ...$

(d) $\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36}, ...$

(e) $\frac{3}{7}, \frac{5}{10}, \frac{7}{13}, \frac{9}{16}, \frac{11}{19}, ...$

(f) Let $a_n = \left\{\dfrac{4n}{3-2n}\right\}$. List out the first five terms, then estimate $\lim\limits_{n\to\infty} a_n$.

DEFINITION OF THE LIMIT OF A SEQUENCE
Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as, $$\lim_{n\to\infty} a_n = L$$ if for each $\epsilon > 0$, there exists $M > 0$ such that $\lvert a_n - L\rvert < \epsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges**.

I know what you are thinking: *translation please!1??!?*

Convergent: When a sequence has a limit that approaches some real number L.

Divergent: A sequence that does *not* have a limit.

Possibilities:

1. If $\lim\limits_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity.
2. If $\lim\limits_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity.
3. If $\lim\limits_{n\to\infty} a_n = L$, a real finite number, then $\{a_n\}$ converges to L.
4. If $\lim\limits_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation.

Example: Does the sequence $a_n = \left\{\dfrac{\ln\sqrt{n}}{n}\right\}$ converge or diverge? If convergent, where to?

As we just saw, L'Hopital's Rule becomes a very important part of determining convergence of *interesting* sequences.

Some Important Limits to Know:

$$\lim_{n\to\infty} \frac{c}{n} =$$

$$\lim_{n\to\infty} \frac{\ln n}{n} =$$

$$\lim_{n\to\infty} \sqrt[n]{n} =$$

$$\lim_{n\to\infty} x^n =$$

$$\lim_{n\to\infty} x^{\frac{1}{n}} =$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n =$$

$$\lim_{n\to\infty} \frac{x^n}{n!} =$$

$$\lim_{n\to\infty} \frac{n!}{x^n} =$$

Examples:

Determine whether the following sequences converge or diverge.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(c) $a_n = 3 + (-1)^n$

(d) $a_n = \dfrac{n}{1-2n}$

(e) $a_n = \dfrac{\ln n}{n}$

(f) $a_n = \dfrac{n!}{(n+2)!}$

Series & Convergence

DEFINITION OF CONVERGENT AND DIVERGENT SERIES
For the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by $$S_n = a_1 + a_2 + a_3 + ... + a_n.$$ If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges**. The limit S is called the **sum of the series**. $$S_n = a_1 + a_2 + a_3 + ... + a_n \text{ where } S = \sum_{n=1}^{\infty} a_n$$ If $\{S_n\}$ diverges, then the series **diverges**.

Example: Convergent and Divergent Series
Find the sequence of partial sums S_1, S_2, S_3 and an expression for S_n for each of the following series. Determine its convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ...$

(c) $\displaystyle\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + ...$

A **telescoping series** has the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + ...$ therefore $S_n = b_1 - b_{n+1}$. A telescoping series converges if and only if $b_n \to$ a finite number as $n \to \infty$. In which case, $S = b_1 - \displaystyle\lim_{n\to\infty} b_{n+1}$.

Example: A Telescoping Series In Disguise

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Geometric Series

Specific Example: $3 + 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + ...$

General Form: $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ... + ar^n + ...$

CONVERGENCE OF A GEOMETRIC SERIES

1. A geometric series with ratio r diverges if $|r| \geq 1$.
2. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \ 0 < |r| < 1.$$

Example: Convergent and Divergent Geometric Series
Determine the convergence or divergence of the series. If it converges, find its sum.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

PROPERTIES OF INFINITE SERIES

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B, and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, the the following series converge to the indicated sums

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$

nth-Term Test for Divergence

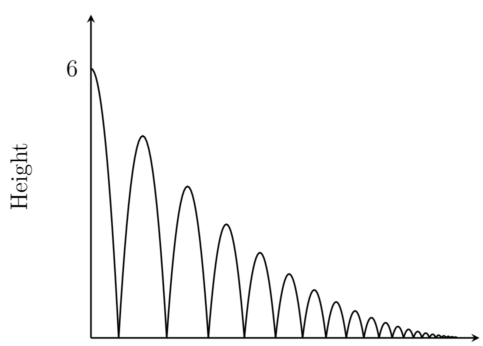
LIMIT OF THE nTH TERM OF A SERIES
Let $\sum a_n$ converge, then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

We commonly abbreviate the name of this theorem to the ”Divergence Test”.

Example: Using the Divergence Test
Determine the convergence or divergence of each series.

(a) $\displaystyle\sum_{n=0}^{\infty} 2^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Example: Bouncing Ball Problem
A ball is dropped from a height of 6 feet and begins bouncing as shown in the figure below. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



Integral Test & p-Series

Let's start this section with an example...

Consider the series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^2} + ...$

First, verify that the Divergence Test is inconclusive and that this is not a geometric series.

Does the series converge?

THE INTEGRAL TEST
If $a_n = f(n)$ and f is continuous, positive, and decreasing on $[1, \infty)$, then $$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_1^{\infty} f(x)\,dx$$ either both converge or diverge.

THREE IMPORTANT FACTS:

1. They do **not** converge to the same value! In the example we just did:
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \neq 1 = \int_1^{\infty} \frac{1}{x^2}\,dx$$
2. We can start at other indices (limits of integration).
3. f does not have to be always decreasing, only **ultimately** decreasing.

Examples: Determine the convergence or divergence of each series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+2}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

DEFINITION OF A p-SERIES

A series of the form
$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$
is a ***p*-series**, where p is a positive constant. For $p = 1$, the series
$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$$
is called the **harmonic series**. A **general harmonic series** is of the form $\sum_{n=1}^{\infty} \frac{1}{an+b}$.

