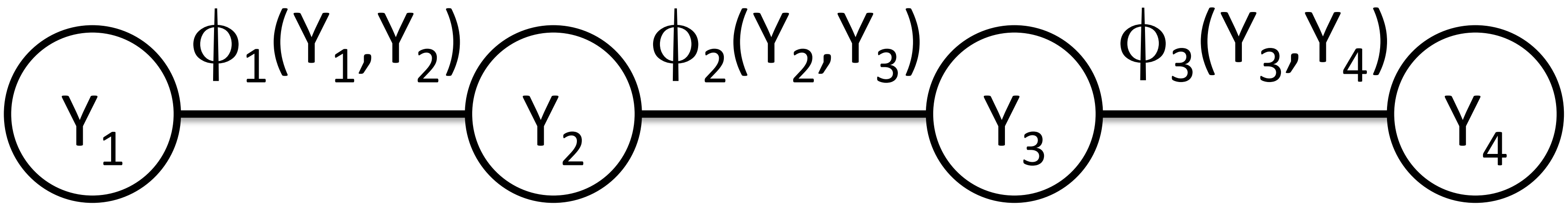
Example: Let's trace the variable elimination algorithm with the elimination ordering Y_4, Y_3, Y_2 for the graph and factors shown below.



$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Step 0: Initialize

$V = \{Y_4, Y_3, Y_2\}$

$F_0 = F =$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\phi_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

$F_1 =$

$\phi_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

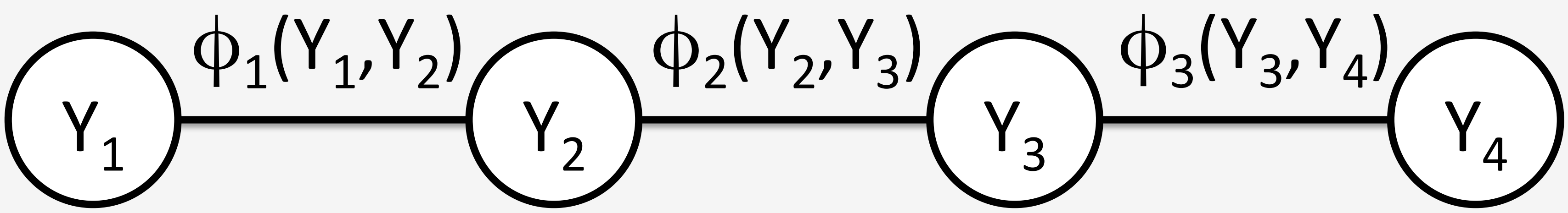
$F'_1 =$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1



Step 1: Eliminate Y_4

$V = \{Y_4, Y_3, Y_2\}$

$F_1 = \{\ \phi_3(Y_3,Y_4)\ \}$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

$F'_1 = \{\ \phi_1(Y_1,Y_2),\ \phi_2(Y_2,Y_3)\ \}$

$\phi_1(Y_1,Y_2)$

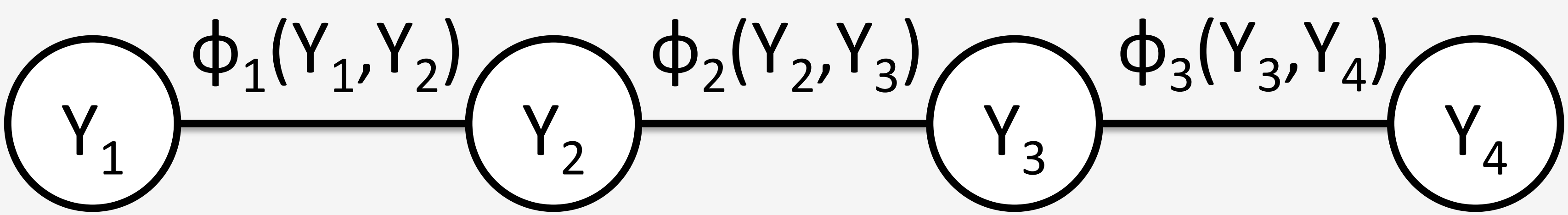
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$$\psi_1(Y_3,Y_4) = \prod_{\phi \,\varepsilon\, F_1} \phi =$$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2



Step 1: Eliminate Y_4

$V = \{Y_4, Y_3, Y_2\}$

$F_1 = \{$ $\phi_3(Y_3, Y_4)$ $\}$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

$F'_1 = \{$ $\phi_1(Y_1, Y_2)$, $\phi_2(Y_2, Y_3)$ $\}$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

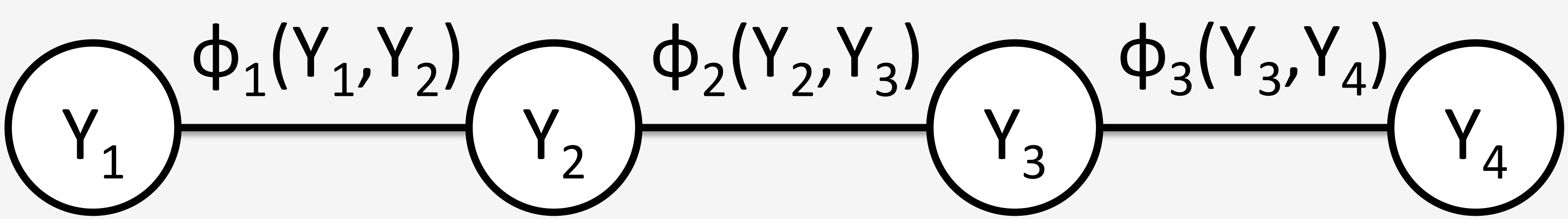
$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$$\tau_1(Y_3) = \sum_{y_4} \psi_1(Y_3, y_4) = \sum_{y_4}$$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

$=$

Y_3	0
0	3
1	3



Step 1: Eliminate Y_4

$V = \{Y_4, Y_3, Y_2\}$

$F_1 = \{ \phi_3(Y_3,Y_4) \}$

$\phi_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

$F'_1 = \{ \phi_1(Y_1,Y_2), \phi_2(Y_2,Y_3) \}$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$F = F'_1 \cup \{\tau_1(Y_3)\} = \{ \phi_1(Y_1,Y_2), \phi_2(Y_2,Y_3), \tau_1(Y_3) \}$

$\phi_1(Y_1,Y_2)$

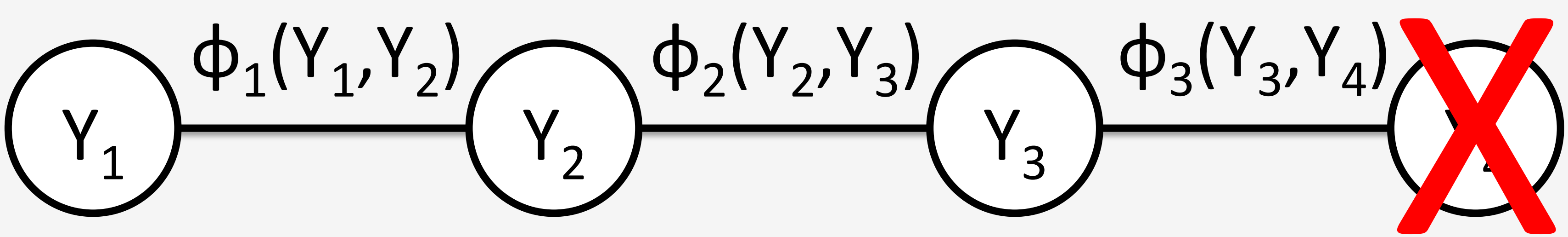
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\tau_1(Y_3)$

