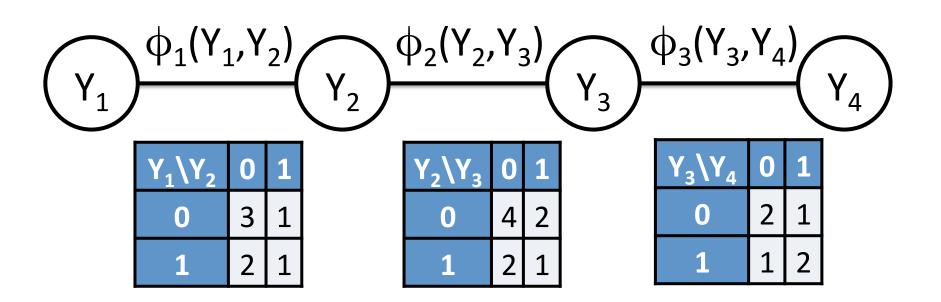
**Example:** Let's trace the variable elimination algorithm with the elimination ordering  $Y_4$ ,  $Y_3$ ,  $Y_2$  for the graph and factors shown below.



#### Step 0: Initialize

$$V = \{Y_4, Y_3, Y_2\}$$

$$\phi_1(Y_1, Y_2) = \begin{cases} \phi_2(Y_2, Y_3) & \phi_3(Y_3, Y_4) \\ Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

$$\phi_3(Y_3, Y_4) = \begin{cases} \phi_3(Y_3, Y_4) & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

$$F_1 = \begin{cases} \phi_3(Y_3, Y_4) & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{cases}$$

$$F_1 = \begin{cases} \phi_1(Y_1, Y_2) & \phi_2(Y_2, Y_3) \\ Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

$$F_1 = \begin{cases} \phi_1(Y_1, Y_2) & \phi_2(Y_2, Y_3) \\ Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

$$\underbrace{ \begin{pmatrix} Y_1 \end{pmatrix}^{\varphi_1(Y_1,Y_2)} \begin{pmatrix} Y_2 \end{pmatrix}^{\varphi_2(Y_2,Y_3)} \begin{pmatrix} Y_3 \end{pmatrix}^{\varphi_3(Y_3,Y_4)} \begin{pmatrix} Y_4 \end{pmatrix}^{\varphi_3(Y_4,Y_4)} \begin{pmatrix} Y_4 \end{pmatrix}^{\varphi_3(Y_4$$

$$V = \{Y_4, Y_3, Y_2\}$$

$$\varphi_3(Y_3, Y_4) \qquad \varphi_1(Y_1, Y_2) \qquad \varphi_2(Y_2, Y_3) \qquad \varphi_2(Y_3, Y_3) \qquad \varphi_2(Y_$$

$$\psi_{1}(Y_{3},Y_{4}) = \prod_{\varphi \in F_{1}} \varphi = \begin{bmatrix} Y_{3} \setminus Y_{4} & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\underbrace{ (Y_1)}^{\varphi_1(Y_1,Y_2)} \underbrace{ (Y_2)}^{\varphi_2(Y_2,Y_3)} \underbrace{ (Y_3)}^{\varphi_2(Y_2,Y_3)} \underbrace{ (Y_3)}^{\varphi_3(Y_3,Y_4)} \underbrace{ (Y_4)}^{\varphi_3(Y_3,Y_4)} \underbrace{ (Y_4)}^{\varphi_3(Y_4,Y_4)} \underbrace{ (Y_4)}^{\varphi_3(Y_4$$

$$V = \{Y_4, Y_3, Y_2\}$$

$$\phi_3(Y_3, Y_4)$$

$$F_1 = \{Y_1 \setminus Y_2 \mid 0 \mid 1 \}$$

$$0 \mid 2 \mid 1 \}$$

$$1 \mid 1 \mid 2 \mid 1 \}$$

$$F'_1 = \{Y_1 \setminus Y_2 \mid 0 \mid 1 \}$$

$$0 \mid 4 \mid 2 \mid 1 \}$$

$$\tau_{1}(Y_{3}) = \sum_{y_{4}} \psi_{1}(Y_{3}, y_{4}) = \sum_{y_{4}} \begin{bmatrix} Y_{3} \setminus Y_{4} & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} Y_{3} & 0 \\ 0 & 3 \\ 1 & 3 \end{bmatrix}$$

$$\underbrace{ \begin{pmatrix} Y_1 \end{pmatrix}} \underbrace{ \varphi_1(Y_1, Y_2)} \underbrace{ \begin{pmatrix} Y_2 \end{pmatrix}} \underbrace{ \varphi_2(Y_2, Y_3)} \underbrace{ \begin{pmatrix} Y_3 \end{pmatrix}} \underbrace{ \begin{pmatrix} Y_3, Y_4 \end{pmatrix}} \underbrace{ \begin{pmatrix} Y_4 \end{pmatrix}$$

$$V = \{Y_4, Y_3, Y_2\}$$

$$F_1 = \{Y_4, Y_3, Y_4\} = \{Y_1, Y_2\} = \{Y_1,$$

$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_5) \xrightarrow{\varphi_3(Y_4, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) (Y_$$

$$V = \{Y_3, Y_2\}$$

$$\phi_1(Y_1, Y_2) \qquad \phi_2(Y_2, Y_3) \qquad \tau_1(Y_3)$$

$$V_1 \setminus Y_2 \quad 0 \quad 1 \qquad Y_2 \setminus Y_3 \quad 0 \quad 1 \qquad Y_3 \quad 0$$

$$0 \quad 3 \quad 1 \quad 0 \quad 4 \quad 2 \quad 0 \quad 3$$

$$1 \quad 2 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 3$$

$$F_2 = \{Y_3, Y_2\}$$

$$V_1 \setminus Y_2 \quad 0 \quad 1 \qquad Y_3 \quad 0$$

$$V_2 \setminus Y_3 \quad 0 \quad 1 \qquad Y_3 \quad 0$$

$$V_2 \setminus Y_3 \quad 0 \quad 1 \qquad Y_3 \quad 0$$

$$V_2 \setminus Y_3 \quad 0 \quad 1 \qquad Y_3 \quad 0$$

$$V_2 \setminus Y_3 \quad 0 \quad 1 \qquad 0 \quad 3 \quad 1$$

$$V_3 \quad 0 \quad 1 \quad 0 \quad 3 \quad 1$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_2 \setminus Y_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 0 \quad 3 \quad 1$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_2 \setminus Y_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 1$$

$$V_4 \setminus Y_2 \quad 0 \quad 1$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_2 \setminus Y_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 0 \quad 3$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_2 \setminus Y_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 0 \quad 3$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_2 \setminus Y_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 0 \quad 3$$

$$V_1 \setminus Y_2 \quad 0 \quad 1$$

$$V_2 \setminus Y_3 \quad 0 \quad 1$$

$$V_3 \quad 0 \quad 0 \quad 0 \quad 0$$

$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_5) \xrightarrow{\varphi_3(Y_4, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) (Y_$$