CONVERGENCE OF A p-SERIES

A series of the form
$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$
converges if $p > 1$ and diverges if $0 < p \leq 1$.

Examples:

1. Determine the convergence or divergence of each series.

   (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

2. Find the positive values of p for which the series converges:

   (a) $\sum_{n=2}^{\infty} \frac{1}{n\left(\ln(n)\right)^p}$ (b) $\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

Comparison of Series

One motif of the series (which you probably didn't notice) that we have been looking at is that they are relatively simple. However, with any slight complications, we can no longer apply the tests in our arsenal. For example:

1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.
3. $a_n = \frac{n}{(n^2+3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2+3)^2}$ is not.

THE DIRECT COMPARISON TEST
Let $0 < a_n \leq b_n$ for all n. 1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. 2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

THREE IMPORTANT FACTS:

1. Your comparison does not always have to be greater than or equal to a_n, just as long as it is **ultimately** greater than (or less than) the original.
2. To prove a series is convergent, your comparison must be **larger**.
3. To prove a series is divergent, your comparison must be **smaller**.

Examples: Determine the convergence or divergence of each series.

(a) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

THE LIMIT COMPARISON TEST

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is *finite* and *positive*. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Examples: Using the Limit Comparison Test

Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$.

Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{4n^3+1}$.

Alternating Series

The simplest of alternating series can be a geometric series. The following example is easy to determine its convergence since we know that the geometric series test can be applied:

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

THE ALTERNATING SERIES TEST
Let $a_n > 0$. The alternating series $$\sum_{n=1}^{\infty}(-1)^n\, a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\, a_n$$ converge if the following conditions are met. 1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n$, for all n

ONE IMPORTANT FACT:

1. The second condition in the Alternating series test can be modified to require only that $0 < a_{n+1} \le a_n$ for all n greater than some integer N.

Examples:
Determine the convergence or divergence of each series.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2+3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$

Example: When the AST Fails

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$

2. $\displaystyle\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$

Alternating Series Remainder:

THE ALTERNATING SERIES REMAINDER THEOREM
If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the reminder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is, $$\lvert S - S_N \rvert = \lvert R_N \rvert \leq a_{N+1}$$

The reason why this is the case is hard to conceptualize, so perhaps a proof might help.

Proof: The series obtained by deleting the first N terms of the given series satisfies the conditions of the conditions of the Alternating series test and has a sum of R_N.

$$\begin{aligned}
R_N = S - S_N &= \sum_{n=1}^{\infty}(-1)^{n+1}a_n - \sum_{n=1}^{N}(-1)^{n+1}a_n \\
&= (-1)^N a_{N+1} + (-1)^{N+1}a_{N+2} + (-1)^{N+2}a_{N+3} + \cdots \\
&= (-1)^N \left(a_{N+1} - a_{N+2} + a_{N+3} - \cdots\right) \\
\lvert R_N \rvert &= a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} - \cdots \\
&= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - \cdots \leq a_{N+1}
\end{aligned}$$

Consequently, $\lvert S - S_N \rvert = \lvert R_N \rvert \leq a_{N+1}$, which establishes the theorem.

Examples:

1. Consider the following series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

   (a) Approximate the sum of the series using the first six terms.

   (b) What is the error associated with this approximation?

   (c) What is the interval for which the true sum of the series, S, must lie within? (Use the previous two answers)

2. Consider the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n \cdot 3^{n+1}}{2^{2n}}$.

   (a) Find the maximum error associated with the approximation of S_4.

   (b) How many terms would it take for the error to be less than $\frac{1}{200}$?

Absolute and Conditional Convergence

Occasionally, we run into series that have both positive and negative terms but is **not** an alternating series. For example:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

In order to obtain any information about the convergence of this series, let's instead consider

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| .$$

ABSOLUTE CONVERGENCE
If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Note that the opposite is **not** true. For instance, the **alternating harmonic series** converges by the alternating series test. Yet the harmonic series diverges. This is called **conditional convergence.**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

DEFINITIONS OF ABSOLUTE AND CONDITIONAL CONVERGENCE
1. $\sum a_n$ is **absolutely convergent** if $\sum \lvert a_n \rvert$ converges. 2. $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum \lvert a_n \rvert$ diverges.

Practice:
Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2} n!}{3^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

Ratio and Root Tests

THE RATIO TEST
Let $\sum a_n$ be a series with nonzero terms. 1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left\|\dfrac{a_{n+1}}{a_n}\right\| < 1$. 2. $\sum a_n$ diverges if $\lim_{n\to\infty}\left\|\dfrac{a_{n+1}}{a_n}\right\| > 1$ or $\lim_{n\to\infty}\left\|\dfrac{a_{n+1}}{a_n}\right\| = \infty$. 3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left\|\dfrac{a_{n+1}}{a_n}\right\| = 1$.

ONE IMPORTANT FACT:

1. This is not the one stop shop for tests of convergence, but *usually* is for series involving factorials or exponentials.

Example:
Determine the convergence or divergence of each series.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$

THE ROOT TEST

Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

ONE IMPORTANT FACT:

1. The Root Test is always inconclusive for a p-series. So don't even bother.

Example:
Test the series for convergence or divergence. Afterwards, *try* the Ratio Test to see why it is not ideal. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

Strategies for Testing Series

You have learned 10 strategies for testing a series for convergence or divergence.