Y_3	0
0	3
1	3



Step 1: Eliminate Y_3

$V = \{Y_3, Y_2\}$

$F = \{$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\tau_1(Y_3)$

Y_3	0
0	3
1	3

$\}$

$F_2 = \{$

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\tau_1(Y_3)$

Y_3	0
0	3
1	3

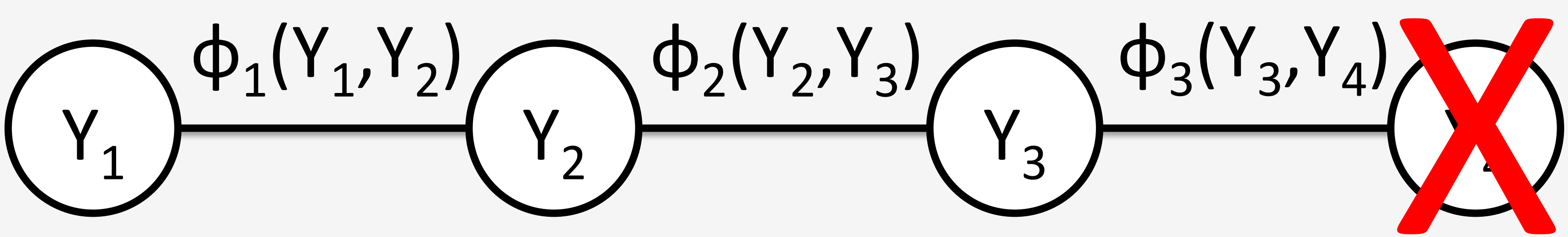
$\}$

$F'_2 = \{$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\}$



Step 2: Eliminate Y_3

$V = \{Y_3, Y_2\}$

$F_2 = \{$

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\tau_1(Y_3)$

Y_3	0
0	3
1	3

$\}$

$F'_2 = \{$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\}$

$$\psi_2(Y_2,Y_3) = \prod_{\phi \,\varepsilon\, F_2} \phi =$$

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

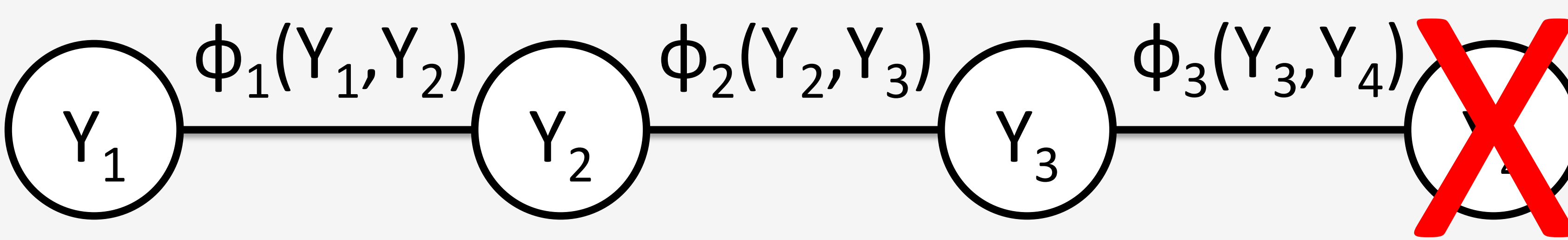
$\times$

$\tau_1(Y_3)$

Y_3	0
0	3
1	3

$=$

$Y_2 \backslash Y_3$	0	1
0	12	6
1	6	3



Step 2: Eliminate Y_3

$V = \{Y_3, Y_2\}$

$F_2 = \{$

$\phi_2(Y_2, Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\tau_1(Y_3)$

Y_3	0
0	3
1	3

$\}$

$F'_2 = \{$

$\phi_1(Y_1, Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

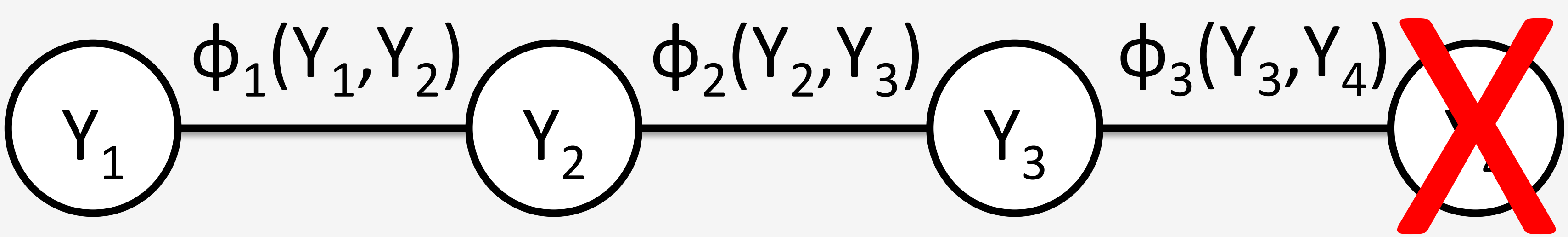
$\}$

$$\tau_2(Y_2) = \sum_{y_3} \psi_2(Y_2, y_3) = \sum_{y_3}$$

$Y_2 \backslash Y_3$	0	1
0	12	6
1	6	3

$=$

Y_2	0
0	18
1	9



Step 2: Eliminate Y_3

$V = \{Y_3, Y_2\}$

$F_2 = \{\phi_2(Y_2,Y_3), \tau_1(Y_3)\}$

$\phi_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\tau_1(Y_3)$

Y_3	0
0	3
1	3

$F'_2 = \{\phi_1(Y_1,Y_2)\}$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

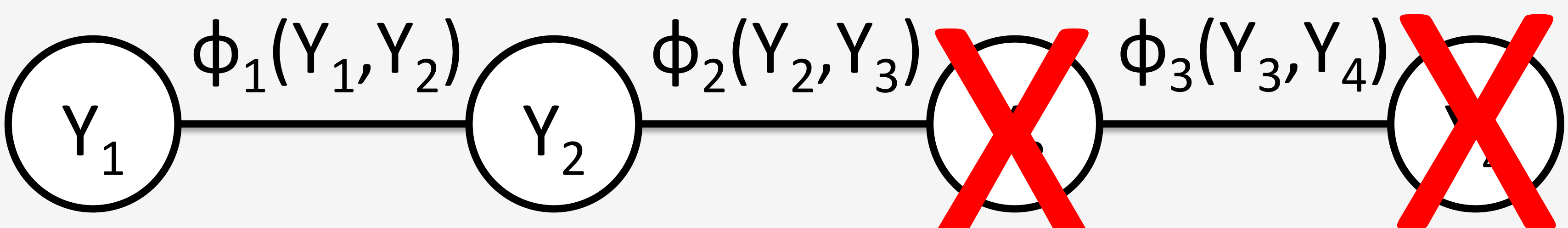
$F = F'_2 \cup \{\tau_2(Y_2)\} = \{\phi_1(Y_1,Y_2), \tau_2(Y_2)\}$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\tau_2(Y_2)$

Y_2	0
0	18
1	9



Step 3: Eliminate Y_2

$V = \{Y_2\}$

$F =$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\tau_2(Y_2)$

Y_2	0
0	18
1	9

$F_3 =$

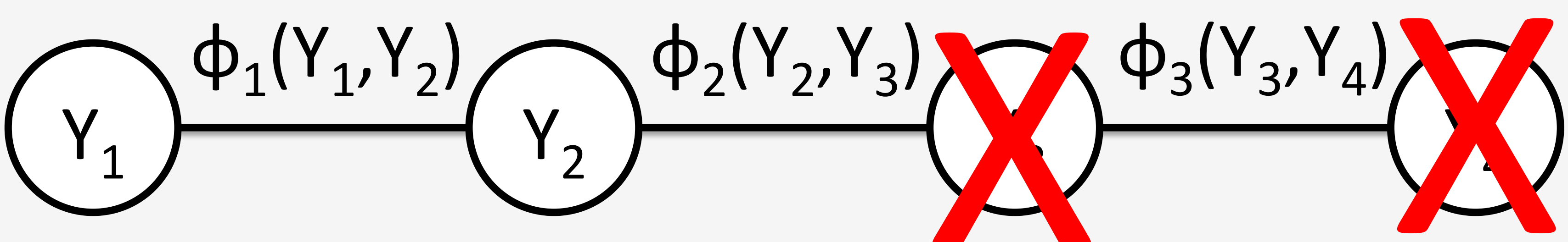
$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\tau_2(Y_2)$

Y_2	0
0	18
1	9

$F'_3 = \{ \quad \}$



Step 3: Eliminate Y_2

$V = \{Y_2\}$

$F_3 = \{\ \phi_1(Y_1, Y_2),\ \tau_2(Y_2)\ \}$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\tau_2(Y_2)$

Y_2	0
0	18
1	9

$F'_3 = \{\ \}$

$$\psi_3(Y_1,Y_2) = \prod_{\phi\, \varepsilon\, F_3} \phi = \phi_1(Y_1,Y_2) \times \tau_2(Y_2) =$$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

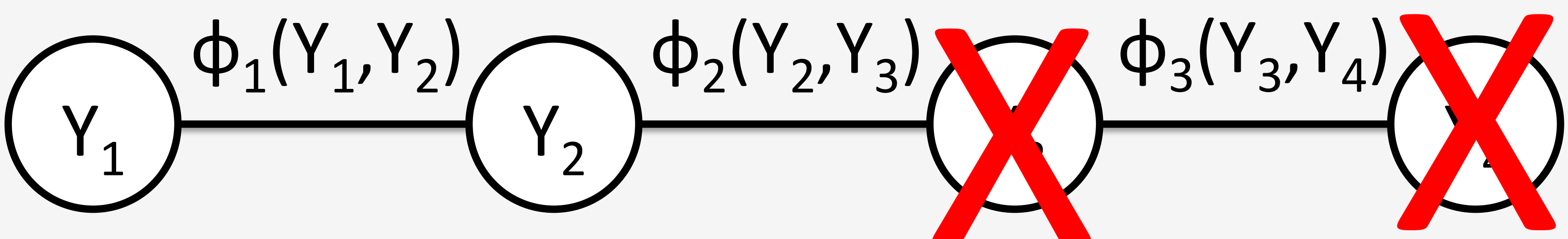
×

$\tau_2(Y_2)$

Y_2	0
0	18
1	9

=

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9



Step 3: Eliminate Y_2

$V = \{Y_2\}$

$F_3 = \{$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\tau_2(Y_2)$

Y_2	0
0	18
1	9

$\}$

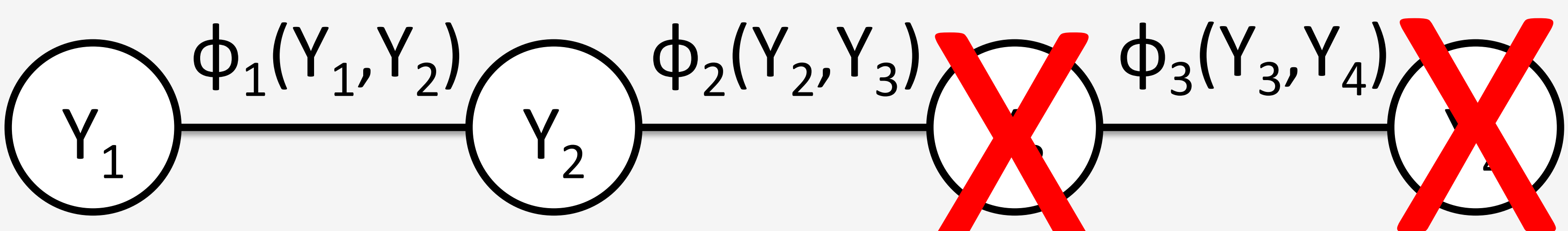
$F'_3 = \{ \quad \}$

$$\tau_3(Y_1) = \sum_{y_2} \psi_3(Y_1, y_2) = \sum_{y_2}$$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9