 $\phi \epsilon F_2$ 

$$V = \{Y_3, Y_2\}$$

$$F_2 = \begin{cases} \phi_2(Y_2, Y_3) & \tau_1(Y_3) \\ 0 & 4 & 2 \\ 1 & 2 & 1 \end{cases}$$

$$Y_3 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

$$F'_2 = \begin{cases} \phi_1(Y_1, Y_2) \\ Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

$$\psi_2(Y_2, Y_3) = \prod \Phi = \begin{cases} \phi_2(Y_2, Y_3) & \tau_1(Y_3) \\ 0 & 4 & 2 \\ 0 & 3 & 1 \end{cases}$$

$$V = \begin{cases} \phi_2(Y_2, Y_3) & \tau_1(Y_3) \\ Y_2 \setminus Y_3 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 1 \end{cases}$$

$$\underbrace{ \begin{pmatrix} \varphi_1(Y_1, Y_2) \\ Y_1 \end{pmatrix}} \underbrace{ \begin{pmatrix} \varphi_2(Y_2, Y_3) \\ Y_2 \end{pmatrix}} \underbrace{ \begin{pmatrix} \varphi_2(Y_2, Y_3) \\ Y_3 \end{pmatrix}} \underbrace{ \begin{pmatrix} \varphi_3(Y_3, Y_4) \\ Y_3 \end{pmatrix}}$$

$$V = \{Y_3, Y_2\}$$

$$F_{2} = \begin{cases} \phi_{2}(Y_{2}, Y_{3}) & \tau_{1}(Y_{3}) \\ Y_{2} \setminus Y_{3} & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 3 \end{cases}$$

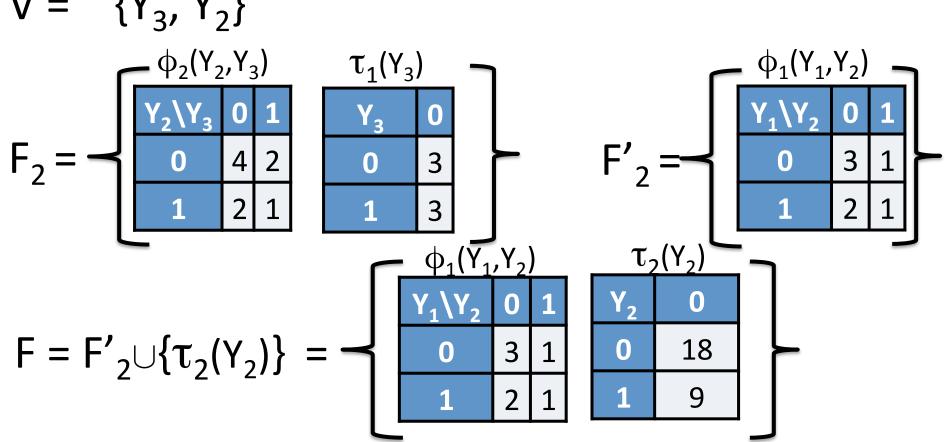
 $\tau_2(Y_2) = \sum \psi_2(Y_2, y_3) = \sum$ 

6

 $\phi_1(Y_1,Y_2)$ 

$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_4, Y_4)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) (Y_$$

$$V = \{Y_3, Y_2\}$$



$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) ($$

$$V = \{Y_{2}\}$$

$$\varphi_{1}(Y_{1},Y_{2}) \qquad \tau_{2}(Y_{2})$$

$$Y_{1}\setminus Y_{2} \qquad 0 \qquad 1$$

$$0 \qquad 3 \qquad 1$$

$$1 \qquad 2 \qquad 1 \qquad 1 \qquad 9$$

$$F_{3} = \{Y_{2}\}$$

$$\varphi_{1}(Y_{1},Y_{2}) \qquad \tau_{2}(Y_{2})$$

$$Y_{2} \qquad 0$$

$$Y_{1}\setminus Y_{2} \qquad 0 \qquad 1$$

$$0 \qquad 3 \qquad 1$$

$$1 \qquad 2 \qquad 1 \qquad 1 \qquad 9$$

$$F'_{3} = \{Y_{2}\}$$

$$F'_{3} = \{Y_{1},Y_{2}\}$$

$$Y_{1}\setminus Y_{2} \qquad 0 \qquad 1$$

$$Y_{2} \qquad 0$$

$$Y_{3} \qquad Y_{2} \qquad 0$$

$$Y_{1}\setminus Y_{2} \qquad 0 \qquad 1$$

$$Y_{2} \qquad 0$$

$$Y_{3} \qquad Y_{4} \qquad Y_{5} \qquad 0$$

$$Y_{5} \qquad Y_{7} \qquad Y_{7} \qquad 0$$

$$Y_{1}\setminus Y_{2} \qquad 0 \qquad 1$$

$$Y_{2} \qquad 0$$

$$Y_{3} \qquad Y_{4} \qquad Y_{5} \qquad 0$$

$$Y_{5} \qquad Y_{7} \qquad Y_{7} \qquad 0$$

$$Y_{1}\setminus Y_{2} \qquad 0 \qquad 1$$

$$Y_{2} \qquad 0$$

$$Y_{3} \qquad Y_{5} \qquad 0$$

$$Y_{4} \qquad Y_{5} \qquad 0$$

$$Y_{5} \qquad Y_{5} \qquad Y_{5} \qquad 0$$

$$Y_{5} \qquad Y_{5} \qquad Y_$$

$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) ($$

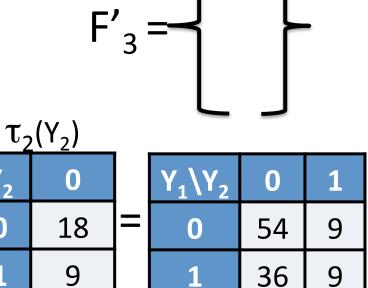
$$V = \{Y_2\}$$

$$F_{3} = \begin{cases} \phi_{1}(Y_{1}, Y_{2}) \\ Y_{1} \setminus Y_{2} & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