Guidelines for Testing a Series:

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types (geo, p-series, telescoping, alternating)?
3. Can the integral test, ratio test, or root test be applied?
4. Can the series be compared favorably to one of the special types?

Example: Applying Strategies

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3}{4n+1}$

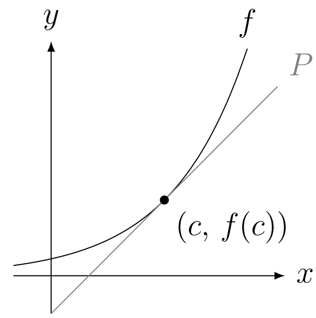
(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Taylor Polynomials & Approximations

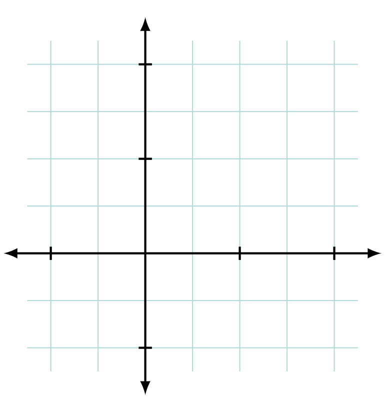
The goal is to show polynomial functions can be used to approximate any elementary function. To do this, we require the following two conditions to be true about our function f and the polynomial approximation P:

$$f(c) = P(c) \qquad\qquad f'(c) = P'(c).$$



The approximating polynomial is said to be **expanded about *c*** or **centered at *c***. It should make sense that we require the slopes to match as there are an infinite number of polynomials that pass through $(c, f(c))$. This constraint, at the very least, makes the approximation look somewhat similar to f at that point.

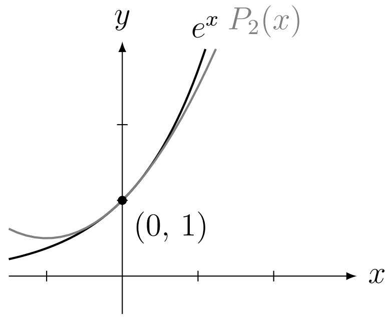
Introductory Example: First-Degree Polynomial Approximation of $\mathbf{f(x) = e^x}$:
For the function $f(x) = e^x$, find a first degree polynomial function $P_1(x) = a_0 + a_1 x$ whose value and slope agree with the value of the slope of f at $x = 0$.



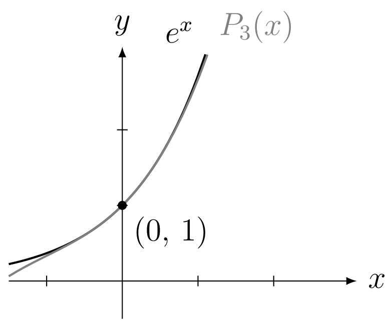
Based on P_1 found in example 1, we can see that our approximation fits relatively well for values of x that are close to $c = 0$. However, the further we move away from $(0, 1)$, it is clear that our approximation will not be accurate whatsoever. How might we improve the accuracy even further?

Let's consider the following:

$$P_2(x) = 1 + x + \frac{1}{2}x^2$$



$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$



Clearly, P_2 is a better approximation than P_1, and P_3 more still. If we continue on, matching each nth derivative of $f(x) = e^x$ at $x = 0$, we obtain the following:

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \approx e^x$$

DEFINITIONS OF THE $\boldsymbol{n}$TH TAYLOR POLYNOMIAL AND $\boldsymbol{n}$TH MACLAURIN POLYNOMIAL

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **$\boldsymbol{n}$th Taylor Polynomial for $\boldsymbol{f}$ at $\boldsymbol{c}$**. If $c = 0$, then

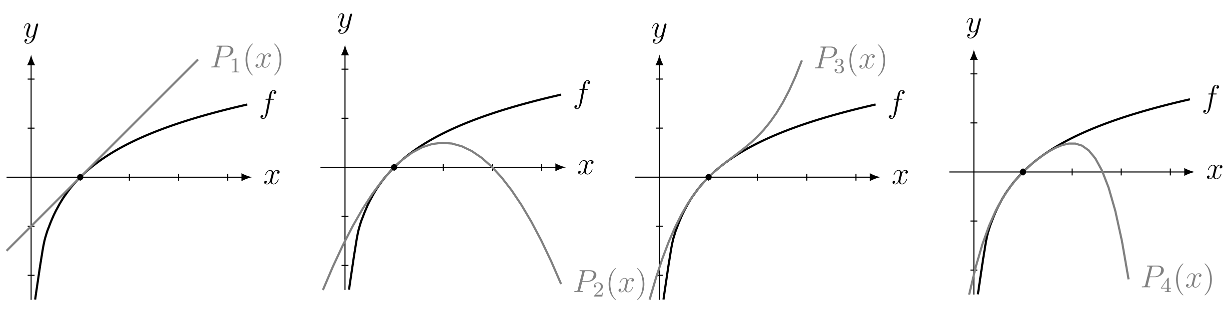
$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **$\boldsymbol{n}$th Maclaurin Polynomial for $\boldsymbol{f}$**.

Examples:

1. Find the nth Maclaurin polynomial for $f(x) = e^x$.

2. Find the Taylor polynomials $P_0,\ P_1,\ P_2,\ P_3,$ and P_4 for $f(x) = \ln x$ centered at $c = 1$.



3. Find the Maclaurin Polynomials P_n for $n = 0$, 2, 4, 6 for $f(x) = \cos x$. Then, use $P_6(x)$ to approximate the value of $\cos(0.1)$.