=

Y_1	0
0	63
1	45



Step 3: Eliminate Y_2

$V = \{Y_2\}$

$F_3 =$

$\phi_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\tau_2(Y_2)$

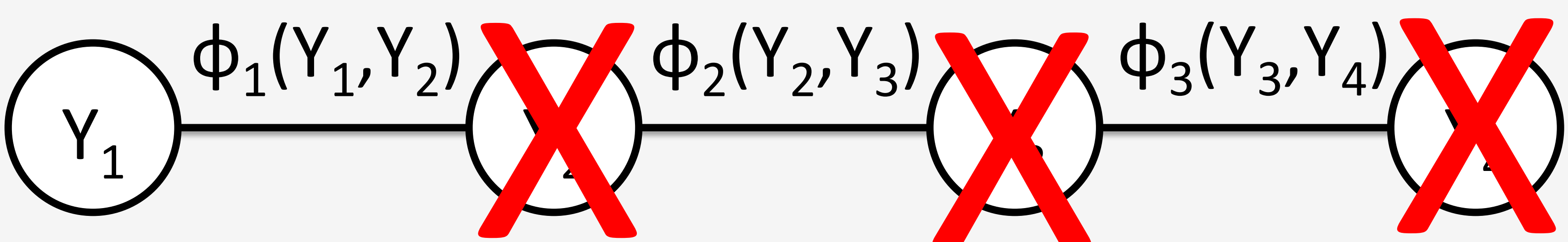
Y_2	0
0	18
1	9

$F'_3 = \{\ \}$

$F = F'_3 \cup \{\tau_3(Y_1)\} =$

$\tau_3(Y_1)$

Y_2	0
0	63
1	45



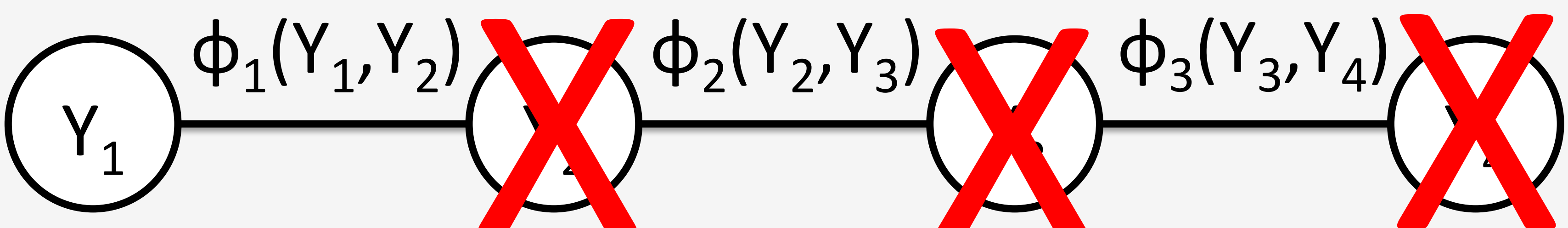
Step 4: Return

Return

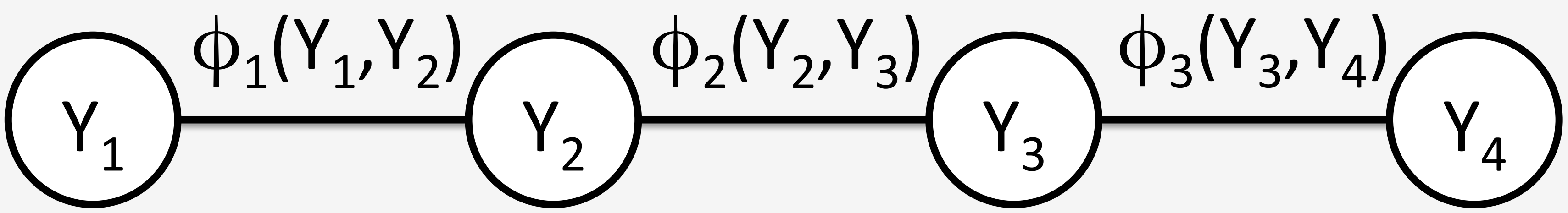
$\tau_3(Y_1)$

Y_2	0
0	63
1	45

$$P(Y_1=1) = \frac{\tau_3(1)}{\tau_3(0) + \tau_3(1)} = 0.4166$$



Question: What happens to the complexity of the algorithm if we start by eliminating Y_3 instead of Y_4?



Step 1: Eliminate Y_3

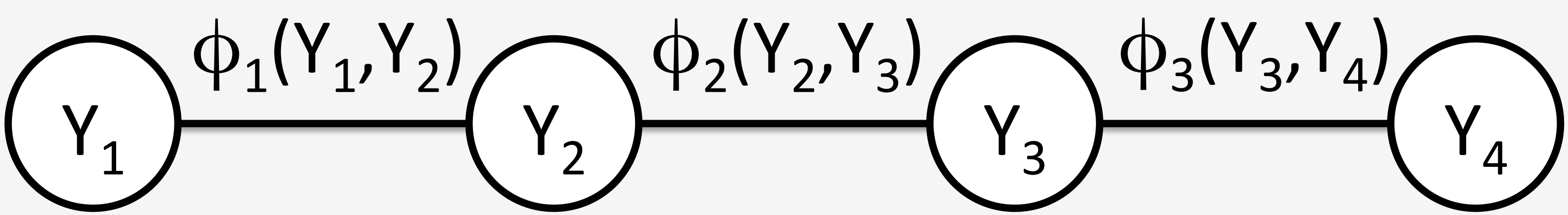
$V = \{Y_3, Y_4, Y_2\}$

$F_0 = \{\phi_1(Y_1, Y_2), \phi_2(Y_2, Y_3), \phi_3(Y_3, Y_4)\}$

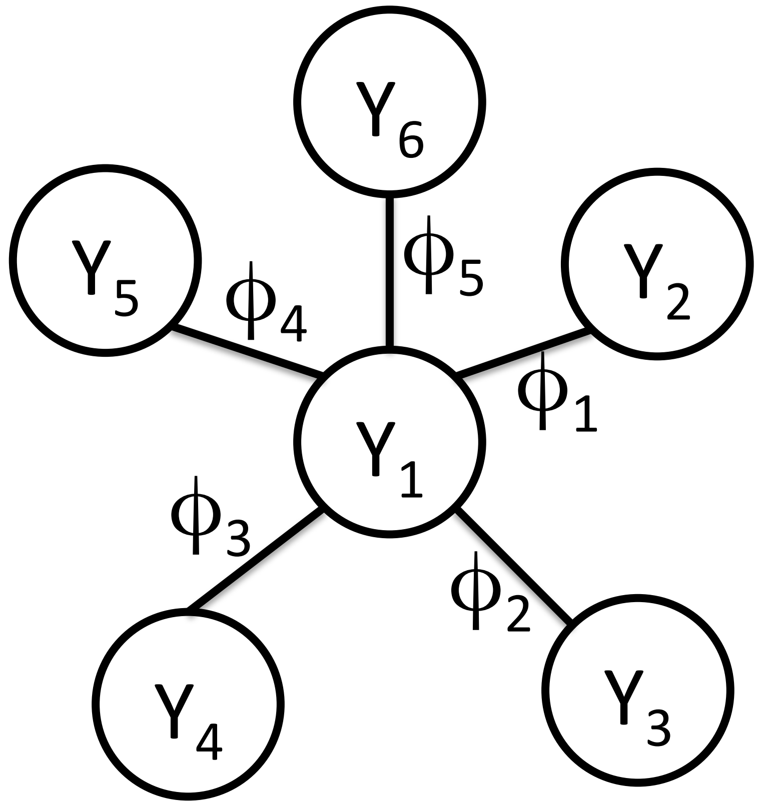
$F_1 = \{\phi_2(Y_2, Y_3), \phi_3(Y_3, Y_4)\}$ $\quad F'_1 = \{\phi_1(Y_1, Y_2)\}$

$$\psi_1(Y_2, Y_3, Y_4) = \prod_{\phi \,\varepsilon\, F_1} \phi = \phi_2(Y_2, Y_3) \times \phi_3(Y_3, Y_4)$$