 $\psi_3(Y_1,Y_2) = \prod \Phi =$ 

 $\phi \epsilon F_3$ 

$$\tau_{2}(Y_{2})$$
 $Y_{2}$ 
 $0$ 
 $0$ 
 $18$ 
 $1$ 
 $9$ 
 $\phi_{1}(Y_{1},Y_{2})$ 



$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_3) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_4) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_5) \xrightarrow{\varphi_3(Y_5, Y_5)} (Y_5) (Y_$$

1

X

0

3

$$V = \{Y_2\}$$

$$F_{3} = \begin{cases} \phi_{1}(Y_{1}, Y_{2}) \\ Y_{1} \setminus Y_{2} & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{cases}$$

| $\tau_2(Y_2)$  |    |  |
|----------------|----|--|
| Y <sub>2</sub> | 0  |  |
| 0              | 18 |  |
| 1              | 9  |  |
|                |    |  |

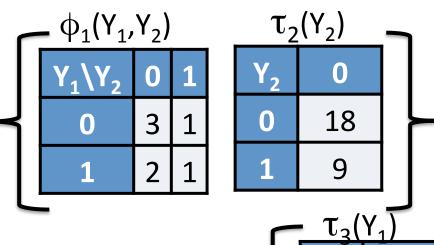
$$F'_3 = \left\{ \right\}$$

$$\tau_3(Y_1) = \sum_{y_2} \psi_3(Y_1, y_2) = \sum_{y_2}$$

$$Y_1 \setminus Y_2 = 0$$
 1  $Y_1 = 0$ 
1 36 9 1

$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2) \xrightarrow{\varphi_2(Y_2, Y_3)} (Y_2) \xrightarrow{\varphi_3(Y_3, Y_4)} (Y_2)$$

$$V = \{Y_2\}$$



$$F'_3 = \left\{ \right\}$$

$$F = F'_3 \cup \{\tau_3(Y_1)\} = \begin{cases} Y_2 & 0 \\ 0 & 63 \\ 1 & 45 \end{cases}$$

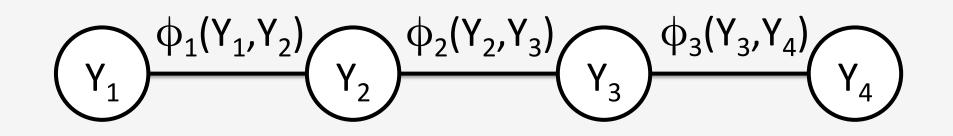
$$(Y_1) \xrightarrow{\varphi_1(Y_1, Y_2)} (Y_2, Y_3) (Y_2, Y_3) (Y_3, Y_4) (Y_1) (Y_1) (Y_1) (Y_2, Y_3) (Y_2, Y_3) (Y_2, Y_3) (Y_3, Y_4) (Y_4) (Y_4, Y_5) (Y_5, Y_5) (Y_5,$$

#### Step 4: Return

$$P(Y_1=1) = \frac{\tau_3(1)}{\tau_3(0) + \tau_3(1)} = 0.4166$$

$$\underbrace{ (Y_1) } \bigoplus_{ \Phi_1(Y_1, Y_2) } \bigoplus_{ \Phi_2(Y_2, Y_3) } \bigoplus_{ \Phi_3(Y_3, Y_4) } \bigoplus_{ \Phi_3(Y_3,$$

**Question:** What happens to the complexity of the algorithm if we start by eliminating  $Y_3$  instead of  $Y_4$ ?



$$V = \{Y_3, Y_4, Y_2\}$$

$$F_0 = \{ \phi_1(Y_1, Y_2), \phi_2(Y_2, Y_3), \phi_3(Y_3, Y_4) \}$$

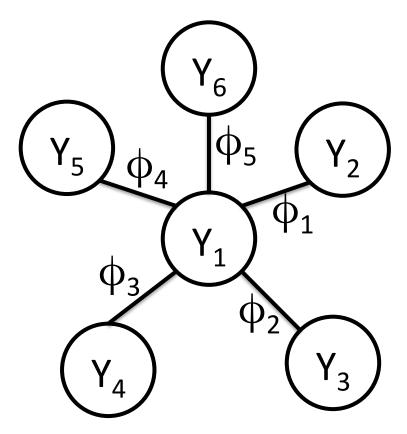
$$F_1 = \{ \phi_2(Y_2, Y_3), \phi_3(Y_3, Y_4) \}$$
  $F'_1 = \{ \phi_1(Y_1, Y_2) \}$ 

$$\psi_1(Y_2,Y_3,Y_4) = \prod_{\phi \in F_1} \phi = \phi_2(Y_2,Y_3) \times \phi_3(Y_3,Y_4)$$