4. Find the third Taylor Polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

Remainder of a Taylor Polynomial

As these are approximations, they will not be exactly equal to the original function. As we did with Alternating Series earlier this unit, we will revisit the concept of the **remainder**.

$$\underset{\text{Exact value}}{f(x)} \;=\; \underset{\text{Approx. value}}{P_n(x)} \;+\; \underset{\text{Remainder}}{R_n(x)}$$

So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation. That is,

$$\text{ERROR} = |R_n(x)| = |f(x) - P_n(x)|.$$

The following theorem is called **Taylor's Theorem**, and the remainder given in the theorem is called the **Lagrange form of the remainder**. You should know both names as synonymous.

TAYLOR'S THEOREM

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

One thing to note when using this theorem is that $|R_n(x)| \le \dfrac{|x-c|^{n+1}}{(n+1)!} \max\left|f^{(n+1)}(z)\right|$ where $\max\left|f^{(n+1)}(z)\right|$ is the maximum value of $\left|f^{(n+1)}(z)\right|$ between x and c.

Example:

The third Maclaurin polynomial for $\sin x$ is given by $P_3(x) = x - \dfrac{x^3}{3!}$. Use Taylor's Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the error of the approximation.

More Examples:

1. Determine the degree of the Talyor Polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

2. Find an upper bound for the error of the 5^{th} degree polynomial approximation of e.

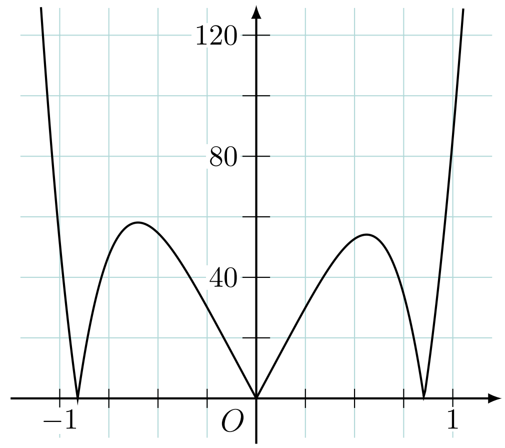
3. Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$ for all x for which the series converges.

   (a) Use the first three terms of the series to approximate $f\left(\dfrac{-1}{3}\right)$.

   (b) Estimate the error involved in the approximation of part (a).

4. Let $P_4(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!}$ be the fourth-degree Taylor Polynomial for f about $x = 0$. Using the information from the graph of $\left|f^{(5)}(x)\right|$ shown to the right, show that

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}.$$



5. The Taylor Series about $x = 3$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 3$ is given by

$$f^{(n)}(3) \leq \frac{(-1)^n\, n!}{5^n\,(n+3)} \text{for all } n \text{ and } f(3) = \frac{1}{3}.$$

Show that $P_3(x)$ approximates $f(4)$ with an error less than $\dfrac{1}{4000}$.

Power Series

In the previous section, we have been approximating some of our elementary functions using Taylor and Maclaurin Polynomials. We examined polynomials of first, second, third (and so on) degree and saw how closely they matched the original curve. The goal is going to be to move away from *approximations* and instead represent a function *exactly*.

For example the function $f(x) = e^x$ can be represented exactly by an infinite series called a **power series**. The representation is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

For each real number x, it can be shown that the infinite series on the right converges to the number e^x.

DEFINITIONS OF POWER SERIES
If x is a variable, then an infinite series of the form $$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$ is called a **power series**. More generally, an infinite series of the form $$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$ is called a **power series centered at c**, where c is a constant.

Radius and Interval of Convergence

A power series in x can be thought of as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where the **domain of *f*** is the set of all x values for which the power series converges. The main goal of this section will be to determine where and when that happens.

CONVERGENCE OF A POWER SERIES
For a power series centered at c, precisely one of the following is true. 1. The series converges only at c. 2. There exists a real number $R > 0$ such that the series converges absolutely for $\lvert x-c \rvert < R$, and diverges for $\lvert x-c \rvert > R$. 3. The series converges absolutely for all x. The number R is called the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, then the radius is $R = \infty$. The set of all values of x for which the series converges is called the **interval of convergence** of the power series.

Examples: Find the Radius of Convergence.

1. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

2. $\displaystyle\sum_{n=0}^{\infty} 3(x-2)^n$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Endpoint of Convergence: When R is a finite value (such as in example 2 from the previous page), our theorem about convergence of power series never specifies what happens at the endpoints. As a result, they must be tested separately.

Examples: Find the interval of Convergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Differentiating and Integrating Power Series

PROPERTIES OF FUNCTIONS DEFINED BY POWER SERIES

If the function given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots \end{aligned}$$

has radius of convergence $R > 0$, then, on the interval $(c-R,\ c+R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} na_n(x-c)^{n-1} \\ &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots \end{aligned}$$

2. $$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} \\ &= C + a_0(x-c) + a_1\frac{(a-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \end{aligned}$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as the original series. The *interval of convergence*, however, may differ because of the behavior of the end points.

Consider the function given by $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$. Find the interval of convergence for each of the following:

(a) $f(x)$

(b) $f'(x)$

(c) $\int f(x)\,dx$

Geometric Power Series

The final type of series involves functions written in the form $f(x) = \dfrac{1}{1-x}$. These series resemble the sum of a Geometric Series from earlier sections:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

We can infer that the power series expansion of $f(x)$ is as follows:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} 1 \cdot x^n.$$

Examples: Find a Power Series and Interval of Convergence for each Function.