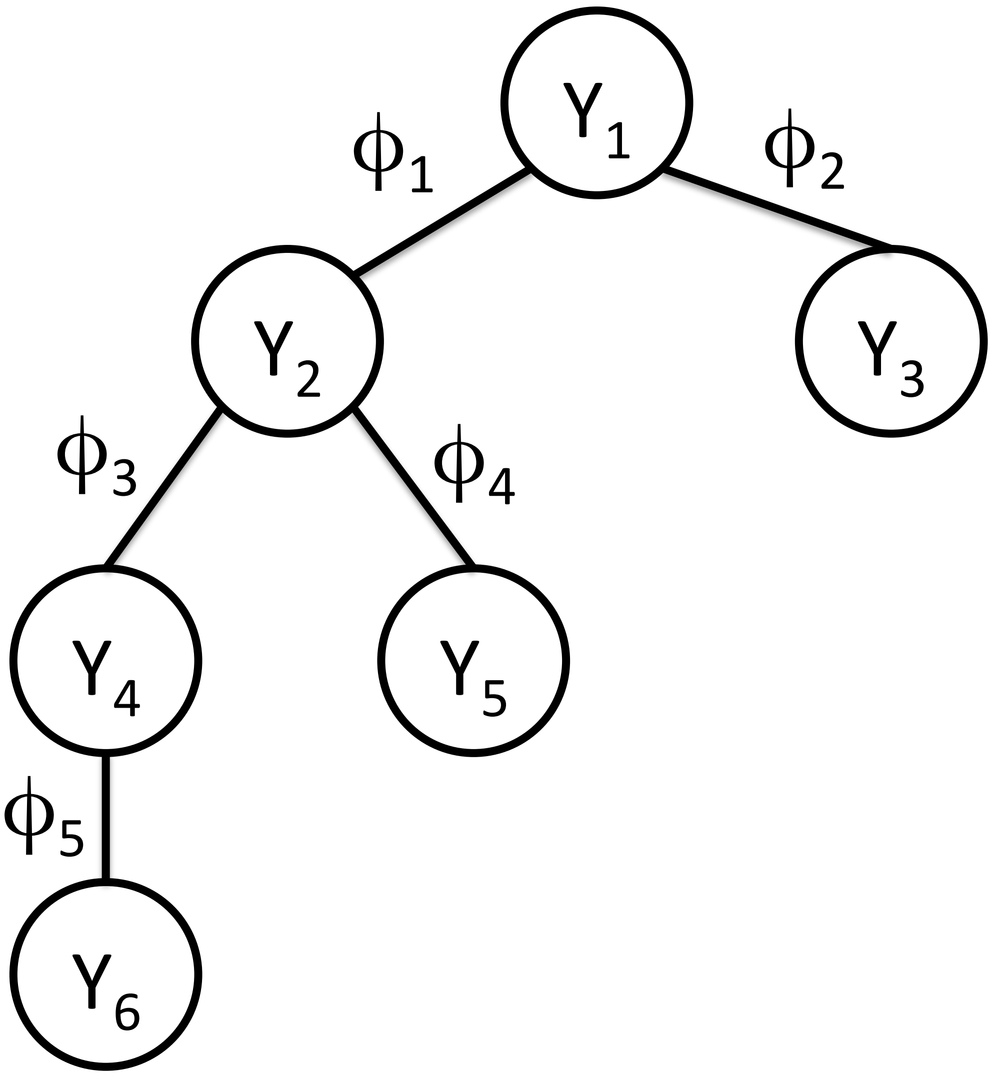
$$\tau_1(Y_2, Y_4) = \sum_{y_3} \psi_1(Y_2, y_3, Y_4) \quad F = \{\phi_1(Y_1, Y_2), \tau_1(Y_2, Y_4)\}$$



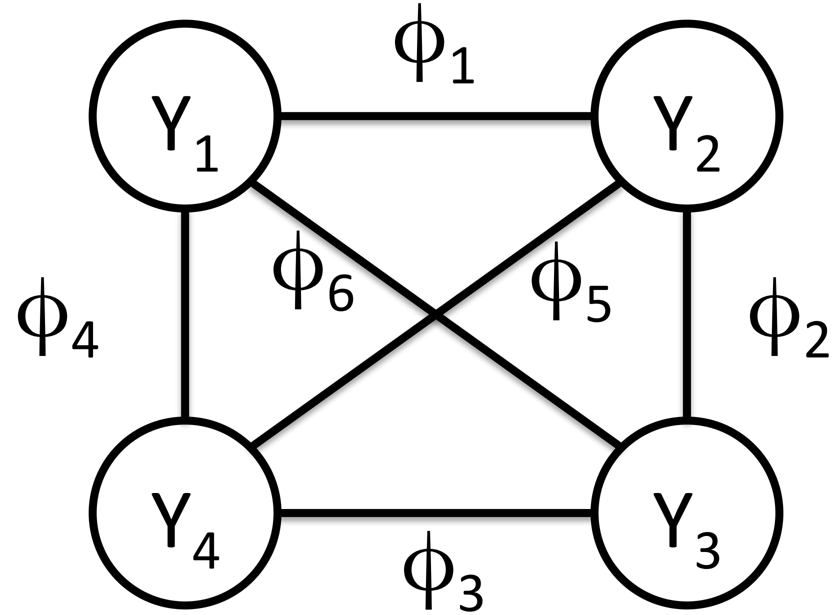
Question: What is the worst elimination ordering for this graph? What is an optimal ordering?



Question: What is an optimal elimination ordering for this graph?



Question: Is there any efficient elimination ordering for the following graph:

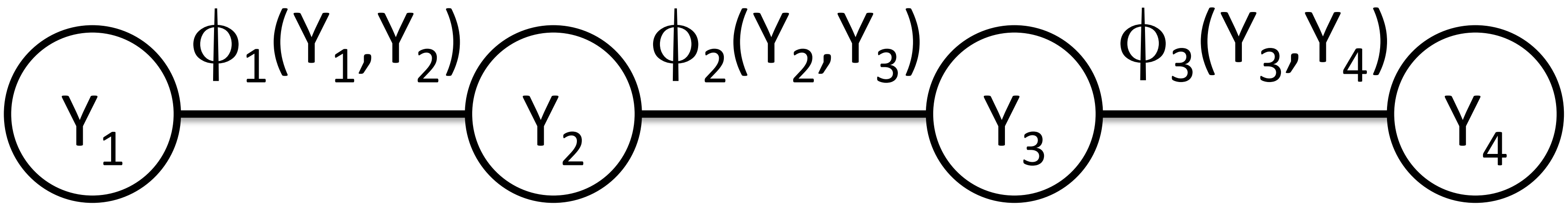


For any elimination ordering we have:

$F_1 = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$ $\quad F'_1 = \{\}$

$$\psi_1(Y_1, Y_2, Y_3, Y_4) = \prod \phi = \phi_1 \times \phi_2 \times \phi_3 \times \phi_4 \times \phi_5 \times \phi_6$$

Example: Let's trace the sum-product algorithm for the graph and factors shown below.

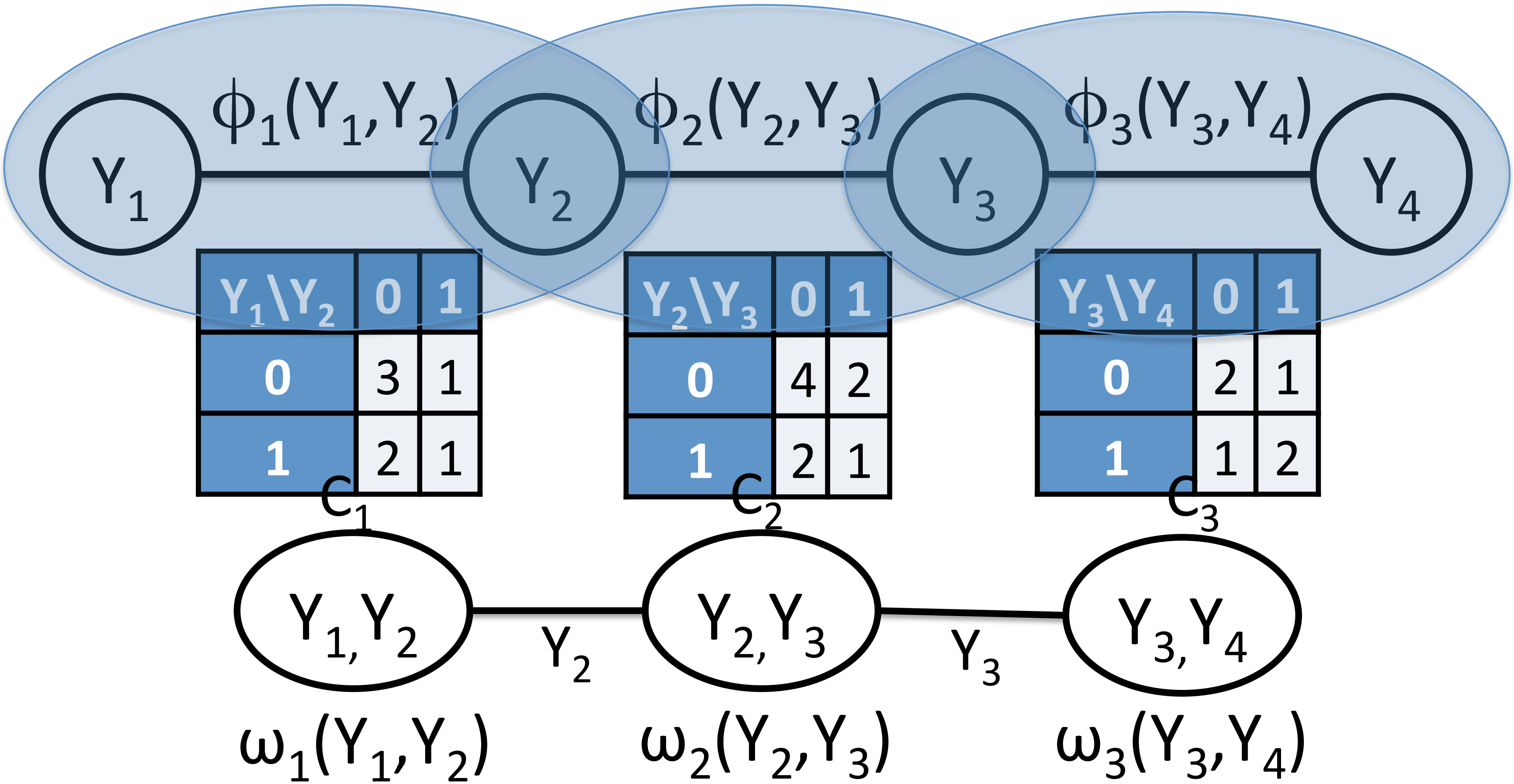


$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product: Forming the Clique Tree