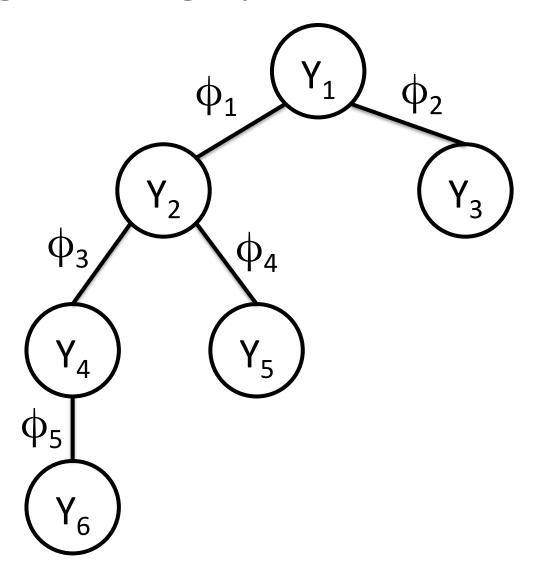
$$\tau_1(Y_2, Y_4) = \sum_{y_3} \psi_1(Y_2, y_3, Y_4) \quad F = \{\phi_1(Y_1, Y_2), \tau_1(Y_2, Y_4)\}$$

$$\underbrace{ (Y_1)}^{\varphi_1(Y_1,Y_2)} \underbrace{ (Y_2)}^{\varphi_2(Y_2,Y_3)} \underbrace{ (Y_3)}^{\varphi_2(Y_2,Y_3)} \underbrace{ (Y_3)}^{\varphi_3(Y_3,Y_4)} \underbrace{ (Y_4)}^{\varphi_3(Y_3,Y_4)} \underbrace{ (Y_4)}^{\varphi_3(Y_4,Y_4)} \underbrace{ (Y_4)}^{\varphi_3(Y_4$$

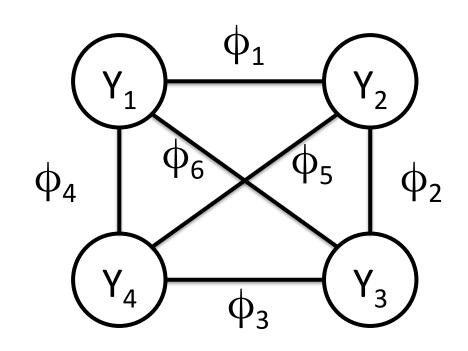
**Question:** What is the worst elimination ordering for this graph? What is an optimal ordering?



**Question:** What is an optimal elimination ordering for this graph?



**Question:** Is there any efficient elimination ordering for the following graph:

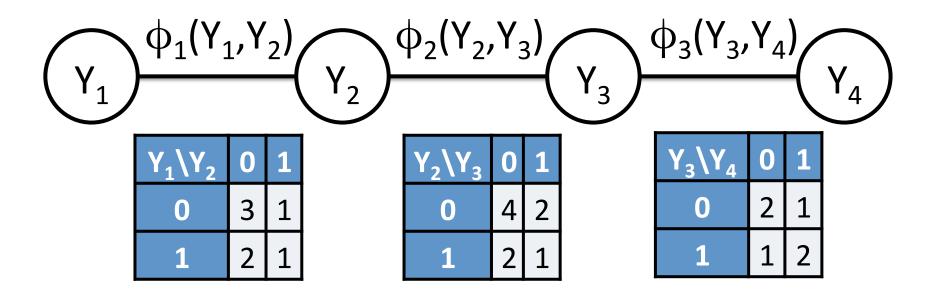


For any elimination ordering we have:

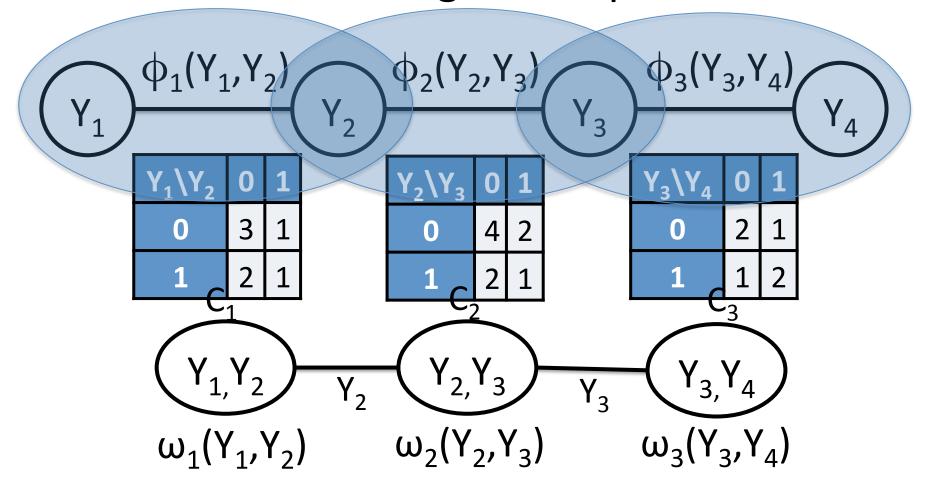
$$F_1 = \{ \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \} \quad F'_1 = \{ \}$$

$$\psi_1(Y_1, Y_2, Y_3, Y_4) = \prod \varphi = \varphi_1 \times \varphi_2 \times \varphi_3 \times \varphi_4 \times \varphi_5 \times \varphi_6$$

**Example:** Let's trace the sum-product algorithm for the graph and factors shown below.

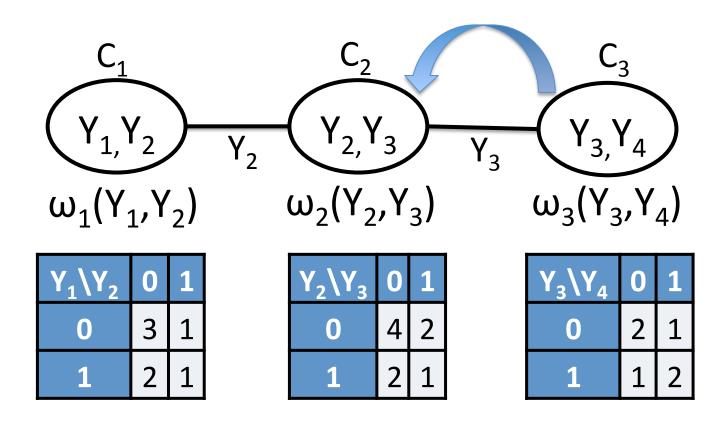


#### **Sum-Product:** Forming the Clique Tree



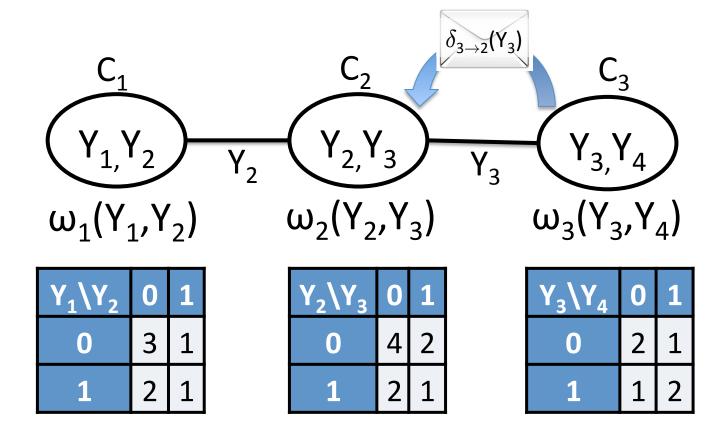
### **Sum-Product Algorithm:** 3→2 Message