1. $f(x) = \dfrac{1}{1-2x}$, $c = 0$
2. $f(x) = \dfrac{1}{4+x}$, $c = 2$

3. $f(x) = \dfrac{3}{3-2x}$, $c = 1$

4. Find the power series representation of the following functions:

   (a) $f(x) = \dfrac{1}{(1-x)^2}$

   (b) $f(x) = \arctan(x)$

Taylor & MacLaurin Series

DEFINITIONS OF TAYLOR AND MACLAURIN SERIES
If a function f has derivatives of all orders at c, then the series $$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$ is called the **Taylor Series for *f* at *c***. If $c = 0$, then $$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$ is also called the **Maclaurin Series for *f***.

1. Form a 4^{th} degree Taylor Polynomial for $f(x) = e^x$ centered at $x = 1$. Also, determine the general term.

2. Form the Maclaurin Series for $\sin x$.

3. Form the Maclaurin Series for $\cos x$.

MEMORIZE KNOWN MACLAURIN SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

Taylor & Maclaurin Manipulation

We can manipulate Taylor (and Maclaurin) series in order to find other series using the following techniques:

- Substitute into the series
- Add or subtract two series
- Multiply or Divide the series by a constant or variable
- Differentiate or integrate a series

There are often many avenues to take that will produce the same answer. Try to deduce the most *efficient* method given the type of problem.

1. Find the first four terms of the Maclaurin series for $f(x) = \sin x \cos x$.

2. Find a Maclaurin series for $\sin^2 x$.

3. Find a Maclaurin series for $g(x) = e^{-2x}$.

4. Find the first 6 non-zero terms of the Maclaurin series for $f(x) = e^x + \cos x$. Approximate the value of $f'(0.5)$.

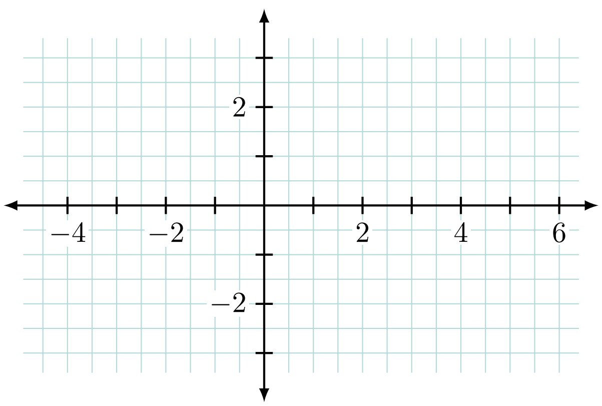
5. Use a 6^{th} degree Maclaurin polynomial to approximate the value of $\int_0^1 \sin(x^2)\,dx$. What is the maximum error of this approximation?

Plane Curves and Parametric Equations

Until now, you have been representing a graph by a single equation involving two variables. Now, we will introduce a third variable to describe some curve in the plane.

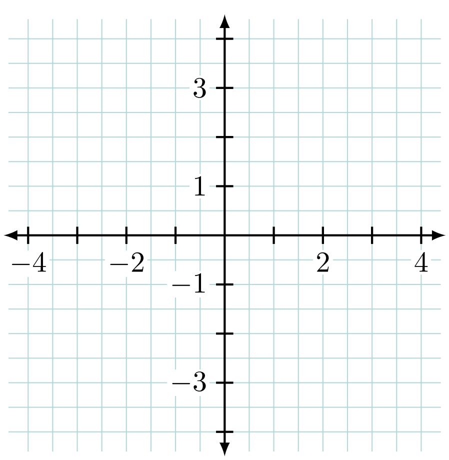
DEFINITIONS OF A PLANE CURVE
If f and g are continuous functions of t on the interval I, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** and t is called the **parameter**. Together, the parametric equations produce the graph called a **plane curve**.

Try sketching the plane curve given by the parametric equations $x = t^2 - 4$ and $y = \dfrac{t}{2}$ for $-2 \leq t \leq 3$.



Now, eliminate the parameter to find the Cartesian equation of the curve.

Sketch the curve represented by $x = 3\cos\theta$ and $y = 4\sin\theta$, $0 \leq \theta \leq 2\pi$. Then, eliminate the parameter.



Parametric Equations and & Calculus

PARAMETRIC FORM OF THE DERIVATIVE
If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$ The second derivative (and every subsequent derivative) is $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}.$$

Example:

1. Given the parametric equations $x = 2\sqrt{t}$ and $y = 3t^2 - 2t$, find dy/dx and d^2y/dx^2.

2. Given the parametric equations $x = 4\cos t$ and $y = 3\sin t$, write the equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

3. Find all points of vertical and horizontal tangency given the parametric equations $x = t^2 + 2$, $y = t^2 - 3t = 5$.

PARAMETRIC FORM OF ARC LENGTH
If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval (a, b), then the arc length of C over the interval (a, b) is given by: $$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

A particle moves along a smooth curve given by $x = t^2 + 1$ and $y = 4t^3 - 1$. How far did the particle travel between $t = 0$ and $t = 5$? BONUS: How far away is the particle from its starting position after the 5 seconds?

Vector Functions and Motion

Parametric, Vector, and (eventually) Polar are all essentially the same. In this section, we introduce the form of a vector valued function, but you will see how it is essentially the same as a parametric one.

The Essentials

If $\vec{r}(t) = \langle x(t),\, y(t)\rangle$ is the position vector of a particle moving along a smooth curve in the xy-plane, then, at any time t,

1. The particle's **velocity vector** $\vec{v}(t) = \langle x'(t),\, y'(t)\rangle$; if drawn from the position point, it is tangent to the points in the direction of increasing t.
2. The particle's **speed** along the curve is the length (magnitude) of the velocity vector:
$$|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2}.$$
3. The particle's **acceleration vector** $\vec{a}(t) = \langle x''(t),\, y''(t)\rangle$, is the derivative of the velocity vector and the second derivative of the position vector.