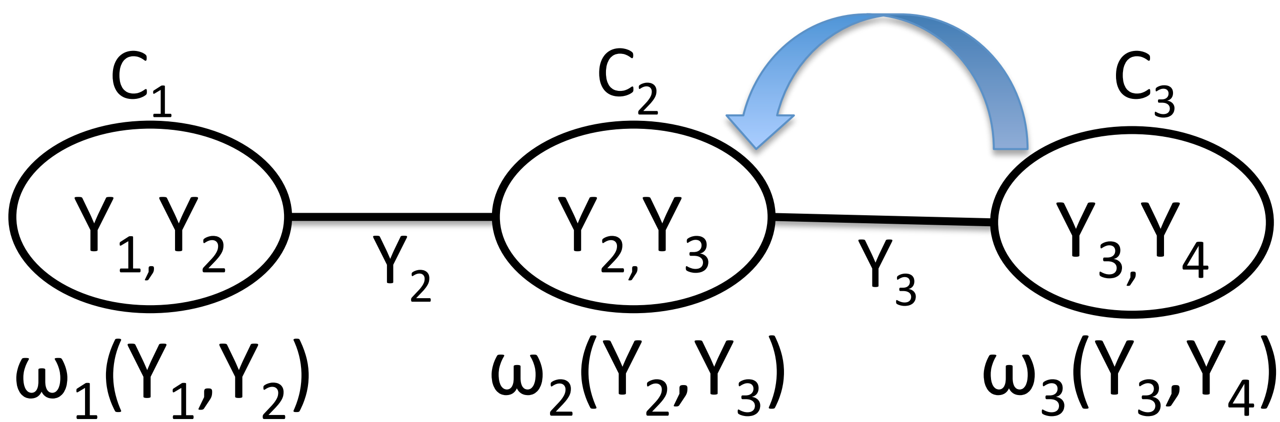
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: 3→2 Message

$$\delta_{3\to 2}(Y_3) \leftarrow \sum_{Y_4} \omega_3(Y_3,Y_4)$$



$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

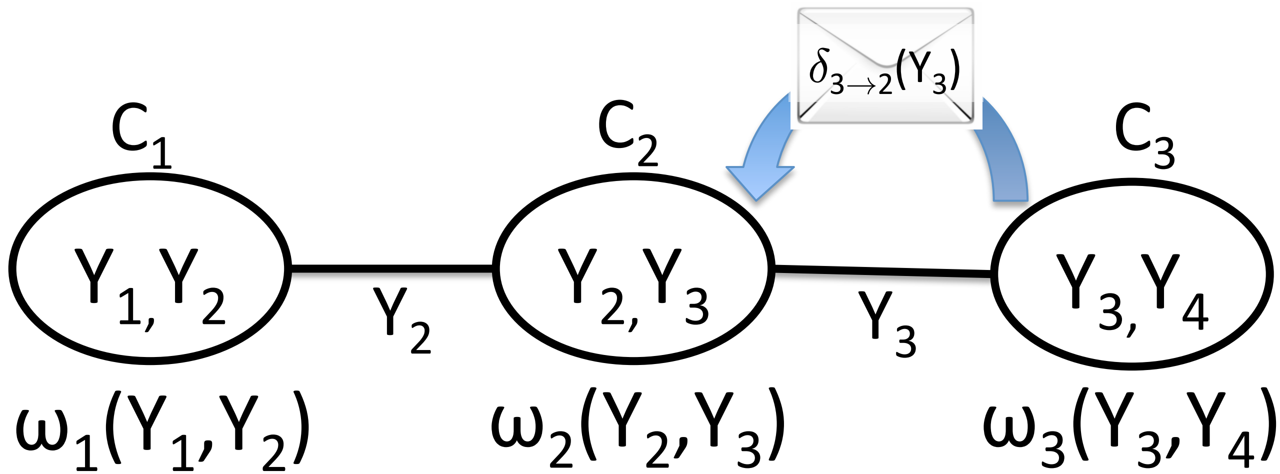
Sum-Product Algorithm: 3→2 Message

$$\delta_{3\to2}(Y_3) \leftarrow \sum_{y_4}$$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

=

Y_3	0
0	3
1	3



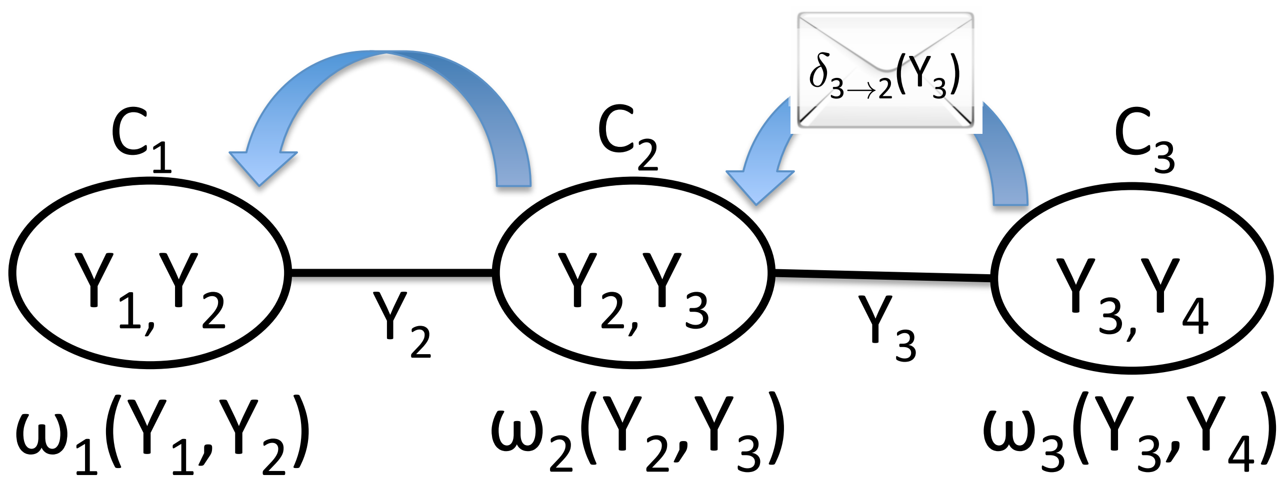
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: 2→1 Message

$$\delta_{2\to1}(Y_2) \;\leftarrow\; \sum_{Y_3} \omega_2(Y_2,Y_3)\; \delta_{3\to2}(Y_3)$$



$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: 2→1 Message

$\delta_{2\to1}(Y_2) \leftarrow \sum_{y_3}$

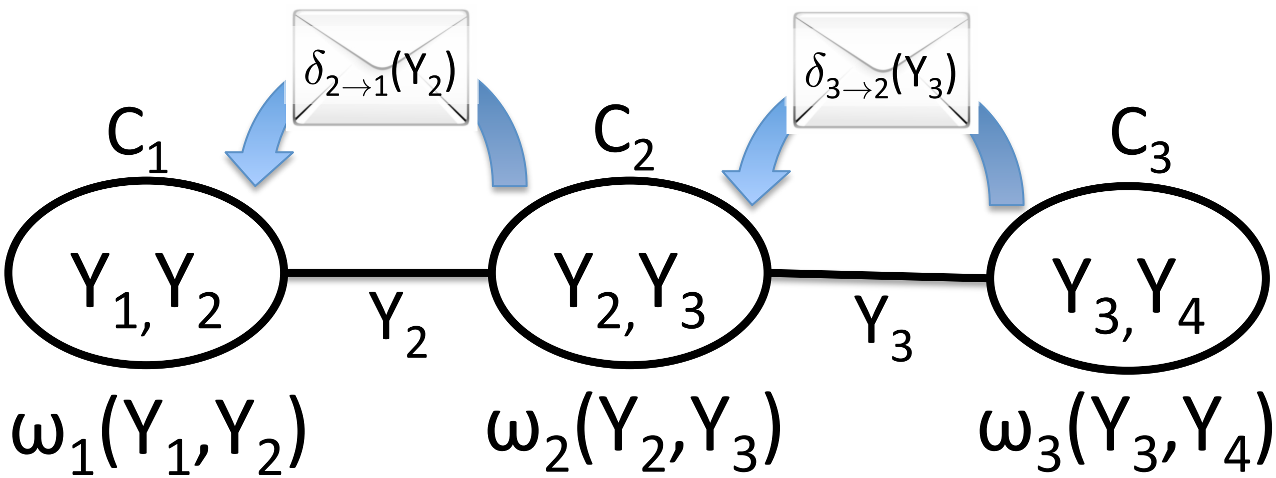
$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