$$\delta_{3\rightarrow 2}(Y_3) \leftarrow \sum_{Y_4} \omega_3(Y_3, Y_4)$$



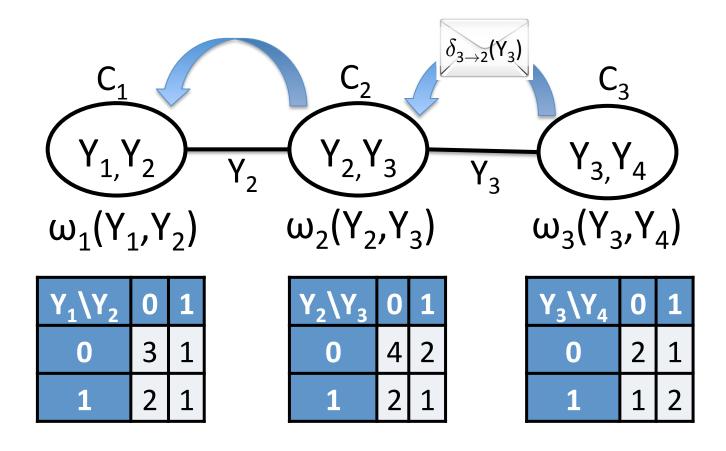
### **Sum-Product Algorithm:** 3→2 Message

$$\delta_{3\to 2}(Y_3) \leftarrow \sum_{y_4} \begin{vmatrix} Y_3 \setminus Y_4 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} Y_3 & 0 \\ 0 & 3 \\ 1 & 3 \end{vmatrix}$$



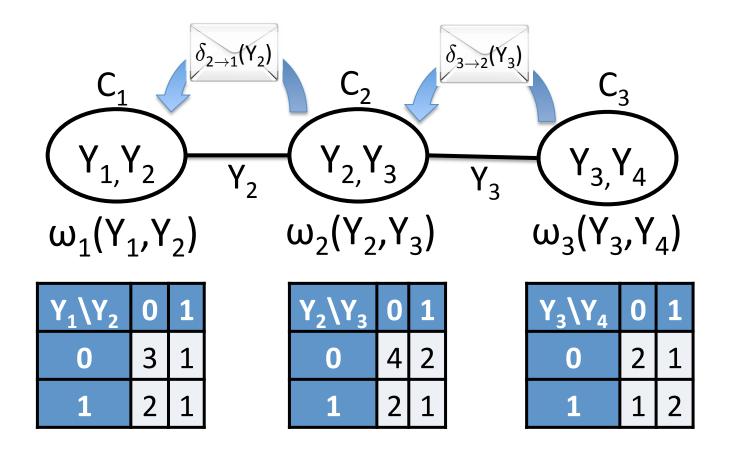
### **Sum-Product Algorithm:** 2→1 Message

$$\delta_{2\rightarrow 1}(Y_2) \leftarrow \sum_{Y_3} \omega_2(Y_2, Y_3) \delta_{3\rightarrow 2}(Y_3)$$



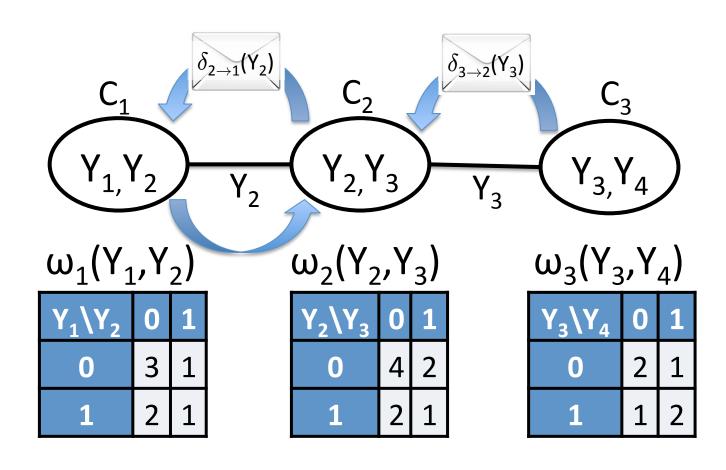
## **Sum-Product Algorithm:** $2 \rightarrow 1$ Message

$$\delta_{2\to 1}(Y_2) \leftarrow \sum_{y_3} \begin{bmatrix} Y_2 \setminus Y_3 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} Y_3 & 0 \\ 0 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} Y_2 & 0 \\ 0 & 18 \\ 1 & 9 \end{bmatrix}$$