If $\vec{v}(t) = \langle x'(t),\, y'(t)\rangle$ is the velocity vector of a particle moving along a smooth curve in the xy-plane, then

1. The **displacement** from $t = a$ to $t = b$ is given by the vector:
$$\left\langle \int_a^b x'(t)\,dt,\ \int_a^b y'(t)\,dt \right\rangle$$
The proceeding vector is added to the position at time $t = a$ to get the position at time $t = b$.
2. The **distance traveled** from $t = a$ to $t = b$ is given by
$$\int_a^b |\vec{v}(t)|\ dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$
Note that this is just arc length for a parametric curve.

Examples:

1. A particle moves in the xy-plane so that at any time t, the position of the particle is given by $x(t) = t^3 + 2t^2$, $y(t) = t^4 - t^3$.

   (a) Find the velocity vector when $t = 1$. (b) Find the $\vec{a}(t)$ when $t = 2$.

2. A particle moves in the xy-plane so that at any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^2 + 3t$, $y(t) = t^3 - 3t^2$. Find the magnitude of the velocity vector when $t = 1$.

3. A particle moves in the xy-plane so that its position is given by

$$\vec{r}(t) = \left\langle \sqrt{3} - 4\cos t,\ 1 - 2\sin t \right\rangle,$$

where $0 \leq t \leq 2\pi$. The path of the particle intersects the x-axis twice. Write an expression that represents the distance traveled by the particle between the two x-intercepts. Do no evaluate.

4. A particle moves with a velocity vector of $\langle 3t^2 - 4t,\ 8t^3 + 5\rangle$. If the position vector at $t = 0$ is $\langle 7,\ -4\rangle$, find the position of the particle at $t = 1$.

Polar Coordinates and Graphs

A Quick Review

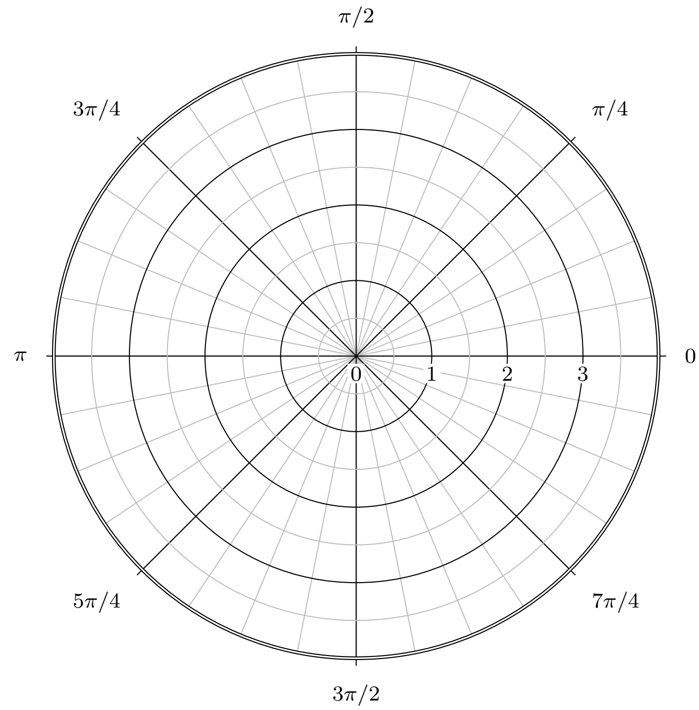
Plot the following points:

1. $A\left(3, \dfrac{7\pi}{6}\right)$

2. $B\left(-2, \dfrac{5\pi}{4}\right)$

3. $C\left(2, \dfrac{-\pi}{6}\right)$

4. $D\,(-3,\,-3\pi)$



COORDINATE CONVERSION EQUATIONS
Let the point P have polar coordinates $(r,\,\theta)$ and rectangular coordinates $(x,\,y)$. Then $x = r\cos\theta \qquad y = r\sin\theta$ $r^2 = x^2 + y^2 \qquad \tan\theta = \dfrac{y}{x}$

Examples: Convert the following points/equations to the other coordinate system.