×

Y_3	0
0	3
1	3

=

Y_2	0
0	18
1	9



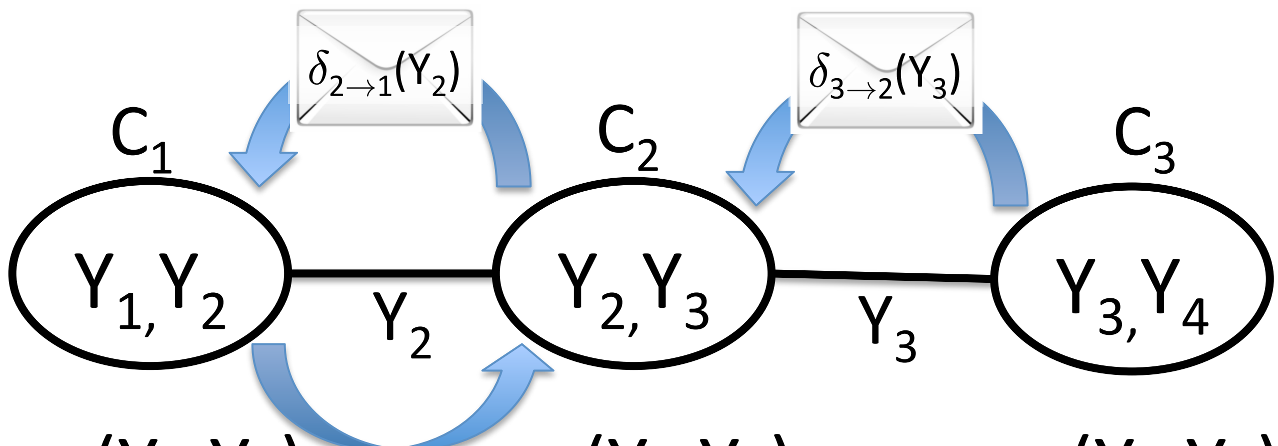
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: 1→2 Message

$$\delta_{1\to 2}(Y_2) \leftarrow \sum_{Y_1} \omega_1(Y_1,Y_2)$$



$\omega_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\omega_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\omega_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

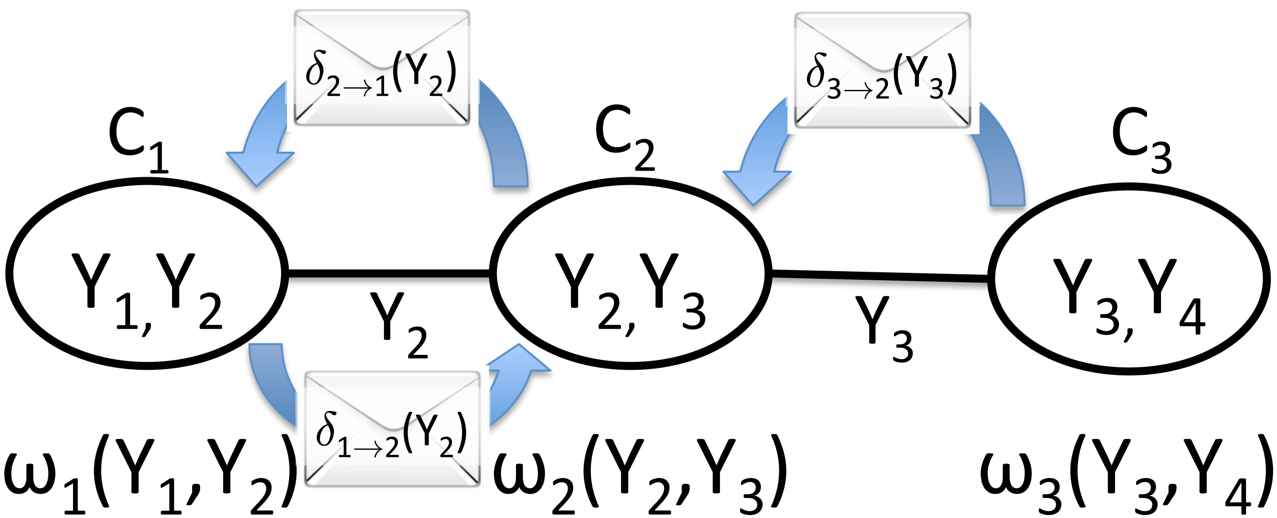
Sum-Product Algorithm: 1→2 Message

$$\delta_{1\to 2}(Y_2) \leftarrow \sum_{Y_1}$$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

=

Y_2	0
0	5
1	2



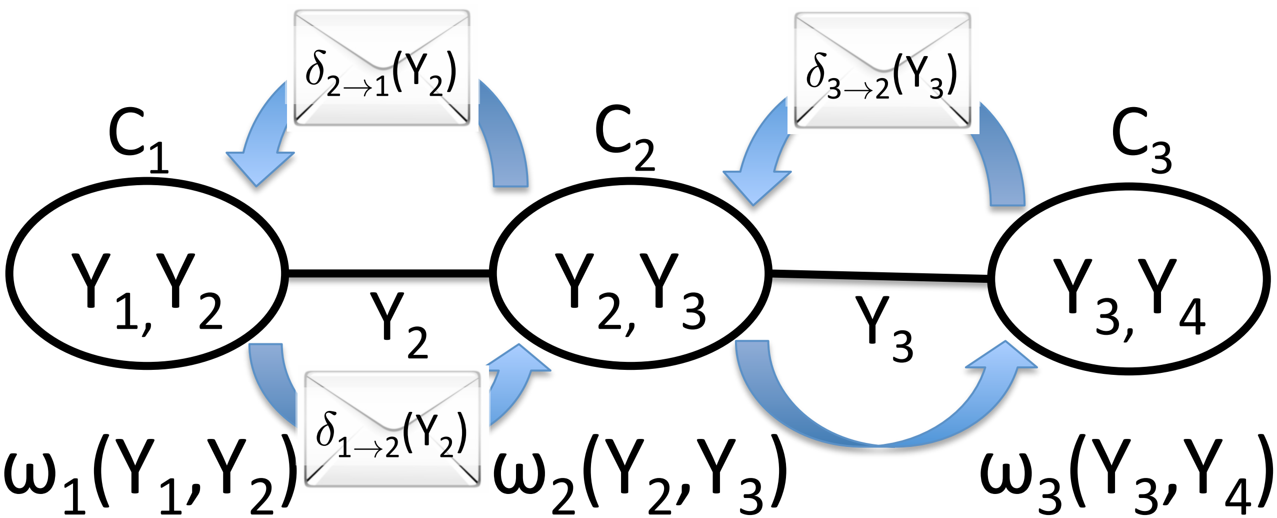
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: 2→3 Message

$$\delta_{2\to3}(Y_3) \leftarrow \sum_{Y_2} \omega_2(C_2)\delta_{1\to2}(Y_2)$$



$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: 2→3 Message

$$\delta_{2\to3}(Y_3) \leftarrow \sum_{Y_2}$$

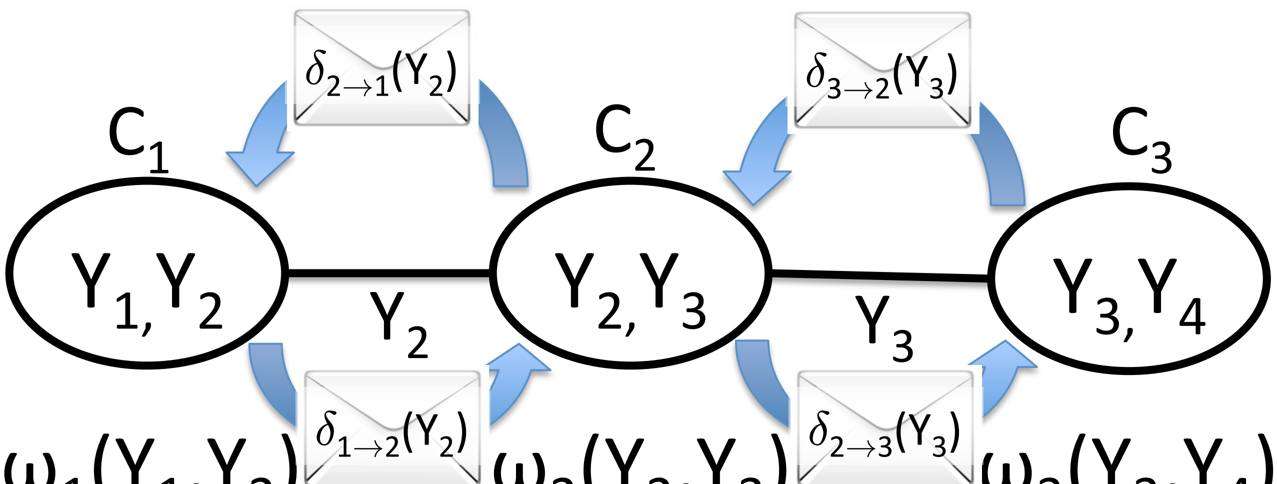
$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

×

Y_2	0
0	5
1	2

=

Y_3	0
0	24
1	12



$\omega_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\omega_2(Y_2,Y_3)$

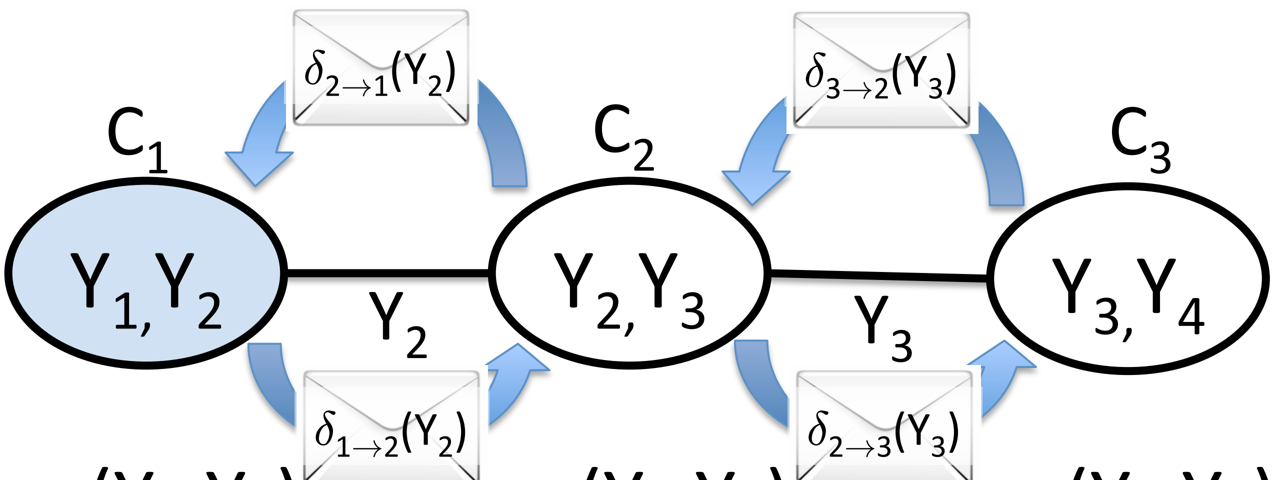
$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\omega_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: C_1 Belief Read-Out