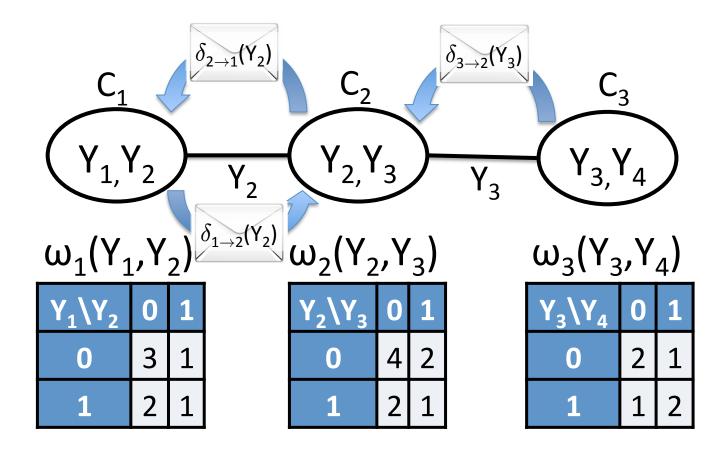
### **Sum-Product Algorithm:** $1\rightarrow 2$ Message

$$\delta_{1\rightarrow 2}(Y_2) \leftarrow \sum_{Y_1} \omega_1(Y_1, Y_2)$$



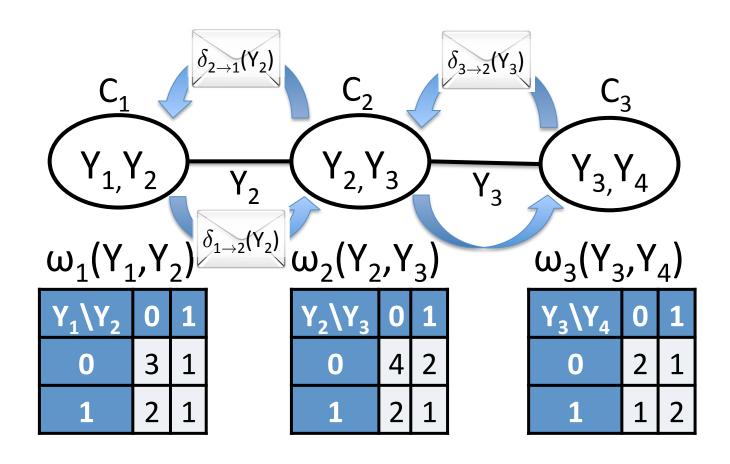
### **Sum-Product Algorithm:** $1\rightarrow 2$ Message

$$\delta_{1\to 2}(Y_2) \leftarrow \sum_{y_1} \begin{vmatrix} Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} Y_2 & 0 \\ 0 & 5 \\ 1 & 2 \end{vmatrix}$$



### **Sum-Product Algorithm:** 2→3 Message

$$\delta_{2\to3}(Y_3) \leftarrow \sum_{Y_2} \omega_2(C_2) \delta_{1\to2}(Y_2)$$



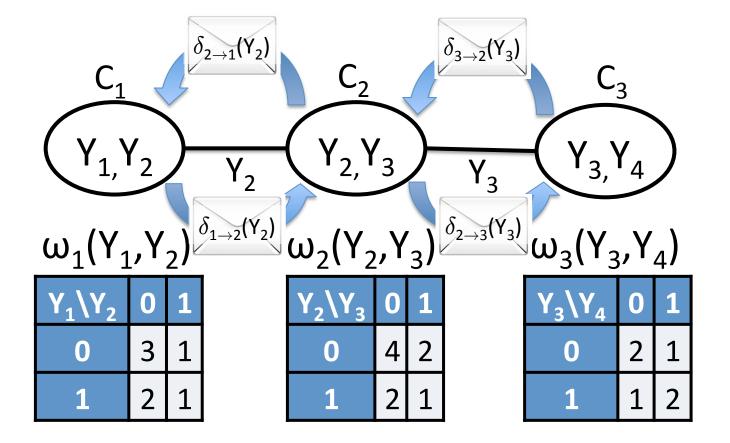
### **Sum-Product Algorithm:** 2→3 Message

$$\delta_{2\rightarrow 3}(Y_3) \leftarrow \sum_{y_2}$$

| $Y_2 \setminus Y_3$ | 0 | 1 |
|---------------------|---|---|
| 0                   | 4 | 2 |
| 1                   | 2 | 1 |

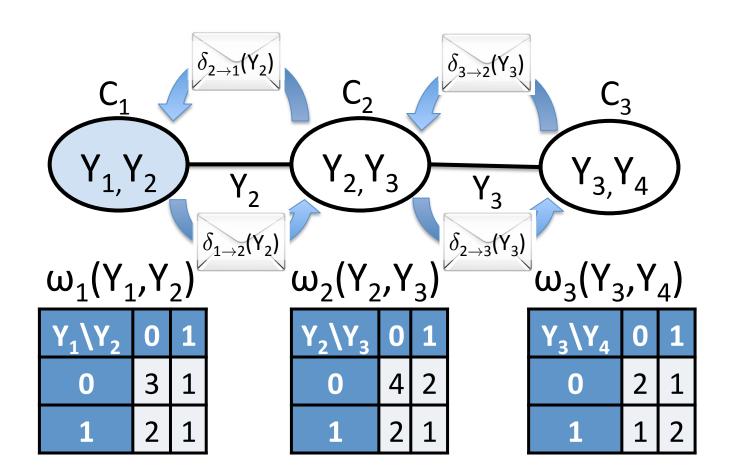
|   | Y <sub>2</sub> | 0 |  |
|---|----------------|---|--|
| • | 0              | 5 |  |
|   | 1              | 2 |  |

| Y <sub>3</sub> | 0  |
|----------------|----|
| 0              | 24 |
| 1              | 12 |



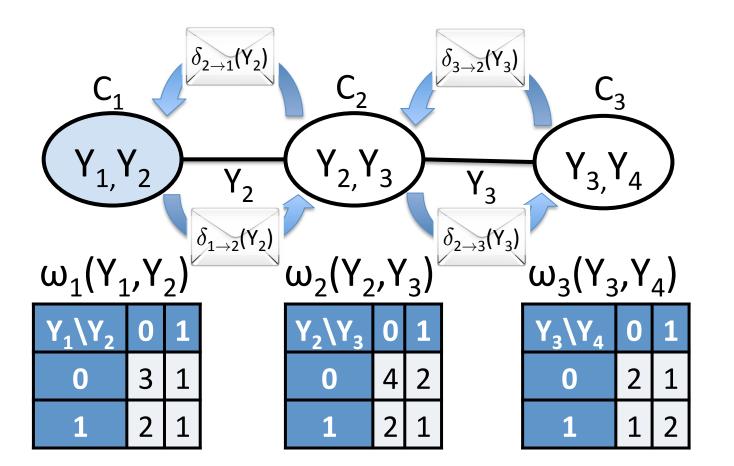
## **Sum-Product Algorithm:** C<sub>1</sub>Belief Read-Out