1. $\left(2, \dfrac{5\pi}{6}\right)$

2. $(3,\,-3)$

3. $y = 4$

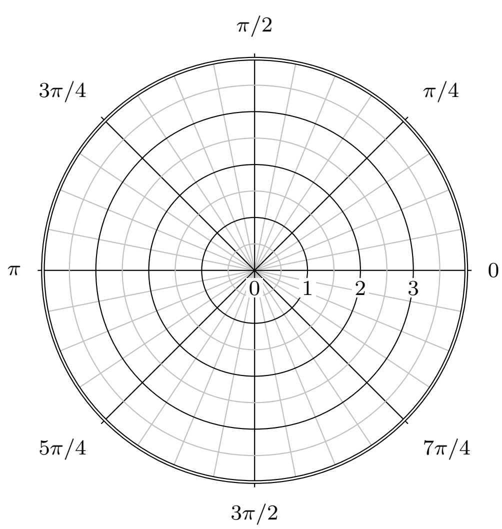
4. $x^2 + y^2 = 25$

5. $r \sin\theta = 3$

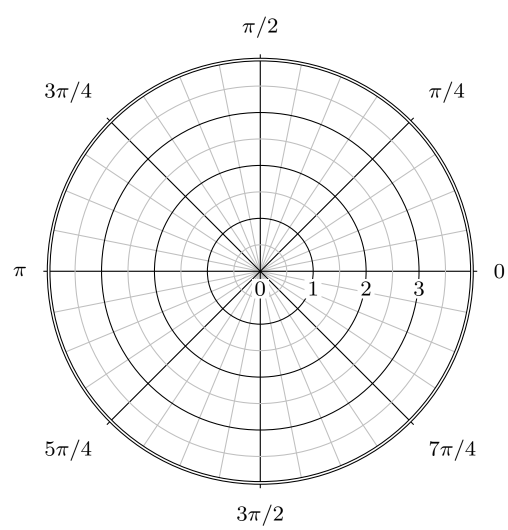
6. $\theta = \dfrac{2\pi}{3}$

Graphing:

1. $r = 2\cos 3\theta$



2. $r = 2(1 - \cos\theta)$



Polar Equations & Calculus

SLOPE IN POLAR FORM
If r is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$ provided that $dx/d\theta \neq 0$ at (r, θ).

From this, we get two main deductions:

1. Solutions to $\dfrac{dy}{d\theta} = 0$ yield horizontal tangents so long as $\dfrac{dx}{d\theta} \neq 0$
2. Solutions to $\dfrac{dx}{d\theta} = 0$ yield vertical tangents so long as $\dfrac{dy}{d\theta} \neq 0$

Examples:

1. Find $\frac{dy}{dx}$ and the slope of the polar curve $r = 3 + 2\sin\theta$ at $\theta = \pi/6$.

2. Find all of the vertical and horizontal tangents to the graph of $r = 2(1 - \cos\theta)$.

3. (Calculator) Sketch the graph of $r = 2\csc\theta + 5$. Find all points of horizontal and vertical tangency.

Polar Area

AREA IN POLAR FORM
If f is a continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ is given by $$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta \end{aligned}$$

Examples: Use a calculator for all of the following.

1. Find the area bounded by the graph of $r = 2 + 2\sin\theta$.

2. Find the area of one petal of $r = 2\sin 3\theta$.

3. Find the area of one petal of $r = 4\cos 2\theta$.

4. Find the area of the inner loop of the graph of $r = 1 + 2\sin\theta$.

5. Find the area of the graph of $r = 1 + 2\sin\theta$

6. Find the area between the two loops of the graph $r = 1 + 2\sin\theta$.

7. Find the area inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$.

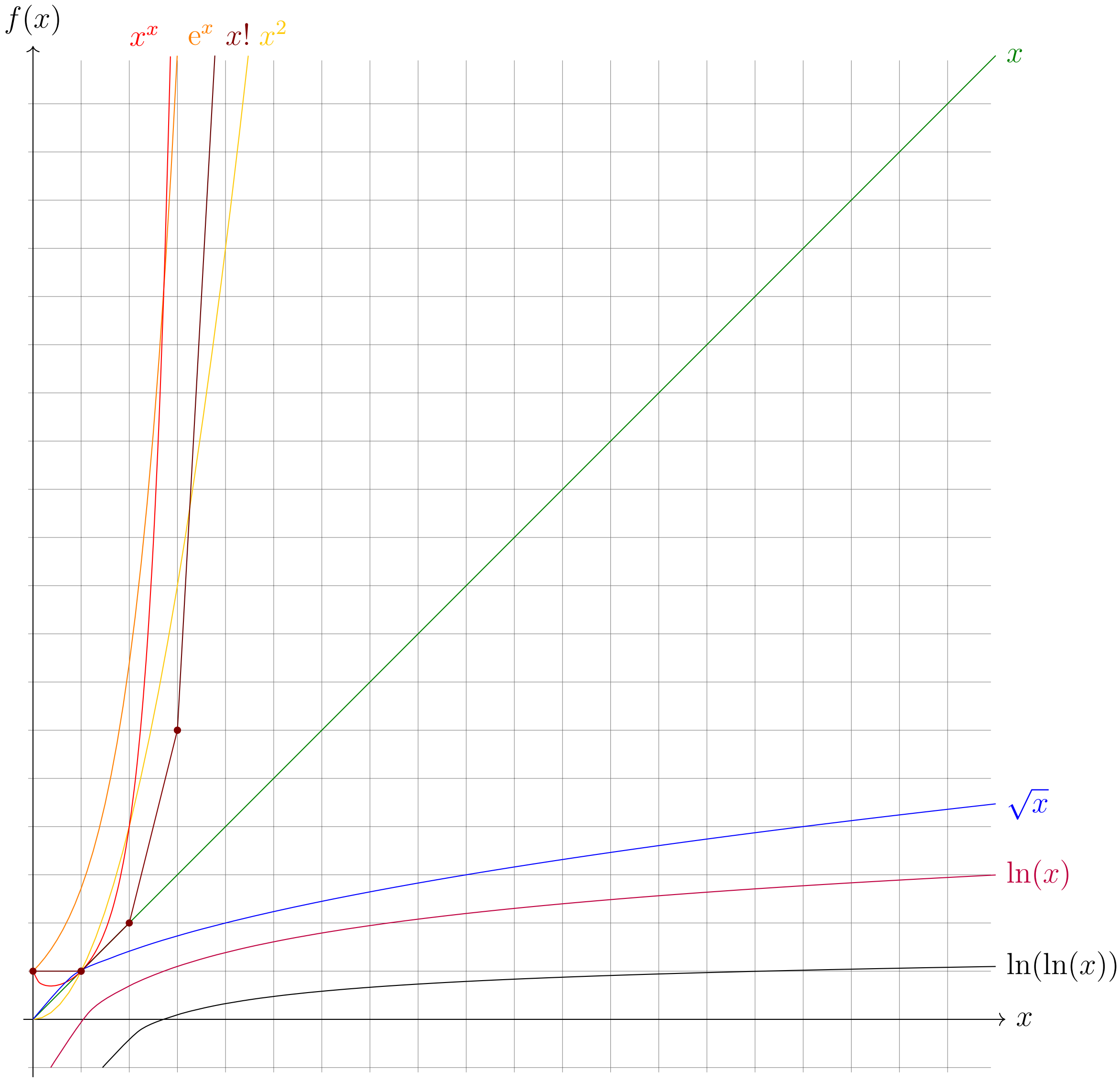
8. Find the area of the common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$.