$\beta_1(Y_1,Y_2) \leftarrow \omega_1(Y_1,Y_2)\ \delta_{2\to1}(Y_2)$



$\omega_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\omega_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\omega_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: C_1 Belief Read-Out

$\beta_1(Y_1, Y_2) \leftarrow$

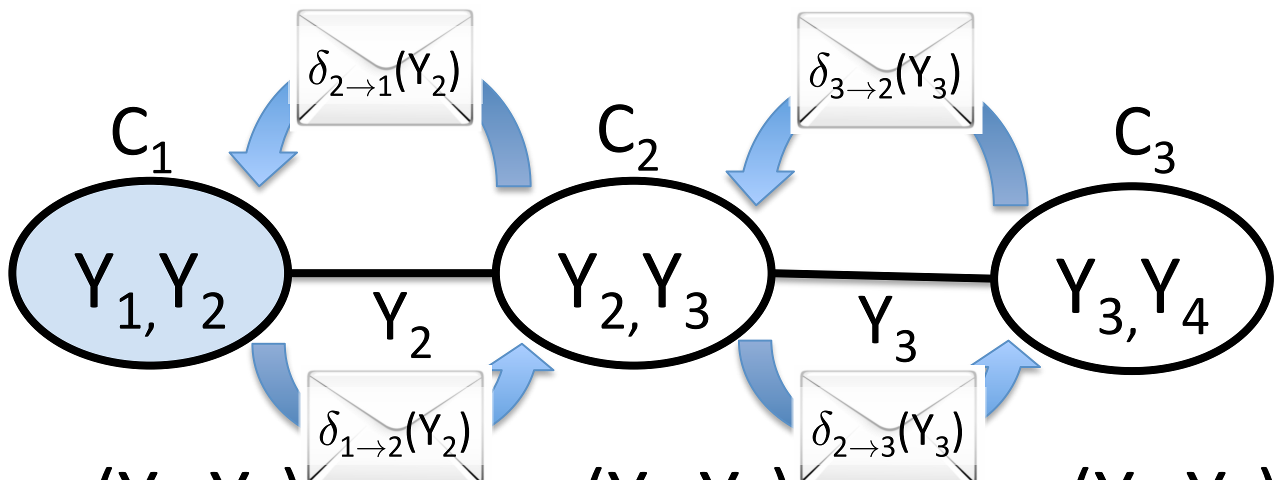
$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\times$

Y_2	0
0	18
1	9

$=$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9



$\omega_1(Y_1, Y_2)$

$Y_1 \backslash Y_2$	0	1
0	3	1
1	2	1

$\omega_2(Y_2, Y_3)$

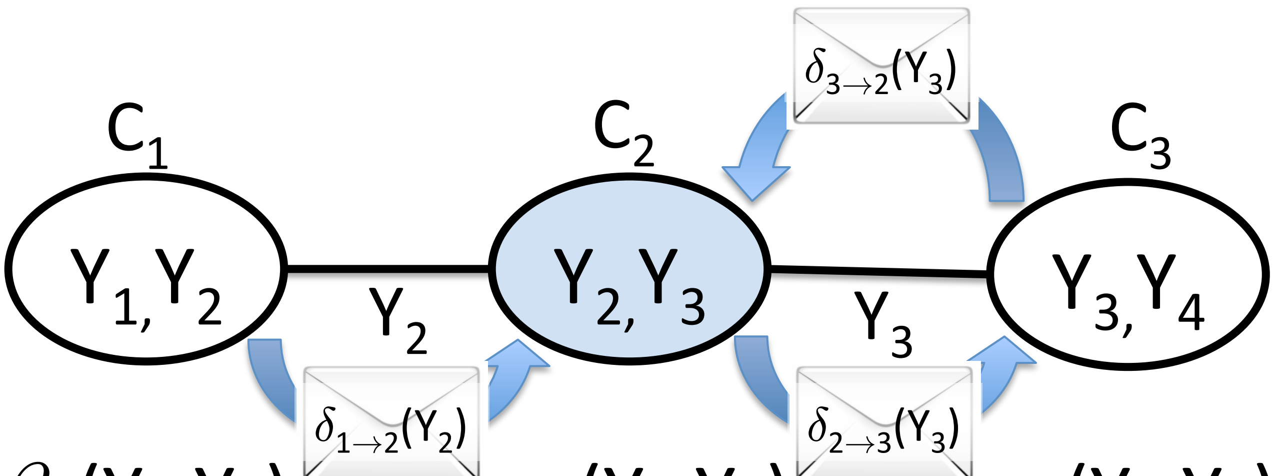
$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\omega_3(Y_3, Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: C_2 Belief Read-Out

$\beta(Y_2,Y_3) \leftarrow \omega_2(Y_2, Y_3)\ \delta_{1\rightarrow 2}(Y_2)\ \delta_{3\rightarrow}(Y_3)$



$\beta_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9

$\omega_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\omega_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: C_2 Belief Read-Out

$\beta(Y_2,Y_3) \leftarrow$

$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\times$

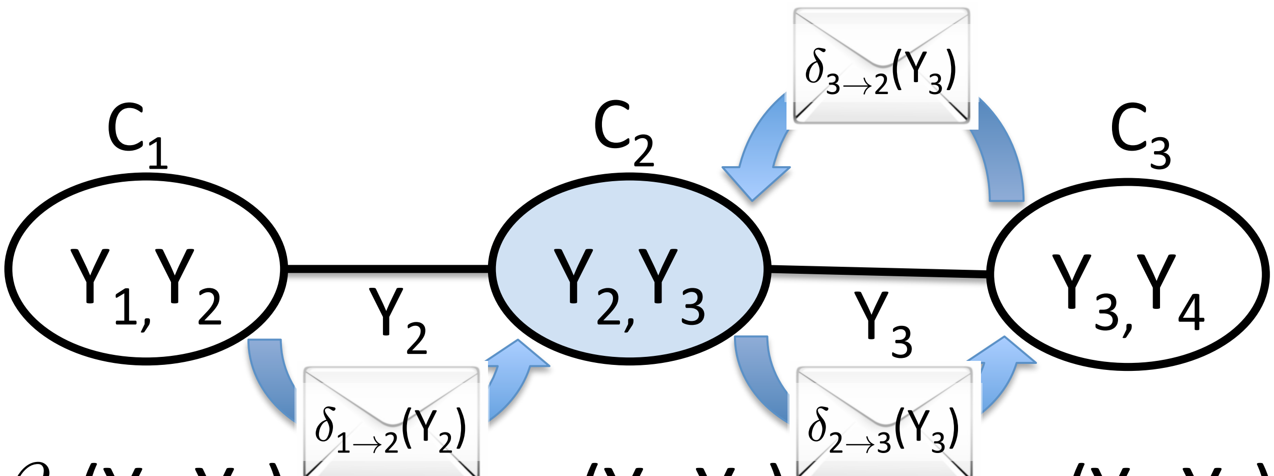
Y_2	0
0	5
1	2

$\times$

Y_3	0
0	3
1	3

$=$

$Y_2 \backslash Y_3$	0	1
0	60	30
1	12	6



$\beta_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9

$\omega_2(Y_2,Y_3)$

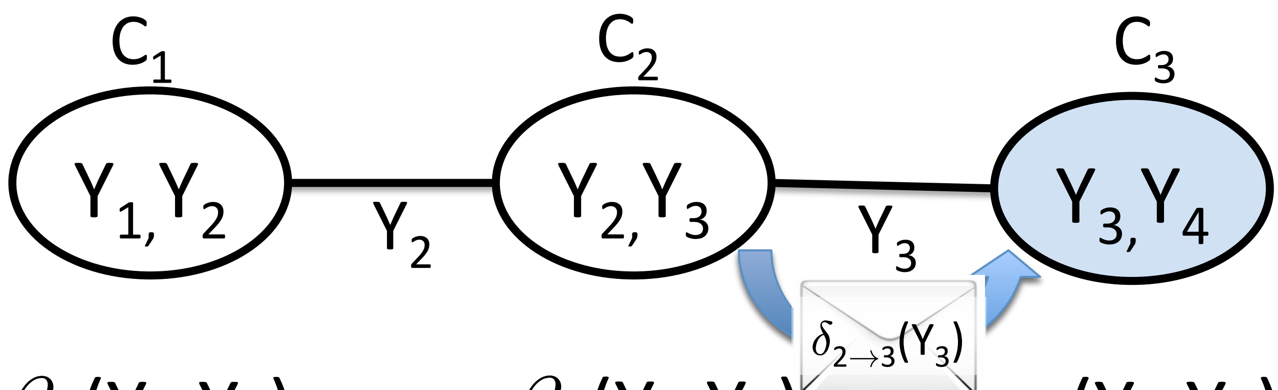
$Y_2 \backslash Y_3$	0	1
0	4	2
1	2	1

$\omega_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: C_3 Belief Read-Out