$$\beta_1(Y_1,Y_2) \leftarrow \omega_1(Y_1,Y_2) \delta_{2\rightarrow 1}(Y_2)$$



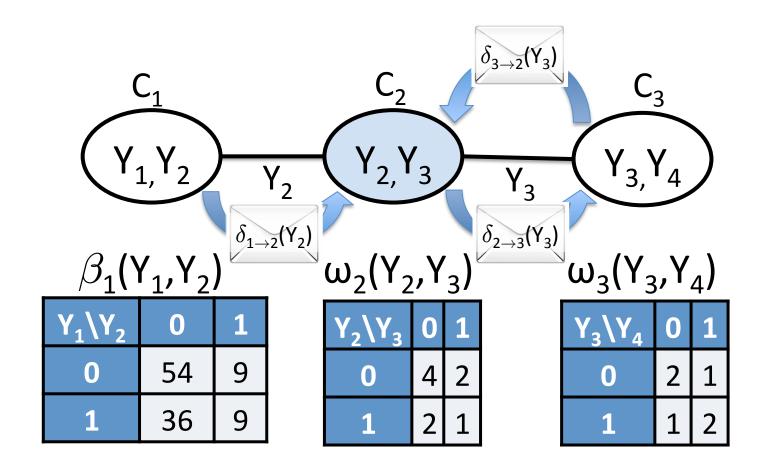
## **Sum-Product Algorithm:** C<sub>1</sub>Belief Read-Out

$$\beta_1(Y_1, Y_2) \leftarrow \begin{bmatrix} Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} Y_2 & 0 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 54 & 9 \\ 1 & 36 & 9 \end{bmatrix}$$



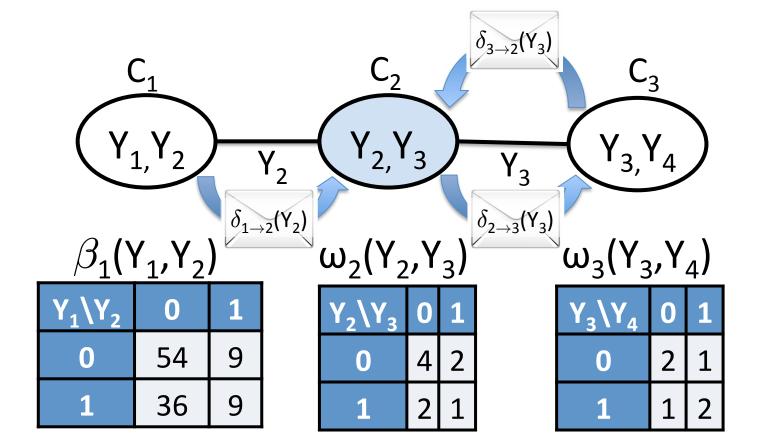
# Sum-Product Algorithm: C<sub>2</sub> Belief Read-Out

$$\beta(Y_2,Y_3) \leftarrow \omega_2(Y_2,Y_3) \delta_{1\rightarrow 2}(Y_2) \delta_{3\rightarrow}(Y_3)$$



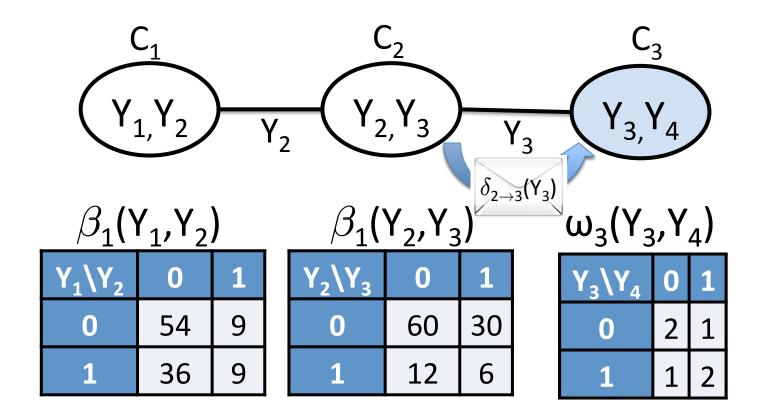
# Sum-Product Algorithm: C<sub>2</sub> Belief Read-Out

$$\beta(Y_2, Y_3) \leftarrow \begin{bmatrix} Y_2 \setminus Y_3 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} Y_2 & 0 \\ 0 & 5 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} Y_3 & 0 \\ 0 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} Y_2 \setminus Y_3 & 0 & 1 \\ 0 & 60 & 30 \\ 1 & 12 & 6 \end{bmatrix}$$