The End

Hierarchy of Functions

We consider how fast different types of functions approach infinity. Here, assume that $c \in \mathbb{R}^+$ and $x \to \infty$.

$$... < \ln(\ln(x)) < \ln(x) < x^{1/c} < x < x^c < c^x < x! < x^x < x^{x^x} < ...$$



The graph above shows some select examples of functions. Their growth helps us conceptualize certain limits involving their quotients.

Continuity A function $f(x)$ is continuous at $x = a$ if all of the following are true: (i) $f(a)$ exists (ii) $\lim_{x\to a} f(x)$ exists (iii) $\lim_{x\to a} f(x) = f(a)$.

Intermediate Value Theorem If function $f(x)$ is continuous on a closed interval $[a, b]$ and if y is any number between $f(a)$ and $f(b)$, then there is at least one number $c \in [a, b]$ such that $f(c) = y$.

Limit Theorem $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

Squeeze Theorem If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$).

Definition of Derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ or $f'(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

Diff. - Cont. Theorem If a function is differentiable at $x = a$, then it is continuous at $x = a$.

Mean Value Theorem If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one number c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Average Value Theorem The average value of f on $[a, b]$ is: $f_ave = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Derivative of Inverse $g'(c) = \frac{1}{f'(g(c))}$.

Linearization $L(x) = f(a) + f'(a)(x-a)$.

Critical Point A number c in the domain of a function f is a critical point of f if either $f'(c) = 0$ or DNE.

Extrema Critical points that are either a local/absolute max/min.

Point of Inflection Points of inflection of $f(x)$ are the point on the domain of f where $f''(x) = 0$ or DNE and f'' changes sign at that point.

Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f has both an absolute max and absolute min value on $[a, b]$.

Fundamental Theorem of Calculus $\int_a^b f(x)\,dx = F(b) - F(a)$ and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Position $s(t) = \int v(t)\,dt$.

Velocity $v(t) = s'(t)$ or $v(t) = \int a(t)\,dt$.

Acceleration $a(t) = v'(t) = s''(t)$.

Speed Speed $= \|v(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$. Speed increases when $v(t)$ and $a(t)$ have the same sign. Speed decreases when $v(t)$ and $a(t)$ have different signs.

Total Distance Total Distance from time a to b is $\int_a^b |v(t)|\,dt$.

Net Distance (Change) $\int_a^b v(t)\,dt$.

Area in the Plane $A = \int_a^b (f(x) - g(x))\,dx$ where $f \ge g$ on $[a, b]$.

Volume by Disks $V = \pi\int_a^b R^2\,dx$ (horizontal) or $V = \pi\int_c^d R^2\,dy$ (vertical).

Volume by Washers $V = \pi\int_a^b (R^2 - r^2)\,dx$ (horizontal) or $V = \pi\int_c^d (R^2 - r^2)\,dy$ (vertical).

Volume by Cross Sections $V = \pi\int_a^b A(x)\,dx$ (if perpendicular to x-axis) or $V = \pi\int_c^d A(y)\,dy$ (if perpendicular to y-axis).

Arc Length $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$ or $L = \int_c^d \sqrt{1 + [f'(y)]^2}\,dy$.

L'Hospital's Rule If $\lim_{x\to c} \frac{f}{g}$ is of the form $0/0$ or ∞/∞, then $\lim_{x\to c} \frac{f}{g} = \lim_{x\to c} \frac{f'}{g'}$.

Improper Integrals An integral $\int_a^b f(x)\,dx$ is improper if any bound is infinite or f has at least one discontinuity on the interval $[a, b]$. Evaluate using limits.

Parametric Slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Parametric 2nd Deriv. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}(dy/dx)}{dx/dt}$.

Parametric Arc Length $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dx$.

Polar Slope $\frac{dy}{dx} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$.

Polar Area $A = \frac{1}{2}\int_a^b r^2\, d\theta$ or $A = \frac{1}{2}\int_a^b (f(\theta))^2\, d\theta$.

Area Between Polar Curves $A = \frac{1}{2}\int_a^b [R^2 - r^2]\, d\theta$.

Vector Position $s(t) = \langle x(t),\, y(t)\rangle$.

Vector Velocity $v(t) = s'(t) = \langle x'(t),\, y'(t)\rangle$.

Vector Acceleration $a(t) = v'(t) = s''(t) = \langle x''(t),\, y''(t)\rangle$.

Vector Speed $|v(t)| = \sqrt{(x't)^2 + (y'(t))^2}$.

Summary of Convergence Tests

A Quote:
Considering how many fools can calculate, it is surprising that it should be thought either a difficult or tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics – and they are mostly clever fools – seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. **Master these thoroughly, and the rest will follow...**