$$\beta_3(Y_3,Y_4) \leftarrow \omega_3(Y_3,Y_4)\ \delta_{2\to3}(Y_4)$$



$\beta_1(Y_1,Y_2)$

$Y_1\backslash Y_2$	0	1
0	54	9
1	36	9

$\beta_1(Y_2,Y_3)$

$Y_2\backslash Y_3$	0	1
0	60	30
1	12	6

$\omega_3(Y_3,Y_4)$

$Y_3\backslash Y_4$	0	1
0	2	1
1	1	2

Sum-Product Algorithm: C_3 Belief Read-Out

$\beta_3(Y_3,Y_4) \leftarrow$

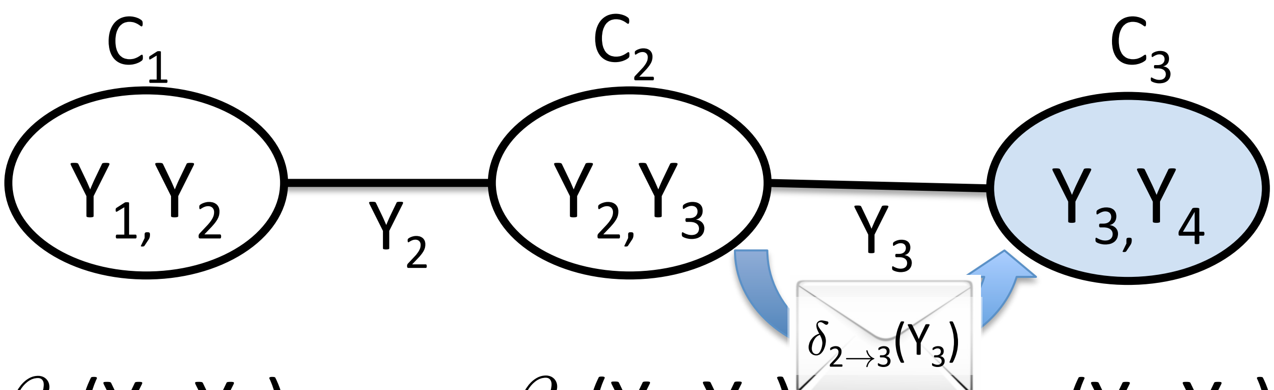
$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

$\times$

Y_3	0
0	24
1	12

$=$

$Y_3 \backslash Y_4$	0	1
0	48	24
1	12	24



$\beta_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9

$\beta_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	60	30
1	12	6

$\omega_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	2	1
1	1	2

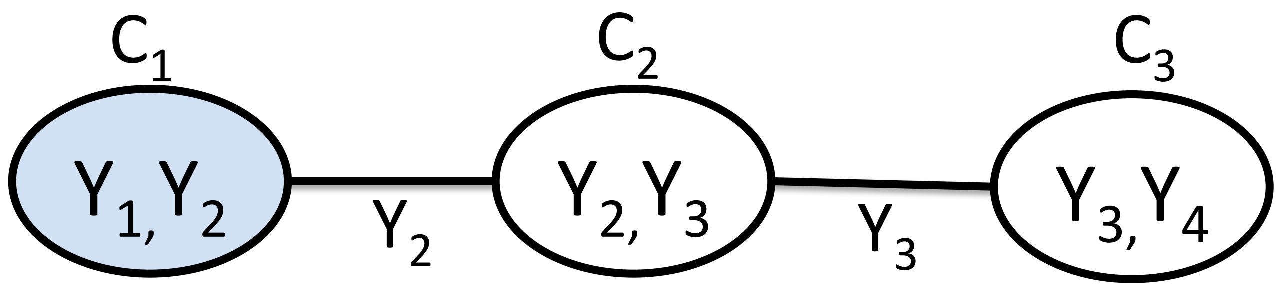
Sum-Product Algorithm: C_1 Probabilities

$$P(Y_1,Y_2) = \frac{\beta_1(Y_1,Y_2)}{\sum_{y_1}\sum_{y_2}\beta_1(y_1,y_2)} =$$

$Y_1 \backslash Y_2$	0	1
0	0.5000	0.0833
1	0.3333	0.0833

$P(Y_1=1) = 0.4166$

$P(Y_2=1) = 0.1666$



$\beta_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9

$\beta_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	60	30
1	12	6

$\beta_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	48	24
1	12	24

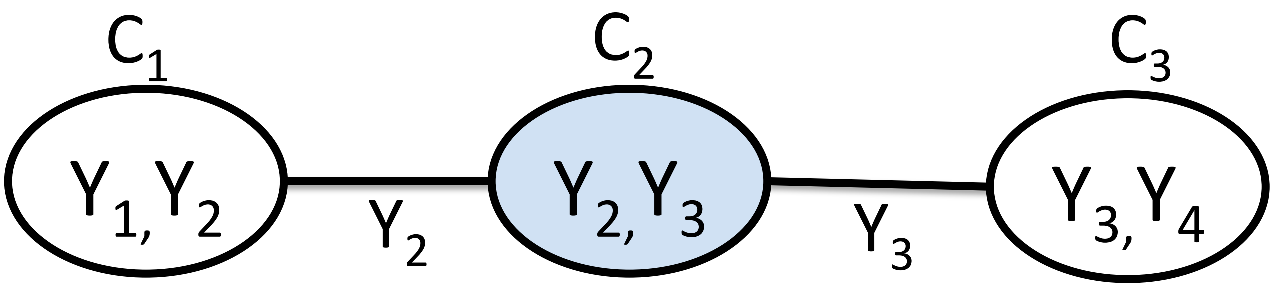
Sum-Product Algorithm: C_2 Probabilities

$$P(Y_2,Y_3) = \frac{\beta_2(Y_2,Y_3)}{\sum_{y_2}\sum_{y_3}\beta_2(y_2,y_3)} =$$

$Y_2 \backslash Y_3$	0	1
0	0.5555	0.2777
1	0.1111	0.0555

$P(Y_2=1) = 0.1666$

$P(Y_3=1) = 0.3333$



$\beta_1(Y_1,Y_2)$

$Y_1 \backslash Y_2$	0	1
0	54	9
1	36	9

$\beta_2(Y_2,Y_3)$

$Y_2 \backslash Y_3$	0	1
0	60	30
1	12	6

$\beta_3(Y_3,Y_4)$

$Y_3 \backslash Y_4$	0	1
0	48	24
1	12	24

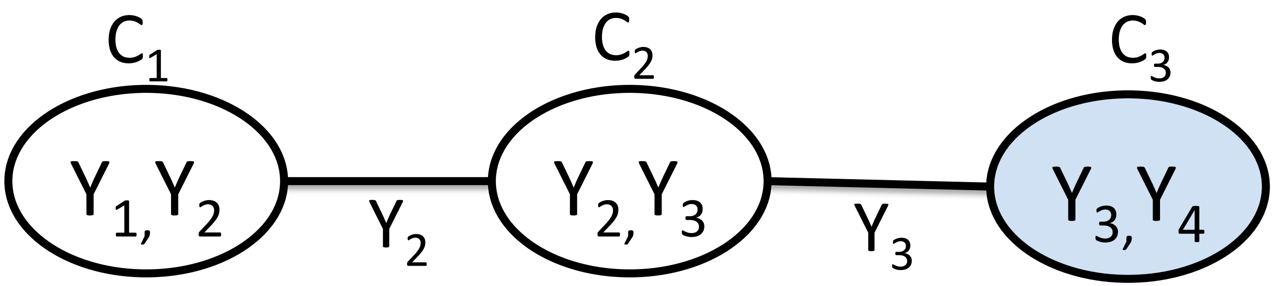
Sum-Product Algorithm: C_3 Probabilities

$$P(Y_3,Y_4) = \frac{\beta_3(Y_3,Y_4)}{\sum_{y_3}\sum_{y_4}\beta_3(y_3,y_4)} =$$

$Y_3\backslash Y_4$	0	1
0	0.4444	0.2222
1	0.1111	0.2222

$P(Y_3{=}1) = 0.3333$

$P(Y_4{=}1) = 0.4444$



$\beta_1(Y_1,Y_2)$

$Y_1\backslash Y_2$	0	1
0	54	9
1	36	9

$\beta_2(Y_2,Y_3)$

$Y_2\backslash Y_3$	0	1
0	60	30
1	12	6

$\beta_3(Y_3,Y_4)$

$Y_3\backslash Y_4$	0	1
0	48	24
1	12	24

Sum-Product: Algorithm Overview

$$\delta_{3\to 2}(Y_3) \leftarrow \sum_{Y_4} \omega_3(Y_3,Y_4)$$

$$\delta_{1\to 2}(Y_2) \leftarrow \sum_{Y_1} \omega_1(Y_1,Y_2)$$

$$\delta_{2\to 1}(Y_2) \leftarrow \sum_{Y_3} \omega_2(Y_2,Y_3)\ \delta_{3\to 2}(Y_3)$$

$$\delta_{2\to 3}(Y_3) \leftarrow \sum_{Y_2} \omega_2(C_2)\delta_{1\to 2}(Y_2)$$

$$\beta_1(Y_1,Y_2) \leftarrow \omega_1(Y_1,Y_2)\ \delta_{2\to 1}(Y_2)$$

$$\beta_2(Y_2,Y_3) \leftarrow \omega_2(Y_{2,}\ Y_3)\ \delta_{1\to 2}(Y_2)\ \delta_{3\to}(Y_3)$$

$$\beta_3(Y_3,Y_4) \leftarrow \omega_3(Y_3,Y_4)\ \delta_{2\to 3}(Y_4)$$

$$P(Y_i,Y_{i+1}) = \frac{\beta_i(Y_i,Y_{i+1})}{\sum_{y_i}\sum_{y_{i+1}} \beta_i(y_i,y_{i+1})}$$