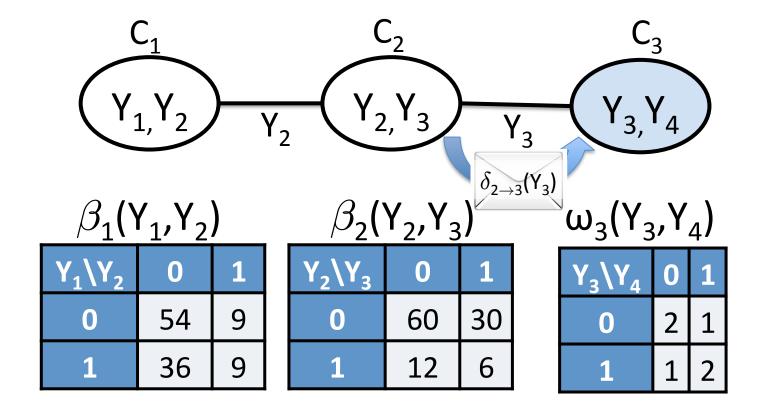


## **Sum-Product Algorithm:** C<sub>3</sub> Belief Read-Out

$$\beta_3(Y_3,Y_4) \leftarrow \omega_3(Y_3,Y_4) \delta_{2\rightarrow 3}(Y_4)$$

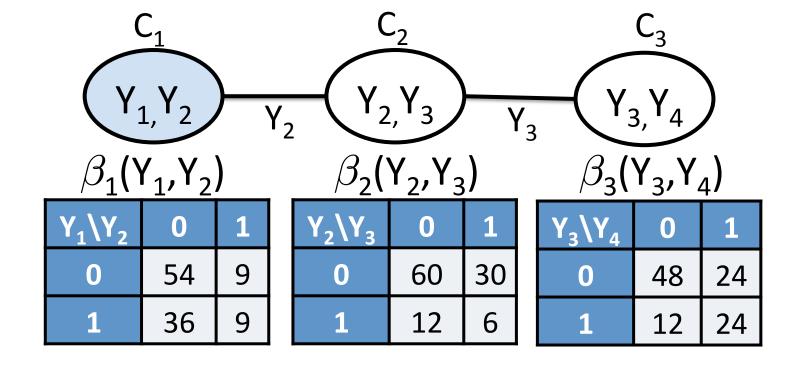


## **Sum-Product Algorithm:** C<sub>3</sub> Belief Read-Out



### Sum-Product Algorithm: C<sub>1</sub> Probabilities

$$P(Y_1,Y_2) = \frac{\beta_1(Y_1,Y_2)}{\sum_{y_1} \sum_{y_2} \beta_1(y_1,y_2)} = \begin{bmatrix} Y_1 \setminus Y_2 & 0 & 1 \\ 0 & 0.5000 & 0.0833 \\ 1 & 0.3333 & 0.0833 \end{bmatrix} P(Y_1=1) = 0.4166$$

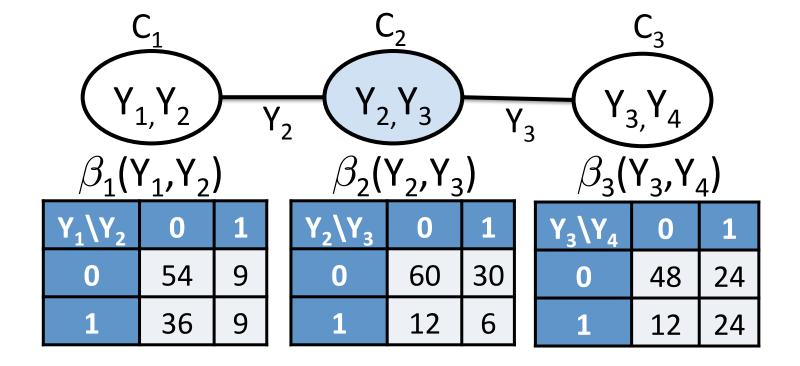


## Sum-Product Algorithm: C<sub>2</sub> Probabilities

$$P(Y_2, Y_3) = \frac{\beta_2(Y_2, Y_3)}{\sum_{y_2 \ y_3}} = \frac{Y_2 \setminus Y_3}{\sum_{y_2 \ y_3}} = \frac{0}{0.5555} = \frac{1}{0.1111} = 0.1666$$

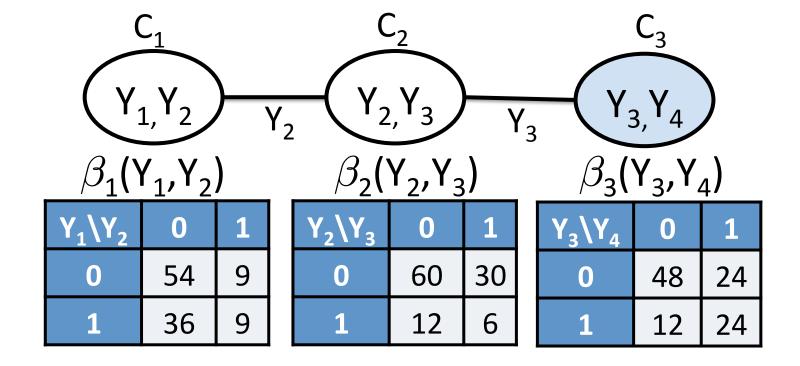
$$P(Y_2 = 1) = 0.1666$$

$$P(Y_3 = 1) = 0.3333$$



## **Sum-Product Algorithm:** C<sub>3</sub> Probabilities

$$P(Y_3, Y_4) = \frac{\beta_3(Y_3, Y_4)}{\sum_{y_3} \beta_3(y_3, y_4)} = \begin{bmatrix} Y_3 \setminus Y_4 & 0 & 1 \\ 0 & 0.4444 & 0.2222 \\ 1 & 0.1111 & 0.2222 \end{bmatrix} P(Y_3 = 1) = 0.3333$$



#### Sum-Product: Algorithm Overview

$$\delta_{3\to2}(Y_3) \leftarrow \sum_{\substack{Y_4 \\ Y_2}} \omega_3(Y_3, Y_4) \qquad \delta_{1\to2}(Y_2) \leftarrow \sum_{\substack{Y_1 \\ Y_2}} \omega_1(Y_1, Y_2)$$

$$\delta_{2\to1}(Y_2) \leftarrow \sum_{\substack{Y_2 \\ Y_3}} \omega_2(Y_2, Y_3) \delta_{3\to2}(Y_3) \qquad \delta_{2\to3}(Y_3) \leftarrow \sum_{\substack{Y_2 \\ Y_2}} \omega_2(C_2) \delta_{1\to2}(Y_2)$$

$$\beta_{1}(Y_{1},Y_{2}) \leftarrow \omega_{1}(Y_{1},Y_{2}) \, \delta_{2\to 1}(Y_{2})$$

$$\beta_{2}(Y_{2},Y_{3}) \leftarrow \omega_{2}(Y_{2},Y_{3}) \, \delta_{1\to 2}(Y_{2}) \, \delta_{3\to}(Y_{3}) \qquad P(Y_{i},Y_{i+1}) = \frac{\beta_{i}(Y_{i},Y_{i+1})}{\sum_{Y_{i}} \sum_{Y_{i+1}} \beta_{i}(y_{i},y_{i+1})}$$

$$\beta_{3}(Y_{3},Y_{4}) \leftarrow \omega_{3}(Y_{3},Y_{4}) \, \delta_{2\to 3}(Y_{4})$$

